Cognitivism about Epistemic Modality and Hyperintensionality

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Abstract

This essay aims to vindicate the thesis that cognitive computational properties are abstract objects implemented in physical systems. I avail of Voevodsky's Univalence Axiom and function type equivalence in Homotopy Type Theory, in order to specify an abstraction principle for two-dimensional (hyper-)intensions. The homotopic abstraction principle for two-dimensional (hyper-)intensions provides an epistemic conduit for our knowledge of (hyper-)intensions as abstract objects. Higher observational type theory might be one way to make first-order abstraction principles defined via inference rules, although not higher-order abstraction principles, computable. The truth of my first-order abstraction principle for two-dimensional hyperintensions is grounded in its being possibly recursively enumerable i.e. Turing computable and the Turing machine being physically implementable. Epistemic modality and hyperintensionality can thus be shown to be both a compelling and a materially adequate candidate for the fundamental structure of mental representational states, comprising a fragment of the language of thought.

1 Introduction

This essay aims to vindicate the thesis that cognitive computational properties are abstract objects implemented in physical systems.[1] A recent approach to the foundations of mathematics is Homotopy Type Theory.[2] In Homotopy Type Theory, homotopies can be defined as equivalence relations on intensional functions. In this essay, I argue that homotopies can thereby figure in abstraction principles for epistemic (hyper-)intensions, i.e. functions from epistemically possible worlds or states to extensions.[3] Homotopies for epistemic hyperintensions

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[1]See Turing (1950); Putnam (1967); Newell (1973); Fodor (1975); and Pylyshyn (1978).

[2]See The Univalent Foundations Program (2013).

[3]For the first proposal to the effect that abstraction principles can be used to define abstracta such as cardinal number, see Frege (1884/1980: 68; 1893/2013: 20). For the locus classicus of the contemporary abstractionist program, see Hale and Wright (2001).

thus comprise identity criteria for some cognitive mechanisms. The philosophical significance of the foregoing is twofold. First, the proposal demonstrates how epistemic modality and hyperintensionality are viable candidates for fragments of the language of thought.[4] Second, the proposal serves to delineate one conduit for our epistemic access to epistemic hyperintensions as abstract objects.[5]

In Section **2**, I provide an abstraction principle for epistemic hyperintensions, by availing of Voevodsky's Univalence axiom and function type equivalence relations countenanced in Homotopy Type Theory. In Section **3**, I describe how models of Epistemic Modal Algebra are availed of when perceptual representational states are modeled in Bayesian perceptual psychology; when speech acts are modeled in natural language semantics; and when knowledge, belief, inten-

[4]Given a metalanguage, a precedent to the current approach – which models thoughts and internal representations via possible worlds semantics – can be found in Wittgenstein (1921/1974: 2.15-2.151, 3-3.02).

[5]The proposal that epistemic intensions might be sui generis abstract objects, not reducible to sets, is proffered by Chalmers (2011: 101) who writes: 'It is even possible to introduce a special sort of abstract object corresponding to these intensions. Of course these abstract objects cannot be sets of ordered pairs. But we might think of an intension formally as an abstract object which when combined with an arbitrary scenario yields a truth value (or an extension).'

Bealer (1982) proffers a *non-modal* algebraic logic for intensional entities – i.e., properties, relations, and propositions – which avails of a λ-definable variable-binding abstraction operator (op. cit.: 46-48, 209-210). Bealer reduces modal notions to logically necessary conditions-cum-properties, as defined in his non-modal algebraic logic (207-209). The present approach differs from the foregoing by: (i) countenancing a *modal* algebra, on an epistemic interpretation thereof; (ii) availing of Voevodsky's Univalence Axiom in Homotopy Type Theory – which collapses identity and isomorphism – in order to provide an equivalence relation for the relevant abstraction principle; and (iii) demonstrating how the model is availed of in various branches of the cognitive sciences, such that Epistemic Modal Algebra may be considered a viable candidate for the language of thought.

Katz (1998) proffers a view of the epistemology of abstracta, according to which the syntax and the semantics for the propositions are innate (35). Katz suggests that the proposal is consistent with both a Fregean approach to propositions, according to which they are thoughts formed by the composition of senses, and a Russellian approach, according to which they are structured tuples of non-conceptual entities (36). He endorses an account of senses according to which they are correlated to natural language sentence types (114-115). One difference between Katz's proposal and the one here presented is that Katz rejects modal approaches to propositions, because the latter cannot distinguish between distinct contradictions (38fn.6). Following, Lewis (1973: I.6), the present approach does not avail of impossible worlds which distinguish between distinct contradictions. For approaches to epistemic space and conceivability which do admit of impossible worlds, see Rantala (1982); Jago (2009; 2014); Berto (2014); Berto and Schoonen (2018); and Priest (2019). However, Elohim (2024) advances an epistemic two-dimensional truthmaker semantics, such that impossible states can be constructed which distinguish between distinct contradictory states (see Fine, 2021, for further discussion). A second difference is that, on Katz's approach, the necessity of mathematical truths is argued to consist in reductio proofs, such that the relevant formulas will be true on all interpretations, and thus true of logical necessity (39). Elohim (2024) argues that modal axiom K, i.e. epistemic closure is invalid for reductio proofs. However, the endeavor to reduce the necessity of mathematical truths to the necessity of logical consequence would result in the preclusion, both of cases of informal proofs in mathematics, which can, e.g., involve diagrams (see Azzouni, 2004; Giaquinto, 2008: 1.2), and of mathematical truths which obtain in axiomatizable, yet non-logical mathematical languages such as Euclidean geometry. Finally, Katz rejects abstraction principles, and thus implicit definitions for abstract objects (105-106).

tional action, and rational intuition are modeled in philosophical approaches to the nature of propositional attitudes. This provides abductive support for the claim that Epistemic Modal Algebra is both a compelling and materially adequate candidate for a fragment of the language of thought. In Section **4**, I argue that the proposal resolves objections to the relevant abstraction principles advanced by both Dean (2016) and Linnebo and Pettigrew (2014). Section **5** provides concluding remarks.

2 An Abstraction Principle for Epistemic (Hyper-)Intensions

In this section, I specify a homotopic abstraction principle for epistemic (hyper-)intensions. Intensional isomorphism, as a jointly necessary and sufficient condition for the identity of intensions, is first proposed in Carnap (1947: §14). The isomorphism of two intensional structures is argued to consist in their logical, or L-, equivalence, where logical equivalence is co-extensive with the notions of both analyticity (§2) and synonymy (§15). Carnap writes that: '[A]n expression in S is L-equivalent to an expression in S' if and only if the semantical rules of S and S' together, without the use of any knowledge about (extra-linguistic) facts, suffice to show that the two have the same extension' (p. 56), where semantical rules specify the intended interpretation of the constants and predicates of the languages (4).[6] The current approach differs from Carnap's by defining the equivalence relation necessary for an abstraction principle for epistemic (hyper-)intensions on Voevodsky's (2006) Univalence Axiom, which collapses identity with isomorphism in the setting of intensional type theory.[7]

Topological Semantics

In the topological semantics for modal logic, a frame is comprised of a set of points in topological space, a domain of propositions, and an accessibility relation:
$\mathrm{F} = \langle \mathrm{X}, \mathrm{R} \rangle$;
$\mathrm{X} = (\mathrm{X}_x)_{x \in X}$; and

[6]For criticism of Carnap's account of intensional isomorphism, based on Carnap's (1937: 17) 'Principle of Tolerance' to the effect that pragmatic desiderata are a permissible constraint on one's choice of logic, see Church (1954: 66-67).

[7]Note further that, by contrast to Carnap's approach, epistemic intensions are here distinguished from contextual linguistic intensions (see Elohim, 2024, for further discussion of the difference between epistemic and contextual intensions), and the current work examines the philosophical significance of the convergence between epistemic hyperintensions and formal, rather than natural, languages. For a translation from type theory to set theory – which is of interest to, inter alia, the definability of epistemic hyperintensions in the setting of set theory – see Linnebo and Rayo (2012). For topological Boolean-valued models of Epistemic Set Theory – i.e., a variant of ZF with the axioms augmented by epistemic modal operators interpreted as informal provability and having a background logic satisfying S4 – see Scedrov (1985), Flagg (1985a), and Goodman (1990). For Epistemic Type Theory, see Flagg (1985b).

$R = (Rxy)_{x,y \in X}$ iff $R_x \subseteq X_x$ x X_x, s.t. if Rxy, then $\exists o \subseteq X$, with $x \in o$ s.t. $\forall y \in o(Rxy)$,
where the set of points accessible from a privileged node in the space is said to be open.[8] A model defined over the frame is a tuple, $M = \langle F,V \rangle$, with V a valuation function from subsets of points in F to propositional variables taking the values 0 or 1. Necessity is interpreted as an interiority operator on the space:
$M,x \Vdash \Box\phi$ iff $\exists o \subseteq X$, with $x \in o$, such that $\forall y \in o$ $M,y \Vdash \phi$.

Homotopy Theory

Homotopy Theory countenances the following identity, inversion, and concatenation morphisms, which are identified as continuous paths in the topology. The formal clauses, in the remainder of this section, evince how homotopic morphisms satisfy the properties of an equivalence relation.[9]

Reflexivity

$\forall x,y{:}A \forall p(p : x =_A y) : \tau(x,y,p)$, with A and τ designating types, 'x:A' interpreted as 'x is a token of type A', p $\bullet$ q is the concatenation of p and q, $\texttt{refl}_x$: $x =_A x$ for any x:A is a reflexivity element, $\prod_{x:A} B(x)$ is a dependent function type, and $e{:}\prod_{x:A}\tau(a, a, \texttt{refl}_\alpha)$ is a dependent function[10]:
$\forall \alpha{:}A \exists e(\alpha) : \tau(\alpha, \alpha, \texttt{refl}_\alpha)$;
$p,q : (x =_A y)$
$\exists r \in e : p =_{(x =_A y)} q$
$\exists \mu : r =_{(p =_{(x =_A y)} q)} s$.

Symmetry

$\forall A \forall x,y{:}A \exists H_\Sigma(x{=}y \rightarrow y{=}x)$
$H_\Sigma := p \mapsto p^{-1}$, such that
$\forall x{:}A(\texttt{refl}_x \equiv \texttt{refl}_x{}^{-1})$.

Transitivity

$\forall A \forall x,y{:}A \exists H_T(x{=}y \rightarrow y{=}z \rightarrow x{=}z)$
$H_T := p \mapsto q \mapsto p \bullet q$, such that
$\forall x{:}A[\texttt{refl}_x \bullet \texttt{refl}_x \equiv \texttt{refl}_x]$.

[8]In order to ensure that the Kripke semantics matches the topological semantics, X must further be Alexandrov; i.e., closed under arbitrary unions and intersections. Thanks here to Peter Milne.

[9]The definitions and proofs at issue can be found in the Univalent Foundations Program (2013: 2.0-2.1). A homotopy is a continuous mapping or path between a pair of functions.

[10]A dependent function is a function type 'whose codomain type can vary depending on the element of the domain to which the function is applied' (Univalent Foundations Program (op. cit.: §1.4).

Homotopic Abstraction

For all type families A,B, there is a homotopy:

H := [(f $\sim$ g) :$\equiv$ $\prod_{x:A}$(f(x) = g(x)], where
$\prod_{f:A\to B}$[(f $\sim$ f) $\wedge$ (f $\sim$ g $\to$ g $\sim$ f) $\wedge$ (f $\sim$ g $\to$ g $\sim$ h $\to$ f $\sim$ h)],
such that, via Voevodsky's (op. cit.) Univalence Axiom, for all type families A,B:U, there is a function:
`idtoeqv` : (A $=_U$ B) $\to$ (A $\simeq$ B),
which is itself an equivalence relation:
(A $=_U$ B) $\simeq$ (A $\simeq$ B).

Epistemic hyperintensions take the form,
$\mathtt{pri}(x) = \lambda s.[\![x]\!]^{s,s}$,
with s an epistemically possible state.

Abstraction principles for epistemic hyperintensions take, then, the form of a function type equivalence:

- $\forall$x[**A**f(x) = **A**g(x)] $\simeq$ [f(x) $\simeq$ g(x)].

Observational type theory countenances 'structure identity principles' which are type equivalences between identification types, and the theory is said to be observational because the type formation rules satisfy structure-preserving definitional equality. Higher observational type theory holds for propositional equality. 'The idea of higher observational type theory is to make these and analogous structural characterizations of identification types be part of their definitional inference rules, thus building the structure identity principle right into the rewrite rules of the type theory' (2023: https://ncatlab.org/nlab/show/higher+observational+type+theory). Shulman (2022) argues that higher observational type theory is one way to make the Univalence Axiom computable. Wright (2012c: 120) defines Hume's Principle as a pair of inference rules, and higher observational type theory might be one way to make first-order abstraction principles defined via inference rules, although not higher-order abstraction principles, computable. The Burali-Forti paradox could be circumvented, because the target abstraction principles would not be based on isomorphism like the Univalence Axiom. See Burali-Forti (1897/1967). Hodes (1984) and Hazen (1985) note that abstraction principles based on isomorphism with unrestricted comprehension entrain the paradox. I avoid the Burali-Forti paradox in my abstraction principle for two-dimensional hyperintensions because the definition is not augmented to second-order logic like in the abstractionist foundations of mathematics, is instead taken in isolation, and the definition defines functions from classes of epistemic states taken as actual to classes of metaphysical states to extensions.

3 Examples in Philosophy and Cognitive Science

The material adequacy of epistemic modal algebras as a fragment of the the language of thought is witnessed by the prevalence of possible worlds semantics – the model theory for which is algebraic (see Blackburn et al., 2001: ch. 5) – in cognitive psychology. Possible worlds model theory is availed of in the computational theory of mind, Bayesian perceptual psychology, and natural language semantics.

Marcus (2001) writes that: 'A multilayer perceptron consists of a set of *input nodes*, one or more sets of *hidden nodes*, and a set of *output nodes* ... These nodes are attached to each other through *weighted connections*; the weights of these connections are generally adjusted by some sort of *learning algorithm* ... *Nodes* are *units* that have activation [real] values ... *Input* and *output nodes* also have *meanings* or *labels* that are assigned by an external programmer ... The *meanings* of nodes (their labels) play no direct role in the computation: a network's computations depend only on the activation values of nodes and not on the labels of those nodes' (7-8). Both a single and multiple nodes can serve to represent the variables for a target domain. A target domain for variables is universally quantified over and the function is one-one, mapping a number of inputs to an equivalent number of outputs (35-36). Models of the above algebraic rules can be defined in both classical and weighted, connectionist systems (42-45). Temporal synchrony or dynamic variable-bindings are stored in short-term memory (56-57), while information relevant to long-term variable-bindings are stored in 'binary registers' i.e. 'bits' (41, 54-56). Marcus writes of bits that: 'Operations are defined in parallel over these sets of binary bits. When a programmer issues a command to copy the contents of variable **x** into variable **y**, the computer copies in parallel each of the bits that represents variable **x** into the corresponding bits that represent variable **y**' (41). Examples of the foregoing algebraic rules on variable-binding include both the syntactic concatenation of morphemes and noun phrase reduplication in linguistics (37-39, 70-72), as well as learning algorithms (45-48). Conditions on variable-binding are further examined, including treating the binding relation between variables and values as tensor products – i.e., an application of a multiplicative axiom for variables and their values treated as vectors (53-54, 105-106). In order to account for recursively formed, complex representations, which he refers to as structured propositions, Marcus argues instead that the syntax and semantics of such representations can be modeled via an ordered set of registers, which he refers to as 'treelets' (108).

A strengthened version of the algebraic rules on variable-binding can be accommodated in models of epistemic modal algebras, when the latter are augmented by cylindrifications, i.e., operators on the algebra simulating the treatment of quantification, and diagonal elements.[11] By contrast to Boolean

[11] See Henkin et al (op. cit.: 162-163) for the introduction of cylindric algebras, and for the axioms governing the cylindrification operators.

Algebras with Operators, which are propositional, cylindric algebras define first-order logics. Intuitively, valuation assignments for first-order variables are, in cylindric modal logics, treated as possible worlds of the model, while existential and universal quantifiers are replaced by, respectively, possibility and necessity operators ($\Diamond$ and $\Box$) (Venema, 2013: 249). For first-order variables, $\{v_i \mid i < \alpha\}$ with α an arbitrary, fixed ordinal, $v_i = v_j$ is replaced by a modal constant $\mathbf{a}_{i,j}$ (op. cit: 250). The following clauses are valid, then, for a model, M, of cylindric modal logic, with $E_{i,j}$ a monadic predicate and T_i for i,j $< \alpha$ a dyadic predicate:

M,w $\Vdash$ p $\iff$ w$\in$V(p);

M,w $\Vdash$ $\mathbf{a}_{i,j}$ $\iff$ w$\in E_{i,j}$;

M,w $\Vdash$ $\Diamond_i\psi$ $\iff$ there is a v with wT_iv and M,v $\Vdash$ ψ (252).

Cylindric frames need further to satisfy the following axioms (op. cit.: 254):

1. p $\rightarrow \Diamond_i$p
2. p $\rightarrow \Box_i\Diamond_i$p
3. $\Diamond_i\Diamond_i$p $\rightarrow \Diamond_i$p
4. $\Diamond_i\Diamond_j$p $\rightarrow \Diamond_j\Diamond_i$p
5. $\mathbf{a}_{i,i}$
6. $\Diamond_i(\mathbf{a}_{i,j} \wedge \text{p}) \rightarrow \Box_i(\mathbf{a}_{i,j} \rightarrow \text{p})$

[Translating the diagonal element and cylindric (modal) operator into, respectively, monadic and dyadic predicates and universal quantification: $\forall$xyz[($T_i$xy $\wedge$ $E_{i,j}$y $\wedge$ $T_i$xz $\wedge$ $E_{i,j}$z) $\rightarrow$ y = z] (op. cit.)]

7. $\mathbf{a}_{i,j} \iff \Diamond_k(\mathbf{a}_{i,k} \wedge \mathbf{a}_{k,j})$.

Finally, a cylindric modal algebra of dimension α is an algebra, $\mathbb{A} = \langle$A, +, $\bullet$, $-$, 0, 1, $\Diamond_i$, $\mathbf{a}_{ij}\rangle_{i,j<\alpha}$, where $\Diamond_i$ is a unary operator which is normal ($\Diamond_i 0 = 0$) and additive [$\Diamond_i$(x + y) = $\Diamond_i$x + $\Diamond_i$y] (257).

The philosophical interest of cylindric modal algebras to Marcus' cognitive models of algebraic variable-binding is that the valuation assignments to variables in the Epistemic Modal Algebra are epistemically possible worlds, while universal quantification is interpreted as epistemic necessitation. The interest of translating universal generalization into operations of epistemic necessitation is, finally, that − by identifying epistemic necessity with apriority − both the algebraic rules for variable-binding and the recursive formation of structured propositions can be seen as operations, the implicit knowledge of which is apriori.

In Bayesian perceptual psychology, the problem of underdetermination is resolved by availing of a gradational possible worlds model. The visual system is presented with a set of possibilities with regard, e.g., to the direction of a light source. So, for example, the direction of light might be originating from above, or it might be originating from below. The visual system computes the constancy, i.e. the likelihood that one of the possibilities is actual.[12] The computation of the perceptual constancy is an unconscious statistical inference,

[12]See Mamassian et al. (2002).

as anticipated by Helmholtz's (1878) conjecture.[13] The constancy places, then, a condition on the accuracy of the attribution of properties – such as boundedness and volume – to distal particulars.[14]

In the program of natural language semantics in empirical and philosophical linguistics, the common ground or 'context set' is the set of possibilities presupposed by a community of speakers.[15] Kratzer (1979: 121) refers to cases in which the above possibilities are epistemic as an 'epistemic conversational background', where the epistemic possibilities are a subset of objective or circumstantial possibilities (op. cit.). Modal operators are then defined on the space, encoding the effects of various speech acts in entraining updates on the context set.[16] So, e.g., assertion is argued to provide a truth-conditional update on the context set, whereas there are operator updates, the effects of which are not straightforwardly truth-conditional and whose semantic values must then be defined relative to an array of intensional parameters (including a context – agent, time, location, et al. – and a tuple of indices).

Finally, Epistemic Modal Algebra, as a fragment of the language of thought, is able to delineate the fundamental structure of the propositional attitudes targeted in 20th century philosophy; notably knowledge, belief, intentional action, and rational intuition. In Elohim (2024) I argue, e.g., that the types of intention – acting intentionally; referring to an intention as an explanation for one's course of action; and intending to pursue a course of action in the future – can be modeled as modal operators, whose semantic values are defined relative to an array of intensional parameters. E.g., an agent can be said to act intentionally iff her 'intention-in-action' receives a positive semantic value, where a necessary condition on the latter is that there is at least one world in her epistemic modal space at which – relative to a context of a particular time and location, which constrains the admissibility of her possible actions as defined at a first index, and which subsequently constrains the outcome thereof as defined at a second index – the intention is realized:

$[\![\text{Intenton-in-Action}(\phi)]\!]_w = 1$ only if $\exists$w'$[\![\phi]\!]^{w',c(=t,l),a,o} = 1$.

The agent's intention to pursue a course of action at a future time – i.e., her 'intention-for-the-future' – can receive a positive value only if there is a possible world and a future time, relative to which the possibility that a state, ϕ, is realized can be defined. Thus:

$[\![\text{Intention-for-the-future}(\phi)]\!]_w = 1$ only if $\exists$w'$\forall$t$\exists$t'[t< t' $\wedge$ $[\![\phi]\!]^{w',t'} = 1$].

In the setting of epistemic logic, epistemic necessity can further be modeled in a relational semantics encoding the properties of knowledge and belief (see

[13]For the history of the integration of algorithms and computational modeling into contemporary visual psychology, see Johnson-Laird (2004).

[14]See Burge (2010), and Rescorla (2013), for further discussion. A distinction ought to be drawn between unconscious perceptual representational states – as targeted in Burge (op. cit.) – and the inquiry into whether the properties of phenomenal consciousness have accuracy-conditions – where phenomenal properties are broadly construed, so as to include, e.g., color-phenomenal properties, as well as the property of being aware of one's perceptual states.

[15]See Stalnaker (1978).

[16]See Kratzer (op. cit.); Stalnaker (op. cit.); Lewis (1980); Heim (1992); Veltman (1996); von Fintel and Heim (2011); and Yalcin (2012).

Hintikka, 1962; Fagin et al., 1995; Meyer and van der Hoek, 1995; Williamson, 2009; Elohim, 2024). In Elohim (2024), I treat Gödel's (1953) conception of rational intuition as a modal operator in the setting of dynamic logic, and demonstrate how – via correspondence theory – the notion of 'intuition-of', i.e. a property of awareness of one's cognitive states, can be shown to be formally equivalent to the notion of 'intuition-that', i.e. a modal operator concerning the value of the propositional state at issue. The correspondence results between (fixed point) modal propositional and bisimulation-invariant first-order logic and monadic second-order logic are advanced in van Benthem (1983; 1984/2003) and Janin and Walukiewicz (1996). Availing of correspondence theory in order to account for the relationship between the notions of 'intuition-of' and 'intuition-that' resolves the inquiry about the foregoing posed by Parsons (1993: 233). As a dynamic interpretational modality, rational intuition can further serve as a guide to possible reinterpretations both of quantifier domains (see Fine, 2005) and of the intensions of mathematical vocabulary such as the membership-relation (see Uzquiano, 2015). This provides an account of Gödel's (op. cit.; 1961) suggestion that rational intuition can serve as a guide to conceptual elucidation.

4 Objections and Replies

Dean (2016) raises two issues for a proposal similar to the foregoing, namely that algorithms – broadly construed – can be defined via abstraction principles which specify equivalence relations between implementations of computational properties in isomorphic machines.[17] Dean's candidate abstraction principle for algorithms as abstracts is: that the algorithm implemented by M_1 = the algorithm implemented by M_2 iff $\mathrm{M}_1 \simeq \mathrm{M}_2$.[18] Both issues target the uniqueness of the algorithm purported to be identified by the abstraction principle.

The first issue generalizes Benacerraf (1965)'s contention that, in the reduction of number theory to set theory, there must be, and is not, a principled reason for which to prefer the identification of natural numbers with von Neumann ordinals (e.g., $2 = \{\emptyset,\{\emptyset\}\}$), rather than with Zermelo ordinals (e.g., $2 = \{\{\emptyset\}\}$). The issue is evinced by the choice of whether to define algorithms as isomorphic *iterations* of state transition functions (see Gurevich, 1999), or to define them as isomorphic *recursions* of functions which assign values to a partially ordered set of elements (see Moschovakis, op. cit.). Linnebo and Pettigrew (2014: 10) argue similarly that, for two 'non-rigid' structures which admit of non-trivial automorphisms, one can define a graph which belies their isomorphism. E.g., let an abstraction principle be defined for the isomorphism

[17] Fodor (2000: 105, n.4) and Piccinini (2004) note that the identification of mental states with their functional roles ought to be distinguished from identifying those functional roles with abstract computations. Conversely, a computational theory of mind need not be committed to the identification of abstract, computational operations with the functional organization of a machine. Identifying abstract computational properties with the functional organization of a creature's mental states is thus a choice point, in theories of the nature of mental representation.

[18] See Moschovakis (1998).

between S and S*, such that

$\forall$S,S*[$\mathbf{A}$S = $\mathbf{A}$S* iff $\langle$S, $R_1 \ldots R_n\rangle \simeq \langle$S*, $R^*_1 \ldots R^*_n\rangle$].

However, if there is a graph, G, such that:

S = $\{v_1, v_2\}$, and R = $\{\langle v_1, v_2\rangle, \langle v_2, v_1\rangle\}$,

then one can define an automorphism, f : G $\simeq$ G, such that $f(v_1) = v_2$ and $f(v_2) = v_1$, such that S* = $\{v_1\}$ while R* = $\{\langle v^*_1, v^*_1\rangle\}$. Then S* has one element via the automorphism, while S has two. So, S and S* are not, after all, isomorphic.

The second issue is that complexity is crucial to the identity criteria of algorithms. Two algorithms might be isomorphic, while the decidability of one algorithm is proportional to a deterministic *polynomial* function of the size of its input – with k a member of the natural numbers, N, and TIME referring to the relevant complexity class: $\bigcup_{k\in N}\mathrm{TIME}(n^k)$ – and the decidability of the second algorithm will be proportional to a deterministic *exponential* function of the size of its input – $\bigcup_{k\in N}\mathrm{TIME}(2^{n^k})$. The deterministic polynomial time complexity class is a subclass of the deterministic exponential time complexity class. However, there are problems decidable by algorithms only in polynomial time (e.g., the problem of primality testing, such that, for any two natural numbers, the numbers possess a greatest common divisor equal to 1), and only in exponential time (familiarly from logic, e.g., the problem of satisfiability – i.e., whether, for a given formula, there exists a model which can validate it – and the problem of validity – i.e. whether a satisfiable formula is valid).[19]

Both issues can be treated by noting that Dean's discussion targets abstraction principles for the very notion of a computable function, rather than for abstraction principles for cognitive computational properties. It is a virtue of homotopic abstraction principles for cognitive intensional functions that both the temporal complexity class to which the functions belong, and the applications of the model, are subject to variation. Variance in the cognitive roles, for which Epistemic Modal Algebra provides a model, will crucially bear on the nature of the representational properties unique to the interpretation of the intensional functions at issue. Thus, e.g., when the internal representations in the language of thought – as modeled by Epistemic Modal Algebra – subserve perceptual representational states, then their contents will be individuated by both the computational constancies at issue and the external, environmental properties – e.g., the properties of lightness and distance – of the perceiver.[20]

The examples of instances of Epistemic Modal Algebra – witnessed by the possible worlds models in Bayesian perceptual psychology, linguistics, and philosophy of mind – provide abductive support for the existence of the intensional functions specified in homotopic abstraction principles. The philosophical significance of independent, abductive support for the existence of epistemic modalities in the philosophy of mind and cognitive science is that the latter permits a circumvention of the objections to the abstractionist foundations of number

[19] For further discussion, see Dean (2021).

[20] The computational properties at issue can also be defined over non-propositional information states, such as cognitive maps possessed of geometric rather than logical structure. See, e.g., O'Keefe and Nadel (1978); Camp (2007); and Rescorla (2009).

theory that have accrued since its contemporary founding (see Wright, 1983). Eklund (2006) suggests, e.g., that the existence of the abstract objects which are the referents of numerical term-forming operators might need to be secured, prior to assuming that the abstraction principle for cardinal number is true. While Hale and Wright (2009) maintain, in response, that the truth of the relevant principles will be prior to the inquiry into whether the terms defined therein refer, they provide a preliminary endorsement of an 'abundant' conception of properties, according to which identifying the sense of a predicate will be sufficient for predicate reference.[21] One aspect of the significance of empirical and philosophical instances of models of Epistemic Modal Algebra is thus that, by providing independent, abductive support for the truth of the homotopic abstraction principles for epistemic hyperintensions, the proposal remains neutral on the status of 'sparse' versus 'abundant' conceptions of properties.[22] The truth of my first-order abstraction principle for hyperintensions is grounded in its being possibly recursively enumerable i.e. Turing computable and the Turing machine being physically implementable. Another aspect of the philosophical significance of possible worlds semantics being availed of in Bayesian vision science and empirical linguistics is that it belies the purportedly naturalistic grounds for Quine's (1963/1976) scepticism of *de re* modality.[23]

5 Concluding Remarks

In this essay, Voevodsky's Univalence Axiom and function type equivalence in Homotopy Type Theory were availed of, in order to specify an abstraction principle for hyperintensional, computational properties. The homotopic abstraction principle for epistemic hyperintensions provides an epistemic conduit for our knowledge of hyperintensions as abstract objects. The truth of my first-order abstraction principle for hyperintensions is grounded in its being possibly recursively enumerable i.e. Turing computable and the Turing machine being physically implementable. Because hyperintensions in Epistemic Modal Algebra are deployed as core models in the philosophy of mind, Bayesian visual psychology, and natural language semantics, there is independent abductive support for the truth of homotopic abstraction. Epistemic modality and hyperintensionality may thereby be recognized as both a compelling and a materially adequate

[21]For identity conditions on abundant properties – where the domain of properties, in the semantics of second-order logic, is a subset of the domain of objects, and the properties are definable in a metalanguage by predicates whose satisfaction-conditions have been fixed – see Hale (2013). For a generalization of the abundant conception, such that the domain of properties is isomorphic to the powerset of the domain of objects, see Cook (2014).

[22]Finding abductive support for abstraction principles is suggested by Rayo (2003). Hale and Wright (2009) and Wright (2012, 2014, 2016) argue that there is prima facie, default non-evidential entitlement to accept that abstraction principles are true.

[23]See Barcan Marcus (1993: 66-67), for a defense of Aristotelian essentialism, according to which essentialist modalities are temporal and 'causal and physical modalities'. Barcan Marcus writes, too, that 'What has gone wrong in recent discussions of essentialism is the assumption of surface synonymy between "is essentially" and *de re* occurrences of "is necessarily"' (60), and examines the distinction in various systems of quantified modal logic (Ch. 4, §III).

candidate for the fundamental structure of mental representational states, and as thus comprising a fragment of the language of thought.

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