THE GENEALOGY OF '∨'

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Abstract. The use of the symbol ∨ for disjunction in formal logic is ubiquitous. Where did it come from? The paper details the evolution of the symbol ∨ in its historical and logical context. Some sources say that disjunction in its use as connecting propositions or formulas was introduced by Peano; others suggest that it originated as an abbreviation of the Latin word for "or," *vel*. We show that the origin of the symbol ∨ for disjunction can be traced to Whitehead and Russell's pre-*Principia* work in formal logic. Because of *Principia*'s influence, its notation was widely adopted by philosophers working in logic (the logical empiricists in the 1920s and 1930s, especially Carnap and early Quine). Hilbert's adoption of ∨ in his *Grundzüge der theoretischen Logik* guaranteed its widespread use by mathematical logicians. The origins of other logical symbols are also discussed.

> No topic which we have discussed approaches closer to the problem of a uniform and universal language in mathematics than does the topic of symbolic logic. The problem of efficient and uniform notations is perhaps the most serious one facing the mathematical public. No group of workers has been more active in the endeavor to find a solution of that problem than those who have busied themselves with symbolic logic ... Each proposed a list of symbols, with the hope, no doubt, that mathematicians in general would adopt them. That expectation has not been realized.
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> Cajori (1929, p. 314)

§1. The mystery of '∨'. The symbol '∨' for inclusive disjunction is almost universally accepted. This contrasts with every other logical operator: negation is symbolized by '¬', '∼', '–', or an overline, conjunction is symbolized by '∧', '&', and '.', and (material) implication is symbolized by '→' and '⊃'. So where do we get '∨' from and why have logicians more or less settled on this one symbol? We

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answer this question here and show that we get the ubiquitous '∨' for disjunction from Whitehead and Russell's pre-*Principia* work in formal logic.

Our answer to this question differs from that given by earlier studies of formal logic and its notations. The earliest use of '∨' for disjunction recorded in the encyclopedic list given by Cajori (1929, p. 307) is in Whitehead and Russell's 1910 *Principia*. The Kneales say that "the system [which includes '∨'] is that introduced by Peano in his *Notations de logique mathématique* of 1894, developed in the successive editions of his *Formulaire de mathématiques*, and then perfected by Whitehead and Russell in their *Principia Mathematica* of 1910" (Kneale & Kneale, 1962, p. 520). In fact, however, Peano always used '∪' for inclusive disjunction and did not use '∨' at all. So one might suspect that the symbol for disjunction is one of the improvements introduced by Whitehead and Russell in *Principia Mathematica*. But is that true? Why was '∨' chosen specifically? And why was it then universally adopted even as logicians introduced alternatives for all of *Principia*'s other logical symbols?

The story has two parts: The first part concerns the introduction of '∨' by Whitehead and Russell, who were working within the notational and formal logical tradition of Peano. We suggest here that the initial introduction of '∨' was motivated by a shared commitment to two design principles for logical symbolisms (Schlimm, 2018, 2021; Toader, 2021).[1] The first principle is that symbols should be *unambiguous*— one shouldn't use the same symbol in different meanings. The second principle is that analogous notions—better, notions satisfying analogous logical laws (say, associativity)—should get *similar* symbols: similar notions should have a similar, though not identical, notation.[2]

As we suggest below, this first design principle led Peano to adopt a new symbol, namely '∪', to indicate a kind of logical union or addition, and to avoid the then-prevalent symbol '+' for this use. The point of this was to distinguish arithmetic and logical addition. Russell followed Peano's practice here, as we will see below.[3] Subsequently, in the early logical investigations which ultimately led to *Principia*,

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[1] For an extended discussion of these two design principles, see Schlimm (2018) on Frege, and Schlimm (2021) on Peano. We take the phrase "design principles" from Schlimm. Toader (2021) highlighted the second principle.

[2] Peano puts this in terms of a *principle of permanence*: "…when establishing a new system of notations, or a new calculus, it is convenient to do it such that the new calculus be similar *as much as possible* with old calculi, so that the student does not have to learn a whole new calculus, but only the differences from the theory known to him" (quoted from Toader, 2021, p. 85).

[3] Although Whitehead and Russell's pre-*Principia* work in formal logic is our source for '∨', we will sometimes speak of Whitehead and of Russell alone because the manuscripts or publications that we discuss are single-authored. Often we speak of Russell alone since Russell's manuscripts comprise most of the pre-*Principia* materials that survive; indeed, the surviving Whitehead manuscript that we discuss below was kept in Russell's papers. Many of the pre-*Principia* draft materials no longer survive. Yet in Russell's case (but sadly not in Whitehead's) the surviving materials provide a detailed record of his evolution both in notations and in design principles for notations. We do not think it is wrong to take the more extensive surviving textual record left behind by Russell as a record of Whitehead's evolving views on notation and design principles for notation pre-*Principia* because Whitehead and Russell of course had to reach agreement on such points before *Principia* was published. See also the discussion in Section 5. Note that, as Russell (1948, pp. 137–138) recalls, Whitehead invented a great deal of the symbolism in *Principia*. See also the remarks by Urquhart (1994, sec. V) and Moore (2014, sec. XI).

Russell introduced the symbol '∨' motivated by that same design principle that Peano held: Russell wanted to distinguish notationally between two kinds of logical addition, namely, propositional disjunction (a.k.a. "propositional addition") and class union (a.k.a. "class addition"). In this, Russell out-Peanoed Peano, who used '∪' for both.[4] But Russell followed the second design principle, that analogous notions should have similar symbols, in adopting '∨' as a sharpening of '∪' so as to stress the analogy between propositional sum and class union. Russell did not think that this analogy was perfect. He held that the formal laws for propositional addition and class addition differed; hence he felt a need for a different, yet similar, symbol.[5] But even where the formal laws are identical, Russell wanted to mark the distinction between operations on different terms, as in the case of class union and relation union. This pattern is noticeable in, for instance, Russell's development of many symbols of his logic of relations, which are usually distinguished only by an accenting mark, suggesting that Russell tacitly followed the second design principle, albeit a weaker version of it than Peano apparently endorsed.

All of that is the first part of the story. The second part of the story is that of the adoption of ∨ in the period after *Principia*, when a number of logical symbolisms (and indeed, logical methodologies) were in use simultaneously. The main rivals of *Principia* were the algebraic approach of the tradition of Boole, Peirce, and Schröder, and the axiomatic logic of Hilbert, with their respective notations. Despite being heavily influenced by both *Principia* and the algebraic notation of Schröder (which goes back to Peirce and the Boolean school), Hilbert introduced different symbols for all the logical connectives—except disjunction.

One upshot of the above findings is to uproot a misconception about the origins of the modern symbol for disjunction. A common story in textbooks is that ∨ really is a version of the letter 'v', the initial letter of *vel*, the Latin word for "or." We wanted to find out whether this was a typographical accident, and later writers simply thought the similarity between '∨' and the initial of *vel* should be suggested as a mnemonic device to students, or whether it actually played a role in the choice of the symbol. Here we argue that although there is a single use of 'v' as a logical symbol abbreviating *vel* in Leibniz, this played no known role in Peano's choice or in Whitehead and Russell's choice of notation for disjunction. However, this historical connection may, at least in part, have lead Hilbert to adopt it.

As usual in the history of mathematics, there is no simple answer to the question of who deserves credit for '∨'. In one sense, it can be answered simply by recording the earliest use of the symbol. But the earliest occurrence is often not the use to which the current practice of the use of the symbol is historically connected. For instance, Peirce in fact used '∨' for disjunction before Whitehead or Russell did—but this use was independent and played no role in the historical development of logical notation systems. (In this case we know this for a fact because Peirce's relevant writings were not published until long after the use of '∨' was already entrenched, in 1933.) So it is

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[4] It should be noted that neither Peano nor Russell followed this principle quite perfectly. Peano used '⊃' for both implication and deductive consequence (Schlimm, 2021, sec. 3.2.1). And *Principia* used square dots both for conjunction and for scope indications.

[5] Bernstein (1932, secs. 2–3) criticized the view that these two notions are disanalogous in the way that *Principia* suggests in comments on ∗4·78·79.

only by tracing the adoption of Russell's symbol that we can establish Russell as the originator of "our" '∨'.

There is a related question, namely that of what should count as a symbol for propositional disjunction in the first place. Is Peano's '∪' a symbol for disjunction, or is it a symbol for something else, perhaps an ambiguous combination of disjunction and union? Who invented disjunction in the sense we now use it—as a connective between truth-apt expressions? This is certainly a deeper and more interesting question, but one we cannot answer here. Our contribution is more modest: to trace the history of the symbol '∨' only, and not the concept it expresses.[6] We do this not by merely recording its first use, but by placing it in the context of the development of logical symbolisms. Of course, even this is a much larger context which we cannot do justice here—we focus only on the parts that are relevant to '∨'.

Both of these considerations lead us to reject the claim that Leibniz's use of 'v' should be considered the origin of '∨'. As we will show, (a) it had no known influence on the eventual introduction and development of '∪' and '∨' by Peano, Whitehead, or Russell, and (b) Leibniz's 'v' was not a symbol for propositional disjunction.

We settle the questions of who invented the use of '∨' for disjunction (Whitehead or Russell) and whether this use of '∨' was in fact introduced as an abbreviation for *vel* (it was not). We suggest, however, that even though these questions may seem trivial, they are not uninteresting, and we will mention, along the way, other interesting aspects of the development of logical symbolism. Further, the story serves as a case study in how the choice and adoption of symbols and the organization of a symbolic system (a kind of conceptual framework) can be determined by methodological commitments but also by extrinsic factors such as the availability or cost of printer's type or the influence that textbooks have on the adoption of symbolisms. In this instance, in addition to the commitment to unambiguous notations in Peano and Russell mentioned above, another factor concerns the reasons that led Russell to choose disjunction as a primitive of the logical system of *Principia*.

§2. Leibniz.   Leibniz deserves some credit for the first use of 'v' as a symbol for 'or' in some sense.[7] But Leibniz's use of 'v' is not for propositional disjunction, and occurs in only one manuscript, his 1679 "Matheseos universalis pars prior: De terminis incomplexis" (Figure 1):

> Also, in the way that + is a conjunctive mark or sign of aggregation corresponding to the word *and*—as in $a + b$, that is, a and b together—so a disjunctive mark or sign of alternation is also given, which corresponds to the word *or* [*vel*], so $a \hat{v} b$ signifies a or b to me. It also has a use in calculus, for if we have $xx + ab = \overline{a + b}x$, then x will be equal to $a \hat{v} b$, whether it signifies a or whether it signifies b. In this

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[6] Our use of 'the concept' in the singular is a stylistic convenience: we do not mean to imply that there is a unique concept throughout the history of logic that correlates naturally to propositional disjunction and that is now widely symbolized using '∨'. For an in-depth treatment of the concepts of disjunction (see Jennings, 1994).

[7] We make no claim here about Leibniz's chronological precursors. We did, however, consult three medievalists who specialize in history of logic about antecedent uses of ∨ for inclusive disjunction, who did not recall any earlier uses.
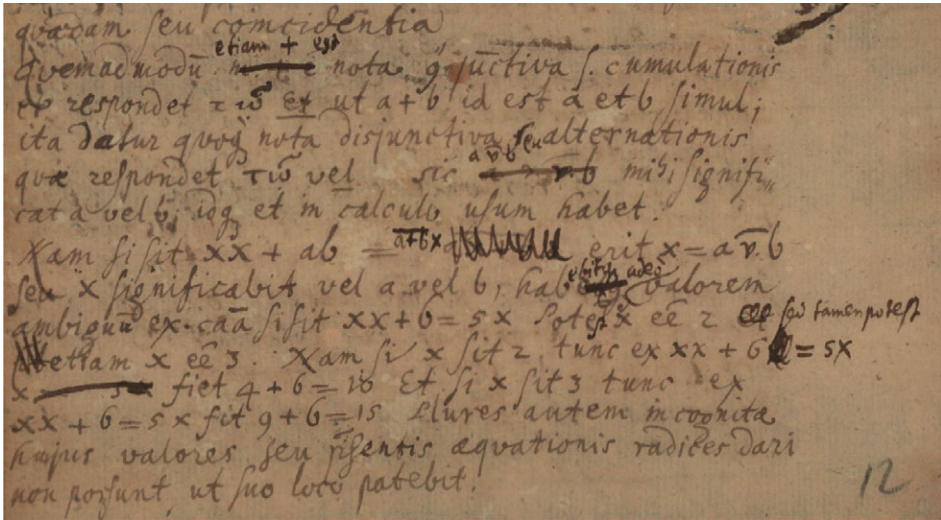
Fig. 1. Use of v̂ in Leibniz 1679).

respect it will have an ambiguous value. For example, if $xx + 6 = 5x$, x can be 2, but x can also be 3. For if x is 2, then from $xx + 6 = 5x$, we get $4 + 6 = 10$; and if x is 3, then from $xx = 6 = 5x$, we get $9 + 6 = 15$. However, many values of this unknown [variable] or of the present equation cannot be given roots, as will be clear in the appropriate place below.

Hence, ambiguous signs also have a use, and this will be clear in the appropriate place below, ambiguity being the font of irrationality in calculus; and so when I write $x = 3 + \sqrt{4}$, this can be explicated by $3 + \sqrt{4}$ or $3 + 2$ or 5, just as when I write $3 - \sqrt{4}$, it is explicated either by $3 - 2$ or 1; thus, we get $x = 5 \, \hat{v} \, 1$. For in order for us to remove the irrationality, let $x - 3 = \sqrt{4}$; therefore $xx - 6x + 9 = 4$, whether $xx - 6x + 5 = 0$ or $xx + 5 = 6x$, where it is clear that it would satisfy 5 as much as 1. For if x were given the value 5, we would get $25 + 5 = 30$; if x is given the value 1, we get $1 + 5 = 6$.[8]

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[8] "Quemadmodum etiam + est nota conjunctiva seu cumulationis et respondet τῷ et, ut $a + b$ id est a et b simul, ita datur quoque nota disjunctiva seu alternationis quae respondet τῷ vel, sic $a \, \hat{v} \, b$ mihi significat a vel b. Idque et in calculo usum habet, nam si sit $xx + ab = \overline{a + b}x$, erit $x = a \, \hat{v} \, b$ seu x significabit vel a vel b habebitque adeo valorem ambiguum. Ex. causa si sit $xx + 6 = 5x$, potest x esse 2, sed tamen potest etiam x esse 3. Nam si x sit 2, tunc ex $xx + 6 = 5x$ fiet $4 + 6 = 10$; et si x sit 3, tunc ex $xx + 6 = 5x$ lit $9 + 6 = 15$. Plures autem incognitae hujus valores seu praesentis aequationis radices dari non possunt, ut suo loco patebit.

Hinc usum quoque habent signa ambigua, et suo loco patebit, ambiguitatem in calculo esse fontem irrationalitatis; itaque cum scribo $x = 3 + \sqrt{4}$, tunc id potest explicari tam per $3 + \sqrt{4}$ seu $3 + 2$ seu 5, quam per $3 - \sqrt{4}$ seu $3 - 2$ seu 1, adeoque erit $x = 5 \, \hat{v} \, 1$. Nam ut tollamus irrationalitatem, sit $x - 3 = \sqrt{4}$; ergo $xx - 6x + 9 = 4$ seu $xx - 6x + 5 = 0$ seu $xx + 5 = 6x$, ubi patet satisfacere tam 5 quam 4. Nam si x valeat 5, fiet $25 + 5 = 30$; sin [sic] x valeat 4, fit $1 + 5 = 6$" (Leibniz, 1679, 9v and 9r; 1863, pp. 57–58), translation courtesy of

This occurrence of '$\hat{v}$' is the first use of a '$\vee$'-like symbol for 'or' (in some sense) that we know of.[9] However, Leibniz does not seem to use the symbol here as a sign for disjunction of (truth-apt) propositions as is usually the case today. Indeed, the surrounding context does not include any uses of 'a' or 'b' for propositions. Rather, Leibniz seems to be using 'a' and 'b' here as standing for (truth-inapt) terms, and using '$a \hat{v} b$' to indicate *ambiguity*, that is, one of two alternatives without a specific one being indicated, as in phrases like "either chocolate or vanilla" or "7 or 8."[10] That use of '$\hat{v}$' for alternation between truth-inapt terms is consistent with the other two uses '$a \hat{v} b$' and '$5 \hat{v} 1$', where the symbol is flanked by (variables for) numbers and serves to produce an ambiguously referring term.[11]

These three uses, in this one manuscript, seem to be the only uses of the symbol '$\hat{v}$' in the entirety of his *Mathematische Schriften* (Leibniz, 1863). We also found it nowhere else in Leibniz's writings. So this manuscript may record the only uses of '$\hat{v}$' for alternation of (truth-inapt) terms in Leibniz's corpus. The manuscript was not edited or published until 1863. It is perhaps unsurprising, then, that Leibniz's use of '$\hat{v}$' for disjunction (which, so far as we can tell, was confined to this one manuscript) found no imitators: even Arnauld's (1683) *La logique ou l'art de penser*, which was an especially influential book in the 1700s and in the 1800s before Boole's work, omits 'v' for disjunction. Logicians in the 1700s and 1800s largely used symbols, when they did at all, for the relations between terms in categorical logic and not propositional disjunction; indeed, most logicians either did not symbolize propositional disjunction and argued it was eliminable as a form of reasoning, or else did not discuss it all (see Hailperin, 2004, p. 339, for discussion).[12]

To qualify this slightly, a close textual reading of one or another figure may extract, induce, or produce a propositional logic in some sense.[13] Even so, our interest lies in the story of how '$\vee$' became so widespread as a symbol for disjunction in modern usage. It is our contention that Leibniz and other logicians working primarily with categorical logics—between Leibniz and Boole, we might say—did not inspire or cause the usage that we now have today.

Jack Zupko). Leibniz seems to be using the Greek article $\tau\tilde{\omega}$ to indicate mention rather than use; Leibniz underlined the corresponding occurrences of 'et' and 'vel' in the first sentence.

9 Why did he put an inverse breve over 'v'? We conjecture that Leibniz wanted to distinguish the constant near-nonce symbol '$\hat{v}$' from the variable 'v' that he used frequently (e.g., Leibniz, 1679, pp. 57 and 59). Another possibility is that it indicates an abbreviation, like a scribal *siglum*. However, the inverse breve was not usually used for this purpose, and the standard abbreviation for *vel* was a lowercase 'l' (ell), not 'v', combined with a vertical stroke (ł) or macron (ī).

10 We thank the reviewer for helpfully suggesting that we put this Leibnizian use of '$\hat{v}$' in terms of ambiguity.

11 See also Mugnai (2018, p. 195) on disjunction in Leibniz, and Mugnai (2010) more generally on logic and mathematics in the seventeenth century.

12 Whately (1831, sec. IV.4), for example, argues that all disjunctions of categorical propositions are conditional in meaning due to the equivalence of 'A is B or C is D' with 'if A is not B, then C is D; and if 'C is not D, then A is B' (see also Mill, 1843, chap. IV, p. 110).

13 To take one notable example, Lenzen (2004) argues that a system of propositional logic can be extracted from one of Leibniz's categorical logics, and that many of its axioms are so-formulated that they might reasonably be recognized by us today as commonplace principles in logic textbooks treating the propositional calculus.

§3. The nineteenth century. In the course of mathematically analyzing logical principles using algebraic equations, Boole used '+' as a symbol for something akin to '∨'. In Boole's algebra, however, '+' is not a symbol for disjunction in our sense, since the expressions that flank '+' are not signs for propositions or statements, but for concept expressions,[14] and so has rightly been thought of as an algebra of classes (as we might put it), or as an algebra of concepts in extension (as Boole seems to put it).

Boole can appropriately use '1' for the universe, multiplication for class intersection, and subtraction for complementation (Boole, 1847, p. 20). Although Boole's operators '+' and '·' applied primarily to classes, he also considered symbols 'x' for propositions as expressing the "cases" where the proposition X in question is true. The expression '$x + y$' then expresses the union of the cases where X and Y are true—assuming, however, that these classes are disjoint. Claims about the propositions are made using equations involving 1 and 0 (the empty class). Multiplication is re-interpreted to define (the algebraic analogue of) propositional conjunction; similarly subtraction is re-interpreted as (the algebraic analogue of) propositional negation. Then these operations so-interpreted are similarly used to define (the algebraic analogues of) propositional implication and propositional disjunction. In Boole's system, that either X is true or Y is true would be expressed using the equation $x + y - xy = 1$ (Boole, 1847, p. 51).

Of course, Boole's choice of '+' for disjunction (Boole, 1847, 1854) had a more immediate influence on notation for disjunction in the 1800s.[15] Subsequent choices of symbols for operations on classes were influenced by Boole even when logicians departed from Boole's philosophical program to varying degrees. For example, Jevons (1883) uses '·|·' for non-disjoint union, to distinguish it from Boole's '+'.[16] Peirce (1870, p. 9) similarly uses '+' for disjoint union and adopts '+,' for non-disjoint

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[14] "Literal symbols, as x, y, &c., representing things as subjects of our conceptions" (Boole, 1854, p. 27). It should be noted that we do not mean to imply that Boole's intended interpretation of his symbols was fixed throughout his life. Indeed, Peckhaus (1997) and others have argued that there are significant differences between Boole (1847) and (1854) in their view of what terms pick out (concepts versus extensions of concepts). These differences are important, but incidental to our discussion since, either way, the terms are not truth-apt (even though they can be used to induce or extract a propositional logic, or can be applied in a manner that mirrors propositional logics, much as one can do with Leibniz's categorical logic).

[15] In an 1848 manuscript "The Nature of Logic," Boole instead used '+' for (the algebraic analogue of) propositional conjunction, adding parenthetically, "which indeed was originally a contraction of the Latin *et*" (Boole, 1997, p. 6). In an 1856 manuscript "On the Foundations of the Mathematical Theory of Logic and on the Philosophical Interpretation of Its Methods and Processes," he repeats this point and suggests that the use of '+' for *et* and of '–' for *minus* originates "in medieval Latin manuscripts" (Boole, 1997, p. 87).

[16] In his 1864 "Pure Logic," Jevons followed Boole in using '+' for disjoint union (Jevons, 1864, p. 80). He changes the notation in *The Principles of Science* for a similar reason that leads Peano to adopt ∪ (see below): "[T]his sign [+] should not be employed unless there exists exact analogy between mathematical addition and logical alternation. We shall find that the analogy is imperfect, and that there is such profound difference between logical and mathematical terms as should prevent our uniting them by the same symbol. Accordingly I have chosen a sign ·|·, which seems aptly to suggest whatever degree of analogy may exist without implying more" (Jevons, 1883, p. 68).

union.[17] In "The Simplest Mathematics" (1902), which was not published until 1933, Peirce actually used '∨' interchangeably with his '⊽' symbol for inclusive or. By 1933, of course, the use of '∨' for disjunction was already entrenched. Other important writers on the algebra of logic, such as Schröder (1877, 1890), MacColl (1877), and Ladd-Franklin (1883), all use '+' for union, and so does Whitehead (1898) originally. Even Nagy (1890), writing in Italian and familiar with Peano's work, sticks to Boole's notation.

Common to the followers of Boole is not just the choice of symbols, but also the exclusive focus on equations. Using equations, they express relations between class terms, and provide translations of syllogistic reasoning into such equations. However, MacColl (1877), Peirce (1885), Ladd-Franklin (1883), and Schröder (1891) took the further step of showing that terms could translate propositionally valid inferential relationships. They allowed operators like '+' to connect not just terms for classes or concepts but also terms for propositions or statements, and developed systems of formulas, not just of equations. In their work, the symbol '+' serves as a genuine propositional disjunction (in the inclusive sense).[18]

Working alongside the algebraic logicians were Frege and Peano, who both were important sources for Russell. Frege had his own two-dimensional notation (first introduced in *Begriffsschrift* and later developed in *Grundgesetze*). He was able to define disjunction using the content strokes for implication and negation, but did not introduce a separate symbol for it (Frege, 1879, p. 11). Regarding Frege's notational influence on the story of '∨', the record shows that Frege's actual notations and two-dimensional style were rarely adopted (and sometimes were ignored or even maligned) by others (with the exception of his popular turnstile '⊢'). But Frege's notational influence on Russell was nonetheless felt: indeed, Frege's views on *design principles* for notation seem to have been quite influential on Russell, even though his choice of actual symbols was not. Russell's engagement with Frege's writings seems to have led Russell to later distinguish propositional sum and class sum, for which purpose he introduced our '∨'—as we will see. First, however, we consider the influence of Peano's notations and logical framework on the development of *Principia*.

§4. Peano.   Peano is responsible for much of the notations in *Principia*, and Russell's notational imitations of Peano likely led him to introduce '∨' for disjunction. Indeed, Peano seems to be responsible for suggesting to later readers that '∨' was a sensible (Latin-inspired) choice of symbol for disjunction.

Peano, however, did not use '∨' (or the letter 'v') for disjunction or class union. From his *Calcolo geometrico* onward, he instead used '∪' for both (and '∩' for conjunction and class intersection).

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[17] Peirce (1883, pp. iii–iv) noted that the debate over whether '+' should be reserved for disjoint union or not had been decided: he notes that the majority of authors "declared in favor of using the sign of addition to unite different terms into one aggregate, whether they be mutually exclusive or not." Only Boole & Venn (1881) insisted on disjoint union.

[18] On the development of algebraic approaches to logic in the nineteenth century, see Peckhaus (1997, 2009); on Schröder's logic, see Dipert (1991) and Peckhaus (2004); on Peirce, Jevons, and Schröder, see Dipert (1978); on Peirce and Peano's influence on Schröder, see Peckhaus (2014); on MacColl, see Anellis (2011) and Peckhaus (1999).

> 2. By the expression $A \cap B \cap C \cap \dots$, or $ABC \dots$, we mean the largest class contained in the classes A, B, C, ... or the class formed by all the entities which are at the same time in A and B and C, etc. The sign $\cap$ is read *and*; the operation is indicated by the sign $\cap$ is logical *conjunction*. We shall also call it *logical multiplication*, and say that the classes A, B, ... are *factors* of the *product AB* ...
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> 3. By the expression $A \cup B \cup C \dots$, we mean the smallest class which contains the classes A, B, C, ... or the class formed by all the entities which are in A or B or C, etc. The sign $\cup$ is read *or*; the operation is indicated by the sign $\cup$ is logical *disjunction*. We shall also call it *logical addition*, and say that the classes A, B, ... are *terms* of the *sum $A \cup B \cup \dots$* (Peano, 1888, pp. 1–2; 1973, p. 76)

Similarly, Peano uses '$-A$' and '$\overline{A}$' for set complementation and interprets it also as negation; he uses '$\bigcirc$' for the empty class and interprets $A = \bigcirc$ as "there is no A"; and he uses '$\circledcirc$' for the universal class and interprets $A \cup B = \circledcirc$ as "everything is an A or a B" (Peano, 1888, pp. 76–77).

Peano was of course aware of the then-entrenched use of '+' and '·' or '×' for both disjunction and conjunction and for union and intersection in the algebra of logic tradition. He deliberately avoided using the same symbols in his own work, however:

> It seemed useful to substitute the symbols $\cap$, $\cup$, $-A$, $\bigcirc$, $\circledcirc$ for the logical symbols $\times$, $+$, A_i, 0, 1 used by Schröder, in order to forestall any possible confusion between the symbols of logic and those of mathematics (a thing otherwise advised by Schröder himself).[19]

The symbols '$\cup$' and '$\cap$' are most likely adopted as typographical variants of '$\smile$' and '$\frown$' introduced by Grassmann (1844). Grassmann had used '$\frown$' as both a sign for an arbitrary operation (*Verknüpfung*) and as the sign for a kind of geometrical product. Leibniz had also sometimes used '$\frown$' as a symbol for multiplication (e.g., Leibniz, 1690, p. 5; reprinted in Leibniz, 1858, p. 15), which Peano cites in *Calcolo*. Although Peano does not say, at least not in *Calcolo*, these uses may have suggested to him that '$\cap$' (or '$\frown$') would be suitable to express logical multiplication (conjunction). He mentions Leibniz's use of '$\frown$' for multiplication a decade later in the *Formulaire*:

> The sign $\frown$ was adopted by Leibniz to indicate the arithmetical product. We adopt it for the logical product, since the arithmetical product is today indicated by $\times$.[20]

The propositional (re)interpretation of set-theoretic operators is purposeful: he extends (really, reaffirms) his propositional interpretation of the set-theoretic symbols in his treatment of propositions in Sections 1.4–1.11 on deductive logic, as was common

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[19] "Credetti utile di sostituire i segni $\cap$, $\cup$, $-A$, $\bigcirc$, $\circledcirc$, ai segni di logica $\times$, $+$, A_i, 0, 1, usati dallo Schröder, affine d'imedire una possibile confusione fra i segni della logica e quelli della matematica (cosa del resto avvertita dallo Schröder stesso)" (Peano, 1888, p. x; translation from Peano, 1973).

[20] "Le signe $\frown$ a été adopté par Leibniz pour indiquer le produit arithmétique. Nous l'adoptons pour le produit logique; car le produit arithmétique est aujourd'hui indiqué par $\times$" (Peano, 1897, p. 32, our translation).

practice among (other than Frege) logicians in the 1800s. Still, Peano explicitly uses '∪' for propositional disjunction, e.g.:

> $\alpha \cup \beta$ expresses the condition that α is true or β is true. (Peano, 1888, p. 82)

Peano's lower-case Greek letters may be (re)interpreted such that $\alpha \cup \beta$ is a union of classes wherever convenient. This sort of dual usage of symbols for classes or propositions, and for class operations or propositional connectives, allowed him to double-count proofs for analogous theorems.

   Peano persists in this dual usage in *The Principles of Arithmetic* (Peano, 1889), where he used '∪' as disjunction and class union, and '⊃' for both subset and implication (Peano, 1889, pp. 105 and 108). Such dual usage drew a lengthy impeachment from Frege (1896, pp. 242–247). But in fairness to Peano, it leveraged Boole-style algebraic treatments in multiple domains at once, and Peano had philosophical reasons to so-use symbols.[21] By his principle of permanence, since the formal laws for propositional and class operators were identical (in his view), Peano rightly used the same notation for these different operators. This does not violate the first design principle of symbolism prohibiting ambiguous symbols: any symbol in a theorem or its proof must be unambiguous and the syntax is unimpeachable on this score. The intended interpretation, on the other hand, may vary in those limited cases such that the formal laws—even the theorems and their proofs—do not differ. Indeed, Peano was generally very thoughtful in his selection and use of symbols (see Schlimm, 2021, for an insightful discussion). The "design principles" Peano employed explain, for instance, why he chose pairs of inverted symbols for dual operations like '∪' and '∩'.[22]

   Peano's habit of using one symbol for different notions—that is, for allowing the intended interpretation to vary in order to economize the amount of thought required to master the new symbols—is important to note because it helps explain the genealogy of '∨'. Russell will at first copy Peano's practice, although Russell was more uncomfortable than Peano with using the same symbol under two distinct intended interpretations. Even where the formal laws were identical, as in the case of class notions and their relational analogues, *Principia* avoids using the exact same symbol for two different notions. Rather, the distinction is marked in the syntax of the system by using a different symbol entirely for different kinds of terms, as we will see in the next section. Still, Russell did use similar symbols for notions whose formal laws were close analogues, distinguishing these different symbols often by a small change, like a dot or an accent, in a Peano-like fashion. Thus Russell first used '∨' for a set-theoretic notion (relational product) and for propositional disjunction, and then later uses it just for propositional disjunction.[23] What Russell never does is to use the exact same

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21 Peano thought that it had been established by "Boole, Schröder, and others" that deductive logic, like arithmetic algebra, was part of algebra, that is, the "calculus of operations" (Peano, 1891, p. 153). He even wanted to argue that mathematical analysis produces results that mere logic would not (Peano, 1891, p. 154, n. 3).
22 Compare also the discussion of Frege's symbolism in Schlimm (2018).
23 Peano would also write 'ab' for '$a \cap b$'. This notation is combined with the dot notations for scope. Thus, Peano (1901, p. 3) sometimes uses '$a \boldsymbol{.} bc$' for '$a(bc)$,'' which itself expands into "$a \cap (b \cap c)$."

symbol for distinct intended interpretations: these are always marked in the formal grammar by a different, even if highly similar, symbol.

This is not to impeach Peano's practice. Peano may have felt justified in his ambiguous use of '∪' and '∩' by the principle of permanence, according to which operations obeying the same laws in different domains *should* be symbolized by the same notation (e.g., '+' and '·' applied to different sets of numbers, say, integers and reals), and if not, they should be symbolized differently. Since the laws governing class union and intersection, and the laws for propositional disjunction and conjunction are the same, this justifies the use of the same symbols for both. By contrast, the arithmetical operations + and · obey different laws, e.g., idempotence fails (see Toader, 2021, for discussion). Moreover, context disambiguates between the different meanings of '∪' and '∩'. By contrast, Peano's class-theoretic construction of numbers in *Arithmetices principia* and later the *Formulaire* has him using both '+' and '∪' between class terms, and so using the same symbol here would lead to ineliminable ambiguity.[24]

We have already mentioned that Peano adapted his symbol '∪' from Grassmann's '⌣'. Is there any evidence that Peano was influenced in his choice also by the similarity to the initial *v* of *vel*, or specifically by Leibniz's use of 'v̂' for "or"? Peano was already aware of Leibniz's use of 'v̂' as a symbol for disjunction when he introduced his own: he cites the very page of *Mathematische Schriften* where it occurs in *Calcolo* (Peano, 1888, p. x). In his subsequent writings on logic, he often mentions the fact that Leibniz had used 'v' this way (though Peano omits the accent), and specifically that he did it because 'v' is the initial letter of *vel*:[25]

> Instead of $a \cup b$ Leibniz has a u b (where u is the initial of *uel*)...[26]
> The sign ∘ corresponds to Latin *aut*; the sign ∪ to *vel*.[27]
> Leibniz, in his *Schriften*, vol. VII p. 57, indicates logical addition by 'a u b'; the letter u is the initial of the word *uel*.[28]

However, Peano avoided a symbol reminiscent of 'v' for disjunction to avoid confusion with other symbols: indeed, Peano already used 'v' for the universal class (Peano, 1895,

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[24] Schröder (1905, p. 460) considers the typographic contrast between Peano's logical and arithmetical operators a benefit, as well as the horizontal symmetry between the symbols for dual notions. He also suggests that one might best remember that '⌣' stands for disjunction by imagining the symbol as a see-saw between two alternatives.

[25] Peano deserves great credit for his keen eye: we found only three occurrences of 'v̂' for disjunction in the entire 393-page Volume 7 of Leibniz's mathematical writings (and we found none elsewhere in the Gerhardt edition of Leibniz's mathematical and philosophical writings). Leibniz is not explicit about why he chose 'v̂', but Peano's guess is plausible and supported by Leibniz's writing in Latin.

[26] "Invece di $a \cup b$ si scrisse da Leibniz a u b (ove u 'e l'iniziale di *uel*), da Jevons a .|. b, e dal maggior numero di Autori $a + b$" (Peano, 1891, p. 9; translation from Peano, 1973, p. 158). Note that Peano uses the variant spelling of *vel* with an initial 'u'—classical Latin did not have separate letters for 'U' and 'V'. Presumably he did so because 'u' is more suggestive of Peano's chosen symbol ∪. When writing in Latin (Peano, 1889) or later in his simplified *Latino sine flexione* (see Kennedy, 1980, chap. 15, for discussion) Peano always spelled *vel* with a 'v' whenever he used the word.

[27] "Le signe ∘ correspond au Latin *aut*; le signe ∪ à *vel*" (Peano, 1894, p. 10).

[28] "Leibniz, dans ses Œuvres, t. VII p. 57 indique l'addition logique par ≪a u b≫; la lettre u est l'initiale du mot ≪uel≫." (Peano, 1897, p. 42, our translation).

p. 139, 1897, p. 13). Peano merely suggests that '∪' for disjunction can be thought of as a deformation of the 'v' in *vel*:[29]

> ∗1·2 Let *p* and *q* be P [propositions]; $p \cup q$ means "at least one of these P is true"
>
> One may read the sign "∪" as "or"; this operation is called logical addition....
>
> *Leibniz* has indicated the operation ∪ by the sign + (Leibniz, 1890, p. 229); later (p. 237) by the same sign inside a circle. We cannot use a common sign for arithmetical and logical additions without producing ambiguities ...
>
> One may consider the sign ∪ as a deformation of *v*, the initial letter of "vel," used by Leibniz as well for the same purpose. However, in the "Arithmetices principia" (Peano, 1889), which contains the first theory rendered in symbols, I have chosen the shape of logical signs in such a way as to avoid any confusion.[30]

So even though Peano often suggested a connection between the symbol for disjunction and the Latin *vel*, it is unlikely that his choice of '∪' for "or" was originally motivated by this connection. Russell likely got the idea of using '∨' for disjunction from Peano, whom he read closely.[31] This is not terribly surprising: indeed, it is hard to overstate Peano's influence on *Principia*'s notations. Even Whitehead (1898) followed Boole and symbolized disjunction and conjunction as addition and multiplication before encountering Peano's work. After becoming thoroughly familiar with Peano's work through Russell, Whitehead (1902) also adopted Peano's '∩' and '∪'. Whitehead

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[29] A 'v' symbol was also liable to be confused with 'v' for *verum* used by Peirce (1885) and Boole et al.'s use of *v* as a kind of existential quantifier.

[30] "Soient *p* et *q* des P; $p \cup q$ signifie ≪l'une, au moins, des ces P est vraie≫.

On peut lire ≪ou≫ le signe ∪; cette operation s'appelle addition logique.

Ex. de la somme de Cls: §Np P2 . 1...; de P: §× P1 . 5 §Np P1 . 2...

Leibniz a indiqué l'opération ∪ d'abord par le signe + (p. 229); ensuite (p. 237) par la même signe dans un cercle. Nous ne pouvons pas représenter par un même signe les additions logique et arithmétique, sans produire des ambiguïtés. Voir §+ P 5.

On peut considérer le signe ∪ comme une déformation de *v*, lettre initiale de ≪vel≫, aussi employée par Leibniz pour le même but. Mais dans les ≪Arithmetices principia, a.1889≫, qui contient la première théorie réduite en symboles, j'ai fixé la forme des signes de Logique de façon à éviter toute confusion." (Peano, 1899, p. 16, our translation).

Note that here Peano does use 'v' and the spelling *vel*. Leibniz's use of '+' referred to here by Peano is for class union not propositional disjunction.

[31] "In July 1900, there was an International Congress of Philosophy in Paris... The Congress was a turning point in my intellectual life, because I there met Peano. I already knew him by name and had seen some of his work, but had not taken the trouble to master his notation. In discussions at the Congress I observed that he was always more precise than anyone else, and that he invariably got the better of any argument upon which he embarked. As the days went by, I decided that this must be owing to his mathematical logic. I therefore got him to give me all his works, and as soon as the Congress was over I retired to Fernhurst to study quietly every word written by him and his disciples. It became clear to me that his notation afforded an instrument of logical analysis such as I had been seeking for years, and that by studying him I was acquiring a new and powerful technique for the work that I had long wanted to do" (Russell, 1967, pp. 217–218).

& Russell (1910, p. vii) fully acknowledge Peano's notational impact: "In the matter of notation, we have as far as possible followed Peano..."

Yet in using '∨' for disjunction, *Principia* departs from Peano's notations. As we will see in the next section, Whitehead and Russell are jointly responsible for this. The earliest surviving use of '∨' for disjunction we could find is in a Whitehead manuscript, but Russell may well be the one who introduced the symbol. He first separated the notations for class union and propositional disjunction. Russell also used '∨' for disjunction in solo published work some years before the co-authored *Principia* was published. The next section supplies the evidence for this summary.

§5. Whitehead and Russell.   We give Whitehead and Russell joint credit for first using '∨' for disjunction. The relevant surviving manuscripts from Whitehead and Russell were written in close proximity, but the relative dating is uncertain. Moreover, many manuscripts from the time of the writing of *Principia* are lost. In any case, one of Whitehead and Russell, or both jointly, first used '∨' for disjunction. Their use of this notation, through the influence of *Principia*, is the reason for today's wide acceptance of it (as we argue in the next section).

After framing the timeline for our discussion, we trace the genealogy of '∨' in the surviving manuscripts. We will start from what is definitely known and conclude this section with what is merely probable. In our genealogy of Whitehead and Russell's design principles for notation we also focus on Russell's publications and manuscripts because few of Whitehead's pre-*Principia* manuscripts from this period survive.[32] Our aim is to correctly trace the evolution of *Principia*'s notations, and the design principles for choosing them. Crediting either Russell or Whitehead with any specific decision is difficult for the above reasons. Thus, even where we consider a single-authored manuscript or publication and write of, e.g., as Russell introducing a symbol, we do not mean to imply that Russell alone deserves the credit. We have no doubt that Whitehead and Russell agreed on the design principles for notations as they are manifested in *Principia*.

Here is the time frame for the remainder of this section: in his 1901 "The Logic of Relations" (1901), Russell took implication as a primitive symbol. In Whitehead and Russell's 1910 *Principia*, '∨' for disjunction is now primitive (Whitehead & Russell, 1910, p. 93). This raises two questions about Russell's development: (1) When and why was disjunction taken as primitive? and (2) When and why was '∨' introduced for disjunction?

We first take up (1). To answer the "why" part: disjunction is taken as primitive because the resulting system is simpler in that it has fewer primitive propositions.[33] In

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[32] Of course, Whitehead no less than Russell was responsible for much of the notation in *Principia* and both authors gave notation a great deal of thought. We also do not mean to suggest that their thoughts on notations and design principles for notations were always identical in the pre-*Principia* years; see Russell (1948, pp. 137–138) for some indication of notational divergences between the two. See also the letters quoted by Urquhart (1994, sec. V) and Moore (2014, sec. XI).

[33] As to what "simpler" means here: Bellucci, Moktefi, & Pietarinen (2018, pp. 82–83) have discussed the various notions of simplicity at play in discussions of Peano, Frege, and Peirce in their work on logic: in one sense of "simpler" (as applied to logical systems), a simpler system has fewer total primitive symbols. Another sense of "simpler" is that there are fewer

Principia, Whitehead & Russell (1910, p. 130) comment on ∗5·25, which is ⊢∴ $p \lor q$. ≡∶ $p \supset q$. ⊃ . q:

> From ∗5·25 it appears that we might have taken implication, instead of disjunction, as a primitive idea, and have defined "$p \lor q$" as meaning "$p \supset q$. ⊃ . q." This course, however, requires more primitive propositions than are required by the method we have adopted.

If we set aside inference rules and consider just propositional axioms or axiom schemata, the texts bear out this claim. "The Theory of Implication" (Russell, 1906) gives seven axioms for propositional logic; *Principia* instead has five.[34]

The "when" part of question (1) can be answered using the Couturat–Russell letters translated in the headnote to Russell (1906). The first known evidence of the switch from '⊃' being primitive to '∨' being primitive is from August 21, 1906.

Russell wrote "The Theory of Implication" in the summer of 1905, and it was published in the *American Journal of Mathematics* in April 1906. Again, Russell adopts (material) implication as a primitive therein. Russell justifies '⊃' being primitive explicitly on much the same grounds as he later justifies replacing '⊃' with '∨' as primitive: "it avoids hypotheses which are otherwise necessary" (Russell, 1906, p. 162). What hypotheses does Russell avoid in the 1906 paper? As a footnote makes clear (Russell, 1906, p. 162), these hypotheses are to the effect that for any p and q, if $p \supset q$, then p and q are propositions, and so are truth-apt. These hypotheses are adopted in *The Principles of Mathematics* (Russell, 1903c) to prevent the relation of material implication from obtaining between any non-propositional (truth-inapt) entities.

Recall that Russell's "calculus of propositions," also known as the *substitutional theory*, allowed for variables to range over everything without restriction (Linsky, 2016, p. 224). Landini (1996, p. 558) calls this "the doctrine of the unrestricted variable" because on this theory variables range over all entities whatever (rather than, say, different sorts of variables ranging over objects and predicates, as is now common in presentations of second-order logic). This was only possible because the notions of '⊃' in Russell (1903c, 1906) were quite different from the modern one and that of

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primitive constants involved in the expression of some formulas. Whitehead and Russell in *Principia* are pursuing simplicity in the sense of reducing the total number of primitive symbols and primitive principles, and so are prioritizing simplicity in the first sense. Note that unanalyzability is not required for simplicity in Whitehead and Russell's sense, for Whitehead & Russell (1910, p. vi) openly demur on whether "the analysis could not have been carried farther: we have no reason to suppose that it is impossible to find simpler ideas and axioms by means of which those with which we start could be defined and demonstrated." So it cannot be that their sense of simplicity here is tied to being unanalyzable. See also the quotes discussed in this section, and especially the joy with which Whitehead & Russell (1925, pp. xiii–xiv) welcomed the analysis of the 1910 set of primitives by Sheffer (1913) and Nicod (1917).

34 However, a judgment as to overall simplicity somewhat depends on how we count. If we are counting by *primitive propositions*—including rules of inference like *modus ponens*, well-formedness rules, and uniform substitution rules—then both systems have 10 starred primitive propositions. Hence, Linsky (2016, p. 224) has a list of nine primitive propositions in the 1906 system (by counting *modus ponens* and merging the two rules for uniform substitution); Linsky (2016, pp. 226–227) also has a list of seven total primitive propositions in *Principia*, which omits the rules of well-formedness ∗1·7·71·72.

the 1910 *Principia*. On the earlier substitutional theory, the sign '⊃' is a relation sign: it connects terms to make a term, so that "Socrates ⊃ Socrates" was well-formed in Russell's early systems. The modern sign for implication '→', like the symbol '⊃' in *Principia*, behaves very differently: it connects well-formed formulas to make a well-formed formula, so that "Socrates → Socrates" is syntactically ill-formed (much like "$a → a$" is ungrammatical in most modern systems).

Of course, the primitive postulates in Russell (1903c, sec. 18) precluded any material implication relations obtaining between non-propositional entities even though such strings could be syntactically well-formed: one does not want to allow that "Socrates implies Socrates" is a truth of one's formal system (and this is not a theorem in either early system). How did Russell prevent, say, "Socrates ⊃ Socrates" from being even vacuously true in the 1903 system while preserving the doctrine of the unrestricted variable under which any term is substitutable for a variable? Russell built into some primitive propositions of the 1903 system the antecedent condition that p is a proposition. For example, in place of the axiom $p ⊃ p$, Russell had $p ⊃ q . ⊃ . p ⊃ p$, which on Russell's intended interpretation is tantamount to adding the condition "whatever implies anything is a proposition" (Russell, 1903c, sec. 18).

In "Theory of Implication," (Russell, 1906, ∗1·2) instead interprets '⊃' in such a way that it may be just vacuously false whenever truth-inapt terms flank it, making the further hypothesis that whatever implies anything is a proposition redundant.[35] Russell's intended interpretation of the 1906 system effects some simplification of the primitive propositions in the 1906 system as compared with the 1903 one.[36]

At any rate, in 1906 Russell still takes '⊃' as primitive. Russell finally introduces '∨' for disjunction in ∗4·11 (Russell, 1906, p. 176), which defines $p ∨ q$ as $\sim p ⊃ q$. The first passage of the section makes clear that '∨' means disjunction ("propositional sum"):

> In this section we shall be concerned with the fundamental properties of the *propositional product* and the *propositional sum* of two entities p and q... The *propositional sum* of p and q is practically "either p is true or q is true." We avoid a new primitive idea by taking as the propositional sum $\sim p ⊃ q$, i.e., "if p is not true, then q is true." There are some advantages in taking the propositional sum as a primitive idea instead of '$p ⊃ q$'; we then define '$p ⊃ q$' as the propositional sum of $\sim p$ and q. The choice is a matter of taste. (Russell, 1906, p. 175)

As this quotation shows, Russell was obviously not wedded to one or another choice of primitive in his 1906 paper and allows that either one can define the other. This is a significant shift from his view in *Principles*, where he explicitly argues that implication cannot be defined, even in terms of equivalent formulas using other connectives (Russell, 1903c, sec. 16).

Shortly after publication of "Theory of implication," Russell found a reason to prefer disjunction as a primitive notion over implication: it reduced the number of axioms needed for propositional logic. Couturat and Russell exchanged letters about

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[35] Russell (1906, p. 23) credits Frege's *Grundgesetze*, vol. I with this shift in interpretation of '⊃'.
[36] Note that the number of primitive propositions—again, depending on whether one counts axioms and rules of inference—is still 10 in the 1903 system.

Russell's 1906 paper. On August 21, 1906, Russell wrote to Couturat to announce his discovery that taking '∨' as primitive simplifies the development of his logic:

> One can take as a primitive idea $p \vee q$, while defining
>
> $$p \supset q \;.\!=\!.\; \sim p \vee q.$$
>
> In this way two primitive propositions are made superfluous. (Russell, 2001, #216)

This answers question (1): by August 21, 1906, Russell took disjunction as primitive instead of material implication, and he did this because it was logically simpler, that is, it reduced the number of axioms needed in propositional logic.

Now for question (2): when and why did Whitehead and/or Russell first choose the symbol '∨' for disjunction? Unlike Peano, Whitehead and Russell never explicitly explained this choice of notation. So we will have to reconstruct their reasons.

The two natural conjectures are that (A) Whitehead and Russell adopted '∨' because Leibniz used 'v̂' and (B) Whitehead and Russell adopted '∨' as a modification of the already available notation '∪'. Given the available evidence, we endorse (B). First let us consider why (A) is improbable.

If one supposes that Whitehead and Russell got '∨' from Leibniz, the next question to answer is, 'How did Whitehead and Russell come across it?' Here there are two natural possibilities: (A1) Whitehead and Russell found a use of 'v̂' in Leibniz independently, or (A2) Whitehead and Russell found out about this possibility through Peano, who, as we saw, did notice Leibniz's use of 'v̂'.

Against (A1), there is no textual evidence, for instance, in Russell (1900a), that Whitehead or Russell encountered, on their own, Leibniz's sole manuscript in which 'v̂' is used for disjunction. Indeed, there is no evidence that Whitehead or Russell ever owned *Leibnizens mathematische Schriften*. No notes on these volumes occur in Russell's Leibniz notebooks (Arthur & Griffin, 2017). *Leibnizens mathematische Schriften* are also not in Russell's library, which is held at the Bertrand Russell Research Centre. By contrast, Russell's library has all seven volumes of the *Philosophischen Schriften*, which he annotated heavily; these annotations were published by Arthur, Galaugher, & Griffin (2017). Furthermore, Russell kept a list of books he was reading from 1891 to 1902 (Russell, 1891). None of *Leibnizens mathematische Schriften* appear on this list of books (O'Briant, 1979, pp. 165–166). Finally, Russell's citations of *Leibnizens mathematische Schriften* are sparse enough to have been pulled from other authors who quoted from them directly.[37] So it is probable that Russell never saw the manuscript in question, or even the volume in which it was available.[38] Similar reasoning applies to Whitehead, whose 1898 *A Treatise on Universal*

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[37] Russell's appendix to the index of Leibniz quotes includes just 18 citations to the entire *Mathematische Schriften*, and not even 10 to any single volume in that series. By contrast, there are 492 citations to the *Philosophischen Schriften* series, and, to each of the seven volumes, 15, 143, 39, 63, 108, 62, and 62 citations, respectively.

[38] Russell's Leibniz book has two citations to *Leibnizens mathematische Schriften*, Volume VII, but none to the relevant manuscript (Russell, 1900a, 109, n. 1, 247). And these two quotes were probably pulled from Cohen (1883, pp. 64–65), whom Russell cites in the footnote in question. Similarly, in *Principia*'s introduction and treatment of propositional logic (∗1–∗5), Volume VII is cited exactly once, in a footnote to ∗3·47, but again, the relevant manuscript is

Algebra cites Leibniz (as "Leibnitz") just once, discussing a theorem about resultant forces. And Whitehead reports, "My knowledge of Leibniz's investigations [when I wrote the *Treatise*] was entirely based on Couturat's book, *La Logique de Leibniz*, published in 1901" (Whitehead, 1941, p. 10). In summary, it is consistent with the available evidence that neither Whitehead nor Russell ever directly saw Leibniz's use of 'v̂'.

It is of course true that Whitehead and Russell had some first-hand acquaintance with Latin and Greek thanks to their pre-college education. Russell was educated in Latin and Greek, in addition to being fluent in French and German. Might it be possible that Russell wanted to associate propositional disjunction with '*vel*' even if he never saw the occurrence in Leibniz? We find this quite unlikely because Russell despised Latin (and Greek), writing in his (Russell, 1967, p. 36), "I hated Latin and Greek, and thought it merely foolish to learn a language that nobody speaks." (Happily, he did not feel similarly about logical languages.) Moreover, as Lenz (1987, p. 105) notes, "The learning of Latin and Greek played only a small role in [Russell's] tutelage." Even if Peano's remarks about Leibniz may have caused Russell to think of 'v' when deciding on a symbol to represent disjunction, there is no reason to think that a connection to *vel* motivated Russell's choice of 'v', and some reason to think that Latin associations would have been, if anything, wholly repugnant to him. Whitehead also studied Latin and Greek early on, but there is little evidence to suggest this connection with *vel* would have occurred to Whitehead (or have felt apt to impress the new symbol's meaning upon the mathematical community). Indeed, Whitehead's education early on let Latin take a backseat to mathematics: "I was excused in the composition of Latin Verse and the reading of some Latin poetry, in order to give more time for mathematics" (Whitehead, 1941, p. 6).

On the other hand, in support of (A2), Whitehead and Russell were close readers of Peano's works, including the *Formulaire*, throughout the period from 1900 to 1903. In multiple works, Peano explicitly noted Leibniz's use of 'v̂'. So if one is to accept the view that 'v' comes to us from Leibniz, Peano is the most probable intermediary source of Whitehead and Russell's using 'v' for disjunction, even if Peano never adopted Whitehead and Russell's notation for disjunction. But against this proposal, there was a years-long delay between Whitehead and Russell's close study of Peano's works and their first uses of 'v'. Russell studied Peano's works extensively in September 1900, just after the July 1900 Congress (Russell, 1891, p. 363). Whitehead references the first three volumes of the *Formulaire* and praises the "admirable work" of Peano's school in a footnote to his April 1901 "Memoir on the Algebra of Symbolic Logic" (Whitehead, 1901, p. 140). However, as we will see, Russell first used 'v' in his spring 1902 "On Likeness" manuscript, and Whitehead first uses it in his contemporaneous "The Logic of Propositional Functions" manuscript. So neither Whitehead nor Russell used this notation until many months after their deep study of Peano. This suggests it was not Leibniz's notational choices that led Whitehead and Russell to adopt 'v' for disjunction, but some other development. We describe just such a development, consistent with hypothesis (B), that explains why Whitehead and Russell modified '∪' to use 'v' for disjunction. Accordingly, we reject hypothesis (A).

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not cited. This citation in *Principia* is also identical to an earlier citation (see Russell, 1906, p. 182).

We saw above that Peano used '∪' for both disjunction and class union—not ambiguously in a problematic sense, but so that one could follow a different intended interpretation of the symbol and retain all the formal laws governing '∪' and the proofs involving occurrences of '∪'. This accords with the "design principle" for notations, the principle of permanence, discussed above. Russell seems to have accepted a similar principle, but not quite the same one. Indeed, even where the formal laws are identical, *Principia* distinguishes between different kinds of terms. So in *Principia* we find '$\alpha \cup \beta$' for class union in ∗22 and '$R \cup S$' for relational union (i.e., a union of ordered couples) in ∗23. This is despite the fact that "the definitions and propositions of this number [∗23] are to be exact analogues of those of ∗22" (Whitehead & Russell, 1910, p. 226).

So Russell cannot have held that different notions should get the same notation even when the formal laws are identical. In rejecting Peano's design principle here, Russell was likely further influenced by Frege's sharp criticisms of Peano's conceptual notations.[39] Indeed, as we will see below, the very manuscript wherein Russell first introduces '∨' for disjunction is written after Russell's engagement with Frege's writings (although Russell seems to have been moving in this direction already; see Urquhart, 1994, p. xvii). Furthermore, Frege explicitly criticizes Peano's alleged "twofold use" of symbols for propositional and class notions.[40] So if Russell introduced '∨' to distinguish propositional sum and class sum, as we think he did, the fact that Russell had just taken extensive notes on Frege's writings (and Schröder's writings—Schröder likewise criticized Peano for alleged dual uses of notation for propositional and class notions), it is no coincidence (Linsky, 2004).[41] All this supports conjecture (B): Russell introduced '∨' to separate notations for two different notions—propositional sum and class sum—and was further influenced by Frege's writings to do this.

Still, '∨' did not appear ex nihilo. Rather, this symbol was a modification of the symbol '∪'. Russell's apparent practice, even after engaging with Frege's work, was that *similar* notions should have *similar* notations. Admittedly, Russell never explicitly lays this down as a design principle for symbolism—he is far less explicit than Peano in this regard—but we know that Russell was, as he described himself, "fussy" about even minute aspects of how his writing appeared, such as where punctuation and paragraph breaks occurred, whether a colon or semicolon should be used, and formatting issues

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[39] We say "further influenced" by Frege because Russell was already headed in the direction of separating notations for different notions. By spring 1902 Russell separated '∨' for class union and '∪' for propositional disjunction *before* engaging deeply with Frege's writings in the summer of 1902.

[40] "Not least disturbing, it seems to me, is its falling apart into the calculus of classes and the calculus of judgments, as it is customary to put it. And this separation is already less marked in Mr. Peano's work" (Frege, 1984, p. 242).

[41] Nor is it a coincidence that Russell, having just finished his detailed study of Frege (and of Schröder), writes in *Principles* a Fregean (Schröderian?) broadside against Peano for not using separate symbols for propositional and class notions: "The subject of Symbolic Logic consists of three parts, the calculus of propositions, the calculus of classes, and the calculus of relations. Between the first two, there is, within limits, a certain parallelism, which arises as follows: In any symbolic expression, the letters may be interpreted as classes or as propositions, and the relation of inclusion in the one case may be replaced by that of formal implication in the other ... A great deal has been made of this duality, and in the later editions of the *Formulaire*, Peano appears to have sacrificed logical precision to its preservation" (Russell, 1903c, sec. 13).

like italics (Blackwell, 1983). Needless to say, Russell was a consciously conscientious designer of notations, even if he never made his design principles for notations explicit.

That Russell embraced this design principle can be inferred from his notational design principles of (1) using similar symbols for similar notions (ones obeying analogous laws), and of (2) using distinct (even if similar) notations for distinct notions. There are many examples in *Principia* supporting that these two design principles were Russell's apparent (if implicit) practice. For example, we find '⊃' and '⊂' used in *Principia* for implication and class inclusion, respectively. We find '2' and '$\dot{2}$' for the cardinal number 2 and the ordinal number 2, respectively. The formal laws governing these pairs of notions differ; nonetheless, *Principia* uses similar notations for them because they are analogous. Crucially, *Principia* never uses the same symbol for distinct notions whose formal laws differ.

This fact helps us answer the 'why' part of question (2): we maintain that Russell chose '∨' for disjunction because he was keen to separate symbols for propositional and class notions because he held that their formal laws differed. In this Russell was unlike Peano who deliberately wed them together, since he held that the formal laws that each obeyed were the same. Russell's choice of '∨' thus mimics Peano's practice of using some symbols dually for propositional connectives and class operators: Russell wants to analogize the notions expressed by '∪' and '∨'. But because the analogy was imperfect, in Russell's view, Russell chose a new but similar symbol for propositional disjunction, namely, a sharpened, pointed '∪'.

Why did Russell think that '∨' and '∪' obeyed distinct formal laws? In "The Theory of Implication" (Russell, 1906, p. 190) he remarks:

> The analogues, for classes, of ∗5·76·79, are false. Take, *e.g.*, ∗5·78, and put p = English people, q = men, r = women. Then p is contained in q or r, but is not contained in q and is not contained in r.

This exact remark is repeated in *Principia* following ∗4·78·79. And Russell's distinction of the propositional and class calculus goes back to *Principles*. In Section 25 of *Principles* (Russell, 1903c), titled "Relation [of the calculus of classes] to propositional calculus," Russell gives an example of a formula that "can only be truly interpreted in the propositional calculus: in the class-calculus it is false." Russell then says this about disjunction:

> The disjunction is what I shall call a *variable* disjunction, as opposed to a constant one: that is, in some cases one alternative is true, in others the other, whereas in a constant disjunction there is one of the alternatives (though it is not stated which) that is always true. Wherever disjunctions occur in regard to propositional functions, they will only be transformable into statements in the class-calculus in cases where the disjunction is constant.

Though the vocabulary is unfamiliar, the underlying point is familiar to modern readers, who might put it as follows: a disjunction of $p \vee q$ might be such that p is always true, as when p is replaced by '0 = 0', or might be such that one of p or q is true, though neither is always true, as when we have p replaced by 'a is human' or 'a is non-human'. The relevance of this passage is that it shows Russell holds that the formal laws in the calculus of classes and those in the calculus of propositions differ.

Accordingly, as early as his 1902 manuscripts where '∨' is introduced, Russell wants a new symbol to distinguish disjunction and class union.

The hypothesis that Russell chose '∨' to logically distinguish disjunction as obeying different formal laws, while still preserving its analogy with '∪', is further supported by his letter to Frank Morley, the editor of the *American Journal of Mathematics*, concerning the publication of his 1906 "The Theory of Implication." The symbol '∨' and others were so unfamiliar to Morley that he wrote to Russell about it and noting, perhaps ominously, that authors occasionally had to bear "any *heavy* expense for new type." Russell wrote back:

> Thank you for your letter. I do not think that any new type ought to be needed, seeing that you have already printed Whitehead's memoirs. I suppose the sign '∨' is doubtful. It is very desirable that it should be pointed, not round; but otherwise it doesn't matter much whether it is large or small, though I think it would be better small. Of course I will share the expense willingly if it is heavy. (Russell, 2014, p. 15)

Perhaps we owe to the editorial largess of Morley the modern symbol for disjunction, for, as Russell intimates, '∨' does not occur in any of Whitehead's publications prior to Russell's 1906 piece. Consequently, one reckons that Morley needed new type (of unknown cost) for '∨'. This amusing episode in the history of '∨' teaches us that editors should not scrimp on notations, and further, that an erstwhile boon of co-authorship was sharing the cost of new type.[42]

The crucial point, however, is that Russell expresses to Morley the importance of not printing the disjunction sign as a round symbol. He does not care about its size, but definitely wants it to be pointed. A clear rationale for this is that Russell wants readers to readily distinguish disjunction from class union, a distinction that Russell had been concerned to make going back to Russell (1903c, chap. II). If the exact same sign was used for both, this could confuse the reader concerning the logical distinction between disjunction and union. This is why the size, in contrast, does not matter, so long as the symbol is pointed: it has to be clearly distinguishable on sight from its class theoretic analogue.

The 1906 paper accounts for the first published use of '∨' for disjunction, but it is worth mentioning that the first unpublished (and so the very first known) systematic uses of '∨' for disjunction (and only for disjunction) occur somewhat earlier in Whitehead and Russell's writings. Initially, Russell's practice was the reverse of the modern one: he used the symbol '∨' for class union and '∪' for "propositional addition" (disjunction). In his spring 1902 "On Likeness," (Russell, 1902, p. 440) which was only published after Russell's death, we find:[43]

$$*1\cdot1\ (R)L(R')\ .=.\ R,\ R'\varepsilon\mathrm{Rel}\ .\ \boxplus 1\to 1\ .\ S\varepsilon(\sigma=\rho\vee\breve{\rho}\ .\ R'=\breve{S}RS)\quad\mathrm{Df.}$$

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[42] For an in-depth discussion of the difficulties of typesetting Frege's much more complex *Begriffsschrift* and the importance of available type there (see Green, Rossberg, & Ebert, 2015).

[43] The definition may be read as, "*R* and *R'* are like relations (order isomorphic)" means that there exists a relation in the intersection of one–one relations and *S*s such that *S*'s domain σ is the union of the domains of *R* and *R'*, and is such that *R'* is equal to the relational product of $\breve{S}$, *R*, and *S*.
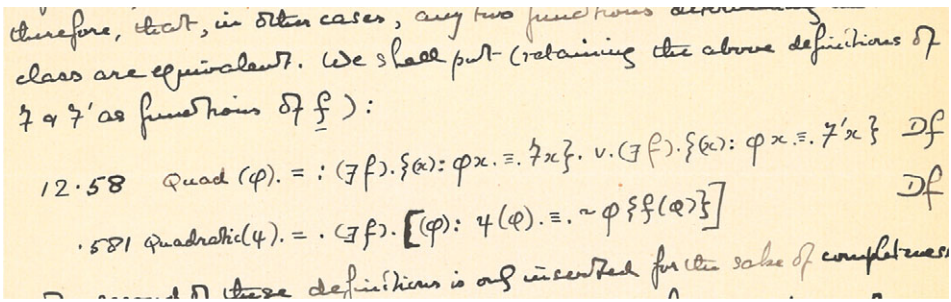
Fig. 2. Use of ∨ for disjunction in Russell (1903a).

This definition mashes notation from Peano's *Formulaire* with some symbols Russell independently devised in his 1901 "The Logic of Relations" where he insisted on distinguishing relational products and relational intersections (Russell, 1901b, p. 315). The key point is that in his 1902 "On Likeness" manuscript, Russell is using the symbol '∨' for the first time, although not for disjunction. Note that he nonetheless distinguishes propositional disjunction from its analogues for classes and relations. Again, this distinction could be maligned or, as Peano would have it, exploited, but Russell is keen to symbolically separate the two (formally different) notions.

Russell switches the symbols for disjunction and union in his early 1903 manuscript "Classes," some of which is now lost. Here we find Russell's first known use of '∨' for disjunction (and only for disjunction) (Russell, 1903a, p. 9; Figure 2):

$$*12\cdot58 \ \text{Quad}(\varphi) . =: (\exists f) . \{(x): \varphi x . \equiv . Fx\} . \vee . (\exists f) . \{(x): \varphi x . \equiv . F'x\} \quad \text{Df.}$$

Comments on a subsequent definition show that Russell here uses '∨' *only* for disjunction:

$$*14\cdot56 \ a \cup b = x \, \varepsilon \, (x \, \varepsilon \, a . \vee . x \, \varepsilon \, b).$$

> If *a* and *b* are classes, $a \cup b$ is their logical sum, *i.e.*, the class formed by adding the two. If *x* is either a member of *a* or a member of *b*, then *x* is a member of $a \cup b$, but not otherwise. The symbol '$a \cup b$' is read '*a* or *b*'... The function $a \cup b$ is the analogue for classes of $p \vee q$ for propositions. (Russell, 1903a, p. 19)

Thus, '∨' for disjunction was born in early 1903 (although Russell used it for class union in 1902), and '∨' for disjunction made its first public appearance in April 1906.[44] This answers the 'when' part of question (2). The 'why' part, as we saw, stemmed from Russell's desire to symbolically distinguish class union and propositional disjunction.

There are two complications about this period between Russell's spring 1902 "On Likeness" (wherein the symbol '∨' is first introduced) and the 1903 "Classes"

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[44] Unfortunately, we cannot provide a photograph of the manuscript page where Russell first defines '$p \vee q$' because this page is lost. We know it existed because Russell (1903a, p. 20) says, "The definition of $p \vee q$ in *4·1 cannot be extended to a class of propositions...." But at least what remains of the manuscript allows us to give a literal picture of the earliest known use of '∨' for disjunction in Russell's corpus.
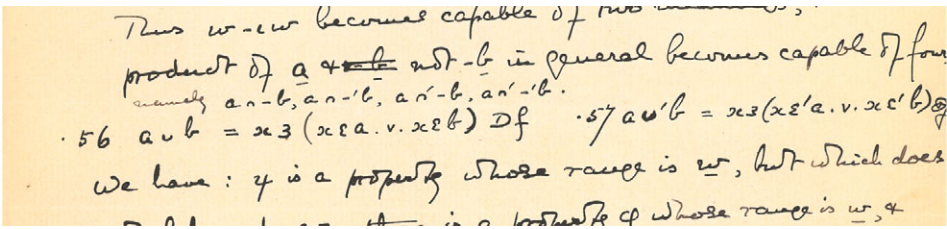
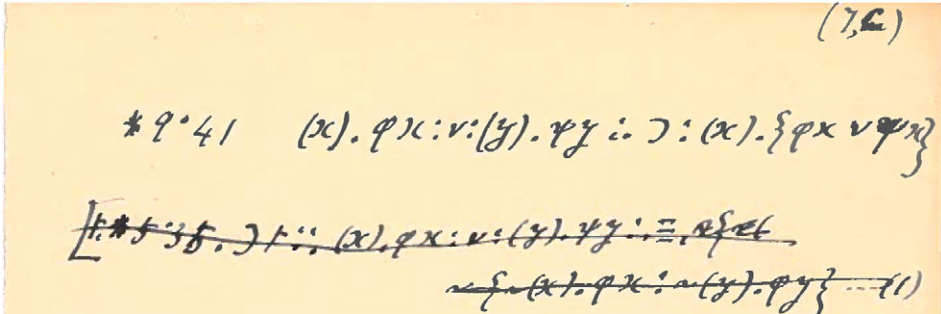Fig. 3. Use of ∨ for disjunction in Russell (1903b).

Fig. 4. Use of ∨ for disjunction in Whitehead (1903).

manuscript. First, as was pointed out to us by Urquhart, in a manuscript contemporaneous with the early 1903 "On Classes," Russell uses '∨' for disjunction (and only for disjunction) as well. This manuscript consists of notes Russell made on Frege's *Grundgesetze* Vol. II and its attempted solution to the contradiction—hence their title, "Frege on the contradiction." In one of the last definitions in this manuscript, Russell defines class union using propositional sum, symbolized by '∨' (Figure 3):

$$\cdot 56 \quad a \cup b = x \ni (x \, \varepsilon \, a \, . \vee . \, x \, \varepsilon \, b) \quad \text{Df.}$$

Urquhart's dating of this manuscript is supported by a letter that Russell wrote to Frege, dated February 20, 1903, thanking him for sending Volume II of *Grundgesetze* and asking about its solution to the contradiction. Given the content of these notes, it is likely they were written at nearly the same time as the "On Classes" manuscript. So we strictly speaking have two candidates for the first use of '∨' for disjunction—and imperfect information as to which one came first!

The other complicating factor is that Whitehead also uses '∨' for disjunction (and only for disjunction) in a manuscript that is also from early 1903, "The Logic of Propositional Functions" (Whitehead, 1903; Figure 4):

This is a familiar quantification theorem:

$$*9 \cdot 41 \quad (x) \, . \, \varphi x : \vee : (y) \, . \, \psi y :. \supset : (x) \, . \, \{\varphi x \vee \psi x\}$$

(In modern notation: $\forall x \, \varphi(x) \vee \forall y \, \psi(y) \supset \forall x(\varphi(x) \vee \psi(x))$). It is not known for certain when this manuscript was written, though the internal evidence suggests it was either late 1902 or early 1903.[45] Either way, Whitehead's manuscript dates from

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[45] See Urquhart's headnote to Russell (1903a) and page 3 of Russell (1994).

after "On Likeness" wherein '∨' is first used by Russell (though not for disjunction). It is difficult beyond that to assign priority for using '∨' for disjunction. The earlier parts of Russell (1903a) were manuscripts on propositional logic and likely included a definition of '∨' in terms of '⊃' and '∼'. But it is not known whether Whitehead's or Russell's manuscript came first. For this reason, we here give joint credit to Whitehead and Russell.[46]

Russell's use of '∨' for disjunction persists in Russell's writings and through the 1910 publication of the co-authored *Principia*. The rest, as they say, is history: the symbol became widely adopted following *Principia*'s publication. In the next section, we sketch how this state of affairs, which persists today, came about.

As a coda to this part of the story, our suggestion that Russell chose '∨' as a modification of '∪' is further supported by the surprisingly similar genealogy of '∧' and of dots for conjunction. Russell uses '∧' for the first time in "On Likeness" for class intersection rather than for propositional product, as in the above definition ∗1·1. But the story curiously diverges in the 1903 "Classes" manuscript: there Russell introduces '∧'*m*' for a class-operator—the conjunction of a class *m* of propositions (∗14·8)— and similarly introduces '∨'*m*' for its dual class-operator—for the disjunction of a class *m* of propositions (∗14·81). Despite this, even though Russell also uses '∨' for propositional disjunction, he uses Peano's square dots for conjunction as well as scope by 1903. Russell was thus quite close to also giving us '∧' for conjunction, having used it briefly himself, but chose an alternative.

Oddly, Russell seems not to have used '∧' for conjunction anywhere else. Why did Russell not seize this opportunity? A partial explanation is provided by the fact that the '∧' notation "rarely occurs" (Russell, 1903a, p. 20). So the '∧' symbol was not much on Russell's mind even once he introduced it. Perhaps more importantly, Russell had used dots for conjunction well before he introduced either '∨' or '∧'. For example, in his "Logic of Relations" (Russell, 1901a) Russell uses dots for conjunction and for scope, reserving concatenation for relational products. In contrast, Peano never used dots for conjunction: he only used dots for scope (Schlimm, 2021, pp. 118– 119). Peano (1895, sec. 1) instead uses concatenation for propositional conjunction and '∩' for class intersection. So Russell perhaps inserted dots for conjunction to distinguish conjunction from relational products. By the time he thought to use '∨' for anything, he had an established practice of using dots for propositional conjunction. This squeezed out using '∧' for that notion. This episode similarly shows that Russell was keen to introduce new symbols to distinguish logical notions—like propositional and relational "conjunction"—and further supports our hypothesis that this common feature of Russell's notational design led to introduction of '∨' rather than any love for Latin or Leibniz.

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[46] One might argue that Russell is more likely responsible because Russell in August 1902 drafted material corresponding to *Principia*'s propositional logic (∗1–∗5) (Moore, 1993, p. 691). Whitehead read this material on propositional logic (and sharply criticized it). It would not be surprising if Whitehead got the use of '∨' for disjunction from Russell's 1902 draft. Additionally, Russell (1948, pp. 137–138) wrote, "In the early part of *Principia*, Whitehead contributed the treatment of apparent variables and the notation $(x) \cdot \varphi x$. Chapters 10, 11, and 13 of the *Principia* are in the main his work." It would be peculiar if Whitehead had contributed notation to the propositional logic like '∨' and Russell omitted this point even while being sure to mention Whitehead's contribution to the notation of *Principia*'s quantifier theory.

The treatment of logic in both Hilbert's lectures and in Behmann's work is already axiomatic (as in *Principia*), and not algebraic.

The idiosyncratic notation of 1920 was replaced, in the lecture notes to the 1921/22 course "Foundations of mathematics" (1922), by the Hilbert–Ackermann notation, which used the symbols '∨', '&', '→', and overlining for negation. A handwritten note by Bernays in the copy of Hilbert (1920) filed in the Göttingen mathematics library, headed "Suggestions for nomenclature [*Vorschläge zur Bezeichnung*]," reads in part:

> $X \vee Y$: X or Y (abbreviation) ("∨" is Russell's symbol, v initial letter of "vel").[50]

It thus seems likely that the motivation to adopt '∨' for disjunction *in the Hilbert school* was two-fold: first, to follow Russell's established usage, and second, because the symbol is reminiscent of *vel*.

The first use of this system of logical notation in print is in Hilbert (1923), and it was adopted in all subsequent publications on logic in Hilbert's school.[51] Hilbert & Ackermann (1928) proved extremely influential. With it, Hilbert's way of writing formulas became widely adopted, especially after Gödel (1930, 1931)—even Skolem used it exclusively from the late 1930s onward (see, e.g., Skolem, 1935).

Tarski was catholic in his use of notations: he used whatever seemed most appropriate. In his dissertation on the logic of *Principia*, he applied the notation of *Principia* (Tajtelbaum, 1923). Writing on propositional calculi with Łukasiewicz, he used Polish notation (Łukasiewicz & Tarski, 1930). Writing for an audience of set theorists, he used algebraic notation (Kuratowski & Tarski, 1931). Writing for an audience of philosophers (in *Erkenntnis*, the journal of the Vienna Circle), he used the notation of *Principia* (Tarski, 1935). And when he wrote for his mathematical colleagues in Vienna (Hahn, Gödel), his logical language used the Hilbert–Ackermann symbols (Tarski, 1933).[52]

Bernays (1926) mentions that '∨' is to have the inclusive meaning of the Latin "*vel*." Bernays & Schönfinkel (1928) and Hilbert & Ackermann (1928, p. 4) likewise mention this, and also note that '∨'—typeset in the latter as a lowercase 'v'—is not to be confused with exclusive or "in the sense of the Latin 'aut–aut'." The identification of *vel* with inclusive and *aut* with exclusive or is already made in Schröder (1890, p. 226), who however used '+' and never '∨'.[53]

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[50] "$X \vee Y$: X oder Y (Abkürzung) („∨" Zeichen von Russell, v Anfangsbuchstabe von „vel")" (Ewald & Sieg, 2013, p. 334). On page 2, where conjunction and disjunction are introduced as $X + Y$ and $X \times Y$, respectively, marginal annotations record the symbols later preferred: $X \& Y$, $X \vee Y$ (Ewald & Sieg, 2013, p. 300).

[51] Gentzen (1935) is an exception: he used Russell's '⊃' for the conditional and '→' for the sequent arrow. For more on the development of propositional logic in the Hilbert school (see Zach, 1999).

[52] Notably, Tarski (1933) also used ∧ for conjunction; see below.

[53] Jennings (1994) argues that it is a myth that *aut* has the sense of exclusive disjunction. Incidentally, Schröder (1890) discusses Leibniz extensively, but only cites the Erdmann edition (Leibniz, 1840) as well as Květ (1857), neither of which mention 'v' as a symbol for disjunction or union.

Quine's *Mathematical logic* of 1940 may be the source of the connection between
'∨' and the initial letter of *vel* in English textbooks.[54] He first discusses the exclusive
and inclusive use of 'or' in English:

> We must decide whether 'or' is to be construed in an *exclusive*
> sense, corresponding to the Latin 'aut', or in an *inclusive* sense,
> corresponding to the Latin 'vel'. …When 'or' is used in the inclusive
> sense, on the other hand, the compound is regarded as true if at least
> one of the components is true; joint truth of the components verifies
> the compound. … In mathematical logic the ambiguity of ordinary
> usage is resolved by adopting a special symbol '∨', suggestive of 'vel',
> to take the place of 'or' in the inclusive sense. Alternation is identified
> with this usage … The exclusive use of 'or' is not frequent enough
> in technical developments to warrant a special name and symbol.
> (Quine, 1940, pp. 12–13)

From *Mathematical Logic*, the idea that *aut* and *vel* in Latin represent the exclusive
and inclusive sense of 'or', and the use of the symbol '∨' for the latter, percolates into
other introductory logic texts, first Quine's own *Methods of logic*:

> Latin has distinct words for the two senses of 'or': *vel* for the
> nonexclusive and *aut* for the exclusive. In modern logic it is customary
> to write '∨', reminiscent of *vel*, for 'or' in the nonexclusive sense:
> '$p \vee q$'. It is this mode of compounding statements, and only this,
> that is called *alternation*. (Quine, 1950, p. 5)

Quine adds a section on "Alternation and duality" to the revised edition of *Elementary
logic*:

> Special symbols are commonly added for 'or' and 'if'. For 'p or q' the
> notation is '$p \vee q$'; here '∨' stands for the Latin *vel*, which means 'or'
> in the inclusive sense. (Quine, 1965, p. 52)[55]

As we saw above, though, Quine's alleged connection between '∨' and *vel* is likely
spurious: given the textual evidence that we have, the only way this conjecture would
hold good would be if Russell saw Peano's mention of the "∨ is for *vel*" story about
Leibniz and, for that reason, Russell decided to use that notation himself. We argued
above that this was a very remote possibility for which there is no surviving evidence,
and that there is a highly plausible alternative explanation that fits with a common
feature of Whitehead and Russell's notational design practices.

The work of Whitehead and Russell, and of Hilbert and his students, was defining for
the further development of logic in the twentieth century. This gives us an answer to the
further question, "why do *we* (still) use '∨'?" The answer is that we have inherited our
logical notations from two influential research projects, namely that of *Principia* and the
project of formalizing first-order logic and investigating its properties (completeness,

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54 Other early textbooks in English that use symbolic logic (Langer, 1937; Lewis, 1918; Nagel &
    Cohen, 1934; Tarski, 1941) do not make this connection even though they use the *Principia*
    notation.
55 The first edition (Quine, 1941) symbolized everything with only negation '∼' and conjunction
    '·' and introduced alternation as the denial of 'neither…nor'.

undecidability, etc.) begun by Hilbert. Both of these used '∨' for disjunction. Textbook presentations cemented this use further, and Quine's texts were likely mainly responsible for the spread of the myth that '∨' abbreviates *vel*.[56]

§7. Other symbols.   The symbol '∨' is special in that it now is almost universally used for disjunction. But it has been used for other purposes as well; and other symbols also have interesting stories.

Ladd-Franklin (Ladd, 1883, pp. 25–26) used '∨' and '∇' in her dissertation, but only for the copula, that is, for predications among classes and not for propositional connectives. So, Ladd says that '$A \vee B$' means "A is in part B," that is, "some A is B," and '$A \nabla B$' means "A is-not B," or "No A is B." The uses of '∨' and '∇' thus correspond to, as Ladd puts it, (Boolean class) "inclusions" and "exclusions" (Ladd, 1883, p. 26). She deals, rather, with an analogue of propositional disjunction from a Boolean perspective, where the symbol occurs between categorical terms or concepts rather than between truth-apt formulas.[57]

Tarski (1933) was one of the first to combine '∨' with '∧' for conjunction. Heyting (1930) predates this use, however, and is perhaps the first to use '∧' in print as the symbol for conjunction. The first use of '∧' for conjunction in a textbook is probably Tarski (1941) (the original Polish and German editions did not use logical symbols).

Sheffer (1913) used '∧' as the symbol for "neither … nor" (NOR), i.e., the logical connective *corresponding* to the operator in Boolean algebra which *he* symbolized '|'. Nicod (1917) used '|' instead for "not both" (NAND), and gave a reduction of the logical constants of *Principia* to it (the reason being that the resulting definitions of other connectives turn out to be much simpler than with Sheffer's NOR). This has stuck; "Sheffer stroke" now universally means NAND. We should point out here, however, that the first published discussion of NAND and NOR, including proofs that the usual primitives can be expressed by either, and an axiomatization of Boolean algebra in terms of them, is neither Sheffer (1913) nor Nicod (1917), but Stamm (1911).[58]

The "neither … nor" connective (NOR) is now commonly called the Peirce arrow, and written '↓'. However, Peirce himself did not use '↓' for NOR, rather, it was introduced by Quine (1940):

> The fact that $\ulcorner(\phi \downarrow \psi)\urcorner$ denies $\ulcorner(\phi \vee \psi)\urcorner$ is reflected in ordinary language, indeed, by the cancellatory 'n' which turns 'either—or' into 'neither—nor'. The vertical mark in '↓' may be thought of as having the same cancellatory effect upon the sign '∨'; it is like the vertical mark in the inequality sign '≠' of arithmetic. (p. 46)

Although Quine himself does not call his '↓' the 'Peirce arrow', he does credit Peirce for his work on it in footnote 1 on page 49:

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[56]   This is not to say that alternative notations were not used after, say, 1940. Polish notation, in particular, remained popular through much of the twentieth century.

[57]   See Uckelman (2021) for a discussion of Ladd-Franklin's dissertation, and Wege et al. (2020) for a discussion of Ladd-Franklin's hypothesis that commutative operators should be represented by symmetric symbols.

[58]   Stamm used ∽ for NAND and ∗ for NOR. His paper seems to have excited little interest at the time, or since. Löwenheim belittled the results as "not especially useful" in his review in the *Jahrbuch für die Fortschritte der Mathematik* 42.0079.02.

> The definability of denial, conjunction, and alternation in terms of joint denial [NOR, the Peirce arrow] was first pointed out by Sheffer (1913); and similarly for alternative denial [NAND, the Sheffer stroke]. The adequacy of joint denial was known to Peirce in 1880, and both facts were known to him in 1902; but his notes on the subject remained unpublished until 1933 (4.12, 4.264).

As Quine remarks, Peirce's writings on NOR had only recently become available. In the manuscripts mentioned by Quine, Peirce first simply used juxtaposition: "Two propositions written in a pair are considered both denied. Thus, AB means that the propositions A and B are both false" (Peirce 1880, p. 13). In Peirce (1902), he introduced the symbol 'ᴪ', which he called "ampheck" (p. 215).

Quine (1940) also gave an origin story for '∼':[59]

> In mathematical logic the denial of a statement is formed by prefixing the tilde '∼', which is a modified 'n' and is conveniently read 'not'. (p. 14)

The history recounted in this paper shows that we should be wary of just-so stories about the origins of notations like Quine tells here. If we had to make a conjecture, we would agree with Moore (2014, p. lxi) that Russell began using a tilde for negation for reasons that conform to his well-attested to practice of distinguishing different notions in his symbolism whenever the formal laws differed, so that '∼' was introduced to distinguish propositional negation from class (or arithmetic) subtraction.

The two standard symbols for the material conditional (implication) are '⊃' and '→'. The latter, as noted above, was introduced by Hilbert (1918, 1920) and become widely adopted after it was used in the influential *Grundzüge* textbook (Hilbert & Ackermann, 1928). The origin of '⊃' is a bit more complicated: Peano introduced 'ɔ' as a symbol for both the conditional and for class containment, as the rotated version of 'c', abbreviating "*est consequentia*" and "*continet*," Latin for "is a consequence of" and "contains":

> [The sign c means *is a consequence of*; thus b c a is read b *is a consequence of the proposition a*. But we never use this sign.]

> The sign ɔ means *one deduces*; thus a ɔ b means the same as b c a.[60]

> The sign ɔ means *is contained*. Thus a ɔ b means *the class a is contained in the class b*. …The signs Λ and ɔ have meanings here which are slightly different from the preceding, but no ambiguity will arise, for if propositions are being considered, the signs are read

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[59] In Quine (1950), he prefers '–' for negation, and '$\bar{p}$' for negated atoms. Incidentally, Quine (1950) also seems to be the origin of '⊤' and '⊥' as constants for truth and falsity: "A convenient graphic method of imposing interpretations, of the second of the above varieties, is simply to supplant the letters in a schema by the mark '⊤' for truths and '⊥' for falsehoods" (p. 23).

[60] "[Signum c significat *est consequentia*; ita b c a legitur b *est consequentia propositionis a*. Sed hoc signo nunquam utimur.]
    Signum ɔ significat *deducitur*; ita a ɔ b idem significat quod b c a" (Peano, 1889, p. viii; translation from Peano, 1973, p. 105).

absurd and *one deduces*, but if classes are being considered, they are read *empty* and *is contained*.[61]

Beginning with Peano (1898), the symbol 'ɔ' is replaced with '⊃' but with the same ambiguous use.[62] Russell of course derived his '⊃' from Peano and used it as early as Russell (1901a). This was a French translation of a 1900 manuscript (Russell, 1900b), published in Peano's *Revue de mathématiques*. Therein the symbol was typeset interchangeably as '⊃' and as 'ɔ'.[63] Later, including in *Principia*, Russell had it typeset as '⊃'.

§8. Conclusion. Logicians and historians of logic have suggested various origin stories for 'v' as the symbol for disjunction. As we saw above, Peano suggests that Leibniz chose 'v' for the Latin *vel*. Kneale and Kneale suggest that Peano first chose it. Cajori attributes the first use to *Principia*. According to Quine, 'v' stands for, or at least suggests, the Latin *vel*. The evidence discussed in this paper indicates that 'v' for disjunction is not a straightforward Latin abbreviation, and that logicians and historians of logic have often attributed the first use of 'v' for disjunction to the wrong author or to the wrong text.

The appealing story often told that our 'v' for disjunction comes from the Latin *vel* is not well-supported by the textual record. It was Whitehead and Russell who first systematically used 'v' for disjunction, and it is more likely given Whitehead and Russell's notational habits (and Russell's disgust for Latin) that this choice of notation was to stress the analogy of propositional addition with class union.

This may all seem like nothing more than logical *Trivial Pursuit*. However, our discussion also highlights features of the trajectory of the development of logical notations that is not trivial. We have seen that the considerations for or against the choice of a symbol used for 'or' were often deeper than just whether or not it serves as

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[61] "Signum ɔ significat *continetur*. Ita *a* ɔ *b* significat *classis a continetur in classis b*. ...Hic signa Λ et ɔ significationem habent quae paullo a praecedenti differt; sed nulla orietur ambiguitas. Nam si de propositionibus agatur, haec signa legantur *absurdum* et *deducitur*; si vero de classibus, *nihil* et *continetur*" (Peano, 1889, p. xi; translation from Peano, 1973, p. 108).

[62] As with Peano's '∪' and Leibniz's 'v', one may wonder if Peano came up with 'ɔ' on his own or if he adopted someone else's use. In the *Formulaire*, Peano was very generous with giving credit and listing the symbols used by others. However, of 'ε' and 'ɔ' he wrote, in Peano (1897, p. 29), that they are symbols that he introduced, just before he surveys symbols others used for the same purpose. Only sometime around 1899 did he became aware of prior uses of 'ɔ'. He mentions a use of 'ɔ' in the sense of "therefore" between theorems by Abel (1881, p. 36) in Peano (1899, p. 8). In Peano (1903), he also mentions Gergonne (1816–1817), who used '*C*' and '*Ɔ*' for "contains" and "is contained in." Gergonne does not use them as propositional or class operators, however, but markers of different kinds of subject–predicate propositions. Schröder (1890, p. 129) independently introduced signs similar to '⊂' and '⊃' as signs for proper concept containment in the same directions we now use them (e.g., "gold ⊂ metal"), and uses '∈' for contains-or-equals. His comparison with the '<' symbol suggests he also was following the design principle that analogous notions should be symbolized with similar symbols.

[63] Russell was displeased with the typesetting of this article, as he complained to Couturat in a letter of July 27, 1901; see the headnote to Russell (1901a). Curiously, the mishmash of '⊃' and 'ɔ' is still present in the collection Russell (1956, pp. 1–38), despite Russell correcting and approving that English translation of Russell (1901a). Note that the reprints in Russell (1993) correct these misprints, and render implication uniformly as '⊃'.

a good mnemonic device (a symbol reminiscent of *vel*). The mathematical tradition of the algebra of logic stressed the similarity of class union and propositional (or truth-value) disjunction with algebraic operations, and thus often chose the same symbols used in algebra ('+' or '×'). Other writers were concerned with separating arithmetical or algebraic notations (or defining them) from more general logical notations. This, e.g., led Peano to use a *different* symbol for (logical) union and disjunction than the algebraic '+', and it led Whitehead and Russell to use distinct symbols for class union and propositional disjunction. These concerns for mathematical uniformity and philosophically relevant distinctions, of course, extend to symbols and notions other than "or."

The story connects with other important concerns that influenced the development of a system of notation which, since Cajori penned the passage we opened the paper with, has become *almost* standardized. These include the motivations for choosing the particular set of primitives that have become commonplace, e.g., why we take inclusive and not exclusive disjunction as a primitive. It also connects with the question of why and when different approaches to logic (and notation) became more widespread or lost appeal: the choice of notations of the pioneering writers of the early to mid-twentieth century provides insight into their influences and methodological commitments. We have only touched upon these deeper topics, but they suggest fruitful directions for further research.

If nothing else, we now have an answer to the question: "Why $\vee$?" We owe it to Whitehead and Russell, who in spring 1902 wanted a symbol different from Peano's (and Grassmann's) symbol '$\cup$' to distinguish union and propositional disjunction in his symbolism. After, and in part because Hilbert endorsed Whitehead and Russell's choice to use '$\vee$' for disjunction in *Principia*, the symbol became ubiquitous.

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