Probability Theory with Superposition Events: A Classical Generalization in the Direction of Quantum Mechanics

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Abstract

In finite probability theory, events are subsets $S \subseteq U$ of the outcome set. Subsets can be represented by 1-dimensional column vectors. By extending the representation of events to two dimensional matrices, we can introduce "superposition events." Probabilities are introduced for classical events, superposition events, and their mixtures by using density matrices. Then probabilities for experiments or 'measurements' of all these events can be determined in a manner exactly like in quantum mechanics (QM) using density matrices. Moreover the transformation of the density matrices induced by the experiments or 'measurements' is the Lüders mixture operation as in QM. And finally by moving the machinery into the n-dimensional vector space over $\mathbb{Z}_2$, different basis sets become different outcome sets. That 'non-commutative' extension of finite probability theory yields the pedagogical model of quantum mechanics over $\mathbb{Z}_2$ that can model many characteristic non-classical results of QM.

Contents

1 Introduction: Probability Theory with Superposition Events

The purpose of this paper is to introduce new concepts such as "superposition events" into finite probability theory. Let $U = \{u_1, ..., u_n\}$ be the outcome set or sample space of outcomes with the respective point probabilities of $p = (p_1, ..., p_n)$. Classical events are represented by subsets $S \subseteq U$ with probabilities $\Pr(S) = \sum_{u_i \in S} p_i$ where the conditional probability of the event T given the event S is $\Pr(T|S) = \frac{\Pr(S \cap T)}{\Pr(S)}$.

2 The Density Matrix Representations

To generalize classical events to superposition events, we need a richer mathematical representation than just the notion of a subset. The mathematical information in a 'classical' event S (for convenience, always non-empty) could be represented in a (normalized) column vector $|S\rangle$ with i^{th} entry being $\sqrt{\frac{p_i}{\Pr(S)}}\chi_S(u_i)$ (where $\chi_S : U \to \{0,1\}$ is the characteristic or indicator function for S, $\chi_S(u_i) = 1$ if $u_i \in S$ and 0 otherwise). The same information could be represented in two dimensions by the diagonal $n \times n$ matrix $\rho(\Delta S)$ with the diagonal entries $\frac{p_i}{\Pr(S)}\chi_S(u_i)$, i.e.,

$$\rho(\Delta S)_i = \frac{p_i}{\Pr(S)}\chi_S(u_i).$$

But the richer two-dimensional matrices allows us to define the *superposition event* ΣS associated with S as being represented by the $n \times n$ matrix $\rho(\Sigma S)$ (writing the transpose $|S\rangle^t = \langle S|$) by multiplying the $n \times 1$ column vector $|S\rangle$ times the $1 \times n$ transpose $|S\rangle^t = \langle S|$:

$$\rho(\Sigma S) = |S\rangle\langle S| \text{ with the entries } \rho(\Sigma S)_{ik} = \sqrt{\frac{p_i}{\Pr(S)}\frac{p_k}{\Pr(S)}}\chi_S(u_i)\chi_S(u_k).$$

Note that singleton events $S = \{u_i\}$ have no distinct elements to superpose and accordingly $\rho(\Delta\{u_i\}) = \rho(\Sigma\{u_i\})$.

Both $\rho(\Delta S)$ and $\rho(\Sigma S)$ are examples of *real density matrices* which can be defined abstractly as symmetric matrices $\rho = \rho^t$ over the reals with trace (sum of diagonal elements) $\operatorname{tr}[\rho] = 1$, and with non-negative eigenvalues. But for practical purposes, density matrices (over the reals unless otherwise stated) may be taken to be any probabilistic mixtures of matrices of the form $\rho(\Sigma S)$. That is, for any probability distribution $q = (q_1, ..., q_m)$ and classical events $S_j \subseteq U$ for $j = 1, ..., m$, the convex combination $\sum_{j=1}^m q_j \rho(\Sigma S_j)$ is also a density matrix.

A density matrix ρ is said to be *pure* if $\rho^2 = \rho$, and otherwise *mixed*. For instance, $\rho(\Sigma S)$ is pure while $\rho(\Delta S)$ is a mixture unless S is a singleton event $\{u_i\}$ since $\rho(\Delta\{u_i\})^2 = \rho(\Sigma\{u_i\})^2 = \rho(\Sigma\{u_i\}) = \rho(\Delta\{u_i\})$ trivially.

A *partition* $\pi = \{B_1, ..., B_m\}$ on U is a set of non-empty mutually disjoint subsets $\{B_j\}_{j=1}^m$ whose union is U. [2] The partition π is represented by the density matrix:

$$\rho(\pi) = \sum_{j=1}^m \Pr(B_j)\rho(\Sigma B_j)$$

Density matrix associated with a partition π on U

that is mixed unless π is the *indiscrete partition* $\mathbf{0}_U = \{U\}$ since $\rho(\mathbf{0}_U) = \rho(\Sigma U)$. With a suitable interchange of rows and columns, any density matrix $\rho(\pi)$ defined by a partition would be block-diagonal according to the partition blocks $B_j \in \pi$. For the *discrete partition* $\mathbf{1}_U = \{\{u_1\}, ..., \{u_n\}\}$ on U, $\rho(\mathbf{1}_U) = \rho(\Delta U)$. Thus the two extreme partitions at the top (discrete partition $\mathbf{1}_U$) and bottom (indiscrete partition $\mathbf{0}_U$) in the lattice of partitions (ordered by refinement) on U correspond to the two extreme density matrices $\rho(\Delta U)$ and $\rho(\Sigma U)$, and all the intermediate partitions π have density matrices that are mixtures of the pure density matrices $\rho(\Sigma B_j)$ for their blocks.

For the discrete partition on a subset S, $\mathbf{1}_S = \{\{u_i\}\}_{u_i \in S}$ and the indiscrete partition $\mathbf{0}_S = \{S\}$ on a subset S, $\rho(\mathbf{1}_S) = \rho(\Delta S)$ and $\rho(\mathbf{0}_S) = \rho(\Sigma S)$. The discrete partition $\mathbf{1}_S$ on a set $S \subseteq U$ distinguishes all the elements of S from each other in singleton blocks, and thus the density matrix $\rho(\mathbf{1}_S)$ associated with that partition is the statistical mixture of the singleton events for elements of S: $\rho(\mathbf{1}_S) = \rho(\Delta S) = \sum_{u_i \in S} \frac{p_i}{\Pr(S)}\rho(\Delta\{u_i\})$. In contrast, the superposition event ΣS associated with S represented by $\rho(\Sigma S)$ blurs, blobs, or coheres together, i.e., superposes, the elements of S. For equal probabilities $\frac{1}{|S|}$, the elements of S are equally superposed. Otherwise, we may say $u_i, u_k \in S$ are superposed with an *amplitude* of $\rho(\Sigma S)_{ik} = \sqrt{\frac{p_i}{\Pr(S)}\frac{p_k}{\Pr(S)}}$. The entries in the density matrices associated with S, namely $\rho(\Delta S)$ and $\rho(\Sigma S)$, have the same diagonal elements and differ only in the off-diagonal elements. When an off-diagonal entry $\rho(\Sigma S)_{ik}$ is non-zero, then it indicates that the corresponding elements $u_i, u_k \in S$ are cohered together with that non-zero amplitude. All the off-diagonal elements in $\rho(\Delta S)$ are zero indicating that the elements of S are completely distinguished or decohered from each other.

For a suggestive visual example, consider the outcome set U as a pair of isosceles triangles that are distinct by the labels on the equal sides and the opposing angles.

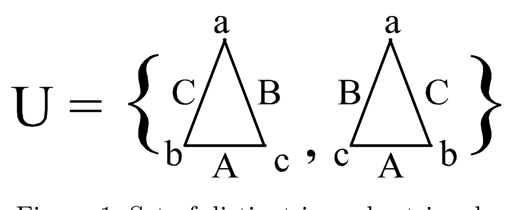

Figure 1: Set of distinct isosceles triangles

The superposition event ΣU is definite on the properties that are common to the elements of U, i.e., the angle a and the opposing side A, but is indefinite where the two triangles are distinct, i.e., the two equal sides and their opposing angles.

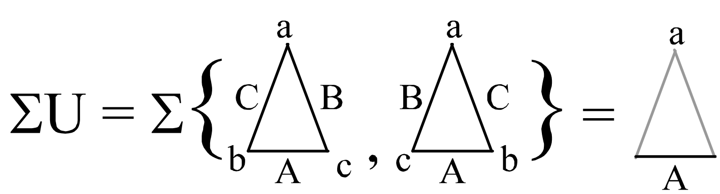

Figure 2: The superposition event ΣU.

Consider the partition $\pi = \{B_1, B_2\} = \{\{\diamondsuit, \heartsuit\}, \{\clubsuit, \spadesuit\}\}$ on the outcome set $U = \{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$ with equiprobable outcomes like drawing cards from a randomized deck. For instance, the superposition event associated with $B_1 = \{\diamondsuit, \heartsuit\}$, is pure since (rows and columns labelled in the order $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$):

$$\rho(\Sigma B_1) = \frac{1}{\Pr(\{\diamondsuit, \heartsuit\})} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \Pr(\{\diamondsuit\}) & \sqrt{\Pr(\{\diamondsuit\})\Pr(\{\heartsuit\})} & 0 \\ 0 & \sqrt{\Pr(\{\heartsuit\})\Pr(\{\diamondsuit\})} & \Pr(\{\heartsuit\}) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

equals its square, but density matrix for the half-half mixture of the two suit-color pure events:

$$\frac{1}{2}\rho(\Sigma B_1) + \frac{1}{2}\rho(\Sigma B_2)$$
$$= \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ 0 & \frac{1}{4} & \frac{1}{4} & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

is a mixture since it does not equal its square.

Intuitively, the interpretation of the superposition event represented by $\rho(\Sigma B_1) = \rho(\Sigma\{\diamondsuit, \heartsuit\})$ is that it is definite on the properties common to its elements, e.g., in this case, being a red suite, but indefinite on where the elements differ. The indefiniteness is indicated by the non-zero off-diagonal elements that indicate that the diamond suite $\diamondsuit$ is blurred, cohered, or superposed with the hearts suite $\heartsuit$ in the superposition state $\Sigma\{\diamondsuit, \heartsuit\}$.

3 Computing 'Measurement' or Trial Probabilities with Density Matrices

A (real-valued) *random variable* on the outcome space U is a function $f : U \to \mathbb{R}$ with values of $\{\phi_1, ..., \phi_m\}$. The inverse image of f is a partition $\pi = \{B_j\}_{j=1}^{m}$ where $B_j = f^{-1}(\phi_j)$. In ordinary classical probability theory, the conditional probability of getting the value ϕ_j given the event S in a trial is $\Pr(\phi_j|S) = \frac{\Pr(B_j \cap S)}{\Pr(S)}$. But now we have two versions of S, the classical event and the superposition event. Since they have different density matrices, we can take the given conditioning event as a density matrix ρ. Let P_T for $T \subseteq U$ be the diagonal projection matrix with the diagonal entries $(P_T)_{ii} = \chi_T(u_i)$. Projection matrices are idempotent, i.e., $P_T P_T = P_T$ and equal their transpose $P_T = P_T^t$. The usual conditional probability of the classical event T given the classical event S can be computed as:

$$\Pr(T|S) := \frac{\Pr(S \cap T)}{\Pr(S)} = \operatorname{tr}[P_T \rho(\Delta S)].$$

In general, the probability of getting the value ϕ_j conditioned by the density matrix ρ is defined as:

$$\Pr(\phi_j|\rho) := \operatorname{tr}\left[P_{B_j}\rho\right].$$

In particular, starting with the conditioning event being the superposition event corresponding to S, that probability is:

$$\Pr(\phi_j|\rho(\Sigma S)) = \operatorname{tr}\left[P_{B_j}\rho(\Sigma S)\right] = \frac{\Pr(B_j \cap S)}{\Pr(S)}.$$

This yields the perhaps surprising result that the probabilities for the values of a random variable (or any given event T) are the same if the conditioning event is the classical event S represented by the mixed $\rho(\Delta S)$ or the superposition event ΣS represented by the pure $\rho(\Sigma S)$:

$$\Pr(\phi_j|\rho(\Sigma S)) = \operatorname{tr}\left[P_{B_j}\rho(\Sigma S)\right] = \frac{\Pr(B_j \cap S)}{\Pr(S)} = \operatorname{tr}\left[P_{B_j}\rho(\Delta S)\right] = \Pr(\phi_j|\rho(\Delta S)).$$

But the interpretation is quite different. The classical trial starting with the subset S represented by $\rho(\Delta S)$ picks out the subset $B_j \cap S$ represented by $\rho(\Delta(B_j \cap S))$ with probability $\Pr(\phi_j|S) = \operatorname{tr}\left[P_{B_j}\rho(\Delta S)\right]$. However, the 'measurement' of the superposition event ΣS represented by $\rho(\Sigma S)$ 'sharpens' or projects that indefinite event to the more definite superposition event $\Sigma(B_j \cap S)$ represented by $\rho(\Sigma(B_j \cap S))$ with probability $\Pr(\phi_j|S) = \operatorname{tr}\left[P_{B_j}\rho(\Sigma S)\right]$. In either case, the follow-up trial or 'measurement' returns the same value ϕ_j with probability 1, i.e., $\Pr(\phi_j|B_j \cap S) = \operatorname{tr}\left[P_{B_j}\rho(\Delta(B_j \cap S))\right] = \operatorname{tr}\left[P_{B_j}\rho(\Sigma(B_j \cap S))\right] = 1$. In the classical case, all the elements of $B_j \cap S$ have the value ϕ_j so the conditioning classical event $B_j \cap S$ occurs with probability 1. In the superposition case, the property of having the value ϕ_j is definite on the superposition event $\Sigma(B_j \cap S)$ represented by $\rho(\Sigma(B_j \cap S))$, so no 'sharpening' occurs and projection P_{B_j} restricted to $B_j \cap S$ is the identity so the measurement returns the same event $\Sigma(B_j \cap S)$ with probability 1.

Let us illustrate this result with the case of flipping a fair coin. The classical set of outcomes $U = \{H, T\}$ is represented by the density matrix:

$$\rho(\Delta U) = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}.$$

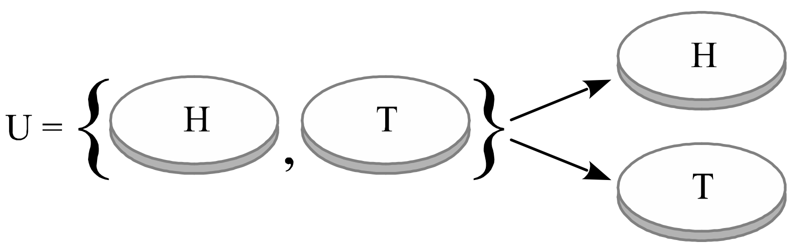

Figure 3: Classical event: trial picks out heads or tails

The superposition event ΣU, that blends or superposes heads and tails, is represented by the density matrix:

$$\rho(\Sigma U) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$

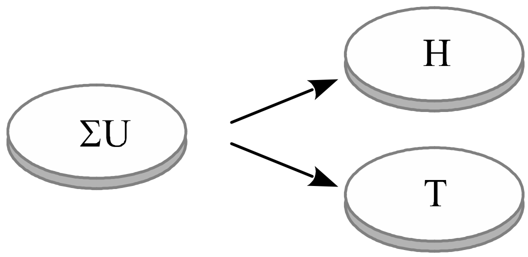

Figure 4: Superposition event: Measurement sharpens to heads or tails.

The probability of getting heads in each case is:

$$\Pr(H|\rho(\Delta U)) = \operatorname{tr}\left[P_{\{H\}}\rho(\Delta U)\right] = \operatorname{tr}\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\right] = \operatorname{tr}\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 0 \end{bmatrix} = \frac{1}{2}$$
$$\Pr(H|\rho(\Sigma U)) = \operatorname{tr}\left[P_{\{H\}}\rho(\Sigma U)\right] = \operatorname{tr}\left[\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}\right] = \operatorname{tr}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{bmatrix} = \frac{1}{2}$$

and similarly for tails. Thus the two conditioning events U and ΣU cannot be distinguished by performing an experiment or measurement that distinguishes heads and tails. But this actually should not be too surprising since the same thing occurs in quantum mechanics. For instance, a spin measurement along, say, the z-axis of an electron cannot distinguish between the superposition state $\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle)$ with a density matrix like $\rho(\Sigma U)$ and a statistical mixture of half electrons with spin up and half with spin down with a density matrix like $\rho(\Delta U)$ [1, p. 176]. The states can only be distinguished by measuring in a different basis, and we will show in a later section how probability theory with superposition events can be further enriched to demonstrate that possibility.

It might be further noticed that the average value of a random variable can also be computed in that same manner as in QM. If $\mathcal{O}_f$ is the $n \times n$ diagonal matrix with diagonal entries $f(u_i)$ which represents $f : U \to \mathbb{R}$, then the average value of the random variable restricted to a subset S, $\sum_{u_i \in S} \Pr(\phi_i|S) f(u_i)$, is:

$$\langle f \rangle_S = \operatorname{tr}[\mathcal{O}_f \rho(\Delta S)] = \operatorname{tr}[\mathcal{O}_f \rho(\Sigma S)].$$

Average value of random variable f on S.

The probability $\Pr(T|S) = \operatorname{tr}[P_T \rho(\Delta S)] = \operatorname{tr}[P_T \rho(\Sigma S)]$ is just the average value of the characteristic function $\chi_T : U \to \{0, 1\}$ on S considered as a random variable on U, i.e., $\mathcal{O}_{\chi_T} = P_T$. In particular,

$$\Pr(S) = \operatorname{tr}[P_S \rho(\Delta U)] = \operatorname{tr}[P_S \rho(\Sigma U)]$$

is the average value of χ_S on U.

4 How 'Measurement' Transforms Density Matrices

Since events, classical or superposition and any probability mixture thereof, are now dealt with using density matrices, we need to define the resulting change in the density matrix when a trial, an experiment, or a measurement of a random variable occurs. Since the density matrix $\rho(\Sigma S)$ is constructed as $|S\rangle$ times its transpose $\langle S|$, the corresponding transformation by the projection matrix P_T is:

$$P_T\rho(\Sigma S)P_T^t = P_T|S\rangle\langle S|P_T = \frac{\Pr(T\cap S)}{\Pr(S)}\rho(\Sigma(T\cap S))$$

since the pre- and post-multiplying by P_T zeros all the entries in $|S\rangle\langle S|$ except the ones $\sqrt{\frac{p_i}{\Pr(S)}\frac{p_k}{\Pr(S)}} = \frac{1}{\Pr(S)}\sqrt{p_ip_k}$ for $u_i, u_k \in T\cap S$, and $\rho(\Sigma(T\cap S))$ has the entries $\frac{1}{\Pr(T\cap S)}\sqrt{p_ip_k}$ for the same $u_i, u_k \in T\cap S$, so $\frac{\Pr(T\cap S)}{\Pr(S)}\frac{1}{\Pr(T\cap S)}\sqrt{p_ip_k} = \frac{1}{\Pr(S)}\sqrt{p_ip_k}$ giving the result. When $T = B_j = f^{-1}(\phi_j)$,

$$P_{B_j}\rho(\Sigma S)P_{B_j} = \frac{\Pr(B_j\cap S))}{\Pr(S)}\rho(\Sigma(B_j\cap S)).$$

When the outcome of the experiment is ϕ_j with probability $\Pr(\phi_j|S) = \frac{\Pr(B_j\cap S)}{\Pr(S)}$, then the superposition event ΣS represented by the density matrix $\rho(\Sigma S)$ is transformed into the superposition event $\Sigma(B_j\cap S)$ represented by the density matrix $\rho(\Sigma(B_j\cap S))$. The partition induced on S by $\pi = \{B_j\}_{j=1}^m = \left\{f^{-1}(\phi_j)\right\}_{j=1}^m$ is $\pi\upharpoonright S$, the partition of all the non-empty blocks $B_j\cap S$ for $j = 1,...,m$. The density matrix associated with all the probabilistic results is the mixed sum of the density matrices $\rho(\Sigma(B_j\cap S))$ weighted by their probabilities $\Pr(\phi_j|S) = \frac{\Pr(B_j\cap S)}{\Pr(S)}$ which is denoted by $\rho(\pi\upharpoonright S)$. Thus we have:

$$\begin{gathered}\rho(\pi\upharpoonright S) := \sum_{j=1}^m \Pr(\phi_j|S)\rho(\Sigma(B_j\cap S))\\ = \sum_{j=1}^m \frac{\Pr(B_j\cap S)}{\Pr(S)}\rho(\Sigma(B_j\cap S)) = \sum_{j=1}^m P_{B_j}\rho(\Sigma S)P_{B_j}\\ \text{The Lüders mixture operation: } \rho(\Sigma S)\rightsquigarrow\rho(\pi\upharpoonright S).\end{gathered}$$

The operation of experimenting with or 'measuring' the random variable $f : U\to\mathbb{R}$ starting with the superposition event ΣS represented by the pure density matrix $\rho(\Sigma S)$ transforms it into the mixture $\rho(\pi\upharpoonright S) = \sum_{j=1}^m P_{B_j}\rho(\Sigma S)P_{B_j}$, and that transformation is called the *Lüders mixture operation* [1, p, 279] in quantum mechanics.

As an example, let us take $S = \{\clubsuit,\diamondsuit,\spadesuit\}\subseteq U = \{\clubsuit,\diamondsuit,\heartsuit,\spadesuit\}$ and take $f : U\to\{0,1\}\subseteq\mathbb{R}$ as a random variable that distinguished the color of the suits so $\pi = \{B_1, B_2\} = \left\{f^{-1}(0), f^{-1}(1)\right\} = \{\{\diamondsuit,\heartsuit\},\{\clubsuit,\spadesuit\}\}$. Then we have:

$$\rho(\Sigma S) = \begin{bmatrix}\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\\ 0 & 0 & 0 & 0\\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\end{bmatrix}.$$

And the probability in an experiment of getting a black suite where $B_2 = f^{-1}(1) = \{\clubsuit,\spadesuit\}$ is:

$$\Pr(1|S) = \operatorname{tr}[B_2\rho(\Sigma S)] = \operatorname{tr}\left[\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\\ 0 & 0 & 0 & 0\\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\end{bmatrix}\right] = \operatorname{tr}\begin{bmatrix}\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\\ 0&0&0&0\\ 0 & 0 & 0 & 0\\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\end{bmatrix} = \frac{2}{3}.$$

The experiment of measuring the suite-colors starting with ΣS transforms the density matrix $\rho(\Sigma S)$ according to the Lüders mixture operation:

$$\begin{aligned}\rho(\pi\upharpoonright S) = \sum_{j=1}^2 P_{B_j}\rho(\Sigma S)P_{B_j} = &\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\\ 0 & 0 & 0 & 0\\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\end{bmatrix}\begin{bmatrix}0&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}\\ &+\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}\frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\\ 0 & 0 & 0 & 0\\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3}\end{bmatrix}\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&1\end{bmatrix}\end{aligned}$$

$$= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{bmatrix}.$$

5 Measurement and Logical Entropy

The *logical entropy of a partition* [4] $\pi = \{B_1, ..., B_m\}$ on U is:

$$h(\pi) := \sum_{j=1}^{m} \Pr(B_j)(1 - \Pr(B_j)) = 1 - \sum_{j=1}^{m} \Pr(B_j)^2 = \sum_{j \neq j'} \Pr(B_j)\Pr(B_{j'})$$

and the logical entropy of any probability distribution $q = \{q_1, ..., q_m\}$ is similarly:

$$h(q) = 1 - \sum_{j=1}^{m} q_j^2 = \sum_{j \neq j'} q_j q_{j'} = 2\sum_{j<j'} q_j q_{j'}.$$

The interpretation of the logical entropy of π is the probability in an ordered pair of independent draws or trials to get elements distinguished by π (i.e., elements from different blocks of π) or different q_j's. The logical entropy of any density matrix ρ is:

$$h(\rho) = \operatorname{tr}[\rho(1 - \rho)] = 1 - \operatorname{tr}[\rho^2].$$

The trace of any density matrix squared is the sum of all the squared entries (or the absolute squares in complex density matrices of QM): $\operatorname{tr}[\rho^2] = \sum_{i,k=1}^{n} |\rho_{ik}|^2$ [6, p. 77]. When the partition π is represented by the density matrix $\rho(\pi) = \sum_j \Pr(B_j)\rho(\Sigma B_j)$, then a simple calculation shows that:

$$h(\rho(\pi)) = 1 - \operatorname{tr}\left[\rho(\pi)^2\right] = 1 - \sum_{j=1}^{m} \Pr(B_j)^2 = h(\pi).$$

Since the trace of any density matrix is 1 and for any pure density matrix, $\rho^2 = \rho$, $\operatorname{tr}[\rho^2] = \operatorname{tr}[\rho] = 1$ so the logical entropy of any pure density matrix is 0. Logical entropy measures distinctions, and in a pure superposition event ΣS, there are no distinctions between the superposed or cohered outcomes. When an off-diagonal element of a density matrix is non-zero, that means the corresponding diagonal elements cohere together or are superposed in a superposition. But when the experiment or 'measurement operation' distinguishes (or decoheres) those elements, the corresponding off-diagonal elements are zeroed. Since the logical entropy measures distinctions, the logical entropy created by the measurement operation can be computed as the squares of the off-diagonal elements zeroed in the Lüders mixture operation on the density matrices.

Theorem 1 *The logical entropy created in the measurement of $\rho(\Sigma S)$ by π, i.e. $h(\rho(\pi \upharpoonright S)) - h(\rho(\Sigma S))$ [which equals $h(\rho(\pi \upharpoonright S))$ since $\rho(\Sigma S)$ is pure], is the sum of the squares of the off-diagonal elements in $\rho(\Sigma S)$ that are zeroed in the Lüders mixture operation $\rho(\Sigma S) \rightsquigarrow \rho(\pi \upharpoonright S)$.*

Proof: All elements in the density matrix $\rho(\Sigma S)$ either have the same value (e.g., all diagonal elements and some off-diagonal elements) or are zeroed (e.g., some off-diagonal elements) by the projections in the Lüders mixture operation. Hence the sum of squares of the off-diagonal elements that are zeroed is:

$$\begin{aligned} &\sum_{i,k=1}^{n} \rho(\Sigma S)_{ik}^2 - \sum_{i,k=1}^{n} \rho(\pi \upharpoonright S)_{ik}^2 = \operatorname{tr}\left[\rho(\Sigma S)^2\right] - \operatorname{tr}\left[\rho(\pi \upharpoonright S)^2\right] \\ &= \left(1 - \operatorname{tr}\left[\rho(\pi \upharpoonright S)^2\right]\right) - \left(1 - \operatorname{tr}\left[\rho(\Sigma S)^2\right]\right) = h(\pi \upharpoonright S) - h(\rho(\Sigma S)). \ \square \end{aligned}$$

This theorem holds, *mutatis mutandis*, for quantum logical entropy and the Lüders mixture operation in quantum information theory where the squares are absolute squares [5].

To illustrate the theorem, consider the previous suite-color measurement where $S = \{\clubsuit, \diamondsuit, \spadesuit\}$, The logical entropy of the pure $\rho(\Sigma S)$ is 0, and:

$$\rho(\pi \upharpoonright S)^2 = \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} \frac{2}{9} & 0 & 0 & \frac{2}{9} \\ 0 & \frac{1}{9} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{2}{9} & 0 & 0 & \frac{2}{9} \end{bmatrix}$$

so $h(\rho(\pi \upharpoonright S)) = 1 - \operatorname{tr}\left[\rho(\pi \upharpoonright S)^2\right] = 1 - \frac{5}{9} = \frac{4}{9}$. Comparing the before and after matrices,

$$\rho(\Sigma S) = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 0 & 0 & 0 \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \end{bmatrix} \rightsquigarrow \begin{bmatrix} \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & \frac{1}{3} \end{bmatrix} = \rho(\pi \upharpoonright S),$$

we see that four entries of $\frac{1}{3}$ are zeroed (since the different colors were distinguished by the color measurement) and the sum of their squares is also $\frac{4}{9}$ as per the theorem. For illustrative purposes, we might represent the matrix associated with the superposition event ΣS for $S = \{\clubsuit, \diamondsuit, \spadesuit\}$ represented by $\rho(\Sigma S)$ as:

$$\begin{bmatrix} \{\clubsuit,\clubsuit\} & \{\clubsuit,\diamondsuit\} & 0 & \{\clubsuit,\spadesuit\} \\ \{\diamondsuit,\clubsuit\} & \{\diamondsuit,\diamondsuit\} & 0 & \{\diamondsuit,\spadesuit\} \\ 0 & 0 & 0 & 0 \\ \{\spadesuit,\clubsuit\} & \{\spadesuit,\diamondsuit\} & 0 & \{\spadesuit,\spadesuit\} \end{bmatrix}$$

so it is clear that the four off-diagonal elements zeroed by the measurement (that distinguished color) are the four that cohered different colored suites together in the superposition.

The suit-color partition $\pi = \{\{\diamondsuit, \heartsuit\}, \{\clubsuit, \spadesuit\}\}$ restricted to $S = \{\clubsuit, \diamondsuit, \spadesuit\}$ is $\pi \upharpoonright S = \{\{\diamondsuit\}, \{\clubsuit, \spadesuit\}\}$. In two independent ordered draws from S, the probability of getting elements from different blocks of $\pi \upharpoonright S$ is $\frac{1}{3}\frac{2}{3} + \frac{2}{3}\frac{1}{3} = \frac{4}{9} = h(\rho(\pi \upharpoonright S))$,and that is the general interpretation of $h(\pi)$, the probability in two ordered draws of getting elements in distinct blocks of π.

6 The Pedagogical Model of Quantum Mechanics over $\mathbb{Z}_2$

The previous results including the fundamental theorem connecting measurement and logical entropy hold–*mutatis mutandis*–in quantum mechanics (QM) when superposition states are being measured using a given (orthonormal) basis $U = \{u_1, ..., u_n\}$ of an observable.[5] But many results in QM require consideration of different bases. The above results about probabilities using superposition events can be extended in the pedagogical model of quantum mechanics over $\mathbb{Z}_2$ (QM/Sets) [3] where the state space is $\mathbb{Z}_2^n$ and where the n-ary zero-one vectors are considered as subsets of the basis set with equiprobable outcomes. Then U is just one basis which could be taken as the computational basis, but there are many other bases. By Gauss's formula [7, p. 71], the number of ordered bases for $\mathbb{Z}_2^n$ are: $(2^n - 1)(2^n - 2^1) \dots (2^n - 2^{n-1})$ and the number of unordered bases is obtained by dividing by $n!$.

For $n = 2$, there are $(2^2 - 1)(2^2 - 2^1)\frac{1}{2!} = 3$ (unordered) bases of $\mathbb{Z}_2^2$. In the coin-flipping example where $U = \{H, T\}$ was taken as the outcome set, there is another basis $U' = \{H', T'\}$ where $\{H'\} = \{H, T\}$ and $\{T'\} = \{T\}$ which is a basis since $\{H'\} + \{T'\} = \{H, T\} + \{T\} = \{H\}$ (mod 2 addition) and $\{T'\} = \{T\}$. The third basis is for $U'' = \{H'', T''\}$ where $\{H''\} = \{H\}$ and $\{T''\} = \{H, T\}$. Since we have different bases for $\mathbb{Z}_2^2$, we can consider a ket as an abstract vector that can be represented in different bases, e.g., $\{H\}$, $\{H', T'\}$, and $\{H''\}$ all represent the same abstract vector in different bases. Then we can form a ket-table where each row represents a ket. In $\mathbb{Z}_2^2$, there are $2^2 - 1 = 3$ non-zero abstract vectors, each corresponding to a row in the ket-table.

U-basis	U'-basis	U''-basis
$\{H,T\}$	$\{H'\}$	$\{T''\}$
$\{H\}$	$\{H',T'\}$	$\{H''\}$
$\{T\}$	$\{T'\}$	$\{H'',T''\}$

Figure 5: ket-table for $\mathbb{Z}_2^2$.

Each ket or abstract vector is a superposition in one basis and a singleton event in the other two bases.

We saw previously that we could not distinguish the classical mixture event U associated with $\rho(\Delta U)$ from the superposition event ΣU associated with $\rho(\Sigma U)$ when measured in the U-basis. For instance, the probability of getting heads in the two cases is:

$$\Pr(H|\rho(\Delta U)) = \operatorname{tr}\left[P_{\{H\}}\rho(\Delta U)\right] = \operatorname{tr}\left[\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2} & 0\\ 0 & \frac{1}{2}\end{bmatrix}\right] = \operatorname{tr}\begin{bmatrix}\frac{1}{2} & 0\\ 0 & 0\end{bmatrix} = \tfrac{1}{2}$$

$$\Pr(H|\rho(\Sigma U)) = \operatorname{tr}\left[P_{\{H\}}\rho(\Sigma U)\right] = \operatorname{tr}\left[\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}\right] = \operatorname{tr}\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ 0 & 0\end{bmatrix} = \tfrac{1}{2}.$$

But the two events can be distinguished when measured in a different basis such as the U'-basis.

The vector $\{H\}$ is expressed in the U-basis by the column vector $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]_U$ (the subscript indicating the basis) and in the U'-basis by the column vector $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]_{U'}$. The basis conversion matrix is

$$C_{U\to U'} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} \text{ so } \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}_U = \begin{bmatrix}1\\ 1\end{bmatrix}_{U'}.$$

Hence converting the superposition $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]_U$ or $\{H,T\}$ to the U'-basis gives:

$C_{U\to U'}\begin{bmatrix}1\\ 1\end{bmatrix}_U = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}_U = \begin{bmatrix}1\\ 0\end{bmatrix}_{U'}$ or $\{H'\}$ so its density matrix (computing in the reals) is $\begin{bmatrix}1\\ 0\end{bmatrix}_{U'}\begin{bmatrix}1 & 0\end{bmatrix}_{U'} = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}_{U'}$. The classical mixed event U is the half-half mixture of $\{H\}$ and $\{T\}$. The basis conversion for $\{H\}$ gives $C_{U\to U'}\begin{bmatrix}1\\ 0\end{bmatrix}_U = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}_U = \begin{bmatrix}1\\ 1\end{bmatrix}_{U'}$ so the associated real density matrix is:

$$\begin{bmatrix}\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}}\end{bmatrix}_{U'}\begin{bmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{bmatrix}_{U'} = \begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}_{U'}$$

and for $\{T\}$, $C_{U\to U'}\begin{bmatrix}0\\ 1\end{bmatrix}_U = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}0\\ 1\end{bmatrix}_U = \begin{bmatrix}0\\ 1\end{bmatrix}_{U'}$ so its real density matrix is:

$$\begin{bmatrix}0\\ 1\end{bmatrix}_{U'}\begin{bmatrix}0 & 1\end{bmatrix}_{U'} = \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}_{U'}.$$

Their half-half mixture has the density matrix in the U'-basis:

$$\tfrac{1}{2}\begin{bmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}_{U'} + \tfrac{1}{2}\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}_{U'} = \begin{bmatrix}\frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{3}{4}\end{bmatrix}_{U'}.$$

We then measure by the partition $\sigma = \{\{H'\},\{T'\}\}$ with half-half probabilities so the probability of H' for the superposition event $\{H,T\}$ or $\{H'\}$ in the U'-basis is:

$$\operatorname{tr}\left[P_{\{H'\}}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}_{U'}\right] = \operatorname{tr}\left[\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}_{U'}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}_{U'}\right] = \operatorname{tr}\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}_{U'} = 1$$

and for the classical mixture of half $\{H\}$ and half $\{T\}$which in the U'-basis is the mixture of half $\{H',T'\}$and half $\{T'\}$, is:

$$\operatorname{tr}\left[P_{\{H'\}}\begin{bmatrix}\frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{3}{4}\end{bmatrix}_{U'}\right]=\operatorname{tr}\left[\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}_{U'}\begin{bmatrix}\frac{1}{4} & \frac{1}{4}\\ \frac{1}{4} & \frac{3}{4}\end{bmatrix}_{U'}\right]=\operatorname{tr}\begin{bmatrix}\frac{1}{4} & \frac{1}{4}\\ 0 & 0\end{bmatrix}_{U'}=\frac{1}{4}.$$

The first calculation makes intuitive sense since the superposition $\{H,T\}$ in the U-basis is the singleton event $\{H'\}$ in the U'-basis, so measuring in the U'-basis for the event $\{H'\}$ will give $\{H'\}$ with probability 1. The second calculation makes intuitive sense since it is half-half in the mixture whether we get the $\{T'\}$ event or the $\{H',T'\}$ event and then the probability of getting H' is zero for the $\{T'\}$ event and $\frac{1}{2}$ for the $\{H',T'\}$ event so the overall probability of $\{H'\}$ is $\frac{1}{4}$. Thus the two events, the classical mixture of half $\{H\}$ and half $\{T\}$, and the superposition $\{H,T\}$, which cannot be distinguished by measurements in the U-basis, can be distinguished by measurement in the U'-basis.

7 Concluding Remarks

Ordinary finite probability theory can be extended to include superposition events by using the two-dimensional representations of:

- $\rho(\Delta S)$ for the classical event $S\subseteq U$, where the outcomes in S are kept discrete and completely decohered, and
- $\rho(\Sigma S)$ for the superposition event ΣS that superposes or coheres together the outcomes in S.

The calculation of probabilities for classical events in ordinary finite probability theory can be computed using the density matrices in the form $\rho(\Delta S)$ for classical events S. Thus the extension to include superposition events just extends to using density matrices of the form $\rho(\Sigma S)$, and the density matrix formalism also represents classical mixtures of superposition events.

Ordinary finite probability theory sticks with one outcome or sample space U. But the whole machinery can be developed in $\mathbb{Z}_2^n$ where U is just one among many basis sets and then it is part of the pedagogical model of quantum mechanics over $\mathbb{Z}_2$ or QM/Sets. That pedagogical model of QM over $\mathbb{Z}_2$ could also be viewed as just the *non-commutative extension of finite probability theory with superposition events* (since the bases do not in general commute in QM/Sets). Many characteristic QM results can be modeled in this non-commutative probability theory such as the double-slit experiment, the indeterminacy principle, quantum statistics for identical particles, and even Bell's Theorem.[3]

Our purpose has been to illustrate, in a rather classical setting, the notion of a superposition event, where all the outcomes in the event cohere together (with various amplitudes), so the event is objectively indefinite between those outcomes. The notions of objective-indefiniteness and superposition are the essentials in what Abner Shimony called the "Literal" or objectively-indefinite interpretation of QM, an interpretation that is routinely neglected in the literature that focuses on fantasies about many worlds or hidden variables.

> From these two basic ideas alone – indefiniteness and the superposition principle – it should be clear already that quantum mechanics conflicts sharply with common sense. If the quantum state of a system is a complete description of the system, then a quantity that has an indefinite value in that quantum state is objectively indefinite; its value is not merely unknown by the scientist who seeks to describe the system. [8, p. 47]
> But the mathematical formalism ... suggests a philosophical interpretation of quantum mechanics which I shall call "the Literal Interpretation." ...This is the interpretation

> resulting from taking the formalism of quantum mechanics literally, as giving a representation of physical properties themselves, rather than of human knowledge of them, and by taking this representation to be complete. [9, pp. 6-7]

To understand or interpret QM, one needs to better understand the notions of objective indefiniteness and superposition as well as the related notion of a (distinguishing) measurement that sharpens an indefinite superposition event to a mixture of more definite ones. We have shown that the concepts of superposition, objective-indefiniteness, and measurement can be illustrated in a very small extension of classical finite probability theory–which should help to intuitively understand those notions in quantum mechanics.

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