

Partitions and Objective Indefiniteness in Quantum Mechanics

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Abstract

Classical physics and quantum physics suggest two meta-physical types of reality: the classical notion of an objectively definite reality with properties "all the way down," and the quantum notion of an objectively indefinite type of reality. The problem of interpreting quantum mechanics (QM) is essentially the problem of making sense out of an objectively indefinite reality. These two types of reality can be respectively associated with the two mathematical concepts of subsets and quotient sets (or partitions) which are category-theoretically dual to one another and which are developed in two dual mathematical logics, the usual Boolean logic of subsets and the more recent logic of partitions. Our sense-making strategy is "follow the math" by showing how the mathematics of set partitions can be transported in a natural way to complex vector spaces where it yields the mathematical machinery of QM. And then we show how the machinery of QM can be transported the other way down to set-like vector spaces over \mathbb{Z}_2 yielding a rather fulsome "toy" or pedagogical model of "quantum mechanics over sets." In this way, we try to make sense out of objective indefiniteness and thus to interpret quantum mechanics.

Contents

1	Two types of reality	2
1.1	Objective indefiniteness	2
1.2	Mathematical description of indefiniteness = partitions	3
1.3	Mathematical description of definiteness = subsets	4
1.4	Two dual logics for the two types of reality	4
1.5	Some imagery for objective indefiniteness	5
1.6	The two lattices	8
2	Whence set partitions?	9
2.1	Set partitions from set attributes	9
2.2	Set partitions from set representations of groups	10
2.3	Set partitions from other set partitions	10
3	Partition concepts: from sets to vector spaces	11
3.1	The basis principle	11
3.2	What is a vector space partition?	11
3.3	What is a vector space attribute?	12

4	Whence vector-space partitions?	13
4.1	Vector-space partitions from vector-space attributes	13
4.2	Vector-space partitions from vector-space representations of groups	13
4.3	Vector-space partitions from other vector-space partitions	14
5	"Quantum Mechanics" over sets	16
5.1	Toy models of QM over finite fields	16
5.2	Vector spaces over \mathbb{Z}_2	16
5.3	The brackets	17
5.4	Ket-bra resolution	18
5.5	The norm	18
5.6	The Born Rule	18
5.7	Spectral decomposition on sets	19
5.8	Completeness and orthogonality of projection operators	19
5.9	Measurement in QM/sets	20
5.10	Summary of QM/sets	21
5.11	A glance back at full QM	21
6	Final remarks	22

1 Two types of reality

1.1 Objective indefiniteness

From the beginning of quantum mechanics, there has been the problem of interpretation, and, even today, the variety of interpretations continues to multiply [21]. Our thesis in this paper is that mathematics (including logic) can be used to attack the problem of interpretation since mathematics itself contains a very basic duality that can be associated with two meta-physical types of reality:

1. the common-sense notion of objectively definite reality assumed in classical physics, and
2. the notion of objectively indefinite reality suggested by quantum physics.

The "problem" of interpreting quantum mechanics (QM) *is* essentially the problem of making sense out of the notion of objective indefiniteness.

The approach taken here is to follow the lead of the mathematics of partitions, first for sets (where things are relatively "clear and distinct") and then for complex vector spaces where the mathematics of full QM resides.

There has long been the notion of subjective or epistemic indefiniteness ("cloud of ignorance") that is slowly cleared up with more discrimination and distinctions (as in the game of Twenty Questions). But the vision of reality that seems appropriate for quantum mechanics is *objective or ontological indefiniteness*. The notion of objective indefiniteness in QM has been most emphasized by Abner Shimony ([34], [35], [36]).

From these two basic ideas alone – indefiniteness and the superposition principle – it should be clear already that quantum mechanics conflicts sharply with common sense. If the quantum state of a system is a complete description of the system, then a quantity that has an indefinite value in that quantum state is objectively indefinite; its value is not merely unknown by the scientist who seeks to describe the system. [34, p. 47]

The fact that in any pure quantum state there are physical quantities that are not assigned sharp values will then mean that there is *objective indefiniteness* of these quantities. [36, p. 27]

The view that a description of a superposition quantum state is a *complete* description means that the indefiniteness of a superposition state is in some sense objective or ontological and not just subjective or epistemic.

In addition to Shimony's "objective indefiniteness" (the phrase used here), other philosophers of physics have suggested related ideas such as:

- Peter Mittelstaedt's "incompletely determined" quantum states with "objective indeterminateness" [31],
- Paul Busch and Greg Jaeger's "unsharp quantum reality" [4],
- Paul Feyerabend's "inherent indefiniteness" [16],
- Allen Stairs' "value indefiniteness" and "disjunctive facts" [37],
- E. J. Lowe's "vague identity" and "indeterminacy" that is "ontic" [28],
- Steven French and Decio Krause's "ontic vagueness" [18],
- Paul Teller's "relational holism" [39], and so forth.

Indeed, the idea that a quantum state is in some sense "blurred" or "like a cloud" is now rather commonplace even in the popular literature. The problem of making sense out of quantum *reality* is the problem of making sense out of the notion of *objective* indefiniteness that "conflicts sharply with common sense."

1.2 Mathematical description of indefiniteness = partitions

How can indefiniteness be depicted mathematically? The basic idea is simple; start with what is taken as full definiteness and then factor or quotient out the "surplus" definiteness using an equivalence relation or partition.

Starting with some universe set U of fully distinct and definite elements, a partition $\pi = \{B_i\}$ (i.e., a set of disjoint blocks B_i that sum to U) collects together in a block (or cell) B_i the distinct elements $u \in U$ whose distinctness is to be ignored or factored out, but the blocks are still distinct from each other. Each block represents the elements that are the same in some respect (since each block is an equivalence class in an equivalence relation on U), so the block is indefinite between the elements within it. But different blocks are still distinct from each other in that aspect.

Example 1 Consider the calculation of the binomial coefficient $\binom{N}{m} = \frac{N!}{m!(N-m)!}$. The idea is to count the number of m -ary subsets of an N -ary set ($m \leq N$) where the different orderings of the otherwise same m -ary subset are surplus that need to be factored out. The method of calculation is to first count the number of possible orderings of the whole N -ary subset which is $N! = N(N-1) \dots (2)(1)$. Then we want to quotient out the cases that are distinct only because of different orderings. For any given ordering of the N elements, there are $m!$ ways to permute the first m elements in the given ordering—leaving the last $N-m$ elements the same. Thus we take the first quotient by identifying any two of the $N!$ different orderings if they differ only in a permutation of those first m elements. Since there are $m!$ such permutations, there are now $N!/m!$ equivalence classes or blocks in the resulting partition of the $N!$ orderings. But these equivalence classes still count as distinct the different orderings of the last $N-m$ elements so we further identify blocks which just have a permutation of the last $N-m$ elements to make larger blocks. Then the result is $\binom{N}{m} = \frac{N!}{m!(N-m)!}$ blocks in the partition which is the number of m -element subsets (which equals the number of $N-m$ -element subsets) out of an N -ary set disregarding the ordering of the elements.

In this example, the set of fully determinate alternatives are the $N!$ orderings of the N -element set. Then to consider the subsets of determinate or definite cardinality m (and thus the complementary subsets of definite cardinality $N - m$), we must quotient out the number of possible orderings $m!$ and $(N - m)!$ to render the ordering of the elements in the subsets indefinite or indeterminate.

Example 2 *To be concrete, consider a set $\{a, b, c, d\}$ of $N = 4$ elements so the universe U for fully distinct orderings has $4! = 24$ elements $\{abcd, abdc, \dots\}$. How many 2-element subsets are there? The first quotient groups together or identifies the orderings which only permute the first $m = 2$ elements so two of the blocks in that partition are $\{abcd, bacd\}$ and $\{abdc, badc\}$, and there are $N!/m! = 24/2 = 12$ such blocks. Each block has the same final $N - m = 2$ elements in the ordering so we further identify the blocks that differ only in a permutation of those last $N - m$ elements. One of the blocks in that final partition is $\{abcd, bacd, abdc, badc\}$ and there are $\frac{N!}{m!(N-m)!} = \frac{24}{(2)(2)} = 6$ such blocks with four elements in each block. Each block is distinct from the other blocks in the first m elements and in the last $N - m$ elements of the orderings in the block so the block count is just the number of subsets of m elements (which equals the number of $N - m$ elements as well) where each block is indefinite as to the ordering of elements within the first m elements and within the last $N - m$ elements.*

A similar example within QM is the treatment of the indefiniteness due to the indistinguishability of quantum particles of the same type. The idea is to artificially treat them as distinct and then collect together or superpose the permutations of the particles that factors out their supposed distinctness (see any QM text such as [7]).

But our point in this section is the general mathematical theme that indefiniteness is described by taking a partition or quotient of the set of definite entities. A partition is a mixture of indefiniteness and definiteness. Each block is indefinite between the elements within it, but the blocks of the partition are distinct from one another.

1.3 Mathematical description of definiteness = subsets

The common-sense classical view of reality is that it is completely definite or determined and fully propertied "all the way down." Every entity or thing definitely has a property P or definitely has the property $\neg P$. Peter Mittelstaedt quotes Immanuel Kant's treatment of the idea of complete determinateness:

Every thing as regards its possibility is likewise subject to the principle of complete determination according to which if all possible predicates are taken together with the contradictory opposites, then one of each pair of contradictory opposites must belong to it. [Kant quoted in: [31, p. 170]]

Given a universe set U , a predicate P is represented by the subset $S \subseteq U$ of elements that have the property, and the complement subset $S^c = U - S$ represents the elements that have the property $\neg P$.

1.4 Two dual logics for the two types of reality

The two mathematical concepts of subsets and partitions are thus associated with two metaphysical types of reality:

1. the common-sense notion of objectively definite reality assumed in classical physics, and
2. the notion of objectively indefinite reality suggested by quantum mechanics.

Subsets and quotient sets (or partitions) are mathematically *dual* concepts in the reverse-the-arrows sense of category-theoretic duality, e.g., a subset is the direct image of a set monomorphism (or injection) while a set partition is the inverse image of an epimorphism (or surjection). This duality is familiar in abstract algebra in the interplay of subobjects (e.g., subgroups, subrings, etc.) and quotient objects. William Lawvere calls the general category-theoretic notion of a subobject a *part*, and then he notes: "The dual notion (obtained by reversing the arrows) of 'part' is the notion of partition." [27, p. 85]

The logic appropriate for the usual notion of fully definite reality described by subsets is the ordinary Boolean logic of subsets [2] (usually mis-specified as the special case of "propositional" logic). We have seen that the other vision of objectively indefinite reality suggested by QM is mathematically described by quotients set, partitions, or equivalence relations. The Boolean logic of subsets has an equally fundamental dual logic of quotient sets, equivalence relations, or partitions ([10] and [14]). The dual logics are associated with the two visions of reality.

Since our topic is to better understand objective indefiniteness, and thus to interpret QM, we will be developing partitional concepts. There is a natural bridge between set concepts and vector-space concepts. We will transport partitional concepts across that bridge in both directions. We will see that the mathematics of set partitions can be lifted or transported to complex (inner product) vector spaces where it yields essentially the mathematical machinery of QM (of course, not the specifically physical postulates such as the Hamiltonian or the DeBroglie relationships). The vector space concepts of full QM can be transported back to set-like vector spaces over \mathbb{Z}_2 to yield a "toy" or pedagogical model of "quantum mechanics over sets" or QM/sets [13]. The traffic in both directions supports the idea of interpreting QM in terms of objective indefiniteness as illuminated by the logic and mathematics of partitions [12].

1.5 Some imagery for objective indefiniteness

In subset logic, each element of the universe set U either definitely has or does not have a given property P (represented as a subset of the universe). Moreover an element has properties "all the way down" so that two numerically distinct entities must differ by some property as in Leibniz's principle of the identity of indiscernibles. [26] Change takes place by the definite properties changing. For a hound to go from point A to point B , there must be some trajectory of definite ground locations from A to B .

In the logic of partitions, a partition $\pi = \{B_i\}$ is made up of disjoint blocks B_i whose union is the universe set U (the blocks are also thought of as the equivalence classes in an equivalence relation). The blocks in a partition have been distinguished from each other, but the elements within each block have not been distinguished from each other by that partition. Hence each block can be viewed as the set-theoretic version of a superposition of the distinct elements in the block. When more distinctions are made (the set-version of a measurement), the blocks get smaller and the partitions (set-version of mixed states) become more refined until the discrete partition $\mathbf{1} = \{\{u\} : \{u\} \subseteq U\}$ is reached where each block is a singleton (the set-version of a non-degenerate measurement). Change takes place by some attributes becoming more definite and other (incompatible) attributes becoming less definite. For a hawk to go from point A to point B , it would go from a definite perch at A into a flight of indefinite ground locations, and then would have a definite perch again at B .¹

¹The "flights and perchings" metaphor is from William James [24, p. 158] and according to Max Jammer, that description "was one of the major factors which influenced, wittingly or unwittingly, Bohr's formation of new conceptions in physics." [25, p. 178] The hawks and hounds pairing comes from Shakespeare's Sonnet 91.

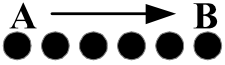
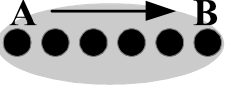
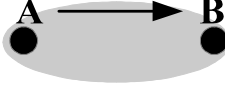
<p>Classical trajectory from A to B. How a hound goes from A to B.</p>	
<p>Subjective indefiniteness about classical position ("cloud of ignorance").</p>	
<p>Objective indefiniteness of quantum trajectory: definite position at A, indefinite position in transition, and definite position at B. How a hawk goes from A to B.</p>	

Figure 1: How a hound and a hawk go from A to B

The imagery of having a sharp focus versus being out of focus could also be used if one is clear that it is the reality itself that is in-focus or out-of-focus, not just the image through, say, a microscope. A classical trajectory is like a moving picture of sharp or definite in-focus realities, whereas the quantum trajectory starts with a sharply focused reality, goes out of focus, and then returns to an in-focus reality (by a "measurement").

The idea of a quantum superposition as being a blurred or indefinite state has been missing the "back story" to make sense out that conception of reality. That back story is provided, in part, by the logic of partitions, equally fundamental from the mathematical viewpoint as Boolean subset logic, and by the logical information theory built on top of partition logic ([9] and [11]).

In the objective indefiniteness interpretation, a subset $S \subseteq U$ of a universe set U should be thought of as a single indefinite element S that is only *represented* as a subset of fully definite elements $\{u : u \in S\}$ —just as a single superposition vector is represented in a certain basis of eigen (= definite) vectors. Abner Shimony ([34] and [35]), in his description of a superposition state as being objectively indefinite, sometimes used Heisenberg's [22] language of "potentiality" and "actuality" to describe the relationship of the eigenvectors that are superposed to give an objectively indefinite superposition. This terminology could be adapted to the case of the sets. The singletons $\{u\} \subseteq S$ are "potential" in the objectively indefinite "superposition" S , and, with further distinctions, the indefinite element S might "actualize" to $\{u\}$ for one of the "potential" $\{u\} \subseteq S$. Starting with S , the other $\{u\} \not\subseteq S$ (i.e., $u \notin S$) are not "potentialities" that could be "actualized" with further distinctions.

This terminology is, however, somewhat misleading since the indefinite element S is perfectly "actual" (in the objectively indefinite interpretation); it is only the multiple eigen-elements $\{u\} \subseteq S$ that are "potential" until "actualized" by some further distinctions. In a "non-degenerate measurement," a single actual indefinite element becomes a single actual definite element. Since a distinction-creating "measurement" goes from actual indefinite to actual definite, the potential-to-actual language of Heisenberg should only be used with proper care—if at all.

Note that there are two conceptually distinct connotations for the mathematical subset $S \subseteq U$. In the "classical" interpretation, it is a set of fully definite elements of $u \in S$. In the "quantum" interpretation of a subset S , it is a single indefinite element that with further distinctions could become one of the eigen-elements $\{u\} \subseteq S$.

Consider a three-element universe $U = \{a, b, c\}$ and a partition $\pi = \{\{a\}, \{b, c\}\}$. The block $S = \{b, c\}$ is objectively indefinite between $\{b\}$ and $\{c\}$ so those singletons are its "potentialities" in the sense that a distinction could result in either $\{b\}$ or $\{c\}$ being "actualized." However $\{a\}$ is not

a "potentiality" when one is starting with the indefinite element $\{b, c\}$.

Note that this objective indefiniteness of $\{b, c\}$ is *not* well-described as saying that indefinite pre-distinction element is "simultaneously both b and c " (like the common misdescription of the undetected particle "going through both slits" in the double-slit experiment); instead it is indefinite between b and c . That is, a "superposition" of two sharp eigen-alternatives should *not* be thought of like a double-exposure photograph which has two fully definite images (e.g., simultaneously a picture of say b and c). Instead of a double-exposure photograph, the superposition should be thought of as representing a blurred or incomplete reality that with further distinctions could sharpen to either of the sharp realities. But there must be some way to indicate which sharp realities could be obtained by making further distinctions ("measurements"), and that is why the blurred or cloud-like indefinite reality is represented by mathematically superposing the sharp "potentialities."

This point might be illustrated using some Guy Fawkes masks.





Eigenstate 1: Guy Fawkes with goatee	
Eigenstate 2: Guy Fawkes with mustache	
Objectively indistinct state before (facial hair) distinctions were made is the pre-distinction state.	
But that objectively indistinct state may be represented by superposition of possible distinct alternatives, the set $\{\text{goatee, mustache}\}$ or vector $ \text{goatee}\rangle + \text{mustache}\rangle$	

Figure 2: Objectively indefinite pure state *represented* as superposition of distinct eigen-alternatives

Instead of a double-exposure photograph, a superposition representation might be thought of as "a photograph of clouds or patches of fog." (Schrödinger quoted in: [17, p. 66]) Schrödinger distinguishes a "photograph of clouds" from a blurry photograph presumably because the latter might imply that it was only the photograph that was blurry while the underlying objective reality was sharp. The "photograph of clouds" imagery for a superposition connotes a clear and complete photograph of an objectively "cloudy" or indefinite reality. Regardless of the (imperfect) imagery, one needs some way to indicate what are the definite eigen-elements that could be "actualized" from a single indefinite element S , and *that* is the role in the set case of conceptualizing a subset S as a collecting together or "superposing" certain "potential" eigen-states $\{u\} \subseteq S$.

1.6 The two lattices

The two dual subset and partition logics are modeled by the two lattices (or, with more operations, algebras) of subsets and of partitions. The conceptual duality between the lattice of subsets (the lattice part of the Boolean algebra of subsets of U) and the lattice of partitions could be described (again following Heisenberg) using the rather meta-physical notions of *substance*² and *form* (as in in-form-ation)—which might be compared to the terms "matter" or "objects" and "structure" respectively in some modern metaphysical discussions.³

For each lattice where $U = \{a, b, c\}$, start at the bottom and move towards the top.

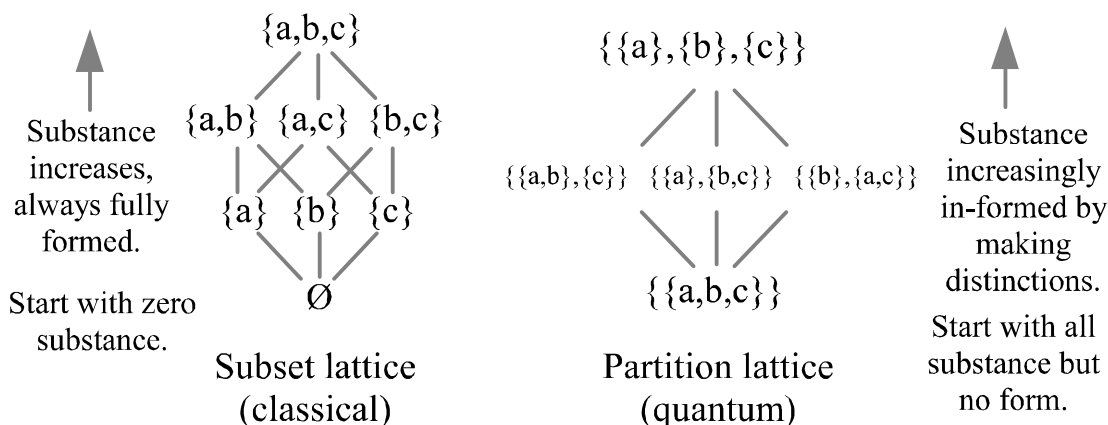


Figure 3: Conceptual duality between the subset and partition logics

At the bottom of the Boolean lattice is the empty set \emptyset which represents no substance. As one moves up the lattice, new elements of substance always with fully definite properties are created until finally one reaches the top, the universe U . Thus new substance is created in moving up the lattice but each element is fully formed and distinguished in terms of its properties.

At the bottom of the partition lattice is the indiscrete partition or "blob" $\mathbf{0} = \{U\}$ (where the universe set U makes one block) which represents all the substance but with no distinctions to in-form the substance.⁴ As one moves up the lattice, no new substance is created but distinctions objectively in-form the indistinct elements as they become more and more distinct, until one finally reaches the top, the discrete partition $\mathbf{1}$, where all the eigen-elements of U have been fully distinguished from each other.⁵ It was previously noted that a partition combines indefiniteness (within blocks) and definiteness (between blocks). At the top of the partition lattice, the discrete partition $\mathbf{1} = \{\{u\} : \{u\} \subseteq U\}$ is the result making all the distinctions to eliminate the indefiniteness. Thus one ends up at the "same" place (macro-universe of distinguished elements) either way, but by two

²Heisenberg identifies "substance" with energy.

Energy is in fact the substance from which all elementary particles, all atoms and therefore all things are made, and energy is that which moves. Energy is a substance, since its total amount does not change, and the elementary particles can actually be made from this substance as is seen in many experiments on the creation of elementary particles. [22, p. 63]

³See McKenzie [30] and the references therein to ontic structural realism.

⁴The "blob" might be thought of as the set-version of a pure state in QM prior to a distinctions-creating measurement that creates non-blob partition analogous to a mixed state (see [13] for spelling this out using density matrices).

⁵This notion of logical in-formation as distinctions is based on partition logic just as logical probability is based on subset logic ([9] and [11]). That is, the *logical entropy* of a partition is the normalized counting measure of the distinctions of a partition (represented as a binary relation) just as the Laplace-Boole *logical probability* of a subset is the normalized counting measure on the subsets (events) of the finite universe set (set of equiprobable outcomes).

totally different but dual ways.⁶

The progress from bottom to top of the two lattices could also be described as two creation stories.

- *Subset creation story*: “In the Beginning was the Void”, and then elements are created, fully propertied and distinguished from one another, until finally reaching all the elements of the universe set U .
- *Partition creation story*: “In the Beginning was the Blob”, which is an undifferentiated “substance,” and then there is a "Big Bang" where elements (“its”) are created by the substance being objectively in-formed (objectified information) by the making of distinctions (e.g., breaking symmetries) until the result is finally the singletons which designate the elements of the universe U .

These two creation stories might also be illustrated as follows.

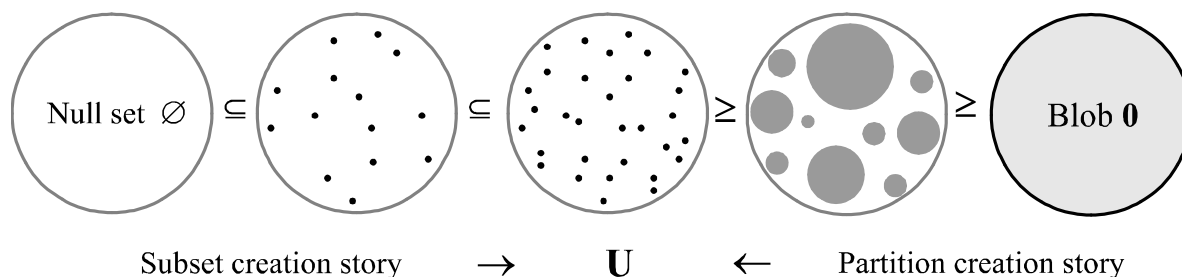


Figure 4: Two creation stories

One might think of the universe U (in the middle of the above picture) as the macroscopic world of definite entities that we ordinarily experience. Common sense and classical physics assumes, as it were, the subset creation story on the left. But *a priori*, it could just as well have been the dual story, the partition creation story pictured on the right, that leads to the *same* macro-picture U .

Since partitions are the mathematical expression of indefiniteness, our strategy is to first show where set partitions come from and then to "lift" or "transport" the partitional machinery to vector spaces. The result is essentially the mathematical machinery of quantum mechanics—all of which shows how quantum mechanics can be interpreted using the objective indefiniteness conception of reality that is associated at the logical level with partition logic.

2 Whence set partitions?

2.1 Set partitions from set attributes

Take the universe set as some specific set of people, say in a room. People have numerical attributes like weight, height, or age as well as non-numerical attributes with other values such place of birth, family name, and country of citizenship. Abstractly an *attribute on a universe set U* is a function $f : U \rightarrow R$ from U to some set of values R (usually the reals \mathbb{R}). In subset logic, an element $u \in U$ either has a property represented by a subset $S \subseteq U$ or not; in partition logic, an attribute f assigns a value $f(u)$ to each $\{u\} \subseteq U$. The two concepts overlap for binary attributes where the attribute might be represented by the characteristic function $\chi_S : U \rightarrow 2$ of a subset $S \subseteq U$.

Each attribute $f : U \rightarrow R$ on a universe U determines the *inverse-image partition* $f^{-1} = \{f^{-1}(r) \neq \emptyset : r \in R\}$. Attributes are one way to define a partition on a set U .

⁶In treating the universe $U = \{u, u', \dots\}$ and the discrete partition $\mathbf{1} = \{\{u\}, \{u'\}, \dots\}$ as the "same" we are neglecting the distinction between u and $\{u\}$ for $u \in U$.

2.2 Set partitions from set representations of groups

Another way to define a partition on U is to map the elements $u \in U$ to "similar" (i.e., same block) elements u' by some set of transformations $G = \{t : U \rightarrow U\}$. This defines a binary relation: uGu' if there exists a $t \in G$ such that $t(u) = u'$. In order to define a partition, the binary relation uGu' has to be an equivalence relation so the blocks of the partition are the equivalence classes. The three requirements for an equivalence relation are reflexivity, symmetry, and transitivity.

- For the relation to be reflexive, i.e., uGu for all $u \in U$, it is sufficient for the set of transformations G to contain the identity transformation $1_U : U \rightarrow U$.
- For the relation to be symmetric, i.e., uGu' implies $u'Gu$, it is sufficient for each $t \in G$ to have an inverse $t^{-1} \in G$ where $U \xrightarrow{t} U \xrightarrow{t^{-1}} U = 1_U = U \xrightarrow{t^{-1}} U \xrightarrow{t} U$.
- For the relation to be transitive, i.e., uGu' and $u'Gu''$ imply uGu'' , it is sufficient for each $t, t' \in G$ that $t't : U \xrightarrow{t} U \xrightarrow{t'} U$ is also in G .

These three conditions, the existence of the identity, the existence of an inverse, and closure under composition, define a *transformation group* $G = \{t : U \rightarrow U\}$, i.e., a *group action* on a set U . Equivalently, a *set representation* of a group G is given by a group homomorphism $T : G \rightarrow S(U)$, where $S(U)$ is the symmetric group of permutations t of the set U (and where the transformation group $\{t : U \rightarrow U\} \subseteq S(U)$ is the image of the map). An *abstract group* satisfies these three conditions where the composition is also required to be associative in the sense that for any $t, t', t'' \in G$, $(t''t')t = t''(t't)$. For a transformation group, the composition is automatically associative.

This connection between groups and equivalence relations or partitions has long been known, e.g., [6]. Instead of elements $u, u' \in U$ being collected in the same block by have the same attribute value $f(u) = f(u')$, the group transformations take any element u to a "similar" or "symmetric" element $t(u) = u'$. A subset $S \subseteq U$ is *invariant under G* if for any $t \in G$, $t(S) \subseteq S$. A minimal invariant subset is an *orbit*, and the partition defined by the transformation group G is the *set partition of orbits*.

What is the significance of the blocks in the partition of minimal invariant subsets? Often the treatment of symmetry groups focuses on what is invariant or conserved, e.g., the perspective of Noether's theorem [3].

There is another perspective with which to view the representations of symmetry groups. To represent an indefinite reality, there is first some notion of the fully definite eigen-alternatives that are then collected together or superposed to represent something indefinite between those alternatives. *What determines the set of eigen-alternatives?* One might think in more metaphysical terms about a principle of plenitude. Given a set of symmetries on a set, in how many different ways can there be distinct subsets that still satisfy the constraints of the symmetry operations? The minimal invariant subsets or orbits of a set representation of a symmetry group provide the answer to that question about the plenitude of "atomic" eigen-forms consistent with the symmetries.

This question and the answer become more significant when we move beyond structure-less sets to linear vector spaces. As the minimal invariant sub-sets, the orbits are the set-version of the minimal invariant sub-spaces, the irreducible subspaces, which are the carriers of the irreducible representations or irreps in vector space representations of groups.

2.3 Set partitions from other set partitions

In the foregoing, we have frequently referred to the making of distinctions as the set version of a measurement. What is the operation for making distinctions? It is the join operation from partition logic. But before two set partitions can be joined to form a more refined partition with more distinctions, they must be *compatible* in the sense of being defined on the same universe set. If two set

partitions $\pi = \{B\}$ and $\sigma = \{C\}$ are compatible, i.e., are partitions of the same universe U , then their *join* $\pi \vee \sigma$ is the set partition whose blocks are the non-empty intersections $B \cap C$.⁷

Since two set attributes $f : U \rightarrow \mathbb{R}$ and $g : U' \rightarrow \mathbb{R}$ define two inverse image partitions $\{f^{-1}(r)\}$ and $\{g^{-1}(s)\}$ on their domains, we need to extend the concept of compatible partitions to the attributes that define the partitions. That is, two attributes $f : U \rightarrow \mathbb{R}$ and $g : U' \rightarrow \mathbb{R}$ are *compatible* if they have the same domain $U = U'$.

Given two compatible set attributes $f : U \rightarrow \mathbb{R}$ and $g : U \rightarrow \mathbb{R}$, the join of their "eigenspace" partitions has as blocks the non-empty intersections $f^{-1}(r) \cap g^{-1}(s)$. Each block in the join of the "eigenspace" partitions could be characterized by the ordered pair of "eigenvalues" (r, s) . An "eigenvector" of f , $S \subseteq f^{-1}(r)$, and of g , $S \subseteq g^{-1}(s)$, would be a "simultaneous eigenvector": $S \subseteq f^{-1}(r) \cap g^{-1}(s)$.

A set of compatible set attributes is said to be *complete* if the join of their partitions is discrete, i.e., the blocks have cardinality 1. A *Complete Set of Compatible Attributes* or CSCA characterizes the singletons $\{u\} \subseteq U$ by the ordered n -tuple (r, \dots, s) of attribute values.

All this machinery of set partitions can be lifted or transported to vector spaces to give the mathematical machinery of QM.⁸

3 Partition concepts: from sets to vector spaces

3.1 The basis principle

There is a natural bridge or ladder connecting set concepts to vector-space concepts. The basic idea is that a vector $v = \sum_i \alpha_i b_i$ represented in terms of a set $\{b_i\}$ of basis vectors is a set but where each element b_i takes a value c_i in the base field K . Given a set concept, the *basis principle* is that one can generate the corresponding vector-space concept by applying the set concept to a basis set and seeing what it generates. Starting with the set concept of cardinality, one arrives at the corresponding vector-space concept by applying the set concept to a basis set to arrive at the cardinality of the basis set. After checking that all bases have the same cardinality, this yields the vector-space notion of *dimension*. Thus the cardinality of a set lifts not to the cardinality of a vector space but to its dimension.

Some of the lifting is accomplished by the free vector space functor from the category of sets to the category of vector spaces over a given field K . A set U is carried by this functor to the vector space K^U spanned by the Kronecker delta basis $\{\delta_u : U \rightarrow K\}_{u \in U}$ where $\delta_u(u') = 0$ for $u' \neq u$ and $\delta_u(u) = 1$. A set U of a certain cardinality thus generates a vector space K^U of the same dimension.

3.2 What is a vector space partition?

In categorical terms, a partition $\pi = \{B\}$ on a set U is a set of subsets whose direct sum (i.e., disjoint union) is the whole set, i.e., a direct sum decomposition of the set. The corresponding vector space concept is a set of subspaces of a vector space whose direct sum is the vector space, i.e., a *direct sum decomposition* of the vector space. In terms of the basis principle, we could apply the set partition $\pi = \{B\}$ of a set U to a basis set $\{b_u\}_{u \in U}$, then each block B generates a subspace V_B and the set of subspaces $\{V_B\}_{B \in \pi}$ is a direct sum decomposition of the vector space spanned by the

⁷Technically, a "distinction" of a partition $\pi = \{B\}$ on U is an ordered pair (u, u') of elements of U in different blocks of the partition. The set of distinctions, $\text{dit}(\pi)$, of a partition π is called a *partition relation* (or *apartness relation* in computer science) and is just the complement of the partition as a binary equivalence relation. The notion of a distinction of a partition is the partition logic analogue of an element of a subset in subset logic. For instance, given two partitions $\pi = \{B\}$ and $\sigma = \{C\}$ on a universe set and two subsets S and T of a universe set, the partition join $\pi \vee \sigma$ combines the distinctions of the partitions, i.e., $\text{dit}(\pi \vee \sigma) = \text{dit}(\pi) \cup \text{dit}(\sigma)$, just as the subset join or union $S \cup T$ combines the elements of the subsets (see [10] or [14] for further developments).

⁸In QM, the extension of concepts on finite dimensional Hilbert space to infinite dimensional ones is well-known. Since our expository purpose is conceptual rather than mathematical, we will stick to finite dimensional spaces.

basis set. Thus the lift or transport of the concept of a set partition is a direct sum decomposition of a vector space. In particular, it is not a set partition of a vector space that is compatible with the vector space operations, i.e., a quotient space V/W as would be defined by each subspace $W \subseteq V$ with the equivalence relation $v \sim v'$ if $v - v' \in W$. While a partition on a set is essentially the same as a quotient set (or equivalence relation on the set), the vector-space lift of a set partition is not a quotient space but a direct sum decomposition of a vector space. Thus there are choices to be made in lifting or transporting the partitional concepts for sets to vector spaces, and we are making the choices that yield the mathematical machinery of quantum mechanics.

Hermann Weyl is one of the few quantum physicists who, in effect, outlined the lifting program by first considering an attribute on a set, which defined the set partition or "grating" [41, p. 255] of elements with the same attribute-value. Then he moved to the quantum case where the set or "aggregate of n states has to be replaced by an n -dimensional Euclidean vector space" [41, p. 256].⁹ The appropriate notion of a partition or "grating" is a "splitting of the total vector space into mutually orthogonal subspaces" so that "each vector \vec{x} splits into r component vectors lying in the several subspaces" [41, p. 256], i.e., a direct sum decomposition of the space, where the subspaces are the eigenspaces of an observable operator.

Weyl's grating metaphor also lends itself to (our own example of) seeing measurement of the, say, 'regular polygonal shape' of an 'indefinite blob of dough' as it randomly falls through a opening in a grating to take on that 'polygonal shape' (with the attribute-value or "eigenvalue" being the number of regular sides $\lambda = 3, 4, 5, 6$).

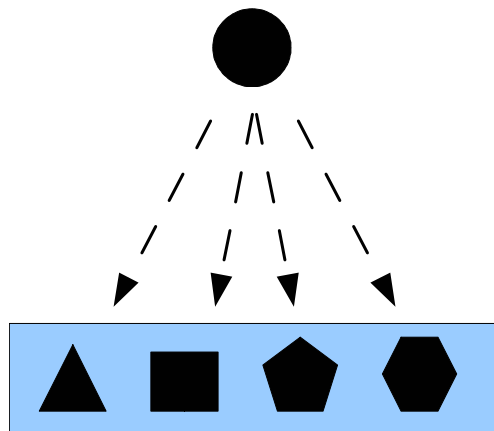


Figure 5: Imagery of measurement as randomly giving an indefinite blob of dough a definite eigen-shape.

Note how the blob of dough is "objectively indefinite" between the regular polygonal shapes and does not "simultaneously" have all those shapes even though it might be mathematically represented as the set $\{\blacktriangle, \blacksquare, \dots\}$ or the superposition vector $\blacktriangle + \blacksquare + \dots$ in a certain space.

3.3 What is a vector space attribute?

A set attribute is a function $f : U \rightarrow \mathbb{R}$ (where the set of values is taken as the reals). The inverse-image $f^{-1}(r) \subseteq U$ of each value $f(u) = r$ is a subset where the attribute has the same value, and those subsets form a set partition. Given a basis set $\{b_u\}_{u \in U}$ of a vector space V over a field K , we can apply a set attribute $f : \{b_u\}_{u \in U} \rightarrow K$ to the basis set and see what it generates. One possibility

⁹Note the lift from sets to vector spaces using the basis principle where the cardinality n becomes the dimension n .

is to linearly extend the function to the whole space to obtain a linear functional $f^* : V \rightarrow K$. But a linear functional defines a quotient space $V/f^{-1}(0)$, not a vector space partition.

The same information $f : \{b_u\}_{u \in U} \rightarrow K$ also defines $\hat{f}(b_u) = f(b_u)b_u$ which linearly extends to a *linear operator* $\hat{f} : V \rightarrow V$. The given basis vectors $\{b_u\}$ are eigenvectors of the operator \hat{f} with the eigenvalues $f(b_u)$, and the eigenspaces are the subspaces where the operator has the same eigenvalue. The eigenvectors span the whole space so we see that the lift or transport of a set attribute is a vector space linear operator whose the eigenvectors span the whole space, i.e., a *diagonalizable linear operator*.

4 Whence vector-space partitions?

4.1 Vector-space partitions from vector-space attributes

Given a diagonalizable linear operator $L : V \rightarrow V$, where V is a finite-dimensional vector space over a field K and where $\lambda_1, \dots, \lambda_k$ are the distinct eigenvalues, then there are projection operators P_i for $i = 1, \dots, k$ such that:

1. $L = \sum_{i=1}^k \lambda_i P_i$;
2. $I = \sum_{i=1}^k P_i$;
3. $P_i P_j = 0$ for $i \neq j$; and
4. the range of P_i is the eigenspace V_i for the eigenvalue λ_i for $i = 1, \dots, k$. [23, Theorem 8, p. 172]

Hence the set of eigenspaces $\{V_i\}$ is a direct-sum decomposition. It is the vector-space partition determined by a vector-space attribute $L : V \rightarrow V$. This standard linear algebra result holds for any base field, but for QM, the base field is the complex numbers \mathbb{C} . In order for the eigenvalues to always be real, the diagonalizable linear operators are required to be *Hermitian* (i.e., equal to their conjugate transposes). Thus the vector-space attributes that represent real-valued observables are given by Hermitian operators.

4.2 Vector-space partitions from vector-space representations of groups

A *vector-space representation* of an abstract group G is a group homomorphism $T : G \rightarrow GL(V)$ where $GL(V)$ is the group of invertible linear transformations $V \rightarrow V$ of a vector space V over the complex numbers. Here again, the idea is to define a (vector-space) partition by a (linear) group of transformations $T_g : V \rightarrow V$ that map elements $v \in V$ to "similar" or "symmetric" elements $T_g(v)$. A subspace $W \subseteq V$ is *invariant* if $T_g(W) \subseteq W$ for all $g \in G$. And again, it is the minimal invariant subspaces, the *irreducible subspaces*, that are of interest. The irreducible subspaces $\{W_\alpha\}$ are the carriers for the *irreducible representations* $T \upharpoonright : W_\alpha \rightarrow W_\alpha$ or *irreps*. And the representation space V is a direct sum of irreducible subspaces so the vector-space representation of a group defines a vector-space partition of the space.

However, due to the linear nature of the representations and the algebraic completeness of the base field \mathbb{C} , the irreps have much more significant structure than in the case of set representations restricted to the minimal invariant subsets or orbits. When considering a given representation $T : G \rightarrow GL(V)$, it is not clear whether or not the irreducible subspaces and their irreps are dependent on the particular properties of the vector space V . As usual, to abstract from those particular properties, an equivalence relation is defined so that we may say an representation of G is the "same" or equivalent across different vector spaces.

Suppose T is a representation of G acting on a space V and T' is a representation of the same G acting on a space V' . Then a linear map $\phi : V \rightarrow V'$ is said *morphism of representations* or *intertwining map* if for all $g \in G$ and all $v \in V$:

$$\begin{array}{ccc} \phi(T_g(v)) = T'_g(\phi(v)), \text{ i.e.,} & & \\ V & \xrightarrow{T_g} & V \\ \phi \downarrow & & \downarrow \phi \\ V' & \xrightarrow{T'_g} & V' \\ & \text{commutes.} & \end{array}$$

If ϕ is also invertible, then ϕ is said to be an *isomorphism* of representations, and the representations are said to be *isomorphic* or *equivalent*.

The remarkable fact is that each group has a fixed set of inequivalent irreps, so the distinct irreps are a characteristic of the group itself, not of a particular representation. Each space carrying a representation of the same group is the direct sum of irreducible subspaces, and, by extending the notion of direct sum to representations, each representation is a direct sum of the group's irreps, perhaps with repetitions.

Now we can return to our previous discussion of the significance of the minimal invariant subsets or now subspaces. To represent indefiniteness, we first need to specify the "universe" of fully definite eigen-alternatives, and then indefiniteness can be described by putting together the "potential" eigen-alternatives (1) in the set case by a subset like a block in a partition or quotient set, or (2) in the vector-space case by a superposition vector. Group representation theory answers the question of whence the plenitude of possible eigen-alternatives; they are given by the minimal invariant subspaces that are the carriers for the irreducible representations of a symmetry group.

For state-dependent attributes of a quantum particle like the linear momentum or angular momentum, the fully definite eigenstates are determined by the irreducible representations of the linear-translation or rotational-translation symmetry groups respectively. For the state-independent attributes of quantum particles, like the mass, charge, and spin of an electron, they are determined in particle physics by the irreducible representations of the appropriate symmetry groups.¹⁰

4.3 Vector-space partitions from other vector-space partitions

The set notion of compatibility lifts to vector spaces, via the basis principle, by defining two vector space partitions $\omega = \{W_\lambda\}$ and $\xi = \{X_\mu\}$ on V as being *compatible* if there is a basis set for V so that the two vector space partitions are generated by two set partitions of that common or simultaneous basis set.

If two vector space partitions $\omega = \{W_\lambda\}$ and $\xi = \{X_\mu\}$ are compatible, then their *vector space join* $\omega \vee \xi$ is defined as the vector space partition whose subspaces are the non-zero intersections $W_\lambda \cap X_\mu$. And by the definition of compatibility, we could also generate the subspaces of the join $\omega \vee \xi$ by the blocks in the set join of the two set partitions of the common basis set.

Since real-valued set attributes lift to Hermitian linear operators, the notion of compatible set attributes just defined would lift to two linear operators being *compatible* if their eigenspace partitions are compatible. It is a standard fact of linear algebra [23, p. 177] that two diagonalizable linear operators $L, M : V \rightarrow V$ (on a finite dimensional space V) are compatible (i.e., have a basis of simultaneous eigenvectors) if and only if they commute, $LM = ML$. Hence the *commutativity* of linear operators is the lift of the compatibility (i.e., defined on the same set) of set attributes. Thus the join of two eigenspace partitions is defined iff the operators commute. As Weyl put it: "Thus combination [join] of two gratings [eigenspace partitions of two operators] presupposes commutability...". [41, p. 257]

¹⁰The classic paper in this group-theoretic treatment of particles is Wigner [42]. For recent overviews, see the group-theoretical definition of particles in Falkenburg [15] or Roberts [32].

Two commuting Hermitian linear operators L and M have compatible eigenspace partitions $W_L = \{W_\lambda\}$ (for the eigenvalues λ of L) and $W_M = \{W_\mu\}$ (for the eigenvalues μ of M). The blocks in the join $W_L \vee W_M$ of the two compatible eigenspace partitions are the non-zero subspaces $\{W_\lambda \cap W_\mu\}$ which can be characterized by the ordered pairs of eigenvalues (λ, μ) . The nonzero vectors $v \in W_\lambda \cap W_\mu$ are *simultaneous eigenvectors* for the two commuting operators, and there is a basis for the space consisting of simultaneous eigenvectors.¹¹

A set of commuting linear operators is said to be *complete* if the join of their eigenspace partitions is nondegenerate, i.e., the blocks have dimension 1. The join operation gives the results of compatible measurements so the join of a complete set of compatible vector space attributes (i.e., commuting Hermitian operators) gives the possible results of a non-degenerate measurement. The eigenvectors that generate those one-dimensional blocks of the join are characterized by the ordered n -tuples (λ, \dots, μ) of eigenvalues so the eigenvectors are usually denoted as the eigenkets $|\lambda, \dots, \mu\rangle$ in the Dirac notation. These *Complete Sets of Commuting Operators* are Dirac's CSCOs [8] (which are the vector space version of our previous CSCAs).¹²

The partitional mathematics for sets and vector spaces is summarized in the following table.

Lifting Summary	Set concept	Vector space concept
Partition	Direct sum decomposition $\pi = \{B\}$ of $U: U = \uplus B$	Direct sum decomposition $\{W_i\}$ of $V: V = \sum \oplus W_i$
Real-valued Attribute	Function $f: U \rightarrow \mathbb{R}$	Hermitian operator $L: V \rightarrow V$
Partition of attribute	Inverse-image partition $\{f^{-1}(r)\}$ for $f: U \rightarrow \mathbb{R}$	Eigenspace partition $W_L = \{W_\lambda\}$ for $L: V \rightarrow V$
Compatible partitions	Partitions π, σ on same set U	Vector space partitions $\{W_i\}$ and $\{X_j\}$ with common basis
Compatible attributes	Attributes $f, g: U \rightarrow \mathbb{R}$ defined on same set U	Commuting operators $LM = ML$, i.e., common basis of simultaneous eigenvectors.
Join of compatible attribute partitions	$f^{-1} \vee g^{-1} = \{f^{-1}(r) \cap g^{-1}(s)\}$ for $f, g: U \rightarrow \mathbb{R}$	$W_L \vee W_M = \{W_\lambda \cap W_\mu\}$ for $LM = ML$
CSCO	Singleton blocks of $\vee f_i^{-1}$ for compatible attributes $\{f_i^{-1}\}$	One-dim. blocks of $\vee W_{L_i}$ for commuting operators $\{L_i\}$

Figure 6: Summary of partition concepts for sets and vector spaces

¹¹One must be careful not to assume that the simultaneous eigenvectors are the eigenvectors for the operator $LM = ML$ due to the problem of degeneracy.

¹²For more analysis using the partitional approach but beyond the scope of this paper, see [12].

5 "Quantum Mechanics" over sets

5.1 Toy models of QM over finite fields

In the tradition of "toy models" for quantum mechanics, Schumacher and Westmoreland [33], Hanson et al. [19], and Takeuchi, Chang, et al. [38], have recently investigated models of quantum mechanics over finite fields. One finite field stands out over the rest, \mathbb{Z}_2 , since vectors in a vector space over \mathbb{Z}_2 have a natural interpretation, namely as *sets* that are subsets of a universe set. But in any vector space over a finite field, there is no inner product so the first problem in constructing a toy model of QM in this context is the definition of Dirac's brackets. Which aspects of the usual treatment of the brackets should be retained and which aspects should be dropped?

Schumacher and Westmoreland chose to have their brackets continue to have values in the base field, e.g., $\mathbb{Z}_2 = \{0, 1\}$, so their "theory does not make use of the idea of probability." [33, p. 919] Instead, the values of 0 and 1 are respectively interpreted modally as *impossible* and *possible* and hence their name of "modal quantum theory." A number of results from full QM carry over to their modal quantum theory, e.g., no-cloning, superdense coding, and teleportation, but without a probability calculus, other results such as Bell's Theorem do not carry over: "in the absence of probabilities and expectation values the Bell approach will not work." [33, p. 921] Similar remarks apply to the other aforementioned toy models all of which have the brackets taking values in the base field.

But all these limitations can be overcome by the different treatment of the brackets based on crossing the sets-to-vector-space bridge in the other direction (essentially using the basis principle in reverse). That yields a full probability calculus for a model of *quantum mechanics over sets* (QM/sets) using the \mathbb{Z}_2 base field. QM/sets yields a full probability calculus—and it is a familiar calculus, logical probability theory for a finite universe set of outcomes developed by Laplace, Boole, and others. The only difference from that classical calculus is the vector space formulation which allows different (equicardinal) bases or universe sets of outcomes and thus it is "non-commutative." This allows the development of the QM/sets version of QM results such as Bell's Theorem, the indeterminacy principle, double-slit experiments, and much else in the "clear and distinct" context of finite sets.

By developing a sets-version of QM, the concepts and relationships of full QM are represented in a pared-down ultra-simple version that can be seen as representing the essential "logic" of QM. It represents the "logic of QM" in that old sense of "logic" as giving the basic essentials of a theory (even reduced to "zero-oneness"), not in the sense of giving the behavior of propositions in a theory which is the usual "quantum logic" [1] that was, in effect, based on the usual misdescription of Boolean subset logic as the special case of "propositional" logic.

5.2 Vector spaces over \mathbb{Z}_2

QM/sets is said to be "over \mathbb{Z}_2 " or "over sets" since the power set $\wp(U) \cong \mathbb{Z}_2^U$ (for a finite non-empty universe set U) is a vector space over $\mathbb{Z}_2 = \{0, 1\}$ where the subset addition $S + T$ is the *symmetric difference* (or inequivalence) of subsets, i.e., $S + T = S \neq T = S \cup T - S \cap T$ for $S, T \subseteq U$. Given a finite universe set $U = \{u_1, \dots, u_n\}$ of cardinality n , the U -basis in \mathbb{Z}_2^U is the set of singletons $\{u_1\}, \{u_2\}, \dots, \{u_n\}$ and a vector in \mathbb{Z}_2^U is specified in the U -basis by its \mathbb{Z}_2 -valued characteristic function $\chi_S : U \rightarrow \mathbb{Z}_2$ for an subset $S \subseteq U$ (e.g., a string of n binary numbers). Similarly, a vector v in \mathbb{C}^n is specified in terms of an orthonormal basis $\{|v_i\rangle\}$ by a \mathbb{C} -valued function $\langle _ | v \rangle : \{|v_i\rangle\} \rightarrow \mathbb{C}$ assigning a complex amplitude $\langle v_i | v \rangle$ to each basis vector. One of the key pieces of mathematical machinery in QM, namely the inner product, does not exist in vector spaces over finite fields but basis-dependent "brackets" can still be defined (see below) and a norm or absolute value can be

defined to play a similar role in the probability algorithm of QM/sets.¹³

Seeing $\wp(U)$ as the vector space \mathbb{Z}_2^U allows different bases in which the vectors can be expressed (as well as the basis-free notion of a vector as a ket, since only the bra is basis-dependent). Consider the simple case of $U = \{a, b, c\}$ where the U -basis is $\{a\}$, $\{b\}$, and $\{c\}$. But the three subsets $\{a, b\}$, $\{b, c\}$, and $\{a, b, c\}$ also form a basis since: $\{a, b\} + \{a, b, c\} = \{c\}$; $\{b, c\} + \{c\} = \{b\}$; and $\{a, b\} + \{b\} = \{a\}$. These new basis vectors could be considered as the basis-singletons in another equicardinal universe $U' = \{a', b', c'\}$ where $a' = \{a, b\}$, $b' = \{b, c\}$, and $c' = \{a, b, c\}$.

In the following *ket table*, each row is a ket of $\mathbb{Z}_2^U \cong \mathbb{Z}_2^3$ expressed in the U -basis, the U' -basis, and a U'' -basis.

$U = \{a, b, c\}$	$U' = \{a', b', c'\}$	$U'' = \{a'', b'', c''\}$
$\{a, b, c\}$	$\{c'\}$	$\{a'', b'', c''\}$
$\{a, b\}$	$\{a'\}$	$\{b''\}$
$\{b, c\}$	$\{b'\}$	$\{b'', c''\}$
$\{a, c\}$	$\{a', b'\}$	$\{c''\}$
$\{a\}$	$\{b', c'\}$	$\{a''\}$
$\{b\}$	$\{a', b', c'\}$	$\{a'', b''\}$
$\{c\}$	$\{a', c'\}$	$\{a'', c''\}$
\emptyset	\emptyset	\emptyset

Vector space isomorphism: $\mathbb{Z}_2^3 \cong \wp(U) \cong \wp(U') \cong \wp(U'')$ where row = ket.

5.3 The brackets

In a Hilbert space, the inner product is used to define the amplitudes $\langle v_i | v \rangle$ and the norm $|v| = \sqrt{\langle v | v \rangle}$ where the probability algorithm can be formulated using this norm. In a vector space over \mathbb{Z}_2 , the Dirac notation can still be used but in a basis-dependent form (like matrices as opposed to operators) that defines a real-valued norm even though there is no inner product. The kets $|S\rangle$ for $S \subseteq U$ are basis-free but the corresponding bras are basis-dependent. For a basis element $\{u\} \subseteq U$, the "bra" $\langle \{u\} |_U : \wp(U) \rightarrow \mathbb{R}$ is defined by the "bracket":

$$\langle \{u\} |_U S \rangle = \begin{cases} 1 & \text{if } u \in S \\ 0 & \text{if } u \notin S \end{cases} = \chi_S(u)$$

Then $\langle \{u_i\} |_U \{u_j\} \rangle = \chi_{\{u_j\}}(u_i) = \chi_{\{u_i\}}(u_j) = \delta_{ij}$ is the set-version of $\langle v_i | v_j \rangle = \delta_{ij}$ (for an orthonormal basis $\{|v_i\rangle\}$). Assuming a finite U , the "bracket" linearly extends to the more general basis-dependent form (where $|S|$ is the cardinality of S):

$$\langle T |_U S \rangle = |T \cap S| \text{ for } T, S \subseteq U.^{14}$$

The basis principle can be run in reverse to transport a vector space concept to sets. Consider an orthonormal basis set $\{|v_i\rangle\}$ in a finite dimensional Hilbert space. Given two subsets $T, S \subseteq \{|v_i\rangle\}$ of the basis set, consider the unnormalized superpositions $\psi_T = \sum_{|v_i\rangle \in T} |v_i\rangle$ and $\psi_S = \sum_{|v_i\rangle \in S} |v_i\rangle$. Then their inner product in the Hilbert space is $\langle \psi_T | \psi_S \rangle = |T \cap S|$, which transports (crossing the bridge in the other direction) to $\langle T |_U S \rangle = |T \cap S|$ for subsets $T, S \subseteq U$ of the U -basis of \mathbb{Z}_2^U . In both cases, the bracket gives the size of the overlap.

¹³Often scare quotes, as in "brackets," are used to indicate the named concept in QM/sets as opposed to full QM—although this may also be clear from the context.

¹⁴Thus $\langle T |_U S \rangle = |T \cap S|$ takes values outside the base field of \mathbb{Z}_2 just like the Hamming distance function $d_H(T, S) = |T + S|$ on vector spaces over \mathbb{Z}_2 in coding theory. [29]

5.4 Ket-bra resolution

The basis-dependent "ket-bra" $|\{u\}\rangle\langle\{u\}|_U$ is the "one-dimensional" projection operator:

$$|\{u\}\rangle\langle\{u\}|_U = \{u\} \cap () : \wp(U) \rightarrow \wp(U)$$

and the "ket-bra identity" holds as usual:

$$\sum_{u \in U} |\{u\}\rangle\langle\{u\}|_U = \sum_{u \in U} (\{u\} \cap ()) = I : \wp(U) \rightarrow \wp(U)$$

where the summation is the symmetric difference of sets in \mathbb{Z}_2^U . The overlap $\langle T|_U S \rangle$ can be resolved using the "ket-bra identity" in the same basis: $\langle T|_U S \rangle = \sum_u \langle T|_U \{u\}\rangle\langle\{u\}|_U S \rangle$. Similarly a ket $|S\rangle$ for $S \subseteq U$ can be resolved in the U -basis;

$$|S\rangle = \sum_{u \in U} |\{u\}\rangle\langle\{u\}|_U S \rangle = \sum_{u \in U} \langle\{u\}|_U S \rangle |\{u\}\rangle = \sum_{u \in U} |\{u\} \cap S| \{u\}$$

where a subset $S \subseteq U$ is just expressed as the sum of the singletons $\{u\} \subseteq S$. That is ket-bra resolution in sets. The ket $|S\rangle$ is the same as the ket $|S'\rangle$ for some subset $S' \subseteq U'$ in another U' -basis, but when the basis-dependent bra $\langle\{u\}|_U$ is applied to the ket $|S\rangle = |S'\rangle$, then it is the subset $S \subseteq U$, not $S' \subseteq U'$, that comes outside the ket symbol $| \rangle$ in $\langle\{u\}|_U S \rangle = |\{u\} \cap S|$.¹⁵

5.5 The norm

Then the (basis-dependent) U -norm $\|S\|_U : \wp(U) \rightarrow \mathbb{R}$ is defined, as usual, as the square root of the bracket:¹⁶

$$\|S\|_U = \sqrt{\langle S|_U S \rangle} = \sqrt{|S|}$$

for $S \in \wp(U)$ which is the set-version of the basis-free norm $|\psi\rangle = \sqrt{\langle\psi|\psi\rangle}$ (since the inner product does not depend on the basis). Note that a ket has to be expressed in the U -basis to apply the basis-dependent norm definition so in the above example, $\|\{a'\}\|_U = \sqrt{2}$ since $\{a'\} = \{a, b\}$ in the U -basis.

5.6 The Born Rule

For a specific basis $\{|v_i\rangle\}$ and for any nonzero vector v in a finite dimensional complex vector space, $|v|^2 = \sum_i \langle v_i|v\rangle\langle v_i|v\rangle^*$ (* is complex conjugation) whose set version would be: $\|S\|_U^2 = \sum_{u \in U} \langle\{u\}|_U S \rangle^2$. Since

$$|v\rangle = \sum_i \langle v_i|v\rangle |v_i\rangle \text{ and } |S\rangle = \sum_{u \in U} \langle\{u\}|_U S \rangle |\{u\}\rangle,$$

applying the Born Rule by squaring the coefficients $\langle v_i|v\rangle$ and $\langle\{u\}|_U S \rangle$ (and normalizing) gives the probability sums for the eigen-elements v_i or $\{u\}$ given a state v or S respectively in QM and QM/sets:

$$\sum_i \frac{\langle v_i|v\rangle\langle v_i|v\rangle^*}{|v|^2} = 1 \text{ and } \sum_u \frac{\langle\{u\}|_U S \rangle^2}{\|S\|_U^2} = \sum_u \frac{|\{u\} \cap S|}{|S|} = 1$$

where $\frac{\langle v_i|v\rangle\langle v_i|v\rangle^*}{|v|^2}$ is a 'mysterious' quantum probability while $\frac{\langle\{u\}|_U S \rangle^2}{\|S\|_U^2} = \frac{|\{u\} \cap S|}{|S|}$ is the unmysterious Laplacian equal probability $\Pr(\{u\} | S)$ rule for getting u when sampling S .¹⁷

¹⁵The term " $\{u\} \cap S''$ " is not even defined since it is the intersection of subsets of two different universes. One of the luxuries of having a basis independent inner product in QM over \mathbb{C} is being able to ignore bases in the bra-ket notation.

¹⁶We use the double-line notation $\|S\|_U$ for the norm of a set to distinguish it from the single-line notation $|S|$ for the cardinality of a set, whereas the customary absolute value notation for the norm of a vector in full QM is $|v|$.

¹⁷Note that there is no notion of a normalized vector in a vector space over \mathbb{Z}_2 (another consequence of the lack of an inner product). The normalization is, as it were, postponed to the probability algorithm which is computed in the rationals.

5.7 Spectral decomposition on sets

An observable, i.e., a Hermitian operator, on a Hilbert space has a home basis set of orthonormal eigenvectors. In a similar manner, a real-valued attribute $f : U \rightarrow \mathbb{R}$ defined on U has the U -basis as its "home basis set." The connection between the numerical attributes $f : U \rightarrow \mathbb{R}$ of QM/sets and the Hermitian operators of full QM can be established by "seeing" the function f as a formal operator: $f \upharpoonright () : \wp(U) \rightarrow \wp(U)$. Applied to the basis elements $\{u\} \subseteq U$, we may write $f \upharpoonright \{u\} = f(u) \{u\} = r \{u\}$ as the set-version of an eigenvalue equation applied to an eigenvector, where the multiplication $r \{u\}$ is only formal (read $r \{u\}$ as: **the function f takes the value r on $\{u\}$**). Then for any subset $S \subseteq f^{-1}(r)$ where f is constant, we may also formally write: $f \upharpoonright S = rS$ as an "eigenvalue equation" satisfied by all the "eigenvectors" S in the "eigenspace" $\wp(f^{-1}(r))$, a subspace of $\wp(U)$, for the "eigenvalue" r . The "eigenspaces" $\wp(f^{-1}(r))$ give a direct sum decomposition (i.e., a vector-space partition) of the whole space $\wp(U) = \sum_r \oplus \wp(f^{-1}(r))$, just as the set partition $f^{-1} = \{f^{-1}(r)\}_r$ gives a direct sum decomposition of the set $U = \bigsqcup_r f^{-1}(r)$. Since $f^{-1}(r) \cap () : \wp(U) \rightarrow \wp(U)$ is the projection operator¹⁸ to the "eigenspace" $\wp(f^{-1}(r))$ for the "eigenvalue" r , we have the "spectral decomposition" of a U -attribute $f : U \rightarrow \mathbb{R}$ in QM/sets analogous to the spectral decomposition of a Hermitian operator $L = \sum_\lambda \lambda P_\lambda$ in QM:

$$f \upharpoonright () = \sum_r r (f^{-1}(r) \cap ()) : \wp(U) \rightarrow \wp(U)$$

$$L = \sum_\lambda \lambda P_\lambda : V \rightarrow V$$

Spectral decomposition of operators in QM/sets and in QM.

When the base field increases from \mathbb{Z}_2 to \mathbb{C} , then the formal multiplication $r (f^{-1}(r) \cap ())$ is internalized as an actual multiplication, and the projection operator $f^{-1}(r) \cap ()$ on sets becomes a projection operator on a vector space over \mathbb{C} . Thus the operator representation $L = \sum_\lambda \lambda P_\lambda$ of an observable numerical attribute is just the internalization of a numerical attribute made possible by the enriched base field \mathbb{C} . Similarly, the set brackets $\langle T|_U S \rangle$ taking values outside the base field \mathbb{Z}_2 become internalized as an inner product with the same enrichment of the base field.

It is the comparative "poverty" of the base field \mathbb{Z}_2 that requires the QM/sets "brackets" to take "de-internalized" or "externalized" values outside the base field and for a formal multiplication to be used in the "operator" representation $f \upharpoonright () = \sum_r r (f^{-1}(r) \cap ())$ of a numerical attribute $f : U \rightarrow \mathbb{R}$. Or put the other way around, the only numerical attributes that can be internally represented in $\wp(U) \cong \mathbb{Z}_2^U$ are the characteristic functions $\chi_S : U \rightarrow \mathbb{Z}_2$ that are internally represented in the U -basis as the projection operators $S \cap () : \wp(U) \rightarrow \wp(U)$.

In the engineering literature, eigenvalues are seen as "stretching or shrinking factors" but that is *not* their role in QM. The whole machinery of eigenvectors [e.g., $f \upharpoonright S = rS$ for $S \subseteq f^{-1}(r)$ in sets], eigenspaces [e.g., $\wp(f^{-1}(r))$], and eigenvalues [e.g., $f(u) = r$] in full QM is a way of representing a numerical attribute [e.g., $f : U \rightarrow \mathbb{R}$ in the set case] *inside* a vector space that has a rich enough base field.

5.8 Completeness and orthogonality of projection operators

The usual completeness and orthogonality conditions on eigenspaces also have set-versions in QM over \mathbb{Z}_2 :

1. completeness: $\sum_\lambda P_\lambda = I : V \rightarrow V$ has the set-version: $\sum_r f^{-1}(r) \cap () = I : \wp(U) \rightarrow \wp(U)$,
and

¹⁸Since $\wp(U)$ is now interpreted as a vector space, it should be noted that the projection operator $T \cap () : \wp(U) \rightarrow \wp(U)$ is not only idempotent but linear, i.e., $(T \cap S_1) + (T \cap S_2) = T \cap (S_1 + S_2)$. Indeed, this is the distributive law when $\wp(U)$ is interpreted as a Boolean ring.

2. orthogonality: for $\lambda \neq \lambda'$, $P_\lambda P_{\lambda'} = 0 : V \rightarrow V$ (where 0 is the zero operator) has the set-version: for $r \neq r'$, $[f^{-1}(r) \cap ()] [f^{-1}(r') \cap ()] = \emptyset \cap () : \wp(U) \rightarrow \wp(U)$.¹⁹

5.9 Measurement in QM/sets

The Pythagorean results (for the complete and orthogonal projection operators):

$$|v|^2 = \sum_\lambda |P_\lambda(v)|^2 \text{ and } \|S\|_U^2 = \sum_r \|f^{-1}(r) \cap S\|_U^2,$$

give the probabilities for measuring attributes. Since

$$|S| = \|S\|_U^2 = \sum_r \|f^{-1}(r) \cap S\|_U^2 = \sum_r |f^{-1}(r) \cap S|$$

"Pythagorean Theorem" for sets

we have in QM and in QM/sets:

$$\sum_\lambda \frac{|P_\lambda(v)|^2}{|v|^2} = 1 \text{ and } \sum_r \frac{\|f^{-1}(r) \cap S\|_U^2}{\|S\|_U^2} = \sum_r \frac{|f^{-1}(r) \cap S|}{|S|} = 1$$

where $\frac{|P_\lambda(v)|^2}{|v|^2}$ is the quantum probability of getting λ in an L -measurement of v while $\frac{|f^{-1}(r) \cap S|}{|S|}$ has the rather unmysterious interpretation of the probability $\Pr(r|S)$ of the random variable $f : U \rightarrow \mathbb{R}$ having the "eigen-value" r when sampling $S \subseteq U$. Thus the set-version of the Born Rule is not some weird "quantum" notion of probability on sets but the perfectly ordinary Laplace-Boole rule for the conditional probability $\Pr(r|S) = \frac{|f^{-1}(r) \cap S|}{|S|}$, given $S \subseteq U$, of a random variable $f : U \rightarrow \mathbb{R}$ having the value r .

The collecting-together of some eigen-elements $\{u\} \subseteq U$ into a subset $S \subseteq U$ to form an "indefinite element" S has the vector sum $|S\rangle = \sum_{u \in U} \langle \{u\} |_U S \rangle \{u\}$ in the vector space $\wp(U)$ over \mathbb{Z}_2 giving the superposition version of the indefinite element. This "cements" the interpretation of "collecting together" in sets as superposition in vector spaces.

The indefinite element S is being "measured" using the "observable" f where the probability $\Pr(r|S)$ of getting the "eigenvalue" r is $\frac{|f^{-1}(r) \cap S|}{|S|}$ and where the "damned quantum jump" goes from S to the "projected resultant state" $f^{-1}(r) \cap S$ which is in the "eigenspace" $\wp(f^{-1}(r))$ for that "eigenvalue" r .

The partition operation in QM/sets that describes "measurement" is the partition join of the partition $\{S, S^c\}$ and $f^{-1} = \{f^{-1}(r)\}$ so that the initial "pure state" S (as a mini-blob) is refined into the "mixture" $\{f^{-1}(r) \cap S\}$ of possible resultant states. The other states $\{f^{-1}(r) \cap S^c\}$ in the join $f^{-1} \vee \{S, S^c\}$ are not possible or "potential" states starting from S . The "state" resulting from the "measurement" represents a more-definite element $f^{-1}(r) \cap S$ that now has the definite f -value of r —so a second measurement would yield the same "eigenvalue" r with probability $\Pr(r|f^{-1}(r) \cap S) = \frac{|f^{-1}(r) \cap [f^{-1}(r) \cap S]|}{|f^{-1}(r) \cap S|} = \frac{|f^{-1}(r) \cap S|}{|f^{-1}(r) \cap S|} = 1$ and the same vector $f^{-1}(r) \cap [f^{-1}(r) \cap S] = f^{-1}(r) \cap S$ using the idempotency of the set-version of projection operators—all analogous to the standard Dirac-von-Neumann treatment of measurement.²⁰

¹⁹Note that in spite of the lack of an inner product, the orthogonality of projection operators $S \cap ()$ is perfectly well defined in QM/sets where it boils down to the disjointness of subsets, i.e., the cardinality of their overlap (instead of their inner product) being 0.

²⁰See [12] and [13] for a more extensive treatment of measurement using density matrices in both full QM and QM/sets.

5.10 Summary of QM/sets

These set-versions are summarized in the following table for a finite U and a finite dimensional Hilbert space V with $\{|v_i\rangle\}$ as any orthonormal basis.

Vector space over \mathbb{Z}_2 : QM/sets	Hilbert space case: QM over \mathbb{C}
Projections: $S \cap () : \wp(U) \rightarrow \wp(U)$	$P : V \rightarrow V$
Spectral Decomp.: $f \uparrow () = \sum_r r (f^{-1}(r) \cap ())$	$L = \sum_\lambda \lambda P_\lambda$
Compl.: $\sum_r f^{-1}(r) \cap () = I : \wp(U) \rightarrow \wp(U)$	$\sum_\lambda P_\lambda = I$
Orthog.: $r \neq r', [f^{-1}(r) \cap ()] [f^{-1}(r') \cap ()] = \emptyset \cap ()$	$\lambda \neq \lambda', P_\lambda P_{\lambda'} = 0$
Brackets: $\langle S _U T \rangle = S \cap T = \text{overlap of } S, T \subseteq U$	$\langle \psi \varphi \rangle = \text{"overlap" of } \psi \text{ and } \varphi$
Ket-bra: $\sum_{u \in U} \{u\}\rangle \langle \{u\} _U = \sum_{u \in U} (\{u\} \cap ()) = I$	$\sum_i v_i\rangle \langle v_i = I$
Resolution: $\langle S _U T \rangle = \sum_u \langle S _U \{u\}\rangle \langle \{u\} _U T \rangle$	$\langle \psi \varphi \rangle = \sum_i \langle \psi v_i\rangle \langle v_i \varphi \rangle$
Norm: $\ S\ _U = \sqrt{\langle S _U S \rangle} = \sqrt{ S }$ where $S \subseteq U$	$ \psi = \sqrt{\langle \psi \psi \rangle}$
Pythagoras: $\ S\ _U^2 = \sum_{u \in U} \langle \{u\} _U S \rangle^2 = S $	$ \psi ^2 = \sum_i \langle v_i \psi \rangle^* \langle v_i \psi \rangle$
Laplace: $S \neq \emptyset, \sum_{u \in U} \frac{\langle \{u\} _U S \rangle^2}{\ S\ _U^2} = \sum_{u \in S} \frac{1}{ S } = 1$	$ \psi\rangle \neq 0, \sum_i \frac{\langle v_i \psi \rangle^* \langle v_i \psi \rangle}{ \psi ^2} = \frac{ \langle v_i \psi \rangle ^2}{ \psi ^2} = 1$
Born: $ S = \sum_{u \in U} \langle \{u\} _U S \rangle \langle \{u\} _U S \rangle, \Pr(\{u\} S) = \frac{\langle \{u\} _U S \rangle^2}{\ S\ _U^2}$	$ \psi\rangle = \sum_i \langle v_i \psi \rangle v_i\rangle, \Pr(v_i \psi) = \frac{ \langle v_i \psi \rangle ^2}{ \psi ^2}$
$\ S\ _U^2 = \sum_r \ f^{-1}(r) \cap S\ _U^2 = \sum_r f^{-1}(r) \cap S = S $	$ \psi ^2 = \sum_\lambda P_\lambda(\psi) ^2$
$S \neq \emptyset, \sum_r \frac{\ f^{-1}(r) \cap S\ _U^2}{\ S\ _U^2} = \sum_r \frac{ f^{-1}(r) \cap S }{ S } = 1$	$ \psi\rangle \neq 0, \sum_\lambda \frac{ P_\lambda(\psi) ^2}{ \psi ^2} = 1$
Measurement: $\Pr(r S) = \frac{\ f^{-1}(r) \cap S\ _U^2}{\ S\ _U^2} = \frac{ f^{-1}(r) \cap S }{ S }$	$\Pr(\lambda \psi) = \frac{ P_\lambda(\psi) ^2}{ \psi ^2}$

Probability mathematics for QM over \mathbb{Z}_2 and for QM over \mathbb{C}

5.11 A glance back at full QM

QM/sets is more than just a pedagogical model in the sense that when some particularly "mysterious" process like measurement can be clearly and distinctly modeled in QM/sets, then it casts some sense-making light back on full QM. A good example is von Neumann's distinction between Type 1 measurement-like processes and Type 2 processes of unitary evolution described by Schrödinger's equation [40]. In QM/sets, we have seen that "measurement" is a distinction-making process described by the partition join operation. In terms of the lattice of set partitions, such a "Type 1" process moves up in the lattice to more refined partitions.²¹ This means in QM/sets that a "Type 2" evolution would be a distinction-preserving process that, as it were, moves horizontally in the lattice of partitions.

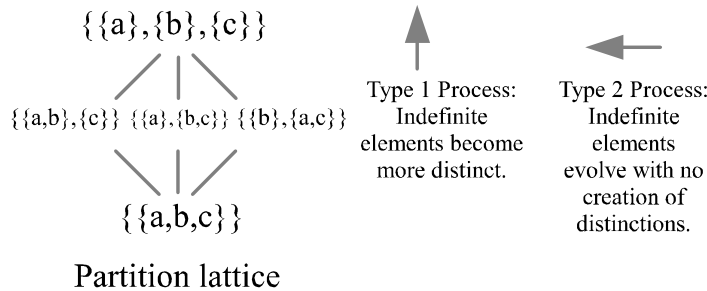


Figure 7: "Type 1" distinction-making and "Type 2" distinction-preserving processes in QM/sets

²¹The usual notion of refinement of partitions, i.e., $\pi = \{B\}$ is *more (or equally) refined* than $\sigma = \{C\}$, denoted $\sigma \preceq \pi$, if for each $B \in \pi$, there is a $C \in \sigma$ such that $B \subseteq C$, is just the inclusion relation on distinctions, i.e., $\sigma \preceq \pi$ iff $\text{dit}(\sigma) \subseteq \text{dit}(\pi)$.

A linear transformation $\wp(U) \rightarrow \wp(U)$ that keeps distinct vectors distinct (i.e., preserves distinctions) is just a non-singular transformation.²² This means that a Type 2 process in full QM should be a process that preserves the degree of distinctness and indistinctness. Given two normalized quantum states ψ and φ , the brackets $\langle\psi|\varphi\rangle$ can be interpreted as the degree of indistinctness of the states with the extreme values of $\langle\psi|\varphi\rangle = 1$ for full indistinctness, i.e., $\psi = \varphi$, and $\langle\psi|\varphi\rangle = 0$ for zero indistinctness, i.e., the full distinctness of orthogonality. Hence under this partitional approach to understanding or making sense of QM, the Type 2 processes are the ones that preserve the degree of indistinctness $\langle\psi|\varphi\rangle$, i.e., the unitary transformations. Thus the clear distinction between "Type 1" distinction-making and "Type 2" distinction-preserving processes in QM/sets helps to make sense of the von Neumann Type 1 distinction-making measurements and Type 2 distinction-preserving unitary transformations in full QM.

6 Final remarks

There are two meta-physical visions of reality suggested by classical physics (objectively definite reality) and by quantum physics (objectively indefinite reality). The problem of interpreting QM is essentially the problem of making sense out of the notion of objective indefiniteness. Our sense-making strategy was to follow the lead of the mathematics.

The definiteness of classical physics is associated with the notion of a subset and is expressed in the classical Boolean logic of subsets. The indefiniteness of quantum physics is associated with the notion of a quotient set, equivalence relation, or partition, and the corresponding logic is the recently developed logic of partitions [10]. Moreover, those associated notions of subsets and quotient sets are category-theoretically dual to one another, so from that viewpoint, those are *the only* two possible frameworks to describe reality. Common sense and classical physics assumes the objectively definite type of reality, but quantum physics strongly indicates an objectively indefinite reality at the quantum level. Hence our approach to interpreting quantum mechanics is not flights of fantasy (e.g., about many worlds or realms of hidden variables) but is trying to make sense out of objective indefiniteness.

Our sense-making strategy was implemented by developing the mathematics of partitions at the connected conceptual levels of sets and vector spaces. Set concepts are transported to (complex) vector spaces to yield the mathematical machinery of full QM, and the complex vector space concepts of full QM are transported to the set-like vector spaces over \mathbb{Z}_2 to yield the rather fulsome pedagogical model of quantum mechanics over sets or QM/sets.

In this manner, we have tried to use partition concepts to make sense of objective indefiniteness and thus to interpret quantum mechanics.

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²²Thus the gates in quantum computing over \mathbb{Z}_2 are the non-singular linear transformations ([33] and [13]).

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