The Relevant Logic E and Some Close Neighbours: A Reinterpretation

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Abstract

This paper has two aims. First, it sets out an interpretation of the relevant logic E of relevant entailment based on the theory of situated inference. Second, it uses this interpretation, together with Anderson and Belnap's natural deduction system for E, to generalise E to a range of other systems of strict relevant implication. Routley–Meyer ternary relation semantics for these systems are produced and completeness theorems are proven.

Keywords: entailment, relevant logic, strict implication, situated inference, ternary relation semantics

1 Introduction

The logic E is supposed to be the logic of relevant entailment. E incorporates intuitions concerning both relevance and necessity. In the 1960s, Alan Anderson and Nuel Belnap constructed two central relevant logics. E and the logic of contingent relevant implication, R. They viewed the entailment connective of E as the strict version of the implication of R. To show that the two logics had this relationship,

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Robert Meyer constructed a modal version of R, $R^{\Box}$, that adds to R a necessity operator and relevant versions of the axioms for S4 [13]. It was hoped that the conjunction, disjunction, negation, and strict implication fragment of $R^{\Box}$ was the same as E. But Larisa Maksimova showed that these two logics are distinct [10].

In *Entailment*, volume 1, Anderson and Belnap wrote:

> we predict that if in fact it is found that $R^{\Box}$ and E diverge, then we shall, with many a bitter tear, abandon E. [1, p. 351]

The logic E has not, however, been completely abandoned. It continues to be studied. *Entailment* volume 2 has both very interesting technical and historical information about E, including Belnap's elegant display logic proof theory for it [2, §62.5.3]. Mark Lance defends E over R as the central relevant logic [8] and Lance and Philip Kremer have developed a theory of linguistic commitment that had E as its logic [9]. Despite all of this, however, E seems to be ignored by most contemporary relevant logicians.

In this paper, we focus on E and some logics that closely resemble it. We give E an interpretation based on the theory of situated inference of [11] and generalise the intuitions behind this interpretation to develop a small class of *entailment logics*. These entailment logics are formulated first in terms of Fitch-style natural deduction systems which make clear both the components of relevance and modality incorporated into them. The logics are then formulated in terms of traditional axiom systems and these are shown complete with respect to classes of Routley–Meyer ternary relation models. The indices of these models are taken to be situations and the ternary relations in these models are interpreted in terms of the theory of situated inference.

The logics that we examine in this paper are negation-free. Although negation is easily added to the semantics of these systems using the Routley star operator, available treatments of it in the natural deduction system are not illuminating in the way that we desire. We promise to investigate the role and representation of negation in entailment logics in a future paper, but we do not do so here.

The plan of the paper is as follows. We begin by reviewing the theory of situated inference as it is applied to the logic R. We show how this theory can be used to understand the Routley–Meyer semantics for that logic. We then modify the theory to apply to the logic E. We examine E through its axiomatic formulation, natural deduction system, and Routley–Meyer semantics. The situated inference interpretation of E employs both situations and worlds. This interpretation is then generalised to treat a small class of other systems that incorporate principles from various modal logics. An E-like logic that is similar to the modal logic K, which we call E.K is explored, and so are its extensions E.KT, E.K4, and E.KT4 (which is just E itself). We end by exploring a suggestion of Alasdair Urquhart for an axiomatisation of an S5-ish entailment logic, E5.

2 Situated Inference

Routley–Meyer models for relevant logics are indexical models, that is, in them the truth or falsity of formulas is relativised to points. We call these points *situations.* A situation is a potential part or state of a possible world. Some situations actually obtain in some worlds, and so can be called possible situations and some do not and can be thought of as impossible situations. Impossible situations are important for the analysis of negation, which is not our main concern here and so we will not mention impossible situations further.

A situation contains or fails to contain particular pieces of information. For example, a situation that includes all the information available at a given time in a lecture room (in which no one is connected to the internet) may not contain information about the weather outside or about the current polls in the American presidential race. Whether a situation satisfies a formula in a model is given by an information condition rather than a truth condition. These information conditions abstract from the canonical ways in which information is made available in actual situations. For example, the way in which we usually tell that an object is not red is that it is of some colour that is incompatible with its being red. The information condition for negation is a more general representation of this sort of information condition. It states that $a \vDash \neg A$ if and only if a is incompatible with any situation b that satisfies A. This means that a contains the information that there is no situation in the same world as a that contains the information that A.

One feature of the informational interpretation of relevant logic is that the satisfaction conditions for the connectives need not be homomorphic. As we can see, we do not in every case set $a \vDash \neg A$ iff $a \nvDash A$. The requirement that satisfaction conditions be homomorphisms between the object language and semantic metalanguage does not hold for information. The way in which we find information structured in our environment need not mirror the structure of way that we express that information linguistically.

The main focus of this paper is the notion of entailment. We approach it through the closely related concept of relevant implication. As we have said, Anderson and Belnap think of entailment as modalised implication and implication as a contingent form of entailment. The Routley–Meyer satisfaction condition for implication is

$$a \vDash A \to B \quad \text{iff} \quad \forall x \forall y ((Raxy \wedge x \vDash A) \supset y \vDash B).$$

The information in one situation can be applied to the information in another situation. For example, if $C \to D$ is in a and C is in b, then combining this information we obtain D. One way of understanding this condition is to think of the information in a combined with the information in b in this way always results in information that is in c.[1]

The theory of situated inference explains why one would care about combining information in this way. Suppose that one is in a situation a and in a world w. She might hypothesise that another situation b also exists in w. Then she can combine the information in a and b to determine that other sort of situations are in w. For example, suppose that a person has available to him in situation a information that there are absolutely no ticks in New Zealand. Then, on the hypothesis that a particular woodland is in New Zealand, he has the licence to infer that there are no ticks in it. The theory of situated inference breaks this inference as being an inference from there being one situation in which the park is in New Zealand and (perhaps) another in which it contains no ticks. The theory of situated inference connects relevant implications in a direct way with ordinary inferences.

What the theory of situated inference tells us is that the ternary relation R relates a and b to a set of situations (call it Rab), such that given the information in a, on the hypothesis that b is in the same world as a, there is a situation c in Rab also in that world. We sometimes say that $Rabc$ says that some situation *like* c can be inferred from that application of a to b. The word 'like' here is not being used in any technical sense. It is just an abbreviation to say that c is one of the set of situations that contains all the information that can be inferred from the combination of the information in a with that in b.

The demodalised nature of relevant implication, as it is characterised by the logic R, is made explicit in the theory of situated inference by the reading of situated inferences as being inferences about situations and information all contained in a single world. As we have said $Rabc$ means that the hypothesis of a and b in the same world allows us to infer the existence of a situation like c in that world. The reading of inferences as being in a single world, together with a rather liberal notion of application, enables justifications of certain particular postulates of the Routley–Meyer semantics for R.

For example, consider the *permutation postulate* of the Routley–Meyer semantics for R: for any situations a, b, c, if $Rabc$ then $Rbac$. This postulate tells us that the

[1] A rather sophisticated reading of the notion of combination is in [3]. In that paper, combination is understood in terms of the application of one situation to another in the sense that functions are applied to arguments. The application reading of R works well for weaker relevant logics but can be used to interpret the logic R as well. It just takes quite a bit of effort to show how it fits with the Routley–Meyer semantics for R and so we do not use it here.

result of combining the information in a with that in b is the same as combining the information in b with that in a. This seems natural as the common notion of combination is symmetric. As we shall see later, however, in the semantics for E the permutation postulate fails, and it fails (on our reading) because E is a modal logic.

The permutation postulate makes valid the thesis of *assertion*: $A \to ((A \to B) \to B)$.

Derivation 1. The following is a proof of assertion in the Anderson–Belnap Fitch system for R:

1	A_1	hypothesis
2	$A \to B_2$	hypothesis
3	$A_{\{1\}}$	1, reiteration
4	$B_{\{1,2\}}$	2, 3, $\to$E
5	$(A \to B) \to B_{\{1\}}$	2–4, $\to$I
6	$A \to ((A \to B) \to B)_{\emptyset}$	1–5, $\to$I

The subscripts are to be understood as referring to situations. When we make a hypothesis, say, A_1, we are postulating the existence of a situation, say, a_1 that contains the information that A. When the subscript is the empty set, the formula is proven to hold in every *normal situation*. (We discuss normal situations in Section 4.) The expression $B_{\{1,2\}}$ is read as saying that an arbitrary situation c that is in the result of combining the information in a_1 with that in a_2. The use of permutation can be seen in this proof through the fact that it does not matter when a number is added to the subscript (in an application of the rule of implication elimination) nor when it is removed (in an application of implication introduction) which order the numbers are in. If we were to reject permutation, the subscript the minor premise in an implication elimination would have to be added at the end of the new subscript and likewise, when a hypothesis is discharged its number could only be removed from the end of the subscript. Having permutation allows us to commute the order the numbers in subscripts.

We can generalise the R relation to be an n-place relation for any positive integer n by taking products of R. We say that $Rabcd$ if and only if $\exists x(Rabx \wedge Rxcd)$ and more generally (for $n \geq 3$) $Ra_1 \dots a_n c$ if and only if $\exists x(Ra_1 \dots a_{n-1}x \wedge Rxa_n c)$. We read $Ra_1 \dots a_n c$ as saying that the hypothesis of $a_1, \dots, a_n$ all in the same world justifies the inference to there being a situation like c also in that world. The generalised R relation and its interpretation justifies a more general permutation postulate: if $Ra_1 \ldots a_m a_{m+1} \ldots a_n c$, then $Ra_1 \ldots a_{m+1} a_m \ldots a_n c$, for any m, $1 \leq m \leq n$. This generalised permutation postulate justifies the derivation of various theses such as the permutation of antecedents $((A \to (B \to C)) \to (B \to (A \to C)))$.

Before we leave R, let us look at a semantic postulate that is related to situated inference in a more complicated manner: the *contraction postulate.* The simple version of the contraction postulate says

$$Rabc \implies Rabbc.$$

The generalised version of contraction says that $Ra_1 \ldots a_m \ldots a_n c$ implies that $Ra_1 \ldots a_m a_m \ldots a_n c$. This generalisation follows from the simple version. We read contraction as saying that if we hypothesise that $a_1, \ldots, a_m, \ldots, a_n$ in a world to infer that there is a situation like c is also present in that world, in an inference we can really use the information in a_m twice as part of the inference the presence of a situation like c. We will return to the topic of contraction in our discussion of entailment logics weaker than E.

3 E

This paper is not about R, but about the logic of relevant entailment, E and some similar systems. The implication of E is usually understood as a form of strict relevant implication. One way of thinking about strict relevant implication is through combining relevant implication with modality. If we think of it that way, then it is natural to represent strict relevant implication in a modal extension of R. But we suggest that the notion of entailment be considered a unified notion that has the properties of being relevant and necessary.

The obvious difference between R and E is in their conditionals. The conditional of R is a contingent implication and that of E is entailment. We can compare the two logics in terms of axiomatisations of their conditional only fragments, $R_\to$ and $E_\Rightarrow$. Here are some axiom schemes that, together with the usual modus ponens rule, generate $E_\Rightarrow$:

1. $A \Rightarrow A$ (Identity);

2. $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ (Suffixing);

3. $(B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ (Prefixing);

4. $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$ (Contraction);

5. $((A \Rightarrow A) \Rightarrow B) \Rightarrow B$ (EntT).

The axiom EntT tells us that if any formula is entailed by a theorem it is true. It is, in effect, a form of the T-axiom from modal logic that tells us that any necessary formula is true. For contrast, consider a set of axioms for $R_{\rightarrow}$:

R1 $A \rightarrow A$ (Identity);

R2 $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$ (Suffixing);

R3 $(A \rightarrow (A \rightarrow B)) \rightarrow (A \rightarrow B)$ (Contraction);

R4 $(A \rightarrow (B \rightarrow C)) \rightarrow (B \rightarrow (A \rightarrow C))$ (Permutation of Antecedents).

The permutation axiom of $R_{\rightarrow}$ "demodalises" its implication. From $(A \rightarrow B) \rightarrow (A \rightarrow B)$, which is an instance of identity, it allows us infer that $A \rightarrow ((A \rightarrow B) \rightarrow B)$. The latter clearly makes $\rightarrow$ into a non-strict form of implication. If we read $\rightarrow$ as $\prec$, even in the sense of S5, this formula is not a logical truth.

The axiomatic basis for conjunction and disjunction are the same for both logics, if we replace $\Rightarrow$ with $\rightarrow$ throughout to obtain the axioms for positive R:

7. $A \Rightarrow (A \vee B)$; $\quad B \Rightarrow (A \vee B)$;
8. $(A \wedge B) \Rightarrow A$; $\quad (A \wedge B) \Rightarrow B$;
9. $((A \Rightarrow B) \wedge (A \Rightarrow C)) \Rightarrow (A \Rightarrow (B \wedge C))$;
10. $((A \Rightarrow C) \wedge (B \Rightarrow C)) \Rightarrow ((A \vee B) \Rightarrow C)$;
11. $(A \wedge (B \vee C)) \Rightarrow ((A \wedge B) \vee (A \wedge C))$.

The rules for positive R and positive E are just modus ponens and adjunction.

E also has the axiom,

$$(\blacksquare A \wedge \blacksquare B) \Rightarrow \blacksquare(A \wedge B) \text{ (Agg}\blacksquare\text{)}$$

The axiom Agg$\blacksquare$ (Aggregation for $\blacksquare$) is a translation of the usual aggregation thesis into the idiom of E. Here $\blacksquare A$ is defined as $(A \Rightarrow A) \Rightarrow A$.

We also add the Ackermann constant t, to facilitate the formulation of another, but more easily used, notion of necessity, $\Box$. This notion of necessity is defined as follows:

$$\Box A =_{\text{df}} t \Rightarrow A$$

The operator $\blacksquare$ is extremely difficult to use in proofs. Consider, for example, the axiom Agg$\blacksquare$. Written in primitive notation it is $((((A \Rightarrow A) \Rightarrow A)) \wedge ((B \Rightarrow B) \Rightarrow B)) \Rightarrow (((A \wedge B) \Rightarrow (A \wedge B)) \Rightarrow (A \wedge B))$. Using this formula to prove other modal theses can be quite difficult. Agg$\Box$, i.e., $(\Box A \wedge \Box B) \Rightarrow \Box(A \wedge B)$, however, is just $((t \Rightarrow A) \wedge (t \Rightarrow B)) \Rightarrow (t \Rightarrow (A \wedge B))$, and this is just an instance of axiom 9.

The axiom and rule for t are the following:

12. $(t \Rightarrow A) \Rightarrow A$ (Tt).

Rule Nt (necessitation for t)

$$\frac{\vdash A}{\vdash t \Rightarrow A}$$

$\vdash t$ follows from axiom 1, i.e., $t \Rightarrow t$, and axiom 12, $(t \Rightarrow t) \Rightarrow t$.

In the context of E, $\Box$ and $\blacksquare$ are equivalent. Here is a proof.

Lemma 3.1. *In E, it is a theorem that* $\Box A \Leftrightarrow \blacksquare A$.

Proof. First, the left-to-right direction of the biconditional:

1.	$(t \Rightarrow A) \Rightarrow ((A \Rightarrow A) \Rightarrow (t \Rightarrow A))$	Suffixing
2.	$((t \Rightarrow A) \Rightarrow A) \Rightarrow (((A \Rightarrow A) \Rightarrow (t \Rightarrow A)) \Rightarrow ((A \Rightarrow A) \Rightarrow A))$	Prefixing
3.	$(t \Rightarrow A) \Rightarrow A$	Tt
4.	$((A \Rightarrow A) \Rightarrow (t \Rightarrow A)) \Rightarrow ((A \Rightarrow A) \Rightarrow A)$	2, 3, MP
5.	$(t \Rightarrow A) \Rightarrow ((A \Rightarrow A) \Rightarrow A)$	1, 4, Suffixing, MP
6.	$\Box A \Rightarrow \blacksquare A$	5, def $\Box$, def $\blacksquare$.

Now, the right-to-left direction:

1.	$t \Rightarrow (A \Rightarrow A)$	Axiom 1 and Nt
2.	$(t \Rightarrow (A \Rightarrow A)) \Rightarrow (((A \Rightarrow A) \Rightarrow A) \Rightarrow (t \Rightarrow A))$	Suffixing
3.	$((A \Rightarrow A) \Rightarrow A) \Rightarrow (t \Rightarrow A)$	1, 2, MP
4.	$\blacksquare A \Rightarrow \Box A$	3, def $\blacksquare$, def $\Box$.

Lemma 3.1 shows that Agg$\blacksquare$ is redundant in the logic with t. Moreover, the definition of $\blacksquare$ does not determine a modality with natural properties in some of the weaker systems we discuss later. These facts allow us to ignore $\blacksquare$ for the remainder of this paper.

Natural deduction proofs for E differ from those for R in the way in which subproofs are understood. In Anderson and Belnap's system [1, 2], only implicational formulas can be reiterated into subproofs. We modify that rule in order to produce proof systems for our other logics. We eliminate the reiteration rule altogether and change the implication eliminate proof to allow that the major premise be a previous step in a superior proof. We use $\Rightarrow$ for relevant entailment.

Derivation 2. The following is a derivation of the thesis of suffixing in the natural deduction system for E:

1	$\mid\ \underline{A \Rightarrow B_1}$	hypothesis
2	$\mid\ \mid\ \underline{B \Rightarrow C_2}$	hypothesis
3	$\mid\ \mid\ \mid\ \underline{A_3}$	hypothesis
4	$\mid\ \mid\ \mid\ B_{\{1,3\}}$	1, 3, ⇒E
5	$\mid\ \mid\ \mid\ C_{\{1,2,3\}}$	2, 4, ⇒E
6	$\mid\ \mid\ A \Rightarrow C_{\{1,2\}}$	3–5, ⇒I
7	$\mid\ (B \Rightarrow C) \Rightarrow (A \Rightarrow C)_{\{1\}}$	2–6, ⇒I
8	$(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))_{\emptyset}$	1–7, ⇒I

At step 4, the first hypothesis is used as the major premise of an implication elimination and the third hypothesis as its minor premise. We can think of scope lines as introducing new possible worlds at which situations indicated by the subscripts (as in proofs in R) hold. Thus, we can rewrite the above proof as:

1	$\mid\ \underline{A \Rightarrow B_1}$	w_1	hypothesis
2	$\mid\ \mid\ \underline{B \Rightarrow C_2}$	w_2	hypothesis
3	$\mid\ \mid\ \mid\ \underline{A_3}$	w_3	hypothesis
4	$\mid\ \mid\ \mid\ B_{\{1,3\}}$	w_3,	1, 3, ⇒E
5	$\mid\ \mid\ \mid\ C_{\{1,2,3\}}$	w_3	2, 4, ⇒E
6	$\mid\ \mid\ A \Rightarrow C_{\{1,2\}}$	w_2	3–5, ⇒I
7	$\mid\ (B \Rightarrow C) \Rightarrow (A \Rightarrow C)_{\{1\}}$	w_1	2–6, ⇒I
8	$(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))_{\emptyset}$	0,	1–7, ⇒I

The world parameters on the right indicate worlds that are hypothesised in each of the subproofs. (0 in the final line indicates that the formula is true at every normal situation. We will discuss the relationship between normal situations and worlds in Section 4.)

E has a ternary relation semantics like the semantics for R, but we wish to read it in a somewhat different way. We use E for the ternary relation in E-frames. The expression '$Eabc$' means that, for any worlds w_1, w_2, if a is in w_1, w_2 is *modally accessible* from w_1, and b is in w_2, then a situation like c is also in w_2. In order to understand the accessibility relation E, we appeal to a second accessibility relation, in this case a binary modal accessibility relation. The status of worlds and the modal accessibility relation are best explained by appealing to the formal semantics for E. Let us move on, then, to this semantics.

4 Routley–Meyer Models for E

A positive E frame is a structure $(S, 0, E)$ such that S is a set (of situations), 0 is a non-empty subset of S, and $E \subseteq S^3$ such that all the following definitions and conditions hold. Where a, b, c, d are situations,

$$a \leq b \ =_{\text{df}} \ \exists x(x \in 0 \wedge Exab)$$

$$E^2abcd =_{\text{df}} \ \exists x(Eabx \wedge Excd)$$

1. if $a \in 0$ and $a \leq b$, then $b \in 0$;
2. if $a \leq b$ and $Ebcd$ then $Eacd$; if $c \leq d$ and $Eabc$ then $Eabd$;
3. there is a $b \in 0$ such that $Eaba$;
4. if E^2abcd then $\exists x(Eacx \wedge Ebxd)$;
5. if $Eabc$ then E^2abbc.

Semantic postulates 3, 4, and 5 need some explanation. Below, we define a modal accessibility relation on *situations*, M, as $Mab =_{\text{df}} \exists x(x \in 0 \wedge Eaxb)$. Postulate 3 tells us that this accessibility relation on situations is reflexive. Postulate 4, however, has to do with the modal accessibility relation on *worlds*. E^2abcd says that if a is in a world w_1, b is in a world w_2, c is in w_3, then there is a situation like d in w_3 and w_2 is accessible from w_1 and w_3 is accessible from w_2. Since the modal accessibility relation on worlds is transitive, w_3 is accessible from w_1. $\exists x(Eacx \wedge Ebxd)$ tells us, in this instance, that there is a situation x such that if a is in w_1 and d is in w_3, then a situation like x is in w_3 and if b is in w_2 and x is in w_3, then a situation like d is also in w_3. The fact that the modal accessibility relation on worlds is transitive allows us to make sense of this postulate.

Semantic postulate 5 relies on reflexivity rather than transitivity. It tells us that if $Eabc$, then there if a is in w_1, b is in w_2, b is in w_3, then we can infer that there is a situation like c in w_3. This seems unintelligible, unless we read this as saying that if $Eabc$, then if a is in w_1, b is in w_2, b is in w_2, then we can infer that c is also in w_2. This makes so much sense as to seem obvious. What does the work here is identifying the worlds in which the first and second instance of b are located. We can do this if the modal accessibility relation is reflexive. To labour the point slightly, we read E^2abbc as saying that there is some situation x such that if a is in w_1 and b is in w_2, we can infer that there is a situation like x in w_2 and if x is in w_2 and b is in w_2 then there is a situation like c also in w_2, such that w_2 is accessible from w_1 and w_2 is accessible from itself.

At this point, we reflect on the nature of the set 0 of normal situations. In the natural deduction system, we can use $A_\emptyset$ at any stage, if A has been proved. In allowing this, we assume that every world contains at least one normal situation. A possible world (one that contains no contradictions) is *covered* by a normal situation. In full models for E, that contain mechanisms to deal with negation as well as the other connectives, normal situations are all bivalent. They make true the law of excluded middle. In this way, we can think of normal situations to some extent as surrogates for worlds in models. We do not, however, want to identify worlds with normal situations, at least as the latter are characterised in frames, since the way in which we understand the E relation in terms of worlds is not made explicit in frames.[2]

A positive E model is a quadruple $(S, 0, E, V)$ such that $(S, 0, E)$ is a positive E frame and V assigns sets of situations to propositional variables such that for any propositional variable p, $V(p)$ is closed upwards under $\leq$. Each value assignment V determines a satisfaction relation $\models_V$, between situations and formulas by means of the following inductive definition:

- $a \models_V p$ if and only if $a \in V(p)$;
- $a \models_V t$ if and only if $a \in 0$;
- $a \models_V A \wedge B$ if and only if $a \models_V A$ and $a \models_V B$;
- $a \models_V A \vee B$ if and only if $a \models_V A$ or $a \models_V B$;
- $a \models_V A \Rightarrow B$ if and only if $\forall b \forall c((Eabc \ \wedge \ b \models_V A) \ \supset \ c \models_V B)$.

We write $\models$ instead of $\models_V$ where no confusion will result.

The following satisfaction condition for $\Box$ can be derived:

$$a \models \Box A \quad \text{iff} \quad \forall b \forall c((Eabc \wedge b \in 0) \supset c \models A)$$

We can extract from this condition a definition of a modal accessibility relation:

$$Mab \ =_{\text{df}} \ \exists x(x \in 0 \wedge Eaxb)$$

[2]For contrast, see Urquhart's semantics discussed in Section 5.

Now we can state a Kripke-style satisfaction condition for necessity:

$$a \vDash \Box A \quad \text{iff} \quad \forall b(Mab \supset b \vDash A)$$

The relation M might not seem as if it is the right relation to represent necessity in E models. After all, this is a relation between situations, not worlds. But we have certain situations that can be treated as worlds. They might be mere proxies of "real" worlds or they might be the worlds themselves. This is a matter for metaphysicians to ponder. We set it aside here. These worlds are the members of the set 0. This is the set of normal situations — the situations at which all of the theorems of E are true under all interpretations.

The M relation, as defined above, has the properties of the accessibility relation in Kripke models for S4:

Proposition 4.1. *In any E-frame, M is transitive and reflexive.*

Proof. Suppose that Mab and Mbc. Then there is an $x \in 0$ and a $y \in 0$ such that $Eaxb$ and $Ebyc$. Therefore, E^2axyc. Thus, by semantic postulate 4, there is some situation z such that $Eayz$ and $Exzc$. Thus, Maz and $z \leq c$. Thus, by semantic postulate 2, Mac. Generalising, M is transitive.

Reflexivity follow directly from semantic postulate 3 and the definition of M. □

In later sections of this paper, we will examine systems with weaker M relations.

5 Urquhart Semantics for E

In his PhD thesis [17] and in [16], Alasdair Urquhart gives a semantics for the implicational fragment of E. This semantics has a lot in common with the semantics for $R^{\Box}$. In particular, it is an extension of his semantics for the implicational fragment of R. The semantics uses a set of pieces of information and a semi-lattice join operator, $\cup$, between pieces of information. The information condition for relevant implication is

$$x \vDash A \to B \quad \text{iff} \quad \forall y(y \vDash A \ \supset \ x \cup y \vDash B).$$

Urquhart modifies this semantics to fit E by adding a set of worlds and a binary relation, N, on them. He also has formulas' being satisfied at a pair of a piece of information and a world. The condition for entailment becomes

$$(x, w_0) \vDash A \Rightarrow B \quad \text{iff} \quad \forall y \forall w_1((Nw_0w_1 \wedge (y, w_1) \vDash A) \ \supset \ (x \circ y, w_1) \vDash B).$$

The debt our interpretation owes to Urquhart's semantics is clearly extensive. The difference is that in Urquhart's semantics an R structure is present under the surface of the E structure.

In order to see what formal difference it makes to have an R structure underlying E models, let us consider using Urquhart's idea to model all of E. Consider a model $(S, 0, R, W, N, V)$ where $(S, 0, R)$ is a positive R frame, W is a non-empty set (of worlds), and N is a reflexive and transitive binary relation on W. Then we set

$$(a, w_0) \vDash A \Rightarrow B \quad \text{iff} \quad \forall b \forall c \forall w_1((Nw_0w_1 \wedge Rabc \wedge (b, w_1) \vDash A) \;\supset\; (c, w_1) \vDash B).$$

The conditions for the other connectives are the obvious ones.

We can prove that this model satisfies the formula that Maksimova constructed to show that NR is a proper extension of E — $((A \Rightarrow (B \Rightarrow C)) \wedge (B \Rightarrow (A \vee C))) \Rightarrow (B \Rightarrow C)$.

Proof. Suppose that $(a, w_0) \vDash (A \Rightarrow (B \Rightarrow C)) \wedge (B \Rightarrow (A \vee C))$. Also assume that $Rabc$, Nw_0w_1, and $(b, w_1) \vDash B$. We show that $(c, w_1) \vDash C$. Since $(a, w_0) \vDash B \Rightarrow (A \vee C)$ and $(b, w_1) \vDash B$, $(c, w_1) \vDash A \vee C$. $Rccc$, so R^2abcc. Suppose that $(c, w_1) \vDash A$. **By the Pasch postulate**, R^2acbc, and so there is some situation x such that $Racx$ and $Rxbc$. Since $(a, w_0) \vDash A \Rightarrow (B \Rightarrow C)$, $(x, w_1) \vDash B \Rightarrow C$. $Rxbc$, $Nw_1w_1w_1$, and $(b, w_1) \vDash B$, so $(c, w_1) \vDash C$. Thus, if either $(c, w_1) \vDash A$ or $(c, w_1) \vDash C$, $(c, w_1) \vDash C$. Thus, $(a, w_0) \vDash B \Rightarrow C$. □

The bolded step is not available, in general, for Routley–Meyer models for E. The Pasch Postulate — $\exists x(Rabx \wedge Rxcd) \supset \exists x(Racx \wedge Rxbd)$ — is in Routley–Meyer frames for R in order to make valid $(A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow (A \Rightarrow C))$ (among other things), which would demodalise E. It would allow the inference from $(A \Rightarrow A) \Rightarrow (A \Rightarrow A)$, which is valid in E, to $A \Rightarrow ((A \Rightarrow A) \Rightarrow A)$, which is not.

We suggest, however, that Urquhart's semantics be used as a guide for the construction of entailment logics. It provides an intuitive treatment of modality in relevant logics. Although the semantics proves too much for E and for the generalisations of E that we examine below, it gives us upper bounds on the logics that we are to consider. It shows us what stronger forms of these logics (that incorporate elements from R) look like and our constructions remain weaker than these logics, but somewhat similar to them.

5.1 Note on $\mathrm{R}^{\Box}$

The system $\mathrm{R}^{\Box}$ — sometimes called "NR" — is a modal extension of R, formulated with a necessity operator and some axioms and rules taken from the modal logic S4.

The axioms are: $(\Box A \wedge \Box B) \to \Box(A \wedge B)$, $\Box(A \to B) \to (\Box A \to \Box B)$. And the additional rule is the rule of necessitation. The definition of a model for $\mathrm{R}^{\Box}$ adds a second accessibility relation, N, for the necessity operator and the usual satisfaction condition for statements of the form $\Box A$ applies:

$$a \vDash \Box A \quad \text{iff} \quad \forall x(Nax \supset x \vDash A)$$

The difficulty in extending Urquhart's semantics to a semantics for all of E is replicated in the proof that $\mathrm{R}^{\Box}$ is not a conservative extension of E. The underlying R frame in $\mathrm{R}^{\Box}$ models creates the conditions for the proof of the Maksimova formula.

One difference between $\mathrm{R}^{\Box}$ and E is that, viewed in terms of situated inference, the two logics represent different standards of information content. Consider the disjunction elimination rules for the two systems written in standard form. They look the same:

$$\begin{array}{|lll}
A \vee B_{\alpha} & & \\
\quad \begin{array}{|l}
A_k \\ \hline
\vdots \\
C_{\beta \cup \{k\}}
\end{array} & & \text{hypothesis} \\
\quad \begin{array}{|l}
B_k \\ \hline
\vdots \\
C_{\beta \cup \{k\}}
\end{array} & & \text{hypothesis} \\
C_{\alpha \cup \beta} & & \vee\mathrm{E}
\end{array}$$

where $k \notin \beta$. This similarity, however, is rather superficial. Given our situated informational interpretation of them, we can see a real difference here. We are licensed to make an inference from a disjunction by this rule in R when we have *contingent relevant implications* from both A and B. According to E, we can only make a similar inference when we have *entailments* from those two propositions. E places a stronger demand on what counts as the information available in a situation than does R. If we add an R semantic structure to an E frame, as happens in the Urquhart semantics and the semantics for $\mathrm{R}^{\Box}$, then we undermine the E demand of stricter relations between the states of affairs of a situation and the further information that they carry. We can think of this distinction as a normative one. The two logics E and $\mathrm{R}^{\Box}$ warrant different claims about what information is available in situations.

6 Two Notions of Necessity

In the semantics for E, there are really two notions of necessity. The first is the one that is incorporated into the entailment connective, $\Rightarrow$. Having $A \Rightarrow B$ true at a world (i.e., having the information that $A \Rightarrow B$ in some situation in that world) means that in any accessible world, if there is a situation that contains the information that A, there is also one that contains the information that B. We call this *closure necessity*, since it expresses closure conditions for worlds. The other sort of necessity is *fill necessity*. This sort of necessity is represented by $\Box$.

Fill necessity can be understood both in terms of relationships between situations and relationships between worlds. As we said in Section 4, we place a modal accessibility relation between situations that acts in terms of necessity in the same way as accessibility relations in the standard worlds semantics do. A formula $\Box A$ is satisfied by a situation a if and only if A is satisfied by all situations M-accessible to a. In terms of worlds, suppose that $\Box A$ is true at a world w_1. Let's suppose that w_2 is accessible from w_1. As we said in Section 4, in each world there is at least one normal situation. Thus, there is some normal situation b in w_2 and there is at least one situation c in w_2 such that $Eabc$. Since $Eabc$, $c \vDash A$. Hence there is a situation in w_2 that contains the information that A, that is, A is true in w_2.

In E, both closure and fill necessity are formulated in terms of entailment. In $R^{\Box}$, they are both formulated in terms of $\Box$. This turns out to be a very important difference between the two logics. As we have seen, they do not give us logically equivalent systems. They are also conceptually quite different. For E, and the associated logics that we will turn to presently, closure necessity is primary. In $R^{\Box}$, fill necessity is more important.

7 E.K

The foregoing analysis of necessity in E suggests that we look at logics in which the virtual accessibility relation between worlds has different properties. We call this relation virtual because it is present only in a very shadowy sense (in terms of the M relation between normal situations) in the formal semantics. As we have seen in Section 4, the modal accessibility relation of E is reflexive and transitive. It seems reasonable to look at systems in which the modal accessibility relation has different properties. The modal accessibility relation, however, is defined in terms of the ternary accessibility relation, which concerns entailment. Thus, we must adjust the ternary relation to modify the binary accessibility relation.

Our strategy is to formulate the properties of modality in terms of the way it is represented in the natural deduction system and then to modify this representation to incorporate different properties for modality. Then we axiomatise the resulting system and construct a Routley–Meyer semantics for it.

We begin with a logic we call E.K, to indicate that it is the basic system in much the same way that K is the basic normal modal logic.

The natural deduction rule is changed to allow applications of ⇒E for cases in which the subproof in which the major premise is contained to be adjacent to the subproof in which the minor premise resides:

$$\begin{array}{|l} A \Rightarrow B_\alpha \\ \vdots \\ \begin{array}{|l} \vdots \\ A_\beta \\ B_{\alpha\cup\beta} \end{array} \end{array}$$

There is one exception to this. If the major premise has an empty-set subscript, then we allow ⇒E to be applied to premises in the same subproof. We have to change the rule ∨E in a similar way:

$$\begin{array}{|l} A \Rightarrow C_\alpha \\ B \Rightarrow C_\alpha \\ \vdots \\ \begin{array}{|l} \vdots \\ A \vee B_\beta \\ C_{\alpha\cup\beta} \end{array} \end{array}$$

Again, we allow the exception that the entailment formulas can be in the same subproof as the disjunction if the subscript on the entailment formulas is the empty set. In addition, we add a rule to allow the closure of worlds under provable implications:

$$\begin{array}{|ll} A \Rightarrow B_\emptyset & \\ A_\alpha & \\ \vdots & \\ B_\alpha & \qquad \text{Th}{\Rightarrow} \end{array}$$

Otherwise, the rules are the same as for E.

The following is an axiomatisation of E.K:

Axioms

1. $A \Rightarrow A$
2. $A \Rightarrow (A \vee B)$; $B \Rightarrow (A \vee B)$
3. $(A \wedge B) \Rightarrow A$; $(A \wedge B) \Rightarrow B$
4. $(A \wedge (B \vee C)) \Rightarrow ((A \wedge B) \vee (A \wedge C))$
5. $((A \Rightarrow B) \wedge (A \Rightarrow C)) \Rightarrow (A \Rightarrow (B \wedge C))$
6. $((A \Rightarrow C) \wedge (B \Rightarrow C)) \Rightarrow ((A \vee B) \Rightarrow C)$
7. t

Rules

$$\frac{\begin{array}{c}\vdash A \Rightarrow B\\ \vdash A\end{array}}{\vdash B}\ \text{(MP)} \qquad \frac{\begin{array}{c}\vdash A\\ \vdash A\end{array}}{\vdash A \wedge B}\ \text{(ADJ)} \qquad \frac{\vdash A}{\vdash \Box A}\ \text{N}$$

$$\frac{\vdash B \Rightarrow C}{\vdash (A \Rightarrow B) \Rightarrow (A \Rightarrow C)}\ \text{(PR)} \qquad \frac{\vdash A \Rightarrow B}{\vdash (B \Rightarrow C) \Rightarrow (A \Rightarrow C)}\ \text{(SR)}$$

$$\frac{\begin{array}{c}\vdash A^m \Rightarrow (A^{m+1} \Rightarrow \cdots (A^{n-1} \Rightarrow (B \Rightarrow C))\ldots)\\ \vdash A^p \Rightarrow (A^{p+1} \Rightarrow \cdots (A^n \Rightarrow B)\ldots)\end{array}}{\vdash A^1 \Rightarrow (A^2 \Rightarrow \cdots (A^n \Rightarrow C)\ldots)}\ \text{(RK)}$$

where $1 \leq m \leq n-1$, $1 \leq p \leq n$, and at least one of $m = 1$ or $p = 1$.

The proof that all the axioms are provable in the natural deduction system and that the rules, with the exception of RK, are admissible in it is straightforward. To prove that the axiom system includes all the theorems provable in the natural deduction system, we show that in a given proof, if $C_{\{i_1,\ldots,i_n\}}$ is provable, then $A_{i_1} \Rightarrow (\cdots (A_{i_n} \Rightarrow C)\ldots)$ is provable in the axiom system, where $A_{i_1},\ldots,A_{i_n}$ are the i_1th, …, i_nth hypotheses in the proof, respectively. Before we can prove this, we need to prove a crucial lemma.

Lemma 7.1. *If A_α is a step in a valid natural deduction proof in the system for E.K, then either $\alpha = \emptyset$ or the numbers in α are numerically consecutive and if α is non-empty, then α includes the numeral of the hypothesis of the subproof in which A_α occurs.*

Proof. By induction on the length of the proof of A_α. If α is empty, then the lemma follows. If A_α is a hypothesis, then it follows as well. The cases for conjunction introduction and elimination and disjunction introduction are straightforward, as is the case for implication introduction. Implication elimination and disjunction elimination are similar to one another. Suppose that we have a proof segment of the following form:

$$\begin{array}{l|l}
 & A \Rightarrow B_\beta \\
 & \quad \vdots \\
 & \quad A_\gamma \\
 & \quad B_{\beta\cup\gamma} \qquad \Rightarrow\text{E}
\end{array}$$

By the inductive hypothesis, the numbers in β and γ are consecutive. By the entailment elimination rule the maximal number in γ is one higher than the maximal number in β. Let n be the maximal number in γ. Then $\gamma - \{n\}$ is either a subset of β or a proper superset. If it is a proper superset then $\gamma \cup \beta = \gamma$. If it is a subset, then $\gamma \cup \beta = \beta \cup \{n\}$. In either case, the lemma follows. As we said, the disjunction elimination case is similar. □

The following proposition shows that E.K is at least as strong as the logic DJ, which in its class of theorems is the same as Ross Brady's logic of meaning containment, MC [4].

Proposition 7.2. $((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$ *is a theorem of* E.K.

Proof. Let $(A \Rightarrow B) \wedge (B \Rightarrow C)$ be A^1 and A be A^2.

1. $((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (B \Rightarrow C)$ axiom 3
2. $((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow B)$ axiom 3
3. $((A \Rightarrow B) \wedge (B \Rightarrow C)) \Rightarrow (A \Rightarrow C)$ 1, 2, RK

□

Given the definition of $\Box A$ as $t \Rightarrow A$, Proposition 7.2 also shows that the following version of the K axiom is a theorem of E.K:

$$(\Box A \wedge (A \Rightarrow B)) \Rightarrow \Box B$$

Moreover, the following aggregation principle is an instance of axiom 5:

$$(\Box A \wedge \Box B) \Rightarrow \Box(A \wedge B)$$

Thus, E.K contains a good deal of what are relevant counterparts of the key theorems of the modal logic K.

For the following theorem, we use an abbreviation. Where $\{1, \ldots, n\}$ is a set of subscripts of hypotheses in a derivation, $A^1, \ldots, A^n$, respectively, $\{1, \ldots, n\} \Rightarrow C$ is the formula $A^1 \Rightarrow (A^n \Rightarrow C)$.

Theorem 7.3. *For any formula C, if $C_{\{i_1,\ldots,i_n\}}$ is a step in a valid natural deduction proof then $A_{i_1} \Rightarrow (\cdots (A_{i_n} \Rightarrow C) \ldots)$ is provable in the E.K axiom system.*

Proof. By induction on the length of the proof $C_{\{i_1,\ldots,i_n\}}$.

Base Case. Suppose that C_i is a hypothesis. By axiom 1, $C \Rightarrow C$ is a theorem of the axiom system.

The conjunction and negation cases are straightforward, as are the cases for the entailment and disjunction introduction rules. Thus, we prove only the cases for entailment and disjunction elimination.

Entailment Elimination. Suppose that $C_{\{i_1,\ldots,i_n\}}$ is proven from $A \Rightarrow C_\alpha$ and A_β. Then, by the rules of the natural deduction system, the maximal number in β is 1 greater than the maximal number in α. By the inductive hypothesis, $\vdash \alpha \Rightarrow (A \Rightarrow C)$ and $\vdash \beta \Rightarrow C$. By RK, then, $\vdash (\alpha \cup \beta) \Rightarrow C$.

Disjunction Elimination. Suppose that $C_{\{i_1,\ldots,i_n\}}$ is proven from $A \Rightarrow C_\alpha$ and $B \Rightarrow C_\alpha$ and $A \vee B_\beta$ by $\vee$E. Then $\alpha \cup \beta = \{i_1, \ldots, i_n\}$ and by the inductive hypothesis, $\vdash \alpha \Rightarrow (A \Rightarrow C)$, $\vdash \alpha \Rightarrow (B \Rightarrow C)$, and $\vdash \beta \Rightarrow (A \vee B)$. Thus, by axiom 5 and repeated applications of the prefixing rule, $\vdash \alpha \Rightarrow ((A \Rightarrow C) \wedge (B \Rightarrow C))$ and so, by Proposition 7.2 and repeated applications of the prefixing rule, $\vdash \alpha \Rightarrow ((A \vee B) \Rightarrow C)$. Thus, by $\vdash \beta \Rightarrow (A \vee B)$ and RK, we obtain $\vdash (\alpha \cup \beta) \Rightarrow C$. $\square$

We need the following lemmas for the completeness proof.

Lemma 7.4. *The following rule is derivable in E.K:*

$$\frac{\begin{array}{c}\vdash A^1 \Rightarrow (\cdots (A^{n-1} \Rightarrow (A^n \Rightarrow C)) \ldots) \\ \vdash B \Rightarrow A^n\end{array}}{\vdash A^1 \Rightarrow (\cdots (A^{n-1} \Rightarrow (B \Rightarrow C)) \ldots)}$$

Proof.

$$
\begin{array}{lll}
1. & \vdash A^1 \Rightarrow (\cdots (A^{n-1} \Rightarrow (A^n \Rightarrow C)) \ldots) & \text{Premise} \\
2. & \vdash B \Rightarrow A^n & \text{Premise} \\
3. & \vdash (A^n \Rightarrow C) \Rightarrow (B \Rightarrow C) & 2,\ \text{PR} \\
4. & \vdash (A^{n-1} \Rightarrow (A^n \Rightarrow C)) \Rightarrow (A^{n-1} \Rightarrow (B \Rightarrow C)) & 3,\ \text{PR} \\
5. & \ldots & \\
6. & \vdash (A^1 \Rightarrow (\cdots (A^{n-1} \Rightarrow (A^n \Rightarrow C)) \ldots)) \Rightarrow & \\
 & \qquad (A^1 \Rightarrow (\cdots (A^{n-1} \Rightarrow (B \Rightarrow C)) \ldots)) & \\
7. & \vdash A^1 \Rightarrow (\cdots (A^{n-1} \Rightarrow (B \Rightarrow C)) \ldots) & 1,6,\ \text{MP}
\end{array}
$$

□

Lemma 7.5. *The following rule is derivable in E.K. Where $m \leq n$,*

$$
\frac{\begin{array}{c}\vdash (A^1 \Rightarrow (\cdots (A^n \Rightarrow (D \Rightarrow E)) \ldots)) \\ \vdash (B^1 \Rightarrow (\ldots (B^m \Rightarrow (C \Rightarrow D)) \ldots))\end{array}}{\vdash (A^1 \Rightarrow (\cdots ((A^{n-m} \wedge B^1) \Rightarrow ((A^{(n-m)+1} \wedge B^2) \Rightarrow (\cdots ((A^n \wedge B^m) \Rightarrow (C \Rightarrow E)) \ldots))) \ldots))}
$$

Proof.

$$
\begin{array}{lll}
1. & \vdash (A^1 \Rightarrow (\cdots (A^n \Rightarrow (D \Rightarrow E)) \ldots)) & \text{Premise} \\
2. & \vdash (B^1 \Rightarrow (\cdots (B^m \Rightarrow (C \Rightarrow D)) \ldots)) & \text{Premise} \\
3. & \vdash (A^n \wedge B^m) \Rightarrow A^n & \text{Axiom 3} \\
4. & \vdash (A^1 \Rightarrow (\cdots ((A^n \wedge B^m) \Rightarrow (D \Rightarrow E)) \ldots)) & 1,3,\ \text{Lemma 7.4} \\
5. & \ldots & \\
6. & \vdash (A^1 \Rightarrow (\cdots ((A^{n-m} \wedge B^1) \Rightarrow (\cdots ((A^n \wedge B^m) \Rightarrow & \\
 & \qquad (D \Rightarrow E)) \ldots)) \ldots)) & \\
7. & \vdash (B^1 \Rightarrow (\cdots ((A^n \wedge B^m) \Rightarrow (C \Rightarrow D)) \ldots)) & 2,3,\ \text{Lemma 7.4} \\
8. & \ldots & \\
9. & \vdash (A^{n-m} \wedge B^1) \Rightarrow (\cdots ((A^n \wedge B^m) \Rightarrow (C \Rightarrow D)) \ldots) & \\
10. & \vdash (A^1 \Rightarrow (\cdots ((A^{n-m} \wedge B^1) \Rightarrow ((A^{(n-m)+1} \wedge B^2) \Rightarrow & \\
 & \qquad (\cdots ((A^n \wedge B^m) \Rightarrow (C \Rightarrow E)) \ldots))) \ldots)) & 6,9\ \text{RK}
\end{array}
$$

□

8 E.K Models

A positive E.K frame is a triple $(S, 0, E)$ just as for E frames, except that the semantic postulates are now the following:

1. $\leq$ is a partial order;
2. if $a \in 0$ and $a \leq b$, then $b \in 0$;
3. if $a \leq b$ and $Ebcd$ then $Eacd$;   if $c \leq d$ and $Eabc$ then $Eabd$;
4. if $Ea_1 \dots a_n c$, then $\exists x \exists y(Ea_m \dots a_{n-1}x \wedge Ea_p \dots a_n y \wedge Exyc)$ (where $n-m \geq 2$ and $n - p \geq 1$).

Here we use an extension of the definition of E^2 given in Section 4. We define

$$E^{n+1}a_1 \dots a_n a_{n+1} a_{n+2} c \quad \text{as} \quad \exists x(E^n a_1 \dots a_{n+1} x \wedge E x a_{n+2} c).$$

For convenience, we drop the superscript from E^n and merely write $Ea_1 \dots a_{n+1}c$.

A positive E.K model is a quadruple $(S, 0, E, V)$ such that $(S, 0, E)$ is a positive E.K frame and V assigns sets of situations to propositional variables such that for any propositional variable p, $V(p)$ is closed upwards under $\leq$. Each value assignment V determines a satisfaction relation $\models_V$, applying the same clauses as for E models. We write '$\models$' instead of '$\models_V$' where no confusion will result.

The meaning of entailment in E.K is, on one level, the same as it is for E: $A \Rightarrow B$ says that if, in an accessible world, there is a situation that contains the information that A, then there is a situation that contains B. Without reflexivity or transitivity, one use of E.K's implication could be to represent a form of doxastic entailment. An agent might be said to hold $A \rightarrow B$ in the sense of E.K if and only if she believes that B follows from A.

Excluding the rule RK, the axiomatic basis for E.K is the same as that of the minimal relevant logic B. The soundness of B over the Routley–Meyer semantics is well known [14, 15]. Thus it is sufficient to show that the class of E.K frames satisfy RK.

Lemma 8.1. *In any E.K model, if* $\models A^m \Rightarrow (\cdots(A^{n-1} \Rightarrow (B \Rightarrow C))\dots)$ *and* $\models A^p \Rightarrow (\cdots(A^n \Rightarrow B)\dots)$, *then* $\models A^1 \Rightarrow (\cdots(A^n \Rightarrow C)\dots)$.

Proof. Suppose that $\models A^m \Rightarrow (\cdots(A^{n-1} \Rightarrow (B \Rightarrow C))\dots)$ and $\models A^p \Rightarrow (\cdots(A^n \Rightarrow B)\dots)$ and suppose that $Ea_1 \dots a_n c$ and $a_i \models A_i$ for each i, $1 \leq i \leq n$. Let $m = 1$. The case in which $p = 1$ is similar. By semantic condition 4, there is a situation x such that $Ea_1 \dots a_{n-1}x$ and a situation y such that $Ea_p \dots a_n y$ and $Exyc$. By assumption and the information condition for implication, $x \models B \Rightarrow C$ and $y \models B$. So $c \models C$. □

Thus, we can now state the following soundness theorem:

Theorem 8.2 (Soundness)**.** *All the theorems of E.K are valid in the class of E.K frames.*

8.1 Completeness of E.K

In order to construct the canonical model, we define a form of logical consequence for a logic L:

$$\Gamma \vdash_L \Delta \text{ iff } \exists G_1, \ldots, G_m \in \Gamma\, \exists D_1, \ldots, D_n \in \Delta(\vdash_L (G_1 \wedge \ldots \wedge G_m) \Rightarrow (D_1 \vee \ldots \vee D_n)).$$

We use this consequence relation for a wide variety of purposes in what follows, first to define the notion of a theory.

Definition 8.3 (Theory)**.** An L-theory Γ is a set of formulas such that if $\Gamma \vdash_L \{A\}$, then $A \in \Gamma$.

It is easy to show that if Γ is an L-theory, $A \in \Gamma$, and $B \in \Gamma$, then $A \wedge B \in \Gamma$. A theory Γ is said to be *prime* if and only if for all formulas $A \vee B \in \Gamma$, either $A \in \Gamma$ or $B \in \Gamma$. Γ is said to be *regular* if and only if $t \in \Gamma$.

We also use the consequence relation to define the notion of L-consistency: a pair of sets of formulas (Γ, Δ) is said to be *L-consistent* if and only if $\Gamma \nvdash_L \Delta$. The form of Lindenbaum extension theorem that is used for relevant logics employs L-consistency, rather than the more standard notion of negation consistency. This lemma was originally proven by Nuel Belnap and Dov Gabbay (see [5]).

Theorem 8.4. *If (Γ, Δ) is L-consistent, then there is a prime theory $\Gamma' \supseteq \Gamma$ such that (Γ', Δ) is L-consistent.*

Corollary 8.5. *A formula A is a theorem of L if and only if $A \in \Gamma$ for all regular prime L-theories Γ.*

Proof. If $\vdash_L A$, then $\vdash_L t \Rightarrow A$, by RN. If Γ is regular, then, by definition, $t \in \Gamma$, hence $\Gamma \vdash_L \{A\}$. Since Γ is a theory, $A \in \Gamma$. If $\nvdash_L A$ then $(L, \{A\})$ is L-consistent, where L is taken here to be the set of theorems of L. Thus, by the Lindenbaum theorem, there is a prime regular L-theory Γ such that $\Gamma \nvdash_L \{A\}$. □

In order to formulate our canonical model, we utilise a binary fusion operator on theories. Where a and b are L-theories for any of our logics L,

$$a \circ b \;=_{\text{df}}\; \{\, B \in Fml\colon\; \exists A((A \Rightarrow B) \in a \;\wedge\; A \in b) \,\}.$$

It is easy to prove that the fusion of two L-theories is an L-theory. Note, however, that the fusion of two prime theories may not be prime, but we can prove the following lemma.

Lemma 8.6. *(a) If a, b, and c are L-theories, a and c are prime, and $a \circ b \subseteq c$, then there is a prime L-theory $b' \supseteq b$ such that $a \circ b' \subseteq c$; (b) if a, b, and c are L theories, b and c are prime, and $a \circ b \subseteq c$, then there is a prime L-theory $a' \supseteq a$ such that $a' \circ b \subseteq c$; (c) where a, b, and c are L-theories, if $a \circ b \subseteq c$, then there are prime L-theories a', b', and c' such that $a' \circ b' \subseteq c'$.*

Proof. (a) Suppose that a, b, and c are L-theories, a and c are prime and $a \circ b \subseteq c$. Let X be the set of formulas A such that there is some $B \notin c$ and $A \Rightarrow B \in a$.

We show that (b, X) is L-consistent. Suppose that (b, X) is L-inconsistent. Then there are $B_1, \ldots, B_m \in b$ and $C_1, \ldots, C_n \in X$ such that $\vdash_L (B_1 \wedge \cdots \wedge B_m) \Rightarrow (C_1 \vee \cdots \vee C_n)$. By the definition of X, there are $A_1, \ldots, A_n$, not in c such that $C_1 \Rightarrow A_1 \in a, \ldots, C_n \Rightarrow A_n \in a$. By a simple logical derivation, $(C_1 \vee \cdots \vee C_n) \Rightarrow (A_1 \vee \cdots \vee A_n)$. Since c is prime, $(A_1 \vee \cdots \vee A_n) \notin c$. But if $\vdash_L (B_1 \wedge \cdots \wedge B_m) \Rightarrow (C_1 \vee \cdots \vee C_n)$, then $(C_1 \vee \ldots \vee C_n) \in b$. Hence, $a \circ b \not\subseteq c$. Thus, by reductio, (b, X) is L-consistent.

By Theorem 8.4, there is a prime theory b' extending b such that (b', X) is L-consistent. Hence $a \circ b' \subseteq c$.

(b) The proof is similar to that of (a).

(c) Suppose that $a \circ b \subseteq c$. Then, there is a prime L-theory c' extending c such that $a \circ b \subseteq c'$. Now we extend a to a prime L-theory a' such that $a' \circ b \subseteq c'$. We do so by noting that the set $X = \{A \Rightarrow B \colon A \in b \wedge B \notin c'\}$ is such that (a, X) is L-consistent (see the proof of (a) above). Then, by the Lindenbaum lemma, a can be extended to a prime L-theory a' such that $a' \circ b \subseteq c'$. By (a) above, there is a prime L-theory b' such that $a' \circ b' \subseteq c'$. □

Lemma 8.7. *For every L-theory a, there is some regular prime L-theory o such that $o \circ a = a$.*

Proof. Let a be an L-theory. Then $Thm(L) \circ a = a$ and so $Thm(L) \circ a \subseteq a$. By Lemma 8.6(b), there is a prime L-theory o extending $Thm(L)$ such that $o \circ a \subseteq a$. Since o extends $Thm(L)$, o is regular. □

We are now ready to construct the canonical model. The canonical model is a quadruple $\mathfrak{M}_L = (S, 0, E, V)$ such that

- S is the set of prime theories of L;
- 0 is the set of regular prime theories of L;

- $E \subseteq S^3$ is such that $Eabc$ if and only if $a \circ b \subseteq c$;
- V is a function from propositional variables to subsets of S such that $a \in V(p)$ if and only if $p \in a$.

Lemma 8.8. *For all L-theories, $a \leq b$ if and only if $a \subseteq b$.*

Proof. Suppose that $a \leq b$. Then there is some regular prime L-theory o such that $Eoab$. By the definition of E, $o \circ a \subseteq b$. Since $A \Rightarrow A \in o$, $a \subseteq b$.

Suppose now that $a \subseteq b$. By Lemma 8.7, there is a $o \in 0$ such that $o \circ a \subseteq a$, thus by the transitivity of subset, $o \circ a \subseteq b$. Therefore, $a \leq b$. □

Lemma 8.9. *For $2 \leq n$, $Ea_1 \dots a_n b$ if and only if $(\dots(a_1 \circ a_2) \circ \cdots) \circ a_n \subseteq b$.*

Proof. By induction on n.
Base case: $n = 2$. Follows from the definition of E for the canonical model.
Inductive case: Suppose that, for all $b \in S$, $Ea_1 \dots a_n b$ iff $(\dots(a_1 \circ a_2) \circ \cdots) \subseteq b$. We show that for all $a_{n+1}, b \in S$, $Ea_1 \dots a_{n+1} b$ iff $(\dots(a_1 \circ a_2) \circ \cdots) \circ a_{n+1} \subseteq b$.

Suppose that $Ea_1 \dots a_{n+1} b$. By definition, $Ea_1 \dots a_n a_{n+1} b$ if and only if there is some $x \in S$, $Ea_1 \dots a_n x$ and $Exa_{n+1} b$. By hypothesis, $(\dots(a_1 \circ a_2) \circ \cdots) \circ a_n) \subseteq x$. Clearly, for all L-theories x, y, z, if $z \subseteq w$, then $z \circ y \subseteq w \circ y$. So, $(\dots(a_1 \circ a_2) \circ \cdots) \circ a_n) \circ a_{n+1} \subseteq x \circ a_{n+1}$. Since $Exa_{n+1} b$, $x \circ a_{n+1} \subseteq b$. Thus, by the transitivity of subset, $(\dots(a_1 \circ a_2) \circ \cdots) \circ a_{n+1} \subseteq b$.

Suppose now that $(\dots(a_1 \circ a_2) \circ \cdots) \circ a_{n+1} \subseteq b$. $(\dots(a_1 \circ a_2) \circ \cdots) \circ a_n$ is an L-theory. Thus, by Lemma 8.6(b), there is a prime L-theory x such that $(\dots(a_1 \circ a_2) \circ \cdots) \circ a_n \subseteq x$ and $x \circ a_{n+1} \subseteq b$. By hypothesis, $Ea_1 \dots a_n x$ and, by definition of E, $xa_{n+1} b$. Therefore, $Ea_1 \dots a_{n+1} b$. □

Lemma 8.10. *If $Ea_1 \dots a_n b$, then for all p, $1 \leq p \leq n-1$, $Ea_1 \dots a_{n-1} x$ and $Ea_p \dots a_n y$ and $Exyb$.*

Proof. Suppose that $Ea_1 \dots a_n b$. Then, by Lemma 8.9, $(\dots(a_1 \circ a_2) \circ \cdots) \circ a_n \subseteq b$. We show that $(\dots(a_m \circ a_{m+1}) \circ \cdots) \circ a_{n-1}) \circ (\dots(a_p \circ a_{p+1}) \circ \cdots) \circ a_n) \subseteq b$, where either m or p is 1. Case 1. $m = 1$. Suppose that $C \in (\dots(a_1 \circ a_2) \circ \cdots) \circ a_{n-1}) \circ (\dots(a_p \circ a_{p+1}) \circ \cdots) \circ a_n)$. We show that $C \in b$. Then, by the definition of fusion on theories, there is some formula B such that $B \Rightarrow C \in (\dots(a_1 \circ a_2) \circ \cdots) \circ a_{n-1}$ and $B \in (\dots(a_p \circ a_{p+1}) \circ \cdots) \circ a_n)$. Using the same reasoning, we can see that there are $A^2, \dots, A^{n-1}$ such that for $2 \leq i \leq n-1$, $A^i \in a_i$ and

$$A^2 \Rightarrow (\cdots(A^p \Rightarrow (\cdots(B \Rightarrow C)\dots))\dots) \in a_1.$$

Similarly, there are formulas $D^{p+1}, \dots, D^n$ such that for all j, $p+1 \leq j \leq n$, $D^j \in a_j$ and

$$D^p \Rightarrow (\cdots (D^n \Rightarrow B) \dots) \in a_p.$$

Now, we know that E.K proves

$$(A^2 \Rightarrow (\cdots (A^p \Rightarrow (\cdots (B \Rightarrow C) \dots)) \dots)) \Rightarrow (A^2 \Rightarrow (\cdots (A^p \Rightarrow (\cdots (B \Rightarrow C) \dots)) \dots))$$

and

$$\vdash_{E.K} (D^p \Rightarrow (\cdots (D^n \Rightarrow B) \dots)) \Rightarrow (D^p \Rightarrow (\cdots (D^n \Rightarrow B) \dots)).$$

By Lemma 7.5, then, we can derive

$$\begin{aligned} &\vdash_{E.K} (A^2 \Rightarrow (\dots (A^p \Rightarrow (\dots (B \Rightarrow C) \dots)) \Rightarrow \\ &\qquad ((\dots (A^p \wedge (D^p \Rightarrow (\dots (D^n \Rightarrow B) \dots)) \Rightarrow (D^p \Rightarrow (\dots (D^n \Rightarrow B) \dots)) \Rightarrow \\ &\qquad\qquad ((A^{p+1} \wedge D^{p+1}) \Rightarrow ((A^{n-1} \wedge D^{n-1}) \Rightarrow (D^n \Rightarrow C) \dots). \end{aligned}$$

Thus,

$$\begin{aligned} &((\dots (A^p \wedge D^p) \Rightarrow (\dots (D^n \Rightarrow B) \dots)) \Rightarrow (D^p \Rightarrow (\dots (D^n \Rightarrow B) \dots)) \Rightarrow \\ &\qquad ((A^{p+1} \wedge D^{p+1}) \Rightarrow ((A^{n-1} \wedge D^{n-1}) \Rightarrow (D^n \Rightarrow C) \dots) \in a_1. \end{aligned}$$

For all i, $2 \leq i < p$, and all j, $p+1 \leq j \leq n$, $A^i \in a_i$ and $(A^i \wedge D^j) \in a_j$. In addition, $(A^p \wedge D^p) \Rightarrow (\cdots (D^n \Rightarrow B) \dots)) \in a_p$, so

$$C \in (\dots (a_1 \circ a_2) \circ \cdots) \circ a_n).$$

Since $(\dots (a_1 \circ a_2) \circ \cdots) \circ a_n \subseteq b$, $C \in b$, as required. Generalising, $(\dots (a_1 \circ a_2) \circ \cdots) \circ a_{n-1}) \circ (\dots (a_p \circ a_{p+1}) \circ \cdots) \circ a_n) \subseteq b$. Hence, by Lemma 8.6, there is a prime theory x extending $(\dots (a_1 \circ a_2) \circ \cdots) \circ a_{n-1}$ and a prime theory y extending $(\dots (a_p \circ a_{p+1}) \circ \cdots) \circ a_n$ and $x \circ y \subseteq b$, i.e., $Exyb$.

Case 2. $p = 1$. Similar to case 1. □

9 E.KT and E.K4

We now look at two logics between E.K and E. These are E.KT and E.K4. In terms of their natural deduction systems, E.KT adds to E.K modified forms of the entailment and disjunction elimination rules. In E.KT, we can apply a major to a minor premise when the two are in the same subproof:

$$\begin{array}{|ll} A \Rightarrow B_\alpha & \\ A_\beta & \\ B_{\alpha\cup\beta} & \Rightarrow\mathrm{E} \end{array}$$

For E.K4, the ⇒E rule is modified to allow, not premises in the same subproof, but a major premise that is in a subproof separated from the minor by one or more other subproofs.

$$\begin{array}{|l} A \Rightarrow B_{\alpha} \\ \ddots \\ \begin{array}{|ll} A_{\beta} & \\ B_{\alpha\cup\beta} & \qquad \Rightarrow\text{E} \end{array} \end{array}$$

For E.KT we replace the rule RK with the rule RKT:

$$\frac{\begin{array}{c} \vdash A^{m} \Rightarrow (A^{m+1} \Rightarrow \cdots (A^{q} \Rightarrow (B \Rightarrow C)) \ldots) \\ \vdash A^{p} \Rightarrow (A^{p+1} \Rightarrow \cdots (A^{n} \Rightarrow B) \ldots) \end{array}}{\vdash A^{1} \Rightarrow (A^{2} \Rightarrow \cdots (A^{n} \Rightarrow C) \ldots)}$$

where either $p = 1$ or $q = 1$, $n - 1 \leq q \leq n$, and $1 \leq p \leq n$. For E.KT we also need to add the rule RTh:

$$\frac{\begin{array}{c} \vdash A^{1} \Rightarrow (\cdots (A^{n} \Rightarrow (B \Rightarrow C)) \ldots) \\ \vdash B \end{array}}{\vdash A^{1} \Rightarrow (\cdots (A^{n} \Rightarrow C) \ldots)}$$

To obtain E.K4 we replace RK with the rule RK4. We begin with a finite sequence of formulas $\Sigma = \langle A_1, \ldots, A_n \rangle$. Let Γ and Δ be sequences, in which all of the formulas that occur in them occur in Σ and occur in the same order as in Σ. Moreover, for any A_i ($1 \leq i \leq n-1$), the total number of times that it occurs in both Γ and Δ is at least the number of times that it occurs in $\langle A_1, \ldots, A_{n-1} \rangle$. (It follows from this that every formula that occurs in Σ occurs at least once in one of Γ or Δ.)

$$\frac{\begin{array}{c} \vdash \Gamma \Rightarrow (B \Rightarrow C) \\ \vdash \Delta \Rightarrow (A^{n} \Rightarrow B) \end{array}}{\vdash A^{1} \Rightarrow (\cdots (A^{n} \Rightarrow C) \ldots)}$$

Here $\langle D_1, \ldots, D_m \rangle \Rightarrow E$ is defined as $D_1 \Rightarrow (D_2 \Rightarrow (\cdots (D_m \Rightarrow E) \ldots))$.

Proposition 9.1. *Each of (i)* $((A \Rightarrow B) \wedge A) \Rightarrow B$*, (ii)* $(A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$*, and (iii)* $((A \Rightarrow A) \Rightarrow B) \Rightarrow B$ *are theorems of E.KT.*

Proof. (i)

1.	$\vdash ((A \Rightarrow B) \wedge A) \Rightarrow (A \Rightarrow B)$	Axiom 3
2.	$\vdash ((A \Rightarrow B) \wedge A) \Rightarrow A$	Axiom 3
3.	$\vdash ((A \Rightarrow B) \wedge A) \Rightarrow B$	$1, 2,$ RKT

(ii) Taking $A \Rightarrow (A \Rightarrow B)$ to be A^1 and A to be A^2, we get:

1.	$\vdash (A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow (A \Rightarrow B))$	Axiom 1
2.	$\vdash A \Rightarrow A$	Axiom 1
3.	$\vdash (A \Rightarrow (A \Rightarrow B)) \Rightarrow (A \Rightarrow B)$	$1, 2,$ RKT

(iii)

1.	$\vdash ((A \Rightarrow A) \Rightarrow B) \Rightarrow ((A \Rightarrow A) \Rightarrow B)$	Axiom 1
2.	$\vdash A \Rightarrow A$	Axiom 1
3.	$\vdash ((A \Rightarrow A) \Rightarrow B) \Rightarrow B$	$1, 2,$ RTh

□

Proposition 9.2. *(a)* $(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$ *and (b)* $(B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$ *are theorems of E.K4.*

Proof. (a) Let $A \Rightarrow B$ be A^1, $B \Rightarrow C$ be A^2, and A be A^3.

1.	$(A \Rightarrow B) \Rightarrow (A \Rightarrow B)$	axiom 1
2.	$(B \Rightarrow C) \Rightarrow (B \Rightarrow C)$	axiom 1
3.	$(A \Rightarrow B) \Rightarrow ((B \Rightarrow C) \Rightarrow (A \Rightarrow C))$	$1, 2,$ RK4

(b) Let $B \Rightarrow C$ be A^1, $A \Rightarrow B$ be A^2, and A be A^3.

1.	$(B \Rightarrow C) \Rightarrow (B \Rightarrow C)$	axiom 1
2.	$(A \Rightarrow B) \Rightarrow (A \Rightarrow B)$	axiom 1
3.	$(B \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$	$1, 2,$ RK4

□

Proposition 9.2 shows that E.K4 is an extension of TW, which is Anderson and Belnap's system of ticket entailment, T, without the axiom of contraction. The entailment fragment, $\mathrm{TW}_{\Rightarrow}$, is extraordinary because in any case in which an equivalence $A \Rightarrow B$ and $B \Rightarrow A$ is provable, then A and B are the same formula [12, 2]. We do not know at this point in time whether E.K4 is exactly the same logic (i.e., has the same theorems) as TW.

Propositions 9.1 and 9.2 together show that the logic E.KT4 which results from the axiom basis for E.KT together with the rule RK4 yields an extension of E. To show that it is E, it suffices to show that the rules RK, RKT, RTh, and RK4 are all derivable in E. This is easy (although somewhat tedious) to show, and so we can say that E.KT4 is equivalent to E.

Proposition 9.3. $\Box A \Rightarrow \Box\Box A$ *is a theorem of E.K4.*

Proof. Let $t \Rightarrow A$ be A^1 and t be A^2

1.	$(t \Rightarrow A) \Rightarrow ((t \Rightarrow t) \Rightarrow (t \Rightarrow A))$	Proposition 9.2(b)
2.	$t \Rightarrow (t \Rightarrow t)$	axiom 1 and RN
3.	$(t \Rightarrow A) \Rightarrow (t \Rightarrow (t \Rightarrow A))$	$1, 2,$ RK4
4.	$\Box A \Rightarrow \Box\Box A$	$3,$ def. $\Box$

□

Proposition 9.3 shows that the fill necessity of E.K4 is very much like that of the classical modal logic K4.

10 E.KT and E.K4 Models

An E.KT frame is an E.K frame with two additional conditions. The first condition that we add is

$$\text{(SRT)} \quad \text{If } Ea_1 \ldots a_n c, \text{ then } \exists x \exists y (Ea_m \ldots a_n x \ \wedge \ Ea_p \ldots a_n y \ \wedge \ Exyc),$$

where at least one of m or p is 1. We do not require that either $n - m$ or that $n - p$ be at least 0, although we do require that $n \geq 2$. When $m = n$, then we read $Ea_m \ldots a_n x$ as $a_n \leq x$, and similarly for $p = n$.

Suppose that $Eabc$. By SRT we have $Eabx$ and $b \leq y$ and $Exyc$. By semantic condition 3 on E.K frames, we have $Exbc$ and so we have $Eabbc$. Thus, $Eabc$ implies $Eabbc$. This is the condition for contraction from the definition of an E frame.

We can also derive the condition $Eaaa$ for all situations a. Here is the proof. By semantic condition 1 on E.K frames, $\exists x(x \in 0 \wedge Exaa)$. By contraction, $Exaaa$, i.e., there is some y such that $Exay$ and $Eyaa$. By the definition of $\leq$, $a \leq y$ and so by semantic condition 3, $Eaaa$. The condition $Eaaa$ is called *complete reflexivity.*

Complete reflexivity allows us to prove simple instances of RKT such as:

$$\frac{\begin{array}{c}\vdash A \Rightarrow (B \Rightarrow C)\\ \vdash A \Rightarrow B\end{array}}{\vdash A \Rightarrow C}$$

The second condition we add is the following:

$$\text{(T)} \quad \exists x(x \in 0 \wedge Eaxa)$$

The condition T just says that M is reflexive for E.KT, as one would expect.

Lemma 10.1. *The rule RKT is valid in the class of E.KT frames.*

Proof. The only cases that are not covered by the soundness proof for E.K are instances of the rule in which $q = n$. We have already proven the case in which $q = n = 2$. Suppose that $A^1 \Rightarrow (A^{m+1} \Rightarrow \cdots (A^n \Rightarrow (B \Rightarrow C))\ldots)$ and $A^p \Rightarrow (A^{p+1} \Rightarrow \cdots (A^n \Rightarrow B)\ldots)$ are both valid in the class of E.KT frames. Now, consider an E.KT model and situations $a_1, \ldots, a_n$ and c such that $Ea_1 \ldots a_n c$ and $a_i \vDash A^i$ for all i, $1 \leq i \leq n$. By the assumption and SCT, $c \vDash B \Rightarrow C$ and $c \vDash B$. By $Eccc$ and the satisfaction condition, $c \vDash C$. Generalising, $\vDash A^1 \Rightarrow (\cdots (A^n \Rightarrow C)\ldots)$. $\square$

Theorem 10.2. *E.KT is sound over the class of E.KT frames.*

The proof of completeness for E.KT is very like the one for E.K. Lemma 8.10 has to be tweaked slightly, but the proof is essentially the same. Thus we merely state the completeness theorem:

Theorem 10.3. *E.KT is complete over the class of E.KT frames.*

The soundness and completeness theorems for E.KT show that there is an alternative axiomatisation of the logic that includes the axiomatic basis for E.K plus the two axiom schemes PMP and T.

The definition of an E.K4 frame is the same as for an E.K frame except that it includes the following condition. Where $\langle a_1, \ldots, a_n\rangle$ is a sequence of situations and $Ea_1 \ldots a_n c$, there are situations x and y such that $Ea_{i_1} \ldots a_{i_m} x$ and $Ea_{j_1} \ldots a_{j_p} a_n y$ and $Exyc$, where each of the a_is and a_js are in the original sequence and numbered in the same order as in the original sequence.

An E.K4 frame is an E.K frame with the addition of the condition SK4. Let $\sigma = \langle a_1, \ldots, a_{n-1}\rangle$ be a finite sequence of situations. Let γ and δ be sequences of situations taken from σ, such that in γ and δ every situation occurs in the same order as it occurs in σ and between γ and δ each situation occurs at least as many times as it occurs in σ.

$$(SK4) \quad \text{If } Ea_1 \ldots a_{n-1} a_n c, \text{ then } \exists x \exists y(E\gamma x \ \wedge \ E\delta a_n y \ \wedge \ Exyc).$$

Lemma 10.4. *In all E.K4 frames, for all situations a, b, c, d, if $Eabcd$ then there is some situation x such that $Eacx$ and $Ebxd$.*

Proof. Suppose that $Eabcd$. Then, there is some situation y such that $Eacy$ and some situation x such that Ebx and $Exyd$. By definition, Ebx is just $b \leq x$, so by semantic condition 3 for E.K frames, $Ebxd$. □

Lemma 10.4 shows that the condition used in E frames to prove the prefixing axiom is satisfied by E.K4 frames as well. This also shows that the modal accessibility relation M is transitive (see Proposition 4.1).

Lemma 10.5. *The rule RK4 is sound over the class of E.K4 frames.*

Proof. Let $A^1, \ldots, A^n$ be a sequence of formulas such that $\Gamma \Rightarrow (B \Rightarrow C)$ and $\Delta \Rightarrow B$ are valid in the class of E.K4 frames, where Γ is a subset of the sequence not including A^n and Δ is a subset of the sequence that includes A^n and $\Gamma \cup \Delta = \{A^1, \ldots, A^n\}$. Suppose that $Ea_1 \ldots a_n c$, where $a_i \vDash A^i$ for each i, $1 \leq i \leq n$. Let $S(\Gamma)$ be a subset of $\{a_1, \ldots, a_n\}$ such that for each $A^j \in \Gamma$, there is a situation $a_j \in S(\Gamma)$ such that $a_j \vDash A^j$ and similarly for $S(\Delta)$.

Let $\langle a_{j_1}, \ldots, a_{j_m} \rangle$ be the sequence of situations in $S(\Gamma)$ placed in the same order as they appear in $\langle a_1, \ldots, a_n \rangle$, and similarly let $\langle a_{k_1}, \ldots, a_{k_p} \rangle$ be the sequence of situations in $S(\Delta)$ placed in the same order as they appear in $\langle a_1, \ldots, a_n \rangle$. Then by the special semantic condition defining E.K4 frames, there are situations x and y such that $ES(\Gamma)x$, $ES(\Delta)y$ and $Exyc$. Therefore, $c \vDash C$. Generalising, $A^1 \Rightarrow (\cdots (A^n \Rightarrow C) \ldots)$ is valid on the class of E.K4 frames. □

The completeness proof is a slightly more complicated version of the proof for E.K. We do not present it here.

An E.KT4 frame is an E.K frame that satisfies the conditions T and SKT4. The condition SKT4 is the following. Let $\sigma = \langle a_1, \ldots, a_n \rangle$ be a finite sequence of situations. Let γ and δ be sequences of situations taken from σ, such that in γ and δ every situation occurs in the same order as it occurs in σ, and between γ and δ each situation occurs the same number of times it occurs in σ.

$$(SKT4) \quad \text{If } Ea_1 \ldots a_n c, \text{ then } \exists x \exists y (E\gamma x \;\wedge\; E\delta a_n y \;\wedge\; Exyc)$$

We have shown that E.KT frames satisfy the contraction condition and that E.K4 frames satisfy the condition that $Eabcd$ implies $\exists x(Eacx \wedge Ebxd)$. Together with the conditions satisfied by every E.K frame and T, we are justified in stating the following theorem:

Theorem 10.6. *Every E.KT4 frame is an E frame.*

We think the converse is true as well, but we have no proof of this so we leave it open.

11 Symmetry

In order to axiomatise the logic that is characterised by symmetry of the modal accessibility relation, we suggest adding the Urquhart–Fine axiom:

$$\text{(UF)} \quad A \Rightarrow ((A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow C)).$$

Urquhart used this axiom to distinguish between $E_{\Rightarrow}$ and the system $E5_{\Rightarrow}$, which is characterised by his semantics in which the modal accessibility relation is reflexive, transitive, and symmetric [16, 17]. Kit Fine proved that $E_{\Rightarrow}$ together with UF is $E5_{\Rightarrow}$, that is, that it is complete over Urquhart's semantics for it [6].

UF is a relative of the axiom of E sometimes called Restricted Assertion, $(A \Rightarrow B) \Rightarrow (((A \Rightarrow B) \Rightarrow C) \Rightarrow C)$, which is equivalent to E's Permutation axiom, $A \Rightarrow ((A \Rightarrow B) \Rightarrow C) \Rightarrow ((A \Rightarrow B) \Rightarrow (A \Rightarrow C))$. It is, in the presence of the transitivity axioms of Theorem 15, equivalent to a form of Permutation as well.

$$\text{(UF}'\text{)} \quad (A \Rightarrow (B \Rightarrow (C \Rightarrow D))) \Rightarrow (B \Rightarrow (A \Rightarrow (C \Rightarrow D)))$$

UF follows from UF′ and an appropriate instance of axiom 1.

While Fine proved that $E5_{\Rightarrow}$ is complete for Urquhart's semantics where the accessibility relation is reflexive, transitive, and symmetric, E5 with negation and the conditional does not appear to validate the symmetry principle one would expect, namely, the B axiom: $A \Rightarrow \Box\neg\Box\neg A$. This leaves open the possibility that a different axiom is needed for completeness on the symmetric E.K frames, rather than the reflexive, transitive, symmetric frames. The lack of fit between the B axiom and symmetry is not peculiar to the entailment systems. The extension of $R^{\Box}$ with the B axiom is not characterised by the class of models in which the modal accessibility relation is transitive, reflexive and symmetric. Rather, a weaker postulate than symmetry is used [7].

UF does, however, capture a kind of symmetry. The axiom can be recovered in the Fitch system by adding another $\Rightarrow E$ rule.

$$\begin{array}{|l}
A_{\alpha} \\
\ddots \\
\begin{array}{|l}
A \Rightarrow (B \Rightarrow C)_{\beta} \\
B \Rightarrow C_{\alpha\cup\beta}
\end{array}
\end{array}$$

This permits one to use $\Rightarrow$E when the antecedent is in a superior proof, provided the consequent is itself a conditional. If A is true at a situation a in w, Mww', and $A \Rightarrow (B \Rightarrow C)$ is true at a situation b in w' then we can infer that there is a situation c in w' in which $B \Rightarrow C$ is true, justified by the situation in w.

The strengthening of this rule that permits the consequent of the conditional to be a non-conditional, B, would permit the derivation of the R axiom of Assertion, $A \to ((A \to B) \to B)$, which is equivalent to the Permutation axiom whose proof was displayed in Derivation 1. This strengthening is unavailable to us, since it would move us to the non-modal logic R.

It seems that however we axiomatise E5, it should have UF as a theorem. This is perhaps easiest to see on the Urquhart semantics. Suppose that UF is not valid on a reflexive, transitive, symmetric frame, i.e., for some a, w, $a, w \not\models A \Rightarrow ((A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow C))$. Then there is a b, w' with Nww', such that $b, w' \models A$ and $a \circ b, w' \not\models (A \Rightarrow (B \Rightarrow C)) \Rightarrow (B \Rightarrow C)$. There is, then, a c, w'' with $Nw'w''$ such that $c, w'' \models A \Rightarrow (B \Rightarrow C)$ but $a \circ b \circ c, w'' \not\models B \Rightarrow C$. But, by transitivity and symmetry of N, $Nw''w$, so $c \circ a, w \models B \Rightarrow C$. Again by the symmetry of N, Nww'', so, by the properties of $\circ$, $a \circ b \circ c, w'' \models B \Rightarrow C$. We conclude that, contrary to the assumption, UF is valid. The preceding proof used the transitivity of N, which underlines the possibility that a different axiom is needed for symmetry in the absence of transitivity.

As we said in Section 5, we use Urquhart's semantics (and the systems characterised by it) as an upper bound of our E-based systems. E5 is the upper bound, although it should perhaps be called E.KT45. We will leave open the question of whether E.K5, E.K45, and E.KT5, obtained by adding UF to E.K, E.K4, and E.KT, respectively, are complete for the classes of symmetric, symmetric transitive, and symmetric reflexive frames, respectively.

Appendix I: The Natural Deduction System for E

Hypothesis: Any formula can be hypothesised with a new numeral as a subscript and introducing a new subproof.

Repetition: Any formula can be repeated within the same sub-proof.

Theorem: Any formula that has been previously proven or the constant t can be stated anywhere in any proof with the subscript $\emptyset$.

In $\Rightarrow$E the premises can be in the same subproof or the minor premise may be in a (not necessarily immediate) subproof of the proof in which the major premise occurs. The same is true for $\vee$E.

$$
\begin{array}{|l}
A \Rightarrow B_\alpha \\
\ddots \\
\begin{array}{|ll}
A_\beta & \\
\vdots & \\
B_{\alpha\cup\beta} & \Rightarrow\text{E}
\end{array}
\end{array}
\qquad
\begin{array}{|ll}
\begin{array}{|ll}
A_i & \text{hypothesis} \\ \hline
\vdots & \\
B_\alpha &
\end{array} & \\
A \Rightarrow B_{\alpha-\{i\}} & \Rightarrow\text{I}
\end{array}
$$

In $\Rightarrow$I, $i \in \alpha$.

$$
\begin{array}{|ll}
A_\alpha & \\
B_\alpha & \\
\vdots & \\
A \wedge B_\alpha & \wedge\text{I}
\end{array}
\qquad
\begin{array}{|ll}
A \wedge B_\alpha & \\
\vdots & \\
A_\alpha & \wedge\text{E}
\end{array}
\qquad
\begin{array}{|ll}
A \wedge B_\alpha & \\
\vdots & \\
B_\alpha & \wedge\text{E}
\end{array}
$$

$$
\begin{array}{|ll}
A_\alpha & \\
\vdots & \\
A \vee B_\alpha & \vee\text{I}
\end{array}
\qquad
\begin{array}{|ll}
B_\alpha & \\
\vdots & \\
A \vee B_\alpha & \vee\text{I}
\end{array}
\qquad
\begin{array}{|l}
A \Rightarrow C_\beta \\
B \Rightarrow C_\beta \\
\ddots \\
\begin{array}{|ll}
A \vee B_\alpha & \\
\vdots & \\
C_{\alpha\cup\beta} & \vee\text{E}
\end{array}
\end{array}
$$

$$
\begin{array}{|ll}
A_\emptyset & \\
\vdots & \\
t \Rightarrow A_\emptyset & t \Rightarrow
\end{array}
\qquad
\begin{array}{|ll}
A \wedge (B \vee C)_\alpha & \\
\vdots & \\
(A \wedge B) \vee (A \wedge C)_\alpha & \text{Distribution}
\end{array}
$$

Appendix II: Proof of Derivability of RK

Lemma 11.1. *The rule RK is admissible in the natural deduction system for E.K.*

Proof. Suppose that $A^m \Rightarrow (A^{m+1} \Rightarrow \cdots (A^{n-1} \Rightarrow (B \Rightarrow C))\ldots)$ and $A^p \Rightarrow (A^{p+1} \Rightarrow \cdots (A^n \Rightarrow B)\ldots)$ are provable in the natural deduction system. Let $m = 1$. The case in which $p = 1$ is similar. We then can construct a proof of C from $A^1, \ldots, A^n$ as follows:

$$
\begin{array}{ll}
\vert\ A_1^1 & \text{hypothesis} \\
\vert\ A^1 \Rightarrow (A^2 \Rightarrow \cdots (A^{n-1} \Rightarrow (B \Rightarrow C)) \ldots)_{\emptyset} & \text{assumption} \\
\vert\ A^2 \Rightarrow (A^3 \Rightarrow \cdots (A^{n-1} \Rightarrow (B \Rightarrow C)) \ldots)_{\{1\}} & 1{,}2,\ \Rightarrow\text{E} \\
\vert\ \vert\ A_2^2 & \text{hypothesis} \\
\vert\ \vert\ A^3 \Rightarrow (A^4 \Rightarrow \cdots (A^{n-1} \Rightarrow (B \Rightarrow C)) \ldots)_{\{1,2\}} & 3{,}4,\ \Rightarrow\text{E} \\
\vert\ \vert\ \ddots & \\
\vert\ \vert\ \vert\ A_p^p & \text{hypothesis} \\
\vert\ \vert\ \vert\ A^{p+1} \Rightarrow (A^{p+2} \Rightarrow \cdots (A^{n-1} \Rightarrow (B \Rightarrow C)) \ldots)_{\{1,\ldots,p\}} & \ldots \Rightarrow\text{E} \\
\vert\ \vert\ \vert\ A^p \Rightarrow (A^{p+1} \Rightarrow \cdots (A^n \Rightarrow B) \ldots)_{\emptyset} & \text{assumption} \\
\vert\ \vert\ \vert\ A^{p+1} \Rightarrow (A^{p+2} \Rightarrow \cdots (A^n \Rightarrow B) \ldots)_{\{p\}} & \ldots \Rightarrow\text{E} \\
\vert\ \vert\ \vert\ \vert\ A_{p+1}^{p+1} & \text{hypothesis} \\
\vert\ \vert\ \vert\ \vert\ A^{p+2} \Rightarrow (A^{p+3} \Rightarrow \cdots (A^{n-1} \Rightarrow (B \Rightarrow C)) \ldots)_{\{1,\ldots,p\}} & \ldots \Rightarrow\text{E} \\
\vert\ \vert\ \vert\ \vert\ A^{p+2} \Rightarrow (A^{p+3} \Rightarrow \cdots (A^n \Rightarrow B) \ldots)_{\{p\}} & \ldots \Rightarrow\text{E} \\
\vert\ \vert\ \vert\ \vert\ \ddots & \\
\vert\ \vert\ \vert\ \vert\ \vert\ A_{n-1}^{n-1} & \text{hypothesis} \\
\vert\ \vert\ \vert\ \vert\ \vert\ B \Rightarrow C_{\{1,\ldots,n-1\}} & \ldots \Rightarrow\text{E} \\
\vert\ \vert\ \vert\ \vert\ \vert\ A^n \Rightarrow B_{\ 1,\ldots,n-1\}} & \ldots \Rightarrow\text{E} \\
\vert\ \vert\ \vert\ \vert\ \vert\ \vert\ A_n^n & \text{hypothesis} \\
\vert\ \vert\ \vert\ \vert\ \vert\ \vert\ B_{\{1,\ldots,n\}} & \ldots \Rightarrow\text{E} \\
\vert\ \vert\ \vert\ \vert\ \vert\ \vert\ C_{\{1,\ldots,n\}} & \ldots \Rightarrow\text{E} \\
\vert\ \vert\ \vert\ \vert\ \vert\ A^n \Rightarrow C_{\{1,\ldots,n-1\}} & \ldots \Rightarrow\text{I} \\
\vert\ \vert\ \vdots & \\
\vert\ A^2 \Rightarrow (\cdots (A^n \Rightarrow C) \ldots)_{\{1\}} & \ldots \Rightarrow\text{I} \\
A^1 \Rightarrow (A^2 \Rightarrow (\cdots (A^n \Rightarrow C) \ldots))_{\emptyset} & \ldots \Rightarrow\text{I}
\end{array}
$$

$\square$

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