Undefined 1 (2009) 1–5
IOS Press

Investigation of a neutrosophic group

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Abstract. We use a neutrosophic set, instead of an intuitionistic fuzzy because the neutrosophic set is more general, and it allows for independent and partial independent components $\mu(\chi), \gamma(\chi), \zeta(\chi)$, while in an intuitionistic fuzzy set, all components are totally dependent. In this article, we present and demonstrate the concept of neutrosophic invariant subgroups. We delve into the exploration of this notion to establish and study the neutrosophic quotient group. Further, we give the concept of a neutrosophic normal subgroup as a novel concept.

Keywords: Neutrosophic set, invariant sub-groups, normal sub-group.

1. Introduction

In dealing with many uncertainties that have emerged in a variety of real-life domains, such as sociology, economics, medical research, and the environment, traditional mathematical tools may not be suitable. Despite being well-known and frequently helpful methods for describing uncertainty. Therefore, Zadeh [21] initially introduced the notion of a fuzzy set as an alternative to classical sets, aiming to address the limitations in handling uncertainties. According to this definition, a fuzzy set is a function that assigns membership values graded over a unit interval. However, it has been recognized that this definition falls short when considering both membership and non-membership degrees. In order to tackle this inherent ambiguity, Atanassov [1] introduced a new theory known as intuitionistic fuzzy theory, which extends the concept of fuzzy sets. Nevertheless, the application of intuitionistic fuzzy sets has encountered certain challenges. To overcome issues related to ambiguity and inconsistency in information, Smarandache [17,16] proposed the concept of a neutrosophic set (NS). This novel approach aims to provide a solution to the problems arising from uncertain and inconsistent data.

Rosenfeld [13], Mukherjee [10], Biswas [2], Fathi et al. [5], Marashdeh et al. [8], Sharma [15], Smarandache et al. [6,7], Elrawy et al. [4], and several others extended the classical group theory to the fuzzy set, intuitionistic fuzzy set, and neutrosophic set. This is supported by multiple authors who applied the theory of fuzzy sets, intuitionistic fuzzy sets, and neutrosophic sets to various algebraic structures.

The intersection and union relations of neutrosophic sets has been discussed so far from three main perspectives. The first definitions are offered by Smarandache [16,18,19] and are signified by $\cap_1$ and $\cup_1$. The sec-

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ond definitions are given in [20] and are signified by $\cap_2$ and $\cup_2$. The third definitions are offered in [22] and are signified by $\cap_3$ and $\cup_3$. In addition, Vildan et al. [3] presented a new approach to a neutrosophic sub-group that based on the second viewpoint. Moreover, El Rawy et al. [4] have established and studied another approach of a neutrosophic sub-group and level sub-group based on the first viewpoint. We have improved the definition of the neutrosophic subgroup in [3]. Since in paper [3] the authors use $\min / \min / \max$ and $\geqslant / \geqslant / \leqslant$ while we use $\min / \max / \max$ and $\geqslant / \leqslant / \leqslant$, respectively. Our approach is better since the components: μ is considered of positive quality, while γ and ζ of negative quality, so γ and ζ should have the same operation $\max / \max$ and $\leqslant / \leqslant$, respectively.

Motivated by some of the above work, the goal of this paper is to examine the neutrosophic sub-group and normal sub-group through the study and investigation of various relevant properties and theorems. In the next section, we afford some basic, important definitions for this study. In section 3, the notion of neutrosophic invariant sub-groups is presented and studied. In section 4, we define left and right neutrosophic coset. Also, neutrosophic normal sub-group is established and studied. In the last section, we draw some conclusions.

2. Basic concepts

We discuss a few concepts and results that we utilize in the following section in this section.

Definition 2.1 *[17,20] Let E be the universe set. An NS $\mathcal{N}$ on E is defined as:*

$$\mathcal{N} = \{< \chi, \mu(\chi), \gamma(\chi), \zeta(\chi) >: \chi \in E\},$$

with $\mu, \gamma, \zeta : E \to]0, 1[$. Also, an NS $\mathcal{N}$ on E is called single valued neutrosophic set for any $\chi \in E$ one has $0 \leqslant \mu(\chi) + \gamma(\chi) + \zeta(\chi) \leqslant 3$ and of course $\mu(\chi), \gamma(\chi), \zeta(\chi) \in [0, 1]$.

Definition 2.2 *[16,18,19] Let $\mathcal{N}_1$ and $\mathcal{N}_2$ be two NSs on E. Then we define the follows:*

1. $\mathcal{N}_1 \cup_1 \mathcal{N}_2 = \{< \chi, max(\mu_1(\chi), \mu_2(\chi)), min(\gamma_1(\chi), \gamma_2(\chi)), min(\zeta_1(\chi), \zeta_2(\chi)) >: \chi \in E\}$,
2. $\mathcal{N}_1 \cap_1 \mathcal{N}_2 = \{< \chi, min(\mu_1(\chi), \mu_2(\chi)), max(\gamma_1(\chi), \gamma_2(\chi)), max(\zeta_1(\chi), \zeta_2(\chi)) >: \chi \in E\}$.

Definition 2.3 *[4] Let $\mathcal{G}$ be a group. A neutrosophic subset $\Upsilon = \{< \chi, \mu(\chi), \gamma(\chi), \zeta(\chi) >: \chi \in \mathcal{G}\}$ of $\mathcal{G}$ is called a neutrosophic sub-group of $\mathcal{G}$ if the following conditions are satisfied:*

$$(i) \begin{cases} \mu(\chi\varsigma) \geqslant \min(\mu(\chi), \mu(\varsigma)) \\ \gamma(\chi\varsigma) \leqslant \max(\gamma(\chi), \gamma(\varsigma)) \\ \zeta(\chi\varsigma) \leqslant \max(\zeta(\chi), \zeta(\varsigma)) \end{cases}$$

$$(ii) \begin{cases} \mu\left(\chi^{-1}\right) \geqslant \mu(\chi) \\ \gamma\left(\chi^{-1}\right) \leqslant \gamma(\chi) \\ \zeta\left(\chi^{-1}\right) \leqslant \zeta(\chi) \end{cases}$$

where $\chi, \varsigma \in \mathcal{G}$.

In what follows $\mathcal{S}(\mathcal{G})$ will denote a family of all neutrosophic sets in $\mathcal{G}$.

3. Neutrosophic invariant sub-groups

In this section, we define and study the concept of neutrosophic invariant sub-groups.

It is useful to present the next definition before introduce the notion of neutrosophic invariant sub-groups.

Definition 3.1 *Let $\mathcal{G}$ be a group with a binary operation $\star$, then we define $\Upsilon_1 * \Upsilon_2$ as follows*

$$(\mu_1 * \mu_2)(u) = \begin{cases} \sup_{v\star w=u} min(\mu_1(v), \mu_2(w)) & \textit{if } v\star w = u \\ 0 & \textit{if } v\star w \neq u \end{cases}$$

$$(\gamma_1 * \gamma_2)(u) = \begin{cases} \inf_{v\star w=u} max(\gamma_1(v), \gamma_2(w)) & \textit{if } v\star w = u \\ 0 & \textit{if } v\star w \neq u \end{cases}$$

$$(\zeta_1 * \zeta_2)(u) = \begin{cases} \inf_{v\star w=u} max(\zeta_1(v), \zeta_2(w)) & \textit{if } v\star w = u \\ 0 & \textit{if } v\star w \neq u \end{cases}$$

where $u, v, w \in \mathcal{G}$,
$\Upsilon_1 = \{< \chi, \mu_1(\chi), \gamma_1(\chi), \zeta_1(\chi) >: \chi \in \mathcal{G}\}$ *and* $\Upsilon_2 = \{< \chi, \mu_2(\chi), \gamma_2(\chi), \zeta_2(\chi) >: \chi \in \mathcal{G}\}$.

Definition 3.2 *Let Υ be a neutrosophic sub-group of a group $\mathcal{G}$, then we call Υ a neutrosophic invariant sub-group if*

$$\mu(\chi \star \varsigma) = \mu(\varsigma \star \chi),$$

$$\gamma(\chi \star \varsigma) = \gamma(\varsigma \star \chi),$$

$$\zeta(\chi \star \varsigma) = \zeta(\varsigma \star \chi),$$

where $\chi,\ \varsigma \in \mathcal{G}$.

Note that, when $\mathcal{G}$ is abelian group, then we find every neutrosophic sub-group of a group $\mathcal{G}$ is a neutrosophic invariant sub-group.

Example 3.3 Assume that $\mathcal{G} = Z_3$ is a group under a binary operation $\otimes_3$, Define a neutrosophic sub-group $\Upsilon = \{< 0, 0.7, 0.1, 0.2 >, < 1, 0.6, 0.3, 0.5 >, < 2, 0.6, 0.3, 0.5 >\}$ of Z_3. Since Z_3 is abelian group, the Υ is a neutrosophic invariant sub-group.

Proposition 3.4 *Presume Υ_1 is a neutrosophic invariant sub-group,*
(1) $\Upsilon_1 * \Upsilon_2 = \Upsilon_2 * \Upsilon_1$, *for any* $\Upsilon_2 \in \mathcal{S}(\mathcal{G})$,
(2) *When Υ_2 is a neutrosophic sub-group, then also $\Upsilon_2 * \Upsilon_1$ is a neutrosophic sub-group.*

Proof. (1) It is obvious from Definition 3.2.
(2) Suppose that Υ_2 is a neutrosophic sub-group, then we prove $\Upsilon_2 * \Upsilon_1$ is a neutrosophic sub-group by check all axioms of Definition 2.3 as follows

$$(i)\begin{cases}
(\mu_2*\mu_1)(up) = \sup\limits_{v_1v_2\star w_1w_2=up} min(\mu_1(v_1v_2),\mu_2(w_1w_2)) \\
\qquad \geq \sup\limits_{v_1\star v_2\star w_1\star w_2=up} min(\mu_1(v_1)\mu_1(v_2),\mu_2(w_1)\mu(w_2)) \\
\qquad = \mu_2(u)*\mu_1(p). \\
(\gamma_2*\gamma_1)(up) = \inf\limits_{v_1v_2\star w_1w_2=up} max(\mu_1(v_1v_2),\mu_2(w_1w_2)) \\
\qquad \leq \inf\limits_{v_1\star v_2\star w_1\star w_2=up} max(\gamma_1(v_1)\gamma_1(v_2),\gamma_2(w_1)\gamma(w_2)) \\
\qquad = \gamma_2(u)*\gamma_1(p). \\
(\zeta_2*\zeta_1)(up) = \inf\limits_{v_1v_2\star w_1w_2=up} max(\zeta_1(v_1v_2),\zeta_2(w_1w_2)) \\
\qquad \leq \inf\limits_{v_1\star v_2\star w_1\star w_2=up} max(\zeta_1(v_1)\zeta_1(v_2),\zeta_2(w_1)\zeta(w_2)) \\
\qquad = \zeta_2(u)*\zeta_1(p).
\end{cases}$$

$$(ii)\begin{cases}
(\mu_2*\mu_1)(u^{-1}) = \sup\limits_{v\star w=u^{-1}} min(\mu_2(v),\mu_1(w)) \\
\qquad = \sup\limits_{w^{-1}\star v^{-1}=u} min(\mu_2((v^{-1})^{-1}),\mu_1((w^{-1})^{-1})) \\
\qquad \geq \sup\limits_{w^{-1}\star v^{-1}=u} min(\mu_2(v^{-1}),\mu_1(w^{-1})) \\
\qquad = (\mu_2*\mu_1)(u^{-1}). \\
(\gamma_2*\gamma_1)(u^{-1}) = \inf\limits_{v\star w=u^{-1}} max(\mu_2(v),\mu_1(w)) \\
\qquad = \inf\limits_{w^{-1}\star v^{-1}=u} max(\gamma_2((v^{-1})^{-1}),\gamma_1((w^{-1})^{-1})) \\
\qquad \leq \inf\limits_{w^{-1}\star v^{-1}=u} max(\gamma_2(v^{-1}),\gamma_1(w^{-1})) \\
\qquad = (\gamma_2*\gamma_1)(u^{-1}). \\
(\zeta_2*\zeta_1)(u^{-1}) = \inf\limits_{v\star w=u^{-1}} max(\zeta_2(v),\zeta_1(w)) \\
\qquad = \inf\limits_{w^{-1}\star v^{-1}=u} max(\zeta_2((v^{-1})^{-1}),\zeta_1((w^{-1})^{-1})) \\
\qquad \leq \inf\limits_{w^{-1}\star v^{-1}=u} max(\zeta_2(v^{-1}),\zeta_1(w^{-1})) \\
\qquad = (\zeta_2*\zeta_1)(u^{-1}).
\end{cases}$$

In what follows, suppose that $e \in \mathcal{G}$ is an identity, and $\varUpsilon$ is non-empty set belong to $\mathcal{S}(\mathcal{G})$. Also, we define the set $\mathcal{G}_\varUpsilon$ as follows

$$\mathcal{G}_\varUpsilon = \{u \in \mathcal{G} : \ \mu(u) = \mu(e), \ \gamma(u) = \gamma(e), \ \zeta(u) = \zeta(e)\}.$$

Proposition 3.5 *Presume $\varUpsilon$ is a neutrosophic invariant sub-group of $\mathcal{G}$, then $\mathcal{G}_\varUpsilon$ is an invariant sub-group of $\mathcal{G}$.*

Proof. Assume that $\mathcal{G}_\varUpsilon$ is a non-empty set. Let $u \in \mathcal{G}_\varUpsilon$ and $v \in \mathcal{G}$, then we have

$$\begin{aligned}\mu(vuv^{-1}) &= \mu((vu)v^{-1}) \\ &= \mu((uv)v^{-1}) \\ &= \mu(u(vv^{-1})) = \mu(u) = \mu(e).\end{aligned}$$

Similarly, we find $\gamma(vuv^{-1}) = \gamma(e)$ and $\zeta(vuv^{-1}) = \zeta(e)$. Therefore, $vuv^{-1} \in \mathcal{G}_\varUpsilon$.

Next, we investigate a definition of neutrosophic invariant sub-group to define the neutrosophic quotient group.

Definition 3.6 Assume that $\varUpsilon$ is a neutrosophic invariant sub-group of $\mathcal{G}$. Then the quotient $\mathcal{G} \diagup \mathcal{G}_\varUpsilon$ is called the neutrosophic quotient group of $\mathcal{G}$.

Now, by using the definition of a homomorphism between two groups $\mathcal{F} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$, we have the following definition.

Definition 3.7 Let $\varUpsilon_1 \in \mathcal{S}(\mathcal{G}_1)$ and $\varUpsilon_2 \in \mathcal{S}(\mathcal{G}_2)$. Then we define $\mathcal{F}(\varUpsilon_1)$ in $\mathcal{S}(\mathcal{G}_2)$ as follows

$$\begin{cases} (\mathcal{F}(\mu_1))(\varsigma) = \begin{cases} \sup_{\mathcal{F}(\chi)=\varsigma} \mu_1(\chi) & \text{when } \mathcal{F}^{-1}(\varsigma) \neq \Phi \\ 0 & \text{when } \mathcal{F}^{-1}(\varsigma) = \Phi \end{cases} \\ (\mathcal{F}(\gamma_1))(\varsigma) = \begin{cases} \inf_{\mathcal{F}(\chi)=\varsigma} \gamma_1(\chi) & \text{when } \mathcal{F}^{-1}(\varsigma) \neq \Phi \\ 0 & \text{when } \mathcal{F}^{-1}(\varsigma) = \Phi \end{cases} \\ (\mathcal{F}(\zeta_1))(\varsigma) = \begin{cases} \inf_{\mathcal{F}(\chi)=\varsigma} \zeta_1(\chi) & \text{when } \mathcal{F}^{-1}(\varsigma) \neq \Phi \\ 0 & \text{when } \mathcal{F}^{-1}(\varsigma) = \Phi \end{cases} \end{cases}$$

and we define $\mathcal{F}^{-1}(\varUpsilon_2)$ by the following $\mathcal{F}^{-1}(\varUpsilon_2)(\chi) = \varUpsilon_2(\mathcal{F}(\chi))$, where $\chi \in \mathcal{G}_1$ as a fuzzy set [12].

Again, we suppose that $\mathcal{G}_1$ and $\mathcal{G}_2$ are groups and $\mathcal{F} : \mathcal{G}_1 \longrightarrow \mathcal{G}_2$ is an onto group homomorphism, then we get the following:
When $\varUpsilon_1$ is a neutrosophic sub-group of $\mathcal{G}_1$, then also $\mathcal{F}(\varUpsilon_1)$ is a neutrosophic sub-group of $\mathcal{G}_2$ for all $v \in \mathcal{G}_2$.

$$\begin{aligned}(\mathcal{F}(\mu_1))(v^{-1}) &= \sup_{\mathcal{F}(w)=v^{-1}} \mu_1(w)\\ &= \sup_{\mathcal{F}(w^{-1})=v} \mu_1((w^{-1})^{-1})\\ &\geq \sup_{\mathcal{F}(w^{-1})=v} \mu_1(w^{-1})\\ &= \sup_{\mathcal{F}(u)=v} \mu_1(u) = (\mathcal{F}(\mu_1))(v).\\ (\mathcal{F}(\gamma_1))(v^{-1}) &= \inf_{\mathcal{F}(w)=v^{-1}} \gamma_1(w)\\ &= \inf_{\mathcal{F}(w^{-1})=v} \gamma_1((w^{-1})^{-1})\\ &\leq \inf_{\mathcal{F}(w^{-1})=v} \gamma_1(w^{-1})\\ &= \inf_{\mathcal{F}(u)=v} \gamma_1(u) = (\mathcal{F}(\gamma_1))(v).\\ (\mathcal{F}(\zeta_1))(v^{-1}) &= \inf_{\mathcal{F}(w)=v^{-1}} \zeta_1(w)\\ &= \inf_{\mathcal{F}(w^{-1})=v} \zeta_1((w^{-1})^{-1})\\ &\leq \inf_{\mathcal{F}(w^{-1})=v} \zeta_1(w^{-1})\\ &= \inf_{\mathcal{F}(u)=v} \zeta_1(u) = (\mathcal{F}(\zeta_1))(v).\end{aligned}$$

The same way we have $\mathcal{F}^{-1}(\Upsilon_2)$ in $\mathcal{G}_1$, when Υ_2 is a neutrosophic sub-group of $\mathcal{G}_2$.

Proposition 3.8 *Assume that $\ell : \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{G}_{\Upsilon_1}$ is a natural map. When Υ_1 is a neutrosophic invariant group of $\mathcal{G}$, then we get $\ell^{-1}(\ell(\Upsilon_2)) = \mathcal{G}_{\Upsilon_1} * \Upsilon_2$*

Proof. Suppose that $u \in \mathcal{G}$, then we get

$$\begin{aligned}\ell^{-1}(\ell(\mu_2)) &= (\ell(\mu_2))(\ell(u))\\ &= \sup_{\ell(v)=\ell(u)} \mu_2(v)\\ &= \sup_{uv^{-1}\in\mathcal{G}_{\Upsilon_1}} \mu_2(v).\\ \ell^{-1}(\ell(\gamma_2)) &= (\ell(\gamma_2))(\ell(u))\\ &= \inf_{\ell(v)=\ell(u)} \gamma_2(v)\\ &= \inf_{uv^{-1}\in\mathcal{G}_{\Upsilon_1}} \gamma_2(v).\\ \ell^{-1}(\ell(\zeta_2)) &= (\ell(\zeta_2))(\ell(u))\\ &= \inf_{\ell(v)=\ell(u)} \zeta_2(v)\\ &= \inf_{uv^{-1}\in\mathcal{G}_{\Upsilon_1}} \zeta_2(v).\end{aligned}$$

Also we have

$$\begin{aligned}(\mathcal{G}_{\mu_1} * \mu_2)(u) &= \sup_{v\star w=u} min(\mathcal{G}_{\mu_1}(w), \mu_2(v))\\ &= \sup_{w=uv^{-1}\in\mathcal{G}_{\mu_1}} \mu_2(v).\\ (\mathcal{G}_{\gamma_1} * \gamma_2)(u) &= \inf_{v\star w=u} min(\mathcal{G}_{\gamma_1}(w), \gamma_2(v))\\ &= \inf_{w=uv^{-1}\in\mathcal{G}_{\gamma_1}} \gamma_2(v).\\ (\mathcal{G}_{\zeta_1} * \zeta_2)(u) &= \inf_{v\star w=u} min(\mathcal{G}_{\zeta_1}(w), \zeta_2(v))\\ &= \inf_{w=uv^{-1}\in\mathcal{G}_{\zeta_1}} \zeta_2(v).\end{aligned}$$

Therefore, the proposition is proved.

4. Neutrosophic normal sub-group

In this section, we investigate the concept of a neutrosophic sub-group [4] to define left and right neutrosophic cosets. Also, we define and study the neutro-

sophic normal sub-group.
Firstly, it is useful to define and study the right and left neutrosophic cosets.

Definition 4.1 Let Υ be a neutrosophic set of $\mathcal{G}$ and we define the functions ${}_\chi\Gamma : \mathcal{G} \longrightarrow \mathcal{G}$ and $\Gamma_\chi : \mathcal{G} \longrightarrow \mathcal{G}$ where ${}_\chi\Gamma(\varsigma) = \chi\varsigma$ and $\Gamma_\chi(\varsigma) = \varsigma\chi$, receptively. A neutrosophic right (left) coset is defined as follows

$$\begin{cases} \mu\chi = \Gamma_\chi(\mu) \;\; (\chi\mu =_\chi \Gamma(\mu)) \\ \gamma\chi = \Gamma_\chi(\gamma) \;\; (\chi\gamma =_\chi \Gamma(\gamma)) \\ \zeta\chi = \Gamma_\chi(\zeta) \;\; (\chi\zeta =_\chi \Gamma(\zeta)) \end{cases}$$

where $\chi, \varsigma \in \mathcal{G}$.

Remark 4.2 Clearly, in the case of neutrosophic right and left cosets, we find the following:

$$\begin{cases} (\mu\chi)(\varsigma) = \mu(\varsigma\chi^{-1}) & \text{and } (\chi\mu)(\varsigma) = \mu(\chi^{-1}\varsigma) \\ (\gamma\chi)(\varsigma) = \gamma(\varsigma\chi^{-1}) & \text{and } (\chi\gamma)(\varsigma) = \gamma(\chi^{-1}\varsigma) \\ (\zeta\chi)(\varsigma) = \zeta(\varsigma\chi^{-1}) & \text{and } (\chi\zeta)(\varsigma) = \zeta(\chi^{-1}\varsigma) \end{cases}$$

for each $\chi, \varsigma \in \mathcal{G}$.

Proposition 4.3 *Let Υ be a neutrosophic set of $\mathcal{G}$, then the next axioms are equivalent:*

$$(i) \begin{cases} \mu(\chi\varsigma\chi^{-1}) \geq \mu(\varsigma) \\ \gamma(\chi\varsigma\chi^{-1}) \leq \gamma(\varsigma) \\ \zeta(\chi\varsigma\chi^{-1}) \leq \zeta(\varsigma) \end{cases}$$

$$(ii) \begin{cases} \mu(\chi\varsigma\chi^{-1}) = \mu(\varsigma) \\ \gamma(\chi\varsigma\chi^{-1}) = \gamma(\varsigma) \\ \zeta(\chi\varsigma\chi^{-1}) = \zeta(\varsigma) \end{cases}$$

$$(iii) \begin{cases} \mu(\chi\varsigma) = \mu(\varsigma\chi) \\ \gamma(\chi\varsigma) = \gamma(\varsigma\chi) \\ \zeta(\chi\varsigma) = \zeta(\varsigma\chi) \end{cases}$$

$$(iv) \begin{cases} \chi\mu = \mu\chi \\ \chi\gamma = \gamma\chi \\ \chi\zeta = \zeta\chi \end{cases}$$

$$(v) \begin{cases} \chi\mu\chi^{-1} = \mu \\ \chi\gamma\chi^{-1} = \gamma \\ \chi\zeta\chi^{-1} = \zeta \end{cases}$$

for each $\chi, \varsigma \in \mathcal{G}$.

Proof. From Definition 2.3 and Remark 4.2, the proof is straightforward.

Proposition 4.4 *Let Υ be a neutrosophic sub-group of $\mathcal{G}$, then $\chi\Upsilon\chi^{-1}$ is a neutrosophic sub-group of $\mathcal{G}$ $\forall\, \chi \in \mathcal{G}$.*

Proof. Suppose that Υ be a neutrosophic sub-group of $\mathcal{G}$ and $\chi \in \mathcal{G}$, then we check the axioms of Definition 2.3 as follows:

$$(i) \begin{cases} \kappa\mu\kappa^{-1}(\chi\varsigma) = \mu(\kappa^{-1}(\chi\varsigma)\kappa) \\ \qquad = \mu(\kappa^{-1}(\chi\kappa\kappa^{-1}\varsigma)\kappa) \\ \qquad = \mu((\kappa^{-1}\chi\kappa)(\kappa^{-1}\varsigma\kappa)) \\ \qquad \geqslant \min(\mu(\kappa^{-1}\chi\kappa), \mu(\kappa^{-1}\varsigma\kappa)). \\ \kappa\gamma\kappa^{-1}(\chi\varsigma) = \gamma(\kappa^{-1}(\chi\varsigma)\kappa) \\ \qquad = \gamma(\kappa^{-1}(\chi\kappa\kappa^{-1}\varsigma)\kappa) \\ \qquad = \gamma((\kappa^{-1}\chi\kappa)(\kappa^{-1}\varsigma\kappa)) \\ \qquad \leqslant \max(\gamma(\kappa^{-1}\chi\kappa), \gamma(\kappa^{-1}\varsigma\kappa)). \\ \kappa\zeta\kappa^{-1}(\chi\varsigma) = \zeta(\kappa^{-1}(\chi\varsigma)\kappa) \\ \qquad = \zeta(\kappa^{-1}(\chi\kappa\kappa^{-1}\varsigma)\kappa) \\ \qquad = \zeta((\kappa^{-1}\chi\kappa)(\kappa^{-1}\varsigma\kappa)) \\ \qquad \leqslant \max(\zeta(\kappa^{-1}\chi\kappa), \zeta(\kappa^{-1}\varsigma\kappa)). \end{cases}$$

$$(ii)\begin{cases}\kappa\mu\kappa^{-1}\left(\chi^{-1}\right)=\mu(\kappa^{-1}\left(\chi^{-1}\right)\kappa)\\ \qquad\geqslant\mu(\kappa^{-1}\chi\kappa).\\ \kappa\gamma\kappa^{-1}\left(\chi^{-1}\right)=\gamma(\kappa^{-1}\left(\chi^{-1}\right)\kappa)\\ \qquad\leqslant\gamma(\kappa^{-1}\chi\kappa).\\ \kappa\zeta\kappa^{-1}\left(\chi^{-1}\right)=\zeta(\kappa^{-1}\left(\chi^{-1}\right)\kappa)\\ \qquad\leqslant\zeta(\kappa^{-1}\chi\kappa).\end{cases}$$

where $\kappa,\chi,\varsigma\in\mathcal{G}$.

Definition 4.5 When a neutrosophic sub-group of Υ satisfies one of the axioms in Proposition 4.3, then Υ is called a neutrosophic normal sub-group.

Example 4.6 Consider $\mathcal{X}=\{\pm1,\pm i\}$ is a group under usual multiplication and $\mathcal{A}=\{\pm1,\pm i\}$ sub-group of $\mathcal{X}$. Now, we define a neutrosophic sub-group $\Upsilon=\{<1,0.2,0.5,0.3>,<-1,0.2,0.5,0.3>\}$ of $\mathcal{A}$. Also, it is easy to show any axioms in Proposition 4.3. Therefore, Υ neutrosophic normal sub-group.

Proposition 4.7 *Any intersection of two neutrosophic normal sub-groups of $\mathcal{G}$ is also a neutrosophic normal sub-group of $\mathcal{G}$.*

Proof. Suppose that Υ_1 and Υ_2 are any two neutrosophic normal sub-group of $\mathcal{G}$, then firstly we find $\Upsilon_1\cap_1\Upsilon_2$ is a neutrosophic sub-group of $\mathcal{G}$ [4]. Secondly, we show $\Upsilon_1\cap_1\Upsilon_2$ is a neutrosophic normal sub-group of $\mathcal{G}$. Let $\chi,\varsigma\in\mathcal{G}$, then

$$\begin{cases}(i)\quad(\mu_1\cap_1\mu_2)(\chi\varsigma\chi^{-1})=\mu_1(\chi\varsigma\chi^{-1})\wedge\mu_2(\chi\varsigma\chi^{-1})\\ \qquad\geqslant\mu_1(\varsigma)\wedge\mu_2(\varsigma)\\ \qquad\geqslant(\mu_1\cap_1\mu_2)(\varsigma).\\ (ii)\quad(\gamma_1\cap_1\gamma_2)(\chi\varsigma\chi^{-1})=\gamma_1(\chi\varsigma\chi^{-1})\vee\gamma_2(\chi\varsigma\chi^{-1})\\ \qquad\leqslant\gamma_1(\varsigma)\vee\gamma_2(\varsigma)\\ \qquad\leqslant(\gamma_1\cap_1\gamma_2)(\varsigma).\\ (iii)\quad(\zeta_1\cap_1\zeta_2)(\chi\varsigma\chi^{-1})=\zeta_1(\chi\varsigma\chi^{-1})\vee\zeta_2(\chi\varsigma\chi^{-1})\\ \qquad\leqslant\zeta_1(\varsigma)\vee\zeta_2(\varsigma)\\ \qquad\leqslant(\zeta_1\cap_1\zeta_2)(\varsigma).\end{cases}$$

Therefore, $\Upsilon_1\cap_1\Upsilon_2$ is a neutrosophic normal sub-group of $\mathcal{G}$ and the proposition is claim.

Proposition 4.8 *Assume that $\mathcal{F}:\mathcal{G}\longrightarrow\mathcal{G}^{'}$ is a group homomorphism, then*

(1) *$\mathcal{F}^{-1}(\Upsilon)$ is a neutrosophic normal sub-group of $\mathcal{G}$ when Υ is a neutrosophic normal sub-group of $\mathcal{G}^{'}$.*
(2) *$\mathcal{F}(\Upsilon)$ is a neutrosophic normal sub-group of $\mathcal{G}^{'}$ when Υ is a neutrosophic normal sub-group of $\mathcal{G}$ and $\mathcal{F}$ is an onto.*

Proof. (1) Let Υ be a neutrosophic normal sub-group of $\mathcal{G}^{'}$. Firstly, we explain that $\mathcal{F}^{-1}(\Upsilon)$ is a neutrosophic sub-group as follows

$$(i)\begin{cases}\mathcal{F}^{-1}(\mu(\chi\varsigma))=\mu(\mathcal{F}(\chi\varsigma))\\ \qquad=\mu(\mathcal{F}(\chi)\mathcal{F}(\varsigma))\\ \qquad\geqslant\min(\mu(\mathcal{F}(\chi)),\mu(\mathcal{F}(\varsigma)))\\ \qquad=\mathcal{F}^{-1}\min(\mu(\chi),\mu(\varsigma)).\\ \mathcal{F}^{-1}(\gamma(\chi\varsigma))=\gamma(\mathcal{F}(\chi\varsigma))\\ \qquad=\gamma(\mathcal{F}(\chi)\mathcal{F}(\varsigma))\\ \qquad\leqslant\max(\gamma(\mathcal{F}(\chi)),\gamma(\mathcal{F}(\varsigma)))\\ \qquad=\mathcal{F}^{-1}\max(\gamma(\chi),\gamma(\varsigma)).\\ \mathcal{F}^{-1}(\zeta(\chi\varsigma))=\zeta(\mathcal{F}(\chi\varsigma))\\ \qquad=\zeta(\mathcal{F}(\chi)\mathcal{F}(\varsigma))\\ \qquad\leqslant\max(\zeta(\mathcal{F}(\chi)),\zeta(\mathcal{F}(\varsigma)))\\ \qquad=\mathcal{F}^{-1}\max(\zeta(\chi),\zeta(\varsigma)).\end{cases}$$

$$(ii)\begin{cases}\mathcal{F}^{-1}(\mu\left(\chi^{-1}\right))=\mu\mathcal{F}\left(\chi^{-1}\right)\\ \qquad=\mu((\mathcal{F}\left(\chi\right))^{-1})\\ \qquad\geqslant\mu(\mathcal{F}(\chi))\\ \qquad=\mathcal{F}^{-1}(\mu(\chi)).\\ \mathcal{F}^{-1}(\gamma\left(\chi^{-1}\right))=\gamma\mathcal{F}\left(\chi^{-1}\right)\\ \qquad=\gamma((\mathcal{F}\left(\chi\right))^{-1})\\ \qquad\leqslant\gamma(\mathcal{F}(\chi))\\ \qquad=\mathcal{F}^{-1}(\gamma(\chi)).\\ \mathcal{F}^{-1}(\zeta\left(\chi^{-1}\right))=\zeta\mathcal{F}\left(\chi^{-1}\right)\\ \qquad=\zeta((\mathcal{F}\left(\chi\right))^{-1})\\ \qquad\leqslant\zeta(\mathcal{F}(\chi))\\ \qquad=\mathcal{F}^{-1}(\zeta(\chi)).\end{cases}$$

where $\chi,\varsigma\in\mathcal{G}$. Secondly, we show $\mathcal{F}^{-1}(\Upsilon)$ is a neutrosophic normal sub-group

$$\begin{cases}\mathcal{F}^{-1}(\mu)(\chi\varsigma\chi^{-1})=\mu(\mathcal{F}(\chi\varsigma\chi^{-1}))\\ \qquad=\mu(\mathcal{F}(\chi)\mathcal{F}(\varsigma)\mathcal{F}(\chi^{-1}))\\ \qquad=\mu(\mathcal{F}(\chi)\mathcal{F}(\varsigma)\mathcal{F}(\chi)^{-1})\\ \qquad=\mu(\mathcal{F}(\varsigma))\\ \qquad=\mathcal{F}^{-1}(\mu)(\varsigma).\\ \mathcal{F}^{-1}(\gamma)(\chi\varsigma\chi^{-1})=\gamma(\mathcal{F}(\chi\varsigma\chi^{-1}))\\ \qquad=\gamma(\mathcal{F}(\chi)\mathcal{F}(\varsigma)\mathcal{F}(\chi^{-1}))\\ \qquad=\gamma(\mathcal{F}(\chi)\mathcal{F}(\varsigma)\mathcal{F}(\chi)^{-1})\\ \qquad=\gamma(\mathcal{F}(\varsigma))\\ \qquad=\mathcal{F}^{-1}(\gamma)(\varsigma).\\ \mathcal{F}^{-1}(\zeta)(\chi\varsigma\chi^{-1})=\zeta(\mathcal{F}(\chi\varsigma\chi^{-1}))\\ \qquad=\zeta(\mathcal{F}(\chi)\mathcal{F}(\varsigma)\mathcal{F}(\chi^{-1}))\\ \qquad=\zeta(\mathcal{F}(\chi)\mathcal{F}(\varsigma)\mathcal{F}(\chi)^{-1})\\ \qquad=\zeta(\mathcal{F}(\varsigma))\\ \qquad=\mathcal{F}^{-1}(\zeta)(\varsigma).\end{cases}$$

Thus, $\mathcal{F}^{-1}(\Upsilon)$ is a neutrosophic normal sub-group of $\mathcal{G}$.

(2) Suppose that Υ is a neutrosophic normal sub-group of $\mathcal{G}$ and $\mathcal{F}$ is an onto. Firstly, we explaine that $\mathcal{F}(\Upsilon)$ is a neutrosophic sub-group as follows

$$(i)\begin{cases}\mathcal{F}(\mu(\chi\varsigma))\geqslant\mathcal{F}(\min(\mu(\chi),\mu(\varsigma)))\\ \qquad=\min(\mathcal{F}(\mu(\chi)),\mathcal{F}(\mu(\varsigma))).\\ \mathcal{F}(\gamma(\chi\varsigma))\leqslant\mathcal{F}(\max(\gamma(\chi),\gamma(\varsigma)))\\ \qquad=\max(\mathcal{F}(\gamma(\chi)),\mathcal{F}(\gamma(\varsigma))).\\ \mathcal{F}(\zeta(\chi\varsigma))\leqslant\mathcal{F}(\max(\zeta(\chi),\zeta(\varsigma)))\\ \qquad=\max(\mathcal{F}(\zeta(\chi)),\mathcal{F}(\zeta(\varsigma))).\end{cases}$$

$$(ii)\begin{cases}\mathcal{F}(\mu\left(\chi^{-1}\right))\geqslant\mathcal{F}(\mu(\chi))\\ \mathcal{F}(\gamma\left(\chi^{-1}\right))\leqslant\mathcal{F}(\gamma(\chi))\\ \mathcal{F}(\zeta\left(\chi^{-1}\right))\leqslant\mathcal{F}(\zeta(\chi))\end{cases}$$

where $\chi,\varsigma\in\mathcal{G}^{'}$. Secondly, we show $\mathcal{F}(\Upsilon)$ is a neutrosophic normal sub-group

$$\begin{cases} \mathcal{F}(\mu)(\chi\varsigma\chi^{-1}) = \sup\limits_{\mathcal{F}(\varrho)=\chi\varsigma\chi^{-1}}(\mu(\varrho)) \\ \qquad = \sup\limits_{\mathcal{F}(o)=\chi,\mathcal{F}(\varrho)=\varsigma}(\mu(o\varrho o^{-1})) \\ \qquad = \sup\limits_{\mathcal{F}(\varrho)=\varsigma}(\mu(\varrho)) \\ \qquad = \mathcal{F}(\mu)(\varsigma) \\ \mathcal{F}(\gamma)(\chi\varsigma\chi^{-1}) = \inf\limits_{\mathcal{F}(\varrho)=\chi\varsigma\chi^{-1}}(\gamma(\varrho)) \\ \qquad = \inf\limits_{\mathcal{F}(o)=\chi,\mathcal{F}(\varrho)=\varsigma}(\gamma(o\varrho o^{-1})) \\ \qquad = \inf\limits_{\mathcal{F}(\varrho)=\varsigma}(\gamma(\varrho)) \\ \qquad = \mathcal{F}(\gamma)(\varsigma) \\ \mathcal{F}(\zeta)(\chi\varsigma\chi^{-1}) = \inf\limits_{\mathcal{F}(\varrho)=\chi\varsigma\chi^{-1}}(\zeta(\varrho)) \\ \qquad = \inf\limits_{\mathcal{F}(o)=\chi,\mathcal{F}(\varrho)=\varsigma}(\zeta(o\varrho o^{-1})) \\ \qquad = \inf\limits_{\mathcal{F}(\varrho)=\varsigma}(\zeta(\varrho)) \\ \qquad = \mathcal{F}(\zeta)(\varsigma) \end{cases}$$

since $\mathcal{F}$ is onto homomorphism and $o, \varrho \in \mathcal{G}$.

Definition 4.9 Assume that Υ is a neutrosophic sub-group of $\mathcal{G}$ and $\mathcal{H} \subset \mathcal{G}$. then we call $\mathcal{H}$ the normalizer of Υ if $h\Upsilon h^{-1} = \Upsilon \; \forall h \in \mathcal{H}$.

Proposition 4.10 *A normalizer of a neutrosophic sub-group of $\mathcal{G}$ is a sub-group of $\mathcal{G}$.*

5. Conclusions

Here, we have studied some concepts from the point of view of the definition of the neutrosophic group, which was introduced by El Rawy et al. [4]. The concept of neutrosophic invariant sub-groups has been introduced. Also, we have investigated this to define the neutrosophic quotient group. Furthermore, we have defined the neutrosophic normal sub-group. In each part, several related theorems have been constructed, and these are illustrated. The study and development of NS theory will have a new approach opened up by these concepts.

Data Availability Statement

No data were used to support the study.

Declarations

The authors declare that they have no conflicts of interest.

Funding

This research received no external funding.

Authors' contributions

Not applicable.

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