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A fresh look at research strategies in computational cognitive science: The case of enculturated mathematical problem solving

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Abstract
Marr's seminal distinction between computational, algorithmic, and implementational levels of analysis has inspired research in cognitive science for more than 30 years. According to a widely-used paradigm, the modelling of cognitive processes should mainly operate on the computational level and be targeted at the idealised competence, rather than the actual performance of cognisers in a specific domain. In this paper, we explore how this paradigm can be adopted and revised to understand mathematical problem solving. The computational-level approach applies methods from computational complexity theory and focuses on optimal strategies for completing cognitive tasks. However, human cognitive capacities in mathematical problem solving are essentially characterised by processes that are computationally sub-optimal, because they initially add to the computational complexity of the solutions. Yet, these solutions can be optimal for human cognisers given the acquisition and enactment of mathematical practices. Here we present diagrams and the spatial manipulation of symbols as two examples of problem solving strategies that can be computationally sub-optimal but humanly optimal. These aspects need to be taken into account when analysing competence in mathematical problem solving. Empirically informed considerations on enculturation can help identify, explore, and model the cognitive processes involved in problem solving tasks. The enculturation account of mathematical problem solving strongly suggests that computational-level analyses need to be complemented by considerations on the algorithmic and implementational levels. The emerging research strategy can help develop algorithms that model what we call enculturated cognitive optimality in an empirically plausible and ecologically valid way.

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1 Introduction

For more than 30 years, Marr's (1977, 1982) threefold distinction of computational, algorithmic, and implementational levels of analysis has had an enormous influence on research programs in cognitive science (Bechtel and Shagrir 2015; Blokpoel 2017; Cooper and Peebles 2017; Love 2015).[1] While the algorithmic and implementational levels are also widely studied in cognitive science, in computational modelling much research has focused on the computational level of analysis (Isaac et al. 2014; Szymanik 2016). Such research on computational modelling can be largely independent from considerations on algorithmic-level models of certain cognitive tasks and their physical implementation. In terms of Chomsky's (2015/1965) distinction between competence and performance, which Marr (1982) relates to his three-fold distinction of levels of analysis, this mode of research prioritises the computational-level study of cognitive competence at the expense of the study of cognitive performance on algorithmic and implementational levels of analysis. Using the tools of complexity theory, computational-level models can be characterized in terms of optimal algorithms for calculating outputs for inputs to the model, where optimality is a measure of the time or space it takes to generate an output given a certain input.

In this paper, we would like to explore how mathematical problem solving can be understood within this framework. Based on theoretical considerations and empirical evidence, we will argue that the properties of mathematical problem solving and its ontogenetic acquisition require us to take the performance on algorithmic and implementational levels into consideration in order to arrive at a theoretically sound and empirically plausible account of mathematical problem solving. In a second step, this account could inform the development and refinement of computational models of mathematical problem solving in the future.

The paper is structured as follows: In Sect. 2, we will introduce the Marr (1982) distinction of three levels of analysis, its relation to Chomsky's (2015/1965) distinction between competence and performance, and the analytical role of complexity theory in more detail. Next, we present and discuss two examples of mathematical problem solving that seem to be elusive to a computational-level analysis, which focuses on optimality as a measure of the ways in which humans usually solve these problems (Sect. 3). As a consequence, we propose to make a distinction between computationally optimal algorithms and humanly optimal algorithms in Sect. 4. The idea is that humanly optimal algorithms are not necessarily computationally optimal, but *cognitively* optimal for human cognitive agents with a certain learning trajectory, for example in cases of the spatial manipulation of symbols and the construction and interpretation of diagrams. To date, it is an open question how we can specify human cognitive optimality on a theoretical level. Our proposal is to help close this gap by introducing

[1] Both authors have made substantial and direct intellectual contributions to this work in equal terms.

the enculturation account (Sect. 5). According to this account, competence in mathematical problem solving is the result of the socio-culturally structured acquisition of mathematical cognitive practices that capitalise on the bodily manipulation of symbolic and diagrammatic structures. With the conceptual tools and theoretical insights of the enculturation account in place, we consider a wealth of empirical evidence from neuroscientific, eye-tracking, and behavioural studies that support the idea that mathematical problem solving is indeed enculturated and defies computational optimality (Sect. 6). In Sect. 7, we relate our perspective on enculturated mathematical problem solving to the distinction between the study of competence on the computational level and the investigation of performance on the algorithmic and implementational levels of analysis. Under the assumption that competence in mathematical problem solving is enculturated, and therefore shows specific patterns that cannot be modelled by computationally optimal algorithms, we claim that we need to understand enculturated competence by accumulating evidence and combining theoretical considerations on enculturated performance on algorithmic and implementational levels. We discuss the consequences of this perspective for the computational modelling of mathematical problem solving (Sect. 8). Finally, we will briefly conclude and present open questions for future research in Sect. 9.

2 Marr's levels of analysis, competence, and optimal algorithms

Marr (1977, 1982) identifies three levels of analysis for the scientific understanding of information processing tasks. On the *computational* level, we ask what the task is on the level of "abstract problem description" (Love 2015, p. 231), i.e., what is being computed. This is described purely in terms of the input and the output of a function modelling the information processing task. Furthermore, we also ask why something is computed in a to-be-specified way (Bechtel and Shagrir 2015). On the *algorithmic* level, the question is how the computation is carried out by an algorithm, i.e., how the output is determined for each input. On the level of *implementation*, we study how the information processing task is implemented physically.[2]

In this paper, we study mathematical problem solving as a paradigm case of information processing tasks studied in cognitive science. In Marr's three-level classification, mathematical problem solving tasks can be characterised as follows. First, on the computational level, the task consists in taking a mathematical problem as the input and giving a correct solution as the output. Second, on the algorithmic level, we want to identify the problem solving algorithm that transforms the input into the output. Third, on the level of implementation, we consider the physical realisation of this process.

To take a simple example, let us consider a calculating machine designed to solve basic arithmetical problems. On the computational level, the task is to take any input of natural numbers and correctly calculate the output for arithmetical operations (such as addition and multiplication). On the algorithmic level, it needs to be determined how the output is generated for each input. For example, if the task is calculating the

[2] For an insightful overview of the relation of Marr's levels of analysis and relevant research questions in cognitive science, see Varma (2014).

sum of "3 + 2", the calculator needs to use an algorithm that assigns the operator "+" to the input in order to calculate the output "5". On the level of implementation, it is then determined how this algorithm can be implemented physically. In a digital calculator, for example, the algorithm is a binary sequence, which is implemented by electrical signals.

While Marr's model can be used in explaining all information processing tasks—including those carried out by machines—human cognitive tasks are of particular interest. Indeed, Marr's (1982) main aim was to understand human visual perception. His work was influenced by Newell and Simon (Newell and Simon 1976; Newell 1980, 1982), who argued that cognitive science should focus on functional explanations of what they called *physical symbol systems*, i.e., general classes of systems capable of manipulating symbols. Marr, unlike Newell and Simon, emphasised the importance of the functional explanations being abstracted from the actual ways humans complete cognitive tasks (Tamburrini 1997). However, like them, Marr endorsed the view that we should focus on the *computational* level and its functional analyses in order to make progress in cognitive science:

> Although algorithms and mechanisms are empirically more accessible, it is the top level, the level of computational theory, which is critically important from an information-processing point of view. The reason for this is that the nature of computations that underlie perception depends more upon the computational problems that have to be solved than upon the particular hardware in which their solutions are implemented. […] an algorithm is likely to be understood more readily by understanding the nature of the problem being solved than by examining the mechanism (and the hardware) in which it is embodied (Marr 1982, p. 27).

This focus on the computational level can be seen in how some cognitive scientists seek to explain cognitive tasks. Under the influence of Anderson (1990), who emphasised the importance of the analysis of rational and optimal functions on a rational level (which is very similar to Marr's computational level), there has been a tendency in some research programmes in cognitive science to favour computational-level analyses over analyses on lower levels (Isaac et al. 2014; Szymanik 2016; for a recent example, see Piantadosi et al. 2016).[3] This strategy is characterised by Frixione (2001) in the following terms:

> The relation existing between a computational theory and the algorithmic level can be regarded as the relation between a function (in a mathematical sense) that is computable and a specific algorithm for calculating its values. The aim

[3] As an anonymous reviewer pointed out, this analytic strategy is particularly salient in Bayesian models of cognitive phenomena that have been influenced by Anderson's (1990) work. On these accounts, the computational level is of particular importance for analysing cognitive processes in terms of Bayesian conditionalisation (Eberhardt and Danks 2011; Griffiths et al. 2010). A notable exception is recent work on predictive processing, which depicts perception, action, and cognition as cases of Bayesian conditionalisation in a hierarchical generative model that is implemented in the brain (Clark 2016; Hohwy 2013). Predictive processing is committed to the view that explanations need to make specific assumptions about the algorithmic and implementational details of probabilistic prediction error minimisation (for an example, see Friston 2005).

> of a computational theory is to single out a function that models the cognitive phenomenon to be studied. Within the framework of a computational approach, such a function must be effectively computable. However, at the level of the computational theory, no assumption is made about the nature of the algorithms and their implementation (p. 381).

Amongst cognitive phenomena, mathematical problem solving would seem to be a particularly good fit with this approach. In analysing specific mathematical problem solving tasks, the approach amounts to taking a mathematical problem as the input and giving the correct solution as the output. But unlike with many cognitive tasks, in the case of mathematics there is a well-established methodology for characterising the problem solving task computationally. Namely, if we focus only on the computational level, the complexity of the problem solving task will be identified with the *computational complexity* of that problem.

In mathematics and theoretical computer science, characterising computational problems in terms of their complexity is widely studied (Arora and Barak 2009; Papadimitriou 2003). Computational complexity theory is based on the concept of a *Turing machine* and it approaches the complexity of problems based on the amount of computational resources needed for solving them. Turing (1936) presented the machine as a way to assess computational procedures theoretically, and it is thus not tied to any physical machine. The Turing machine can therefore be seen as giving rise to a theoretical framework that connects the study of computational complexity and computational-level analysis in cognitive science. Indeed, when Newell and Simon developed their concept of physical symbol system, it was specified as an "instance of a universal [Turing] machine" (Newell and Simon 1976, p. 117).

The Turing machine takes a problem as the input, runs an algorithm for solving the problem, and gives the correct answer as the output.[4] Based on the time or space it takes to reach the solution, we can then estimate the complexity of the problem. Since the Turing machine is an abstract entity, the time and space are not measured in seconds or bits, but rather as functions of the size of the input.[5] However, the same problem can be solved by several algorithms. For this reason, a key concept in computational complexity theory is the *optimality* of an algorithm (Arora and Barak 2009). The complexity of a problem is defined by an algorithm that takes the *least* amount of time or space to reach the solution. Such an algorithm is called optimal.[6]

[4] The most common way of doing this is to frame the problems as *decision problems*, so that the output for each input is either yes or no (Kozen 2012).

[5] For particular cases of general problems (such as a particular integer multiplication instead of the general problem of integer multiplication), complexity is often characterized in terms of the number of computational steps it takes to reach the solution.

[6] It needs to be noted that there is not one unique optimal algorithm for solving a particular problem. In theoretical computer science, an algorithm is called asymptotically optimal if it never performs more than a constant factor worse than the best possible algorithm. Hence there can be several, even an infinite number of optimal algorithms for solving a problem. Furthermore, Blum's (1967) speedup theorem shows that it is not always possible to define the complexity of an arbitrary problem in terms of an optimal algorithm for solving it. Namely, the theorem says that there are computable functions for which any algorithm computing that function can be sped up so that it demands less computational resources. Although this possibility is important to remember, it does not imply that optimal algorithms cannot be discussed coherently in most cases.

With optimal algorithms it is possible to determine the computational complexity of mathematical problems. In computational approaches to modelling cognitive tasks, the computational complexity of a problem can therefore be used to characterize the complexity of the problem solving task. At first glance, this explanatory strategy seems both fruitful and persuasive. The more complex a problem is, the more complex is also the cognitive task of solving it. This strategy also appears to have the additional strength of focusing on the abstract functional analysis of the problem solving task. The abstract treatment of algorithms can provide important knowledge about the general characteristics of such processes.

This is in line with the analysis of Chomsky (2015/1965), who presents an influential distinction between linguistic *competence* and *performance*, which has later been applied widely to other cognitive phenomena. Chomsky requires us to "make a fundamental distinction between *competence* (the speaker–hearer's knowledge of his language) and *performance* (the actual use of language in concrete situations)" (Chomsky 2015/1965, p. 2; italics in original). On this view, linguistic theory formation should focus on the study of competence, not actual performance. Note that any analysis of linguistic competence operates under some important simplifying assumptions:

> Linguistic theory is concerned primarily with an *ideal* speaker-listener, in a *completely homogeneous* speech-community, who knows its language *perfectly* and is *unaffected* by such grammatically irrelevant conditions as memory limitations, distractions, shifts of attention and interest, and errors (random or characteristic) in applying his knowledge of the language in actual performance (op. cit., p. 1; emphasis added).

We will see in this paper that these simplifying assumptions, which are necessary to establish the Chomskyan account of competence (in the domain of oral language and beyond), are problematic, at least when they are applied to research on mathematical problem solving. However, Chomsky's notion of competence is important for present purposes, because it augments our understanding of how cognitive abilities are analysed on the computational level. Rather than examining and modelling particular properties of individual cognitive abilities, the computational level focuses on an ideal cogniser. Marr explicitly noted that he saw the distinction between competence and performance as corresponding to his distinction between the computational level on the one hand and the algorithmic and implementational levels on the other hand[7]:

[7] It is interesting to note that Anderson (1990) is in disagreement with Marr (1982) about the relationship of competence and performance on the one hand and the computational, algorithmic, and implementational levels of analysis on the other hand. In particular, he argues that "[u]nlike Marr's case, performance is not just a matter of implementing the goals of competence. Indeed, unlike Marr's computational-level, Chomsky's competence is not concerned with the goals of the system" (Anderson, 1990, p. 9). Nevertheless, Anderson continues to concede that "Chomsky used the competence level to serve the same role in theory building as Marr used computational theory. Under both analyses, the scientist should first work out the higher level. Both felt that this was a key to making progress. Also, the lower levels are constrained somehow to reflect the higher levels" (op. cit.). In what follows, we are committed to this correspondence view of Marr's and Chomsky's contributions to the idea that theory formation—with the help of computational models—should proceed in a top-down fashion.

> Chomsky's (1965) theory of transformational grammar is a true computational theory [...] It is concerned solely with specifying what the syntactic decomposition of an English sentence should be, and not at all with how that decomposition should be achieved. Chomsky himself was very clear about this – it is roughly his distinction between competence and performance (Marr 1982, p. 28).

With this focus on competence, we are not interested in the particular characteristics that the performances of individual cognisers may or may not have. Instead, we want to establish the high-level characteristics of the ability of a fully competent, idealised cogniser in a certain target domain.

This gives rise to a paradigm in which Chomsky's notion of competence is combined with Marr's focus on the computational level (Fig. 1). This combination is helpful in systematically revealing how the analysanda (i.e., competence or performance in a given domain) relate to the appropriate research questions and analytical tools that characterise each of Marr's levels of analysis. As the function that models the cognitive task is identified on the computational level, computational complexity theory can be used to study its complexity. Furthermore, in the case of mathematical problem solving, it is straightforward to characterize the complexity of the cognitive task in terms of the computational complexity of the problem in question.

While this paradigm may be both fruitful and reliable in many cases, in this paper we want to raise questions concerning its general adequacy for analysing mathematical problem solving. The problematic part is that in computational complexity theory, complexity—and by implication competence—is focused on optimal algorithms for solving a problem. It is of course possible that with some mathematical problems the human competence in the problem solving task can be characterised by an optimal algorithm. The addition of natural numbers, for example, is a case where the standard

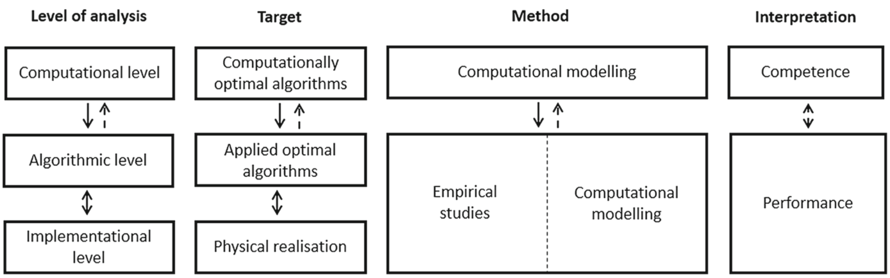

Fig. 1 Combination of Marr's levels of analysis with the Chomskyan notions of competence and performance. The analysis on the level of algorithms and implementation is seen as dealing with performance, whereas the computational level of analysis is associated with competence. It is important to note that computational modelling does not apply exclusively to the computational level of analysis. Both computational modelling and empirical studies are relevant methods on the algorithmic and implementational levels of analysis. As shown by the solid arrows, analyses on the computational level influence the algorithmic and implementational levels. However, according to Marr (1982, p. 25) the three levels are only "loosely coupled". The possible influence from the algorithmic and implementational levels on the computational level is represented by the dotted arrows. By the same token, the underdetermined relationship between competence and performance is indicated by the dotted bidirectional arrow

school algorithm with carrying appears to be also a computationally optimal one. However, we contend that this is not necessarily always the case.

3 Two examples of computationally suboptimal problem solving algorithms

Let us look at a couple of examples, which show that the ways in which competent cognitive agents solve mathematical problems are not algorithmically optimal by necessity. In many cases, cognitive agents solve mathematical problems in two-dimensional space. For example, by manipulating symbols, and arranging and re-arranging them on a piece of paper, cognitive agents successfully complete mathematical tasks (Landy and Goldstone 2007a). Note that this assumption can come in different strengths. It could be argued that the manipulation of symbols in space is a frequently employed, but not indispensable way to complete a cognitive task. Alternatively, it could be argued that symbol manipulation in space is not only a frequent, but often an indispensable component of completing mathematical problem solving tasks. For the moment, it suffices to say that there clearly are cases in which this strategy is successfully employed by cognitive agents. In our first example, this is done by manipulating symbols in space. In the second example, it is done by drawing and interpreting a diagram.

Imagine that your task is to arrive at the sum of the addition of the integers from 1 to 100. How do you proceed? You could, of course, add these integers one after the other:

$$1 + 2 + 3 + 4 + 5 + \cdots + 100 = 5050 \tag{1}$$

But this solution appears to be very error-prone and time-consuming. Maybe there is a better way to solve this problem. According to a famous anecdote about the young Carl Friedrich Gauss, his teacher at school required his pupils to complete exactly this task (Hayes 2006, 2017; Krämer 2016). The many versions of this anecdote agree that the young Gauss ingeniously solved the problem. In these reports, we find at least two different ways in which Gauss is said to have completed the task (adapted from Hayes 2006, 2017; Krämer 2016).[8]

$$\begin{array}{lllllllllll} \underline{(1+100)} & \underline{+} & \underline{(2+99)} & \underline{+} & \underline{(3+98)} & \underline{+ \ldots +} & \underline{(49+52)} & \underline{+} & \underline{(50+51)} & = & 5050 \\ 101 & + & 101 & + & 101 & + \ldots + & 101 & + & 101 & = & 5050 \end{array} \tag{2}$$

In this case, you start to add the first and the last number of the sequence from 1 to 100. In the next step, you add the second and the penultimate number of the sequence, then the third and the antepenultimate number of the sequence, and so on. Hayes (2006) calls this the “folding” strategy. You can also solve the problem in a slightly different

[8] For a detailed account of the different reports of this anecdote—and their inconsistencies—from the late nineteenth to the early twenty-first century, see Hayes (2006, 2017).

way, which Hayes (2006) calls the "two rows" solution. You write the sequence from 1 to 100 in one row. In the second row, you write the sequence from 100 to 1. Then you write down the sum of each pair in a third row and add the numbers in that row. Finally, since each number is now included twice, you divide the sum by 2.

$$\begin{array}{ccccccccccccc} 1 & & 2 & & 3 & \ldots & 99 & & 100 & & & & \\ + & & + & & + & & + & & + & & & & \\ 100 & & 99 & & 98 & \ldots & 2 & & 1 & & & & \\ \hline 101 & + & 101 & + & 101 & + \ldots + & 101 & + & 101 & = & \frac{10100}{2} & = & 5050 \end{array} \tag{3}$$

Formally, for the sequence of integers from 1 to 100 and for any sequence from 1 to n, the "folding" and "two rows" solutions can be expressed as follows[9]:

$$\frac{100}{2}(100+1) = 5050 \tag{2'}$$

$$\frac{n}{2}(n+1) = x \tag{2''}$$

$$\frac{100(100+1)}{2} = 5050 \tag{3'}$$

$$\frac{n(n+1)}{2} = x \tag{3''}$$

Importantly, the two formulas are mathematically equivalent (Hayes 2006, 2017). Furthermore, their computational complexity is also the same. However, solutions (2) and (3) make different use of the spatial arrangement of symbolic representations. In her discussion of the anecdote about the young Gauss, Krämer (2016) indicates that "the *spatial* attributes such as the position, grouping, and re-grouping of signs play a fundamental role" in the solution of the problem (p. 34; italics in original; our translation). The spatially structured symbolic representation of the integers leads to the systematisation of the problem in (2) and (3). It transforms the laborious and error-prone step-by-step addition of the integers from 1 to 100 represented in (1) into efficient and clear arrangements in two-dimensional space.

In computational terms, (2′), (2″), (3′) and (3″) are optimal algorithms to solve the problem. By contrast, (2) and (3) are not algorithmically optimal. And yet, they have clear epistemic advantages, because they transform the relationship between the numbers of the sequence into an easily perceivable symbolic pattern. This symbolic pattern is the result of the active arrangement and manipulation of symbols in space. Perhaps, then, algorithmic optimality is not sufficient to account for the cognitive processes that lead to the solution of this mathematical problem.

[9] Hayes's (2006) detailed survey indicates that three texts report that Gauss employed a different strategy. However, the reported details of the problem differ from the "folding" and "two rows" solutions, because they mention a different interval of integers (i.e., 81,297, 81,495,… 100,899). All three accounts state that Gauss solved the problem by using the following formula: $n\frac{(n+1)}{2} = x$. For the ease of exposition, we focus on the "folding" and "two rows" solutions here.

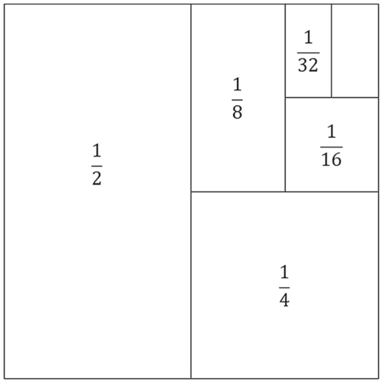

Fig. 2 Diagrammatic proof of the geometric series theorem. Adapted from Brown (2008, p. 37) and Giardino (2017, p. 501)

In addition to the manipulation of symbols in space, the manipulation of diagrams can also be integral to problem solving strategies. To see this, let us consider a second example. Imagine that your task is to provide a proof for the following theorem:

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = 1 \tag{4}$$

A very elegant way to complete the task of proving this geometric series theorem is to draw a diagram (Fig. 2). In the present context, we define diagrams as two-dimensional spatial arrangements of lines, points, and symbols that have the purpose to solve mathematical problems (for similar definitions, see Carter 2017; Giardino 2017).[10]

In the diagram, we start with a square with a side of 1. First, we divide the square in half, getting two areas of $\frac{1}{2}$. Then we divide one of these areas again in half, getting two areas of $\frac{1}{4}$. We continue by dividing one of these areas again in half, getting two areas of $\frac{1}{8}$ and so on. It is easy to see from the diagram that this process, carried out ad infinitum, fills out the square, thus showing that the sum of the series is 1.

An alternative to this diagrammatic proof is a formal proof that makes use of the ε-definition for convergence. A series is said to converge if the sequence of its partial sums converges. In our case, the sequence of the partial sums is as follows:

$$\begin{aligned}
S_1 &= \frac{1}{2} \\
S_2 &= \frac{1}{2} + \frac{1}{4} = \frac{3}{4} \\
S_3 &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} = \frac{7}{8} \\
&\cdots \\
S_n &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}
\end{aligned}$$

[10] For present purposes, we confine ourselves to the exploration of diagrammatic reasoning in mathematical domains other than Euclidean geometry. For insightful considerations on the cognitive role of diagrammatic proofs in Euclidean geometry, see Carter (2017), Giardino (2017), Shin (2012).

The ε-definition for convergence states that a series converges to the number S if for any $\varepsilon > 0$ there exists A so that $|S_n - S| < \varepsilon$ whenever $n > A$. In our series, the sum $\frac{2^n-1}{2^n}$ would appear to have the limit 1. Let us show that this is indeed the case by using the ε-definition. In order for the series to converge to 1, we need to establish that for any $\varepsilon > 0$ there is an A, so that when $n > A$, the following holds:

$$|S_n - S| = \left|\frac{2^n - 1}{2^n} - 1\right| < \varepsilon$$

With simple algebra, we can manipulate the inequation:

$$\begin{aligned}
&\left|\frac{2^n - 1}{2^n} - 1\right| < \varepsilon \\
\Leftrightarrow &\left|\frac{2^n - 1}{2^n} - \frac{2^n}{2^n}\right| < \varepsilon \\
\Leftrightarrow &\left|\frac{2^n - 1 - 2^n}{2^n}\right| < \varepsilon \\
\Leftrightarrow &\left|\frac{-1}{2^n}\right| < \varepsilon \\
\Leftrightarrow &\frac{1}{2^n} < \varepsilon \\
\Leftrightarrow &2^n < \frac{1}{\varepsilon}
\end{aligned}$$

From the definition of logarithms, we get that

$$\log_2 \frac{1}{\varepsilon} < n$$

Now we can choose $A = \log_2 \frac{1}{\varepsilon}$, in which case it holds that whenever $n > A$, it is the case that $\left|\frac{2^n-1}{2^n} - 1\right| < \varepsilon$, that is, $|S_n - S| < \varepsilon$. This concludes the proof that the series converges to 1.

Both the diagrammatic proof and the formal proof are successful in demonstrating the truth of the theorem in (4). However, they do so by different means and possibly also by different standards of rigour for mathematical proofs. In philosophy of mathematics, there is debate whether or not diagrammatic proofs should have the same epistemic status as formal proofs and if not, what their status should be (see e.g., Giaquinto 2008; Mancosu 2008; Shin 2012). There are three main views about the relation between diagrammatic proofs and formal proofs (Giaquinto 2015). First, we can accept that diagrammatic proofs can be just as valid as formal proofs (Brown 2008). Second, we can maintain that diagrams alone cannot be considered as rigorous proofs, but we can accept that they can serve a deductive function as parts of formal proofs (Barwise and Etchemendy 1996; Carter 2010). Third, diagrams cannot serve any deductive function in the proof, but they can work as cognitive tools in constructing and understanding formal mathematical proofs (Tennant 1986). A critic

Fig. 3 Unlike the geometric series, the harmonic series does not converge. However, since the members of the harmonic series approach zero, this kind of diagram may give the false idea that the harmonic series converges

could thus claim that the diagrammatic proof should not be accepted, since it relies on intuitive notions, such as that of convergence. In contrast, the formal proof is based on an explicit definition of convergence.

The lack of explicit definitions can indeed be a concern, because without knowledge of formal notions, our intuitions about concepts such as convergence can lead us astray (see, e.g., Relaford-Doyle and Núñez 2017). For example, while the geometric series indeed converges, the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \cdots$ does not, although the members of the series clearly approach zero. Without having any other explanation of convergence, a diagrammatic representation of the harmonic series might lead us to believe mistakenly that it, too, converges (Fig. 3).

However, the possibility of such misleading inferences does not render rigorous diagrammatic proofs impossible. More importantly for the present topic, it certainly should not be seen to suggest that diagrammatic reasoning does not play an important and conducive role in mathematical problem solving.[11] Looking at the two solutions to proving the geometric series theorem above, even though there may be a difference in the formal rigour of the proofs, it is clear that the general structure of the proofs is the same. We note that as the members of the series become smaller, the slower we approach the sum of 1. In the diagrammatic proof, we grasp that we can never complete the process, but if the process could be continued indefinitely, 1 would be the sum it approaches. The same idea is behind the formal proof. We notice that the partial sums come closer to 1 as the series continues. We then find out a way to show that no matter how small a value we chose for ε, we can always find a value A so that the sum is less than ε away from 1 from that point on.

This way, the cognitive importance of diagrammatic reasoning becomes apparent regardless of the role we assign to diagrams in formal mathematics. Indeed, in mathematical practice this role depends heavily on the context. Instead of line-by-line formal deductive proofs, proofs published in mathematical journals are often designed to omit details that are not seen as important for the structure of the proof, thus facilitating the communication and understanding of the proof (Avigad 2006, 2008). Depending on the subject matter and the intended audience, there can be great variance in the richness of formal details. Similarly, there can be different roles for diagrammatic methods in mathematics, depending on the socio-cultural context. These are important topics studied in the philosophy of mathematical practice (Paolo Mancosu 2008), but for the present purposes it is enough to note that under all interpretations of the role for diagrams in formal mathematics, it is clear that diagrammatic reasoning plays an important role in the mathematical problem solving process.

[11] Here we follow Giardino (2017) and others in not making a clear-cut distinction between problem solving and reasoning when it comes to mathematics. However, we are aware that these two types of processes are distinguished in cognitive psychology (Galotti 1989; Leighton and Sternberg 2012).

How can we specify the cognitive role or roles of the diagrammatic proof? Constructing the diagrammatic proof can make the correctness of (4) accessible to cognitive processes in a way that is not apparent from the formal proof (Giaquinto 2007; Tappenden 2005). Koedinger and Anderson (1990) explain this on the basis that mathematicians use diagrams to formulate key steps in proofs into readily accessible "perceptual chunks", which form an important part of the proof procedure. This way, even if we take the formal proof as the standard for mathematical proofs, the diagrammatic representation of (4) can be seen as an integral cognitive component of the problem solving process. This can happen in at least two ways. First, constructing the diagram can give an informal idea that the sum is indeed 1, which may not be as apparent in the formal presentation of the series. Second, the diagram can help grasp the structure of the formal proof or discovering a proof strategy (Giaquinto 2007). It would thus be integral to the—process leading to the conclusion that (4) is true. It is important to note, however, that from the computational point of view, constructing the diagram adds to the complexity of the algorithm used for the solution. In this case, while the diagram can be cognitively beneficial, an optimal algorithm for solving the problem would not include drawing it.

Our two examples show that both symbolic and diagrammatic representations can be integral components of mathematical problem solving. Furthermore, in the cases at hand, both symbolic and diagrammatic representations visualise and spatialise mathematical problems (Krämer 2014a). In both cases, the visualisation and spatialisation directly contribute to the solution of the problems. Considering the role of diagrams, the solution of cognitive tasks is clearly facilitated in virtue of the property of relationality that is an integral part of diagrammatic representations.[12] This property of diagrams provides the opportunity to discover unknown relations between the representata in virtue of the relationality of the representanda, i.e., configurations of lines, points, and symbols (Krämer 2016).

Returning to Marr's levels of analysis, the upshot of these considerations is that the paradigm focusing exclusively on the computational level is insufficient to understand the role that the active construction of diagrams plays in mathematical problem solving. Importantly, this is the case with all three roles we may see for diagrammatic reasoning. If we agree that diagrams can have the same epistemic status as formal proofs, we need to study diagrams as leading to algorithmically optimal solutions to mathematical problems. This stands in contrast to the computational complexity paradigm based on Turing machines presented in the previous section. The same will apply if we believe that diagrams can play an important deductive role in rigorous proofs, even though they cannot count as proofs proper. In this case we still need to study which diagrams optimally facilitate the reasoning process. Similarly, our first example indicates that the manipulation of symbols in two-dimensional space is an integral component of the solution to the mathematical problem. In contrast to ($2'$), ($2''$), ($3'$), and ($3''$), (2) and (3) are not algorithmically optimal on the received view of computational complexity. However, (2) and (3) appear to be efficient and successful ways to complete

[12] Peirce (1960–1966) is perhaps the most influential proponent of a theoretical account of the representational properties of diagrams and of diagrammatic reasoning more generally. For assessments of the Peircean account of diagrams, see Carter (2017), Giardino (2017), Stjernfelt and Østergaard (2016), Tylén et al. (2014).

the cognitive task. If we are interested in the question how cognitive agents optimally arrive at the correct solution, we need to take the active manipulation of symbols in space into consideration.

Thus, our two examples suggest that an accurate modelling of human mathematical problem solving capacities needs to include considerations on the algorithms used in constructing diagrams and manipulating symbols. Importantly, these algorithms may differ from optimal algorithms which has been in the focus of the computational complexity paradigm. To be clear, we do not want to suggest that human competence in mathematical problem solving could not be computationally modelled. Instead, we want to draw attention to the fact that cognitive agents find efficient ways to provide correct solutions to the problems at hand by making use of symbolic and diagrammatic representations in two-dimensional space. Considering the capacities and limitations of human cognisers and the panoply of mathematical practices, it is conceivable that the integration of spatially arranged symbolic and diagrammatic representations into mathematical problem solving routines is optimal in its own way.

4 Computationally optimal and humanly optimal algorithms

In computational complexity theory, we are usually concerned with general problems rather than particular cases. For example, the multiplication of two integers has been a widely studied problem in theoretical computer science. The ancient method of long multiplication of two n-digit numbers takes at most roughly n^2 computational steps (Boyer 1985; Harvey et al. 2016). However, faster algorithms for multiplication have been subsequently discovered (Karatsuba and Ofman 1962; Schönhage and Strassen 1971). To use the standard "Big O" notation describing the behaviour of functions as the arguments tend to infinity, the schoolbook long multiplication algorithm has the complexity $O(n^2)$. However, the first faster algorithm to be discovered—the Karatsuba algorithm—for example, has the complexity $O(n^{log3/log2}) \approx O(n^{1.585})$. Even faster algorithms are constantly developed (Harvey et al. 2016). Thus, when considering the general problem of integer multiplication, several algorithms are known that are more optimal than the schoolbook algorithm.

However, this does not mean that the faster algorithm is more optimal in particular cases. The Karatsuba algorithm, for example, starts to outperform the long multiplication algorithm only when the digits are hundreds of bits long (Karatsuba and Ofman 1962). For this reason, the asymptotic complexity measures give us limited insight when we want to study human problem solving competence. In integer multiplication problems, the schoolbook algorithm may not be computationally optimal, but it *can* be optimal if we limit our considerations to fixed-size inputs. When studying human cognitive competence, we are of course always working within a finite subset of problems involving infinite domains. For a description of such finite particular cases of problems, instead of the asymptotic complexity measures, it is commonplace to speak of *computational steps* as the measure of complexity for operations such as multiplication (e.g., Aaronson 2013). It should be noted that this notion of computational step is intuitive rather than well-defined, although there are suggestions of definitions in particular contexts (see e.g., Ackerman and Freer 2013). For present purposes, the important point is that computational steps describe a proper difference in the complexities of particular finite problems.[13]

To give an example, let us consider two simple cases of integer multiplication:

45 * 68

832 * 593

With the schoolbook algorithm, (a) takes four multiplication steps, whereas (b) takes nine.[14] In terms of computational complexity theory, both are particular cases of the same computational problem, i.e., the multiplication of integers. But by focusing on computational steps, we can establish a difference between their complexities. Compared to the complexity classes of complexity theory, this gives us a more fine-grained distinction. According to the widely accepted *Cobham's thesis*, we can identify *tractable* (or *efficient*) *algorithms* with those that belong to a particular complexity class **P**, which refers to the class of decision problems that can be solved by a deterministic Turing machine in polynomial time (Cobham 1965; Edmonds 1965). The class **P** is thus thought of as the class of problems that can be effectively solved by computers. Applied to cases of human cognition, this takes the form of the *tractable cognition thesis* (or *P-cognition thesis*). The P-cognition thesis states that only functions belonging to the complexity class **P** can reasonably work as models of human cognitive processes (Frixione 2001; Isaac et al. 2014; Van Rooij 2008).

Such considerations fit well with Marr's computational level of analysis. When we look for a function to model human abilities in mathematical problem solving, according to the tractable cognition thesis we should limit our pursuit to functions in the complexity class **P**. Since human cognitive competence is bounded by physical limitations, it is feasible that there is some limit to the complexity of functions that model those capacities. However, this imposes ultimately quite weak limits to the class of functions that can feasibly model cognitive tasks. Even when restricted to all the functions belonging to **P**, we are still left with an enormous class of functions. Because of such limitations of the computational complexity approach to modelling cognitive tasks, we should look for more fine-grained measures, such as the computational steps taken in particular algorithms for solving a problem.[15]

The great usefulness of this latter approach can be seen in the case of Gauss's strategy for solving the problem of adding the first 100 integers. From the purely computational perspective, the general problem of adding n integers is of some complexity C (depending on the complexity measure), which refers to the complexity of an optimal algorithm that computes the solution to the problem. However, what is

[13] In any case, computational steps for operations like multiplication can be defined explicitly in terms of state transitions of Turing machines.

[14] "Schoolbook algorithm" refers to the pen and paper method of multiplication. For mental arithmetic, different algorithms are often used. To give just one example, a multiplication like $25 * 16$, can be calculated quickly and reliably by breaking it up as $25 * (4 * 4) = 100 * 4 = 400$. While the schoolbook algorithm is used in the same way for all numbers, in mental arithmetic such "shortcuts" can be used when the numbers are appropriate (like in this case the easy multiplication of 25 * 4 making the quicker solution possible). See, e.g., Baroody (1984) for details.

[15] Another potential problem with the P-cognition thesis is that the asymptotic complexity measures may be misleading for the kind of finite inputs we are interested in when modelling human cognitive tasks. Thus, a better solution could be to consider complexity with suitable parameters concerning the size of the input (Van Rooij 2008).

interesting for present purposes is not the complexity of the general problem. Instead, we want to be able to explain how the different strategies (i.e., different algorithms) for solving the problem relate to each other. In part, this can be studied in terms of the computational steps it takes to solve the problem. The solution (1) takes 100 addition steps, which clearly makes it more complex than the solutions (2) and (3).[16]

However, that is not all that we can say about the different algorithms for solving the problem. The solutions (2) and (3) are of equal computational complexity, yet they actively make use of different spatial arrangements. As we have seen, both solutions can be then carried out by a small amount of computational steps by algorithms ($2'$) and ($3'$), respectively, which are also of equal computational complexity. But instead of focusing on ($2'$) and ($3'$), as we would in the purely computational approach, we are interested in the whole cognitive process of human problem solving. Presumably, the problem solving process is different depending on whether the solution uses the "folding" or the "two rows" strategy—not to mention the brute method of adding the numbers one by one.

This approach requires us to re-assess the study of human mathematical problem solving. While the computational approach of estimating the complexity of mathematical problems can be used in analysing the complexity of human problem solving processes, the accuracy and scope of such an analysis must be assessed as part of a wider research strategy. In both examples considered in the previous section, the active bodily manipulation of symbols or diagrams in space is integral to the solutions. In order to study the cognitive complexity of human mathematical problem solving processes in real-world contexts, such manipulations must be included in the analysis. This is the case regardless of the epistemic status we see for diagrammatic methods in mathematics. Regardless of whether or not we accept diagrammatic proof as being on a par with or conducive to formal proofs, an analysis of human mathematical problem solving should include the possibility that the cognitively most favourable path to the solution is one involving diagrammatic reasoning. Different formal solutions may be analysed in terms of their computational complexity, e.g., in terms of the computational steps required to reach the solution. Yet the different formal solutions can be connected to different spatial manipulations of symbols and diagrams. For this reason, we cannot analyse the cognitive complexity of a problem solving process merely in terms of the computational complexity of the problem. We need to take all the relevant cognitive activities that are integral components of the full problem solving process into consideration.

Importantly, this requires us to include the algorithmic level in our analysis. As we will see in the next section, the algorithmic-level analysis should also be complemented by considerations on the implementational level. This is not to suggest that the human problem solving processes could not be computationally modelled (for an example, see Anderson 2005), nor that the computational level of analysis should not be seen as an important stance for evaluating the cognitive complexity of human mathematical problem solving tasks. However, in order to be able to model human

[16] The 100 addition steps solution is the case when each addition of multi-digit numbers is considered as one addition. In reality, for human subjects the addition of multi-digit numbers is processed as several additions of single-digit numbers, with possible carrying increasing the amount of steps (Nuerk et al. 2015). The exact number of addition steps depends (among other factors) on the number base.

cognitive processes involved in mathematical problem solving accurately, we should analyse the properties of problem solving processes in a wider conceptual framework, in which the computational-level analysis is complemented by considerations on the algorithmic and implementational levels.

The upshot is that a computationally optimal algorithm for solving a mathematical problem may not always be also a cognitively optimal one for human problem solvers. Thus, the paradigm of computational complexity theory, i.e., finding the computationally optimal algorithm, needs to be amended by including analyses of *humanly optimal* algorithms (Pantsar under review). As a consequence, we are required to re-assess the concepts of competence and performance. We have argued that competence in mathematical problem solving as characterised by computationally optimal algorithms ignores crucial cognitive components, most importantly the bodily manipulation of symbols and diagrams. To include these aspects in the analysis we must have a notion of competence that is based on the actual algorithms that are cognitively optimal for human problem solvers.

This in an important change in the research strategy and it prompts an important new question. How can we identify which algorithm or algorithms can model human problem solving? For example, can the algorithms (2) and (3) both be humanly optimal for solving the problem of adding the first 100 integers? In the next section, we will propose a framework for theoretical and empirical research on mathematical problem solving that can help provide an answer to this question. The basic assumption of this framework will be that mathematical problem solving is embodied and dependent upon socio-culturally distributed cognitive practices. We will argue that the acquisition of this cognitive capacity is the result of enculturation. By evaluating empirically studied patterns in the human performance of cognitive practices, it is possible to assess the optimality of mathematical problem solving strategies used by human cognisers. In turn, this will enrich our understanding of human competence.

5 An account of enculturated cognition

We have seen in the previous sections that the active bodily manipulation of symbols and diagrams is an integral component of mathematical problem solving. However, the manipulation of symbols and diagrams needs to be modelled by humanly optimal, rather than by computationally optimal algorithms. This requires specifying the properties of mathematical problem solving and how knowledge and skills in mathematical practices come about in the course of ontogenetic cognitive development. Any theoretical account of mathematical problem solving that can meet this requirement has to provide answers to the following questions:

First, how do human cognitive systems acquire the ability to bodily manipulate symbols and diagrams in space? Second, what is the cognitive role of symbols and diagrams in mathematical problem solving? Third, how does our acquisition of skills in symbol and diagram manipulation shape our competence and performance in mathematical problem solving? Fourth, how can we specify the physical realisation of humanly optimal mathematical problem solving on the implementational level given the current state of empirical research in the cognitive sciences? What is needed in order to start developing answers to these questions is an integrative framework for the study of the acquisition and enactment of mathematical practices, which should be conceptually coherent and empirically plausible. In what follows, we will argue that the *enculturation account* meets this job description and can help us explore the cognitive process of mathematical problem solving in a way that allows specifying humanly optimal algorithms and how human cognisers can become competent mathematical problem solvers.

Our starting point is the assumption that human mathematical problem solving ability is the result of enculturation. Enculturation is defined as the acquisition of culturally evolved *cognitive practices* during ontogeny (Fabry 2017, 2018, Menary 2013b, 2015). It transforms human cognitive competence in multiple domains. Cognitive practices are culturally evolved procedures to complete cognitive tasks. In this paper we are interested in the tasks of solving mathematical problems such as the addition of integers and proving mathematical theorems. Cognitive practices are not restricted to individuals. Rather, they are socio-culturally distributed in a community of cognitive agents. For this reason, there are *cognitive norms* that constrain the ways in which cognitive practices can be realised given their *cognitive purposes* (Menary 2007, 2010, 2013a, 2015), e.g., providing solutions to mathematical problems. By definition, cognitive practices involve the manipulation of cognitive tools, which include symbol systems, diagrams, and other representational systems and artefacts.

Enculturation is characterised by plastic changes to the structural, functional, and effective connectivity of brain regions that contribute to the realisation of cognitive practices. This principle is called *learning driven plasticity* (LDP; Menary 2015). LDP is governed by the principle of *neural reuse* (Fabry 2019), according to which brain regions are re-used or re-exploited so as to contribute to multiple neural circuits associated with specific cognitive functions (Anderson 2010, 2015, 2016).[17] Furthermore, enculturation is also constituted by the development and modification of specific motor patterns and action routines. Thus, our bodily abilities to interact with cognitive tools in the local environment are adapted to the processing requirements of cognitive practices. Let us call this principle *learning driven bodily adaptability* (LDBA; Fabry 2018, 2019). In virtue of LDBA, the *bodily manipulation* of cognitive tools (e.g., symbols, diagrams) often plays an important functional role in the completion of cognitive tasks (Menary 2007, 2010; Rowlands 1999). Cases of bodily manipulation are ubiq-

[17] Menary (2015) builds his assumptions about learning-driven plasticity on Dehaene's (2010) neuronal recycling hypothesis. According to this hypothesis, the ontogenetic acquisition of cognitive practices such as arithmetic and reading is rendered possible by the recycling of already existing cerebral regions. The implication is that recycled regions lose their original function and cease to contribute to formerly established neural circuitry. We will see later on in this paper that the neuronal recycling hypothesis is at odds with the empirical evidence of the realisation of arithmetical cognition throughout ontogenetic cognitive development. In the domain of reading, the neuronal recycling hypothesis has led to the claim that the left ventral occipito-temporal region is recycled and contributes to the neural circuit realising visual word recognition (Dehaene, 2010; Dehaene et al. 2005; McCandliss et al. 2003). Importantly, Dehaene and his collaborators have argued that this region ceases to contribute to the realization of processes other than reading, for example to the recognition of faces, objects, and visuo-spatial patterns. However, there is ample evidence suggesting that this is not the case and that the ventral occipito-temporal area continues to be involved in the neuronal realisation of these other processes (Price and Devlin 2003, 2004; Vogel et al. 2012, 2014). In other words, the ventral occipito-temporal area appears to be *reused*, *not recycled*. For this reason, we prefer to focus on the neural reuse account. For details, see Fabry (2019).

uitous in human mathematical problem solving. As we will show, they include such fundamental processes as eye movements targeted at cognitive tools, the writing of symbols, and the drawing of diagrams.

Considering the process of enculturation as it applies to cognitive agents as a whole, we suggest that the acquisition of cognitive practices is realised by *scaffolded cultural learning*. Cultural learning is a specific variant of social learning, which transmits culturally evolved skills and knowledge required for the competence in specific cognitive practices within and across generations (Henrich 2016; Heyes 2012, 2016). One pervasive example of cultural learning is explicit instruction in school contexts, and it requires the active participation of learners and teachers. Cultural learning is scaffolded: it transmits the knowledge and skills required for the successful acquisition of cognitive practices at a rate that accommodates the learner's current cognitive capacities, as well as her anticipated learning trajectory (Clark 1997; Estany and Martínez 2014; Menary 2010; Sterelny 2012; Wood et al. 1976).

The theoretical perspective on enculturation provides an answer to the *evolutionary recency question* about the very possibility of mathematical cognitive practices (Fabry 2019). The first numerical symbols are approximately only 5000 years old (Donald 1991; Olson 1994; Ong 2012), so the first mathematical cognitive practices governing the manipulation of number symbols must have developed in the relatively short period after that. There could not have been sufficient time for the evolutionary emergence of dedicated neural circuitry, motor patterns, and learning mechanisms (Anderson 2010; Dehaene 2011; Menary 2014, 2015). In order to show how most human cognitive agents can become competent mathematical problem solvers, we must therefore look for other explanations. Our view is that competence in mathematical problem solving is rendered possible by the intricate combination of LDP (governed by neural reuse), LDBA, and scaffolded cultural learning.

How does this account of enculturation relate to other accounts of learning that have been developed by research in the cognitive sciences? In a recent review article, Walsh and Lovett (2016) provide an overview of different accounts of learning. Based on this overview, they identify the following conditions that a learning system needs to satisfy: "(1) the system interacts with the environment; (2) the system extracts, processes, and records information from its experiences; and (3) the system selects future behavior based on these records. Common across all definitions is the idea that learning involves change sustained over time. The definitions further specify storage of knowledge" (p. 213). Considering condition (1), the enculturation account defends the view that the interaction of human cognisers with their local environment is characterised by the embodied engagement with symbols, diagrams, and other cognitive tools. As for condition (2), mathematical cognitive practices are acquired in virtue of cultural learning. Cultural learning induces systematicity and goal-directedness in the novice's interaction with informational structures in the local socio-cultural environment. With regard to condition (3), the enculturation account postulates that learning is path-dependent in the sense that acquired knowledge and skills constrain future cognitive activities. Like other accounts of learning in the cognitive sciences, the enculturation account defends the view that learning unfolds across time. However, this account parts company with cognitivist approaches to learning because it emphasises the importance of skillful embodied practices that are socially distributed within a certain community, rather than assuming that the internal mental representation and storage of knowledge at a remove from the local environment play a crucial role in learning.

In comparison to other accounts of learning (Shanks 2010), *cultural learning*, which plays an important role in the enculturation account, operates at a more general level: it describes the inter-individual processes that lead to the acquisition of skills and knowledge. By contrast, mainstream accounts of learning in the cognitive sciences have focused on intra-individual processes that are associated with learning. On the enculturation account, however, processes of cultural learning can be associated with LDP and LDBA on an implementational level. This account thereby captures the intricate relationship of inter-individual and intra-individual processes.

From the present perspective, cultural learning can be understood as a socially distributed process with the purpose to share cognitive practices with novices and proficient cognisers. These practices are culturally evolved and are defined by the manipulation of symbols, diagrams, and other cognitive tools. In this sense, the present account has a restricted scope as it is only concerned with symbol-integrating mathematical practices, reading, writing, and other culturally shaped cognitive techniques. Unlike other accounts of learning, the enculturation provides assumptions about the functional principles that underlie learning on both algorithmic and implementational levels of analysis: LDP and LDBA. These principles are most likely realised by domain-general mechanisms that enable the acquisition of knowledge and skills. However, the enculturation account restricts itself to an exploration of LDP and LDBA as far as the acquisition of cognitive practices is concerned. The upshot is that the enculturation account is strikingly different from mainstream accounts of learning in the cognitive sciences, which range from associative (Shanks 2007) to propositional accounts (De Houwer et al. 2005). At the same time, the enculturation account promises to meet the conditions of learning systems formulated by Walsh and Lovett (2016). To substantiate this claim, we will now proceed to discuss empirical evidence and theoretical considerations that lend support to the idea that mathematical problem solving is the result of enculturation.

6 Enculturated mathematical problem solving

As argued in the previous section, human competence in mathematical problem solving is the result of scaffolded cultural learning. One of the main goals of this kind of learning is to provide cognitive agents with the relevant cognitive norms governing mathematical cognitive practices. During their first years of life, most cognitive agents build on their (possibly innate) proto-arithmetical skills, i.e., numerosity approximation and subitising, by learning to count and to use numerals to refer correctly to cardinalities (Carey 2009; Dehaene 2011; Fabry 2019; Pantsar 2014, 2018, in press; Spelke 2000). This development is highly culturally influenced, as seen in the way children learn to count. The Chinese numeral system, for example, follows the base ten structure more closely than the English one. Whereas English-speaking children need to learn new words “eleven” and “twelve”, in Mandarin the corresponding words are “shí-yī” (literally: ten-one) and “shí-èr” (literally: ten-two). Note that even when English starts to use the base ten structure, it first reverses the order (from thirteen to nineteen) and only starts using the placeholder structure in numerals starting from twenty-one. In Mandarin, this structure is present already from eleven and continues from there. Miller et al. (1995) have suggested this as the reason why Chinese children in comparison to US-American children are faster in learning to count beyond the first few numerals (which show no structure), as well as their consequent higher performance in mathematics.[18]

The cultural influence, however, is not limited to language-related factors. Through explicit instruction and structured novice–teacher interactions, cognitive agents acquire knowledge about cognitive norms that govern mathematical cognitive practices. In the initial stages, for example, they need to acquire norms about the systematic correspondences between quantities, number words, and numerical symbols (Merkley and Ansari 2016). In the case of the Indian-Arabic symbol system, the acquisition and application of the place-value norm is a particularly important developmental step towards mathematical competence. According to the place-value norm, the magnitude of a symbolically represented number (>9) is determined not only by the value of the composite digits, but also by their spatial arrangement. Many other cognitive norms, such as those governing the manipulation of operators (e.g., parenthesis norms, precedence norms) are of vital importance for the developmental trajectory leading to competence in mathematical problem solving.

As a result of enculturation, mathematical cognitive practices are associated with a unique neuronal profile. This is established in virtue of LDP and it is governed by neural reuse in the course of ontogenetic cognitive development. In enculturated cognitive agents, activations in the bilateral intraparietal sulci (IPS) are consistently associated with arithmetical cognitive practices (Ansari 2008; Hannagan et al. 2015; Lyons et al. 2015). Importantly, empirical evidence suggests that these regions also contribute to number approximation (Ansari 2008; Dehaene 2005, 2011; Lyons et al. 2015), which is already detectable in infants, and to counting (Castelli et al. 2006). The idea is that the bilateral IPS are reused for arithmetical cognition.[19] Other crucial cortical regions that make important contributions to neural circuitry associated with enculturated mathematical practices are the bilateral ventral inferior temporal gyri (vITG). These regions are consistently associated with the processing of visually presented numerical symbols (Amalric and Dehaene 2016; Hannagan et al. 2015).[20]

18 It should be noted that many languages are more confusing than English in this sense. In German, for example, the order of tens and ones is always reversed (sixty-four, for example, is *vierundsechzig*, literally "four-and-sixty". In French, ninety-eight is *quatre-vingt-dix-huit*, literally "four-twenty-ten-eight".

19 This suggests the functional changes to the bilateral IPS is a matter of neural reuse, and not neuronal recycling, since these brain regions continue to contribute to neuronal processes associated with number approximation, rather than being exclusively associated with arithmetical processes. This lends additional support for our view that neural reuse is preferable to neuronal recycling as a mechanism that underlies LDP (see footnote 17).

20 As expected, given the large cultural differences in educational systems and mathematical learning strategies, there are culture-specific differences in the neuronal profiles of numerical processing, even when the same number symbols are used. Tang et al. (2006), for example, report an fMRI study according to which native Chinese speakers show more tendency to engage in visuo-premotor association of numbers than native English speakers as evidenced by activation levels in the premotor association area and the supplemental motor area. By contrast, mental calculation processes in native English speakers are associated with increased activation levels in the left perisylvian cortex, which plays an important role in the neuronal

Recent neuroscientific studies have revealed that there are systematic differences in the neuronal activation patterns associated with mathematical cognitive practices in expert mathematicians in comparison to non-expert control participants (Amalric and Dehaene 2016, 2018). The completion of a judgment task that requires the evaluation of mathematical statements in analysis, algebra, topology, and geometry is associated with a reliable activation pattern across mathematical domains in experts, but not in control participants. In addition, comparisons of activation patterns associated with the judgment of mathematical versus non-mathematical statements in experts identify a neural circuit that comprises the bilateral IPS, bilateral inferior temporal regions (including the ITG), as well as dorsal, superior, and mesial parts of the prefrontal cortex in both hemispheres, and the cerebellum (Amalric and Dehaene 2016, 2018). These findings indicate that "advanced mathematics, basic arithmetic, and even the mere viewing of numbers and formulas recruit similar and overlapping cortical sites in mathematically trained individuals" (Amalric and Dehaene 2016, p. 4913; see also Amalric and Dehaene 2018; Ansari 2016).[21]

In sum, current neuroscientific evidence suggests that competence in mathematical problem solving is associated with neuronal activations in brain regions contributing to neural circuitry that is the result of learning driven plasticity governed by neural reuse. To the best of our knowledge, neuroscientific research has focused on symbol-integrating mathematical cognitive practices. Consequently, it remains an open question how diagram-integrating mathematical practices are neuronally realised. However, we would like to offer a conjecture based on the current state of research. Hubbard et al. (2005) review empirical studies that explore the relation between the neuronal realisation of basic mathematical and spatial cognitive capacities. They conclude that "various protocols indicate that numbers automatically elicit task-, modality- and effector-independent spatial representations, even when these

Footnote 20 continued
realization of language processing. Tang and colleagues suggest the wider use of the abacus in China as one reason for this, alongside factors such as shorter number words in Chinese.

[21] Amalric and Dehaene's (2016, 2018) studies are in part motivated by the question whether or not mathematical cognitive practices are derived from phylogenetically older linguistic capacities that capitalize on the recursivity of linguistic structures. They attribute the assumption that mathematics is a derivative of language to Chomsky (2006): "According to Noam Chomsky, 'the origin of the mathematical capacity [lies in] an abstraction from linguistic operations'" (Amalric and Dehaene 2018, p. 1; see also Amalric amd Dehaene 2016, p. 4909). This clearly misconstrues Chomsky's (2006) position. The entire sentence, which is only partly quoted, reads as follows: "Speculations about the origin of the mathematical capacity as an abstraction from linguistic operations are not unfamiliar" (pp. 184-185). Irrespective of this misconstrual of Chomsky's (2006) position, the question about the dependency of mathematical competence on linguistic competence has sparked interest in cognitive neuroscience recently. Contrasting the evaluation of mathematically meaningful and of mathematically meaningless linguistic statements (Monti et al. 2012) and of mathematically meaningful and random symbol strings (Maruyama et al. 2012), the main finding of two independent fMRI studies is that the neuronal activation patterns associated with the evaluation of mathematically meaningful symbolic representations is markedly different from the neuronal activation patterns associated with the evaluation of mathematically meaningless symbolic representations. In both studies, the evaluation of mathematically meaningful symbolic representations is associated with activations in brain regions that have been found to play an important role in mathematical cognition (e.g., the right IPS and other parietal regions). By contrast, the evaluation of mathematically meaningless linguistic statements in the study by Monti et al. (2012) is associated with activations in brain regions that are typically associated with language processing (e.g., left inferior frontal gyrus, left middle and superior temporal gyri). This indicates that the neuronal realization of competence in mathematical problem solving and in written language processing are clearly distinct.

spatial representations are not strictly relevant to the task" (Hubbard et al. 2005, p. 438). They attribute this fact to converging evidence suggesting that the bilateral IPS and other brain regions are activated by basic mathematical *and* spatial cognitive tasks.[22]Hubbard et al. (2005) consider the possibility that "[s]patial-numerical interactions might have been progressively shaped by cultural conventions, such as the orientation of writing or the conventional orientation of mathematical graph axes" (p. 437). This theoretical possibility would lend further support to the important role of neural reuse for enculturated mathematical cognitive practices. Furthermore, it could imply that the integration of diagrams into mathematical problem solving routines can play an important role precisely because spatial and mathematical capacities interact both functionally and neuronally. It can also lend support to the idea that the manipulation of symbols and diagrams in space is beneficial for mathematical problem solving (see Sect. 3), because the manipulations rely on neural mechanisms of mathematical and spatial cognition that are integrated in proficient and expert mathematical problem solvers.

In recent years, the important contribution of the enculturated bodily manipulation of representations in public space has been explored by eye-tracking and behavioural experiments. Schneider et al. (2012) report three consecutive experiments, which investigate the eye movement patterns associated with arithmetical problem solving. For current purposes, the first and second experiments are of particular relevance. In these experiments, arithmetical expressions are either structured explicitly by parentheses (Experiment 1) or structured both explicitly and implicitly by cognitive precedence norms (Experiment 2). Here are examples of the used material (adapted from Schneider et al. 2012, p. 477, Fig. 2):

	Left-branching	Right-branching
Explicit structure (parenthesis norms)	$((3+2)-1)+4=?$	$4+(1-(3+2))=?$
Implicit structure (precedence norms)	$(3*2-5)+4=?$	$4+(5-3*2)=?$

The main finding of both experiments is that the first fixations for each trial are located at the left side of the algebraic expression, which is consistent with the direction of reading of both mathematical and linguistic expressions. After that, fixations are consecutively targeted at arithmetically relevant components of the expression in an

[22] The most robust effect detected by these studies is the *spatial-numerical association of response codes* (SNARC) effect (Dehaene 2011; Everett 2017; Hubbard et al. 2005; Shaki et al. 2009; Tschentscher et al. 2012). When asked to indicate which numeral of a pair is greater by pressing the assigned key on a keyboard, cognitive agents who have been enculturated in a socio-cultural environment in which left-to-right is the prevalent direction of writing is the norm, both for linguistic and mathematical symbols, they are significantly faster in identifying the larger numeral if the appropriate key is on their right hand-side. This effect is reversed in cognitive agents who are enculturated in an environment in which right-to-left writing is the norm (Shaki et al. 2009). Intriguingly, Shaki et al. (2009) also found that Israelis are not subject to the SNARC effect, because in this cultural community "words are read from right to left and number from left to right" (p. 330). Overall, the SNARC effect has been interpreted as evidence for the idea that the relationship between numerical and spatial cognition is often characterized by "cognitive intertwinement" (Everett 2017, p. 207).

ordered fashion. Consider the example of a left-branched expression the combines explicit and implicit structure. The idea is that

$$(3 * 2 - 5) + 4 =?$$

is decomposed by eye movement patterns so that

$3 * 2$

is fixated first, then

$- 5$

and finally

$+ 4$

to arrive at the correct solution:

$(3 * 2 - 5) + 4 =?$

This indicates that the manipulation of symbolic algebraic expressions in terms of targeted eye movement patterns plays a crucial functional role in algebraic mathematical problem solving. These eye movement patterns are the result of learning driven bodily adaptability, which can be further specified by taking the scaffolded acquisition of cognitive norms into consideration. As suggested by Schneider et al. (2012), eye movement patterns are structured by cognitive parenthesis and precedence norms, which are acquired in the course of scaffolded cultural learning. This is in line with the finding reported by Goldstone et al. (2010) that the eye movement patterns of enculturated cognitive agents in an arithmetical problem solving task are governed by implicit precedence norms. Here are two examples of the material used in that study:

$2 \times 3 + 4 =?$

$2 + 3 \times 4 =?$

The main result of the study is that participants' "very first eye movements tended to be toward the multiplication, and gazes to multiplications lasted longer than gaze to additions" (Goldstone et al. 2010, pp. 275–276).

The relevance of the tight relationship between bodily manipulation and the cognitive norms governing mathematical problem solving is further elucidated by a recent study reported by Inglis and Alcock (2012). In an experiment exploring eye movement patterns associated with proof validation in expert mathematicians and undergraduate mathematics students, they find that fixation locations, fixation durations and the sequence of fixations for symbols in consecutive lines differ across expert mathematicians and students. The main difference is that algebraic manipulations receive more and longer fixations in undergraduate students, whereas the parts of the proofs that deliver the logical structure of the proof receive more and longer fixations in expert mathematicians. As evidenced by reading paths, mathematicians moved more back and forth between different lines of the proofs compared to the students. According to Inglis and Alcock, these findings suggest that expert mathematicians are less likely to read proofs line by line and more prone to infer "implicit between-line warrants" (Inglis and Alcock 2012, p. 384). This indicates that expertise in enculturated mathematical problem solving is associated with eye movement patterns that can be distinct even from eye movement patterns associated with the highly proficient competence in enculturated mathematical problem solving that undergraduate students possess. The upshot is that LDBA leads to unique ways to solve mathematical problems as indicated by eye movement patterns. Importantly, mathematical symbolic structures are manipulated by *active looking*, rather than by merely seeing or visually perceiving them (Dewey 1896; Findlay and Gilchrist 2003).

This account of the important functional role of eye movements for mathematical problem solving gives rise to the question whether a similar case can be made for the contribution of eye movement patters to diagrammatic mathematical problem solving. As we have argued in Sect. 3, the manipulation of diagrams can make important contributions to solving mathematical problems. To the best of our knowledge, the question about the properties of eye movement patterns associated with diagrammatic mathematical problem solving has not been explored empirically. However, there is evidence for the important role of eye movement patterns for the solution of a diagrammatically presented insight problem solving task (i.e., Duncker's radiation problem).[23] Experiment 1 of two consecutive experiments reported by Grant and Spivey (2003) shows that the location of fixations in two-dimensional diagrammatic space and their duration directly contributes to the successful solution of the problem. We submit the empirically testable hypothesis that a similar effect should be detectable for the development of diagrammatic proofs. In particular, the hypothesis is that success in providing a diagrammatic proof, or in creating a diagram that directly contributes to the successful development of a formal proof, is dependent upon the location and duration of fixations.

In Sect. 3, we showed that the manipulation and exploitation of symbolic and diagrammatic structures in two-dimensional space can play an important role in mathematical problem solving. In addition to the eye-tracking studies discussed above, psychological research has begun to systematically investigate how enculturated cognitive agents exploit space to solve mathematical problems in a number of behavioural experiments (Goldstone et al. 2010, 2017; Landy and Goldstone 2007a, b, 2010). A series of experiments on algebraic competence in undergraduate students systematically manipulates the spatial arrangement of symbolic algebraic expressions (Landy and Goldstone 2007b, 2010). In particular, the spacing between individual symbols is systematically manipulated. The main finding of all three experiments is that the spatial arrangement of mathematical symbols in two-dimensional space has a significant impact on the accuracy levels of the participants, and thus on their success in solving algebraic mathematical problems. In particular, the compliance of spacing with mathematical cognitive norms was positively associated with accuracy levels.

The above experiments suggest that the bodily manipulation of symbols is associated with an integration of *movement space* and *structural space* (Krämer 2014b). According to Krämer (2014b), structural space is "characterized by the coexistence of places and the relations between them" (p. 10). Movement space "comes into being through the motions of actors, and which is temporally bound to them" (op. cit., pp. 10–11; see also Giardino 2016). Through the integration of movement space

[23] Duncker's (1945) radiation problem is represented by a diagram that shows a tumor which is enclosed by healthy tissue. The healthy tissue is surrounded by the skin, which in turn is surrounded by the outside local environment. Grant and Spivey's (2003) instruction reads as follows: "Given a human being with an inoperable stomach tumor, and lasers which destroy organic tissue at sufficient intensity, how can one cure the person with these lasers and, at the same time, avoid harming the healthy tissue that surrounds the tumor?" (p. 462).

and of structural two-dimensional space on a piece of paper or a computer screen, enculturated cognitive agents are able to competently solve mathematical problems.

To sum up, we argue that competence in mathematical problem solving is the result of enculturation. In virtue of LDP, LDBA, and scaffolded cultural learning, cognitive agents are able to complete mathematical tasks by actively manipulating representations in public two-dimensional space. This explains why the kind of spatial manipulations involved in Sect. 3 are crucial to problem solving. If this account of mathematical problem solving is on the right track, we will have to revise the received conceptions of competence and performance that have guided and inspired computational modelling and the Marrian prioritisation of the computational level of analysis over the algorithmic and implementational levels.

Before moving on, we would like to take stock and summarise why the enculturation account can significantly contribute to our understanding of mathematical problem solving. In the beginning of Sect. 5, we identified four questions that a suitable account of mathematical problem solving should be able to answer. The first question was how human cognitive systems acquire the ability to bodily manipulate symbols and diagrams in space. According to the enculturation account, the acquisition of this ability depends upon scaffolded cultural learning. On a sub-personal level of description, the acquisition of abilities in manipulating symbolic and diagrammatic representations is realised by LDP and LDBA. Considering the second question about the cognitive role of symbols and diagrams in mathematical problem solving, the enculturation account shows that the manipulation of symbols (and diagrams by conjecture) clearly facilitates the solution of mathematical problems as evidenced by the eye-tracking studies reported in this section. This facilitation can be explained in terms of the integration of structural space and movement space. The third question was how our acquisition of skills in symbol and diagram manipulation shapes our competence and performance in mathematical problem solving. According to the enculturation account, the acquisition of competence should be associated with specific neuronal and functional signatures that set expert and proficient mathematicians apart from novices. In this section we have reviewed clear empirical evidence from neuroimaging studies (Amalric and Dehaene 2016, 2018), eye-tracking studies (Goldstone et al. 2010; Inglis and Alcock 2012; Schneider et al. 2012), and behavioural studies (Goldstone et al. 2017; Landy and Goldstone 2007a, b, 2010) in support of this hypothesis. As for the last question concerning the physical realisation of mathematical problem solving, the interpretation of these empirical studies from the perspective of the enculturation account with an emphasis on LDP and LDBA enriches our understanding of the underlying neuronal and ocular-motor processes. In particular, the enculturation account is necessary for showing how culturally evolved cognitive abilities can be acquired and enacted in the ways indicated by empirical research in the cognitive sciences. As we will see in the remainder of this paper, the present account can help us understand how humanly optimal algorithms as developed on the algorithmic level can be informed and constrained by considerations on the implementational level.

7 Enculturated competence and performance in mathematical problem solving

If the considerations presented in the previous sections are largely correct, an account of mathematical problem solving should make room for the idea that the active manipulation of symbols and diagrams is an integral component of cognitive performance *and* competence. This view stands in contrast to Chomsky's considerations on the seminal distinction between competence and performance. To strengthen our position, we would like to discuss a recent addition that Chomsky has made to his previous work. This discussion will bring our main assumption about competent mathematical problem solving into focus, which will be further developed in the remainder of this paper. In his preface to the 50th anniversary edition of *Aspects of the theory of syntax*, Chomsky extends his notion of competence to include general and context-independent capacities that are required for arithmetical problem solving. He describes arithmetical competence as follows:

> Arithmetical competence yields the correct number z for every pair (x, y) under addition or multiplication. But only a small finite subpart of arithmetical competence can be exhibited *without external aids* (*by calculating in one's head*). Obviously, the fact does not imply that arithmetical competence is correspondingly *limited* (Chomsky 2015/1965, p. xii; emphasis added).

For Chomsky, the exhibition of arithmetical competence, i.e., arithmetical performance, is limited in comparison to arithmetical competence itself, because cognitive agents rely on the manipulation of "external aids". However, our considerations in Sect. 3 and our empirically informed treatment of mathematical problem solving (above and beyond arithmetic) strongly suggest that the integration of "external aids", i.e., of cognitive tools, into mathematical problem solving routines often plays a crucial role. If our view is largely correct, we have good empirical and theoretical reasons to assume that the normatively constrained bodily manipulation of cognitive tools is in fact an integral part of mathematical competence. This suggests that mathematical *competence*, and not only performance, is the result of enculturation. On this view, because the possibilities of "calculating in one's head" are limited, as Chomsky correctly notes, we frequently rely on the systematic manipulation of diagrams and numerical symbols in two-dimensional space. We therefore propose that our overall cognitive capacities in mathematical problem solving are constituted by *enculturated competence*. Furthermore, the plethora of empirical studies we have reviewed and discussed above attests to the idea that we need to investigate enculturated competence by systematically studying *enculturated performance*. The accumulation and theoretical integration of empirical data give rise to plausible and robust assumptions that are open to further empirical and theoretical scrutiny.

How does this account of enculturated competence and performance relate to Marr's (1982) postulation of computational, algorithmic, and implementational levels of analysis?

Recall from Sect. 2 that for Chomsky (2015/1965), the analysis of linguistic competence rests on the assumption of an idealised and perfectly capable cognitive agent who is not subject to cognitive limitations or biases. Based on all the considerations above, at least in the domain of mathematical problem solving, this conception of competence is hardly tenable. The reason is that cognitive agents heavily rely on the manipulation of symbols and diagrams as a result of enculturation. Given this, it will be of limited explanatory and descriptive value to assume that competence in mathematical problem solving is a matter of idealised, context-independent cognitive capacities.

As a direct consequence, human competence in mathematical problem solving cannot be analysed exclusively in terms of optimal algorithms on Marr's computational level. Rather, as our examples in Sect. 3 and our theoretical considerations in Sect. 6 strongly suggest, enculturated competence in mathematical problem solving requires us to consider *humanly optimal* algorithms on the algorithmic level. Furthermore, considerations on humanly optimal algorithms need to be integrated with assumptions about their neuronal and bodily realisation on the implementational level. In contrast to Marr (1982), we therefore assume that the analysis of competence does not stop at the computational level, but needs to include assumptions and constraints on the algorithmic and implementational levels. Only then can we hope to work towards a conceptually coherent and empirically plausible account of human competence in mathematical problem solving.

In turn, the multi-level analysis of competence in mathematical problem solving requires us to include empirical data and theoretical assumptions on enculturated performance. The reason is that competence is rendered possible by a scaffolded and normatively constrained developmental trajectory of performance in mathematical problem solving, which ultimately gives rise to enculturated competence. In turn, it manifests itself in a multitude of performances. If we combine this idea with the empirically informed considerations on proficient and expert mathematical problem solving presented above, we will arrive at the suggestion that our approach allows for different types of enculturated competence. This is in contrast to Chomsky's (2015/1965) account of competence, which does not leave room for the empirically supported insight that there are types of competence that are qualitatively and quantitatively different, ranging from proficient to expert competence in mathematical problem solving.

On all levels of expertise in mathematics, both within and across domains, there are differences in the ways in which particular cognitive agents acquire and apply cognitive practices. Already in learning simple operations, such as integer multiplication, some children may have difficulties grasping and using the standard algorithms (Fuson 2003).[24] These children may benefit from alternative algorithms for the operations, which often involve different manipulations of symbols in space (Carroll and Porter 1998; Randolph and Sherman 2001). Perhaps the best-known alternative to the standard long multiplication algorithm is the lattice (or tableau) multiplication method. It originates in the thirteenth century and has been in use during different time periods

[24] Here "standard algorithm" and "alternative algorithm" can be understood either as being descriptive or prescriptive. In the descriptive case, "standard algorithm" refers to the algorithm most commonly used by the members of a certain socio-cultural community. In the prescriptive case, "standard algorithm" refers to the algorithm that is favoured by the cognitive norms governing mathematical problem solving in a certain socio-cultural community.

in the Arab world, India, China and Europe (Chabert 1999). In lattice multiplication, multi-digit integers are arranged spatially in the following fashion, here the example is 235*28 (Fig. 4):

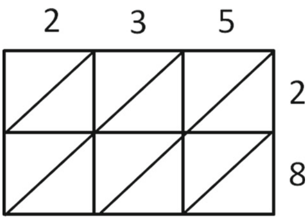

Fig. 4 Lattice multiplication of 235 and 28

At the next stage, each single-digit multiplication is carried out, with the digit for tens placed in the upper-left corner of the square and the digit for ones to the lower-right corner.

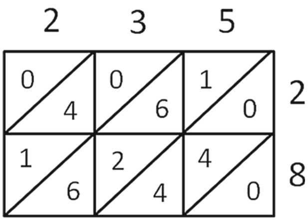

Fig. 5 Single-digit multiplications are carried out

At the final stage, the digits are added diagonally as follows (if the sum is more than 9, the digit is carried to the next diagonal sum, as shown here) (Fig. 5):

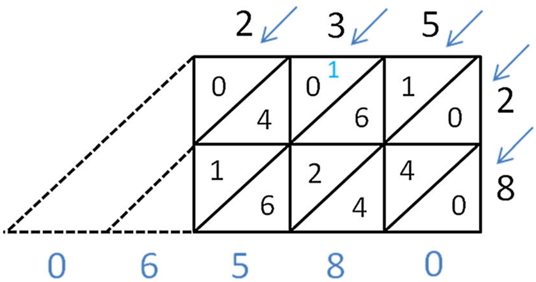

Fig. 6 At the final stage the digits are added diagonally

The answer, 6580, can then be read directly from the lowest row (Fig. 6).

Problem solving strategies like lattice multiplication have been suggested as a tool for children with learning problems (e.g., Gurganus 2007) and the availability of different strategies for children is often seen as facilitating the learning process for students with different knowledge and skill levels (Fuson 2003; Randolph and Sherman 2001). But equally importantly, differences in the way methods are learned may be due to differences in the cultures in which the subjects are enculturated. Instructional practices, textbooks and many other aspects regarding mathematics education can vary a great deal across different socio-cultural communities (see, e.g., Huntly et al. 2012). This variation can also happen across time periods, as seen in the way the lattice multiplication method has been used historically. In fifteenth century Europe, it was standardly presented in arithmetic textbooks (Chabert 1999), while the long multiplication algorithm has taken over the curriculum in modern times.

As far as we know, there exists no systematic study of different multiplication strategies in terms of their cognitive benefit for children (or adults) with different cultural backgrounds or learning abilities. Students who are taught to use one strategy throughout their education may have difficulties grasping a new strategy, due to the unfamiliarity with the manipulations of symbols in space.[25] This can be explained by the enculturation account we have proposed. The culturally set scaffolded learning trajectory, which includes symbol manipulation and space and diagrams, determines which problem solving strategies are available for an individual. Starting from basic number concepts, the developmental trajectory of enculturation continues all the way to the level of expertise in mathematics. On all levels of competence, we should expect the kind of important differences in mathematical cultures that are revealed by research on mathematical practices (see, e.g., Larvor 2016).

What we want to argue for in this paper is that such differences in enculturation cannot be treated as superficial differences in the manifestation of some cross-culturally constant mathematical competence. Based on the above considerations, it is highly unlikely that there exists universal competence in mathematical problem solving. Thus, in order to do justice to the culturally shaped way in which mathematical cognitive practices develop, the focus must be on *enculturated* competence. Enculturated competence includes general limitations to human problem solving capacity due to constraints of our cognitive architecture, such as working memory limitations. But, importantly, it also includes culturally specific ways in which our problem solving abilities have developed.[26]

One of the great epistemic advantages of our account of enculturated competence and performance in mathematical problem solving is that it can shed new light on the

[25] In fact, there is evidence that enculturation plays a crucial role in the development of mathematical cognition already in the acquisition of number concepts. The Amazonian tribe of Pirahã do not have a stable numeral system in their language (Gordon 2004). Frank et al. (2008) have argued that this prevents them from representing exact quantities, which renders learning even basic arithmetic almost impossible for mono-lingual members of the Pirahã (see also Everett 2017).

[26] In this way, the present account can be seen as compatible with, e.g., Simon (1968/1996), according to whom behaviour is optimized to "reflect characteristics largely of the outer environment" (p. 53). In our account, mathematical problem solving strategies should be assessed in terms of limitations due to our cognitive architecture, but they are also determined by cultural factors. Therefore, it would be mistaken to state that the problem solving strategies are optimized to reflect the characteristics of the outer world in a strong realist sense. But if "outer environment" is understood in a culturally sensitive fashion, the optimization of problem solving strategies can be characterized in terms of reflecting the outer environment. We thank an anonymous reviewer for pointing out this connection.

examples presented in Sect. 3. Recall from the first example that the "folding" and the "two rows" solutions to the integer addition problem described by Hayes (2006, 2017), Krämer (2016), and many others heavily rely on the arrangement and manipulation of symbols in two-dimensional space. Considering the second example, we have seen that diagrammatic mathematical problem solving can contribute to the development of a proof of the geometric series theorem, regardless of the epistemic status we ultimately assign to diagrammatic proofs in mathematics. From our new perspective, we are now in a position to specify *why* the manipulation of symbols and diagrams is an integral, often indispensable component of completing mathematical cognitive tasks. The reason is scaffolded cultural learning, which equips novices with socio-culturally shared skills and norms for the completion of mathematical tasks. Given socio-culturally shared cognitive practices and the properties of culturally evolved tools, the manipulation of symbols and diagrams turns out to be an efficient and effective way to solve mathematical problems. This would be unimaginable without learning-driven bodily adaptability and its interaction with learning-driven plasticity as sub-personally realised mechanisms.

Whatever the epistemic status of diagrammatic proofs may be, enculturated competence in diagrammatic mathematical reasoning consists not in "a single manipulation, but rather a set of procedures, not abstract rules, but instructions on how to act on the diagram and to read and interpret it correctly" (Giardino 2010, p. 37). This gives rise to the view that socio-culturally developed and shared cognitive norms constrain the ways in which symbols, lines, and points can be meaningfully arranged and modified.[27] In the case of providing a diagrammatical proof of the geometric series, these norms are shared and maintained within a community of expert mathematicians. Importantly, our discussion of the epistemic status of diagrammatic proofs also shows that these cognitive norms are overtly scrutinised and possibly revised as part and parcel of scientific mathematical practices.[28] By contrast, the cognitive norms governing the solution to the integer addition problem are less restrictive, but still govern the ways in which symbols and lines can be arranged horizontally and vertically by the embodied exploitation of two-dimensional space. In other words, the presented solutions to

[27] This is in line with Krämer's (2016) following assumption: "Embedded in normatively shaped practices, which do not need to be explicit, but are often implicitly anchored in cultural habits, diagrams organize shared epistemic experiences" (p. 80; italics removed; our translation).

[28] It helps sharpen our theoretical understanding of diagrammatic problem solving to briefly compare it to Giardino's (2016, 2017) account. Our and Giardino's account differ in two important respects. First, Giardino (2017) highlights the *heterogeneity* of reasoning and continues to argue that "humans happen to rely on many different sorts of instruments with the aim of *externalizing thought*, diagrams being among them" (p. 500; emphasis added; see also Giardino 2010, 2014, 2016). The important difference between Giardino's account and ours is that we do not subscribe to the view that the bodily manipulation of diagrams *externalises thought*. Rather, our assumption is that the bodily manipulation of diagrams is deeply integrated into mathematical problem solving and directly contributes to the completion of cognitive tasks. Second, we part company with Giardino's (2016) suggestion that "[d]iagrams as well as other kinds of cognitive tools can serve as *convenient inferential shortcuts* if the space they display is correctly interpreted and permissible actions are performed on it" (p. 97; emphasis added). We suggest that the bodily manipulation of diagrams and other cognitive tools accomplishes more than providing "convenient inferential shortcuts". Rather, they augment and transform mathematical competence and have a direct impact on the normatively constrained reasoning capacities of proficient and expert mathematicians. This is in full agreement with Krämer (2014b), who argues that "[k]nowledge is not only represented, transmitted and disseminated through the diagrammatic; it is *produced and expanded* by it" (p. 3; emphasis added).

the integer addition problem perfectly illustrate "the phenomenon of problem-solving through spatialized ordering" (Krämer 2014b, p. 5).[29]

8 Enculturated competence and computational modelling

What are the consequences of our proposal to empirically and conceptually investigate mathematical problem solving in terms of enculturated competence and performance on multiple levels of analysis for attempts to develop computational models? Is it possible, after all, to identify and construct humanly optimal algorithms that model competence in mathematical problem solving? A good starting point is the observation that the importance of the bodily manipulation of public representations for the completion of cognitive tasks has been acknowledged in the philosophical literature on connectionism almost 30 years ago (Clark 1990). Given human cognitive processing needs and biases, enculturated cognitive agents alter current task demands, for example by manipulating symbols and diagrams (see Sect. 3) in ways that facilitate the problem solving process. Discussing Rumelhart's et al. (1986) proposal on connectionist networks, Clark (1990) suggests that "insofar as human beings actually exhibit the full scale classical competence they do so only by deploying *other resources* (for example, a linked symbol processor or real-world structures (like pen and paper) for manipulating symbols)" (pp. 208–209; italics in original). This suggests that it is not only theoretically warranted, but also methodologically feasible to integrate the manipulation of cognitive tools into connectionist models of competence. By implication, this supports the view that computational models can and should be devised to integrate the bodily manipulation of symbols and diagrams into models of competence in mathematical problem solving.

We have argued that human mathematical problem solving processes should be studied within the framework of enculturation. As we have seen in previous sections, this position is supported by results from neuroimaging, eye-tracking, and behavioural studies. The review of neuroimaging studies strongly suggests that mathematical problem solving is associated with neuronal activation patterns that are the result of learning driven plasticity. The physical realisation of mathematical problem solving as evidenced by neuroimaging and eye-tracking studies shows important variation based on different levels of mathematical competence. From these and the other results reviewed in Sect. 6, a consistent picture emerges: Mathematical problem solving is the result of enculturation and the very idea of enculturation suggests that there are different levels of competence in this domain. The characteristics of scaffolded cultural learning can vary across different groups, within and across socio-cultural communities. Sub-personally, learning driven plasticity and learning driven bodily adaptability lead to neural circuitry and embodied action patterns that realise mathematical problem solving in different ways, depending on the level of expertise.

In our view, the identification and modelling of human problem solving processes requires us to be sensitive to both the stable *and* the variable properties of mathematical

[29] The integer addition problem is solved by exploiting what Krämer (2014b) calls the *operative iconicity* of public representations in mathematics and other domains.

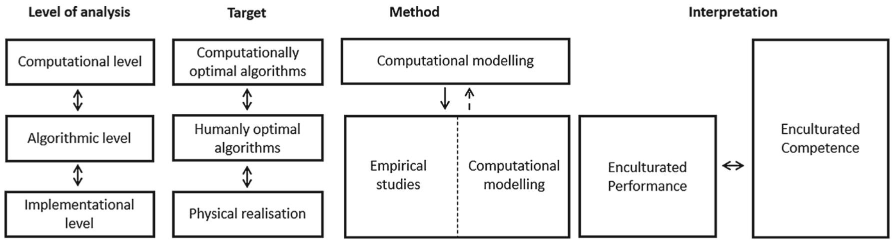

Fig. 7 In analysing competence in the present framework, both the algorithmic and the implementational level are included. Instead of discussing general competence and performance, the framework focuses on enculturated competence and performance in the domain of mathematical problem solving. By studying humanly optimal algorithms and their physical realization, we gain an understanding of enculturated performance, which is used to analyse enculturated competence. Note that also computationally optimal algorithms can play an explanatory role in this model by their connection to humanly optimal algorithms

practices. Our positive proposal is that the enculturation account provides a theoretical perspective that allows us to systematically explore the stability and the variability of mathematical practices on algorithmic and implementational levels of analysis.[30] Crucially, the framework offers conceptual resources to consider the empirically and theoretically supported ramifications of the important role of the bodily manipulation of symbols and diagrams in two-dimensional space. An accurate analysis of this role cannot be limited to the computationally optimal problem solving algorithms. Instead, the focus should be on the question which problem solving algorithms are *humanly* optimal.

One direct consequence of adopting the enculturation framework is that we must extend the analysis beyond Marr's (1982) computational level. In order to accurately model the process of solving a problem *P*, we cannot be restricted to studying computationally optimal algorithms for solving *P*. Instead, our analysis needs to include aspects of the actual problem solving strategies that are not part of an optimal algorithm for solving P. This framework for the analysis of enculturated competence is presented in Fig. 7.

In this paper, we have identified the bodily manipulation of symbols and diagrams in space as a crucial component of human problem solving processes. Computationally, a solution involving the construction of a diagram can be suboptimal. However, it is in many cases cognitively optimal for human problem solvers, given the knowledge and skills that have been acquired in the course of enculturation. In order to identify and model such aspects, the analysis needs to include considerations on the algorithmic level. In turn, algorithmic-level considerations are indispensably influenced by empirical insights into the neuronal and bodily realisation of mathematical problem solving as specified on the implementational level.

A critic of our approach could argue that it appears to be difficult to reconcile our emphasis on the bodily manipulation of symbols and diagrams with the Marr (1982) commitment to representationalism. In Marr's (1982) original treatment, the analyses on the algorithmic level were assumed to be specified by the postulation of

[30] An important task for future research on humanly optimal algorithms will be a reassessment of existing computational models of mathematical problem solving.

representations. However, similar to Ramsey (2017), we deny that a commitment to Marr's levels entails a commitment of representationalism. In the analysis of cognitive processes, it is a viable strategy to include functional specifications like algorithms without assuming that they have to be understood as representational vehicles.

In this vein, we assume that it is possible to develop an algorithmic-level analysis of mathematical problem solving without being committed to a representationalistic view of the mind. For example, it is possible to specify mathematical problem solving (and other cognitive tasks) on the algorithmic level by focusing on the complementarity of cognitive functions and symbolic (and diagrammatic) structures in the local environment without the need to evoke representations as theoretical entities (Clark 1990). Another possibility is to cash out the algorithmic details of a cognitive task within the predictive processing scheme (Clark 2016; Friston 2010; Hohwy 2013). In this case, the computational-level analysis of Bayesian "ideal observer models" (Zednik and Jäkel 2016, p. 3955) would correspond to the algorithmic-level analysis of a cognitive task in terms of "approximate Bayesian inference" (Love 2015, p. 234). The upshot is that we need to identify and model humanly optimal algorithms in order to arrive at an empirically and theoretically plausible account of enculturated competence in mathematical problem solving. The identification and modelling of humanly optimal algorithms does not dependent upon a representationalistic analysis of mathematical problem solving on algorithmic and implementational levels.

There is another potentially controversial aspect to our research strategy. In extending the analysis to include humanly optimal problem solving algorithms, it may be asked whether we suggest to study mathematical *performance* rather than competence. This would be a misinterpretation of our argument. As specified in Sect. 7, the purpose of the research strategy presented in this paper is not to identify all the variations of problem solving algorithms used by individuals. Instead, the aim of the analysis is to identify and model cross-individual *enculturated competence*. The enculturatedness of norm-governed mathematical practices implies that these practices are widely shared within subgroups of populations. Rather than the individual variations in performance, we identify enculturated competence in mathematical problem solving as the relevant target phenomenon.[31]

Critics might maintain that even if we are interested in enculturated competence, doesn't this approach still emphasise studying performance? After all, can we hope to reveal the characteristics of enculturated competence without studying performance?

[31] Although our focus is on enculturated competence, one potential advantage of our account over computational-level explanations is that it might be better at predicting errors that human problem solvers may systematically commit. By studying enculturated competence through enculturated performance, the research can be sensitive to the ways in which performance may be flawed as a negative side-effect of enculturation. The harmonic series case discussed in Sect. 3 would be a good candidate for systematic errors in this context. As with enculturated mathematical problem solving processes in general, there is nothing to suggest that the systematic errors could not be computationally modelled. In fact, research on errors that are predicted by computational models was already conducted decades ago for procedural skills such as arithmetical problem solving (Brown and VanLehn 1980). Errors in carrying and borrowing in arithmetical problem solving by the schoolbook methods, for example, can be predicted by computational models. It is likely that such errors can be traced back to failures in following enculturated problem solving strategies and therefore the enculturation account can be helpful in predicting, and repairing erroneous problem solving methods. This is a highly interesting topic that should be tackled in future research. We thank an anonymous reviewer for making this point.

Indeed, this is an integral aspect of our proposed research strategy. Rather than estimating a priori the optimality of algorithms based on considerations of computational complexity, the present approach is committed to the exploration of the *enculturated cognitive optimality* of algorithms, which is decisively informed by empirical research. Enculturated cognitive optimality is thus a fundamentally distinct notion from computational optimality. Whereas the latter is characterised by algorithms using a minimal amount of computational resources (i.e., time or space), the former is designed to include any aspect relevant for the optimality of human problem solving strategies, e.g., the bodily manipulation of symbols and diagrams in space. Characterising these aspects necessitates moving beyond the computational level paradigm of studying competence. Moreover, it becomes clear that enculturated competence and enculturated cognitive optimality cannot be explored without studying the performance of actual human problem solvers.

While this insight is in conflict with Marr's (1982) focus on the computational level of analysis, it is fully in line with recent empirical studies of mathematical problem solving, such as Goldstone's et al. (2010) attempt to integrate empirical research on algebraic reasoning and computational modelling. As reviewed in Sect. 6, the studies reported by Goldstone et al. (2010, 2017) and Landy and Goldstone (2007a, b, 2010) suggest that the bodily manipulation of symbols is a crucial component of mathematical problem solving. Goldstone et al. (2010) argue that the results of these studies cannot be accommodated by computational models of competence that focus on computationally optimal algorithms. In particular, they argue as follows:

> We should resist the temptation to posit mental representations with forms that match our intellectualized understanding of mathematics. A more apt input representation to give a computational model (D. Landy, unpublished data) would be a visual–spatial depiction of notational elements that includes their absolute positions, spacings, sizes, and accompanying nonmathematical pictorial elements (Goldstone et al. 2010, p. 279).

The computational models influenced by this proposal would be models that include considerations on enculturated cognitive optimality. Such accounts need to take empirical results about the implementational details of mathematical problem solving into account. In this way, it becomes possible to study enculturated competence through enculturated performance in a systematic, computationally tractable, and empirically plausible way.[32] Our task in this paper was to develop a conceptual framework for studying mathematical problem solving. We noted that such a framework must be sensitive to central parts of human problem solving strategies like the manipulation of symbols in space and construction and interpretation of diagrams. This has led us to re-assess the notions of competence and performance relevant to mathematical problem solving. In particular, our conceptual framework helps reveal the relationship between competence and performance from the perspective of enculturation.[33]

[32] It should be noted that this approach of studying competence through performance is nothing new. Clearly, the Chomskyan notion of competence requires that an account of competence is grounded in the empirical and theoretical study of performance (Fitch et al. 2005).

[33] This framework bears important similarities with Clark's (1990) description of *rogue models*. Clark is interested in the ways in which connectionist explanations based on neural networks can provide the

9 Concluding remarks

In this paper, we have proposed to re-consider the seminal distinction between the study of competence on the computational level and the investigation of performance on the algorithmic and implementational levels of analysis in our attempt to understand mathematical problem solving. If our account of enculturated mathematical problem solving is largely correct, one important consequence will be that enculturated cognitive optimality, rather than computational optimality, is key to an empirically plausible perspective that can inform future work on the development and refinement of computational models of mathematical problem solving. We have emphasised the important functional role of the bodily manipulation of symbolic and diagrammatic structures for enculturated competence in mathematical problem solving and we have shown that this functional role is indeed supported by several lines of empirical evidence. This functional role can only be captured, we submit, if enculturated cognitive optimality, rather than computational optimality is the foundation of computational models.

This paper should be seen as a starting point for further considerations on the properties of mathematical problem solving and how they could be modelled computationally. Therefore, many important open questions for future research remain. These questions are of both theoretical and empirical concern. On theoretical grounds, our assumptions on enculturated cognitive optimality should be specified. Relatedly, we seek to elaborate on our suggestions concerning the potential influence of these theoretical assumptions on attempts to computationally model mathematical problem solving. Second, our perspective on enculturated competence in mathematical problem solving has important implications for research on mathematical cognition in philosophy of cognitive science (e.g., Menary 2015). If our account of mathematical problem solving is on the right track, it will lend support to positions that emphasise the important role of our embodied and embedded encounters with our socio-cultural environment. It also broadens the scope of current research efforts to understand mathematical cognition, because it adds important considerations on the role of the bodily manipulation of diagrams in mathematical problem solving. Our account also speaks to current research in philosophy of mathematics. This is the case in the philosophy of

Footnote 33 continued
resources to model human's competence in the completion of a certain cognitive task under ecologically valid conditions. In contrast to *Newtonian models*, in which "the connectionist network is *itself* capable, under idealised conditions, of behaving in all the ways specified by the competence theory", rogue models take the important functional role of the manipulation of symbolic (or diagrammatic) structures into account (Clark, 1990, p. 208; italics in original). More specifically, "[i]n a rogue model […] the basic connectionist network does not itself have the capacity (even under idealizations of processing time and well-posed problems) to produce the full range of results required by (i.e., derivable) the competence theory" (op. cit.). The implication of our conceptual framework for computational modelling is entirely consistent with this idea of rogue models. In rogue models, the cognitive competence postulated by a certain theory is modelled by specifying the complementarity of the model itself and the symbolic (or diagrammatic) structures that serve as input and output of the connectionist network. Unlike computational models in the Marrian paradigm, rogue models encompass computational and algorithmic levels of analysis and are informed by implementational-level considerations, for example on distinct eye movement and neuronal activation patterns in proficient and expert mathematicians reviewed in Sect. 6. In fact, Clark's (1990) consideration of rogue models is an important theoretical precursor of our proposal to explore enculturated cognitive optimality, and to study enculturated competence in mathematical problem solving by accumulating evidence about enculturated performance.

mathematical practice, where diagrammatic reasoning is a key topic (e.g., Mancosu 2008; Carter 2010, 2017; Giardino 2016), but also generally in the epistemology of mathematics. Our considerations on enculturated mathematical problem solving are particularly relevant for approaches that focus on visual thinking in mathematics (e.g., Giaquinto 2007, 2008). In general, empirically-informed epistemological theories of mathematics (e.g., Lakoff and Núñez 2000; Pantsar 2014, 2015, 2016, 2018) can gain a great deal from a better understanding of the enculturated nature of mathematical cognitive practices.

Ultimately, our account of enculturated mathematical problem solving also has implications for educational practice.[34] Based on our considerations on enculturated cognitive optimality, we have argued that there are different, socio-culturally entrenched ways in which mathematical problems can be solved. This allows for inter-individual differences in the optimality of different problem solving strategies, which could inform instruction methods in both elementary and expert mathematics.

In addition to these primarily theoretical questions, there are also desiderata for future empirical research on mathematical problem solving in cognitive science. For example, in comparison to symbolic mathematical problem solving, diagrammatic mathematical problem solving has not received much attention in empirical research using neuroimaging and eye-tracking methodologies. It would therefore be important to test our hypothesis on the cognitive roles of diagrams in mathematical problem solving. Furthermore, following up on the findings reported by Amalric and Dehaene (2016, 2018) and by Inglis and Alcock (2012), it would be desirable to intensify the empirical study of systematic differences in the neuronal and sensori-motor realisation of mathematical problem solving across different levels of expertise. In mathematics education, the enculturation approach can provide a fruitful theoretical framework for empirical research on the suitability of diverse cognitive practices for different domains of mathematics, different levels of instruction, and students with different learning profiles. Although the importance of spatial manipulation of diagrams, symbols and objects is generally accepted in mathematics education, there is a great deal of research to be done on these issues. Establishing an accurate account of enculturated mathematical competence in the context of scaffolded cultural learning would form an important basis for such research.

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[34] Our view is largely consistent with Cobb's (1994) integration of constructivist assumptions on the one hand and assumptions about the ontogenetic cognitive development of mathematical practices that emphasise the importance of socio-culturally situated processes on the other hand. In particular, we share Cobb's (1994) claim that "mathematical learning should be viewed as both a process of active individual construction and a process of enculturation into the mathematical practices of wider society" (p. 13). We are grateful to an anonymous reviewer for bringing this to our attention.

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