# Comparativism and the Measurement of Partial Belief

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**Abstract**

Comparativism is the view that *comparative beliefs* (e.g., believing $p$ to be more likely than $q$) are more fundamental than *partial beliefs* (e.g., believing $p$ to some degree $x$), with the latter explicable as theoretical constructs designed to facilitate reasoning about patterns within systems of comparative beliefs that exist under special conditions. In this paper, I first outline several varieties of comparativism, including two 'Ramseyan' varieties which generalise the standard 'probabilistic' approaches. I then provide a general critique that applies to any and all comparativist views. Ultimately, there are too many things that we ought to be able to say about partial beliefs that comparativism renders unintelligible. Moreover, there are alternative ways to account for the measurement of belief that need not face the same expressive limitations.

## 1 Introduction

Meet Sally, an ordinary human being, and one of the subjects of this paper. Like the rest of us, Sally has beliefs, broadly construed: there's some way she takes the world to be that's generally responsive to her evidence and which guides her intentional behaviour. This paper concerns what Sally's system of beliefs might be like at its most fundamental level, and the relationship between different kinds of beliefs she seems to have. To get the ball rolling, I'll start by making some assumptions.

First, I'll assume that Sally has at least two kinds of belief: *partial* and *comparative*. In the former category, for example, Sally is 100% certain that dropbears exist, between 95% and 99% confident that there's one in the trees above her, quite unsure what will happen if it attacks, but doubtful it'll be good. Each of her partial beliefs comes with some (possibly imprecise) *strength*, which can (at least sometimes) be described numerically; e.g., '99% confident'. And in the category of her comparative beliefs, Sally is, for instance, just as confident that dropbears exist as she is that $2 + 2 = 4$, she's more confident that there's one in the trees above than that there isn't, and so on.

Second, and mostly just to keep complications to a minimum, I'm going to assume that what it is for Sally to have partial and comparative beliefs cannot

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be cashed out in terms of outright beliefs. In fact, I'll assume that anything that can be said about Sally's doxastic state *in toto* can be said in terms of her partial and/or comparative beliefs—were I to specify all of her partial beliefs and all of her comparative beliefs, then I would have said enough to fix all the facts about her doxastic state, with nothing left over. In describing how Sally takes the world to be, there's no need to mention her outright beliefs at all.[1] This is the last I'll speak of outright beliefs in this paper.

Given these two assumptions, it's natural to wonder about the relationship between Sally's partial beliefs and her comparative beliefs. It's clear enough that they are connected. For instance, from the fact that Sally has high confidence that $p$ and low confidence that $q$, it follows that she's more confident that $p$ than she is that $q$. Likewise, if she's takes $p$ and $q$ to be equally likely, then she's certain that $p$ just in case she's certain that $q$. Such inferences have a feel of apriority about them, and it's reasonable to think that they're underwritten by some interesting metaphysical connection.

An obvious and natural thought is that the facts about Sally's comparative beliefs fall out of the facts about the relative strengths of her partial beliefs, and so partial beliefs are the more fundamental of the two belief states. But there are other possibilities. In particular, according to one common position, *comparativism*, the facts about Sally's partial beliefs supervene on, and hold in virtue of, the facts about her comparative beliefs. So, for example, a comparativist might say that Sally's certain that there are dropbears if and only if, and because, she considers the proposition *dropbears exist* to be at least as probable as any other proposition whatsoever; and she's 50% confident that there are hoopsnakes whenever she takes *hoopsnakes exist* to be exactly as probable as its negation.[2]

Comparativism comes in a range of shapes and sizes, with advocates going back at least as far as deFinetti (1931), Koopman (1940), Fine (1973), Zynda (2000), and more recently, Hawthorne (2016) and Stefánsson (2017, 2018). But, from the very beginning, it has been centrally motivated by the need to explain 'where the numbers come from'. As Koopman put it,

> … all the axiomatic treatments of intuitive probability current in the literature take as their starting point a number (usually between 0 and 1) corresponding to the 'degree of rational belief'… Now we hold that such a number is in no wise a self-evident concomitant with or expression of the primordial intuition of probability, but

[1] I recognise that not everyone will agree with these assumptions, and that there are important positions that I am setting to one side. Some may prefer, e.g., the view that Sally's partial and comparative beliefs are really just outright beliefs about objective probabilities. Some may even doubt that partial beliefs exist in any meaningful sense (cf. Horgan 2017). But a discussion has to start somewhere, and I'm hardly alone in thinking that outright beliefs can be dispensed with in favour of a more fundamental, and fundamentally graded, doxastic attitude (e.g., Christensen 2004; Eriksson and Hájek 2007; Clarke 2013).

[2] I emphasise: comparativism, as I'm understanding it here, is *not* the view that partial beliefs supervene on *some comparative thing or other*. It is specifically about the relationship between partial and comparative beliefs. Comparativism can be—and often is—divorced from the thesis that our doxastic states depend on our preferences, and the reader should be careful to keep these ideas separate for what follows. Of course, one could be a comparativist and also think that comparative beliefs supervene in turn on preferences (e.g., Savage 1954), but comparativists in general aren't committed to this. Likewise, if you think partial beliefs supervene on comparative beliefs in combination with preferences, or anything else, then you're not a comparativist for my purposes—though see §5.1 for discussion.

> rather a mathematical construct derived from [comparative beliefs] under very special conditions… (1940, p. 269)

Koopman is right about this at least: the numbers we use to refer to and reason about strengths of beliefs are not essential to them. They belong merely to a conventional system for the representation and measurement of some psychological phenomena which must be fundamentally *qualitative* in nature. As such, anyone who wants to explain what partial beliefs *are* has to be able to say, in purely qualitative terms, just what the strength of belief is a measure of, and why we're justified in measuring it as we do. A primary driving force behind comparativism is its promise to do just this.

However, I don't think comparativists have correctly identified the actual qualitative phenomena that explain the strengths of our partial beliefs. In the sequel, I'll argue that comparativism requires too radical a departure from our ordinary understanding of partial belief, and that it's unable to adequately accommodate the full range of commitments implicit in our partial belief discourse. There's too many things we ought to be able to say that comparativism renders meaningless, and we've yet to be given plausible independent reasons for thinking that they *should* be considered meaningless.

The paper proceeds as follows. In §2, I will outline the 'standard' comparativist explanation for how partial beliefs can be measured. §3 is expository: I discuss four varieties of comparativism, and show how they all purport to explain the measurement of partial belief using essentially the same idea. Then, in §4, I argue that the explanation is misguided, and (moreover) that there are still some things we should like to say about partial beliefs that comparativism, in general, *cannot* accommodate. §5 deals with several objections, and in §6, I argue that despite recent defences, comparativists still lack an adequate account of how we make interpersonal comparisons.

Ultimately, though, comparativism can only be considered relative to its rivals. After all, it may be that the best alternatives to comparativism suffer from the same—or worse!—expressive limitations, or otherwise fail to offer a satisfying an account of what our beliefs are like. Consequently, in §7, I conclude the discussion with a sketch of an alternative *non*-comparativist account of the measurement of belief—one with richer expressive resources and a natural explanation of interpersonal comparisons.

## 2 Comparativist Explanations of Cardinality

It's usually taken for granted that numerical strengths of belief encode more-than-merely-ordinal (a.k.a. *cardinal*) information. For instance, we're generally happy to say that Sally can believe one proposition $p$ *much more* than she believes $q$. Likewise, and stronger, most would take the following as valid:

1. Sally believes $p$ to degree $x$
2. Sally believes $q$ to degree $y$

$\therefore$ If $x = n{\cdot}y$, then Sally believes $p$ $n$ times as much as $q$

And in the other direction (from cardinal comparisons to absolute degrees):

1. Sally believes $p$ $n$ times as much as $q$

$\therefore$ If Sally believes $p$ to degree $y$, then she believes $q$ to degree $x = n{\cdot}y$

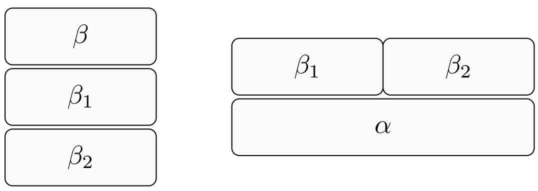
These facts mark a widespread commitment to idea that strengths of belief can be measured on a ratio scale, or at least something much like it. Such commitments need to be explained, or explained away, by any adequate account of what partial beliefs *are*.

Comparativists have a standard strategy for explaining how partial beliefs can be measured on a ratio scale. The basic idea goes at least as far back as (deFinetti 1931). It's discussed in numerous locations, though in particular depth by Fine (1973, pp. 68ff) and Stefánsson (2017, 2018). As Krantz et al. put it, the strategy is 'to treat the assignment of [subjective] probabilities as a measurement problem of the same fundamental character as the measurement of, e.g., mass or [length]' (1971, p. 200). As such, I'll begin by showing how it's possible to construct a ratio scale for the measurement of length from a system of purely ordinal length comparisons.

Let $\alpha$ and $\beta$ be two concrete objects, and compare:

- $O$. $\alpha$ is longer than $\beta$
- $R$. $\alpha$ is twice as long as $\beta$

$R$ obviously contains strictly more information than $O$, which suggests a puzzle: how can the extra cardinal information in $R$ be explained merely by reference to a system of weaker ordinal comparisons like $O$? Well, note that $R$ holds true (roughly) iff, if you were to take two disjoint objects each as long as $\beta$ (call them $\beta_1$ and $\beta_2$, $\beta$'s *duplicates*) and join them end-to-end, the resulting object would be as long as $\alpha$:




Arguably, there's nothing more to the truth of a claim like $R$ than this—that is, '$\alpha$ is twice as long as $\beta$' just means something to the effect of '$\alpha$ is as long as two duplicates of $\beta$ joined end-to-end.'

Call the operation of joining objects end-to-end *concatenation*. By reference, then, to ordinal comparisons between duplicates and their concatenations, we're able to give straightforward *qualitative* meaning to $R$. And we can easily generalise this idea to explain arbitrary (rational) ratio comparisons.[3] For positive integers $n, m$, say that $\alpha$ is $^n/_m$ times as long as $\beta$ whenever there's some object $\gamma$ such that:

(i) $\alpha$ is as long as the concatenation of $n$ duplicates of $\gamma$
(ii) $\beta$ is as long as the concatenation of $m$ duplicates of $\gamma$

Let $x$ designate $\gamma$'s length in whatever units you like—say, furlongs. Intuitively, $\alpha$ must then be $n{\cdot}x$ furlongs long, and $\beta$ $m{\cdot}x$ furlongs long. Hence, $\alpha$ is $^n/_m$ times as long as $\beta$.

[3] It's possible to explain irrational ratio comparisons with a variation on the same basic strategy. But to keep things relatively simple, we'll stick to rational comparisons throughout.

Hiding in the background here is a crucial assumption: that the operation of concatenation behaves as a kind of qualitative analogue of addition. We rely on this assumption to move from, e.g., '$\alpha$ is as long as the concatenation of $n$ duplicates of an object that's $x$ furlongs long' to '$\alpha$ is $n{\cdot}x$ furlongs long'—i.e., the length of a concatenation is just the sum of the lengths of the concatenands. (Imagine if, instead, concatenation worked like quaddition: whenever you concatenate up to 57 duplicates together, things are as usual; but concatenate more and the result is always as long as 5 duplicates. We could then use concatenations to define our way up to one object's being 57 times as long than another, but no further.)

Fortunately, the analogy between concatenation and addition is quite close. Where

$$\begin{aligned}\alpha \succsim^l \beta &\text{ iff } \alpha \text{ is at least as long as } \beta,\\ \alpha \sim^l \beta &\text{ iff } \alpha \text{ is exactly as long as } \beta,\\ \alpha \oplus \beta &= \text{the concatenation of } \alpha \text{ and } \beta,\end{aligned}$$

then $\succsim^l$ is transitive and complete, $\sim^l$ is just the symmetric part of $\succsim^l$, and $\oplus$ satisfies the following properties in relation to $\succsim^l$: for all disjoint objects $\alpha, \beta, \gamma$,

1. $\alpha \oplus \beta \succsim^l \beta$ (*positivity*)
2. $\alpha \oplus \beta \sim^l \beta \oplus \alpha$ (*commutativity*)
3. $\alpha \oplus (\beta \oplus \gamma) \sim^l (\alpha \oplus \beta) \oplus \gamma$ (*associativity*)
4. $\alpha \succsim^l \beta$ iff $\alpha \oplus \gamma \succsim^l \beta \oplus \gamma$ (*monoticity*)

To which we can compare essential properties of $+$ in relation to $\geq$ and $=$, where $n$ and $m$ are non-negative real numbers:

1. $n + m \geq m$ (*positivity*)
2. $n + m = m + n$ (*commutativity*)
3. $n + (m + k) = (n + m) + k$ (*associativity*)
4. $n \geq m$ iff, for any $k$, $n + k \geq m + k$ (*monotonicity*)

Indeed, with a rich enough space of objects and a further 'Archimedean' condition (roughly: no object is infinitely longer than another), we can say something stronger: where $\boldsymbol{\mathcal{O}}$ is the set of concrete objects and $\mathbb{R}^+$ the positive reals, the qualitative system $< \boldsymbol{\mathcal{O}}, \succsim^l, \oplus >$ has (basically) the same formal structure as the numerical system $< \mathbb{R}^+, \geq, + >$. Thus, we can assign numbers to objects in such a way that $\succsim^l$ is modelled by $\geq$, and $\oplus$ is modelled by $+$. And with that in hand, we can start to define up ratios of lengths, differences in length, ratios of differences in length, and so on—i.e., we have the basic resources needed to explain how our assignments of numerical lengths carry ratio information.

The upshot: numerical lengths represent a fully *qualitative* system of ordinal length comparisons with an 'additive' structure over concatenations. We're justified in treating ratios of lengths as meaningful because there exists an operation on objects that is intuitively and formally like 'adding' lengths together. And we can apply the same basic ideas to account for the measurement of other (extensive) quantities. For instance, $\alpha$ has *twice as much mass as* $\beta$ iff $\alpha$ is *as massive as* the concatenation of two mass-duplicates of $\beta$. Likewise, an event $E_1$ has *twice the duration* of $E_2$ iff $E_1$ could be split into two disjoint events each with *the same duration as* $E_2$.

To apply the the same ideas to the measurement of belief, comparativists have historically sought to identify an operation on the relata of a subject's comparative beliefs (i.e., propositions) that behaves, with respect to those beliefs, similarly enough to addition to justify treating it as a qualitative analogue thereof. So let's see how that plays out in practice.

# 3 The Many Faces of Comparativism

In this section, I'll outline four varieties of comparativism of (roughly) increasing generality. This will require some formal machinery, so I'll start by laying down some background assumptions and vocabulary that will be used throughout.

I will assume that for any thinking subject $S$, the propositions regarding which $S$ has beliefs can be modelled as subsets of some space of logically possible worlds, $\Omega$. By 'logically possible', I mean no more than that the worlds are closed under a consequence relation at least as strong as that of classical propositional logic. (So, $\Omega$ can include metaphysically impossible worlds, if that's what floats your boat.) The restriction to possible words will matter for some of my critical points, especially in §4.4, and I'll explain its relevance in §5.4 when I respond to objections.

Next, let $\mathcal{B}_S \subseteq \wp(\Omega)$ denote that set of propositions regarding which $S$ has beliefs (i.e., whether partial or comparative). So, if $S$ thinks $p$ is more likely than $q$, then $p, q \in \mathcal{B}_S$; and if $S$ partially believes $r$ to any degree, then $r \in \mathcal{B}_S$. For simplicity, I'll assume throughout that $\mathcal{B}_S$ is an algebra of sets on $\Omega$:

**Definition 1.** $\mathcal{B} \subseteq \wp(\Omega)$ is an *algebra of sets on* $\Omega$ iff, $\forall p, q \in \wp(\Omega)$,

(i) $\Omega \in \mathcal{B}$
(ii) If $p \in \mathcal{B}$, then $\Omega \setminus p \in \mathcal{B}$
(iii) If $p, q \in \mathcal{B}$, then $p \cup q \in \mathcal{B}$

Given this, I'll assume that every $S$'s full range of comparative beliefs can be modelled with a single binary relation $\succsim_S$ on $\mathcal{B}_S$, where

$$p \succsim_S q \text{ iff } S \text{ believes } p \text{ at least as much as she believes } q$$

I'll refer to $\succsim_S$ as $S$'s *belief ranking*. (From now on, I'll drop the indices from '$\succsim$' and '$\mathcal{B}$' except when needed to avoid ambiguity.) Implicit in this assumption are two commitments that comparativists in general need not accept, which are worth pausing to highlight.

First, where $\succ$ and $\sim$ stand for the doxastic comparatives *more probable* and *equally probable* respectively, I'm assuming that $p \sim q$ iff $(p \succsim q) \,\&\, (q \succsim p)$, and $p \succ q$ iff $(p \succsim q) \,\&\, \neg(q \succsim p)$. In other words, $\sim$ and $\succ$ constitute the symmetric and asymmetric parts of $\succsim$ respectively; hence, $p \succsim q$ iff $(p \succ q) \vee (p \sim q)$. Nothing about this is obvious or trivial. For example, Sally might think that $p$ is at least as likely as $q$, without thinking that $p$ is more likely than $q$, or that $p$ is just as likely as $q$. Nevertheless, the assumption will simplify the discussion considerably, and nothing of critical importance will hang on it.

Second, by assuming that comparative beliefs can be represented by a *binary* relation, I'm ignoring an important class of comparativist views—advocated for example by Koopman (1940) and Hawthorne (2016)—according to which comparative beliefs will be better represented by a quaternary relation $\succsim^\star$,

$$p, q \succsim^\star r, s \text{ iff } S \text{ believes } p \text{ given } q \text{ at least as much as she believes } r \text{ given } s$$

Most of the central critical points raised in what follows have close analogues for (let's call it) *quaternary* comparativism. But, if it turns out that my arguments work only against *binary* comparativism, I'll still consider that a win.

Finally, where a function $\mathcal{C}r$ assigns real numbers to the propositions in $\mathfrak{B}$, I'll say that $\mathcal{C}r$ *almost agrees with* $\succsim$ iff, for all relevant propositions $p, q$,

$$p \succsim q \text{ only if } \mathcal{C}r(p) \geq \mathcal{C}r(q)$$

Furthermore, $\mathcal{C}r$ *agrees with* $\succsim$ just in case

$$p \succsim q \ \text{ iff } \ \mathcal{C}r(p) \geq \mathcal{C}r(q)$$

I'll treat agreement as symmetric: $\succsim$ agrees with $\mathcal{C}r$ iff $\mathcal{C}r$ agrees with $\succsim$. A function $\mathcal{C}r$ agrees with $\succsim$ whenever it assigns values in a way that precisely reflects their order in the belief ranking—in other words, whenever it's an ordinal-scale measure of $\succsim$.

Finally, assuming that $\mathcal{C}r$ agrees with $S$'s belief ranking, say that $\mathcal{C}r$ constitutes a *fully adequate model* of $S$'s beliefs whenever

$$S \text{ believes } p \ {}^{n}\!/_{m} \text{ times as much as she believes } q \ \text{ iff } \ \mathcal{C}r(p) = \frac{n}{m} \cdot \mathcal{C}r(q)$$

We can also say that $\mathcal{C}r$ is *L-to-R adequate* just in case the left-to-right direction of those biconditionals hold, and *R-to-L adequate* just in case the right-to-left directions hold. Only an assignment of numerical strengths that's fully adequate licenses both directions of inference we saw at the beginning of §2—i.e., from claims about cardinal comparisons to numerical strengths, and from claims about numerical strengths to cardinal comparisons. As such, it's *arguably* only fully adequate models that are sufficient to accommodate all of the facts about how we think and talk about partial beliefs.

With that said, individual comparativists may want to reject full adequacy in favour of mere L-to-R or R-to-L adequacy. To keep the discussion from spiralling out of control, however, I'll assume that full adequacy is what we ought to be striving for when developing an account of 'where the numbers come from'. (This won't matter a great deal to my critical points in §4, which generate concerns even if we focus only on L-to-R or R-to-L adequacy.)

## 3.1 Probabilistic Comparativism

We'll start with the most limited, but historically most common, variety of comparativism. First, consider the usual definition of a *probability function*:

**Definition 2.** $\mathcal{C}r : \mathfrak{B} \mapsto \mathbb{R}$ is a *probability function* iff $\forall p, q \in \mathfrak{B}$,

(i) $\mathcal{C}r(\Omega) = 1$
(ii) $\mathcal{C}r(p) \geq 0$
(iii) If $p \cap q = \varnothing$, then $\mathcal{C}r(p \cup q) = \mathcal{C}r(p) + \mathcal{C}r(q)$

It follows immediately from criterion (iii), that *if* a probability function—any probability function—agrees with $\succsim$, then the *union of disjoint sets* will behave just like $+$ with respect to $\succsim$. Great! That's exactly what we were looking for. Moreover, we know the exact conditions under which belief rankings agree with a probability function. Where $\mathfrak{B}$ is finite, the following are individually necessary and jointly sufficient (see Scott 1964):

**A1.** $\succsim$ is complete
**A2.** $\succsim$ is transitive
**A3.** $\Omega \succ \varnothing$
**A4.** $\varnothing$ is minimal
**A5.** Where $\mathbf{1}_p$ denotes the indicator function of $p$, and $(p_i)_{i=1}^n$ and $(q_i)_{i=1}^n$ are finite sequences of propositions from $\mathcal{B}$, then if

(i) $\sum_{i=1}^n \mathbf{1}_{p_i}(\omega) = \sum_{i=1}^n \mathbf{1}_{q_i}(\omega)$ for all $\omega \in \Omega$, and
(ii) $p_i \succsim q_i$, for $i = 1, \ldots, n-1$,

then $q_n \succsim p_n$

For this reason, comparativists have frequently suggested that, *at least when* $\succsim$ *satisfies* A1–A5, partial beliefs are ratio-scale measurable, with the union of disjoint sets playing the role of concatenation.

However, we can say something more general than this, and doing so will be useful in demonstrating continuity with the varieties of comparativism discussed below. First, note that criterion (iii) also implies:

(iv) If $\mathcal{C}r(p \cap q) = 0$, then $\mathcal{C}r(p \cup q) = \mathcal{C}r(p) + \mathcal{C}r(q)$

That is, probability functions are also additive with respect to the union of what we might call *pseudodisjoint* propositions, where $p$ and $q$ are pseudodisjoint for Sally just in case she has zero confidence in their intersection, $p \cap q$. Or,

**Definition 3.** $p$ is *minimal* iff $q \succsim p$ for all $q \in \mathcal{B}$, and *maximal* iff $p \succsim q$ for all $q \in \mathcal{B}$

**Definition 4.** $\mathcal{P} \subseteq \mathcal{B}$ is a *set of pseudodisjoint propositions* iff, for any $\mathcal{P}^\star \subseteq \mathcal{P}$ where $|\mathcal{P}^\star| \geq 2$ and any minimal $q$,

$$\bigcap \mathcal{P}^\star \sim q$$

Furthermore, propositions $p_1, \ldots, p_n$ are *pairwise pseudodisjoint* iff there's a set of pseudodisjoint propositions $\mathcal{P}$ such that $p_1, \ldots, p_n \in \mathcal{P}$

That is, assuming that Sally has exactly zero confidence in $p$ whenever $p$ is minimal, Definition 4 plausibly characterises in comparative terms what it is for Sally to think that at most one proposition from $p_1, \ldots, p_n$ is true.[4]

So with that in hand, we can note that A1–A5 jointly imply that $\succsim$ is Archimedean and, where $p, q, r$ are pairwise pseudodisjoint,

1. $(p \cup q) \succsim q$ (*positivity*)
2. $(p \cup q) \sim (q \cup p)$ (*commutativity*)
3. $(p \cup (q \cup r)) \sim ((p \cup q) \cup r)$ (*associativity*)
4. $p \succsim q$ iff $(p \cup r) \succsim (q \cup r)$ (*monotonicity*)

And that also looks like exactly what we needed: *if* some probability function agrees with $\succsim$—i.e., if A1–A5 are satisfied—*then* the union of pseudodisjoint sets relates to that belief ranking exactly as $+$ relates to $\geq$.

[4] Definition 4 implies that every singleton set $\{p\} \in \mathcal{B}$ is trivially a 'set of pseudodisjoint propositions.' This is a feature, not a bug. The rather tortured definition will be useful later on, when we move away from probability functions.

So let's turn these mathematical points into a philosophical hypothesis. Let *probabilistic comparativism* denote any comparativist theory that's committed to the following conditional:

> **Probabilistic Comparativism.** If $\mathcal{C}r$ is the unique probability function that agrees with $S$'s belief ranking, then $\mathcal{C}r$ is a fully adequate model of $S$'s beliefs

Note the requirement that the probability function be *unique*. This is necessary to avoid contradiction. For any non-trivial algebra $\boldsymbol{\mathcal{B}}$, there will always be some probability functions on $\boldsymbol{\mathcal{B}}$ that agree with the very same belief ranking. And since any two probability functions on the same domain will disagree on at least some ratios, a general pattern of inference from '$\mathcal{C}r(q) = {}^{n}\!/_{m} \cdot \mathcal{C}r(q)$' to '$S$ believes $p$ ${}^{n}\!/_{m}$ times as much as $q$' will be valid *only* when the $\mathcal{C}r$ is unique in the relevant sense. In other words, R-to-L adequacy presupposes uniqueness, which in turn requires further conditions on $\succsim$.

There are several conditions that suffice to establish uniqueness. Of particular note is the following, which Stefánsson (2017, 2018) uses to ensure uniqueness in his recent defences of comparativism:

> **Continuity.** For all non-minimal $p, q$, there are $p', q'$ such that $p \sim p'$, $q \sim q'$, and $p'$ and $q'$ are each the union of some subset of a finite set of disjoint propositions $\{r_1, \ldots, r_n\}$ such that $r_i \sim r_j$ for $i, j = 1, \ldots, n$

The interested reader can see (Krantz et al. 1971, §5.2) and (Fishburn 1986) for a range of other conditions sufficient to ensure uniqueness.

Now, probabilistic comparativism clearly has some resources to put forward an account of ratio comparisons, *in the event that $\succsim$ satisfies the requisite conditions.* (By this, I don't mean that probabilistic comparativism gives the *right* account, just that it's in a position to offer *something.*) Consider the following principle, which is the comparative belief version of how we defined rational ratio comparisons for length:[5]

> **General Ratio Principle.** $S$ believes $p$ ${}^{n}\!/_{m}$ times as much as $q$ if
> (i) For $0 < n \leq m$, there are $m$ non-minimal, equiprobable pairwise pseudodisjoint propositions $r_1, \ldots, r_m$ such that $q \sim (r_1 \cup \cdots \cup r_m)$ and $p \sim (r_1 \cup \cdots \cup r_n)$; or
> (ii) $S$ believes $p$ ${}^{n'}\!/_{m'}$ times as much as $r$, and believes $r$ ${}^{n''}\!/_{m''}$ times as much as $q$, where ${}^{n}\!/_{m} = {}^{n' \cdot n''}\!/_{m' \cdot m''}$

So, for instance, Sally will take $p$ to be twice as probable as $q$ if there is some proposition $q'$ that's obviously inconsistent with $q$ such that $q \sim q'$ and $(q \cup q') \sim p$. In this case, $q$ and $q'$ are acting as 'duplicates' of one another, and $q \cup q'$ is their 'concatenation'.

## 3.2 Imprecise-Probabilistic Comparativism

Say that $\mathcal{C}r$ *coheres* with the General Ratio Principle (henceforth: GRP) just in case, whenever that principle implies that $p$ is believed ${}^{n}\!/_{m}$ times as much as

[5] The first clause of the General Ratio Principle is a close relative of Stefánsson's (2018) 'Ratio Principle.' The second (inductive) clause is new—in the context of Continuity it's redundant, but see §3.3 for it put to work.

$q$, then $\mathcal{C}r(p) = {}^{n}\!/\!_{m} \cdot \mathcal{C}r(q)$; otherwise, it *conflicts* with the GRP. Interestingly, if any probability function *almost* agrees with $\succsim$ and $\varnothing$ is minimal, then that function coheres with the GRP. This means that it's possible to extend the account of ratio comparisons just given to imprecise probabilities and incomplete belief rankings.

For non-ideal agents, completeness (A1) is widely considered highly implausible. We should expect plenty of gaps in $\succsim$. Consider the following example, adapted from (Fishburn 1986):

$p$ = The global population in 2100 will be greater than 13 billion
$q$ = The next card drawn from this old and incomplete deck will be a heart

$p$ and $q$ are sufficiently far removed from one another that it's hard to make a judgement as to which is more likely than the other. Similar examples abound.

There's a natural way of dealing with incompleteness to which comparativists can (and do) appeal. Where $\mathfrak{F}$ is any set of real-valued functions on $\wp(\Omega)$, say this time that the *set* $\mathfrak{F}$ agrees with $\succsim$ just in case for all relevant $p, q$,

$$p \succsim q \text{ iff } \forall \mathcal{C}r \in \mathfrak{F}, \mathcal{C}r(p) \geq \mathcal{C}r(q)$$

The idea behind a set-of-functions model is to recapture the belief ranking by supervaluating over the functions in $\mathfrak{F}$—only what's common to every such is treated as having representative import. Whenever $\succsim$ fails to hold between $p$ and $q$, $\mathfrak{F}$ will contain at least one pair of probability functions that disagree on the relative ordering of $p$ and $q$; hence, we manage to 'numerically' represent $\succsim$.

Where $\mathcal{B}$ is finite, a set of probability functions agrees with $\succsim$ just in case the latter satisfies A2–A4 plus a very slightly stronger version of A5 (see Alon and Lehrer 2014):

**A5′.** Where $(p_i)_{i=1}^n$ and $(q_i)_{i=1}^n$ are finite sequences of propositions, and $(k_i)_{i=1}^n$ is a finite sequence of natural numbers, then if

(i) $\sum_{i=1}^n k_i \cdot \mathbf{1}_{p_i}(\omega) = \sum_{i=1}^n k_i \cdot \mathbf{1}_{q_i}(\omega)$ for all $\omega \in \Omega$, and
(ii) $p_i \succsim q_i$, for $i = 1, \ldots, n-1$,

then $q_n \succsim p_n$

Furthermore, while there will often be more than one set of probability functions $\mathfrak{F}$ that agrees with $\succsim$, the union of all such sets will always agree with $\succsim$. So, whenever $\succsim$ satisfies A1–A5 there's always a unique set of probability functions that agrees with $\succsim$ that's maximal with respect to inclusion.

Hence, if we extend the definitions of full / L-to-R / R-to-L adequacy in the natural way (i.e., by inserting '$\forall \mathcal{C}r \in \mathfrak{F}$' in the appropriate locations), we can characterise *imprecise-probabilistic comparativism* by its commitment to:

**Imprecise-Probabilistic Comparativism.** If a non-empty set of probability functions $\mathfrak{F}$ agrees with $S$'s belief ranking and $\mathfrak{F}$ is maximal with respect to inclusion, then $\mathfrak{F}$ is a fully adequate model of $S$'s beliefs

Imprecise-probabilistic comparativism implies the precise version. That is, if we assume that $\mathfrak{F}$ and $\mathcal{C}r$ are essentially the same model whenever $\mathfrak{F} = \{\mathcal{C}r\}$, then the two varieties of comparativism say exactly the same thing whenever exactly one probability function agrees with $\succsim$. Furthermore, every $\mathcal{C}r$ in a set $\mathcal{F}$ that agrees with $\succsim$ will itself *almost* agree with $\succsim$. So, if we also extend the definition of *coherence* in the natural way to sets of functions, it follows that if a set of probability functions $\mathcal{F}$ agrees with $\succsim$, then $\mathcal{F}$ coheres with the GRP. In short, both versions of comparativism explain cardinal comparisons using *basically* the same idea.

### 3.3 The Ramseyan Alternative

The benefit of imprecise-probabilistic comparativism is that it's able to explain cardinal comparisons in a strictly wider range of conditions. But it's possible to go further still. A2–A5′ are sufficient for the union of pseudodisjoint sets to behave like addition; they're by no means necessary.

I'll end this section with two new and more general 'Ramseyan' varieties of comparativism, and I'll show that they're about as general as comparativism can get so long as it's committed to the foregoing explanation of cardinality in terms of unions of pseudodisjoint sets. Both are inspired by the following remark from Ramsey (1929):

> [...] 'Well, I believe it to an extent $^2/_3$', i.e. (this at least is the most natural interpretation) 'I have the same degree of belief in it as in $p \vee q$ when I think $p, q, r$ equally likely and know that exactly one of them is true'. (p. 256)

The idea is also discussed briefly by Weatherson (2016, pp. 223–4), who likewise puts it forward as a basis for more general variety of comparativism. However, neither Ramsey nor Weatherson take their discussion beyond this very initial suggestion, and (as we'll see) there's a few conditions that need to be met before it can ground a minimally plausible account of partial belief.

First, some more definitions:

**Definition 5.** A set of $n$ pseudodisjoint propositions $\mathcal{P}$ is an *$n$-scale of $p$* iff $\bigcup \mathcal{P} \sim p$ and for all $q, q'$ in $\mathcal{P}$, $q \sim q'$

We can take Definition 5 as a comparativist characterisation of what it is for an agent to think that $q$'s as likely as a disjunction of equiprobable propositions at most one of which is true. So, e.g., if Sally thinks $q$ is as likely as $p_1 \cup p_2$, where $p_1$ and $p_2$ are equiprobable and pseudodisjoint, then $\{p_1, p_2\}$ is a 2-scale of $q$. We'll also need to assume that Sally is *certain* of $p$'s truth just in case $p$ is maximal. This is non-trivial, for reasons we'll discuss below, but it also appears to be unavoidable given the limited resources with which *certainty* might be defined within the framework of (binary) comparativism.

Given this, Ramsey's idea becomes:

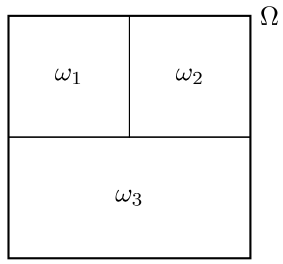
$$\text{Sally believes } p \text{ to degree } {}^n/_m \text{ if } p \sim (q_1 \cup \cdots \cup q_n),$$

where the $q_1, \ldots, q_n$ belong to an $m$-scale $\{q_1, \ldots, q_n, \ldots, q_m\}$ of some maximal proposition $r$. A good start—but there's a natural extension that will be helpful to incorporate into what follows.

Consider the following situation: $\mathcal{B}$ is the powerset of $\Omega = \{\omega_1, \omega_2, \omega_3\}$, $\succsim$ is transitive, and

$$\Omega \succ \{\omega_1, \omega_3\} \sim \{\omega_2, \omega_3\} \succ \{\omega_1, \omega_2\} \sim \{\omega_3\} \succ \{\omega_1\} \sim \{\omega_2\} \succ \varnothing$$

We can represent $\succsim$ as follows, where the relative sizes of the boxes containing the $\omega_i$ correspond to the order of propositions in the belief ranking:

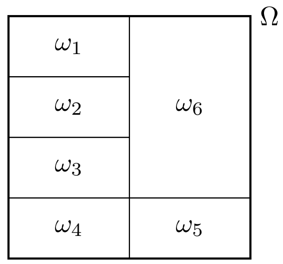



Now $\{\Omega\}$ is a 1-scale of $\Omega$, and $\{\{\omega_3\}, \{\omega_1, \omega_2\}\}$ is a 2-scale of $\Omega$, so Ramsey would say that

$$\mathcal{C}r(\Omega) = 1, \;\; \mathcal{C}r(\{\omega_3\}) = \mathcal{C}r(\{\omega_1, \omega_2\}) = {}^1\!/_2$$

However, the singletons $\{\omega_1\}$ and $\{\omega_2\}$ don't belong to any $n$-scale of $\Omega$, so Ramsey's idea doesn't yet give us any strength with which they're believed. But since $\{\{\omega_1\}, \{\omega_2\}\}$ is a 2-scale of $\{\omega_1, \omega_2\}$, it's only reasonable to say that $\mathcal{C}r(\{\omega_1\}) = \mathcal{C}r(\{\omega_2\}) = {}^1\!/_4$.

Likewise, consider the following case, where $\Omega = \{\omega_1, \ldots, \omega_6\}$:




Here, assume that $\Omega$ is maximal and $\varnothing$ minimal, and $\succsim$ includes:

$$\{\omega_5, \omega_6\} \sim \{\omega_1, \omega_2, \omega_3, \omega_4\} \succ \{\omega_6\} \sim \{\omega_1, \omega_2, \omega_3\} \succ \{\omega_1\}$$

$$\{\omega_1\} \sim \{\omega_2\} \sim \{\omega_3\} \sim \{\omega_4\} \sim \{\omega_5\}$$

This time, $\{\{\omega_5, \omega_6\}, \{\omega_1, \omega_2, \omega_3, \omega_4\}\}$ is a 2-scale of $\Omega$, and $\{\omega_1\}, \{\omega_2\}, \{\omega_3\}$ are three members of the 4-scale $\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}$ of $\{\omega_1, \omega_2, \omega_3, \omega_4\}$; we'd therefore like to say that $\mathcal{C}r(\{\omega_1, \omega_2, \omega_3\}) = {}^3\!/_4 \cdot {}^1\!/_2 = {}^3\!/_8$. We note also that $\{\{\omega_6\}\}$ is a 1-scale of $\{\omega_1, \omega_2, \omega_3\}$; hence, $\mathcal{C}r(\{\omega_6\}) = {}^3\!/_8$.

We can capture the foregoing points by means of the following definition:

**Definition 6.** For integers $n, m$ such that $m \geq n \geq 0$, $m > 0$, $p$ is

(i) ${}^0\!/_m$-*valued* if $p$ is minimal and ${}^m\!/_m$-*valued* if $p$ is maximal

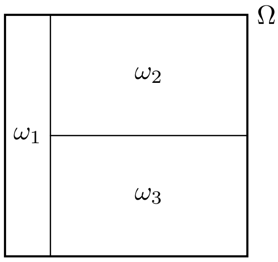
(ii) ${}^n\!/_m$-*valued* if $p \sim (q_1 \cup \cdots \cup q_{n'})$, where the $q_1, \ldots, q_{n'}$ belong to an $m'$-scale of an ${}^{n''}\!/_{m''}$-valued proposition, and ${}^{n' \cdot n''}\!/_{m' \cdot m''} = {}^n\!/_m$

The generalised version of Ramsey's suggestion now amounts to the claim that Sally believes $p$ to degree ${}^n\!/_m$ if $p$ is ${}^n\!/_m$-valued. As such, define a *Ramsey function* as follows:

**Definition 7.** $\mathcal{C}r : \boldsymbol{\mathcal{B}} \mapsto [0,1]$ is a *Ramsey function* (relative to $\succsim$) iff, for all $p \in \boldsymbol{\mathcal{B}}$, if $p$ is ${}^{n}/_{m}$-valued, then $\mathcal{C}r(p) = {}^{n}/_{m}$

The connection between Ramsey functions and the GRP should be immediately apparent. In fact, in present terminology, the first (non-inductive) clause of the GRP states that for $m \geq n$, $p$ is believed ${}^{n}/_{m}$ times as much as $q$ whenever $\boldsymbol{\mathcal{P}}$ is an $m$-scale of $q$, and $\boldsymbol{\mathcal{P}'} \subseteq \boldsymbol{\mathcal{P}}$ is an $n$-scale of $p$. In this case, for any Ramsey function $\mathcal{C}r$, $\mathcal{C}r(p) = {}^{n}/_{m} \cdot \mathcal{C}r(q)$. With respect to ${}^{n}/_{m}$-valued propositions, Ramsey functions cohere with the GRP perfectly.

Essentially, a Ramsey function scales every non-minimal ${}^{n}/_{m}$-valued proposition relative to a maximal proposition, which has a stipulated value. With respect to pairs of propositions that cannot be so scaled, however, a Ramsey function *may* conflict with the GRP. An example where this could occur is:




Where

$$\Omega \succ \{\omega_2, \omega_3\} \succ \{\omega_1, \omega_2\} \sim \{\omega_1, \omega_3\} \succ \{\omega_2\} \sim \{\omega_3\} \succ \{\omega_1\} \succ \varnothing$$

The only non-trivial $n$-scale is the 2-scale $\{\{\omega_2\}, \{\omega_3\}\}$ of $\{\omega_2, \omega_3\}$; but, since $\{\omega_2, \omega_3\}$ can't be scaled relative to $\Omega$, the values of $\{\omega_2\}, \{\omega_3\}$ and $\{\omega_2, \omega_3\}$ are left indeterminate.

Call any proposition that's ${}^{n}/_{m}$-valued 'Ramsey-scalable'. Ramsey says nothing about how to measure propositions that aren't Ramsey-scalable, and this is an important lacuna in his proposal—though perhaps not a very troubling one. One might assume that such cases don't exist. Let $\boldsymbol{\mathcal{N}}$ designate the set of Ramsey-scalable propositions. Then, the assumption would be:

**B1.** $\boldsymbol{\mathcal{N}} = \boldsymbol{\mathcal{B}}$

B1 isn't implied by A1–A5. However, it *is* implied by A4 plus Continuity. (Let $q$ be maximal, $\varnothing$ minimal, and consider any non-minimal $p$; Continuity then states that $\{p\}$ is a 1-scale of the union of $n$ members of an $m$-scale of $q$, which is strictly more than B1 requires.) In other words, probabilistic comparativists have nothing to fear from a condition like B1.

But B1 only ensures that every $p \in \boldsymbol{\mathcal{B}}$ is Ramsey-scalable. It isn't *yet* enough to ground a plausible comparativist story. There's still two additional problems that can arise in the absence of further conditions on $\succsim$.

First: nothing's been said to guarantee that Definition 7 is *consistent*. Without further assumptions, it's entirely possible for, e.g., $p \sim q$, where for some $r$, $p$ belongs to a 2-scale of $r$ and $q$ belongs to a 3-scale of $r$. This is clearly undesirable: Sally can't believe $p$ to the degrees ${}^{1}/_{2}$ and ${}^{1}/_{3}$ simultaneously! If

Ramsey functions are to be well-defined, we'll need to ensure that if $p$ is both ${}^{n}\!/\!_{m}$-valued and ${}^{n'}\!/\!_{m'}$-valued, then ${}^{n}\!/\!_{m} = {}^{n'}\!/\!_{m'}$.

Second: nothing's been said to guarantee that a Ramsey function relative to $\succsim$ will even *agree* with $\succsim$. Indeed, nothing ensures that $\mathcal{C}r(p) \geq \mathcal{C}r(q)$ if *or* only if $p \succsim q$—$p$ could be ${}^{1}\!/\!_{2}$-valued, and $q$ ${}^{1}\!/\!_{4}$-valued, yet $q \succsim p$. This is also undesirable: if the order of the values we assign propositions don't match up to the belief ranking, then there's no natural sense in which those values are a measure of the *strengths* with which those propositions are believed.

In the context of B1, we can kill these two birds with one stone by making the following rather strong assumption:

**B2.** If $p$ is ${}^{n}\!/\!_{m}$-valued and $q$ is ${}^{n'}\!/\!_{m'}$-valued, then $p \succsim q$ iff ${}^{n}\!/\!_{m} \geq {}^{n'}\!/\!_{m'}$

B2 is obviously necessary (and given B1, sufficient) to avoid both worries, as the following establishes:

**Theorem 1.** *$\succsim$ satisfies B2 iff there exists a Ramsey function $\mathcal{C}r$ with respect to $\succsim$; furthermore, $\mathcal{C}r$ is the unique Ramsey function relative to $\succsim$ that agrees with $\succsim$ iff $\succsim$ satisfies B1*

(Proofs for all theorems in this section can be found in the Appendix.)

It's easy to see that B2 is implied already by A1–A5. Indeed, if any probability function $\mathcal{C}r$ agrees with $\succsim$, then $\mathcal{C}r$ is *also* a Ramsey function relative to $\succsim$. Moreover, where B1 plus A1–A5 hold, then the unique probability function that agrees with $\succsim$ *is* the unique Ramsey function that agrees with $\succsim$.

In the other direction, we can also easily see that while B1–B2 jointly imply A1–A2, they don't imply A3–A5/A5′. It's straightforward to find examples where B1–B2 are satisfied but A3–A4 aren't; and for an example where A5/A5′ is falsified, assume again that $\Omega = \{w_1, w_2, w_3\}$, and:

$$\{w_1, w_2\} \succ \Omega \sim \{w_1\} \sim \{w_2\} \succ \{w_3\} \sim \{w_1, w_3\} \sim \{w_2, w_3\} \sim \varnothing$$

As $\{\{w_1\}, \{w_2\}\}$ is a 2-scale of the maximal $\{w_1, w_2\}$, $\mathcal{C}r(\{w_1, w_2\}) = 1$ and $\mathcal{C}r(\{w_1\}) = {}^{1}\!/\!_{2}$. $\Omega$ isn't maximal, but it's as likely as $\{w_1\}$; so $\mathcal{C}r(\Omega) = {}^{1}\!/\!_{2}$.

In short, B2 imposes a very limited kind of qualitative additivity on $\succsim$, specifically with respect to comparative beliefs between propositions constructed out of members of the same $n$-scale of any ${}^{n'}\!/\!_{m'}$-valued proposition. Roughly: *within* an $n$-scale, $\succsim$ behaves probabilistically—but not every proposition is constructible out of the members of an appropriate $n$-scale, and *across* $n$-scales $\succsim$ can behave quite irrationally indeed.

If we wanted to drop B1 out of the picture, we could do so by adopting a set-of-functions representation of $\succsim$. For that, we would need to add at least:

**B3.** $\succsim$ is a preorder

For simplicity, we focus on the case where $\boldsymbol{\mathcal{B}}$ is countable; thus,

**Theorem 2.** *Where $\boldsymbol{\mathcal{B}}$ is countable, $\succsim$ satisfies B2–B3 iff there exists a nonempty set $\boldsymbol{\mathfrak{F}}$ of real-valued functions bounded by 0 and 1 that agrees with $\succsim$, where every $\mathcal{C}r$ in $\boldsymbol{\mathfrak{F}}$ is a Ramsey function relative to $\succsim$*

Furthermore, whenever B2–B3 are satisfied, there will be a unique set $\mathcal{F}$ that's maximal with respect to inclusion.

Given the above, let's characterise two 'Ramseyan' comparativisms:

> **Ramseyan Comparativism.** If $\mathcal{C}r$ is the only Ramsey function relative to $S$'s belief ranking, then $\mathcal{C}r$ is a fully adequate model of $S$'s beliefs
>
> **Imprecise-Ramseyan Comparativism.** If $\mathcal{F}$ is a non-empty set of Ramsey functions with respect to $S$'s belief ranking, which is maximal with respect to inclusion and agrees with $\succsim$, then $\mathcal{F}$ is an R-to-L adequate model of $S$'s beliefs

Note that imprecise-Ramseyan comparativism only claims R-to-L adequacy. This is because (as we've seen) B2–B3 are not sufficient for *total* coherence with the GRP. But both Ramseyan comparativisms agree with probabilistic comparativism whenever A1–A5 and B1 are satisfied.[6]

Finally, we can show that B2 and B3 are individually *necessary* for coherence with the GRP. For B3 this is trivial: the condition is obviously necessary for *any* real-valued function *or* set thereof to agree with $\succsim$. And given some minimal scaling assumptions, violations of B2 imply that any $\mathcal{C}r$ that agrees with $\succsim$ won't cohere with the GRP:

> **Theorem 3.** *If (i) $\mathcal{C}r$ agrees with $\succsim$, (ii) there are $p, q$ such that $p \succ q$, and (iii) $\mathcal{C}r(r) = 0$ whenever $r$ is minimal, then $\mathcal{C}r$ coheres with the GRP only if B2*

Corollary: under the same assumptions, *mutatis mutandis*, a set of functions $\mathcal{F}$ coheres with the GRP only if B2. In other words, assuming just that $\succsim$ is non-trivial and that minimal propositions can be assigned value 0, coherence with the GRP implies B2–B3.

# 4 Inexpressibility Challenges

Let's take stock. The standard comparativist strategy for explaining ratio comparisons is based on a purported continuity with the measurement of extensive quantities like length and mass: to say that $p$ is $n$ times more likely than $q$, we just need to be able to say that $p$ is as likely as the union of $n$ 'duplicates' of $q$, where the 'duplicates' are propositions that are equiprobable and pairwise pseudodisjoint. The Ramseyan comparativisms just outlined offer an accounts of when this kind of 'adding' is meaningful that generalise the conditions assumed by probabilistic comparativism, applying in a wide range of cases that aren't probabilistically representable. Moreover, we know that the union of pseudodisjoint sets behaves like addition only given B2 and B3.

It remains to be seen whether it's correct to say that $p$ is $n$ times more likely than $q$ iff $p$ is as likely as $n$ pseudodisjoint duplicates of $q$. The examples of this section provide reasons to doubt both directions of that biconditional. More generally, they establish that there are some seemingly sensible distinctions between belief states that comparativists, in principle, cannot make.

[6] This is not so obvious in the case of imprecise-Ramseyan comparativism, but consider: if A1–A5 and B1 hold, then the probability function $\mathcal{C}r$ that agrees with $\succsim$ is the Ramsey function that agrees with $\succsim$; from imprecise-Ramseyan comparativism, $\mathcal{C}r$ is R-to-L adequate, so $\mathcal{C}r$ determines a unique ratio comparison for every pair of non-minimal propositions; and finally, $S$ cannot believe $p$ ${}^{n}/{}_{m}$ times as much as $q$ and ${}^{n'}/{}_{m'}$ as much as $q$, for ${}^{n}/{}_{m} \neq {}^{n'}/{}_{m'}$.

## 4.1 Almost Omniscience

Consider Zeus, who knows almost everything:

> **Example 1.** Zeus is *almost omniscient*: his comparative beliefs satisfy A1–A5, and he's certain that the actual world is either $w_1$ or $w_2$. While he's got some confidence in each, he's a little more confident that the actual world is $\omega_1$ than that it's $\omega_2$.

I take it that the notion of *almost omniscience* makes sense. It already exists in the literature, in the case of David Lewis' two gods (Lewis 1979, pp. 520–1). And we could easily imagine each of Lewis' gods having more or less confidence regarding which of the two (centred) worlds they inhabit. However, no variety of comparativism that's been described so far has the resources to accommodate Example 1 as described.

I focus my discussion on the probabilistic comparativisms; the issues are analogous for the Ramseyan varieties. Let $\mathfrak{B}$ be any algebra of sets relative to any $\Omega$, and a probability function $\mathcal{C}r$ will almost agree with Zeus's belief ranking $\succsim$ iff for all $p \in \mathfrak{B}$,

$$\mathcal{C}r(p) = \begin{cases} 0, & \text{if } \omega_1, \omega_2 \notin p \\ x, & \text{if } \omega_1 \in p,\, \omega_2 \notin p \\ y, & \text{if } \omega_2 \in p,\, \omega_1 \notin p \\ 1, & \text{if } \omega_1, \omega_2 \in p, \end{cases}$$

where $1 \geq x \geq y \geq 0$ and $x + y = 1$, with full agreement whenever $1 > x > y > 0$. There's uncountably many probability functions that agree with $\succsim$, so probabilistic comparativism tells us nothing about Zeus's situation.

More importantly, *imprecise* probabilistic comparativism fares no better. To slightly abuse notation, let $\mathcal{C}r(\omega_i)$ pick out Zeus's confidence that the actual world is $\omega_i$. The imprecise-probabilistic comparativist can then say that $\mathcal{C}r(\omega_1)$ takes the 'imprecise' value $[0.5, 1]$, and $\mathcal{C}r(\omega_2)$ takes $[0, 0.5]$. But regardless of how you want to interpret those values—i.e., if you read them as saying that there's *no fact of the matter* about what the strengths of Zeus's beliefs are within that range, or if read them as saying that his beliefs *determinately* have imprecise strengths—you don't get to say that Zeus has a little more (or a lot, or twice as much, etc.) confidence in $\omega_1$ as in $\omega_2$. By virtue of learning almost everything there is to know, Zeus has lost the capacity to believe one thing just a little more than another.

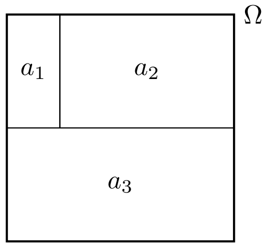
## 4.2 Jack and Jill

Here's a similar case, this time involving interpersonal comparisons:

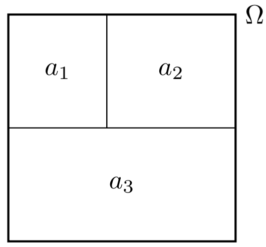
> **Example 2.** Jack and Jill have identical comparative beliefs, each satisfying A1–A5, over a simple finite algebra with atoms $a_1, a_2, a_3$. They agree that $a_3$ is more likely than $a_2$, $a_2$ more likely than $a_1$, and $a_1 \cup a_2$ is just as likely as $a_3$. However, Jack insists that, unlike Jill, he has at least twice the confidence in $a_2$ as in $a_1$.

Jack and Jill's situation should again make intuitive sense. (Nothing much hangs here on the extreme paucity of the algebra, which is only for simplicity of exposition. We'll be able to find similar examples using any non-trivial algebra.)

In pictorial form,




Jack




Jill

Comparativists cannot accept Jack's claim: if Jack and Jill have identical comparative beliefs, then they have identical beliefs *simpliciter*. (Recall the characterisation of comparativism from §1.) The question for us is: can we have good reasons to think that Jack is telling the truth?

I think we can. A probability function $\mathcal{C}r$ agrees with Jack's belief ranking (and hence also Jill's belief ranking) just in case

1. $\mathcal{C}r(a_3) = 0.5 > \mathcal{C}r(a_2) > 0.25 > \mathcal{C}r(a_1) > 0$, and
2. $\mathcal{C}r(a_1) + \mathcal{C}r(a_2) = 0.5$

As with Example 1, there are many probability functions satisfying these conditions. And, crucially, *each* such function predicts a different set of preferences when it's (a) taken to model a possible belief state, and (b) combined with a standard model of rational preferences. Suppose for instance that Jack and Jill each face choices which share a similar structure:

| | $a_1$ | $a_2$ | $a_3$ |
|---|---|---|---|
| $\alpha$ | $-2x$ | $x$ | $x$ |
| $\beta$ | 0 | 0 | $x$ |

Jack prefers $\alpha$ to $\beta$; Jill has the opposite preferences. Assuming $\mathcal{C}r$ is a probability function satisfying the stated conditions, $\alpha$'s expected utility is greater than $\beta$'s just in case $\mathcal{C}r(a_2) > 1/3$. So, a natural explanation for the difference is that Jack believes $a_2$ to a degree greater than $1/3$, and Jill less than $1/3$.

Comparativists are committed to saying that the difference in Jack's and Jill's preferences *cannot* be due to any differences of belief—there are none. Indeed, since their preferences differ but their choices have the same structure, comparativists seem committed to saying that either at least one of Jack or Jill has chosen irrationally, *or* that neither of them ought to choose either of the options. The former looks hard to accept, as there's nothing especially irrational about either pattern of preferences. And the latter fails to offer a satisfying explanation in the event that their choices *consistently* point to Jack believing $a_2$ more than $1/3$, and Jill less than $1/3$. For example, we could imagine that in the following kind of situation, Jack prefers $\delta$ to $\gamma$, and Jill $\gamma$ to $\delta$:

| | $a_1$ | $a_2$ | $a_3$ |
|---|---|---|---|
| $\gamma$ | $2x$ | $-x$ | $x$ |
| $\delta$ | 0 | 0 | $x$ |

And we could generate countless more decision situations to tell them apart. There's a natural explanation for why Jack and Jill would consistently choose 'as if' they believed $a_2$ more than $1/3$ and less than $1/3$ respectively, and it's strictly off-limits to comparativism.

## 4.3 Logical Doubts

This time, we assume that only one probability/Ramsey function agrees with the relevant belief ranking:

> **Example 3.** Although his comparative beliefs satisfy A1–A5 and B1, Agrippa is *not* the perfect Bayesian. After reading a little too much Pyrrhonian literature, he insists it's never rational to be *fully* certain of anything: one should always reserve *some* slight doubt (say, 1%) that even the most firm of logical truths might be false, and that any logical falsehood might be true. Moreover, his preferences consistently reflect this—e.g., he'd prefer \$90 to the gamble $\langle \$100$ *if* $p \vee \neg p$, $-\$1000$ *otherwise*$\rangle$, and he prefers $\langle \$100$ *if* $p \vee \neg p$, $\$100,000$ *otherwise*$\rangle$ to being given \$1000 outright.

Here's a natural explanation of Agrippa's preferences: where $\mathcal{C}r$ is the probability function that uniquely agrees with Agrippa's belief ranking, his *partial* beliefs are in fact faithfully modelled by the *non*-probabilistic function $\mathcal{C}r^\star$, where

$$\mathcal{C}r^\star(p) = 0.98 \cdot \mathcal{C}r(p) + 0.01$$

(Think of $\mathcal{C}r^\star$ as $\mathcal{C}r$ squished down by 1% on either side.) $\mathcal{C}r^\star$ conflicts with the GRP everywhere, but if we plug it into an ordinary account of preference formation it fits perfectly well with his preferences.[7] Say what you will about the rationality of doubting logical truths, at least Agrippa chooses rationally *given* his beliefs.

But, again, such an explanation is off-limits to comparativism. Indeed, all four versions of comparativism I've described are committed to saying that the probabilistic $\mathcal{C}r$ is a fully adequate model of Agrippa's beliefs, and that $\mathcal{C}r^\star$ *misrepresents* him: synchronically, Agrippa *is* an epistemically ideal Bayesian agent, despite the evidence of his preferences and his protestations to the contrary.

## 4.4 Non-Additivity and the GRP

The above examples are fanciful, but it's important not to miss the more general point they're intended to convey. If it sounds too strange to say that Zeus can be almost omniscient, or that Agrippa might have some slight doubt in even the most basic of logical truths despite considering them more probable than anything else, then never fear: whenever any probability/Ramsey function agrees with $\succsim$, no matter how uniquely, there are infinitely many ordinal transformations of that function (with values bounded by 1 and 0) that we could draw

[7] I should note that this example doesn't rely on the error that Joyce (2015, pp. 418–9) discusses, of re-scaling the model of Agrippa's beliefs without making appropriate adjustments to how expected utilities are calculated—thus giving the misleading impression that $\mathcal{C}r^\star$ generates different predictions about Agrippa's preferences when they're plugged into an expected utility model of preference formation. For we get to say that $\mathcal{C}r^\star$ is a mere re-scaling of $\mathcal{C}r$ only under the substantive assumption that whatever Agrippa believes most (least) of all, he believes to the most (least) extent possible. The scale we use to measure belief is a matter of stipulation, to be sure, but we don't get to stipulate that *maximality* equates to *certainty* without argument.

upon to find counterexamples instead. Perhaps *some* of these transformations fail to pick out genuinely distinct, possible belief states, but comparativists will be hard pressed to show this for *all* of them.

Consider, for instance:

> **Example 4.** Descartes doubts every contingent claim, being certain of only the self-evident logical truths. His comparative beliefs satisfy A1–A5 and B1, and $\mathcal{C}r$ is the unique probability function that agrees with them. But his own self-attributions fit better with:

$$\mathcal{C}r^{\dagger}(p) = \begin{cases} {}^{1}\!/\!{}_{10} \cdot \mathcal{C}r(p), & \text{whenever } \mathcal{C}r(p) < 1 \\ \mathcal{C}r(p) & \text{otherwise,} \end{cases}$$

$\mathcal{C}r^{\dagger}$ is ordinally equivalent to $\mathcal{C}r$ but with most values bunched up in the $[0, 0.1]$ range. Having partial beliefs like these is absurd, maybe even impossible. I'm not sure whether it is, but that set *that* issue aside and compare $\mathcal{C}r^{\dagger}$ to another ordinal transformation that more smoothly shifts some of $\mathcal{C}r$'s values a little to the left:

> **Example 5.** Thomas is *almost* an ideal Bayesian. His comparative beliefs satisfy A1–A5 and B1, and $\mathcal{C}r$ is the unique probability function agrees with them. But his own self-attributions fit better with:

$$\mathcal{C}r^{\dagger\dagger}(p) = \begin{cases} {}^{1}\!/\!{}_{2} \cdot \mathcal{C}r(p), & \text{whenever } 0 \leq \mathcal{C}r(p) < 0.05 \\ {}^{3}\!/\!{}_{4} \cdot \mathcal{C}r(p), & \text{whenever } 0.05 \leq \mathcal{C}r(p) < 0.075 \\ {}^{7}\!/\!{}_{8} \cdot \mathcal{C}r(p), & \text{whenever } 0.075 \leq \mathcal{C}r(p) < 0.0825 \\ \quad\vdots & \\ \mathcal{C}r(p) & \text{otherwise} \end{cases}$$

Such a distribution looks neither absurd nor impossible. And absent independent reasons, it's very hard to accept that we should rule out the possibility of partial beliefs like those modelled by $\mathcal{C}r^{\dagger\dagger}$ that fall just shy of probabilistic coherence.

Don't be tempted to think that Example 5 isn't so bad, because at least the cardinal comparisons we get from applying the GRP to Thomas' belief ranking will in most cases *approximate* the comparisons encoded in $\mathcal{C}r^{\dagger\dagger}$. This is a non-sequitor: the GRP gets every cardinal comparison *exactly* right for $\mathcal{C}r$, and $\mathcal{C}r$ and $\mathcal{C}r^{\dagger\dagger}$ are ordinal equivalents. So, if $\mathcal{C}r$ is a fully adequate model of some possible belief state—and of course it is—then comparativists are committed to saying that $\mathcal{C}r^{\dagger\dagger}$ can't be either an L-to-R *or* an R-to-L adequate model of Thomas' (or anyone else's) beliefs. According to comparativism, $\mathcal{C}r^{\dagger\dagger}$ isn't a model of some slightly irrational beliefs for which the GRP is approximately accurate—it's not a model of any doxastic state at all. And *that's* implausible.

$\mathcal{C}r^{\dagger\dagger}$ is an instance of a non-additive *Choquet capacity*, of the kind frequently used to model beliefs in many descriptive-oriented models of partial beliefs and decision-making.[8] On such models, partial beliefs are represented with functions such that $\mathcal{C}r(p \cup q)$ need not *and frequently does not* equal $\mathcal{C}r(p) + \mathcal{C}r(q)$ even when $p$ and $q$ are pseudodisjoint (or genuinely disjoint). The introduction of

[8] A capacity can be defined as any monotonically increasing transformation of a probability function which preserves the values for $\varnothing$ and $\Omega$.

non-additive models was motivated in part by a wealth of evidence suggesting that the absolute and comparative confidence judgements of non-ideal thinkers are non-additive with respect to the union of disjoint sets. See, e.g., (Tversky and Kahneman 1982), (Yates and Carlson 1986), (Carlson and Yates 1989), (Bar-Hillel and Neter 1993), (Tversky and Koehler 1994).[9]

There's been no empirical work done on whether partial and comparative beliefs are additive specifically with respect to the union of equiprobable pseudodisjoint sets—though, Thomas' beliefs hardly seem too far out of the ordinary. Indeed, they're consistent with multiple approaches to explaining non-probabilistic confidence judgements. For example, on the 'probability+noise' model, comparative/partial beliefs *roughly* adhere to the norms of probability theory, *but* for systematic deviations due to random 'noise' in cognitive processing. Such noise can produce super-additive confidence judgements of the kind seen in Example 5 (cf. Costello 2008; Fisher and Wolfe 2014).

Once we allow non-additive models of partial belief, we're going to have models that cannot be squared with the GRP. Indeed, for any non-probabilistic capacity $\mathcal{C}r$, it will *almost always* be the case that:

(i) $\mathcal{C}r$ is ordinally equivalent to some function that coheres with the GRP, but itself conflicts with the GRP; or
(ii) $\mathcal{C}r$ is not ordinally equivalent to any function that coheres with the GRP, i.e., it agrees with a $\succsim$ that violates B2.

Examples 4–5 involve the former kind of capacity; for the latter, consider:

> **Example 6.** Relative to Frank's comparative beliefs, $p_1, \ldots, p_{100}$ and $q_1, \ldots, q_{101}$ are two sequences of pairwise pseudodisjoint, equiprobable propositions, with $p_1 \sim q_1$. But due to a minor accounting error, Frank thinks $p_1 \cup \cdots \cup p_{100}$ is as likely as $r$, which is also as likely as $q_1 \cup \cdots \cup q_{101}$.

According to the GRP, Frank believes $r$ 100 times as much as $p_1$ and 101 times as much as $p_1$, so $r$ is believed 1% more than itself! But Frank's comparative beliefs violate B2. Does this mean that Frank just doesn't have quantifiable beliefs with respect to the $p_i$ and $q_i$? Why should we say this when there are *plenty* of non-additive capacities that agree with his belief ranking quite straightforwardly. (Example: let the capacity be additive as usual with respect to the $p_i$, and sub-additive by $^{100}/_{101}\%$ for significantly large unions of the $q_i$.) All of them look irrational to some extent or other, but most of them look *possible*, and they *all* encode cardinal comparisons that conflict with the GRP.

# 5 Objections and Replies

## 5.1 Supplementing Comparativism

Each of the examples of the previous section involves non-uniqueness in some way. One lesson to draw from them is that comparative beliefs are too coarse-

[9] Of course, the literature on how close ordinary human beings come to being probabilistically coherent is vast, and most of it controversial. I won't attempt to review it here. But even the most committed of 'descriptive' Bayesians will typically only claim that additive probabilities are useful idealising models for analysing the computational problems that humans face in the domains where we should independently expect cognition to be especially well optimised; e.g., vision, motor control, and language processing (Griffiths et al. 2012). Outside these domains, there's widespread agreement that the evidence for frequent fallacies in quantitative and comparative probability judgements is overwhelming.

grained to make sense of distinctions amongst partial belief states that seem at least conceptually possible. So a natural thought would be to enrich the supervenience base beyond what comparativism, strictly speaking, allows. If comparativism *per se* doesn't work, then perhaps something similar will.

Let *supplemented comparativism* denote the view that partial beliefs supervene on comparative beliefs *plus something else*. For example, while multiple probability functions agree with Jill's belief ranking, we may also find that only one of those has maximal *fit* with her preferences and/or life history of evidence. On that basis we might assign it as the uniquely correct model of her beliefs, and thereby distinguish her from Jack. A representation theorem like that of (Joyce 1999) could help in providing the mathematical foundations for such a view. Or we could supplement comparative belief with forms of qualitative judgement, like whether $p$ and $q$ are probabilistically independent, or whether $p$ is evidence for $q$ (cf. Joyce 2010, p. 288).

I have no general argument against supplemented comparativism to present here—it isn't the intended target of this paper. However, I will say this: it's not enough to just fix a unique assignment of numerical values (precise or imprecise) to the agents described in the above cases. *Uniqueness* and *fine-grainedness* are not the whole problem. We also need a plausible explanation of how we can make cardinal comparisons. Any explanation in terms of equiprobable pseudodisjoint propositions will not help us make sense of Zeus's case—there are no such propositions to 'add'. And if non-probabilistic functions really do provide good models of Agrippa's, Descartes', Thomas', and Frank's partial beliefs, then every one of them is a counterexample to the GRP. Or consider Jack and Jill. Perhaps supplemented comparativism could use preference information to determine that Jack believes $a_1$ to a degree greater than $1/3$—but we'd still need a justification for saying that Jack believes $a_2$ *at least twice* as much $a_1$, and Jill believes $a_2$ *less than twice* as much as she believes $a_1$. That explanation *cannot* come from any differences in their belief rankings.

An assignment of numbers without an empirically plausible explanation for how those numbers manage to carry cardinal information isn't a solution to comparativism's expressibility problems, regardless of how unique it is. To deal with the examples of §4, a different explanation of cardinal comparisons is needed. The GRP is false, and supplementing comparativism with further facts to make distinctions where mere belief rankings can't just shifts the bump under the rug.

I suspect that when we finally do explain the cardinality of partial belief, we'll see that it has much to do with the relationship between partial beliefs and preferences under conditions of uncertainty, and the union of (pseudo)disjoint propositions won't have much of a role at all. But more on that in §7.

### 5.2 'I only want to model ideally rational agents'

One common response to the counterexamples of §§4.3–4, which involve non-ideal agents with comparative and/or partial beliefs that flout the orthodox norms of probabilism, is that such agents simply aren't relevant for philosophical modelling purposes. We can safely ignore non-ideal agents, at least for now, because what matters for philosophical purposes is that we have an explanation of cardinality for *ideally rational agents*. And where 'ideally rational agent' is given a probabilist reading, the GRP seems to work well for *those* kinds of agents—at least when we set aside 'extreme' cases like Example 1.

Now, there's a reason that the GRP predicts accurate cardinal comparisons for ideally rational agents, and it has nothing to do with comparativism: the GRP is a good norm of rationality. According to probabilism, you're rational only if you have partial beliefs in alignment with the GRP. So there's no surprise on a probabilist conception of rationality, the principle generates accurate cardinal comparisons for ideally rational agents. This is common ground for probabilist comparativists and non-comparativists alike. The question is whether the GRP also reflects some interesting dependence relationship.

If it does, then presumably that same relationship should hold for non-ideal agents. It would be unreasonable to say that Sally doesn't have partial beliefs encoding interesting cardinal information just because she isn't ideally rational. Even if she were *highly* irrational, Sally could still believe one proposition *much more* than another, or *at least twice* as another. Our capacity to make cardinal comparisons isn't hostage to any presupposition of ideal rationality. Because of this, comparativists should want to show that their explanation of cardinality is plausibly *generalisable*. For if there doesn't seem much hope for generalising to non-ideal agents, then we've got reason to think that the explanation is false—even in the case of ideally rational agents.

Let me emphasise: none of this is to deny the obvious claim that it's usually fruitful to get an explanation of some phenomenon working for an idealised model first, before moving on to less idealised cases. That's how the development of most good explanations go in the sciences, and it's exactly how we should expect things to go for our theories of partial belief. But idealised models have explanatory value only when the conclusions drawn from them are robust under variations to their idealising conditions. Roughly, the model should not 'break down' when their idealising conditions are perturbed. So it would be helpful if we had *some* assurance that comparativism's strategy for explaining cardinality doesn't depend crucially on idealised assumptions.

The examples in §4 are reasons to worry that comparativist explanations of cardinality aren't sufficiently generalisable. Comparativism is *in principle* blind to perfectly sensible distinctions between infinitely many ordinally equivalent partial belief states. And even if we ignore the problem of ordinal equivalents, the comparativist explanation of cardinality requires that (i) there's enough equiprobable pseudodisjoint propositions around to 'add'; (ii) an assumed equivalence of maximality with certainty and minimality with zero confidence; and (iii) that non-ideal agents with quantifiable beliefs will satisfy quite strong rationality conditions like B2—conditions that ordinary humans *do* seem to falsify.

## 5.3 Disjunctivism

A third response to cases involving probabilistically incoherent belief rankings is *disjunctivism*: if the union of equiprobable pseudodisjoint sets doesn't behave additively for some non-ideal agents, then perhaps there's another operation that *does* behave additively with respect to their belief ranking and we can characterise their cardinal comparisons in terms of *that* operation instead.

Note, first of all, that disjunctivism doesn't seem to help with any of Examples 1–5. At best, it helps with cases like Example 6, and even then there are still the worries arising from ordinally equivalent alternative functions. But moreover, if different operations are supposed to play the concatenation role for different agents, each contingent on whatever operation is appropriate for that agent's idiosyncratic belief ranking, then both interpersonal and intrapersonal cardinal comparisons would become quite useless in general. Before we could know what it means for Sally to believe $p$ twice as much as $q$, we would first have to take into account her entire belief ranking, work out what the concatenation operation is, and only then give empirical meaning to the comparison. Without knowledge of the overall structure of her belief ranking, the cardinal comparison would only tell us that

(i) Sally believes $p$ more than $q$, and
(ii) There's *some* binary operation $\oplus$ that shares certain formal properties with addition relative to $\succsim$ such that for $q', q''$ somehow related to $q$, $q' \oplus q'' = p$

The latter is deeply uninformative, and the former we don't need cardinal comparisons to express. Disjunctivism is a non-starter; for non-supplemented comparativists, it's the GRP or bust.

## 5.4 Impossible Worlds

A final objection, this time relating to the assumption that $\Omega$ contains only *possible* worlds (§3). Roughly, the worry is this. In §4.4, I argued that there are quantifiable belief states that cannot be modelled by any $\mathcal{C}r$ that agrees with a belief ranking satisfying B2. But my argument for this assumes that $\mathcal{C}r$ is defined over a space of *possible* worlds. With an algebra defined on a rich enough space of possible *and* impossible worlds, we can construct probability functions that 'mimic' the behaviour of *any* non-probabilistic function on $\boldsymbol{\mathcal{B}}$.[10] So, what looks like partial and comparative beliefs inconsistent with B2 when we use only possible worlds can be *re*-modelled as a probability function, *if* we help ourselves to enough impossible worlds.

This 're-modelling' strategy also suggests that what seemed like very strong rationality conditions, A1–A5, are in fact fully compatible with the belief rankings of highly non-ideal agents. To generalise the comparativist explanation of cardinality, we need not go beyond probability theory—we just need to have enough impossible worlds. And the strategy may even help with Examples 1–5. For this, it would have to be shown that for any sequence of ordinally equivalent $\mathcal{C}r$ on $\boldsymbol{\mathcal{B}}$ whose members we want to distinguish, there are as many ordinally *non*-equivalent probabilities defined over the space with impossible worlds to 'mimic' them. I suspect that this is false for finite $\boldsymbol{\mathcal{B}}$. But the point is moot, because the 're-modelling' strategy will not help comparativism. I'll set out my reasons for saying this very briefly, since the relevant issues are discussed at length in (Elliott, forthcoming).

Once $\Omega$ includes enough impossible worlds for the strategy to work—roughly, for any impossibility, there's an impossible world that verifies it—most subsets of $\Omega$ will be *meaningless*, not representative of any proper contents of thought. More importantly, the set of *meaningful* subsets will have precisely zero interesting set-theoretic structure. For any meaningful subset $p$ of $\Omega$, none of $p$'s

[10] In (Elliott, forthcoming), I show that if $\boldsymbol{\mathcal{B}}$ is a countable subset of $\wp(\Omega)$, where $\Omega$ is a space of possible worlds, $\Omega^+$ is a rich enough extension of $\Omega$ (i.e., has enough impossible worlds), and $\mathcal{C}r$ is any function from $\boldsymbol{\mathcal{B}}$ into $[0, 1]$, then there's a probability function $\mathcal{C}r^+$ on an appropriate algebra of sets $\boldsymbol{\mathcal{B}}^+$ on $\Omega^+$ such that $\mathcal{C}r^+$ assigns $x$ to the subset of $\Omega^+$ that verifies $\varphi$ just in case $\mathcal{C}r$ assigns $x$ to the subset of $\Omega$ that verifies $\varphi$. For more details, see also (Cozic 2006), (Halpern and Pucella 2011).

subsets or supersets will be meaningful, and no subset of $\Omega \setminus p$ will be meaningful either. In short, having too many impossible worlds in $\Omega$ renders useless useful set-theoretic definition of 'concatenation'.

Of course, we don't *have* to define 'concatenations' set-theoretically. But the only other place we'll find the requisite structure is in the *logical* relations amongst the contents that the meaningful subsets of $\Omega^+$ represent. That is, we could define 'concatenations' in terms of the *disjunctions of inconsistent contents*. But doing things this way brings us right back to where we started *vis-à-vis* the generalisability issues associated with A1–A5 and B2.

As I argue in (Elliott, forthcoming), any algebra that's minimally rich enough to represent the contents of belief will contain only meaningful propositions just when the relevant space of worlds is closed under a consequence relation that is, for all intents and purposes, at least as strong as classical propositional logic. Impossible worlds aren't a magical solution to comparativism's generalisability worries. Quite the opposite.

# 6 Interpersonal Comparability

Over and above the need to accommodate *intrapersonal* comparisons, comparativists should also be able to explain *interpersonal* comparisons of strength of belief. (See Meacham and Weisberg 2011, pp. 659–60, for a discussion on why interpersonal comparisons are theoretically important.) In his recent defence of probabilistic comparativism, Stefánsson (2017) argues for the possibility of interpersonal comparisons of belief as follows:

> It is generally assumed that… subjective probabilities (which represent strengths of belief) are interpersonally comparable… The crucial difference between desires and beliefs in this regard is the widely held assumption that any two rational people believe equally strongly whatever they fully believe (such as a tautology), and, similarly, believe equally strongly whatever they believe least of all… (pp. 81–2)

The argument proceeds: Suppose Ann's and Bob's comparative beliefs satisfy A1–A5 and Continuity, so there are unique probability functions $\mathcal{C}r_A$ and $\mathcal{C}r_B$ that agree with $\succsim_A$ and $\succsim_B$ respectively. Moreover (from Continuity), for every non-minimal proposition $p$, $p$ is the union of some finite collection of equiprobable atomic propositions. Thus, according to GRP, for any $p \in \mathcal{B}_A$ and $q \in \mathcal{B}_B$, there's a fact of the matter as to how much less $p$ and $q$ are believed than the maximal $\Omega$. So,

> … we might compare the degree to which Ann believes [$p$] with the degree to which Bob believes [$q$], by comparing the distance between [$p$] and the tautology according to Ann with the distance between [$q$] and the tautology according to Bob.

Moreover,

> The result of the above comparison is the same across different numerical models of Ann's and Bob's comparative beliefs [i.e., positive similarity transformations of $\mathcal{C}r_A$ and $\mathcal{C}r_B$]. That is, if Ann believes

[$p$] more strongly than Bob believes [$q$] according to one of these models, then the same holds according to all of these models. (p. 82)

The idea seems to be that because

(A) $\mathcal{C}r_A$ and $\mathcal{C}r_B$ are ratio-scale measures of Ann's and Bob's beliefs respectively, and
(B) they both sit on the $[0, 1]$ scale, with $\varnothing$ at 0 and $\Omega$ at 1,

we're licensed in saying that $\mathcal{C}r_A$ and $\mathcal{C}r_B$ belong to the 'same model' of partial belief. And because of this, we're licensed in making comparisons between them. If we applied some positive similarity transformation to, say, $\mathcal{C}r_A$ but not $\mathcal{C}r_B$, the result would be an adequate ratio-scale measure of Ann's beliefs, but it would ruin the interpersonal comparisons. To ensure interpersonal comparability, we just need to ensure that $\succsim_A$ and $\succsim_B$ belong to the 'same model.'[11]

Much hangs on the assumption that the minima and maxima of $\succsim_A$ and $\succsim_B$ are comparable across rational agents. It's not clear to me how widely held this really is outside of orthodox Bayesian circles, and it's much less clear how we could generalise the explanation to accommodate non-ideal agents. But set those points aside, and assume that for every agent $S$,

1. $\succsim_S$ satisfies A1–A5 and Continuity
2. $\Omega$ is believed to the fullest extent that $S$ can believe anything, and $\varnothing$ to the least extent

The real worry here is that we've been offered no real explanation of interpersonal comparability, even given these assumptions. Facts (A) and (B) above give us no reason to think that $\mathcal{C}r_A$ and $\mathcal{C}r_B$ measure comparable quantities.

An analogy will help to make this clear. Imagine a universe $\Delta$ that's finite in extent, and consists fundamentally of spherical atoms each with some non-zero diameter and non-zero mass, with no occupiable spaces between them. The non-atomic objects of this universe are the mereological sums of contiguous atoms. Let $\mathfrak{O}$ be set of all such objects in $\Delta$. Included in $\mathfrak{O}$ will be two special objects: $\varnothing$, the 'empty' arrangement of atoms; and $\Delta$ itself. Assume that length is always measured along some privileged axis such that every object has a unique length, and let $\succsim^l$ and $\succsim^m$ denote the *is at least as long as* and *is at least as massive as* relations respectively.

Obviously, $\succsim^l$ and $\succsim^m$ will be correlated in many respects, and they'll share a number of their properties. In fact, given the intuitively additive properties of $\succsim^l$ with respect to concatenations (§2) and precisely analogous properties for $\succsim^m$, it's possible to construct a pair of ratio-scale measures $f_l$ and $f_m$ of $\succsim^l$ and $\succsim^m$ respectively, such that for all $\alpha, \beta \in \mathfrak{O}$,

| | | | |
|---|---|---|---|
| (i$^l$) | $\alpha \succsim^l \beta$ iff $f_l(\alpha) \geq f_l(\beta)$ | (i$^m$) | $\alpha \succsim^m \beta$ iff $f_m(\alpha) \geq f_m(\beta)$ |
| (ii$^l$) | $f_l(\varnothing) = 0$ and $f_l(\Delta) = 1$ | (ii$^m$) | $f_m(\varnothing) = 0$ and $f_m(\Delta) = 1$ |
| (iii$^l$) | If $\alpha, \beta$ share no parts, then $f_l(\alpha \oplus o_2) = f_l(\alpha) + f_l(\beta)$ | (iii$^m$) | If $\alpha, \beta$ share no parts, then $f_m(\alpha \oplus o_2) = f_m(\alpha) + f_m(\beta)$ |

[11] In (Stefánsson 2017) this argument in terms of positive *affine* transformations. In his (2018), though, Stefánsson switches to similarity transformations, as I've done here. This won't make a difference to my discussion.

Indeed, $f_l$ will be unique, in the sense that no other function will satisfy (i$^l$) through (iii$^l$); and likewise for $f_m$, *mutatis mutandis*. It should go without saying that none of this implies that lengths and masses are comparable, in the sense that if $f_l(\alpha) \geq f_m(\beta)$, then $\alpha$ has as at least as much length as $\beta$ has mass.

Suppose we now stipulate that $\Delta$ has as much length as it has mass. Maybe then we could derive additional length-mass comparisons for arbitrary pairs of objects; e.g., if $f_l(\alpha) = 0.5$, then $\alpha$ will have less length than $\Delta$ has mass, because $\alpha$ has less length than $\Delta$ and $\Delta$ has just as much length as it has mass, so $\alpha$ must have less length than $\Delta$ has mass. And furthermore, so long as we stick to making sure length and mass are being measured on 'the same model,' all such length-mass comparisons will remain unchanged. That is, if $f_l(\varnothing) = f_m(\varnothing)$, $f_l(\Delta) = f_m(\Delta)$, and $f_l(\alpha) \geq f_m(\beta)$, then those relations will be preserved under any pair of functions $f_l^\star$ and $f_m^\star$ that we define by applying the same positive similarity transformation to $f_l$ and $f_m$ respectively.

So we've learnt that if you stipulate that length and mass have comparable maxima, then you'll be able to construct a privileged set of pairs of functions according to which lengths and masses can be compared more generally. But this is entirely uninteresting: the stipulation is false, and the resulting comparisons look deeply artificial. The fact that circumstances might conspire such that we can measure the two loosely correlated quantities with functions that share a lot of their formal properties does nothing to change the fact that length-mass comparisons are meaningless. If they *were* meaningful, then there would be some interesting *scale-independent* physical relation that holds between $\alpha$ and $\beta$ whenever $\alpha$ has at least as much length as $\beta$ has mass. But since nothing interesting in physics changes when we, say, hold the scale for length fixed while varying the scale for mass, there's no genuine basis for making such comparisons.

Likewise, to explain interpersonal comparisons of partial belief, we need an interesting scale-independent relation that holds between Ann and Bob when Ann believes $p$ at least as much as Bob believes $q$. We won't find any such relation merely by looking at $\succsim_A$ and $\succsim_B$—as we've seen, Ann and Bob could have identical comparative beliefs yet distinct partial beliefs. Rather, we'll likely find the justification for interpersonal comparisons in the broader role that partial beliefs play in the psychologies of the agents that have them. Once again, we need to take preferences into account.

# 7 Non-Comparativism

It's clear that we need a qualitative account of 'where the numbers come from'. There are no numbers in the head. But there are multiple approaches to explaining qualitatively how we can come to have numerically quantifiable partial beliefs that don't take us through comparative beliefs first. The so-called *anti-realist* or *interpretivist* approaches—according to which partial beliefs are taken to be mere tools for representing rational preferences—tend to receive the most attention in philosophy. I suspect that much of the sympathy for comparativism derives from a distaste for this kind of anti-realism, but there's no reason that non-comparativists should be committed to anything of the sort.

One alternative approach takes *comparative expectations* over random variables as basic, from which partial and comparative beliefs are derived simultaneously (see, e.g., Suppes and Zanotti 1976, Clark 2000). It's well known that these approaches allow for distinctions between probability functions that comparativism cannot make, though as yet there's been no serious work done on the empirical plausibility of the view. Another possibility would be a 'map-like' approach to beliefs, under which neither comparative beliefs nor (individual) partial beliefs are more fundamental, instead both being grounded in one or more holistic representations of uncertainty (cf. Lewis 1982).[12] Such an approach could be given realist credentials; for instance, according to *probabilistic population coding* models, uncertainty can be represented by the activity of populations of neurons that encode parameters of probability density distributions over small sets of random variables (e.g., means and covariances). No specific subset of the neuronal population represents an individual belief state; rather, probability functions are represented by the population as a whole. (See Ma et al. 2006, Pouget et al. 2013, for reviews.)

Once we start looking for qualitative structures that could ground our partial beliefs, we see that there's no shortage of possibilities to consider. I won't try to review them all. Instead, in the remainder of this section, I want to give a rough sketch of a functionalist non-comparativism that I find attractive—one that takes their role in the production of preferences seriously, and (arguably) fits more naturally with how partial beliefs are ordinarily conceptualised.

A metaphor may be helpful to begin with. Imagine that inside Sally's head, perhaps in a little compartment labelled 'partial beliefs', is a collection of barrels. While every barrel is the same size, each has a different marking. Perhaps the markings are in a language we don't recognise, or perhaps they aren't language-like at all—what matters is just that the markings can be used to identify each barrel individually. Furthermore, inside each barrel is a certain amount of *confidence fluid*. After poking around inside Sally's head for a bit, we find that the amount of fluid within the barrel labelled 'ʖ' correlates nicely with Sally having preferences over actions and dispositions to choose across varying decision situations that we'd typically expect if she believed $p$ to different degrees. (Or she does so in normal circumstances: when not intoxicated, stressed, etc.)

In more detail, we find, for example, that when the barrel marked 'ʖ' is 100% full, Sally generally has preferences we'd associate with being certain that $p$. But when the barrel is only partially full, matters are a little more tricky: Sally's preferences in such circumstances often depend heavily on the state of the other barrels in her 'partial belief' compartment, as well as some other barrels in a separate compartment labelled 'utilities'. Indeed, sometimes there can be quite different 'total barrel states' that give rise to the exactly same preferences. As such, there's no simple one-one relationship between Sally's preferences and her 'total barrel state'. This complicates matters, but we have enough know-how to also investigate what happens when we hold the amount of fluid in one barrel fixed while varying the others. After extensive investigation, we find that:

1. Under all (or almost all) perturbations in the states of the other barrels, whenever the barrel marked 'ʖ' is $x$% full, Sally's preferences are consistent with expectations were Sally to believe that $p$ to degree $^x/_{100}$; and

[12] So, for example, $S$ *believes* $p$ *to degree* $x$ iff $S$ is in one of a disjunction of map-like states represented by a distribution $\mathcal{C}r$ such that $\mathcal{C}r(p) = x$; and $S$ *believes* $p$ *more than* $q$ iff she's in a map-like state represented by a $\mathcal{C}r$ such that $\mathcal{C}r(p) > \mathcal{C}r(q)$.

2. For any other proposition $q$ and degree $y$, under most perturbations where the barrel marked 'ʖ' is $x$% full, the resulting preferences are inconsistent with expectations were Sally to believe $q$ to degree $y$

Like any good functionalist, we conclude that Sally believes $p$ to degree $^{x}/_{100}$ when the barrel marked 'ʖ' is $x$% full. We apply a similar strategy to work out what mental types the other barrels correspond to. This gives us access to Sally's partial beliefs, from which we derive her comparative beliefs.

On this picture, the strength of belief is a ratio-scale measure of relative volume, and (crucially) attaches to each partial belief individually. Sally's entire set of partial beliefs can still be represented by a single function $\mathcal{C}r : \boldsymbol{\mathcal{B}} \mapsto [0, 1]$, but that function is not *itself* a measure of anything—it merely summarises the individual measurements of every $p \in \boldsymbol{\mathcal{B}}$. Still, $\mathcal{C}r$ captures some interesting cardinal relationships *between* partial beliefs. Because the barrels are the same size, if $\mathcal{C}r(p) = 2 \cdot \mathcal{C}r(q)$, then we know there's twice as much confidence fluid in the barrel associated with Sally's $p$-beliefs than the barrel associated with her $q$-beliefs. Because Sally is mostly rational, if the $(p \cap q)$-barrel is empty, then the amount of fluid in the $(p \cup q)$-barrel is usually more-or-less equal to the sum of the volumes of the fluid in the $p$- and $q$-barrels. But this isn't necessarily true, and it's no part of the *explanation* for why her partial beliefs are ratio-scale measurable.

There are three key elements to the metaphor:

(i) Partial belief state-types are in principle identifiable independently of their causal roles (and likewise for utilities),
(ii) The actual and potential causal roles of these state-types in normal circumstances are sufficiently rich to allow them to be distinguished, and
(iii) Given these facts, we can understand the cardinality and comparability of partial beliefs by reference to their role in the production of preferences

There's much to be said about (i) and (ii), and justifying either is no easy matter. If necessary, we can make (ii) more plausible by expanding the causal role to include not only 'forward-looking' roles (i.e., in the production of preferences), but also 'backward-looking' roles (i.e., responses to evidence). But spot me those two claims for the sake of argument. The important issue for present purposes is how to cash out (iii). Roughly, the question is: what is it for 'barrel' to be $x$% filled with confidence fluid, and why think that any one barrel's being $x$% full should be comparable to any other barrel's being $x$% full?

Once again, Ramsey—albeit in a different mood than we last encountered him—serves as inspiration:

> [...] the degree of a belief is a causal property of it, which we can express vaguely as the extent to which we are prepared to act on it. (1931, p. 169)

Or to put it another way: the strength of a belief that $p$ is a measure of its influence on preferences over gambles conditional on $p$ under normal conditions. This is easiest to see when considering simple *binary gambles* of the kind that Ramsey focused on, though of course we could easily extend the point to a richer space of gambles represented in the style of, e.g., Savage (1954).

Suppose we've got a fix on Sally's utilities, which we represent using a utility function $\mathcal{U}$ defined over a rich space of *outcomes*, where $\mathcal{U}$ sits on (at least) an interval scale. Given this, I'll make two assumptions about ordinary agents. First, if $\mathcal{C}r$ represents Sally's partial beliefs, then $\mathcal{C}r(p) = 1 - \mathcal{C}r(\overline{p})$, and that when Sally is faced with a pair of gambles,

$$\langle o_1 \textit{ if } p_1,\ o_2 \textit{ otherwise}\rangle, \quad \langle o_3 \textit{ if } p_2,\ o_4 \textit{ otherwise}\rangle,$$

then she weakly prefers the former to the latter iff

$$\begin{gathered}\mathcal{C}r(p_1)\cdot\mathcal{U}(o_1 \cap p_1) + \mathcal{C}r(\overline{p_1})\cdot\mathcal{U}(o_2 \cap \overline{p_1}) \geq \\ \mathcal{C}r(p_2)\cdot\mathcal{U}(o_3 \cap p_2) + \mathcal{C}r(\overline{p_2})\cdot\mathcal{U}(o_4 \cap \overline{p_2})\end{gathered}$$

So suppose that Sally is given a choice between

$$\begin{aligned} g_1 &= \langle o_1 \textit{ if } p,\ o_1 \textit{ otherwise}\rangle, & g_2 &= \langle o_2 \textit{ if } p,\ o_2 \textit{ otherwise}\rangle, \\ g_3 &= \langle o_3 \textit{ if } p,\ o_3 \textit{ otherwise}\rangle, & g_4 &= \langle o_1 \textit{ if } p,\ o_2 \textit{ otherwise}\rangle, \end{aligned}$$

where, for simplicity, we assume that $\mathcal{U}(o_1) = \mathcal{U}(o_1 \cap p) = \mathcal{U}(o_1 \cap \overline{p})$, and similarly for $o_2$, $o_3$. In other words, Sally is given a choice between a gamble conditional on $p$ for either $o_1$ or $o_2$, and three 'trivial' gambles worth exactly as much as $o_2$, $o_2$, and $o_3$ respectively. She'd prefer $g_1$ to $g_4$, and $g_4$ to $g_2$. We'd expect exactly this if Sally preferred $o_1$ to $o_2$, and were somewhat uncertain as to whether $p$.

But by *how much* does she prefer $g_4$ to $g_2$? Well, she's indifferent between $g_3$ and $g_4$, so

$$\mathcal{C}r(p)\cdot\mathcal{U}(o_1) + (1 - \mathcal{C}r(p))\cdot\mathcal{U}(o_2) \ = \mathcal{U}(o_3)$$

Given this, we can quantify the 'effect' that her uncertainty regarding $p$ has had on her preferences regarding $g_4$ by considering where $o_3$ sits in the utility scale between $o_1$ and $o_2$. More perspicuously, and rearranging the above, we get the standard *betting ratios* equation:

$$\mathcal{C}r(p) \ = \ \frac{\mathcal{U}(o_3) - \mathcal{U}(o_2)}{\mathcal{U}(o_1) - \mathcal{U}(o_2)}$$

So, if Sally believes $p$ to degree, say, 0.75, then the utility she associates with the gamble $g_3$ will sit 75% of the way between the utilities of $o_1$ and $o_2$.[13]

There are three important points to note about the foregoing. First, the account doesn't commit us to saying that Sally's partial beliefs *supervene* on her preferences—nor, worse, that they are *nothing over and above* her preferences. Sally could, for example, be indifferent between all possible outcomes (cf. Eriksson and Hájek 2007's 'Zen monk' example), in which case she'd have the same preferences regardless of what her partial beliefs looked like. But that merely indicates that the role of beliefs in generating *actual* preferences are sometimes not enough to distinguish between them; it's no reason to think that their causal potential in counterfactual situations be likewise insufficient. Similarly, there may be agents whose partial beliefs are so far from being probabilistically coherent that their preferences over gambles don't fit the nice patterns we usually expect. For instance, in Descartes' case (Example 4), $\mathcal{C}r^{\dagger}(p) \not\approx 1 - \mathcal{C}r^{\dagger}(\overline{p})$, so his belief that $p$ won't have its usual effects on preferences as represented in the betting ratios equation. But the strength of a belief is only supposed to be indicative

[13] As demonstrated in Elliott (2017a,b), nothing in the above requires that $\mathcal{C}r$ be a probability function—in fact, it can be highly incoherent. We did assume that $\mathcal{C}r(p) = 1 - \mathcal{C}r(\neg p)$ (under normal conditions), which is a far cry from full probabilistic coherence.

of its *normal* causal role—that causal role does not have to be manifest in all cases (cf. Lewis 1980). And it's perfectly reasonable to think that, in most cases, $\mathcal{C}r(p) = 1 - \mathcal{C}r(\overline{p})$, at least to a near approximation.

Second, while $\mathcal{U}$ need not be any stronger than an interval scale, the characterisation we've given of partial beliefs will represent them on a full ratio scale: ratios of differences of utilities will remain unchanged under any positive affine transformation of $\mathcal{U}$. We don't have to assume that utilities are comparable across individuals in order to explain interpersonal comparability of belief—to explain both intrapersonal and interpersonal belief comparisons, we just need to assume that (a) utilities are measurable on an interval scale, and (b) strengths of belief influence preferences over gambles in a similar sort of way within and across ordinary individuals. From that, we get that if $A$ and $B$ both believe $p$ to degree $x$, then their preferences will display similar patterns with respect to gambles conditional on $p$. Conversely, in cases like Jack and Jill's (§4.2), we know that Jack believes $a_2$ more than Jill does precisely *because* their preferences come apart with respect to matters involving $a_2$.

And finally, the account is continuous with the characterisation of other quantities found throughout the sciences. In particular, the pattern applied here, of characterising one quantity in terms of its interaction with other quantities, is familiar from the definition of various *dimensionless* quantities—e.g., refractive index, relative permeability, and Mach number. Consider Mach numbers. Contrary to common opinion, a Mach number is *not* a unit of speed: it is a ratio that represents the speed of an object travelling through a medium *relative to* the speed of sound in that medium. Or, if we let $\mathcal{S}$ be any measure of speed on at least an interval scale, let $o_{\text{stationary}}$ denote a stationary object and $o_{\text{sound}}$ an object travelling at the speed of sound in the medium, then the Mach number of an object $o_1$ can be defined as:

$$Mach(o_1) \;=\; \frac{\mathcal{S}(o_1) - \mathcal{S}(o_{\text{stationary}})}{\mathcal{S}(o_{\text{sound}}) - \mathcal{S}(o_{\text{stationary}})}$$

Interestingly, the language with which we attribute partial beliefs also fits the pattern of dimensionless quantity attributions. To avoid ambiguity, attributions of dimensional quantities like length, mass, and temperature require specification of a unit. For instance, in most contexts we need to say '$o_1$ has a length of 10 *meters* and weighs 10kg'. But because dimensionless quantities have no special units (ratios of differences are absolute) we say, e.g., 'water has a refractive index of 1.33', or 'wood has a relative permeability of 1.0'. Likewise: 'Sally believes $p$ to degree $x$'—not 'Sally believes $p$ with $x$ *credals*', as one might expect if the strength of belief were a dimensional quantity like length.

# 8 Conclusion

Of course, the kind of functionalism I've just sketched faces its own challenges, as do the other non-comparativist approaches I noted. We shouldn't invest an extreme amount of confidence in any of them. But they *do* highlight alternative avenues for a fully qualitative, non-comparativist, and realist account of 'where the numbers come from'. And they no doubt only scratch the surface. Any suggestion that we *need* comparativism to explain the measurement of belief is plainly false, and given comparativism's many limitations, we have every reason to look elsewhere for an account of the fundamental doxastic state.[14]

# 9 Appendix

## Theorem 1

*Existence, left-to-right*: (i) Assume B2. If $p$ is $^n/_m$-valued and $^{n'}/_{m'}$-valued, then $^n/_m = {}^{n'}/_{m'}$; so we're able to assign a unique $r \in [0,1]$ to every $p \in \mathfrak{N}$ and thus define a Ramsey function $\mathcal{C}r$ relative to $\succsim$ on $\mathfrak{N}$. $\mathcal{C}r$ can then obviously be extended to $\mathfrak{B}$. (ii) Suppose for $p, q \in \mathfrak{N}$, $p \succsim q$. Where $p$ is $^n/_m$-valued and $q$ is $^{n'}/_{m'}$-valued, $^n/_m \geq {}^{n'}/_{m'}$. By (i), $\mathcal{C}r(p) \geq \mathcal{C}r(q)$. Next suppose $\mathcal{C}r(p) \geq \mathcal{C}r(q)$. Since $\mathcal{C}r$ is a Ramsey function, $p$ is $^n/_m$-valued and $q$ is $^{n'}/_{m'}$-valued, for $^n/_m \geq {}^{n'}/_{m'}$. So from B2, $p \succsim q$. *Existence, right-to-left* is straightforward and omitted.

*Uniqueness*: The left-to-right is obvious. The restriction of $\mathcal{C}r$ to $\mathfrak{N}$ is the unique Ramsey function relative to the restriction of $\succsim$ to $\mathfrak{N}$; so if $\mathfrak{N} = \mathfrak{B}$ then $\mathcal{C}r$ is unique *simpliciter*. □

## Theorem 2

*Existence, left-to-right*: Assume B2, B3, $\mathfrak{B}$ is countable. We focus on the case where $\mathfrak{N} \subset \mathfrak{B}$ as B1 trivialises the proof. From B3, at least one nonempty set $\mathfrak{G}$ of functions $\mathcal{C}r : \mathfrak{B} \mapsto \mathbb{R}$ agrees with $\succsim$—see (Dubra et al. 2004, p. 556). We need that there's a nonempty $\mathfrak{G}^* \subseteq \mathfrak{G}$ s.t.

1. $\mathfrak{G}^*$ agrees with $\succsim$
2. $\forall \mathcal{C}r \in \mathfrak{G}^*$, there's a strictly increasing transformation $\mathcal{C}r'$ of $\mathcal{C}r$ s.t.
   (a) $\mathcal{C}r'$ is bounded by 1 and 0
   (b) $\mathcal{C}r'$ is a Ramsey function w.r.t. $\succsim$

The set $\mathfrak{F}$ of all such transformations will agree with $\succsim$, completing the proof. There are three cases:

1. $\mathfrak{N}$ is empty
2. $\mathfrak{N}$ contains only the minimal and/or maximal elements of $\mathfrak{B}$
3. $\mathfrak{N}$ contains non-minimal, non-maximal elements of $\mathfrak{B}$

The first two are straightforward and omitted. For the third, if $\mathfrak{G}$ agrees with $\succsim$, and $p \succ q$, then:

(i) $\mathcal{C}r(p) \geq \mathcal{C}r(q)$, for all $\mathcal{C}r \in \mathfrak{G}$
(ii) $\mathcal{C}r(p) > \mathcal{C}r(q)$, for some $\mathcal{C}r \in \mathfrak{G}$

So for any $\mathcal{C}r \in \mathfrak{G}$, if $p \succ q$ then $\mathcal{C}r(p) > \mathcal{C}r(q)$ or $\mathcal{C}r(p) = \mathcal{C}r(q)$. For $p, q \in \mathfrak{N}$, B2 implies that for any Ramsey function, if $p \succ q$, then $\mathcal{C}r(p) > \mathcal{C}r(q)$; so, it's not in general true that if $\mathfrak{G}$ agrees with $\succsim$, then for every $\mathcal{C}r \in \mathfrak{G}$ there's

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a strictly increasing transformation of $\mathcal{C}r$ that's also a Ramsey function with respect to $\succsim$. But define $\mathcal{G}^* \subseteq \mathcal{G}$ as follows:

$$\mathcal{G}^\star = \{\mathcal{C}r \in \mathcal{G} : \text{if } p, q \in \mathcal{N} \text{ and } p \succ q, \text{ then } \mathcal{C}r(p) > \mathcal{C}r(q)\}$$

$\mathcal{G}^\star$ agrees with $\succsim$, and by (ii) above, we know that it's nonempty. Let $\mathcal{G}^R$ denote the set of restrictions of every $\mathcal{C}r \in \mathcal{G}^\star$ to $\mathcal{N}$, now the unique Ramsey function $\mathcal{C}r^R$ on $\mathcal{N}$ is a strictly increasing transformation of every $\mathcal{C}r \in \mathcal{G}^R$. So we just have to show that each $\mathcal{C}r \in \mathcal{G}^\star$ has a strictly increasing transformation that's there's an extension of $\mathcal{C}r^R$ from $\mathcal{N}$ to $\mathcal{B}$. Where $\mathcal{B}$ is countable this is straightforward, given that for every $\mathcal{C}r \in \mathcal{G}^\star$, $\mathcal{C}r(p)$ is rational. $\square$

## Theorem 3

Suppose $\succsim$ violates B2 and $\mathcal{C}r$ agrees with $\succsim$. So there exist $p, q$ s.t. $p$ is $^n/_m$-valued, $q$ is $^{n'}/_{m'}$-valued, and not: $(p \succsim q) \leftrightarrow (^n/_m \geq {}^{n'}/_{m'})$. There are three cases:

1. Neither $p$ nor $q$ is minimal
2. Both $p$ and $q$ are minimal
3. Exactly one of $p$ or $q$ is minimal

Start with case 1. Focus on $p$, and let *max* designate some maximal proposition. (If $p$ is $^n/_m$-valued, non-minimal, then *max* exists.) $p$ is either:

(i) The union of $n$ members of an $m$-scale of *max*, or
(ii) The union of $n''$ members of an $m''$-scale of … the union of $n'''$ members of an $m'''$-scale of *max*

If (i), $\mathcal{C}r$ coheres with the GRP only if $\mathcal{C}r(p) = {}^n/_m \cdot \mathcal{C}r(r)$; if (ii), only if $\mathcal{C}r(p) = {}^{(n'' \cdot \ldots \cdot n''')}/_{(m'' \cdot \ldots \cdot m''')} \cdot \mathcal{C}r(r)$, where ${}^{(n' \cdot \ldots \cdot n'')}/_{(m' \cdot \ldots \cdot m'')} = {}^n/_m$. The same applies to $q$, *mutatis mutandis*, so $\mathcal{C}r$ coheres with the GRP only if $\mathcal{C}r(p) = {}^{n'}/_{m'} \cdot \mathcal{C}r(r)$. Assume for *reductio* that $\mathcal{C}r$ coheres with the GRP. Now suppose $^n/_m \geq {}^{n'}/_{m'}$, so $\mathcal{C}r(p) \geq \mathcal{C}r(q)$, and hence $p \succsim q$. In the other direction, suppose $p \succsim q$; so $\mathcal{C}r(p) \geq \mathcal{C}r(q)$, and $^n/_m \geq {}^{n'}/_{m'}$. So, $p \succsim q \leftrightarrow {}^n/_m \geq {}^{n'}/_{m'}$, which violates our assumptions.

Case 2. Assume there are $p, q \in \mathcal{B}$ such that $p \succ q$, and that if $p$ is minimal, then $\mathcal{C}r(p) = 0$. If $p, q$ are minimal then $p \sim q$, and if $\mathcal{C}r$ agrees with $\succsim$ then $\mathcal{C}r(p) = \mathcal{C}r(q) > \mathcal{C}r(s)$, for any $s$ such that $s \not\sim p$ (and hence $s \succ p$). Since $p, q$ are $^0/_m$-valued by definition, B2 is violated only if $p$ or $q$ is also $^n/_m$-valued, for $n > 0$. Suppose this of $p$; then by the earlier reasoning, $\mathcal{C}r$ coheres with the GRP only if $\mathcal{C}r(p) = {}^n/_m \cdot \mathcal{C}r(r)$. Since $^n/_m > 0$ and $\mathcal{C}r(r) > 0$, this is false; so $\mathcal{C}r$ conflicts with the GRP.

Case 3 is then straightforward, and the proof of the corollary (for sets of functions) follows the same structure. Both are omitted. $\square$

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