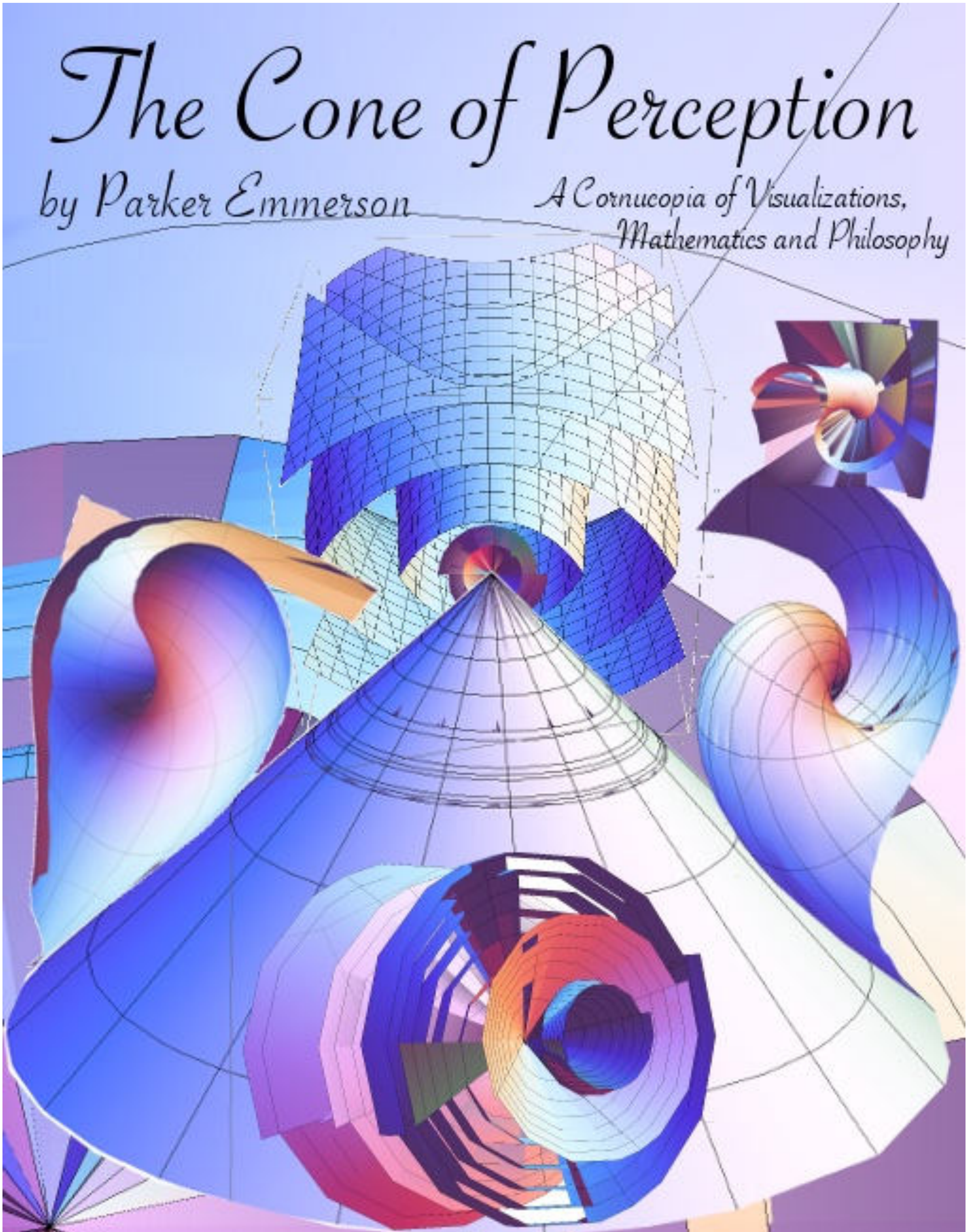

The Cone of Perception
by Parker Emmerson
A Cornucopia of Visualizations,
Mathematics and Philosophy

The Cone of Perception 4th Edition Table of Contents

INTRODUCTION TO THE CONE OF PERCEPTION - 4th EDITION

I write this introduction with much work already completed in this 4th Edition of The Cone of Perception, primarily to frame the work and touch on what might be missing from it. Namely → though it is hardly lacking these, I'd like to add a few insights about the content of the work and its relevance to the subject and future technology. Why is this work still relevant? How can you use it in your work/subject/area/field, and how am I supposed to read all of these equations?!!

This work is relevant, because is shows us that what we may have thought impossible is phenomenological and actualized. That there are, compressed within simple formulations of geometry and parameters of spatio-temporal experiencing, really a complex harmony of relations, coefficients, hexadecimal coefficients, equilibrium, dynamics, resonances, all of which are alternating with each other in a dance of higher dimensionality through many balancing solutions. All of this, and infinitely more is going on between the distance from me to you, and between each particle in between us.

You can use this in your life, work, or area of study, practically no matter what it is that you study, because these equations and insights apply to the fundament of the fabric of reality that we all share and live and the consciousness of it. Whether you are a physicists or a monk, these equations reveal answers and are the answers for the eternal questions of nature and reality.

Harmony with number, visual forms, etc. are a new language. Why have questions about God been so hard to express? Words don't always do justice to the potency of philosophical debate. Much magic mystery and magic have been neglected because of this, but math, equations, visualizations, etc. can do more to reveal insights and converse on topics philosophers have been seeking for ages.

Also, this book reveals that what science is currently doing with particle accelerators, while offering its own set of insights, may actually be less productive, wasteful, unnecessary and dangerous that we think.

What The Cone of Perception offers is a mental scope that will lead you on a journey from what is currently smaller than what physicists currently believe is the smallest mass/time unit all the way out to numbers and equations so large that they are not allowed by God for us to currently visualize their form. The Cone of Perception has revealed that there are particles of the photo electric effects with mass smaller than the quantum foam.

That is a revelation that ought turn the obtuse world of physics on its head. There are many other section in The Cone of Perception all with equally revolutionary material. This work is meant to revolutionize the current perception of the physical world. These equations are useful to those who study dynamics, seek understanding of balance, and those who seek visualizations, mystery and empirical proof of metaphysics and magic in this world.

Specific Insights into Geometric Psychology and Phenomenology of Ontology

Introduction: The Meaning of Now

The following document will attempt to describe an interpretive meaning of the now. The "now" is a "totalized" transformation at a specific given point in space-time. For the purpose of this project, precision in terminology is extremely important. I may use standard definitions of the words moment and instant, both meaning a specific point in time. However, these words can be ambiguous, moment's suggesting a period of time, instant's signifying a point in time. In embarking upon this ontological investigation, I will use mathematics to elucidate a functional definition for "the now" using the idea and reality of a cycle with implicit structure. I propose that, within the now, there is a whole cycle that can be described geometrically and to a certain extent, algebraically. In this project, I wish to distinguish between an exact point in time (a moment or instant) and duration. For this paper, an exact "time point," is defined as a specific, numerical coordinate, though geometry and algebra will elucidate how there is a whole, geometric transformation within this "time point." For example, there is only one point in time or "time point" at which it is 1:00 (not the duration between 1:00 - 1:00:00:00:00… 01).

These assertions and definitions have all been constructed with English and some very simple mathematics. I simply would like to note that if not set by an arbitrary constant of the perceivers selection, ***time*** can otherwise be most purely understood through the functions of the architecture of a given geometric transformation and the spontaneously emerging velocity expressions from the non-assuming, algebraic manipulation of the equations that result from the perceiving and interpreting of the aforementioned geometric architecture when combined with fundamental constants of the universe like the speed of light and the coefficients used for relativistic transformations. We see this potential in the Geometric Pattern of Perception Theorems and it is brought to fruition in Part XXV of the mathematical analysis section of the Cone of Perception (Emmerson, 2009-Present).

However, I propose that a point in time could be described as the “whole moment” as now. I will use geometry to elucidate the meaning of the, “whole moment” and the “whole moment as now.” This project is an attempt at understanding and epistemologically introspecting into being. Even though a single point in time does not have duration, it does have being. Within the being of now, there is an implied geometric transformation. ”To understand the being of the now frees ourselves from the bonds of time,” (LC Davis). Understanding of the transformation of circle into a cone as the circle "folds up" into a right cone and the geometric algebra there in will be very helpful to one coming to understand the points iterated in this paper.

*It can/ought also be understood that depending on the perspective (philosophically) or depending on which variable one uses to measure or correlate with the passing of a concept, for example, time or distance, angle or speed, the philosophical implications will differ/change. If we accept that our understanding of the meaning of a philosophical concept/idea actually changes instead of simply differing, then it is possible that this change occurs immediately as opposed to gradually.

II. Geometric Analysis of the Being of the Now

Circles are not only fascinating geometric shapes with algebraic properties, they are also useful philosophical devices. We are familiar with their properties like circumference. 2π times the radius of the circle is equal to π times the diameter of the circle, and π times the diameter equals the circumference of the circle. We are also familiar with the relationship of the difference in circumferences of two circles to an arc length of the "first circle" (for simplicity's sake, the larger of the two; the minuend). I am, of course referring to that fundamental equation that has been explored in my work *A Geometric Pattern of Perception,* (Emmerson, 2009). 2πr-2πx = (theta) r.

From this equation, a cone can be constructed when applying the Pythagorean theorem. As stated before, for the purposes of outlining the topics discussed in this paper, there is only one “time point” at which it is exactly 1:00. However, in the geometric transformation described in *A Geometric Pattern of Perception* (Emmerson, 2009-2011), when theta would equal 2π rad. (360 degrees), we see that the radius of the minuend circle equals the height of the cone, and thus, there is no angle at 2π radians, because there are no vertices to form an angle, theta. This means that we must consider a synchronous system by which we may measure the position at which the height of the cone equals the initial radius at 2π radians in the synchronous system. It so happens that the perfect candidate for such a synchronous system is the angle which, when multiplied by the subtrahend circle's radius, would equal the difference in the circumferences of the two circles, as stated by an equation, 2πr-2πx = (gamma) x, where gamma is the aforementioned angle that we may use as our synchronous system for measurement.

The synchronous system would be like the hand of a clock, passing around at a constant rate with our normal conception of time as a constant angular velocity, except as the clock hand goes around, it shrinks. *Note: with regard to visual perception and linear perspective, perspective dictates that if the subtrahend circle were to approach the eye as it was shrinking at some specific rate (or change thereof), the distance (space) it outlines (the distance that would be projected onto the retina), would remain constant. At 2π radians (360 degrees) in the "synchronous system," the radius of the initial (minuend) circle described in *A Geometric Pattern of Perception*, would equal the height of the cone. Since there isn't an angle (theta) at 2π radians, there also isn't an angle at 0π radians, but the lack of being of an angle at 0π radians (0 degrees) is different from the lack of there being an angle at 2π radians in the *sense* that there is an angle whose value equals 0π radians at 0π radians, whereas there are no rays or lines at "2π rad.," to form an angle on that plane. At least at 0π radians, there could be thought of two rays occupying the same space, whereas at 2π radians in the synchronous system, there are no rays or lines that intersect to form an angle in the system of a circle transforming into a line through a cone. The whole moment as now is a total transformation of the circle into a cone, which eventually transforms into a line that is orthogonal to the initial circle within a single now point. I submit that within a single now point, this "whole" geometric transformation takes place.

In the "algebraicized," geometric system referred to above (the one with root equation 2πr-2πx=(theta)r), there is no angle at "the now," but "there is no angle" in two different senses, thus there is change in the now - a change in sense. There is a change in sense at the now (in the now (as a total, whole transformation)), though there may be no change in position. Perhaps, the sense-perspective is what changes. I will show how this change of logical sense in the Now is analogous to the two seemingly true statements of paradox. The being of each "true" statement is non-contested as logically viable and deductively factual, though through two different methods - one's being Euclidean geometry, the other's being algebra. Since the results (conclusions) of these two methods contradict each other and occupy the same reality (e.g. both's being equally real and equally verifiable by accepted method), this paradox is similar in nature to the two senses of there not being and angle at 0π radians and 2π radians for separate reasons. Just as there is lacking of consistency between the two deductions (proofs), there is lacking of being of an angle at the now for different reasons (in the $2\pi r-2\pi x=$(theta)r system).

III. The Origin of Paradox

The origin of paradox takes place within the geometric-algebraic-arithmetical system of a circle's transforming into a cone and takes on a couple of philosophical-mathematical meanings. These meanings are:

1. A paradox results when there are two seemingly true statements. In our case, one of the statements is that the difference in circumferences of two circles equals an arc length of the initial circle (minuend circle). That is provable with Euclidean geometry. The other statement is that the radius of the initial circle (the minuend circle) is expressible as an algebraic function in which the only necessary parameter is that angle which when multiplied by the radius of the minuend circle gives the arc length that equals the difference in the circumferences of the two circles. We come to this statement from finding the algebraic expression of the number one and

subtracting from it the numeric value (digit) "one," equating the result with the algebraic expression for the number zero found from subtracting the arc length removed from the initial circle from the difference in circumference of two circles of different sizes.

a) The beginning of a signifying of origin

We use the Pythagorean theorem to solve for the base of the cone , which is the smaller of the two circles and substitute that solution for the base of the cone. We can also use the Pythagorean's theorem to solve for the height of the cone in terms of the difference in the circumferences of the two circles. The process of substitutions is outlined in Addendum 1 attached: *(See addendum one)

b) Processing origin

Having made a substitution for the variable that represents the height of the cone where the algebraic expression (formula) for the height of the cone is in terms of the, "initial radius" (radius of the minuend circle) and the angle, "taken out" of the initial circle, we then solve for the radius of the initial circle. Solving the resulting formula leads to a statement that ought be true through normative algebra, but one that can be dis-proven with Euclidean geometry, because one can draw a circle of any size without specifying a theta quantity. See this process attached at Addendum 2 attached *(See addendum two)

c) A given significance of origin

This statement is that the initial radius is a function of only the angle taken out of the initial circle. We can show that when the initial radius equals zero, the solution to the equation yields one of the previous lemmas of the system (that the interior angle of the cone is a particular function of the angle related to the arc length taken out of the initial circle's circumference). "Tangentially," I can also show that when the initial radius of the circle (minuend radius) equals one, there are more solutions of one angle in terms of the other. More relevant to the discussion, however is the statement that the initial radius is a function of the angle related to the arc length taken out of the initial circle's circumference is one of the statements which is seemingly true in the paradox and is shown to be true at the *origin* of space-time (where the initial radius of the circle equals zero). It is "proven," because the equation of the form 1-1=0 solves to deliver a solution to the initial radius in terms of one of the interior, variable angles of the cone (see attached *), and when you set that solution to the initial radius equal to zero, it solves to deliver one of the proven lemmas of the system.

2. The other meaning of the origin of paradox comes from the form of the equation, 1-1=0, algebraically expressed through the algebraic-geometric transformation of a circle into a cone. The equation, 1-1=0 is the equation that provides the framework for finding the intrinsic, paradoxical statement of the system. It can be considered an "origin" equation, because it equates two expressions of zero, the origin of a number line. There is "embedded" in this the idea that zero has being through algebra, whereas in geometry, zero lacks being. These two conclusions - that zero both has being and lacks being depending upon the mathematical framework one uses to understand it, is the other meaning of paradox with which we are dealing.

IV. The Analogy of Paradox to the Now

The analogy of "paradox is to the now" as "zero is to infinity" is a fitting relationship of the geometric analysis of the being of the now to the origin of paradox.

In the geometric analysis of the being of the now, we find that there are two senses within the now, and theoretically, each is located at a different sense of origin, for each is located at a different sense of there not being an angle. One sense of origin expressed by the geometric analysis of the being of the now described earlier was the being of "no angle" in the sense that there was an angle of no value. This can be understood as an origin location (origin) from which the system expands. However, there is another sense of the now in which there is a different sense of origin. The position at which the height of the cone equals the initial radius of the minuend circle is an origin in the sense that the radius of the "base" of the cone equals at that juncture equals zero, as the circle has "collapsed" into a line. Zero's being the origin on the number line - zero's being the ultimate end that is never reached, yet the point from which all that is once resided. At this point, there is also no angle, theta, but for a different reason - that there are no vertices to make an angle. The location of our numerical zero value for the angle, theta, has correlated within infinitessimally close precision of the philosophical idea of nothing yet at entirely opposite ends of the transformation (start and finish).

The sense of now in which there is not an angle has been correlated to the origin of paradox. The sense of now embodies origin, as well as the being of paradox.

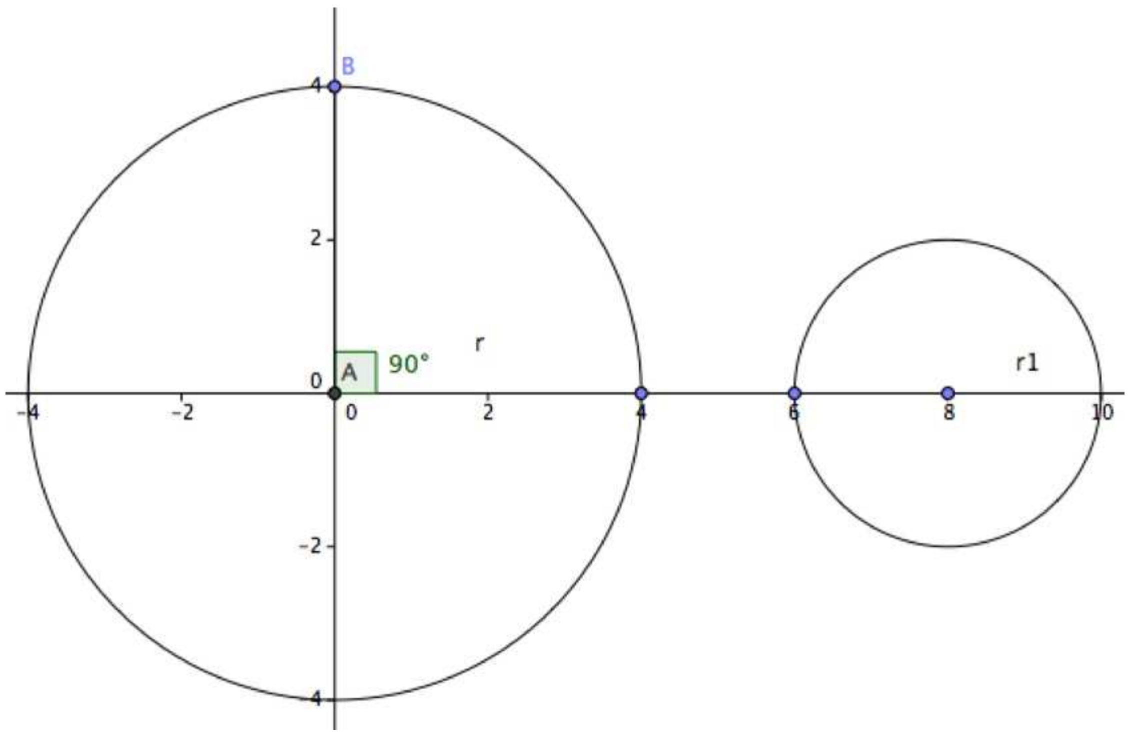
The Geometric Pattern of Perception Theorems

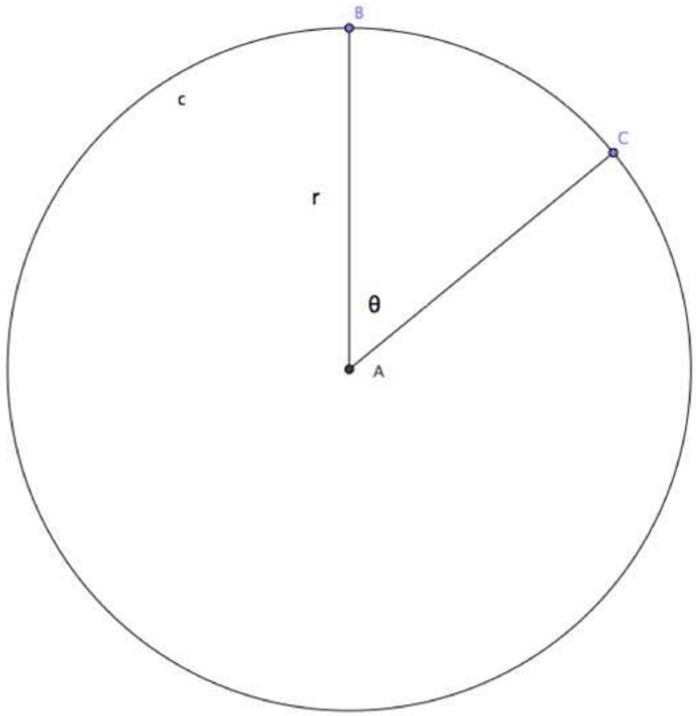
Visualization, Surfaces, and Geometry

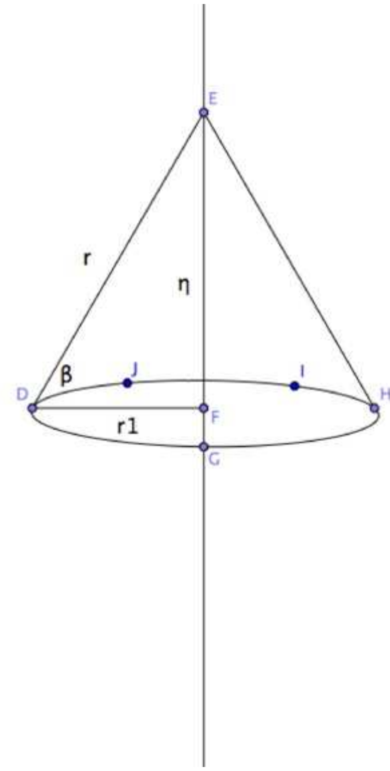
by Parker Emmerson

I. Math for Transforming a Circle into a Cone

by Parker Emmerson







When a sector of a circle is collapsed (removed), we may "fold up" the resulting shape into a cone. The parameters are related by the following theorem :

Theorem 1 *When a sector of angle θ is removed from a circle of radius r and the resulting shape is folded into a cone, then the base of the cone has radius r_1 given by $r_1 = r - \frac{(r\,\theta)}{2\pi}$; and height η, given by $\eta = \sqrt{r^2 - r_1^2}$* = r Sin[$\beta$]

Proof. The circumference of the initial circle is 2π r and the wedge removed has an arc length r θ. Therefore, the remaining circumference is of length r $(2\pi - \theta)$, and after the fold, this is the circumference of the base of the cone.

Establishing the circumference of the base of the cone, from the equation, $\theta\,\text{r} = 2\pi\,\text{r} - 2\pi\,r_1$, we calculate that its radius r_1 is $\frac{2\pi r - r\theta}{2\pi}$, which simplifies to r - $\frac{r\theta}{2\pi}$. Thus, we have proved the first part of the theorem.

To find the height of the cone, η, we apply the Pythagorean theorem to a right triangle formed between the apex of the cone, the center of the base, and a point on the circumference of the base. This gives $\eta = \sqrt{r^2 - r_1^2}$ = r Sin[β], where β is the angle formed by the slant of the cone and the base of the cone. The initial radius is always equal to the slant of the cone, and the height of the cone is always orthogonal to the center of the base of the cone.

Lemma 1 The height of the cone can be caluclated in terms of r and θ.

Proof.

$\theta\,\text{r} = 2\pi\,\text{r} - 2\pi\,r_1$

$\eta = \sqrt{r^2 - r_1^2}$

$\theta\,\text{r} = 2\pi\,\text{r} - 2\pi\sqrt{(r^{\wedge}2 - \eta^{\wedge}2)}$

Solving this equation we find that,

$$\left\{\left\{\eta \to -\frac{\sqrt{4\pi r^2 \theta - r^2 \theta^2}}{2\pi}\right\}, \left\{\eta \to \frac{\sqrt{4\pi r^2 \theta - r^2 \theta^2}}{2\pi}\right\}\right\}$$

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Solve[Limit[(2 Pi) / Sqrt[4 Pi θ - θ^2], {θ → -Infinity, θ → Infinity}] ==
  {2 π r - 2 π x - θ r, 2 π r - 2 π x - θ r}, θ]
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$$\left\{\left\{\theta \to \frac{2\,(\pi r - \pi x)}{r}\right\}\right\}$$

Lemma 2 The angle θ can be calculated in terms of r and η.

Proof

$$\text{Solve}\left[\eta == \frac{\sqrt{4\pi r^2 \theta - r^2 \theta^2}}{2\pi}, \theta\right]$$

$$\left\{\left\{\theta \to \frac{2\pi\left(r^2 - \sqrt{r^4 - r^2\eta^2}\right)}{r^2}\right\}, \left\{\theta \to \frac{2\pi\left(r^2 + \sqrt{r^4 - r^2\eta^2}\right)}{r^2}\right\}\right\}$$

Lemma 3 The initial radius is a function of θ and η.

$$\text{Solve}\left[\frac{\sqrt{4\pi r^2 \theta - r^2 \theta^2}}{2\pi} == \eta, r\right]$$

$$\left\{\left\{r \to -\frac{2\pi\eta}{\sqrt{4\pi\theta - \theta^2}}\right\}, \left\{r \to \frac{2\pi\eta}{\sqrt{4\pi\theta - \theta^2}}\right\}\right\}$$

Lemma 4 *The height of the cone can be calculated in terms of only r and* θ, thus β is a function of θ alone.

Proof. Since we have shown that θ r = 2 π r - 2 π r_1 and $r_1 \to \sqrt{r^2 - \eta^2}$, we can substitute the expression for r_1, calculated from the Pythagorean theorem in terms of the height of the cone and the initial radius of the circle, into the expression for θ r in terms of the change in circumference of the initial circle to the circle that is the base of the cone into which the circle was transformed.

$\theta r = 2\pi r - 2\pi\sqrt{(r\hat{}2 - \eta\hat{}2)}$, thus, $\eta = \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}$ = (r Sin[β]). From $\frac{2\pi\eta}{\sqrt{4\pi\theta - \theta^2}}$ = r, we note that: r = $\frac{2\pi r\,\text{Sin}[\beta]}{\sqrt{4\pi\theta - \theta^2}}$. So we solve the equation,

$$\text{Solve}\left[r == \frac{2\pi r\,\text{Sin}[\beta]}{\sqrt{4\pi\theta - \theta^2}}, \beta\right]$$

$$\left\{\left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{(4\pi - \theta)\,\theta}}{2\pi}\right]\right\}\right\}$$

Lemma 5 *The height of the cone can be calculated in terms of only r and* θ, thus θ is a function of β alone.

Proof. Since we have shown that θ r = 2 π r - 2 π r_1 and $r_1 \to \sqrt{r^2 - \eta^2}$, we can substitute the expression for r_1, calculated from the Pythagorean theorem in terms of the height of the cone and the initial radius of the circle, into the expression for θ r in terms of the change in circumference of the initial circle to the circle that is the base of the cone into which the circle was transformed.

θ r = 2 π r - 2 π $\sqrt{(r\,\hat{}\,2 - \eta\,\hat{}\,2)}$, thus, $\eta = \frac{\sqrt{4 \pi r^2 \theta - r^2 \theta^2}}{2 \pi}$ = (r Sin[β]). From $\frac{2 \pi \eta}{\sqrt{4 \pi \theta - \theta^2}}$ = r, we note that: r = $\frac{2 \pi r \text{Sin}[\beta]}{\sqrt{4 \pi \theta - \theta^2}}$. So we solve the equation,

$$\text{Solve}\left[r == \frac{2 \pi r \text{Sin}[\beta]}{\sqrt{4 \pi \theta - \theta^2}}, \theta\right]$$

$$\left\{\left\{\theta \to 2\left(\pi - \sqrt{\pi^2 - \pi^2 \text{Sin}[\beta]^2}\right)\right\}, \left\{\theta \to 2\left(\pi + \sqrt{\pi^2 - \pi^2 \text{Sin}[\beta]^2}\right)\right\}\right\}$$

- **A note about time passing like a clock.**

The elapse of one unit of time, t, can be expressed by a constant function of the angle θ. The simplest expression is t (seconds) = $\frac{\theta}{2\pi}$; θ = k t , where k is 2π, because one unit of time is equal to one revolution of θ through a circle.

Proof. θ r = 2 π r - 2 π (r - r t) yields $t = \frac{\theta}{2\pi}$.

Theorem 2 When we designate that a single unit of time passes per revolution of the angle through the total number of radians in a circle, instantaneous velocity through the distance of the height of the cone can be found by taking the first derivative of the expression for that distance, which is in terms of r and θ, with respect to t = $\frac{\theta}{2\pi}$. There is also a velocity through the height of the cone, which is equal to wavelength times frequency = $\lambda f = \frac{\eta}{\left(\frac{\theta}{(2\pi)}\right)}$ considered the average velocity through the height of the cone. Under the condition that one unit of time passes with one revolution of the circle, these two velocities are equal to each other at the position where a 30-60-90 triangle is formed between the apex, center of the base of the cone, and point on the circumference of the circle of the base of the cone.

Proof.

To prove this, we can substitute r Sin[β] for the height of the cone in the expression of velocity = ((2 π η)/θ) and find a real and two complex solutions for theta in terms of β , thus from Lemma 4, we can solve for β exactly.

$$\text{Instantaneous Velocity} = \frac{d\eta}{dt} = \frac{d\eta}{d\frac{\theta}{(2\pi)}} = D\left[\frac{\sqrt{4 \pi r^2 \theta - r^2 \theta^2}}{2\pi}, t\right] =$$

$$D\left[k \frac{\sqrt{4 \pi r^2 \theta - r^2 \theta^2}}{2\pi}, \theta\right] = D\left[2\pi \frac{\sqrt{4 \pi r^2 \theta - r^2 \theta^2}}{2\pi}, \theta\right] = \frac{4 \pi r^2 - 2 r^2 \theta}{2\sqrt{4 \pi r^2 \theta - r^2 \theta^2}}$$

$$\text{Average Velocity} = (\eta / (\theta / 2\pi))$$

$$\text{Instantaneous Velocity} = \frac{4 \pi r^2 - 2 r^2 \theta}{2\sqrt{4 \pi r^2 \theta - r^2 \theta^2}} = \text{Average Velocity} = \frac{2 \pi \eta}{\theta}$$

$$\text{Solve}\left[\frac{4 \pi r^2 - 2 r^2 \theta}{2\sqrt{4 \pi r^2 \theta - r^2 \theta^2}} == \frac{2 \pi \eta}{\theta}, \theta\right]$$

$$\left\{\left\{\theta \to \frac{4\pi}{3} + \frac{-4\pi^2 r^4 + 12\pi^2 r^2 \eta^2}{6\pi r^2 \left(r^6 - 18 r^4 \eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11 r^8 \eta^4 + r^6 \eta^6}\right)^{1/3}} - \frac{2\pi\left(r^6 - 18 r^4 \eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11 r^8 \eta^4 + r^6 \eta^6}\right)^{1/3}}{3 r^2}\right\},\right.$$

$$\left\{\theta \to \frac{4\pi}{3} - \frac{\left(1 + i\sqrt{3}\right)\left(-4\pi^2 r^4 + 12\pi^2 r^2 \eta^2\right)}{12\pi r^2 \left(r^6 - 18 r^4 \eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11 r^8 \eta^4 + r^6 \eta^6}\right)^{1/3}} + \frac{\left(1 - i\sqrt{3}\right)\pi\left(r^6 - 18 r^4 \eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11 r^8 \eta^4 + r^6 \eta^6}\right)^{1/3}}{3 r^2}\right\},$$

$$\left.\left\{\theta \to \frac{4\pi}{3} - \frac{\left(1 - i\sqrt{3}\right)\left(-4\pi^2 r^4 + 12\pi^2 r^2 \eta^2\right)}{12\pi r^2 \left(r^6 - 18 r^4 \eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11 r^8 \eta^4 + r^6 \eta^6}\right)^{1/3}} + \frac{\left(1 + i\sqrt{3}\right)\pi\left(r^6 - 18 r^4 \eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11 r^8 \eta^4 + r^6 \eta^6}\right)^{1/3}}{3 r^2}\right\}\right\}$$

$$\text{Solve}\left[\frac{k\left(4\pi r^2 - 2 r^2 \theta\right)}{4\pi\sqrt{4\pi r^2 \theta - r^2 \theta^2}} == \frac{k\, r\, \text{Sin}[\beta]}{\theta}, \theta\right]$$

$$\left\{\left\{\theta \to \frac{4\pi}{3} - \frac{-4\pi^2 + 12\pi^2 \text{Sin}[\beta]^2}{6\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}} + \frac{2}{3}\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}\right\},\right.$$

$$\left\{\theta \to \frac{4\pi}{3} + \frac{\left(1 + i\sqrt{3}\right)\left(-4\pi^2 + 12\pi^2 \text{Sin}[\beta]^2\right)}{12\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}} - \frac{1}{3}\left(1 - i\sqrt{3}\right)\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}\right\},$$

$$\left.\left\{\theta \to \frac{4\pi}{3} + \frac{\left(1 - i\sqrt{3}\right)\left(-4\pi^2 + 12\pi^2 \text{Sin}[\beta]^2\right)}{12\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}} - \frac{1}{3}\left(1 + i\sqrt{3}\right)\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}\right\}\right\}$$

The real solution for θ, solved from equating the instantaneous velocity to the average velocity, can be equated with the real solution for the expression for θ from Lemma 4 to yield an exact solution for β that tells us that when these solutions for theta are equal, a 30-60-90 triangle is formed between the azimuth of the cone, the point on the base of the cone and the center of the base of the cone.

$$\text{Solve}\left[\frac{4\pi}{3} - \frac{-4\pi^2 + 12\pi^2 \text{Sin}[\beta]^2}{6\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}} + \frac{2}{3}\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3} == 2\left(\pi + \sqrt{\pi^2 - \pi^2 \text{Sin}[\beta]^2}\right), \beta\right]$$

$$\left\{\left\{\beta \to -\frac{\pi}{3}\right\}, \left\{\beta \to \frac{\pi}{3}\right\}\right\}$$

We know that the height of the cone is perpendicular to the center of the base of the cone, so this proves a 30-60-90 triangle, because the sum of the angles of the triangle must be 180 degrees or π radians.

Lemma 7 We can show that $\beta = \frac{\pi}{3}$, thus we can show that there are two solutions to θ at which this occurs.

Proof.

$$\text{ArcSin}\left[\frac{\sqrt{(4\pi - \theta)\,\theta}}{2\pi}\right] = \beta$$

$$\text{Solve}\left[\text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\,\theta}}{2\pi}\right] == \frac{\pi}{3},\ \theta\right]$$

$$\{\{\theta \to \pi\},\ \{\theta \to 3\pi\}\}$$

Lemma 8 We can show can show that the position at which instantaneous rate of change of the height of the cone with respect to theta equals average rate of change of the height of the cone, 'per theta measure,' at

$$\left\{\left\{\beta \to -\text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right]\right\},\ \left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right]\right\}\right\}.$$

Proof.

$$\text{Solve}\left[\theta == \frac{4\pi}{3} - \frac{-4\pi^2 + 12\pi^2\,\text{Sin}[\beta]^2}{6\left(-\pi^3 + 18\pi^3\,\text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2 + 11\pi^6\,\text{Sin}[\beta]^4 + \pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}} + \frac{2}{3}\left(-\pi^3 + 18\pi^3\,\text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2 + 11\pi^6\,\text{Sin}[\beta]^4 + \pi^6\,\text{Sin}[\beta]^6}\right)^{1/3},\ \beta\right]$$

$$\left\{\left\{\beta \to -\text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right]\right\},\ \left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right]\right\}\right\}$$

Theorem 3 The "innate velocity," v, within the Lorentz transformation can be solved for in terms of the system of the circle transforming into a cone. If r is multiplied by the Lorenz transformation, then it measures the distance in the prime system, denoted by r '. If t ' equals $\frac{\left(\frac{\theta}{(2\pi)}\right)}{\sqrt{1-\left(\frac{v^2}{c^2}\right)}}$, then the quantity r $\theta = \theta$ ' r '. We are only dealing with algebraic forms and the solutions necessitated by them. Logical, algebraic, reasoning will be given why, when using the exact speed of light, 2.99792458 (10^8) meters per second, the units of the speed of light can be ignored for the purposes of calculation and computation. This theorem states that, although, normal algebra would require the speed of light as a quantity to cancel out, valid expressions for the solutions for the intrinsic velocity, v, can be found in terms of η, r, and θ, or θ and β, depending on the expression used for the height of the cone.

Proof.

$$c = 2.99792458\left(10^8\right)\ \text{meters per second}$$

$$r' = r\sqrt{1 - \frac{v^2}{c^2}}$$

$$t' = \frac{\left(\frac{\theta}{(2\pi)}\right)}{\sqrt{1-\left(\frac{v^2}{c^2}\right)}}$$

$$t' = \frac{\theta'}{2\pi}$$

$$2\pi t' = \theta'$$

$$\Theta' = \frac{\Theta}{\sqrt{1-\frac{v^2}{c^2}}}$$

Therefore,

$$r' * \Theta' = \left(\frac{\Theta}{\sqrt{1-\frac{v^2}{c^2}}}\right)\left(r\sqrt{1-\frac{v^2}{c^2}}\right)$$

$$\left(\frac{\Theta}{\sqrt{1-\frac{v^2}{c^2}}}\right)\left(r\sqrt{1-\frac{v^2}{c^2}}\right) = r\,\Theta$$

$$r' * \Theta' = r\,\Theta = 2\pi r - 2\pi r_1 = 2\pi r - 2\pi\sqrt{r\char`^2 - \eta\char`^2}$$

```
Solve[r Θ == 2 π r - 2 π Sqrt[r^2 - η^2], η]
```

$$\left\{\left\{\eta \to -\frac{\sqrt{4\pi r^2\Theta - r^2\Theta^2}}{2\pi}\right\}, \left\{\eta \to \frac{\sqrt{4\pi r^2\Theta - r^2\Theta^2}}{2\pi}\right\}\right\}$$

```
Solve[r ' θ ' == 2 π r - 2 π Sqrt[r^2 - η^2], η]
```

$$\left\{\left\{\eta \to -\frac{\sqrt{r'}\sqrt{\theta'}\sqrt{4\pi r - r'\theta'}}{2\pi}\right\}, \left\{\eta \to \frac{\sqrt{r'}\sqrt{\theta'}\sqrt{4\pi r - r'\theta'}}{2\pi}\right\}\right\}$$

The argument follows modus ponens, saying that, through commutation, $r'\theta' = \theta r$, therefore $\eta = \frac{\sqrt{r'}\sqrt{\theta'}\sqrt{4\pi r - r'\theta'}}{2\pi} = \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}$

$\eta =$

$$\frac{\sqrt{r'}\ \sqrt{\theta'}\ \sqrt{4\,\pi\,r - r'\,\theta'}}{2\,\pi} = \frac{\sqrt{r\,\sqrt{1-\frac{v^2}{c^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}}\ \sqrt{4\,\pi\,r - r\,\sqrt{1-\frac{v^2}{c^2}}\ \frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}}}{2\,\pi} = \frac{\sqrt{r\,\theta}\ \sqrt{4\,\pi\,r - r\,\theta}}{2\,\pi}$$

where c is its numeric value of the speed of light, its units being shown to cancel out.

$$\frac{\sqrt{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}}{\sqrt{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}}$$

$$\textbf{Solve}\left[\sqrt{\frac{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}},\ \text{meters}\right]$$

{{}}

$$\textbf{Solve}\left[\sqrt{\frac{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}},\ \text{second}\right]$$

{{}}

$$\textbf{Solve}\left[\sqrt{\frac{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}},\ v\right]$$

{{}}

$$\textbf{Solve}\left[\sqrt{\frac{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}},\ c\right]$$

$\text{Solve}\left[\text{True},\ 2.99792\times 10^{8}\right]$

$$\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}}}\ \sqrt{4\pi r - r\theta}}{2\pi} =$$

$$\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}\ \frac{\theta}{\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}}\ \sqrt{4\pi r - r\theta}}}{2\pi} =$$

$$\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c)^2}}\ \frac{\theta}{\sqrt{1-\frac{v^2}{(c)^2}}}\ \sqrt{4\pi r - r\theta}}}{2\pi} = (1)\ \frac{\sqrt{r\sqrt{1-\frac{v^2}{(c)^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{(c)^2}}}}\ \sqrt{4\pi r - r\theta}}{2\pi}$$

Logically, the units of the speed of light will cancel out, therefore, their relationship will be set equal to one and taken out of the equation for the purposes of computational calculation.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}\ \frac{\theta}{\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}}}\ \sqrt{4\pi r - r\theta}}{2\pi} == \eta,\ \text{meters}\right]$$

{{}}

Meters cancel out.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}\ \frac{\theta}{\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}}}\ \sqrt{4\pi r - r\theta}}{2\pi} == \eta,\ \text{second}\right]$$

{{}}

Seconds cancel out.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}\ \frac{\theta}{\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}}}\ \sqrt{4\pi r - r\theta}}{2\pi} == \eta,\ c\right]$$

{{}}

The numeric c cancels out.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\ (\text{meters/second}))^2}}\sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{(c\ (\text{meters/second}))^2}}}}\sqrt{4\pi r - r\theta}}}{2\pi} == \eta,\ r\right]$$

$$\left\{\left\{r \to -\frac{2\pi\eta}{\sqrt{4\pi\theta-\theta^2}}\right\},\ \left\{r \to \frac{2\pi\eta}{\sqrt{4\pi\theta-\theta^2}}\right\}\right\}$$

Radius yields the result from Lemma 3.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c)^2}}\ \frac{\theta}{\sqrt{1-\frac{v^2}{(c)^2}}}\sqrt{4\pi r - r\theta}}}{2\pi} == \eta,\ v\right]$$

{{}}

Velocity cancels out.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c)^2}}\ \frac{\theta}{\sqrt{1-\frac{v^2}{(c)^2}}}\sqrt{4\pi r - r\theta}}}{2\pi} == \eta,\ v\right]$$

{{}}

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c)^2}}}\sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{(c)^2}}}}\sqrt{4\pi r - r\theta}}{2\pi} == \eta,\ v\right]$$

{}

Velocity cancels out. Everything *cancels out*. Only when using the exact speed of light, in scientific notation, can solutions to the innate velocity be found.

We set the speed of light equal to its numeric value for the purpose of making computations, dropping the units, because in the expression for the height of the cone, they would cancel out anyway. It should be noted that this is necessary for computing the function of the velocity and that the exact speed of light is to be used as well as that the numeric value of the speed of light has to be in the form of scientific notation in order to find results to this equation.

Theorem 3 Continued From the expression of the height of the cone of Lemma 1, with the Lorentz transformations implicitly expressed, we can solve for the velocity within the Lorenz coefficient in terms of the height of the cone, the initial radius, and the angle, θ when using the exact speed of light in scientific notation and only when it is its exact (or extremely closely approximated) value expressed in scientific notation.

Proof.

```
c := 2.99792458 * (10^8)
```

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{(v)^2}{c^2}}}\sqrt{\frac{\theta}{\sqrt{1-\frac{(v)^2}{c^2}}}}\sqrt{4\pi r - r\theta}}{2\pi} == \eta,\ v\right]$$

$$\left\{\left\{v \to -\frac{1.\sqrt{3.54814\times 10^{18}\,\eta^2 - 1.12941\times 10^{18}\,r^2\,\theta + 8.98755\times 10^{16}\,r^2\,\theta^2}}{\sqrt{39.4784\,\eta^2 - 12.5664\,r^2\,\theta + r^2\,\theta^2}}\right\},\right.$$

$$\left.\left\{v \to \frac{\sqrt{3.54814\times 10^{18}\,\eta^2 - 1.12941\times 10^{18}\,r^2\,\theta + 8.98755\times 10^{16}\,r^2\,\theta^2}}{\sqrt{39.4784\,\eta^2 - 12.5664\,r^2\,\theta + r^2\,\theta^2}}\right\}\right\}$$

Theorem 3 Continued From the expression of the height of the cone, from Lemma 1 with the Lorentz transformations implicitly expressed, we can solve for the velocity within the Lorenz coefficient in terms of θ and β.

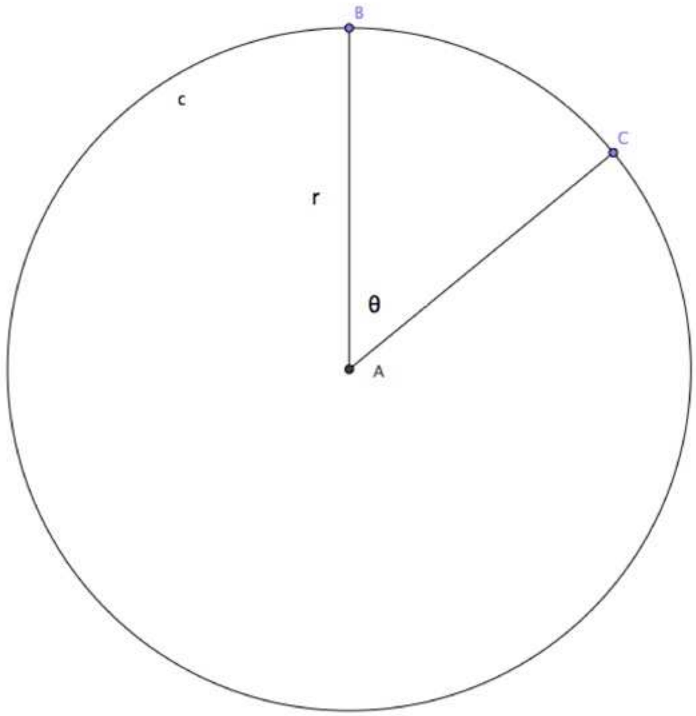
$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{(v)^2}{c^2}}}\sqrt{\frac{\theta}{\sqrt{1-\frac{(v)^2}{c^2}}}}\sqrt{4\pi r - r\theta}}{2\pi} == r\,\text{Sin}[\beta],\ v\right]$$

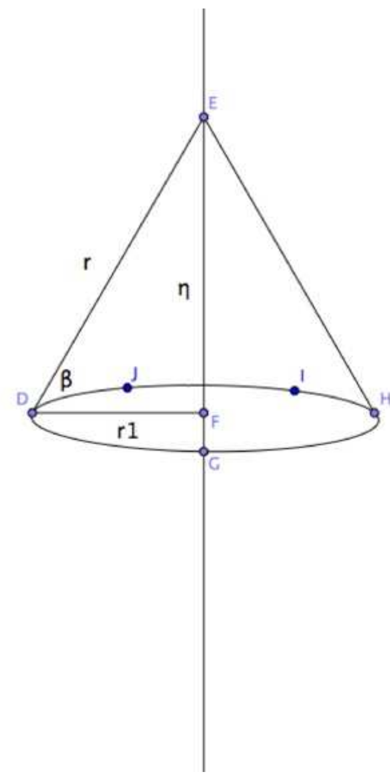
$$\left\{\left\{v \to -\frac{1.\sqrt{-1.12941\times 10^{18}\,\theta + 8.98755\times 10^{16}\,\theta^2 + 3.54814\times 10^{18}\,\text{Sin}[\beta]^2}}{\sqrt{-12.5664\,\theta + \theta^2 + 39.4784\,\text{Sin}[\beta]^2}}\right\},\right.$$

$$\left.\left\{v \to \frac{\sqrt{-1.12941\times 10^{18}\,\theta + 8.98755\times 10^{16}\,\theta^2 + 3.54814\times 10^{18}\,\text{Sin}[\beta]^2}}{\sqrt{-12.5664\,\theta + \theta^2 + 39.4784\,\text{Sin}[\beta]^2}}\right\}\right\}$$

II. The Visualizations of Math for Transforming a Circle into a Cone

by Parker Emmerson





When a sector of a circle is removed, we may "fold up" the resulting shape into a cone. The parameters are related by the following theorem :

Theorem 1 *When a sector of angle θ is removed from a circle of radius r and the resulting shape is folded into a cone, then the base of the cone has radius r_1 given by $r_1 = r - \frac{(r\,\theta)}{2\,\pi}$; and height η, given by* $\eta = \sqrt{r^2 - {r_1}^2}$ =r Sin[β]

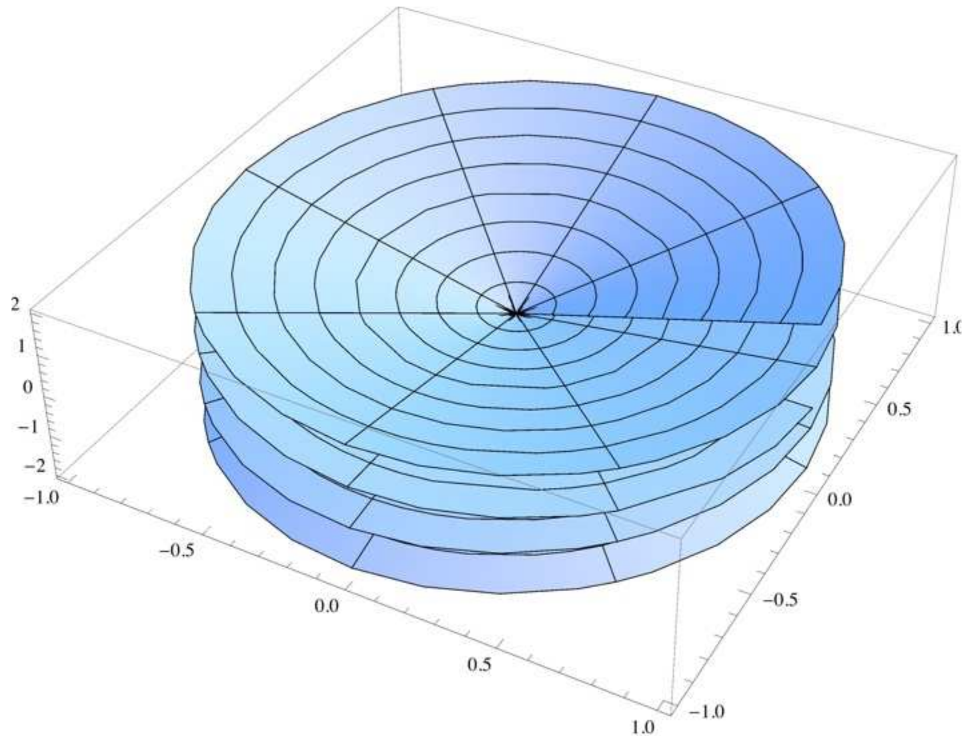
Proof. The circumference of the initial circle is 2 π r and the wedge removed has an arc length r θ. Therefore, the remaining circumference is of length r (2 π - θ), and after the fold, this is the circumference of the base of the cone.

Establishing the circumference of the base of the cone, from the equation, θ r = 2 π r - 2 π r_1, we calculate that its radius r_1 is $\frac{2\,\pi\,r - r\,\theta}{2\,\pi}$, which simplifies to r - $\frac{r\,\theta}{2\,\pi}$. Thus, we have proved the first part of the theorem.

To find the height of the cone, η, we apply the Pythagorean theorem to a right triangle formed between the apex of the cone, the center of the base, and a point on the circumference of the base. This gives $\eta = \sqrt{r^2 - {r_1}^2}$ = r Sin[β], where β is the angle formed by the slant of the cone and the base of the cone. The initial radius is always equal to the slant of the cone, and the height of the cone is always orthogonal to the center of the base of the cone.

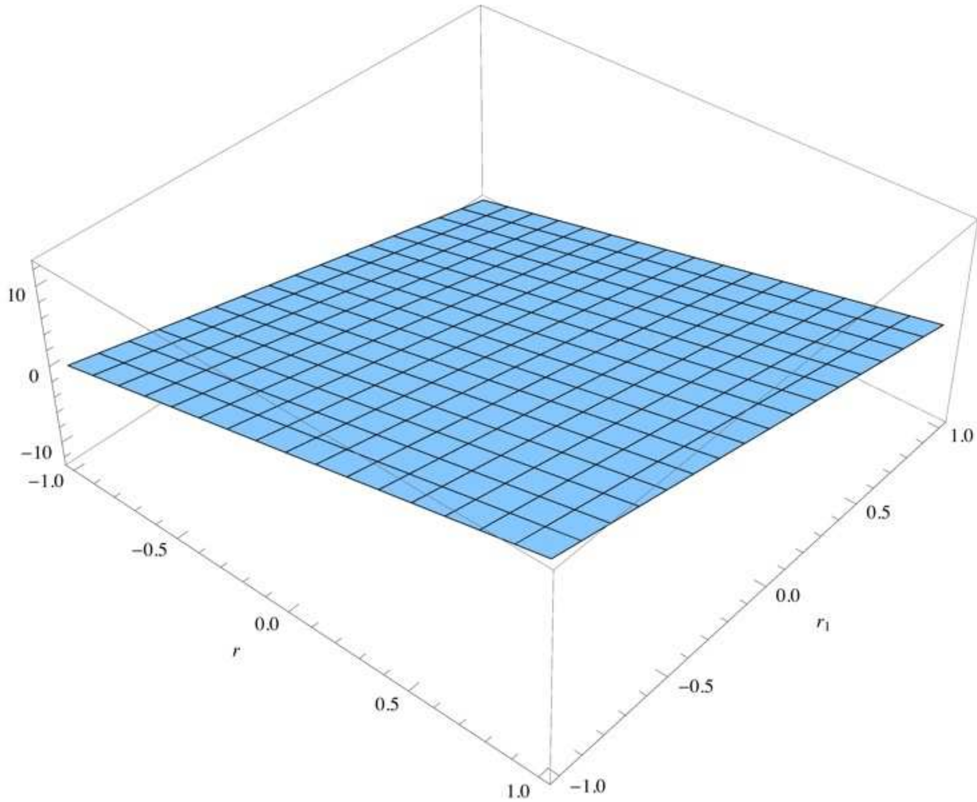
$$r_1 = r - \frac{r\,\theta}{2\,\pi} \qquad (1)$$

```
RevolutionPlot3D[r - rθ/(2π), {r, -1, 1}, {θ, -2π, 2π}]
```



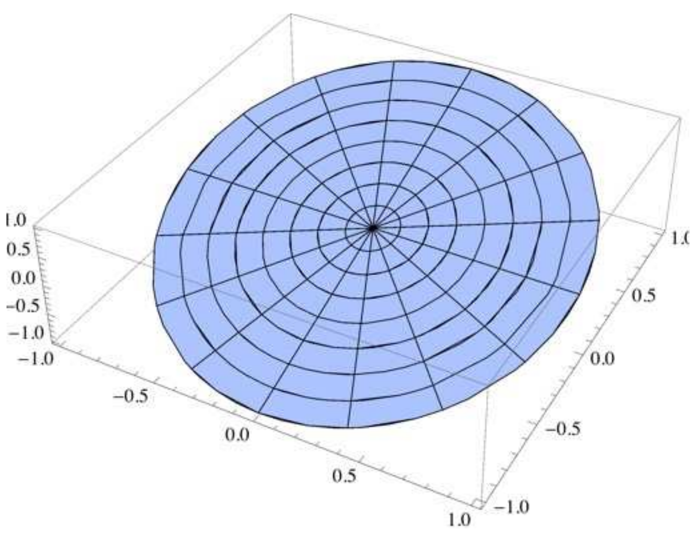
$$\theta r = 2\pi r - 2\pi r_1 \quad (2)$$

```
Plot3D[2 π r - 2 π r₁, {r₁, -1, 1}, {r₁, -1, 1}, AxesLabel → Automatic]
```



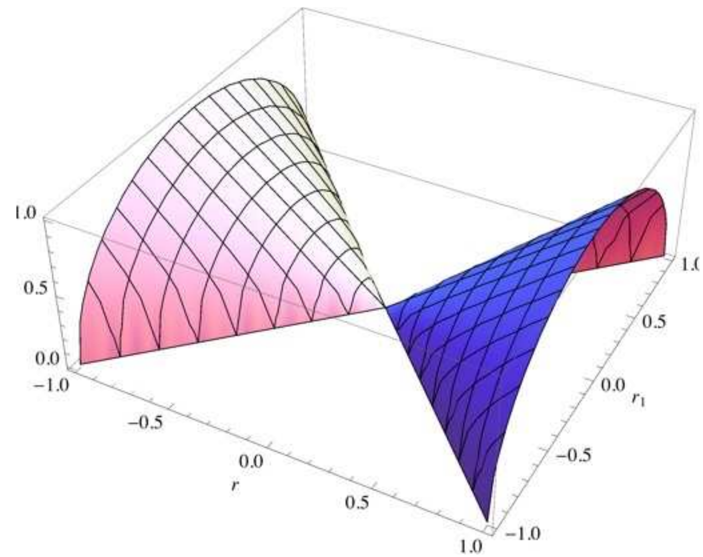
$$\eta = r\ Sin[\beta] \quad (3)$$

```
RevolutionPlot3D[r Sin[β], {r, -1, 1}, {β, -π, π}]
```



$$\eta = \sqrt{r^2 - r_1{}^2} \quad (4)$$

```
Plot3D[√(r² - r₁²) , {r, -1, 1}, {r₁, -1, 1}, AxesLabel → Automatic]
```



Lemma 1 The height of the cone can be caluclated in terms of r and θ.

Proof.

$\theta \, \mathrm{r} = 2\,\pi\,\mathrm{r} - 2\,\pi\,r_{\;1}$

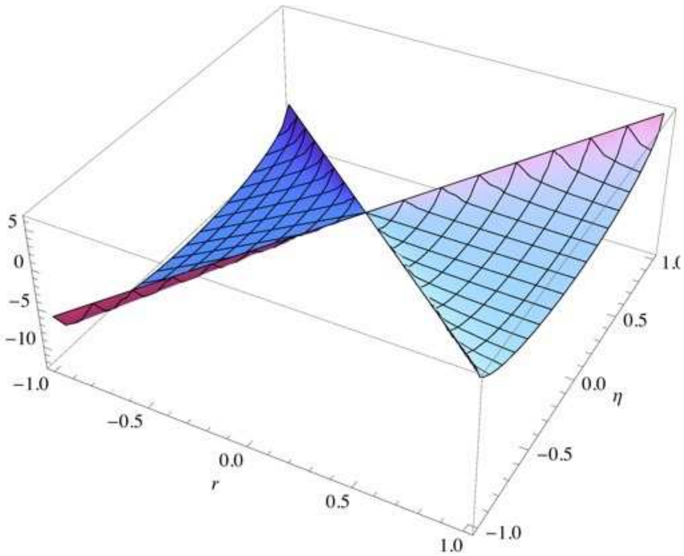
$\eta = \sqrt{r^2 - r_1{}^2}$

$\theta \, \mathrm{r} = 2\,\pi\,\mathrm{r} - 2\,\pi\,\sqrt{(\mathtt{r}\,\hat{}\,2 - \eta\,\hat{}\,2)}$

$$\eta = \frac{\sqrt{4\,\pi\,\mathbf{r}^2\,\theta - \mathbf{r}^2\,\theta^2}}{2\,\pi}$$

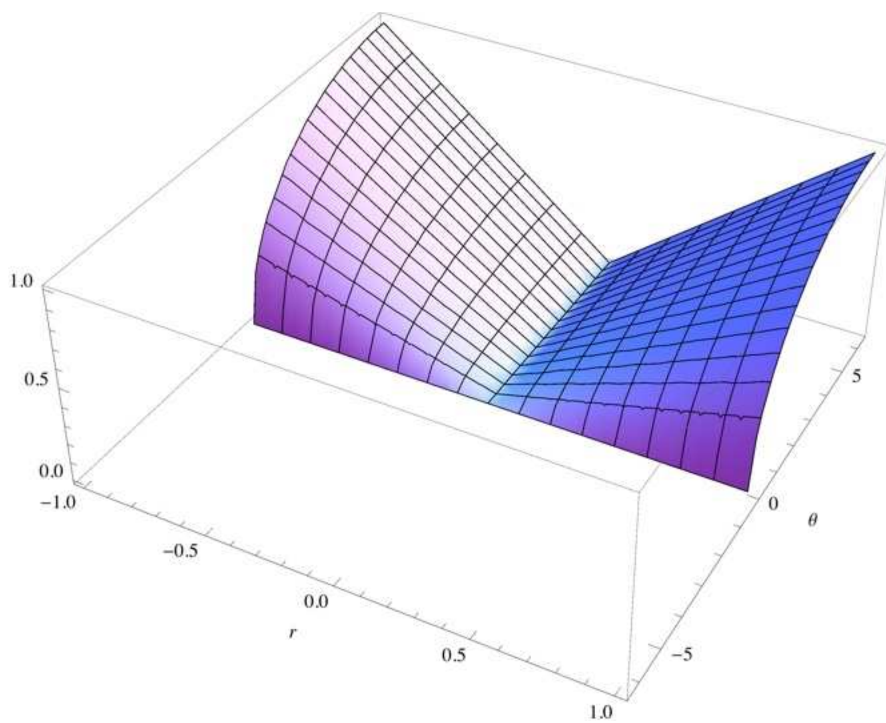
$$\theta\,\mathbf{r} = 2\,\pi\,\mathbf{r} - 2\,\pi\,\sqrt{(\mathtt{r}\,\hat{}\,2 - \eta\,\hat{}\,2)} \tag{5}$$

```
Plot3D[2 π r - 2 π √((r^2 - η^2)), {r, -1, 1}, {η, -1, 1}, AxesLabel → Automatic]
```



$$\eta = \frac{\sqrt{4 \pi r^2 \theta - r^2 \theta^2}}{2 \pi} \tag{6}$$

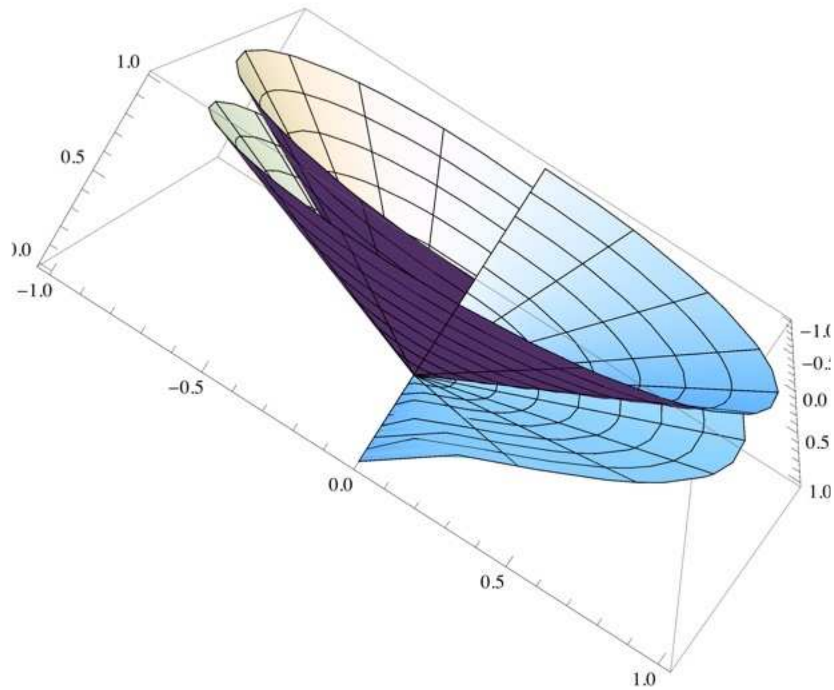
```
Plot3D[√(4 π r² θ - r² θ²) / (2 π), {r, -1, 1}, {θ, -2 π, 2 π}, AxesLabel → Automatic]
```



$$\eta = \frac{\sqrt{4 \pi r^2 \theta - r^2 \theta^2}}{2 \pi} \tag{7}$$

```
RevolutionPlot3D[Sqrt[4 π r^2 θ - r^2 θ^2]/(2 π), {r, -1, 1}, {θ, -2 π, 2 π}]
```

$$\text{RevolutionPlot3D}\left[\frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}, \{r, -1, 1\}, \{\theta, -2\pi, 2\pi\}\right]$$



Lemma 2 The angle θ can be calculated in terms of r and η.

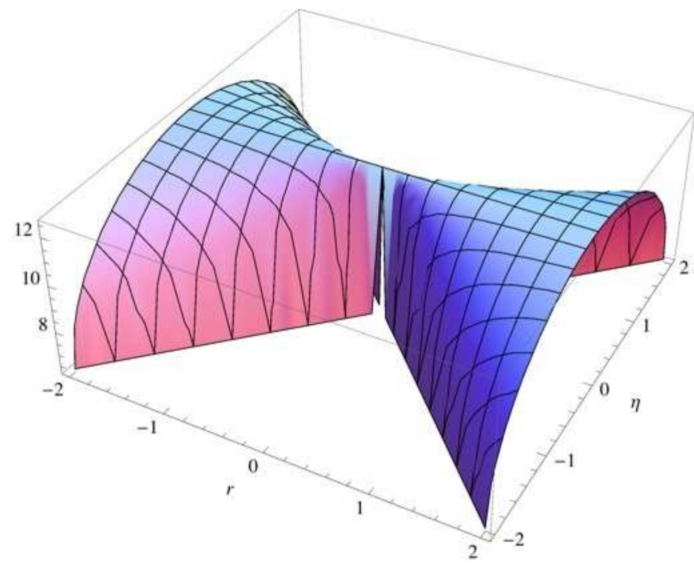
Proof

$$\text{Solve}\left[\eta == \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}, \theta\right]$$

$$\left\{\left\{\theta \to \frac{2\pi\left(r^2 - \sqrt{r^4 - r^2\eta^2}\right)}{r^2}\right\}, \left\{\theta \to \frac{2\pi\left(r^2 + \sqrt{r^4 - r^2\eta^2}\right)}{r^2}\right\}\right\}$$

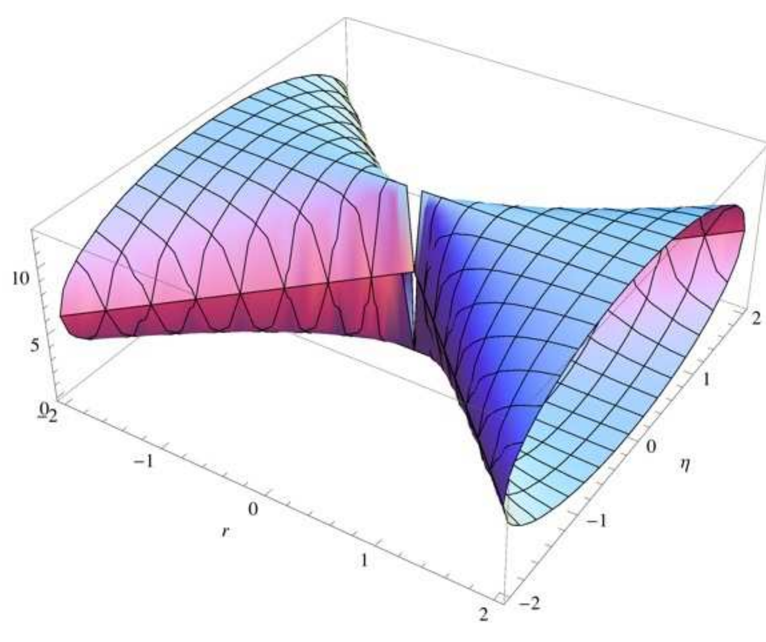
$$\theta = \frac{2\pi\left(r^2 + \sqrt{r^4 - r^2\eta^2}\right)}{r^2} \quad (8)$$

```
Plot3D[(2 π (r^2 + Sqrt[r^4 - r^2 η^2]))/r^2, {r, -2, 2}, {η, -2, 2}, AxesLabel → Automatic]
```



$$\Theta = \frac{2\pi\left(r^2 \pm \sqrt{r^4 - r^2\eta^2}\right)}{r^2} \quad (9)$$

```
Plot3D[{(2 π (r^2 - Sqrt[r^4 - r^2 η^2]))/r^2, (2 π (r^2 + Sqrt[r^4 - r^2 η^2]))/r^2},
 {r, -2, 2}, {η, -2, 2}, AxesLabel → Automatic]
```



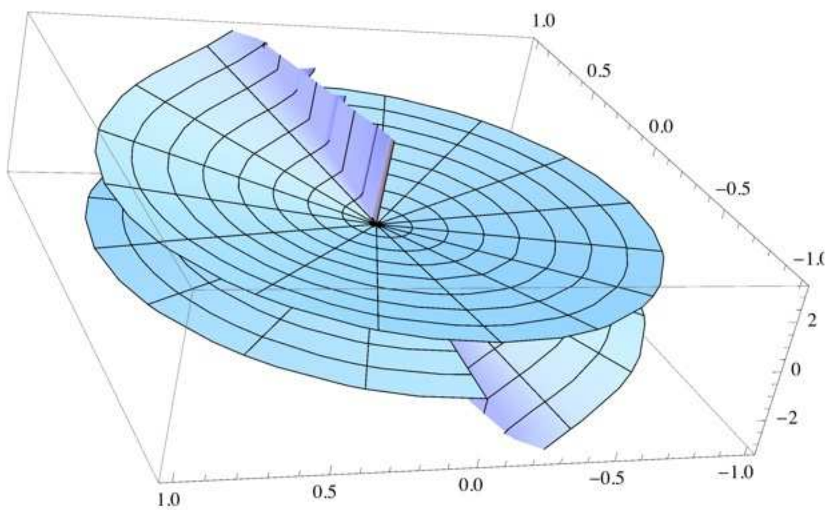
Lemma 3 The initial radius is a function of θ and η.

$$\text{Solve}\left[\frac{\sqrt{4\pi r^2 \theta - r^2 \theta^2}}{2\pi} == \eta, r\right]$$

$$\left\{\left\{r \to -\frac{2\pi\eta}{\sqrt{4\pi\theta - \theta^2}}\right\}, \left\{r \to \frac{2\pi\eta}{\sqrt{4\pi\theta - \theta^2}}\right\}\right\}$$

$$r = \frac{2\pi\eta}{\sqrt{4\pi\theta - \theta^2}} \tag{10}$$

$$\textbf{RevolutionPlot3D}\left[\frac{2\pi\eta}{\sqrt{4\pi\theta - \theta^2}}, \{\eta, -1, 1\}, \{\theta, -2\pi, 2\pi\}\right]$$



Lemma 4 *The height of the cone can be calculated in terms of only r and θ*, thus β is a function of θ alone.

Proof. Since we have shown that θ r = 2 π r - 2 π r_1 and $r_1 \to \sqrt{r^2 - \eta^2}$, we can substitute the expression for r_1, calculated from the Pythagorean theorem in terms of the height of the cone and the initial radius of the circle, into the expression for θ r in terms of the change in circumference of the initial circle to the circle that is the base of the cone into which the circle was transformed.

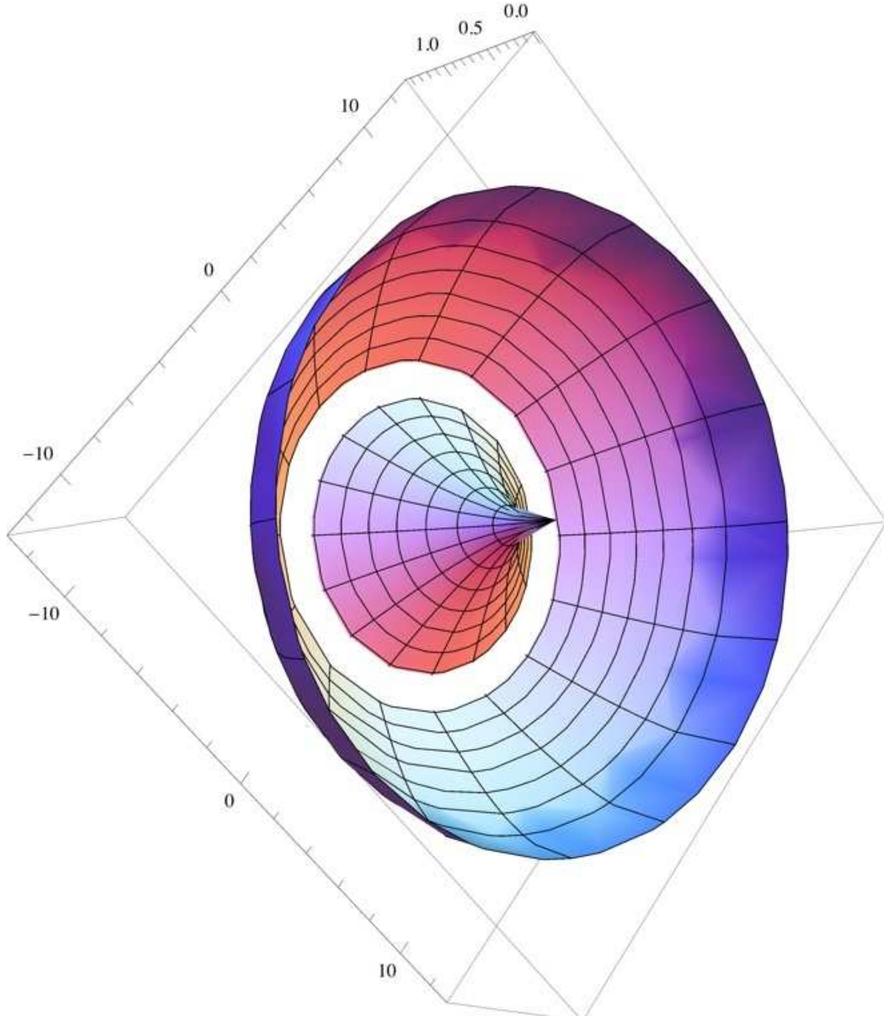
θ r = 2 π r - 2 π $\sqrt{(r\hat{}2 - \eta\hat{}2)}$, thus, $\eta = \frac{\sqrt{4\pi r^2 \theta - r^2 \theta^2}}{2\pi}$ = (r Sin[β]). From $\frac{2\pi\eta}{\sqrt{4\pi\theta-\theta^2}}$ = r, we note that: r = $\frac{2\pi r \,\text{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}}$. So we solve the equation,

$$\text{Solve}\left[r == \frac{2\pi r\,\text{Sin}[\beta]}{\sqrt{4\pi\theta - \theta^2}}, \beta\right]$$

$$\left\{\left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{(4\pi - \theta)\,\theta}}{2\pi}\right]\right\}\right\}$$

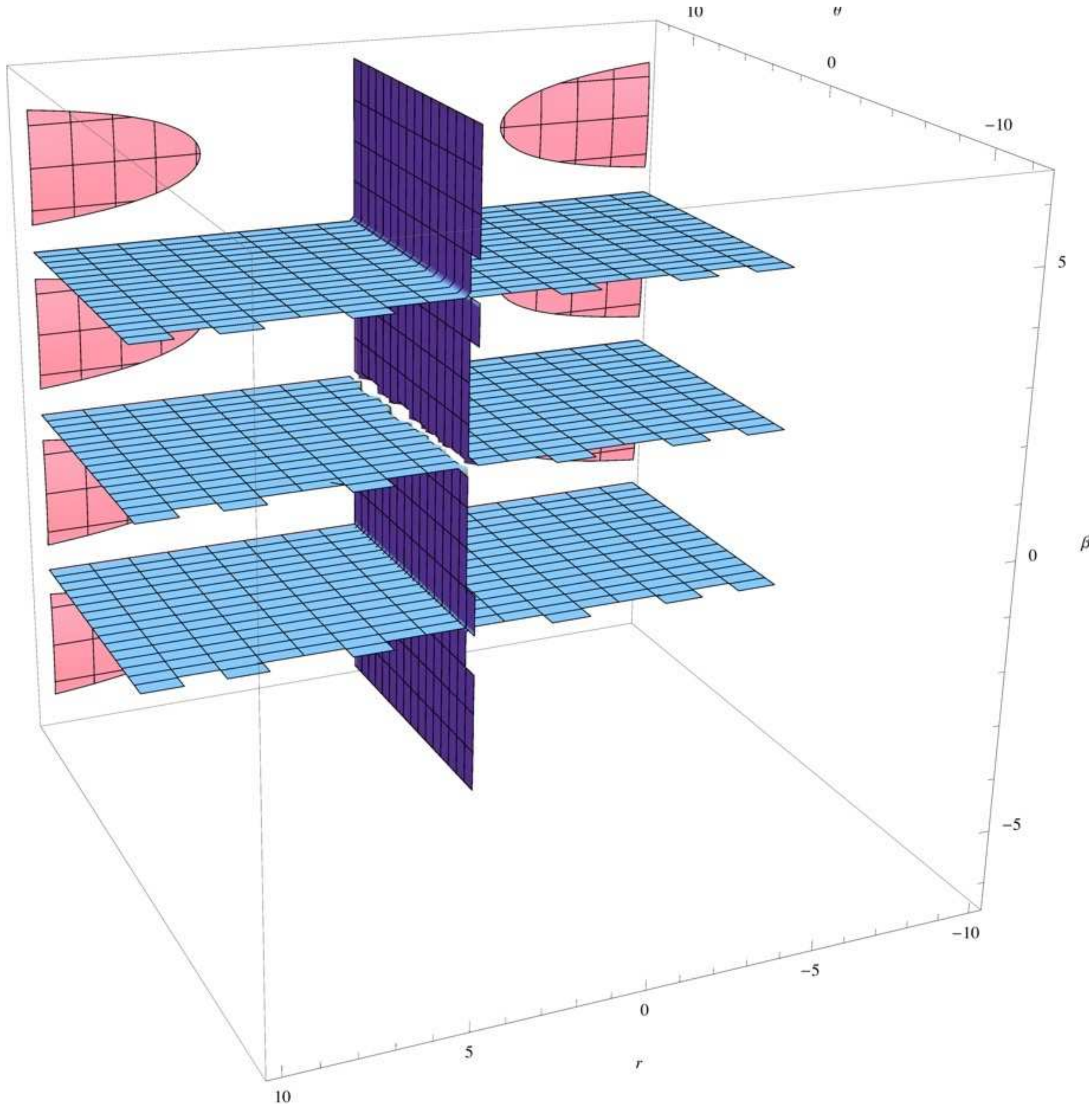
$$\beta = \text{ArcSin}\left[\frac{\sqrt{(4\pi - \theta)\,\theta}}{2\pi}\right] \tag{11}$$

```
RevolutionPlot3D[ArcSin[√((4 π - θ) θ) / (2 π)], {θ, -4 π, 4 π}]
```



$$r = \frac{2\pi r \operatorname{Sin}[\beta]}{\sqrt{4\pi\theta - \theta^2}} \qquad (12)$$

```
ContourPlot3D[2 π (r Sin[β]) / Sqrt[4 π θ - θ^2], {β, -2 π, 2 π}, {θ, -4 π, 4 π}, {r, -10, 10}, AxesLabel → Automatic]
```



Lemma 5 *The height of the cone can be calculated in terms of only r and* θ, thus θ is a function of β alone.

yields,

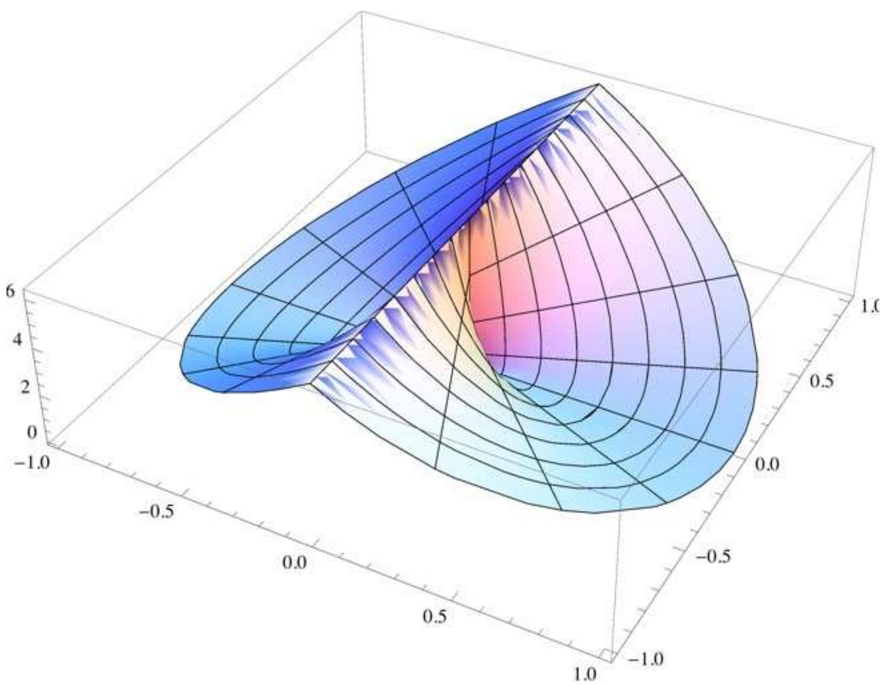
Proof. Since we have shown that θ r = 2 π r - 2 π r_1 and $r_1 \to \sqrt{r^2 - \eta^2}$, we can substitute the expression for r_1, calculated from the Pythagorean theorem in terms of the height of the cone and the initial radius of the circle, into the expression for θ r in terms of the change in circumference of the initial circle to the circle that is the base of the cone into which the circle was transformed.

θ r = 2 π r - 2 π $\sqrt{(r^2 - \eta^2)}$, thus, $\eta = \frac{\sqrt{4 \pi r^2 \theta - r^2 \theta^2}}{2 \pi}$ = (r Sin[β]). From $\frac{2 \pi \eta}{\sqrt{4 \pi \theta - \theta^2}}$= r, we note that: r = $\frac{2 \pi r \, \text{Sin}[\beta]}{\sqrt{4 \pi \theta - \theta^2}}$. So we solve the equation,

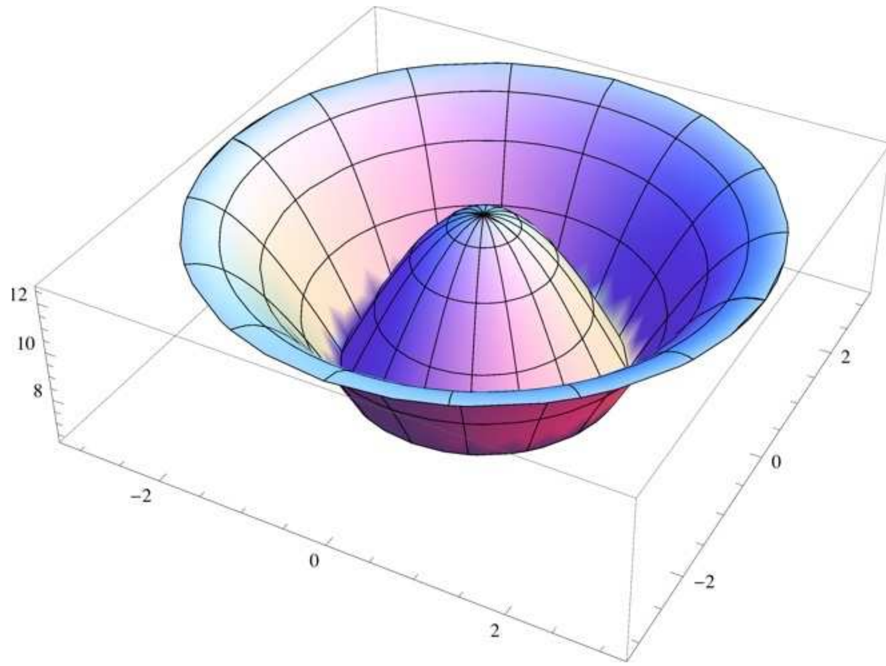
```
Solve[r == 2 π r Sin[β] / Sqrt[4 π θ - θ^2], θ]
```

$$\left\{\left\{\theta \to 2 \left(\pi - \sqrt{\pi^2 - \pi^2 \, \text{Sin}[\beta]^2}\right)\right\}, \left\{\theta \to 2 \left(\pi + \sqrt{\pi^2 - \pi^2 \, \text{Sin}[\beta]^2}\right)\right\}\right\} \quad (13)$$

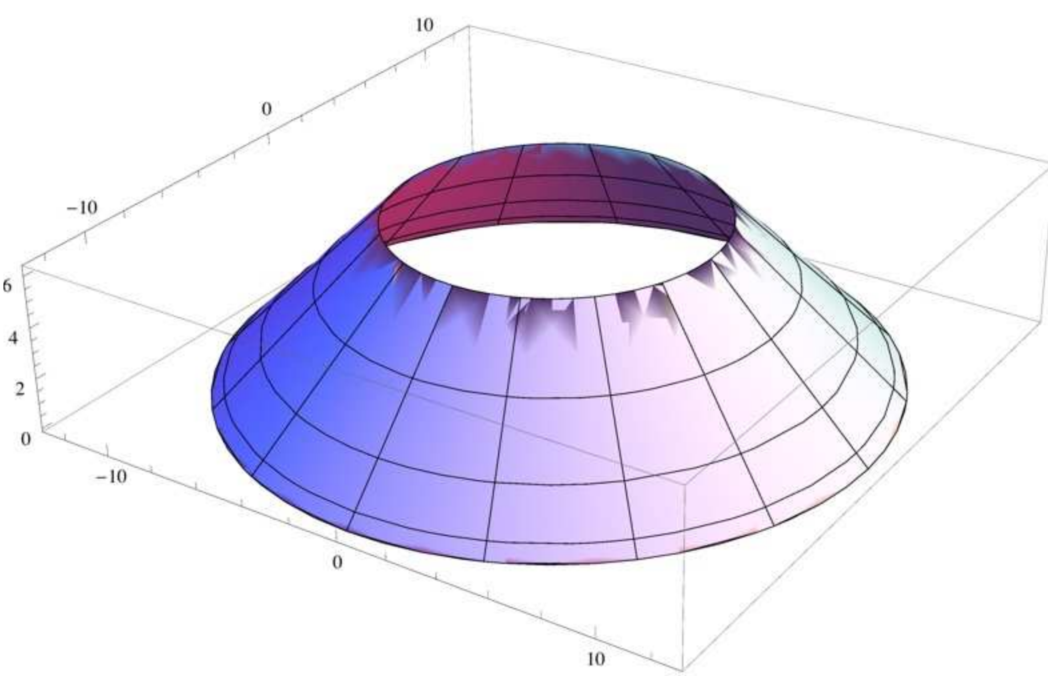
```
RevolutionPlot3D[2 (π - Sqrt[π^2 - π^2 Sin[β]^2]), {r, -1, 1}, {β, -π, π}]
```



```
RevolutionPlot3D[2 (π + Sqrt[π^2 - π^2 Sin[β]^2]), {β, -π, π}]
```

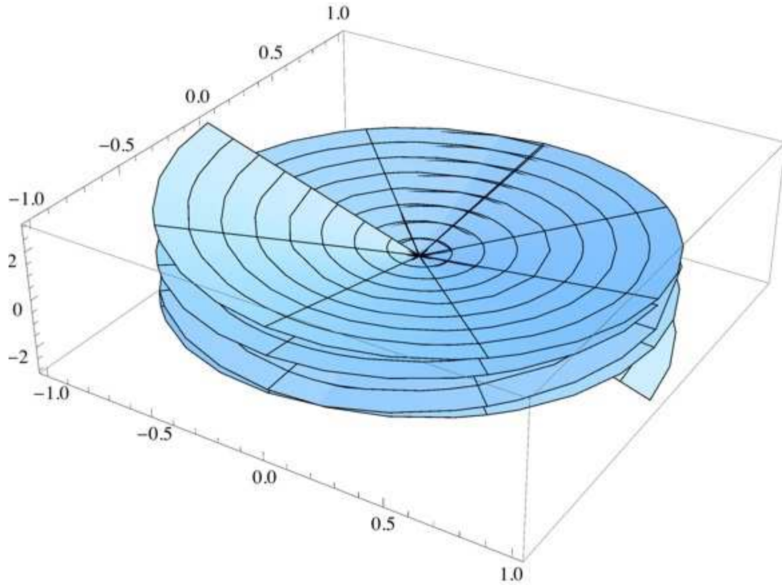



```
RevolutionPlot3D[{2 (π + Sqrt[π^2 - π^2 Sin[β]^2]), 2 (π - Sqrt[π^2 - π^2 Sin[β]^2])}, {β, -π, π}]
```



$$\mathrm{Sin}[\beta] = \frac{r\,(4\,\pi - \Theta)\,\Theta}{4\,\pi^2} \tag{14}$$

```
RevolutionPlot3D[r (4 π - θ) θ / (4 π^2), {r, -1, 1}, {θ, -2 π, 2 π}]
```



$$\text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\,\theta}}{2\pi}\right] = \beta$$

$$\text{Solve}\left[\text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\,\theta}}{2\pi}\right] == \frac{\pi}{3},\ \theta\right]$$

$$\{\{\theta \to \pi\},\ \{\theta \to 3\pi\}\}$$

To place more commentary on the ensuing paradox,

Lemma 8 We can show can show that

$$\left\{\left\{\beta \to -\text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right]\right\},\ \left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right]\right\}\right\}.$$

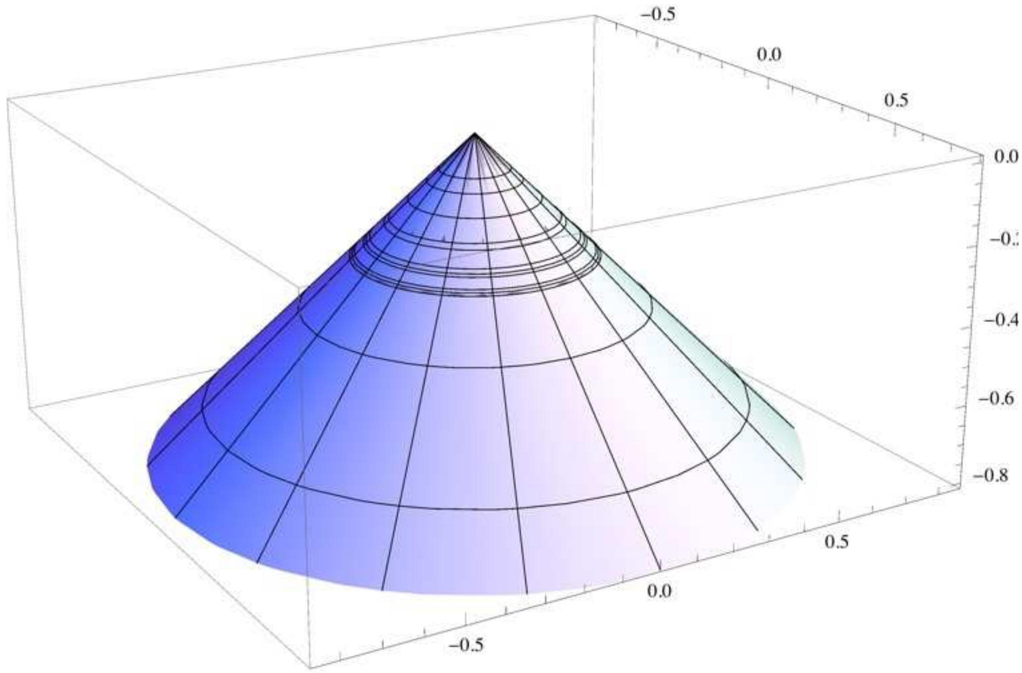
$$\text{Solve}\left[\theta == \frac{4\pi}{3} - \frac{-4\pi^2+12\pi^2\,\text{Sin}[\beta]^2}{6\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}} + \frac{2}{3}\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3},\ \beta\right]$$

$$\left\{\left\{\beta \to -\text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right]\right\},\ \left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right]\right\}\right\}$$

$$\left\{\left\{\beta \to -\text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right]\right\},\ \left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right]\right\}\right\} \quad (15)$$

```
RevolutionPlot3D[{ArcSin[Sqrt[-4 π^2 θ + 4 π θ^2 - θ^3]/(2 π Sqrt[-4 π + θ])], -ArcSin[Sqrt[-4 π^2 θ + 4 π θ^2 - θ^3]/(2 π Sqrt[-4 π + θ])]}, {θ, -4 π, 4 π}]
```

$$\text{RevolutionPlot3D}\left[\left\{\text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right],-\text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right]\right\},\{\theta,-4\pi,4\pi\}\right]$$



III. Computational Results from the Lorentz Transformation

Theorem 3 The "innate velocity," v, within the Lorentz transformation can be solved for in terms of the system of the circle transforming into a cone. If r is multiplied by the Lorenz transformation, then it measures the distance in the prime system, denoted by r '. If t ' equals $\frac{\left(\frac{\theta}{(2\pi)}\right)}{\sqrt{1-\left(\frac{v^2}{c^2}\right)}}$, then the quantity r $\theta = \theta$ ' r '. We are only dealing with algebraic forms and the solutions necessitated by them. Logical, algebraic, reasoning will be given why, when using the exact speed of light, 2.99792458 (10^8) meters per second, the units of the speed of light can be ignored for the purposes of calculation and computation (they canel out - they are equal to one). This theorem states that, although, normal algebra would require the speed of light as a quantity to cancel out, valid expressions for the solutions for the intrinsic velocity, v, can be found in terms of η, r, and θ, or θ and β, depending on the expression used for the height of the cone.

Proof.

$$c = 2.99792458\left(10^8\right)\text{ meters per second}$$

$$r' = r\sqrt{1-\frac{v^2}{c^2}}$$

$$t' = \frac{\left(\frac{\theta}{(2\pi)}\right)}{\sqrt{1-\left(\frac{v^2}{c^2}\right)}}$$

$$t' = \frac{\theta'}{2\pi}$$

$$2\pi t' = \theta'$$

$$\theta' = \frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}$$

Therefore,

$$r' * \theta' = \left(\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}\right)\left(r\sqrt{1-\frac{v^2}{c^2}}\right)$$

$$\left(\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}\right)\left(r\sqrt{1-\frac{v^2}{c^2}}\right) = r\,\theta$$

$$r' * \theta' = r\,\theta = 2\pi r - 2\pi r_1 = 2\pi r - 2\pi\sqrt{r^2 - \eta^2}$$

```
Solve[r θ == 2 π r - 2 π Sqrt[r^2 - η^2], η]
```

$$\left\{\left\{\eta \to -\frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}\right\}, \left\{\eta \to \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}\right\}\right\}$$

```
Solve[r ' θ ' == 2 π r - 2 π Sqrt[r^2 - η^2], η]
```

$$\left\{\left\{\eta \to -\frac{\sqrt{r'}\sqrt{\theta'}\sqrt{4\pi r - r'\theta'}}{2\pi}\right\}, \left\{\eta \to \frac{\sqrt{r'}\sqrt{\theta'}\sqrt{4\pi r - r'\theta'}}{2\pi}\right\}\right\}$$

The argument follows modus ponens, saying that, through commutation, $r'\theta' = \theta r$, therefore $\eta = \frac{\sqrt{r'}\sqrt{\theta'}\sqrt{4\pi r - r'\theta'}}{2\pi} = \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}$

$\eta =$

$$\frac{\sqrt{r'}\,\sqrt{\theta'}\,\sqrt{4\,\pi\,r - r'\,\theta'}}{2\,\pi} = \frac{\sqrt{r\,\sqrt{1-\frac{v^2}{c^2}}}\,\sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}}\,\sqrt{4\,\pi\,r - r\,\sqrt{1-\frac{v^2}{c^2}}\,\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}}}{2\,\pi} = \frac{\sqrt{r\,\theta}\,\sqrt{4\,\pi\,r - r\,\theta}}{2\,\pi}$$

where c is its numeric value of the speed of light, its units being shown to cancel out.

$$\frac{\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}}{\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}}$$

$$\textbf{Solve}\left[\sqrt{\frac{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}},\ \text{meters}\right]$$

{{}}

$$\textbf{Solve}\left[\sqrt{\frac{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}},\ \text{second}\right]$$

{{}}

$$\textbf{Solve}\left[\sqrt{\frac{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}},\ v\right]$$

{{}}

$$\textbf{Solve}\left[\sqrt{\frac{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}},\ c\right]$$

$\text{Solve}\left[\text{True},\ 2.99792\times 10^{8}\right]$

$$\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}\;\sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}}}\;\sqrt{4\pi r - r\theta}}}{2\pi} =$$

$$\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}\;\frac{\theta}{\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}}\;\sqrt{4\pi r - r\theta}}}{2\pi} =$$

$$\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c)^2}}\;\frac{\theta}{\sqrt{1-\frac{v^2}{(c)^2}}}\;\sqrt{4\pi r - r\theta}}}{2\pi} = (1)\;\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c)^2}}\;\sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{(c)^2}}}}\;\sqrt{4\pi r - r\theta}}}{2\pi}$$

Logically, the units of the speed of light will cancel out, therefore, their relationship will be set equal to one and taken out of the equation for the purposes of computational calculation.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c)^2}}\;\frac{\theta}{\sqrt{1-\frac{v^2}{(c)^2}}}\;\sqrt{4\pi r - r\theta}}}{2\pi} == \eta,\ \text{meters}\right]$$

{{}}

Meters cancel out.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c)^2}}\;\frac{\theta}{\sqrt{1-\frac{v^2}{(c)^2}}}\;\sqrt{4\pi r - r\theta}}}{2\pi} == \eta,\ \text{second}\right]$$

{{}}

Seconds cancel out.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}\;\frac{\theta}{\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}}\;\sqrt{4\pi r - r\theta}}}{2\pi} == \eta,\ c\right]$$

{{}}

The numeric c cancels out.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}\sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}}}\sqrt{4\pi r-r\theta}}}{2\pi}==\eta,\ r\right]$$

$$\left\{\left\{r\to-\frac{2\pi\eta}{\sqrt{4\pi\theta-\theta^2}}\right\},\ \left\{r\to\frac{2\pi\eta}{\sqrt{4\pi\theta-\theta^2}}\right\}\right\}$$

Radius yields the result from Lemma 3.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}\sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}}}\sqrt{4\pi r-r\theta}}}{2\pi}==\eta,\ v\right]$$

{}

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}\frac{\theta}{\sqrt{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}}\sqrt{4\pi r-r\theta}}}{2\pi}==\eta,\ v\right]$$

{{}}

Velocity cancels out.

Velocity cancels out. Only when using the exact speed of light (in scientific notation) can solutions be found.

We set the speed of light equal to its numeric value for the purpose of making computations, dropping the units, because in the expression for the height of the cone, they would cancel out anyway. It should be noted that this is necessary for computing the function of the velocity and that the exact speed of light is to be used as well as that the numeric value of the speed of light has to be in the form of scientific notation in order to find results to this equation.

```
c := 2.99792458 (10^8)
```

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{(v)^2}{(2.99792458\,(10\text{^}8))^2}}\sqrt{\frac{\theta}{\sqrt{1-\frac{(v)^2}{(2.99792458\,(10\text{^}8))^2}}}}\sqrt{4\pi r-r\theta}}}{2\pi}==\eta,\ v\right]$$

$$\left\{\left\{v\to-\frac{1.\sqrt{3.54814\times10^{18}\,\eta^2-1.12941\times10^{18}\,r^2\,\theta+8.98755\times10^{16}\,r^2\,\theta^2}}{\sqrt{39.4784\,\eta^2-12.5664\,r^2\,\theta+r^2\,\theta^2}}\right\},\right.$$
$$\left.\left\{v\to\frac{\sqrt{3.54814\times10^{18}\,\eta^2-1.12941\times10^{18}\,r^2\,\theta+8.98755\times10^{16}\,r^2\,\theta^2}}{\sqrt{39.4784\,\eta^2-12.5664\,r^2\,\theta+r^2\,\theta^2}}\right\}\right\}$$

Theorem 3 Continued From the expression of the height of the cone of Lemma 1, with the Lorentz transformations implicitly expressed, we can solve for the velocity within the Lorenz coefficient in terms of the height of the cone, the initial radius, and the angle, θ.

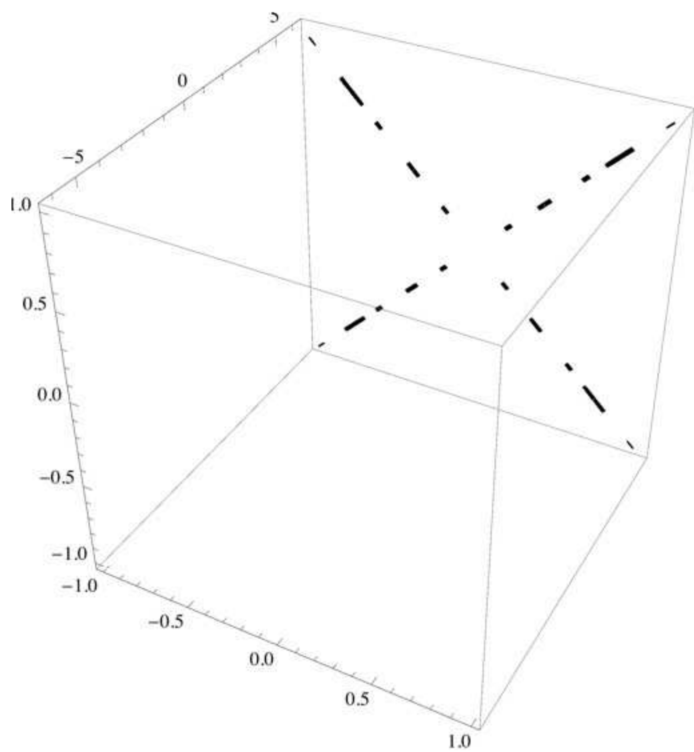
Proof.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{(v)^2}{c^2}}}\sqrt{\frac{\theta}{\sqrt{1-\frac{(v)^2}{c^2}}}}\sqrt{4\pi r - r\theta}}{2\pi} == \eta,\ v\right]$$

$$\left\{\left\{v \to -\frac{1.\sqrt{3.54814\times10^{18}\,\eta^2 - 1.12941\times10^{18}\,r^2\,\theta + 8.98755\times10^{16}\,r^2\,\theta^2}}{\sqrt{39.4784\,\eta^2 - 12.5664\,r^2\,\theta + r^2\,\theta^2}}\right\},\right.$$

$$\left.\left\{v \to \frac{\sqrt{3.54814\times10^{18}\,\eta^2 - 1.12941\times10^{18}\,r^2\,\theta + 8.98755\times10^{16}\,r^2\,\theta^2}}{\sqrt{39.4784\,\eta^2 - 12.5664\,r^2\,\theta + r^2\,\theta^2}}\right\}\right\}$$

```
ContourPlot3D[
 Sqrt[3.5481432270250993`*^18 η^2 - 1.1294090667581471`*^18 r^2 θ + 8.987551787368176`*^16 r^2 θ^2] /
  Sqrt[39.47841760435743` η^2 - 12.566370614359172` r^2 θ + r^2 θ^2],
 {r, -1, 1}, {θ, -2 π, 2 π}, {η, -1, 1}]
```



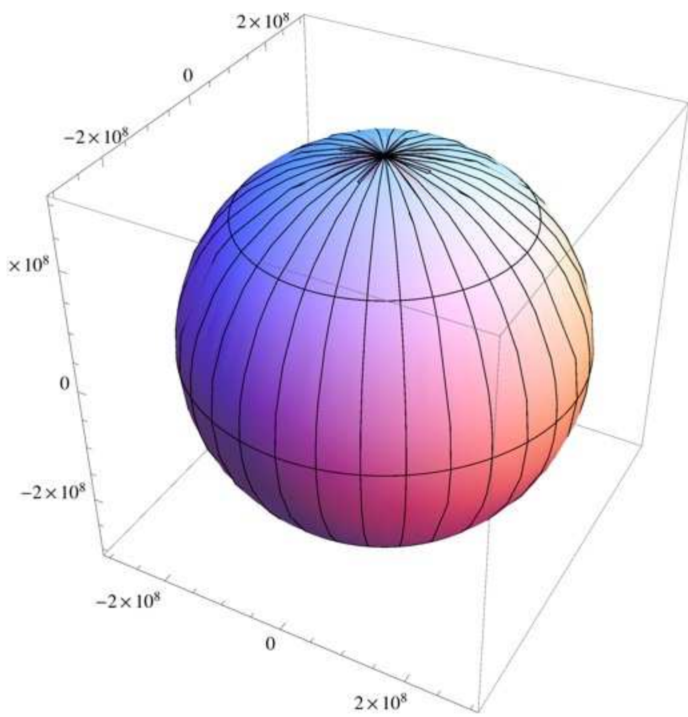
Theorem 3 Continued From the expression of the height of the cone, from Lemma 1 with the Lorentz transformations implicitly expressed, we can solve for the velocity within the Lorenz coefficient in terms of θ and β.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{(v)^2}{c^2}}}\sqrt{\frac{\theta}{\sqrt{1-\frac{(v)^2}{c^2}}}}\sqrt{4\pi r - r\theta}}{2\pi} == r\,\text{Sin}[\beta],\ v\right]$$

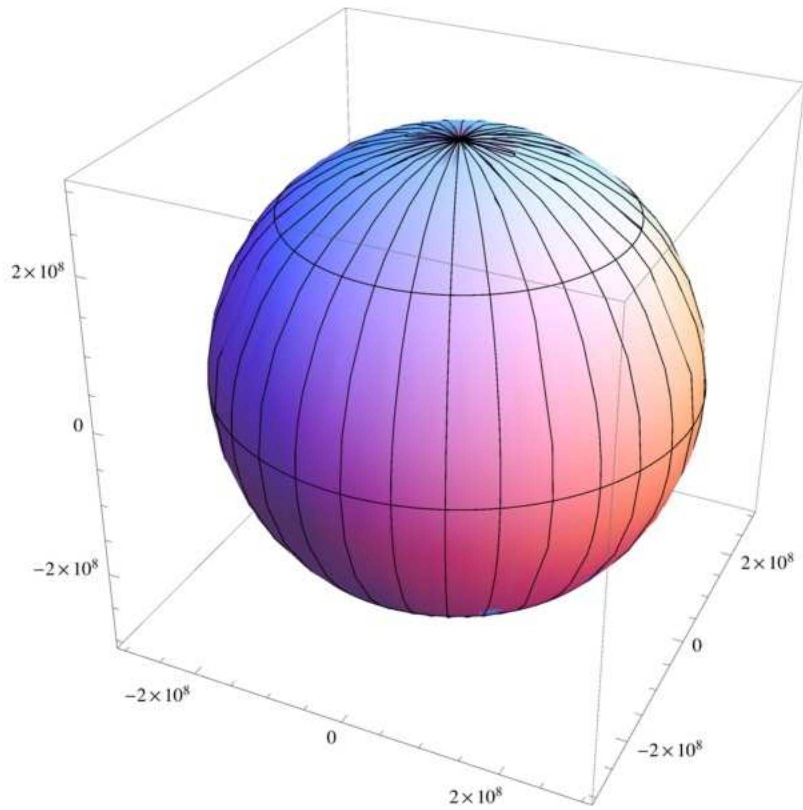
$$\left\{\left\{v \to -\frac{1.\sqrt{-1.12941\times10^{18}\,\theta + 8.98755\times10^{16}\,\theta^2 + 3.54814\times10^{18}\,\text{Sin}[\beta]^2}}{\sqrt{-12.5664\,\theta + \theta^2 + 39.4784\,\text{Sin}[\beta]^2}}\right\},\right.$$

$$\left.\left\{v \to \frac{\sqrt{-1.12941\times10^{18}\,\theta + 8.98755\times10^{16}\,\theta^2 + 3.54814\times10^{18}\,\text{Sin}[\beta]^2}}{\sqrt{-12.5664\,\theta + \theta^2 + 39.4784\,\text{Sin}[\beta]^2}}\right\}\right\}$$

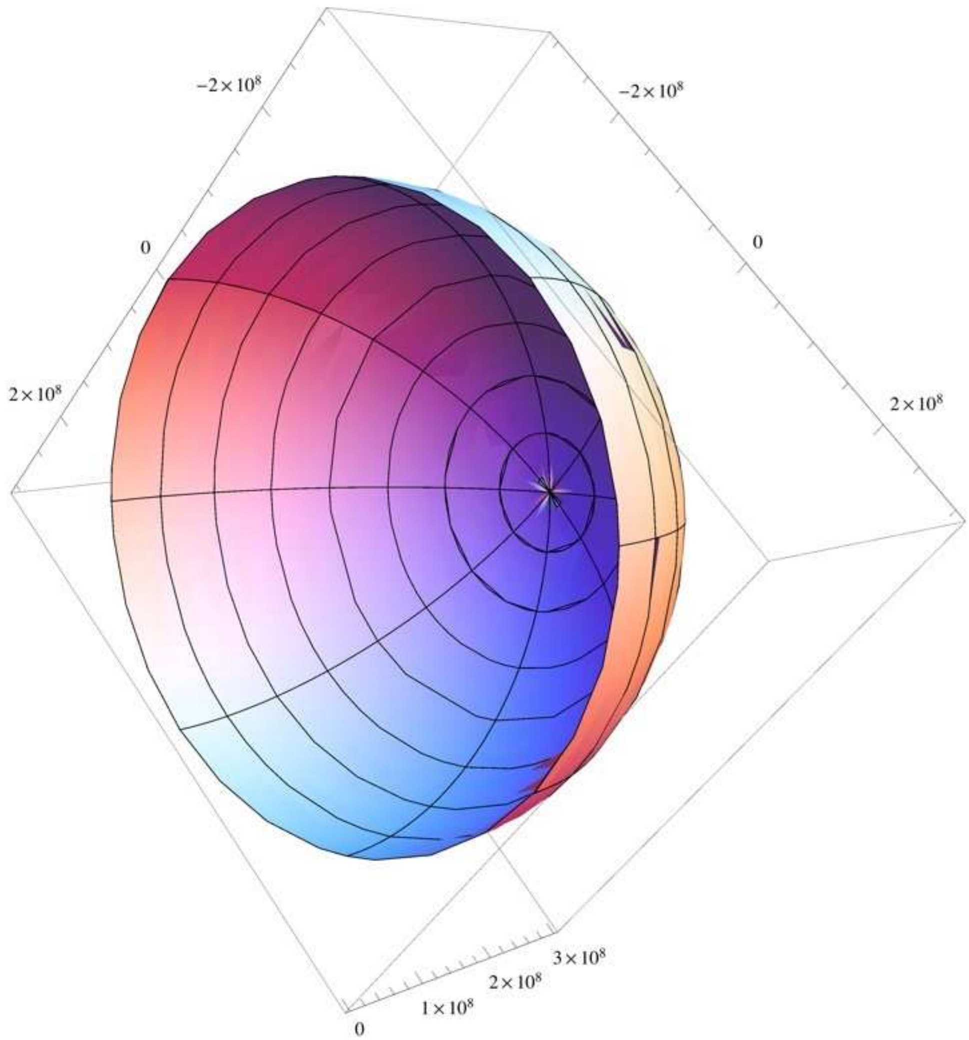
```
SphericalPlot3D[
 {-(1.` √(-1.1294090667581471`*^18 θ + 8.987551787368176`*^16 θ^2 + 3.5481432270250993`*^18
       Sin[β]^2)) / (√(-12.566370614359172` θ + θ^2 + 39.47841760435743` Sin[β]^2)),
  √(-1.1294090667581471`*^18 θ + 8.987551787368176`*^16 θ^2 + 3.5481432270250993`*^18 Sin[β]^2)
   / √(-12.566370614359172` θ + θ^2 + 39.47841760435743` Sin[β]^2)},
 {θ, -2 π, 2 π}, {β, -π / 2, π / 2}]
```



```
SphericalPlot3D[
  Sqrt[-1.1294090667581471`*^18 θ + 8.987551787368176`*^16 θ^2 + 3.5481432270250993`*^18 Sin[β]^2] /
  Sqrt[-12.566370614359172` θ + θ^2 + 39.47841760435743` Sin[β]^2],
  {θ, -2 π, 2 π}, {β, -π / 2, π / 2}]
```

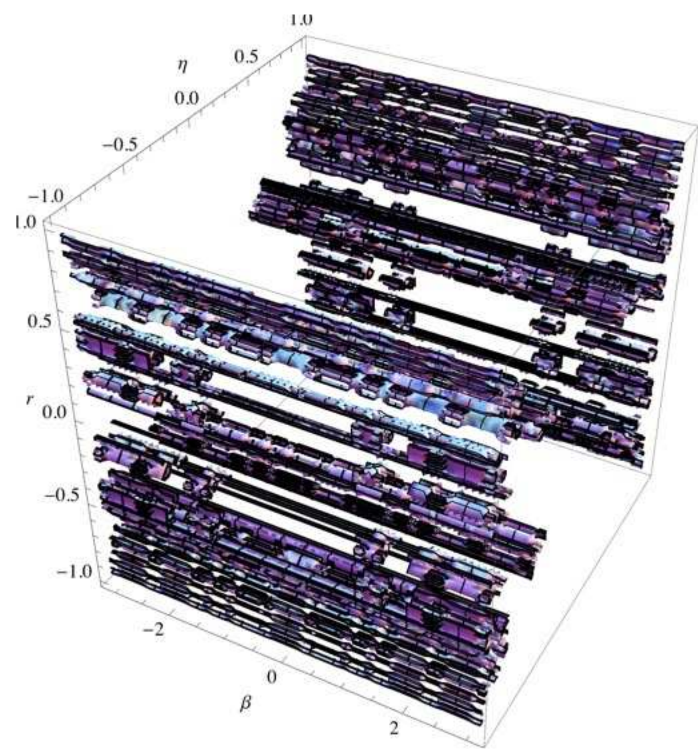


```
SphericalPlot3D[
  Sqrt[-1.1294090667581471`*^18 θ + 8.987551787368176`*^16 θ^2 + 3.5481432270250993`*^18 Sin[β]^2] /
  Sqrt[-12.566370614359172` θ + θ^2 + 39.47841760435743` Sin[β]^2],
  {β, -π / 2, π / 2}, {θ, -2 π, 2 π}]
```



$$\text{Substitute}: \Theta \to \frac{2\pi\left(r^2 + \sqrt{r^4 - r^2\eta^2}\right)}{r^2}$$

```
ContourPlot3D[(Sqrt[(-1.1294090667581471`*^18 (2 π (r^2 + Sqrt[r^4 - r^2 η^2]))/r^2 +
      8.987551787368176`*^16 ((2 π (r^2 + Sqrt[r^4 - r^2 η^2]))/r^2)^2 + 3.5481432270250993`*^18 Sin[β]^2)]) /
   (Sqrt[(-12.566370614359172` (2 π (r^2 + Sqrt[r^4 - r^2 η^2]))/r^2 + ((2 π (r^2 + Sqrt[r^4 - r^2 η^2]))/r^2)^2 +
      39.47841760435743` Sin[β]^2)]),
  {β, -π, π}, {η, -1, 1}, {r, -1, 1}, AxesLabel → Automatic]
```



$$\text{Plot3D}\Bigg[\Bigg(\sqrt{\Bigg(-1.129409066758147\text{`*^}18\left(\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right)+}$$

$$8.987551787368176\text{`*^}16\left(\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right)^2+$$

$$3.5481432270250993\text{`*^}18\,\text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{\left(4\pi-\left(\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right)\right)\left(\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right)}}{2\pi}\right]\right]^2\Bigg)\Bigg)\Bigg/$$

$$\Bigg(\sqrt{\Bigg(-12.566370614359172\text{`}\left(\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right)+\left(\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right)^2+}$$

$$39.47841760435743\text{`}\,\text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{\left(4\pi-\left(\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right)\right)\left(\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right)}}{2\pi}\right]\right]^2\Bigg)\Bigg),$$

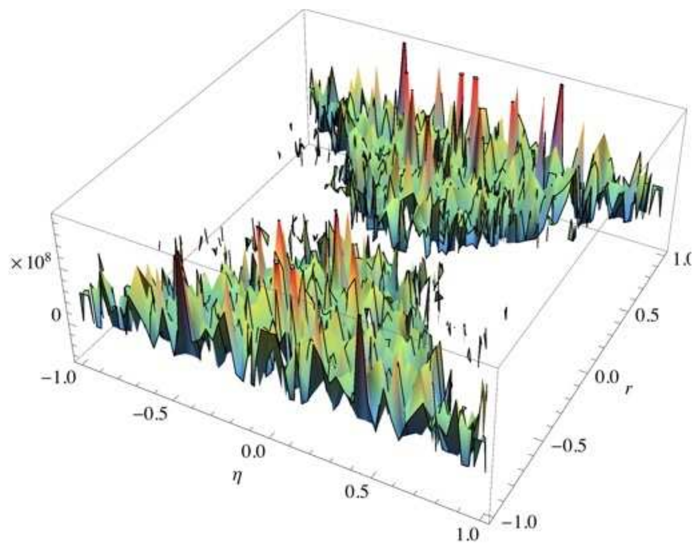
{η, -1, 1}, {r, -1, 1}, ColorFunction → "Rainbow", AxesLabel → Automatic]

$$\texttt{Plot3D}\Bigg[\sqrt{\left(-1.129409066758147\text{`}*\text{^}18\,\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}+8.987551787368176\text{`}*\text{^}16\left(\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right)^2+3.5481432270250993\text{`}*\text{^}18\,\texttt{Sin}\left[\texttt{ArcSin}\left[\frac{\sqrt{\left(4\,\pi-\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right)\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}}}{2\,\pi}\right]\right]^2\right)}\Bigg/$$

$$\sqrt{\left(-12.566370614359172\text{`}\,\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}+\left(\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right)^2+39.47841760435743\text{`}\,\texttt{Sin}\left[\texttt{ArcSin}\left[\frac{\sqrt{\left(4\,\pi-\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right)\left(\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right)}}{2\,\pi}\right]\right]^2\right)},$$

{η, -1, 1}, {r, -1, 1}, ColorFunction → "Rainbow", AxesLabel → Automatic]

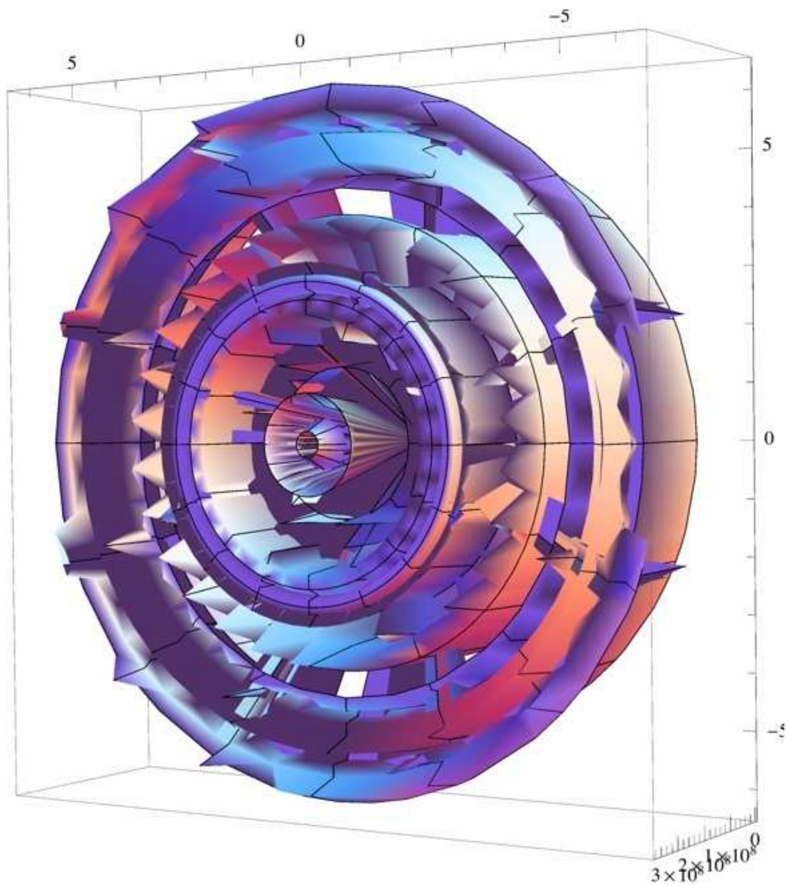
Power::infy : Infinite expression $\frac{1}{0.}$ encountered. ≫



"The Kantian Substitution"

$$\beta = \text{ArcSin}\left[\frac{\sqrt{(4\,\pi - \Theta)\,\Theta}}{2\,\pi}\right]$$

```
RevolutionPlot3D[(Sqrt[(-1.1294090667581471`*^18 θ +
    8.987551787368176`*^16 θ^2 + 3.5481432270250993`*^18 Sin[ArcSin[Sqrt[(4 π - θ) θ]/(2 π)]]^2)]) /
  (Sqrt[(-12.566370614359172` θ + θ^2 + 39.47841760435743` Sin[ArcSin[Sqrt[(4 π - θ) θ]/(2 π)]]^2)]), {θ,
  -2 π, 2 π}]
```

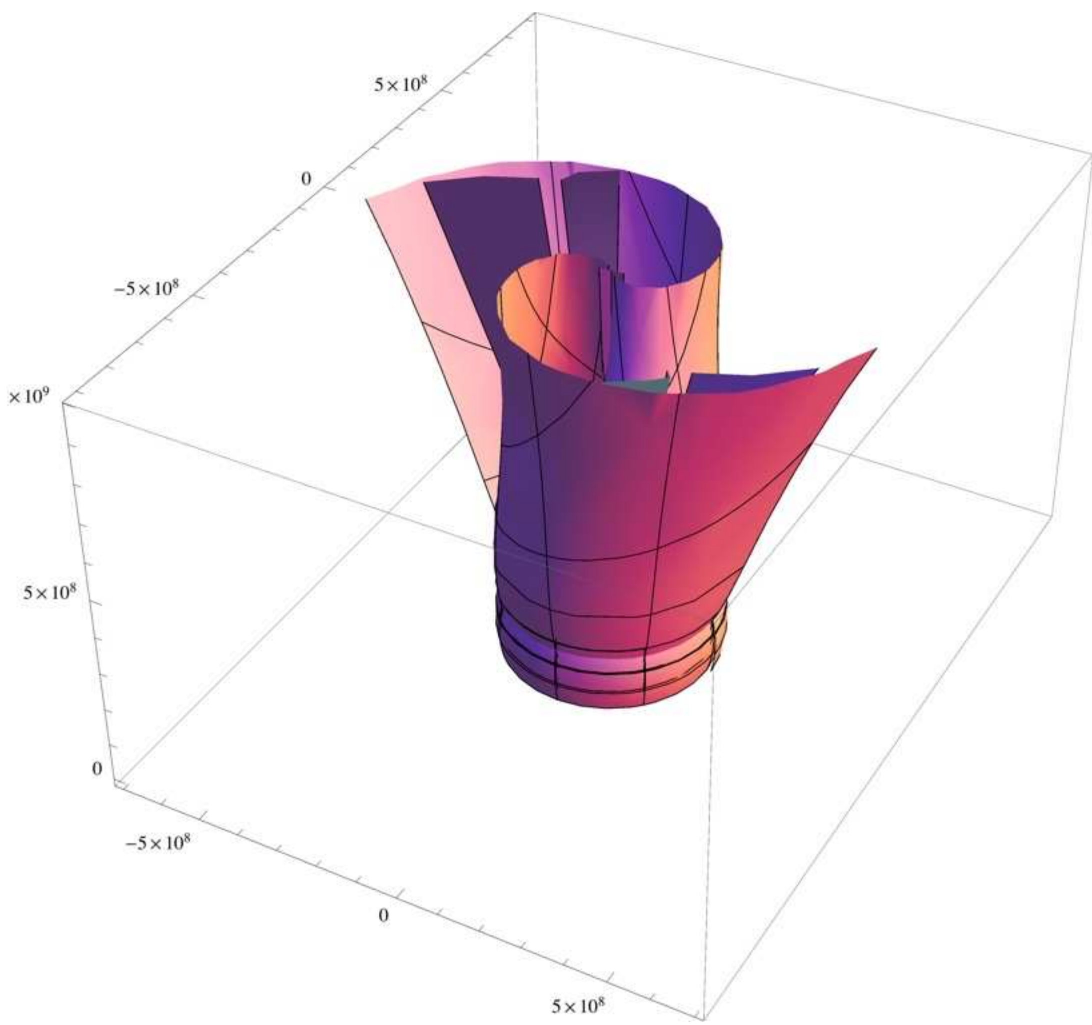


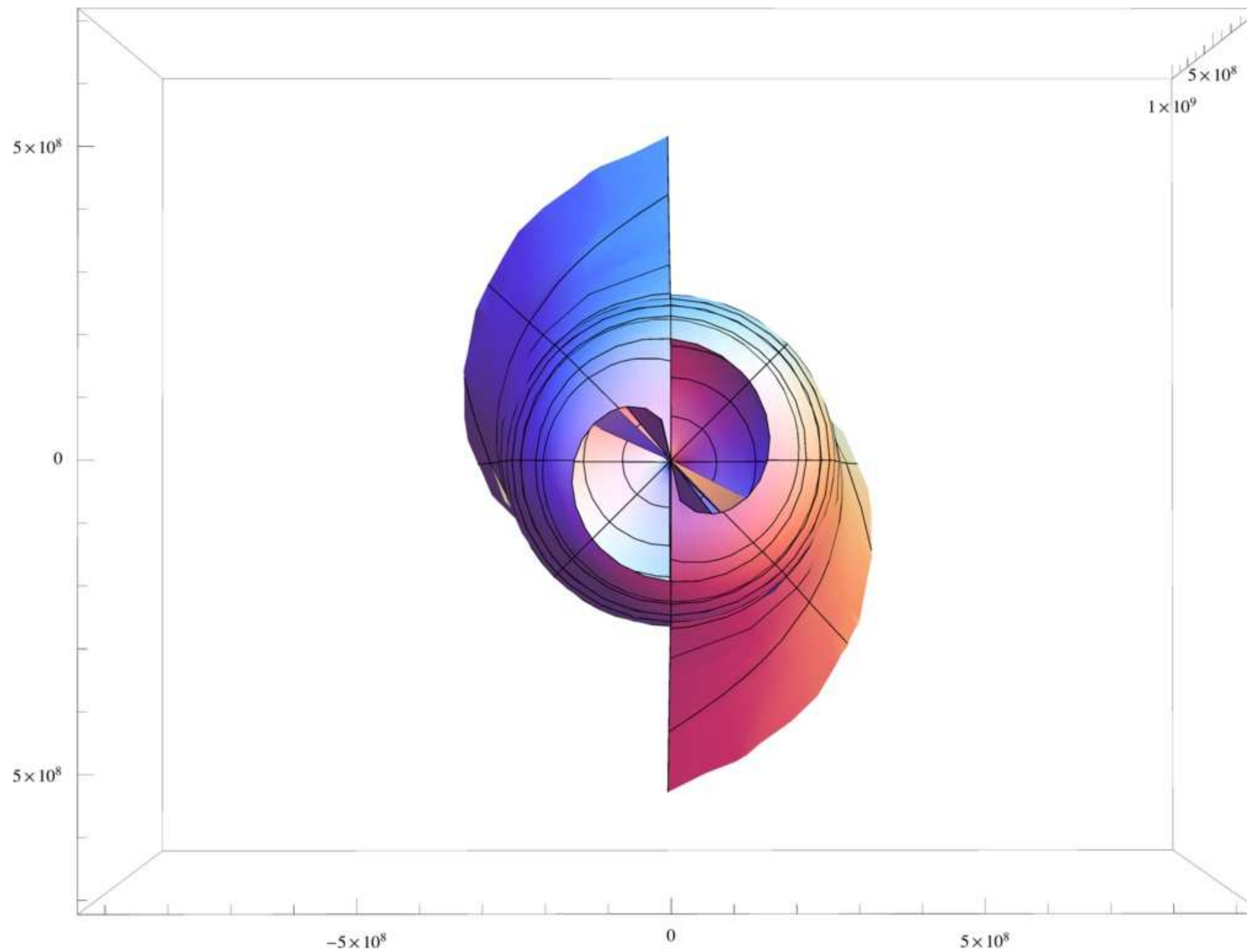
Further Substitutions:

$$\Theta \to 2 \left(\pi + \sqrt{\pi^2 - \pi^2 \, \mathrm{Sin}[\beta]^2}\right)$$

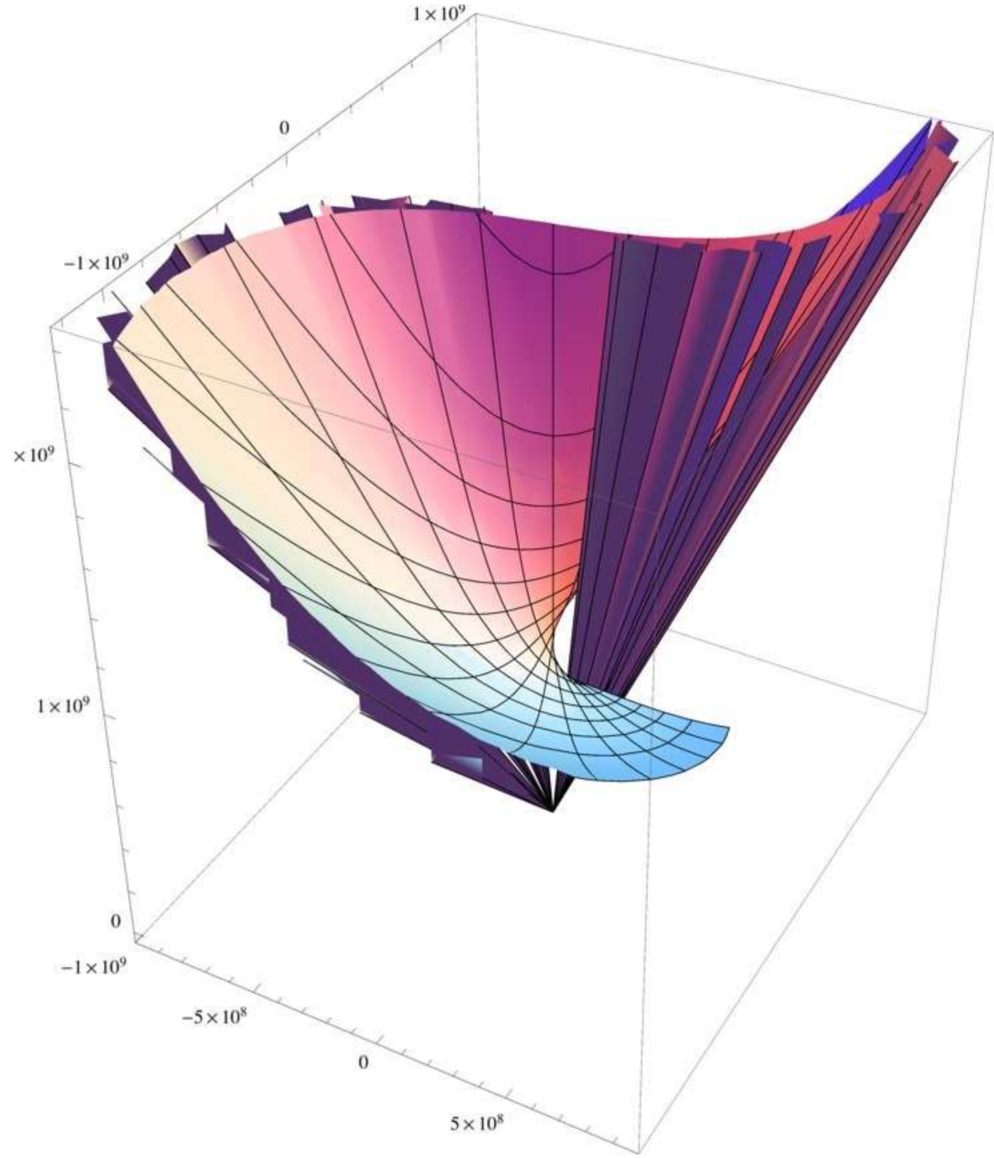
$$\Theta \to \frac{4\pi}{3} - \frac{-4\pi^2 + 12\pi^2 \, \mathrm{Sin}[\beta]^2}{6\left(-\pi^3 + 18\pi^3 \, \mathrm{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \, \mathrm{Sin}[\beta]^2 + 11\pi^6 \, \mathrm{Sin}[\beta]^4 + \pi^6 \, \mathrm{Sin}[\beta]^6}\right)^{1/3}} + \frac{2}{3}\left(-\pi^3 + 18\pi^3 \, \mathrm{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \, \mathrm{Sin}[\beta]^2 + 11\pi^6 \, \mathrm{Sin}[\beta]^4 + \pi^6 \, \mathrm{Sin}[\beta]^6}\right)^{1/3}$$

```
SphericalPlot3D[
 (√(-1.1294090667581471`*^18 θ + 8.987551787368176`*^16 (2 (π + √(π^2 - π^2 Sin[β]^2)))^2 +
      3.5481432270250993`*^18 Sin[β]^2)) /
  (√(-12.566370614359172` θ + (4 π/3 - (-4 π^2 + 12 π^2 Sin[β]^2) /
          (6 (-π^3 + 18 π^3 Sin[β]^2 + 3 √3 √(-π^6 Sin[β]^2 + 11 π^6 Sin[β]^4 + π^6 Sin[β]^6))^(1/3)) +
         2/3 (-π^3 + 18 π^3 Sin[β]^2 + 3 √3 √(-π^6 Sin[β]^2 + 11 π^6 Sin[β]^4 + π^6 Sin[β]^6))^(1/3))^2 +
      39.47841760435743` Sin[β]^2)), {β, -π / 2, π / 2}, {θ, -2 π, 2 π}]
```

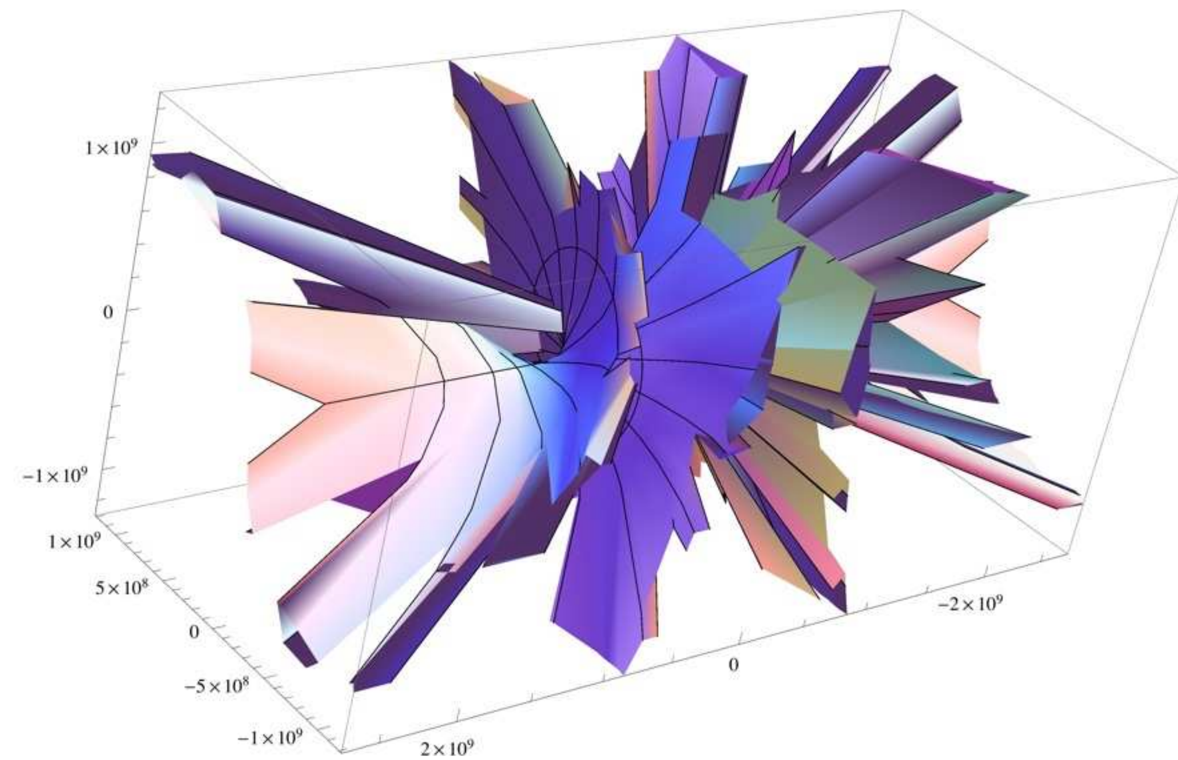



5×10^8
1×10^9
5×10^8
0
5×10^8
−5×10^8
0
5×10^8

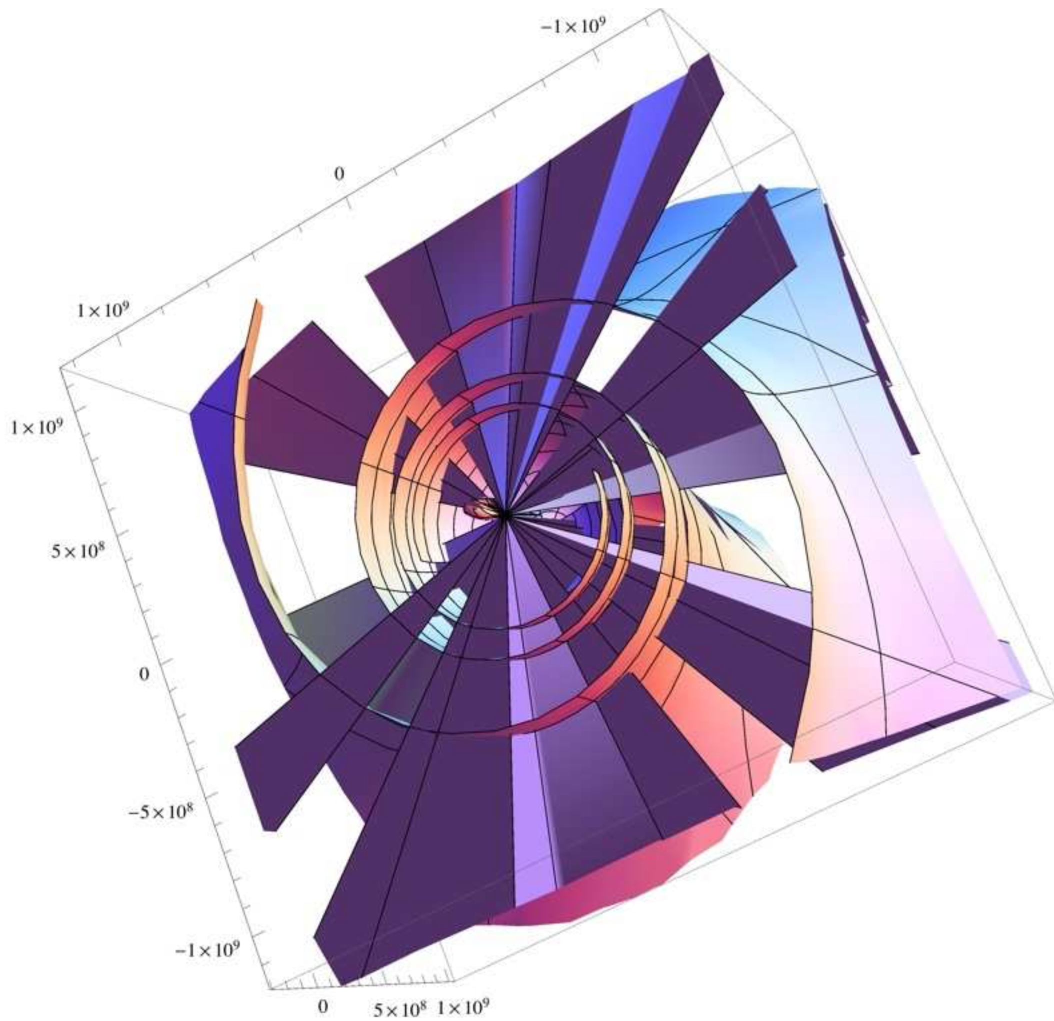
```
SphericalPlot3D[
 (√(-1.1294090667581471`*^18 (2 (π + √(π^2 - π^2 Sin[β]^2))) + 8.987551787368176`*^16 θ^2 +
      3.5481432270250993`*^18 Sin[β]^2)) /
  (√(-12.566370614359172` θ + θ^2 + 39.47841760435743` Sin[β]^2)), {β, -π / 4, π / 4}, {θ, -π, π}]
```

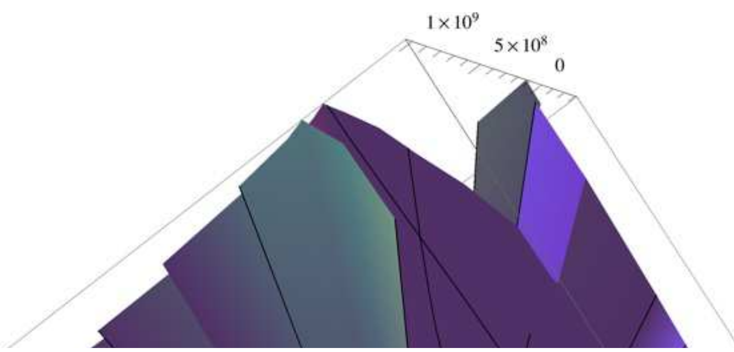


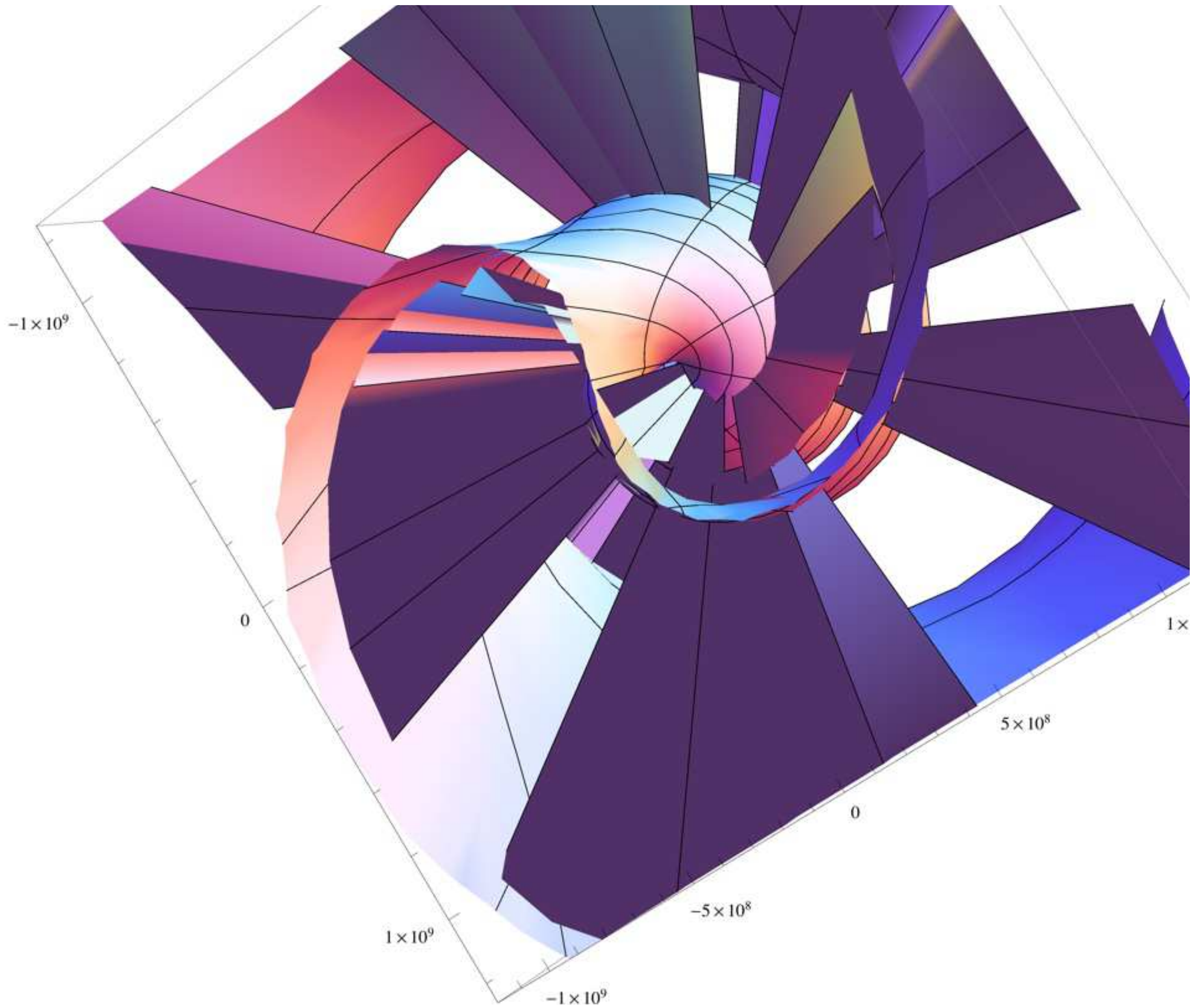
```
SphericalPlot3D[
 (Sqrt[(-1.1294090667581471`*^18 (2 (π + Sqrt[π^2 - π^2 Sin[β]^2])) + 8.987551787368176`*^16 θ^2 +
      3.5481432270250993`*^18 Sin[β]^2)] /
  (Sqrt[-12.566370614359172` θ + θ^2 + 39.47841760435743` Sin[β]^2]), {β, -2 π, 2 π}, {θ, -2 π, 2 π}]
```

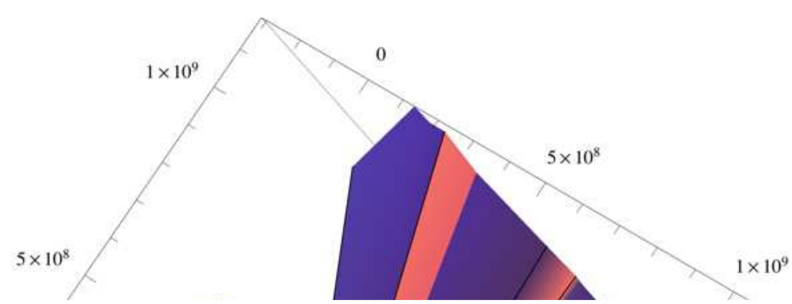


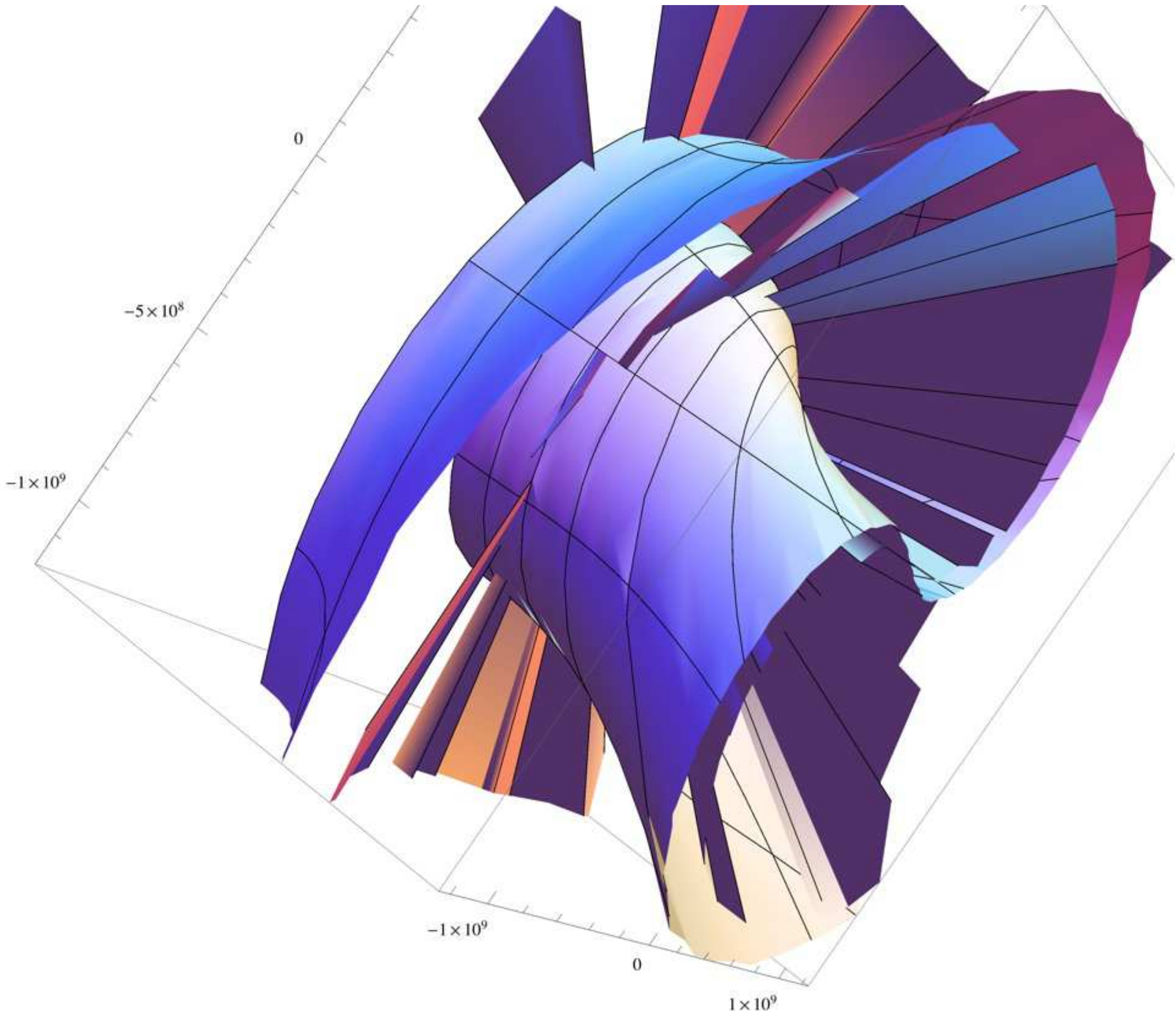

```
SphericalPlot3D[
 (Sqrt[(-1.1294090667581471`*^18 θ + 8.987551787368176`*^16 (4 π/3 - (-4 π^2 + 12 π^2 Sin[β]^2) /
          (6 (-π^3 + 18 π^3 Sin[β]^2 + 3 Sqrt[3] Sqrt[-π^6 Sin[β]^2 + 11 π^6 Sin[β]^4 + π^6 Sin[β]^6])^(1/3)) +
        2/3 (-π^3 + 18 π^3 Sin[β]^2 + 3 Sqrt[3] Sqrt[-π^6 Sin[β]^2 + 11 π^6 Sin[β]^4 + π^6 Sin[β]^6])^(1/3))^2 +
      3.5481432270250993`*^18 Sin[β]^2)] /
  (Sqrt[-12.566370614359172` θ + θ^2 + 39.47841760435743` Sin[β]^2]),
 {β,
  -π /
   2, π /
   2}, {θ, -2
   π, 2
   π}]
```


-1×10^9
0
1×10^9
1×10^9
5×10^8
0
-5×10^8
-1×10^9
0
5×10^8 1×10^9

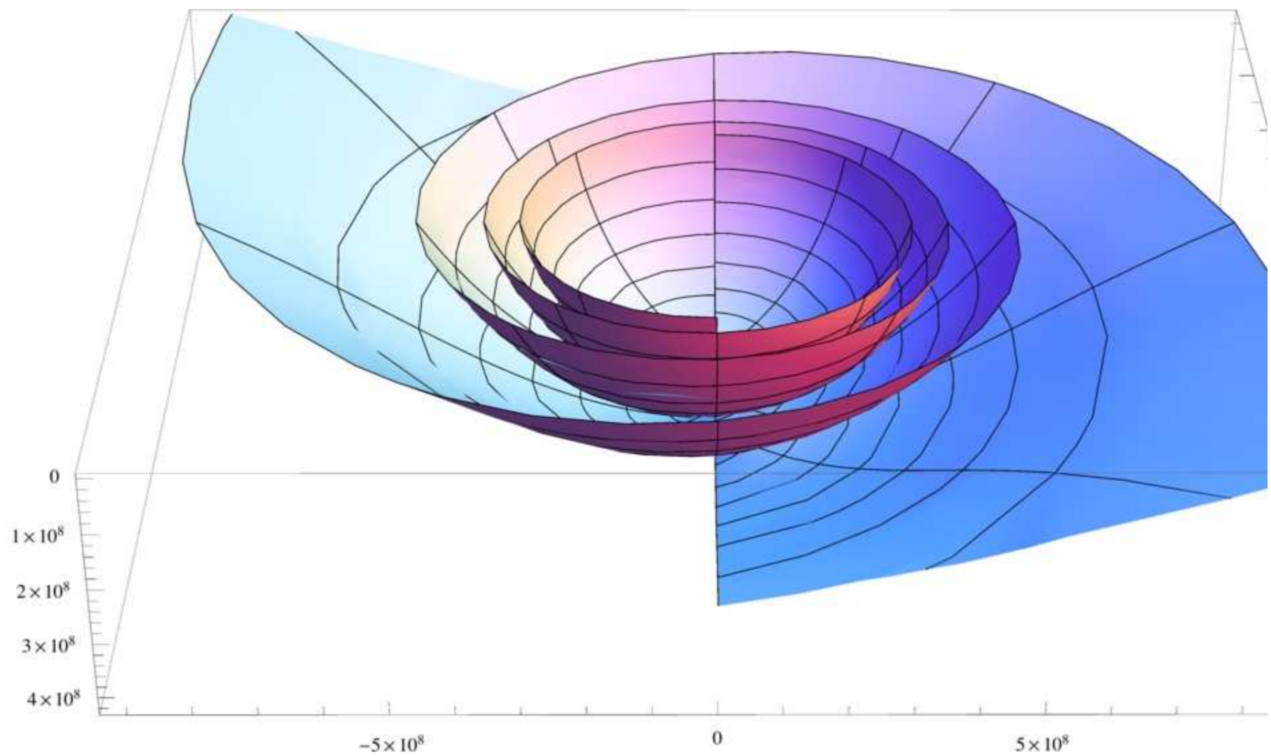

1×10^9
5×10^8
0


-1 × 10^9
0
1 ×
5 × 10^8
0
-5 × 10^8
1 × 10^9
-1 × 10^9

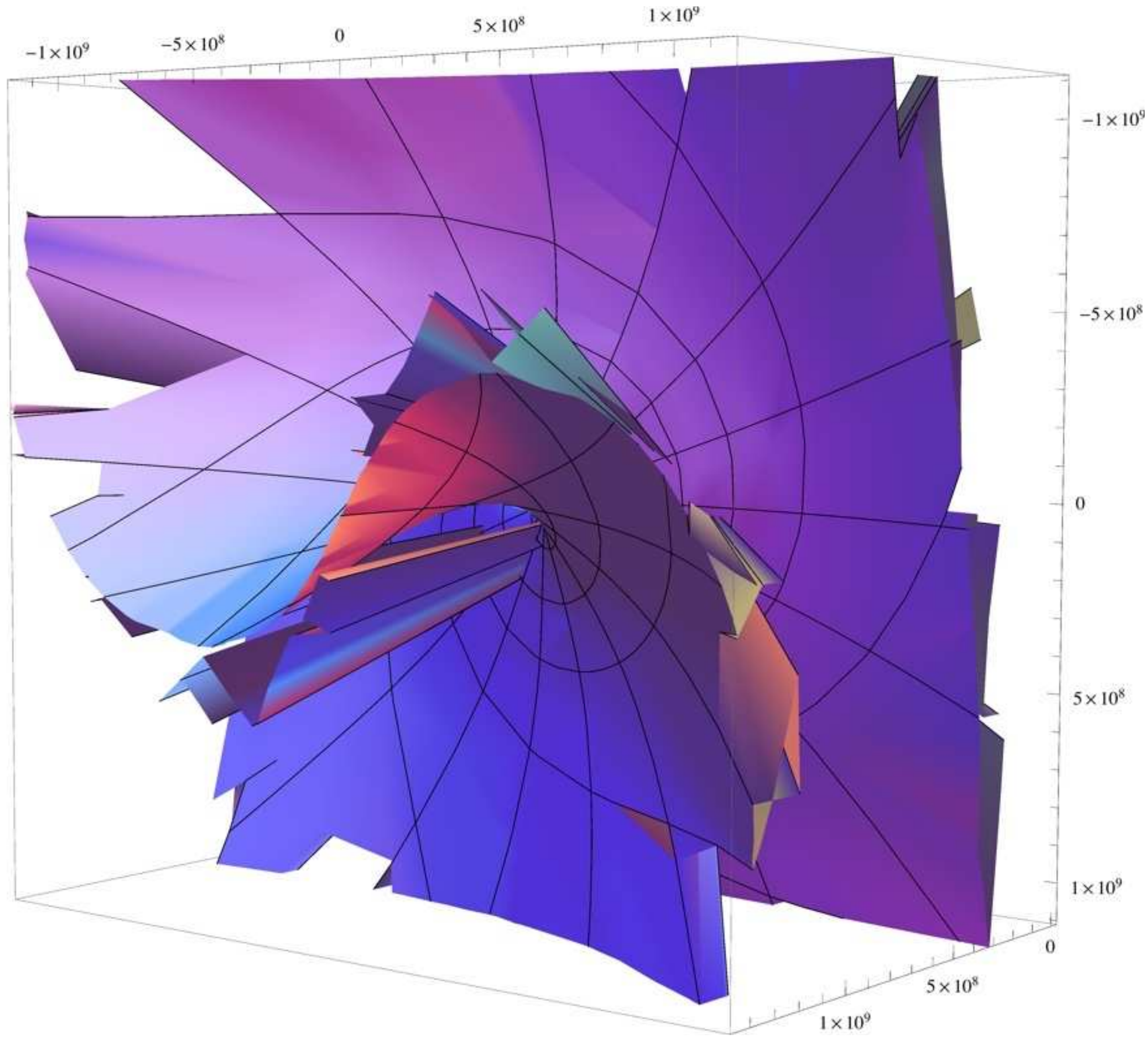

1 × 10^9
0
5 × 10^8
1 × 10^9
5 × 10^8

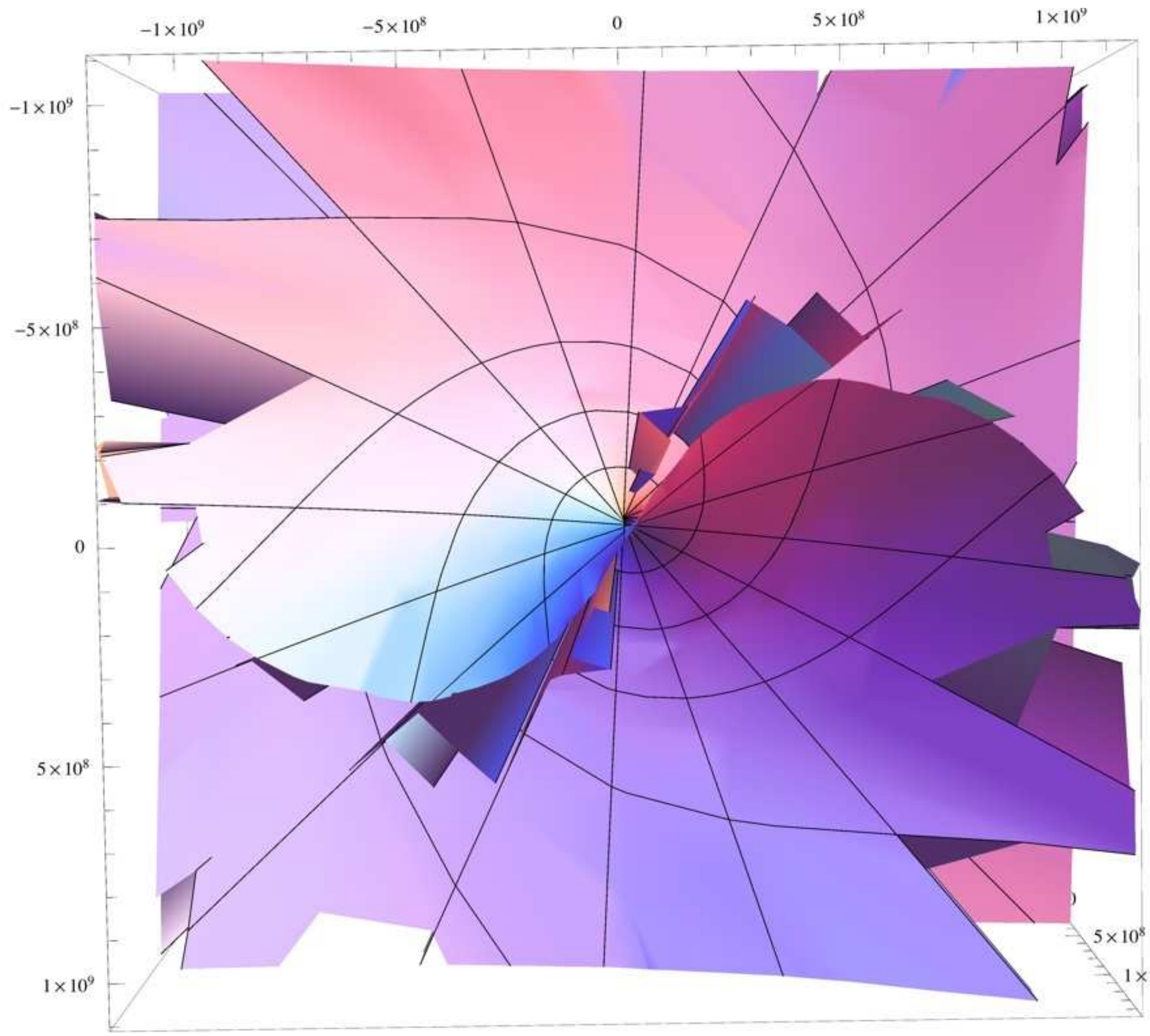



```
SphericalPlot3D[
 (√(-1.1294090667581471`*^18 θ + 8.987551787368176`*^16 (2 (π - √(π^2 - π^2 Sin[β]^2)))^2 +
      3.5481432270250993`*^18 Sin[β]^2)) /
  (√(-12.566370614359172` θ + θ^2 + 39.47841760435743` Sin[β]^2)), {β, -π / 2, π / 2}, {θ, -2 π, 2 π}]
```


0
1×10^8
2×10^8
3×10^8
4×10^8
−5×10^8
0
5×10^8

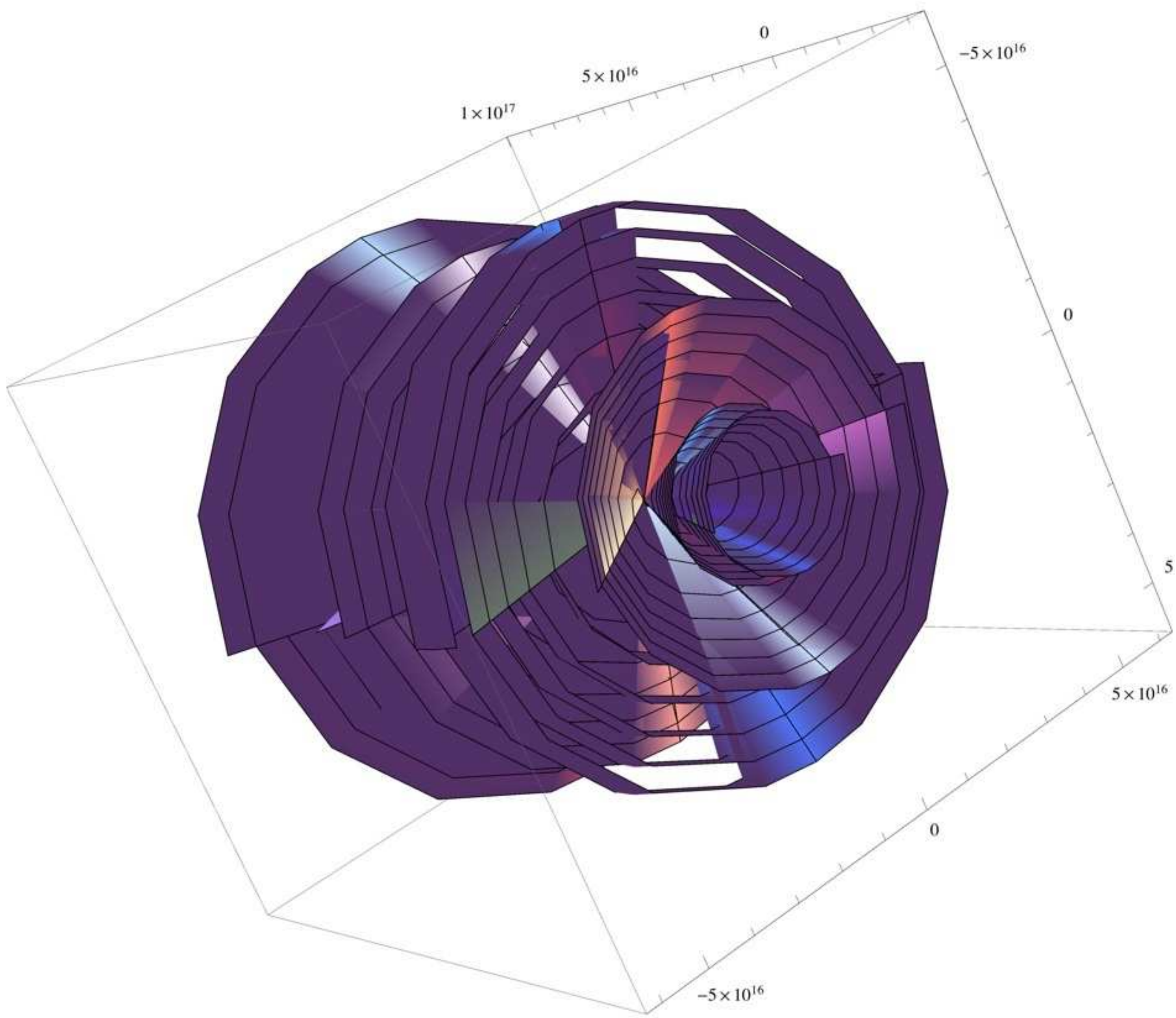
```
SphericalPlot3D[(√(-1.1294090667581471`*^18 θ +
      8.987551787368176`*^16 θ^2 + 3.5481432270250993`*^18 Sin[β]^2)) /
  (√(-12.566370614359172` (2 (π - √(π^2 - π^2 Sin[β]^2))) + θ^2 + 39.47841760435743` Sin[β]^2)),
 {β, -π / 2, π / 2}, {θ, -2 π, 2 π}]
```

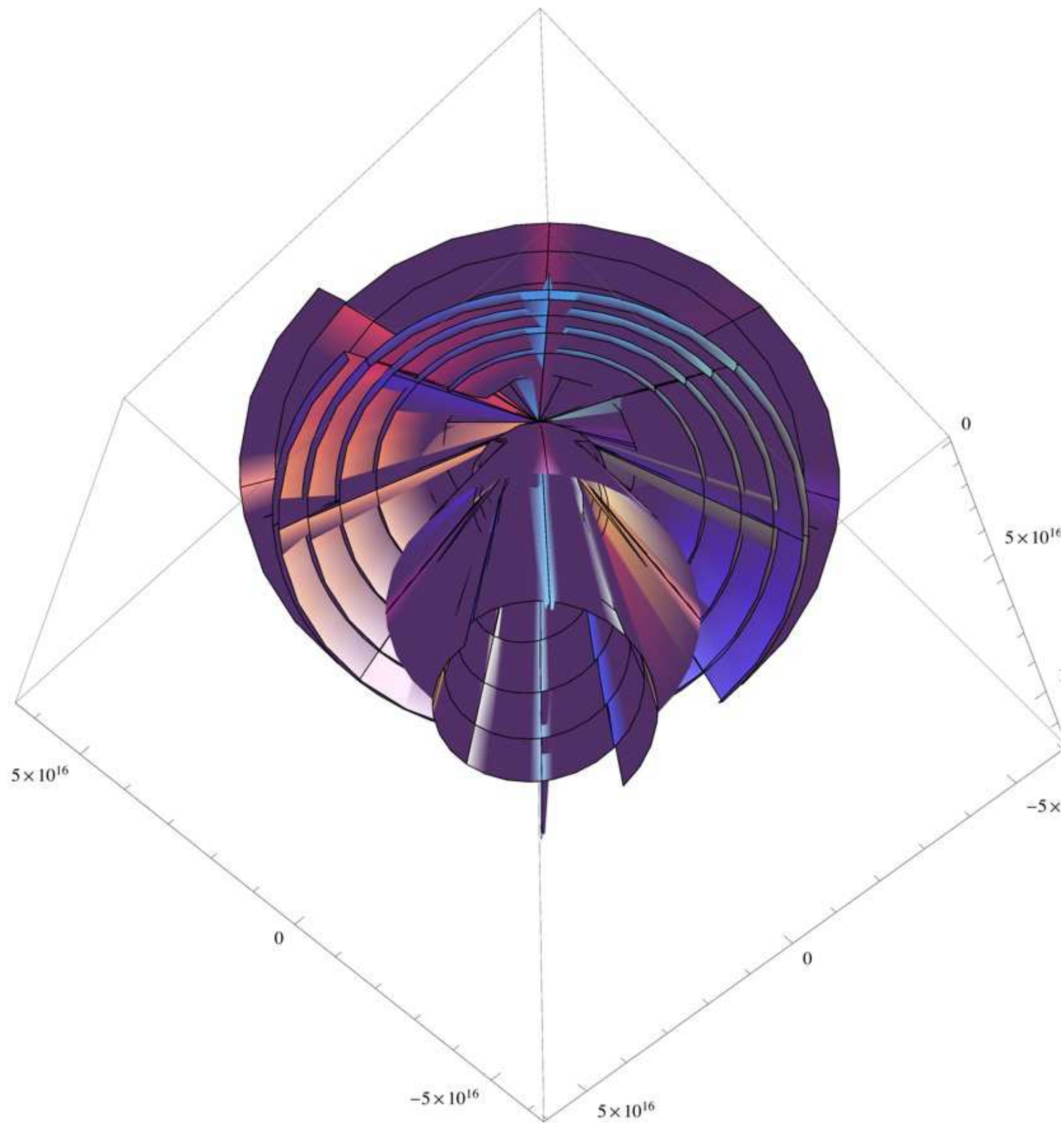




−1×10^9
−5×10^8
0
5×10^8
1×10^9
−1×10^9
−5×10^8
0
5×10^8
1×10^9
5×10^8
1×10^9

```
SphericalPlot3D[
 (Sqrt[(-1.1294090667581471`*^18 θ + 8.987551787368176`*^16 (2 (π + Sqrt[π^2 - π^2 Sin[β]^2]))^2 +
      3.5481432270250993`*^18 Sin[β]^2)]) /
  (Sqrt[(-12.566370614359172` (2 (π + Sqrt[π^2 - π^2 Sin[β]^2])) + (2 (π + Sqrt[π^2 - π^2 Sin[β]^2]))^2 +
      39.47841760435743` Sin[β]^2)]), {β, -π, π}, {θ, -4 π, 4 π}]
```

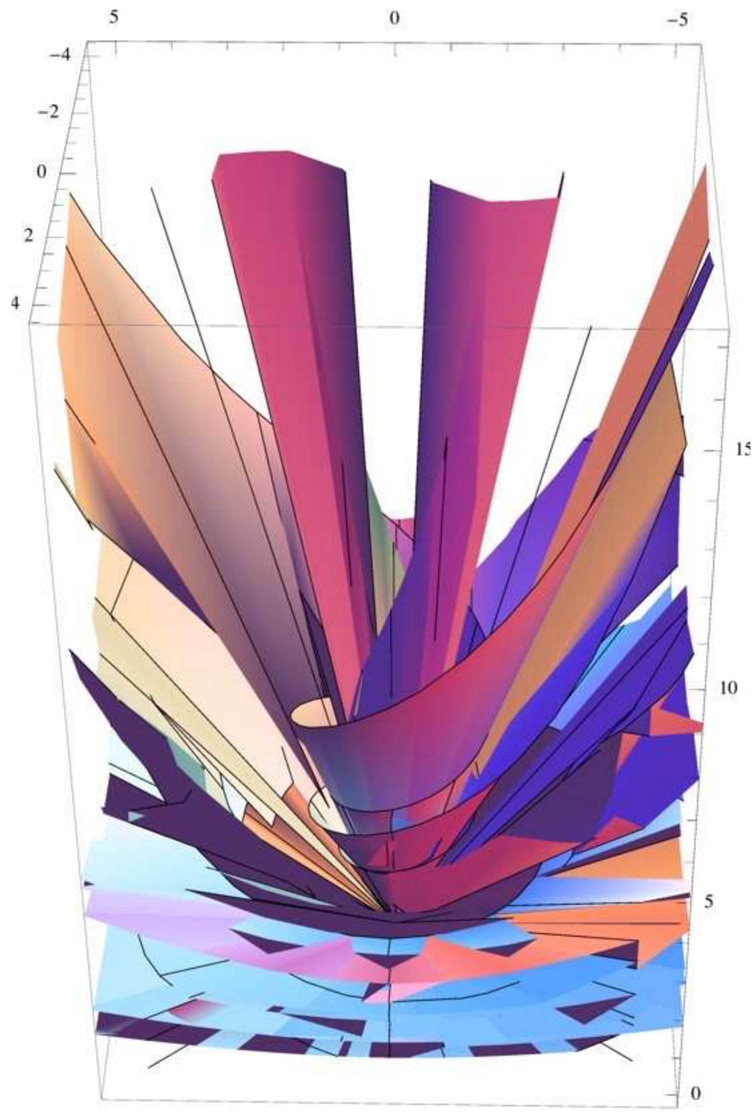



0
5×10^16
-5×
5×10^16
0
-5×10^16
0
5×10^16

SphericalPlot3D$\Big[$

$$\left(\sqrt{\left(-1.1294090667581471\grave{}*\hat{}18\ \left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)\right)+8.987551787368176\grave{}*\hat{}16\ \left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)\right)^2+3.5481432270250993\grave{}*\hat{}18\,\mathrm{Sin}[\beta]^2\right)}\right)\Big/\left(\sqrt{\left(-12.566370614359172\grave{}\ \theta+(\theta)^2+39.47841760435743\grave{}\ \mathrm{Sin}[\beta]^2\right)}\right),$$

$\{\beta, -\pi/2, \pi/2\}, \{\theta, -2\pi, 2\pi\}\Big]$



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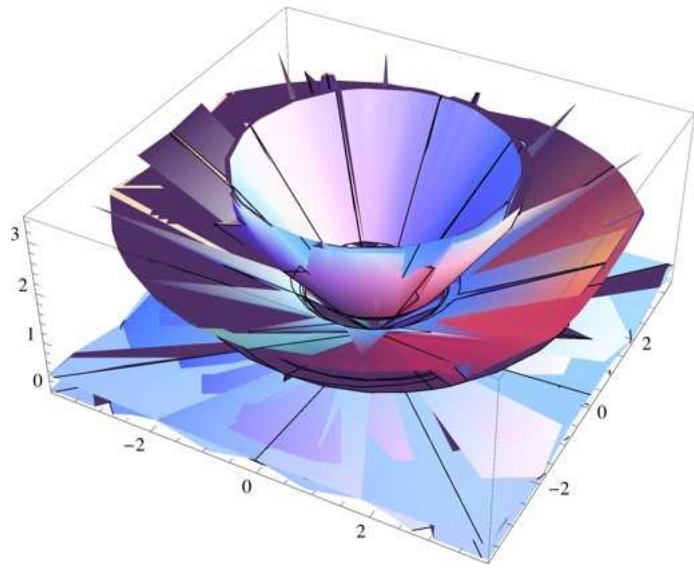
```
SphericalPlot3D[
 (√(-1.1294090667581471`*^18 (2 (π + √(π^2 - π^2 Sin[β]^2))) + 8.987551787368176`*^16
      (2 (π + √(π^2 - π^2 Sin[β]^2)))^2 + 3.5481432270250993`*^18 Sin[β]^2)) /
  (√(-12.566370614359172` (2 (π + √(π^2 - π^2 Sin[β]^2))) + (θ)^2 + 39.47841760435743` Sin[β]^2)),
 {β, -π / 2, π / 2}, {θ, -2 π, 2 π}]
```



The theory is an expansive one. There are many more posible substitutions, and I urge you to find one for yourself. (Just plug in a logically valid substitution (where you see θ, just put a valid θ solution)).

Lemma 9 The position at which the initial radius equals the height of the cone specifies an angle measure, θ to be 2π, however, if we observe the expression for the height of the cone equaling the initial radius, we find that the innate velocity is an indeterminate at 2π. Yet, in the style of phenomenological reduction, if we "bracket" the natural attitude, which takes for granted the existence of the world, or in our system, the existence of the other expressions of the parameters of the cone, which would allow us to show that the initial radius equals 0 when the circle is folded all the way up, or that θ has an exact value of 2π when the circle is folded all the way up, setting the natural attitude and that which it takes for granted aside, then we can visualize the expression for the innate velocity just as it is found as a logical result from this equation alone. This is done for the purpose of mathematically linguistic phenomenological reduction, for which there is philosophical precedence in the works of Edmund Husserl, because he laid the groundwork for transcendental philosophy as all embracing ontology and universal science. Geometry is formal ontology, and combination with logical relations algebraically, allows us to discuss the being of just this equation phenomenologically, for we are studying just the experience of this equation upon visualization.

Proof.

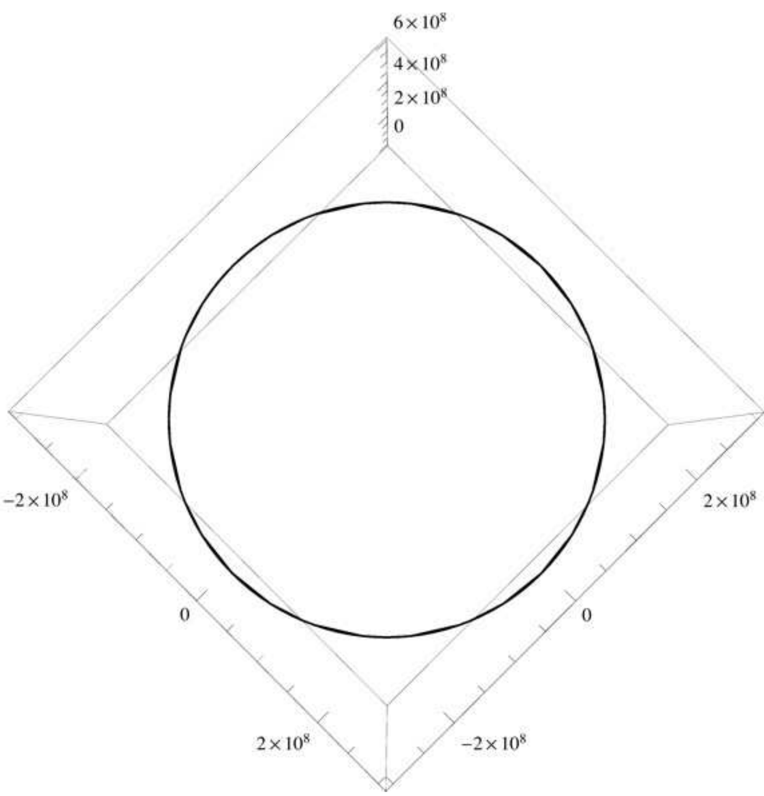
$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{(v)^2}{c^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{(v)^2}{c^2}}}}\ \sqrt{4\pi r - r\theta}}{2\pi} == r,\ v\right]$$

$$\left\{\left\{v \to -\frac{1.\sqrt{3.54814\times10^{18} - 1.12941\times10^{18}\,\theta + 8.98755\times10^{16}\,\theta^2}}{\sqrt{39.4784 - 12.5664\,\theta + \theta^2}}\right\},\right.$$

$$\left.\left\{v \to \frac{\sqrt{3.54814\times10^{18} - 1.12941\times10^{18}\,\theta + 8.98755\times10^{16}\,\theta^2}}{\sqrt{39.4784 - 12.5664\,\theta + \theta^2}}\right\}\right\}$$

$$\text{RevolutionPlot3D}\left[\left\{-\frac{1.\grave{}\ \sqrt{3.5481432270250993\grave{}*\hat{}18 - 1.1294090667581471\grave{}*\hat{}18\,\theta + 8.987551787368176\grave{}*\hat{}16\,\theta^2}}{\sqrt{39.47841760435743\grave{} - 12.566370614359172\grave{}\,\theta + \theta^2}},\right.\right.$$

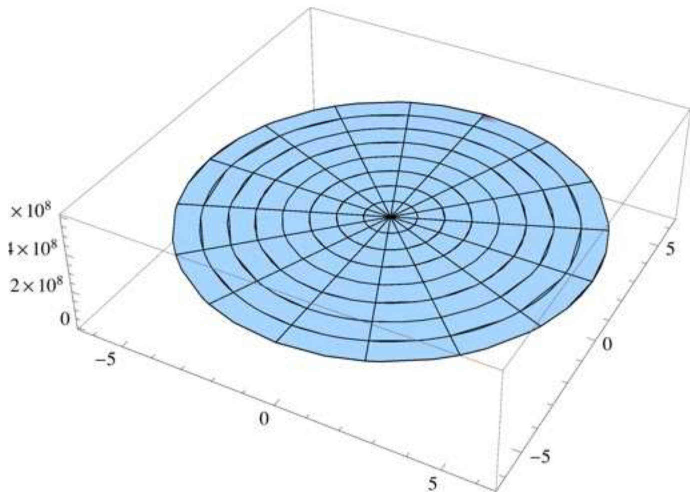
$$\left.\left.\frac{\sqrt{3.5481432270250993\grave{}*\hat{}18 - 1.1294090667581471\grave{}*\hat{}18\,\theta + 8.987551787368176\grave{}*\hat{}16\,\theta^2}}{\sqrt{39.47841760435743\grave{} - 12.566370614359172\grave{}\,\theta + \theta^2}}\right\},\ \{\theta, -2\pi, 2\pi\}\right]$$



The item captions are just my personal interpretations of the shape.

"Particle"

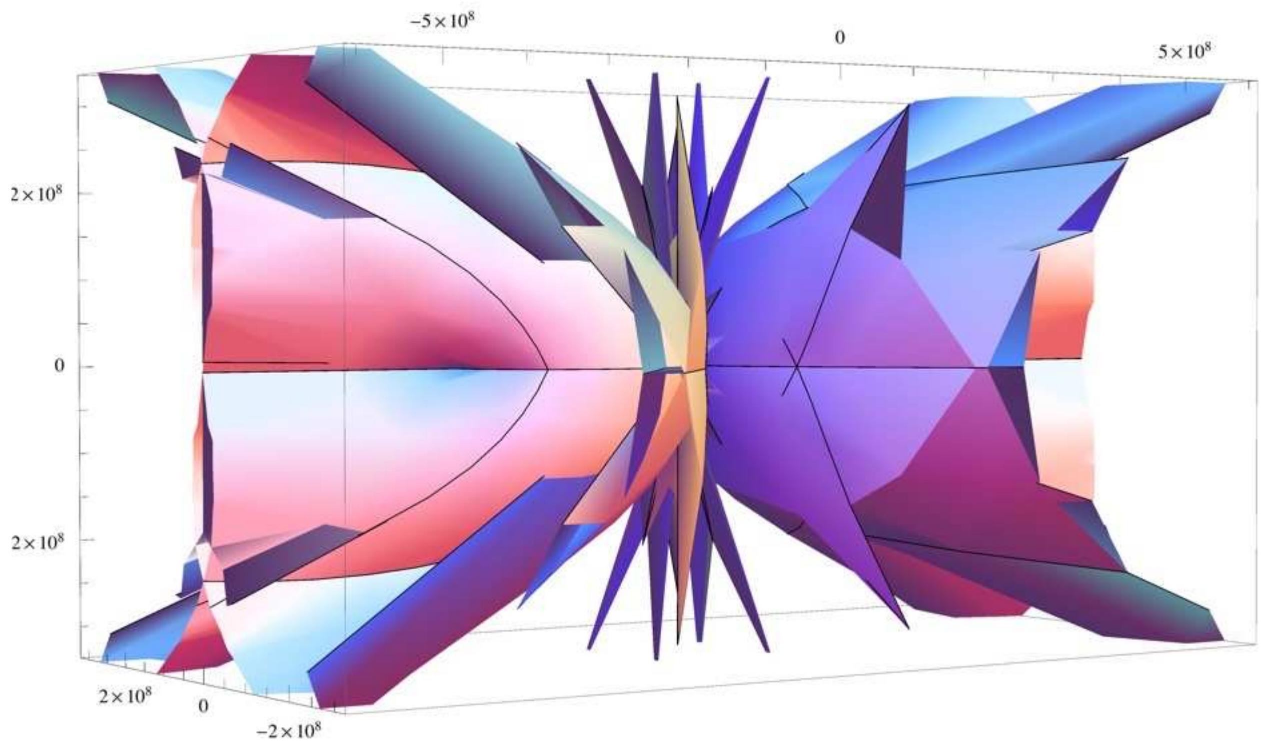
```
RevolutionPlot3D[
 Sqrt[3.5481432270250993`*^18 - 1.1294090667581471`*^18 θ + 8.987551787368176`*^16 θ^2] /
  Sqrt[39.47841760435743` - 12.566370614359172` θ + θ^2],
 {θ, -2 π, 2 π}]
```



There are seven distinct delivered visualizations of this equation when $\Theta = 2\left(\pi + \sqrt{\pi^2 - \pi^2 \operatorname{Sin}[\beta]^2}\right)$ is used in different combinations as a substitution for θ within the equation.

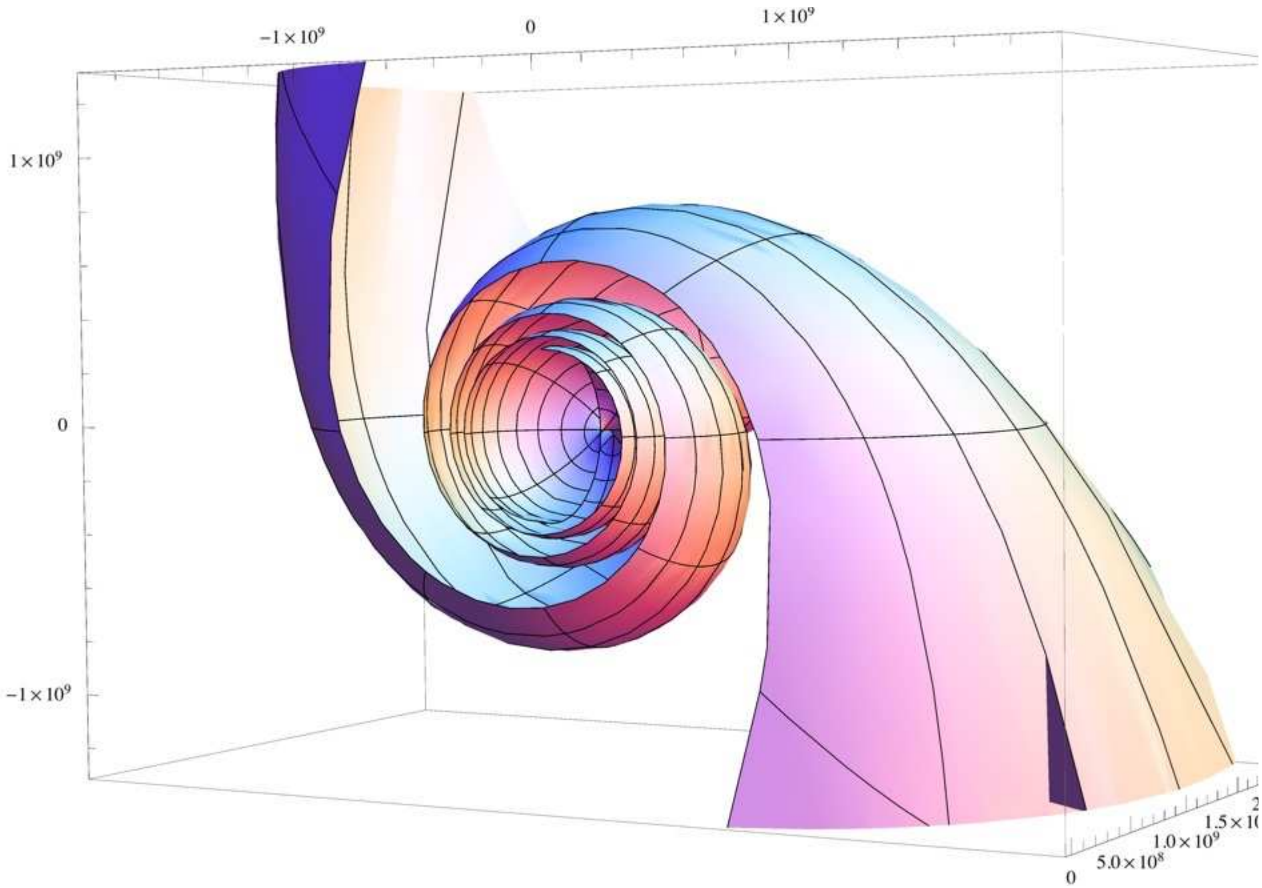
"Intuition"

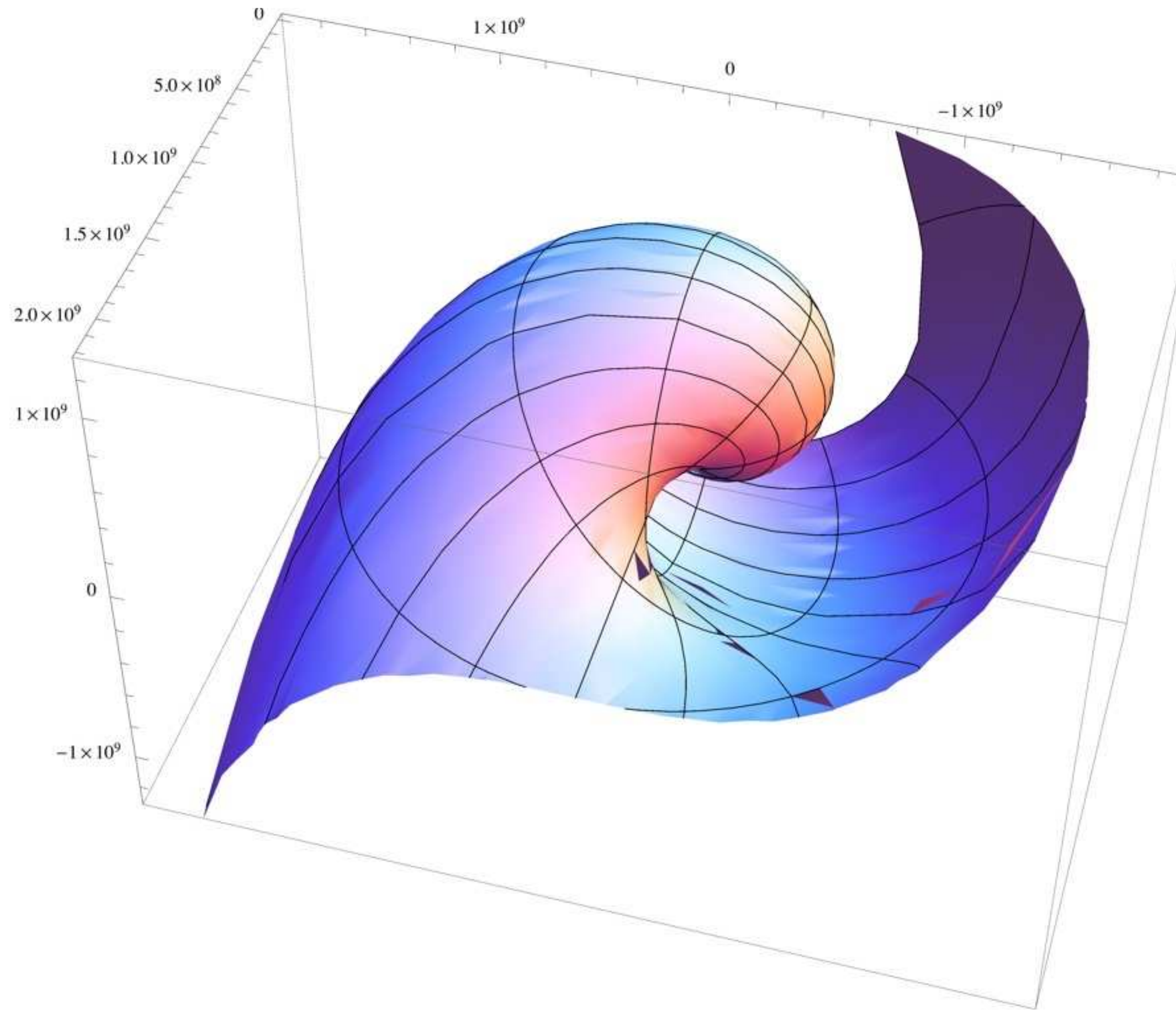
```
SphericalPlot3D[
 (√(3.5481432270250993`*^18 - 1.1294090667581471`*^18 (2 (π + √(π² - π² Sin[β]²))) +
      8.987551787368176`*^16 (2 (π + √(π² - π² Sin[β]²)))²)) /
  (√(39.47841760435743` - 12.566370614359172` (2 (π + √(π² - π² Sin[β]²))) + (θ)²)),
 {β, -π, π}, {θ, -4 π, 4 π}]
```



"Wave"

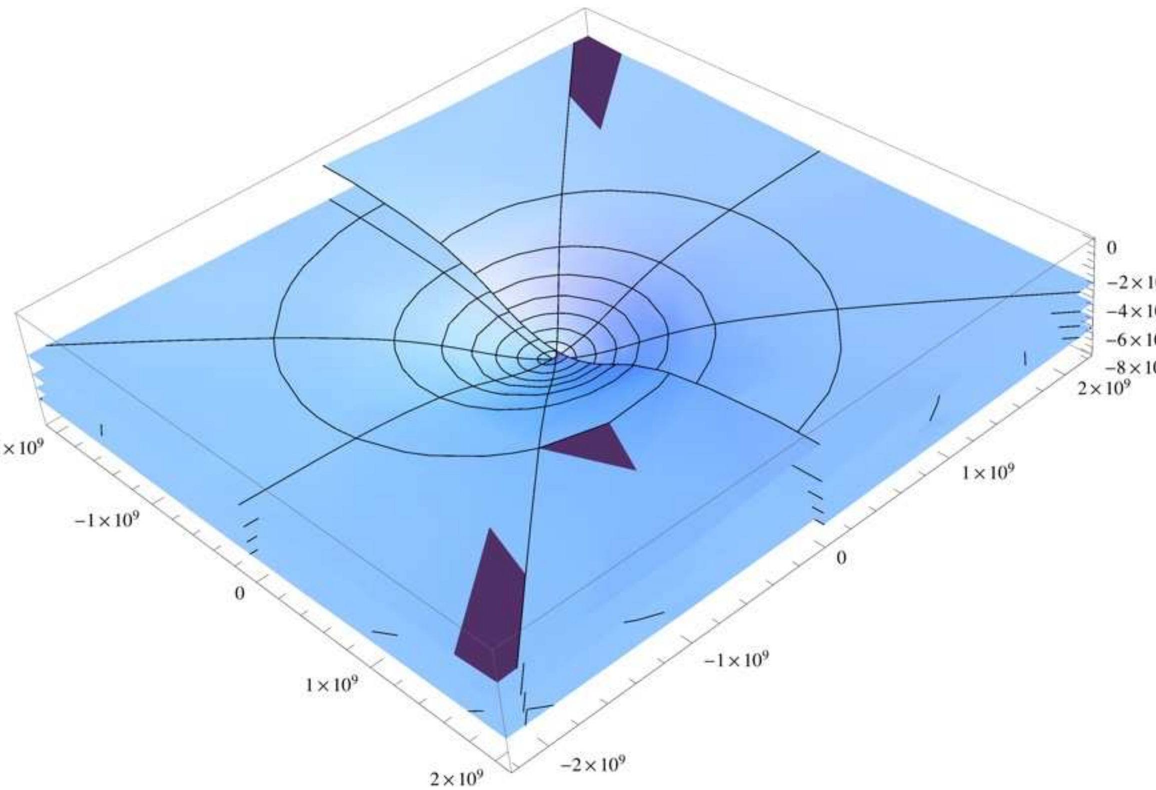
```
SphericalPlot3D[(√(3.5481432270250993`*^18 -
    1.1294090667581471`*^18 θ + 8.987551787368176`*^16 (2 (π + √(π^2 - π^2 Sin[β]^2)))^2)) /
  (√(39.47841760435743` - 12.566370614359172` θ + θ^2)), {β, -π / 2, π / 2}, {θ, -2 π, 2 π}]
```

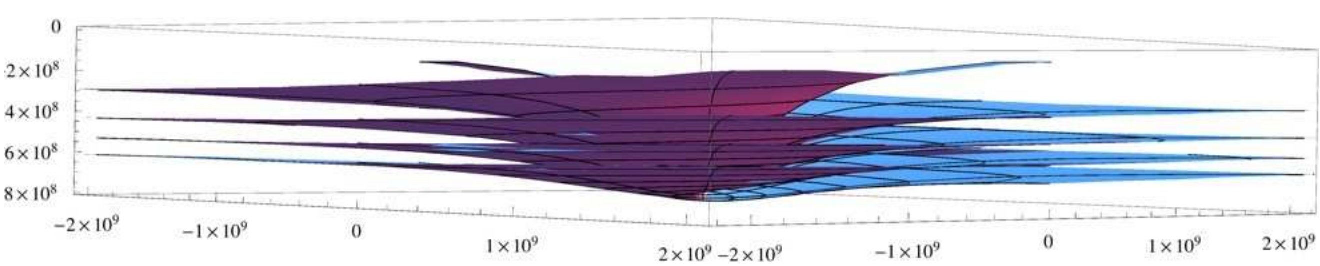




"Manifold"

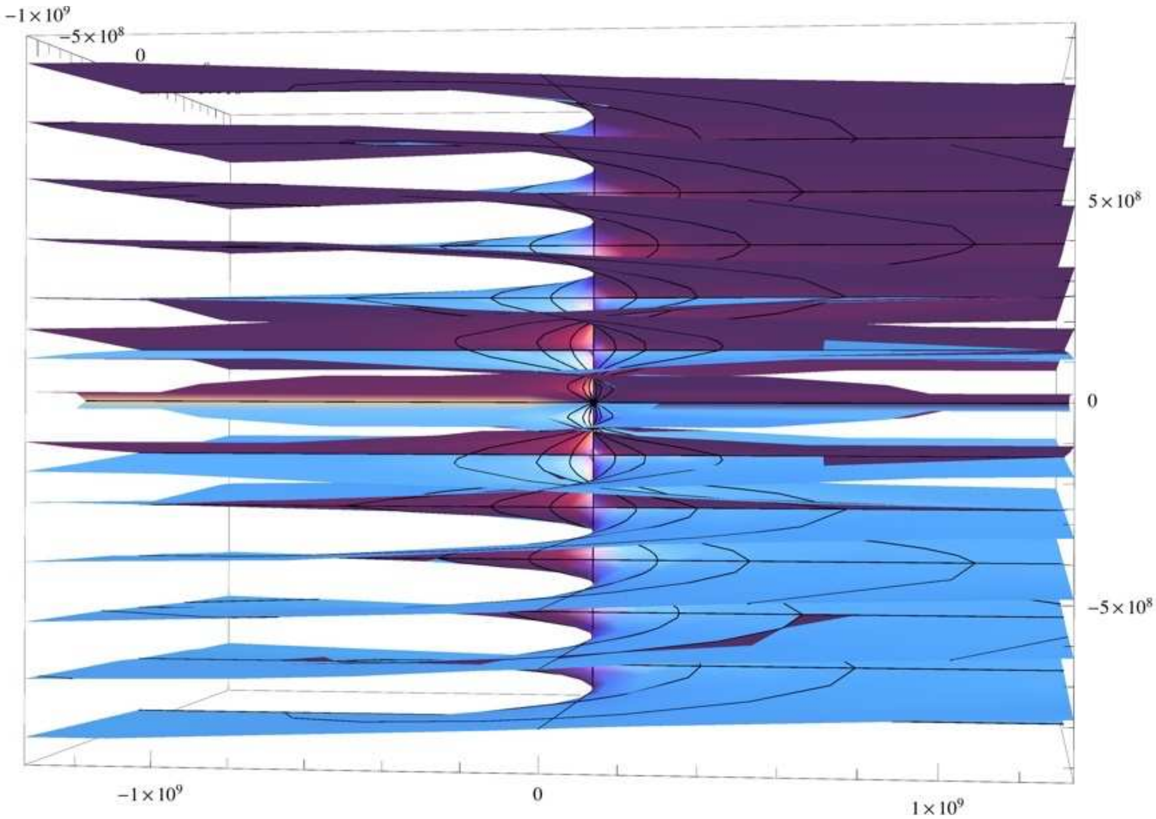
```
SphericalPlot3D[
 -(1.` √((3.5481432270250993`*^18 - 1.1294090667581471`*^18 θ + 8.987551787368176`*^16
       (2 (π + √(π² - π² Sin[β]²)))²)) /
   (√((39.47841760435743` - 12.566370614359172` (2 (π + √(π² - π² Sin[β]²))) +
       (2 (π + √(π² - π² Sin[β]²)))²))), {β, -π / 2, π / 2}, {θ, -2 π, 2 π}]
```





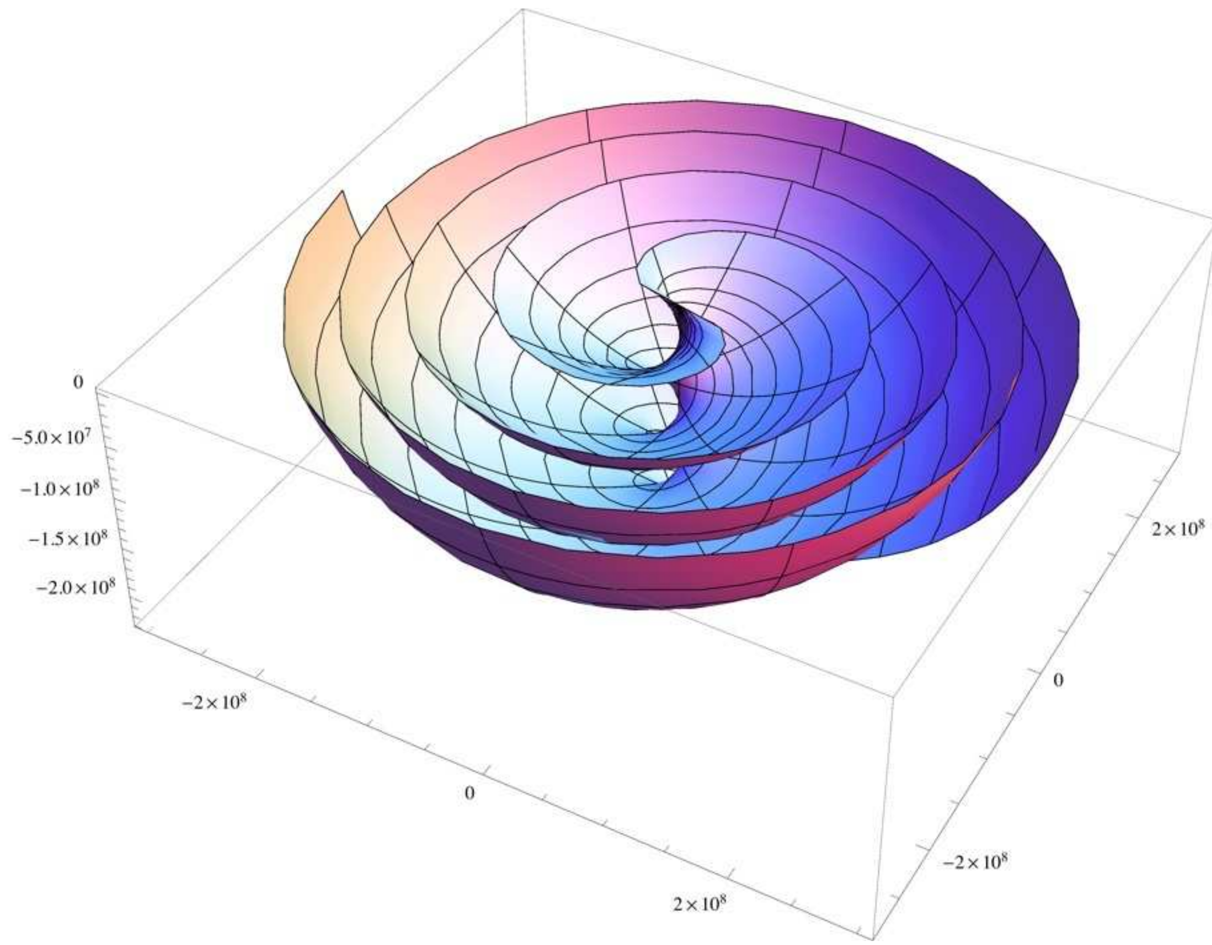
"Genetic"

```
SphericalPlot3D[
 - (1.` √ (3.5481432270250993`*^18 - 1.1294090667581471`*^18 θ + 8.987551787368176`*^16 (θ)^2)) /
  (√ ((39.47841760435743` - 12.566370614359172` (2 (π + √(π^2 - π^2 Sin[β]^2))) +
      (2 (π + √(π^2 - π^2 Sin[β]^2)))^2))), {β, -π, π}, {θ, -4 π, 4 π}]
```



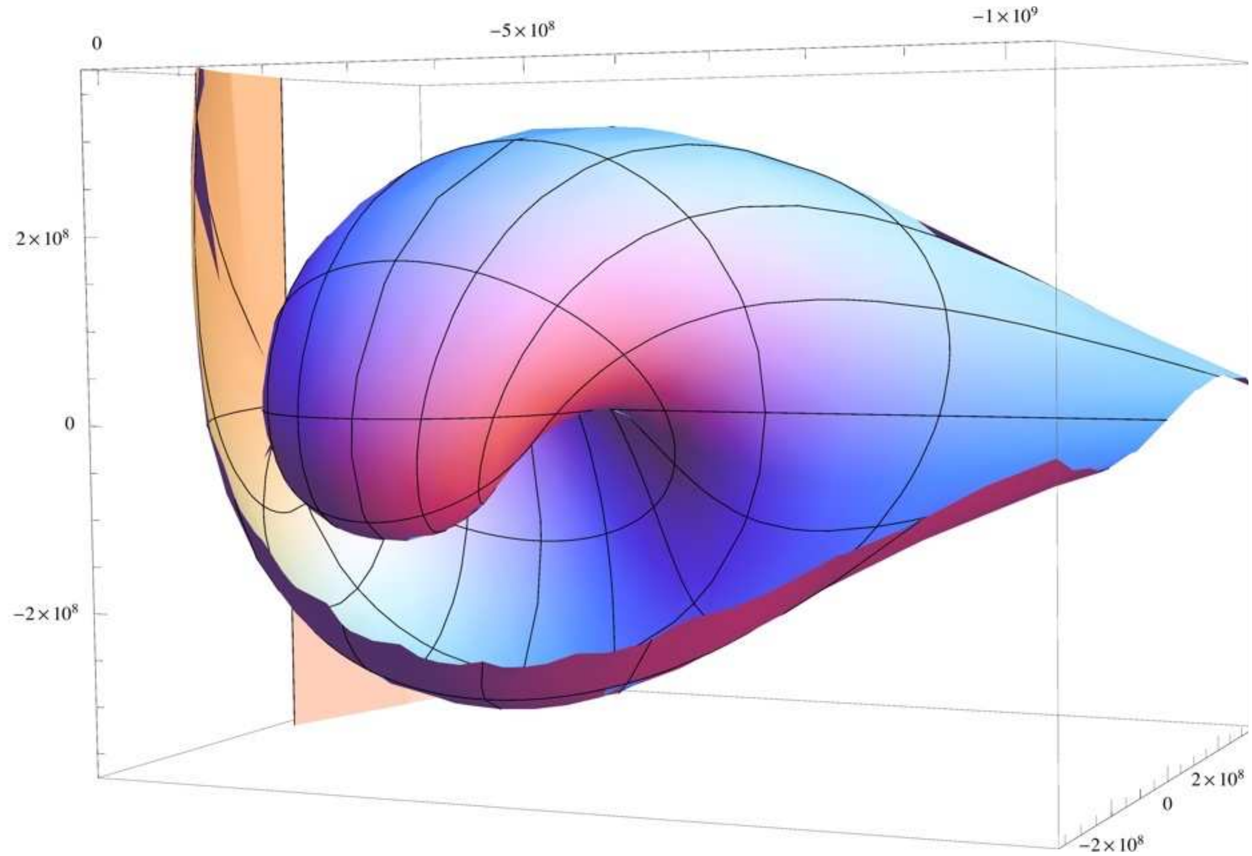
"Noema"

```
SphericalPlot3D[
 - (1.` √ (3.5481432270250993`*^18 - 1.1294090667581471`*^18 θ + 8.987551787368176`*^16 (θ)^2)) /
  (√(39.47841760435743` - 12.566370614359172` (θ) + (2 (π + √(π^2 - π^2 Sin[β]^2)))^2)),
 {β, -π / 2, π / 2}, {θ, -2 π, 2 π}]
```



"Stream of Consciousness"

```
SphericalPlot3D[
 -(1.` √((3.5481432270250993`*^18 - 1.129409066758147`*^18 (2 (π + √(π² - π² Sin[β]²))) +
      8.987551787368176`*^16 ((2 (π + √(π² - π² Sin[β]²))))²)) /
  (√(39.47841760435743` - 12.566370614359172` θ + θ²)), {β, -π / 2, π / 2}, {θ, -2 π, 2 π}]
```



Simply, I will note that through making substitutions of various sorts within equations found from this method, a large number of objects with curved, contoured, symmetrical, and otherwise notable characteristics are found.

IV. Visualizing the Height of the Cone

To describe visual textures of the height of the cone, substitutions can be made for θ or r in multiple ways. Making substitutions, there are certain guidelines if one wishes to stay within the actual factuality of the scenario of a circle ' s transforming into a cone. These guidelines are as follows. Make a selection of positive or negative solutions. In order to stay true to the system, we do not mix and match positive and negative values, e.g. in the case of,

$\frac{\sqrt{4\,\pi\,r^2\,(\theta) - r^2\,\theta^2}}{2\,\pi}$, a substitution like,

$$\frac{\sqrt{4\,\pi\,r^2\left(2\left(\pi + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)\right) - r^2\left(2\left(\pi - \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)\right)^2}}{2\,\pi},$$

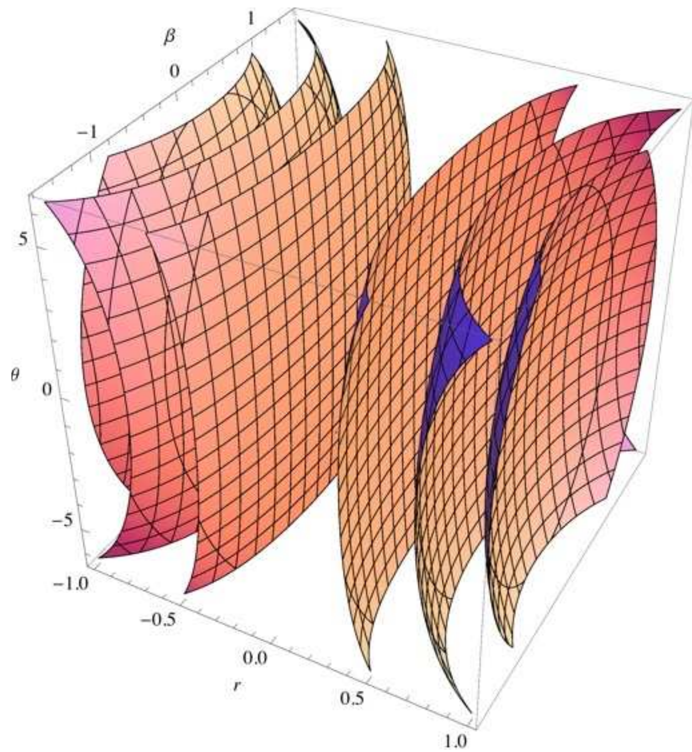
would not be admissible, because it takes the positive solution for η, and says that $\theta = 2\left(\pi + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right) = 2\left(\pi - \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)$, which is not true. This section will only visualize the solutions to theta that are from the initial lemmas of Theorem 1.

$$\eta = \frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi} = \frac{\sqrt{4\,\pi\,r^2\left(2\left(\pi \pm \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)\right) - r^2\,\theta^2}}{2\,\pi} =$$
$$\frac{\sqrt{4\,\pi\,r^2\left(2\left(\pi \pm \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)\right) - r^2\left(2\left(\pi \pm \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)\right)^2}}{2\,\pi} =$$
$$\frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\left(2\left(\pi \pm \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)\right)^2}}{2\,\pi} \qquad (16)$$

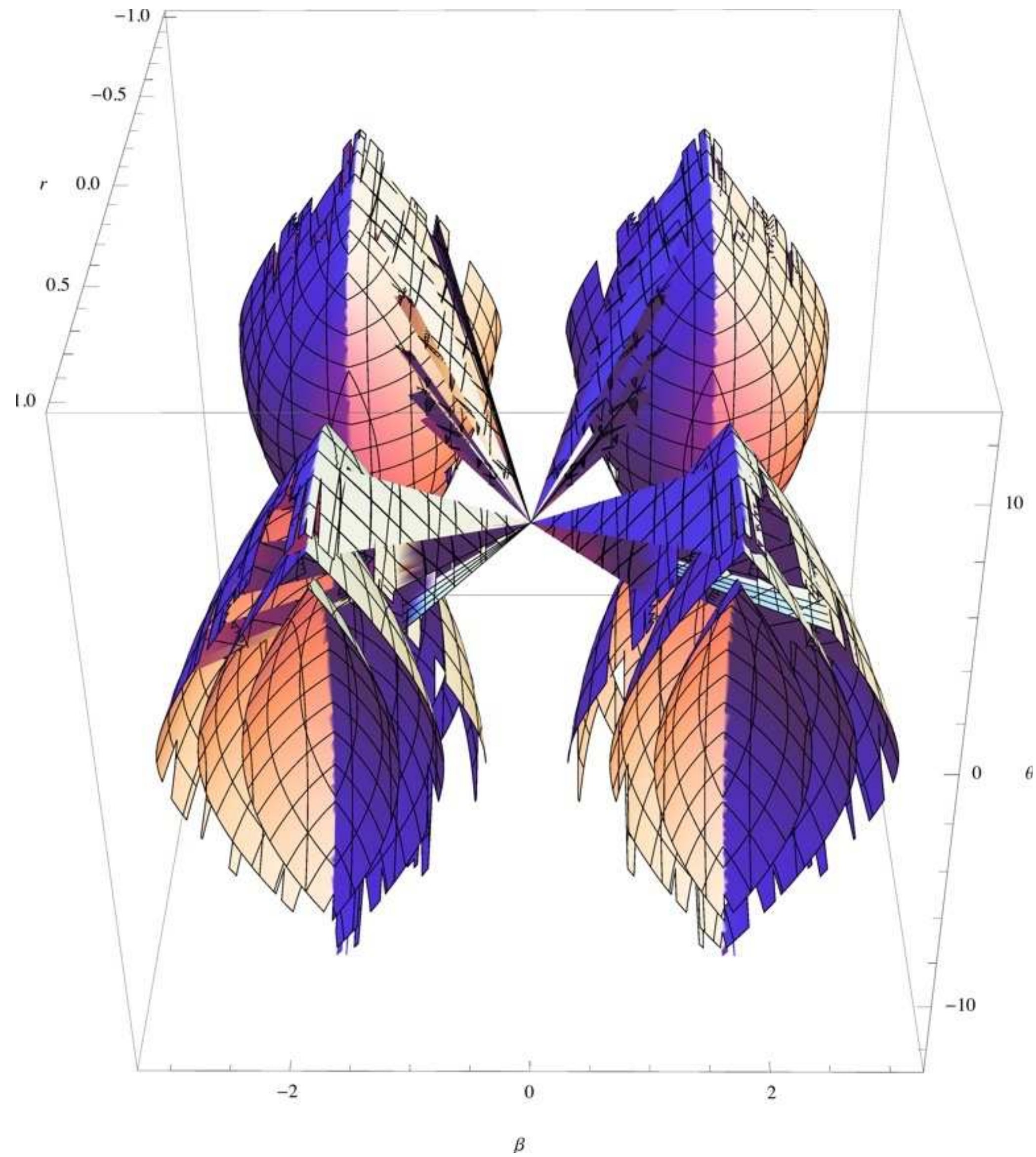
We will visualize these solutions, while in each visualization, theta will either be set equal to $2\left(\pi + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)$, or $2\left(\pi - \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)$ for a given instance of substitution. This is because solutions for theta from the height of the cone yield either $2\left(\pi + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)$, or $2\left(\pi - \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)$, not both at the same time such that they may be mixed when returning to the initial expression to produce visualization of that equation.

```
ContourPlot3D[Sqrt[4 π r^2 (2 (π + Sqrt[π^2 - π^2 Sin[β]^2])) - r^2 θ^2]/(2 π),
 {r, -1, 1}, {β, -π / 2, π / 2}, {θ, -2 π, 2 π}, AxesLabel → Automatic]
```

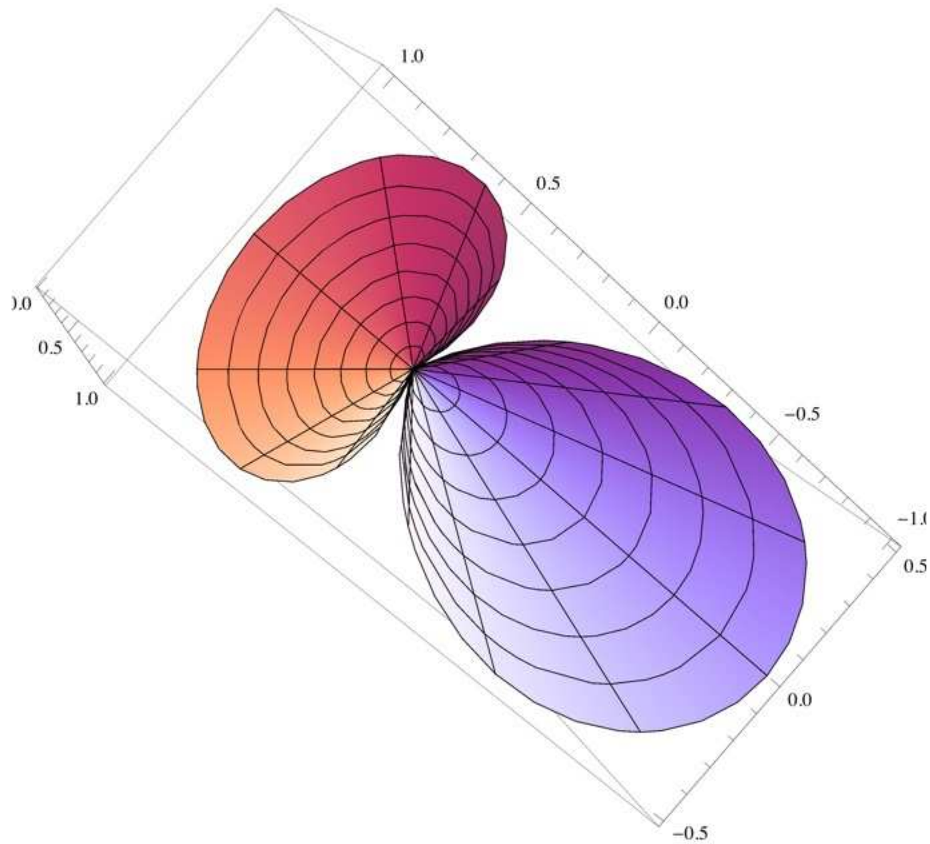
$$\text{ContourPlot3D}\left[\frac{\sqrt{4\,\pi\,r^2\,\left(2\,\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right)-r^2\,\theta^2}}{2\,\pi},\right.$$
$$\left.\{r,-1,1\},\{\beta,-\pi/2,\pi/2\},\{\theta,-2\,\pi,2\,\pi\},\ \text{AxesLabel}\to\text{Automatic}\right]$$

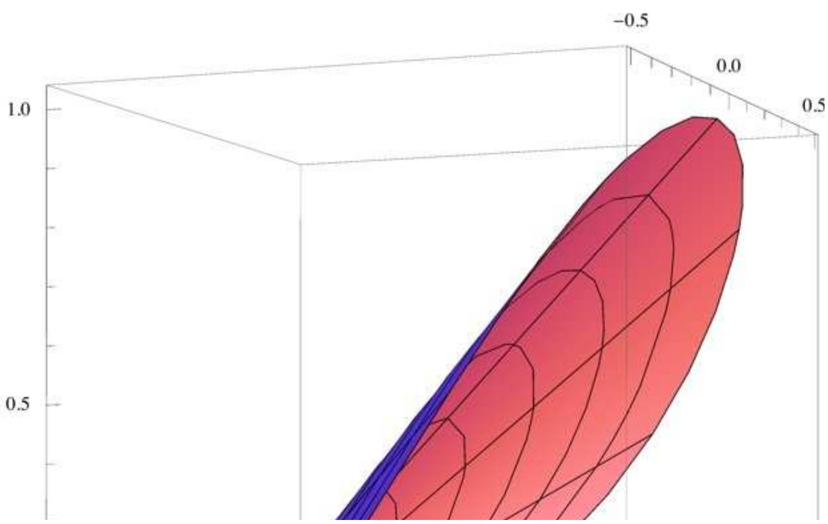


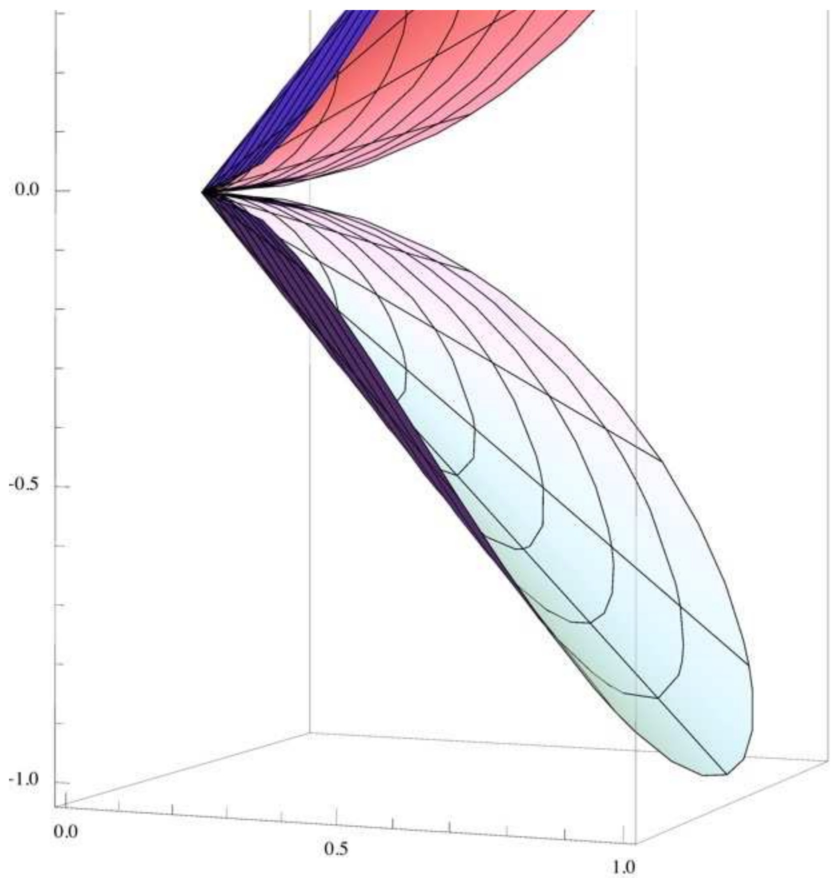
$$\textbf{ContourPlot3D}\left[\frac{\sqrt{4\,\pi\,r^2\left(2\left(\pi-\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)\right)-r^2\,\theta^2}}{2\,\pi},\right.$$
$$\left.\{r,-1,1\},\{\beta,-\pi,\pi\},\{\theta,-4\,\pi,4\,\pi\},\ \mathrm{AxesLabel}\rightarrow\mathrm{Automatic}\right]$$



$$\textbf{RevolutionPlot3D}\left[\left\{\frac{\sqrt{4\,\pi\,r^2\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)\right)-r^2\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)\right)^2}}{2\,\pi},\right.\right.$$
$$\left.\left.\frac{\sqrt{4\,\pi\,r^2\left(2\left(\pi-\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)\right)-r^2\left(2\left(\pi-\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)\right)^2}}{2\,\pi}\right\},\{r,-1,1\},\{\beta,-\pi,\pi\}\right]$$

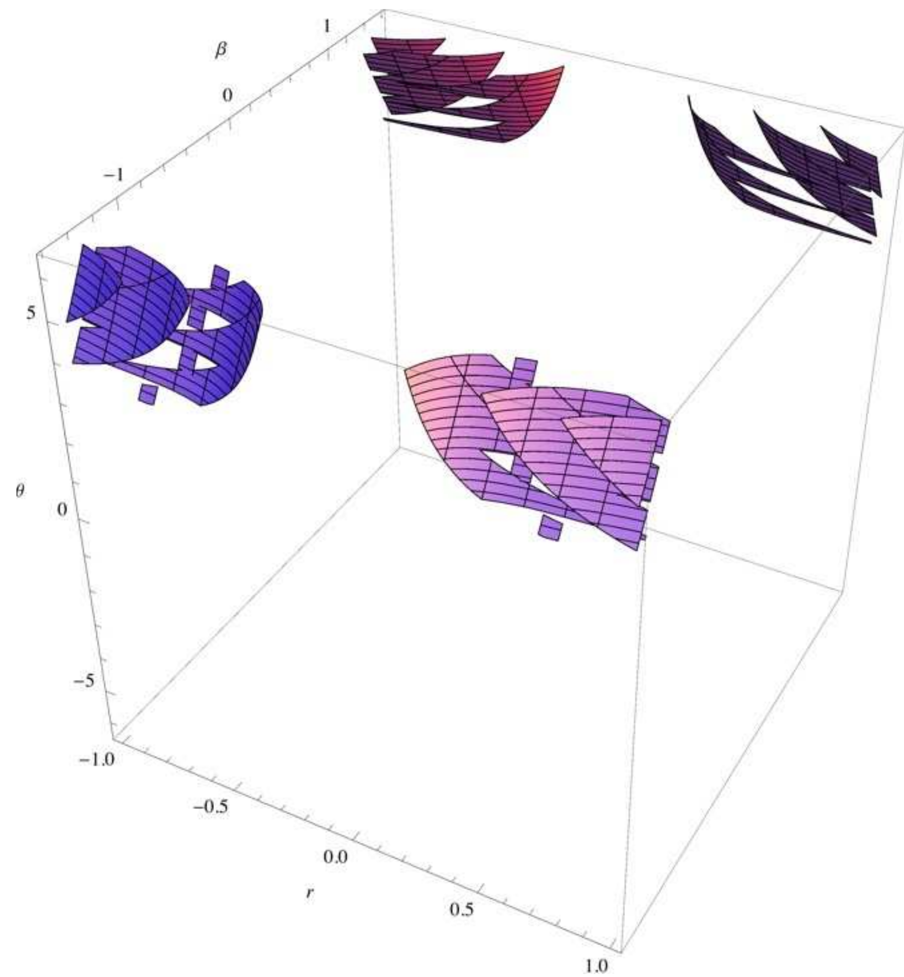

1.0
0.5
0.0
−0.5
−1.0
0.5
1.0
0.5
0.0
−0.5


−0.5
0.0
0.5
1.0
0.5

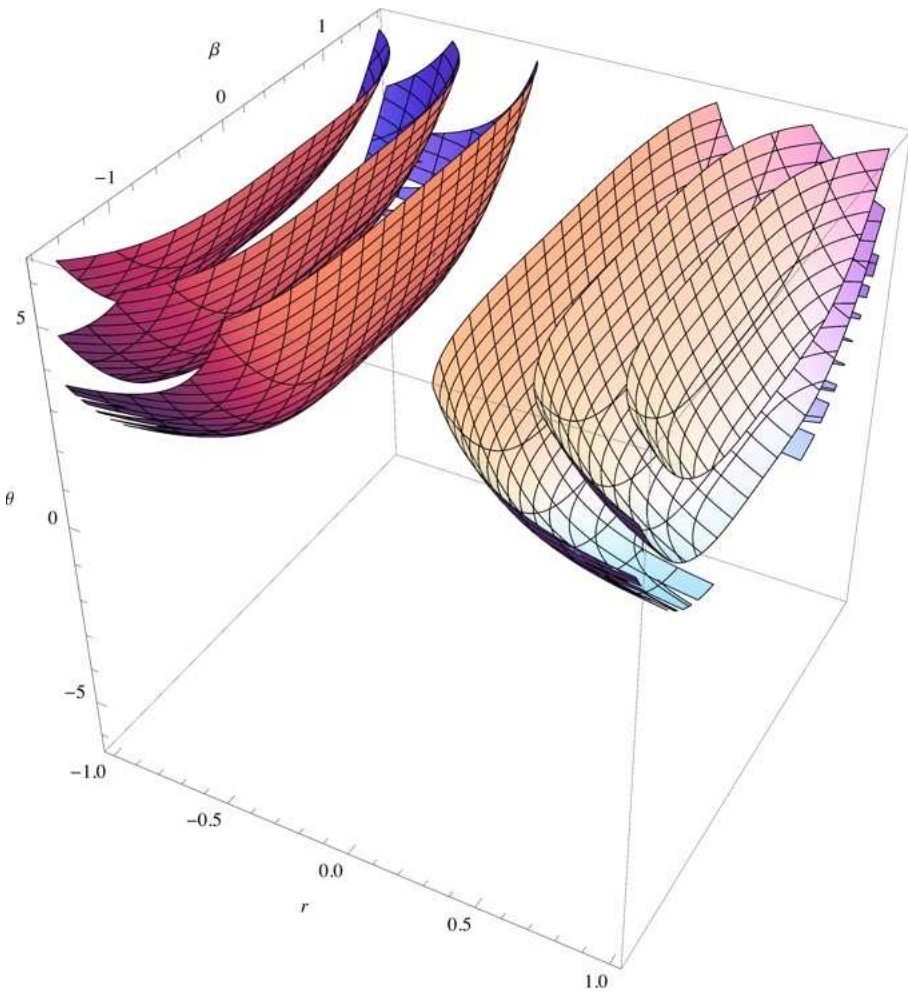


$$\text{ContourPlot3D}\left[\left\{\frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\left(2\,\left(\pi + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)\right)^2}}{2\,\pi}\right\},\right.$$

$$\left.\{r, -1, 1\}, \{\beta, -\pi/2, \pi/2\}, \{\theta, -2\,\pi, 2\,\pi\}, \text{AxesLabel} \to \text{Automatic}\right]$$


β
1
0
−1
5
θ
0
−5
−1.0
−0.5
0.0
r
0.5
1.0

```
ContourPlot3D[{ Sqrt[4 π r^2 Θ - r^2 (2 (π - Sqrt[π^2 - π^2 Sin[β]^2]))^2] / (2 π) },
  {r, -1, 1}, {β, -π/2, π/2}, {Θ, -2 π, 2 π}, AxesLabel → Automatic]
```



$$\Theta = 2\left(\pi + \sqrt{\pi^2 - \pi^2\,\mathrm{Sin}[\beta]^2}\right) = \frac{2\pi\left(r^2 + \sqrt{r^4 - r^2\,\eta^2}\right)}{r^2}$$

$$\eta = \frac{\sqrt{4\,\pi\,r^2\,\Theta - r^2\,\Theta^2}}{2\,\pi} = \frac{\sqrt{4\,\pi\,r^2\left(\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right) - r^2\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)\right)^2}}{2\,\pi} =$$

$$\frac{\sqrt{4\,\pi\,r^2\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)\right) - r^2\left(\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right)^2}}{2\,\pi} =$$

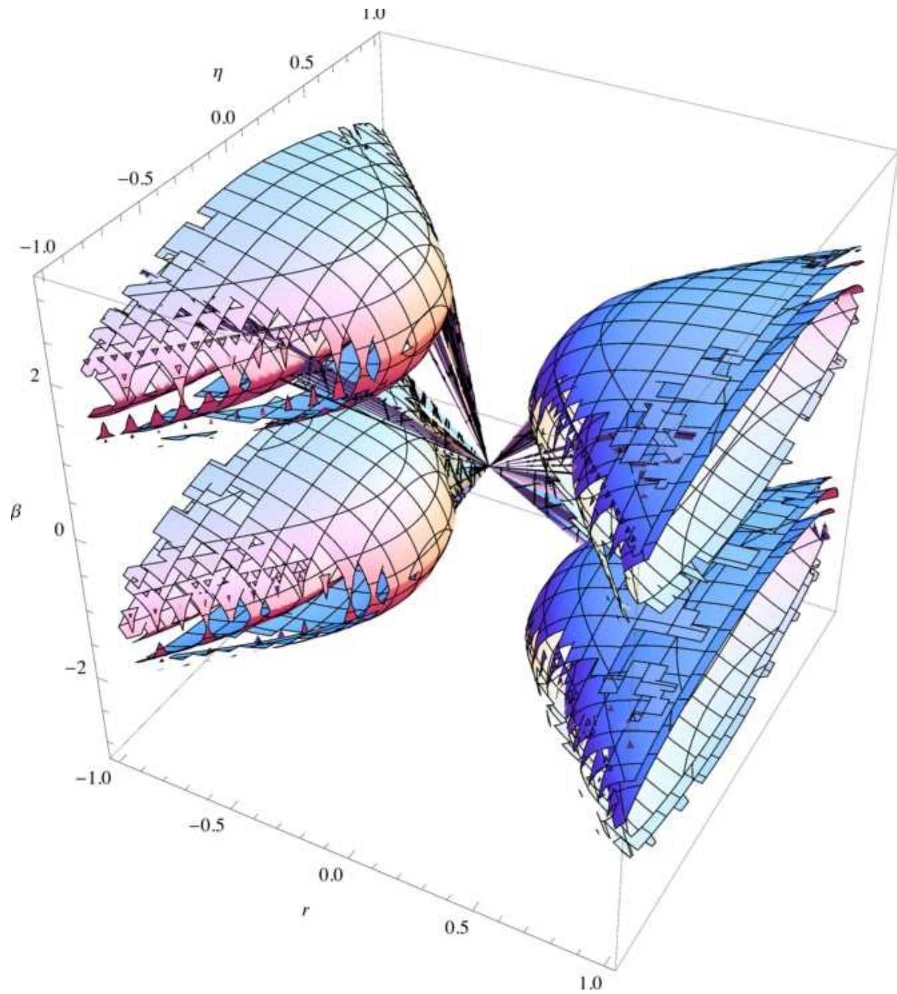
$$\frac{\sqrt{4\,\pi\,r^2\left(\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right) - r^2\,\Theta^2}}{2\,\pi} =$$

$$\frac{\sqrt{4\,\pi\,r^2\left(\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right) - r^2\left(\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right)^2}}{2\,\pi} =$$

$$\frac{\sqrt{4\,\pi\,r^2\,\Theta - r^2\left(\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right)^2}}{2\,\pi}$$

$$\textbf{ContourPlot3D}\left[\frac{\sqrt{4\,\pi\,r^2\left(\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right) - r^2\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)\right)^2}}{2\,\pi},\right.$$

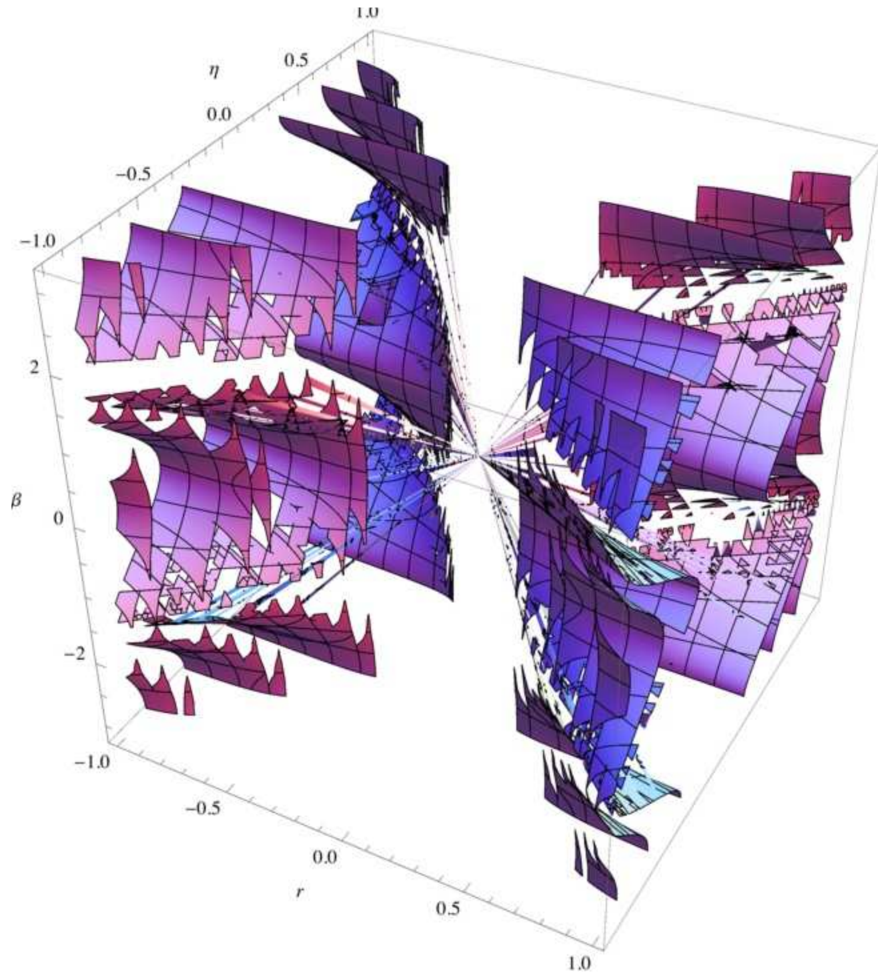
$$\left.\{r, -1, 1\}, \{\eta, -1, 1\}, \{\beta, -\pi, \pi\}, \textbf{AxesLabel} \rightarrow \textbf{Automatic}\right]$$



η
1.0
0.5
0.0
−0.5
−1.0
2
β
0
−2
−1.0
−0.5
0.0
r
0.5
1.0

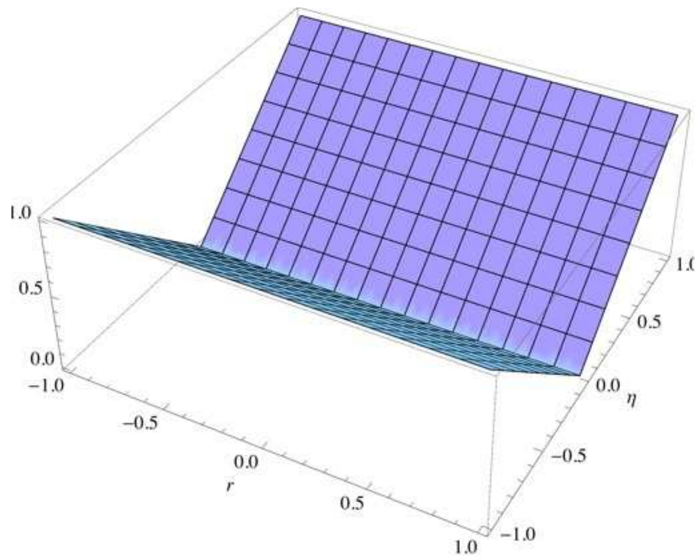
$$\text{ContourPlot3D}\left[\frac{\sqrt{4\,\pi\,r^2\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right)-r^2\left(\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right)^2}}{2\,\pi},\right.$$

$$\left.\{r,-1,1\},\{\eta,-1,1\},\{\beta,-\pi,\pi\},\text{AxesLabel}\rightarrow\text{Automatic}\right]$$



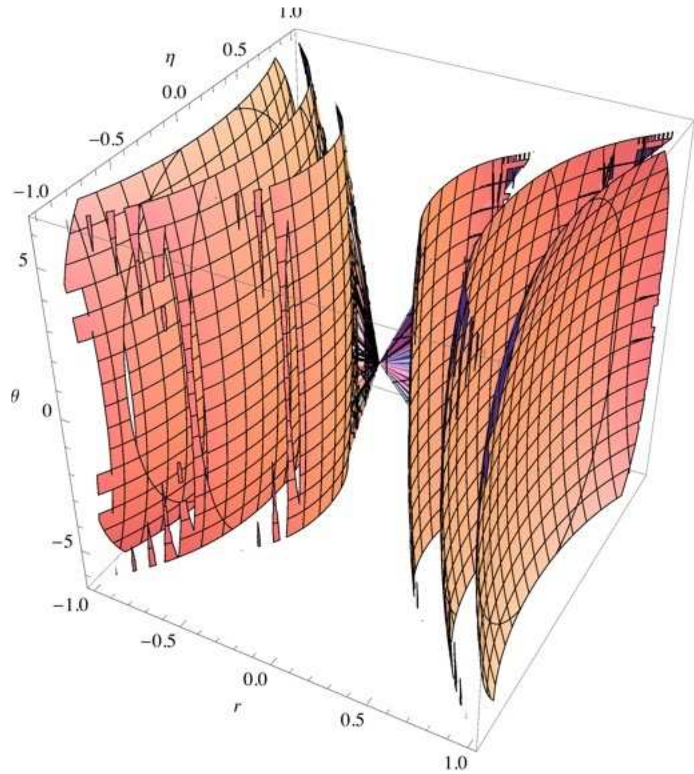
$$\text{Plot3D}\left[\frac{\sqrt{4\,\pi\,r^2\left(\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right)-r^2\left(\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right)^2}}{2\,\pi},\right.$$

$$\left.\{r,-1,1\},\{\eta,-1,1\},\text{AxesLabel}\to\text{Automatic}\right]$$

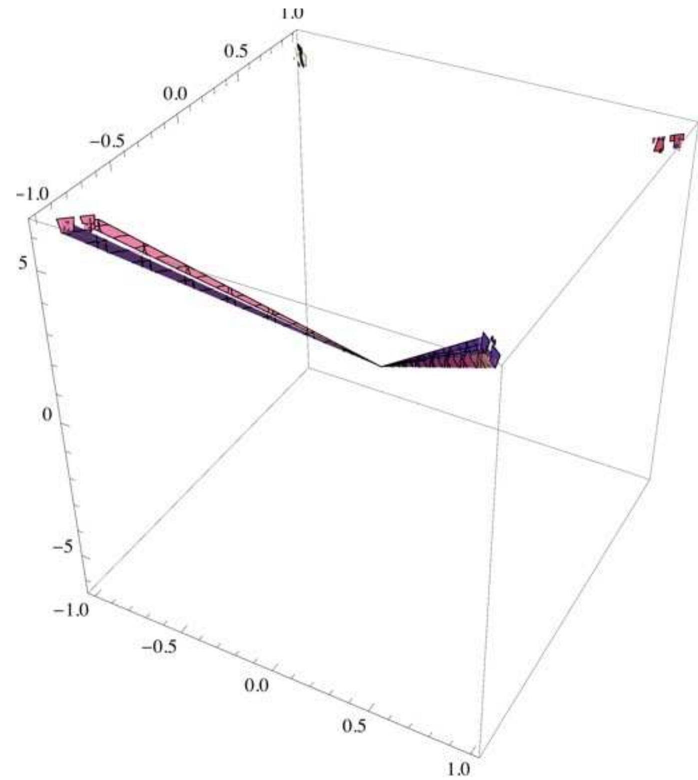


$$\texttt{ContourPlot3D}\left[\frac{\sqrt{4\,\pi\,r^2\left(\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right)-r^2\,\theta^2}}{2\,\pi},\right.$$

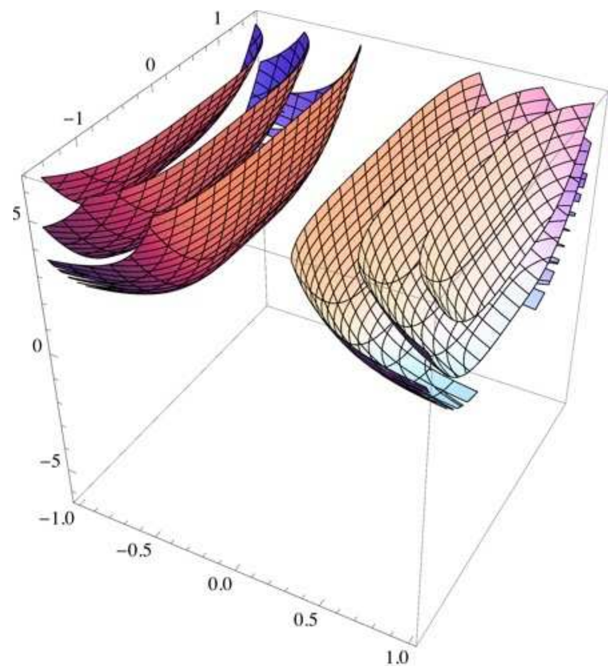
$$\left.\{r,\,-1,\,1\},\,\{\eta,\,-1,\,1\},\,\{\theta,\,-2\,\pi,\,2\,\pi\},\,\texttt{AxesLabel}\rightarrow\texttt{Automatic}\right]$$



$$\text{ContourPlot3D}\left[\frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\left(\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right)^2}}{2\,\pi},\ \{r, -1, 1\},\ \{\eta, -1, 1\},\ \{\theta, -2\,\pi, 2\,\pi\}\right]$$

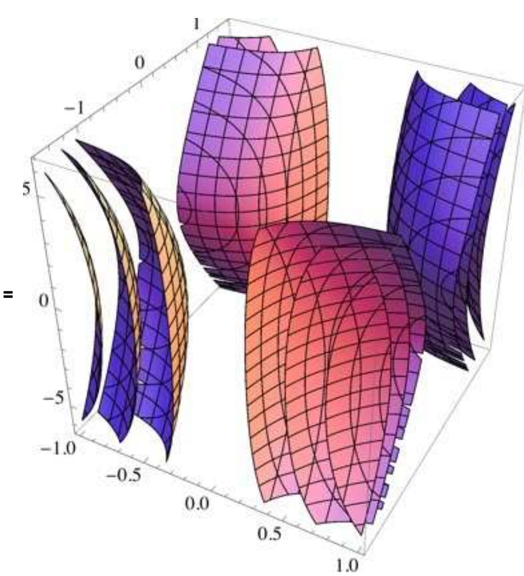


$$\eta = -\frac{\sqrt{4\pi r^2 \theta - r^2 \theta^2}}{2\pi} = -\frac{\sqrt{4\pi r^2 \theta - r^2 \left(2\left(\pi - \sqrt{\pi^2 - \pi^2 \mathrm{Sin}[\beta]^2}\right)\right)^2}}{2\pi} =$$



$$, \ \eta = -\frac{\sqrt{4\pi r^2 \theta - r^2 \theta^2}}{2\pi} =$$

$$-\frac{\sqrt{4\pi r^2\, 2\left(\pi - \sqrt{\pi^2 - \pi^2 \mathrm{Sin}[\beta]^2}\right) - r^2 \theta^2}}{2\pi} =$$

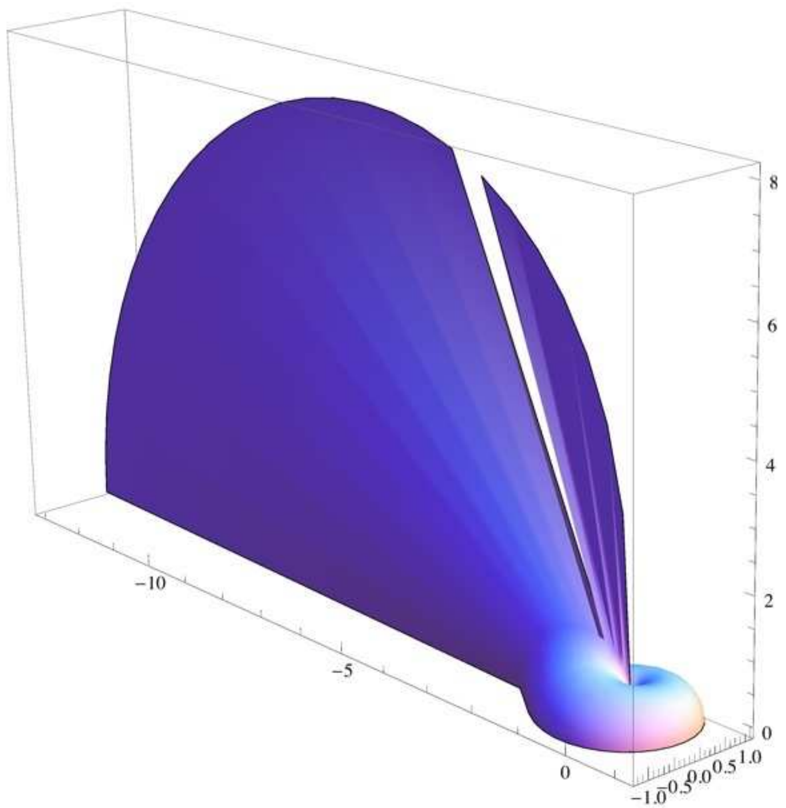


V. The $\sqrt{-1}$

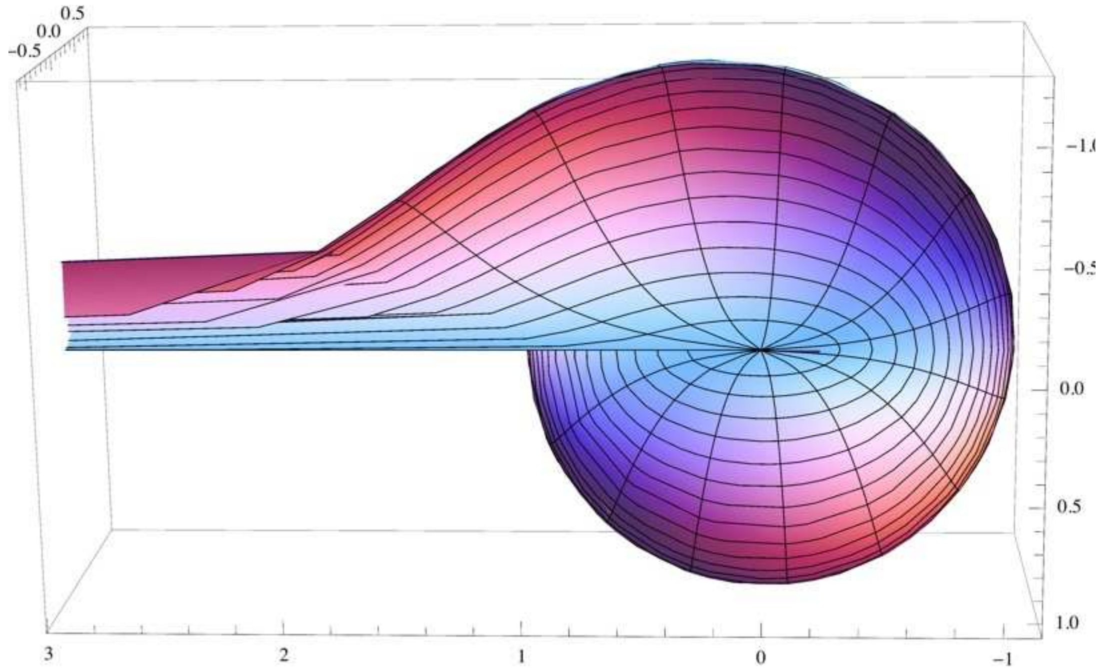
From Lemma 5, $1 == \frac{2\pi \operatorname{Sin}[\beta]}{\sqrt{4\pi\Theta - \Theta^2}}$, therefore,

$-1 = -\frac{2\pi \operatorname{Sin}[\beta]}{\sqrt{4\pi\Theta - \Theta^2}}$, thus, $\sqrt{-1} = \sqrt{-\frac{2\pi \operatorname{Sin}[\beta]}{\sqrt{4\pi\Theta - \Theta^2}}}$

SphericalPlot3D$\left[\sqrt{-\frac{2\pi\,\mathrm{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}}}\right.$**, {β, -π / 2, π / 2}, {θ, -2 π, 2 π}**$\Big]$



```
SphericalPlot3D[Sqrt[-(2 π Sin[β])/Sqrt[4 π θ - θ^2]], {θ, -2 π, 2 π}, {β, -π / 2, π / 2}]
```



VI. Application to the Energy of a Photon

■ For a photon with wavelength equal to the height of the cone:

Theorem 6 The energy of a photon whose wavelength is equal to the height of the cone, where time is equal to the passing of the angle measure, like a clock, produces the diagram used to illustrate illusory, or "virtual" contour in Gestalt Theory.

Proof.

The velocity of a wave with wavelength equal to the height of the cone is equal to the height of the cone times the frequency of the oscillation of the wave.

velocity = (wavelength) (frequency) =

$$\eta\,\nu = \frac{\sqrt{4\pi r^2 \theta - r^2 \theta^2}}{2\pi}\,\nu = \frac{\sqrt{4\pi r^2 \theta - r^2 \theta^2}}{2\pi}\left(\frac{1}{\left(\frac{\theta}{k}\right)}\right) = \frac{\sqrt{4\pi r^2 \theta - r^2 \theta^2}}{2\pi}\left(\frac{1}{\left(\frac{\theta}{2\pi}\right)}\right) = c$$

Thus, we can solve this equation for the initial radius.

$$\text{Solve}\left[\text{Abs}\left[\frac{\sqrt{4\pi r^2 \theta - r^2 \theta^2}}{2\pi}\right]\left(\frac{1}{\left(\frac{\theta}{2\pi}\right)}\right) == c,\ r\right]$$

$$\left\{\left\{r \to -\frac{c\sqrt{\theta}}{\sqrt{4\pi-\theta}}\right\}, \left\{r \to -\frac{i\,c\sqrt{\theta}}{\sqrt{4\pi-\theta}}\right\}, \left\{r \to \frac{i\,c\sqrt{\theta}}{\sqrt{4\pi-\theta}}\right\}, \left\{r \to \frac{c\sqrt{\theta}}{\sqrt{4\pi-\theta}}\right\}\right\}$$

$$\text{Solve}\left[\text{Abs}\left[\frac{\sqrt{4\pi r^2 \theta - r^2 \theta^2}}{2\pi}\right]\left(\frac{1}{\left(\frac{\theta}{2\pi}\right)}\right) == c, \theta\right]$$

$$\left\{\left\{\theta \to \frac{4\pi r^2}{-c^2 + r^2}\right\}, \left\{\theta \to \frac{4\pi r^2}{c^2 + r^2}\right\}\right\}$$

Substituting $r = \frac{c\sqrt{\theta}}{\sqrt{4\pi - \theta}}$, and $\theta = \frac{4\pi r^2}{c^2 + r^2}$

$$E = h\nu = h\left(\frac{1}{\frac{\theta}{2\pi}}\right) = \frac{h}{\frac{(\theta)}{2\pi}} = \frac{h}{\frac{\left(\frac{4\pi r^2}{c^2+r^2}\right)}{2\pi}} = \frac{h\left(c^2 + r^2\right)}{2r^2} = \frac{h\left(c^2 + \left(\frac{c\sqrt{\theta}}{\sqrt{4\pi-\theta}}\right)^2\right)}{2\left(\frac{c\sqrt{\theta}}{\sqrt{4\pi-\theta}}\right)^2}$$, where h is Planck ' s Constant.

From $\left\{\left\{\theta \to 2\left(\pi - \sqrt{\pi^2 - \pi^2 \text{Sin}[\beta]^2}\right)\right\}, \left\{\theta \to 2\left(\pi + \sqrt{\pi^2 - \pi^2 \text{Sin}[\beta]^2}\right)\right\}\right\}$,

I can then make a substitutions for theta into the equation for the energy of a photon.

I can thus form a spherical plot of this equation for the energy of a photon,

$$\text{Energy} = \frac{h\left(c^2 + \left(\frac{c\sqrt{\theta}}{\sqrt{4\pi-\theta}}\right)^2\right)}{2\left(\frac{c\sqrt{\theta}}{\sqrt{4\pi-\theta}}\right)^2},$$

through setting

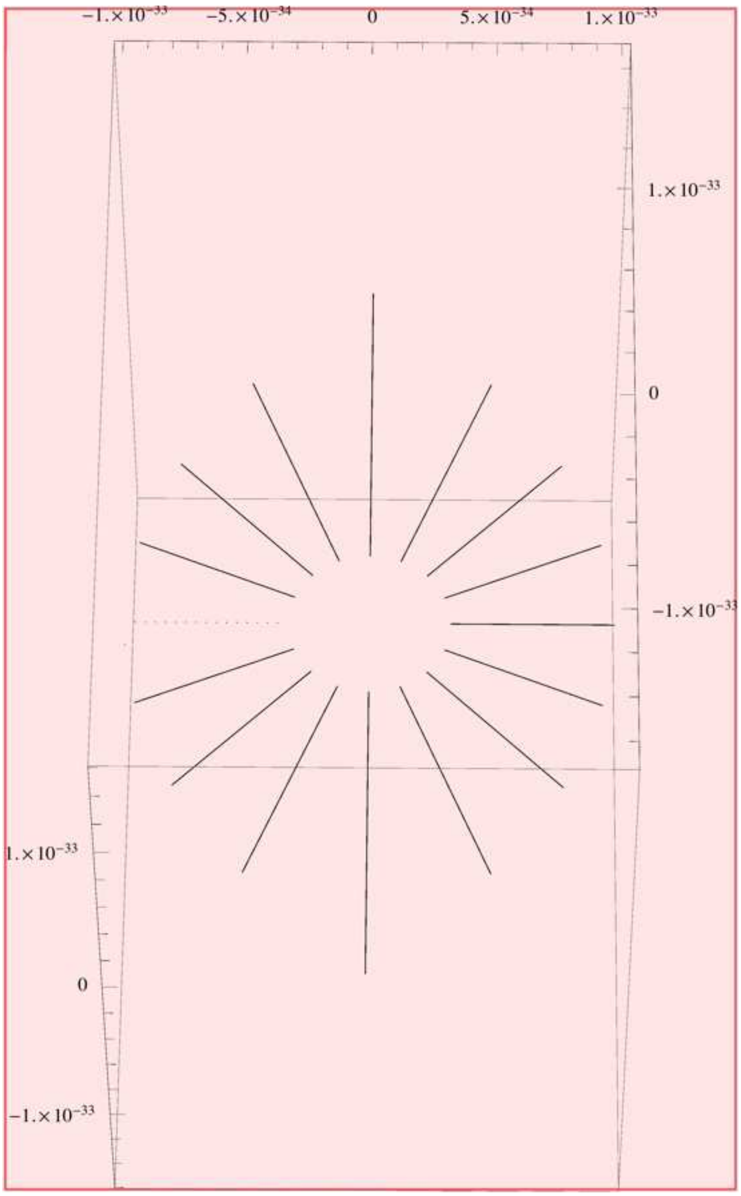
$$\frac{h\left(c^2 + \left(\frac{c\sqrt{\theta}}{\sqrt{4\pi-\theta}}\right)^2\right)}{2\left(\frac{c\sqrt{\theta}}{\sqrt{4\pi-\theta}}\right)^2} = \frac{h\left(c^2 + \left(\frac{c\sqrt{2\left(\pi+\sqrt{\pi^2-\pi^2 \text{Sin}[\beta]^2}\right)}}{\sqrt{4\pi-\theta}}\right)^2\right)}{2\left(\frac{c\sqrt{2\left(\pi+\sqrt{\pi^2-\pi^2 \text{Sin}[\beta]^2}\right)}}{\sqrt{4\pi-\theta}}\right)^2}.$$

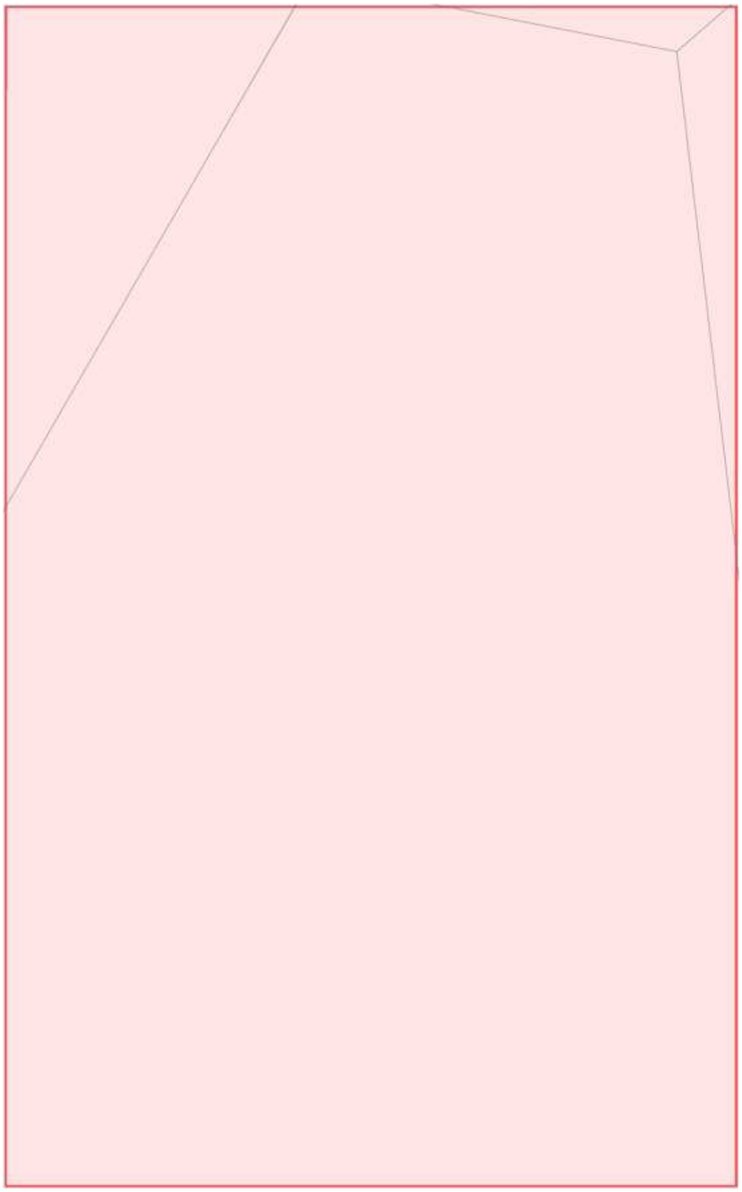
```
h := 6.62606896 * 10^(-34)

c := 2.99792458 * (10^8)
```

$$(\text{Energy of Photon with Wavelength } \eta) = \frac{h\left(c^2 + \left(\frac{c\sqrt{2\left(\pi+\sqrt{\pi^2-\pi^2 \text{Sin}[\beta]^2}\right)}}{\sqrt{4\pi-\theta}}\right)^2\right)}{2\left(\frac{c\sqrt{2\left(\pi+\sqrt{\pi^2-\pi^2 \text{Sin}[\beta]^2}\right)}}{\sqrt{4\pi-\theta}}\right)^2} \text{ Joules} \qquad (18)$$

$$\text{SphericalPlot3D}\left[\frac{h\left(c^2+\left(\frac{c\sqrt{2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)}}{\sqrt{4\,\pi-\theta}}\right)^2\right)}{2\left(\frac{c\sqrt{2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)}}{\sqrt{4\,\pi-\theta}}\right)^2},\ \{\beta,\ -\pi,\ \pi\},\ \{\theta,\ -4\,\pi,\ 4\,\pi\}\right]$$





An interesting phenomenon about this diagram is that when you grab it, it sometimes "flips" into a different mode of controlling the orientation of the graph. It changes from a normal way of maneuvering the orientation (I push up, the box moves up) to a "southpaw" style modality of controlling the orientation of the graph (I push left, it moves right, I push right, it moves left). In this, we see room for a Gestalt interpretation, because, while displaying a diagram of illusory contour, it also displays a kind of multivariance.

Lemma 14 We can find another useful expression for the initial radius in terms of the speed of light and the angle β by solving the equation,

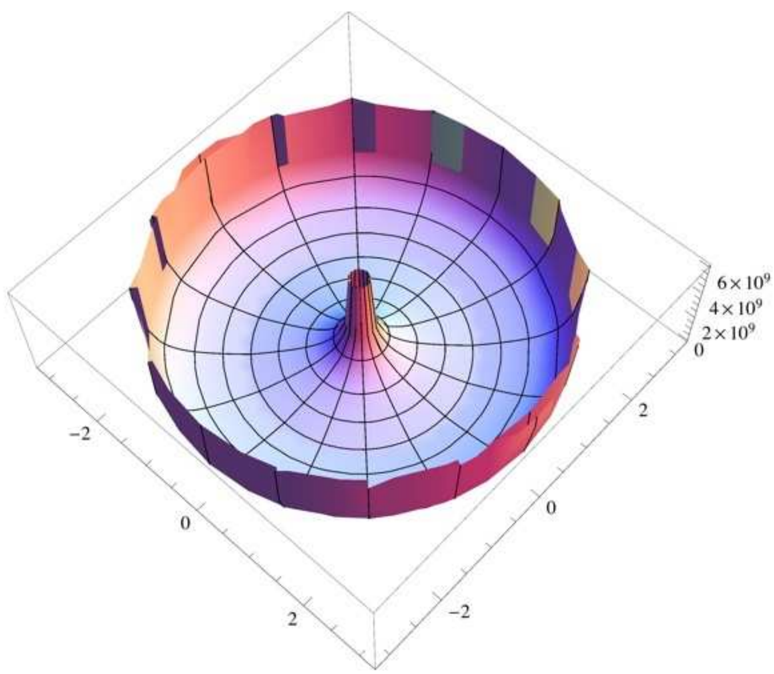
$$\frac{4\pi r^2}{c^2+r^2} = 2\left(\pi + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)$$

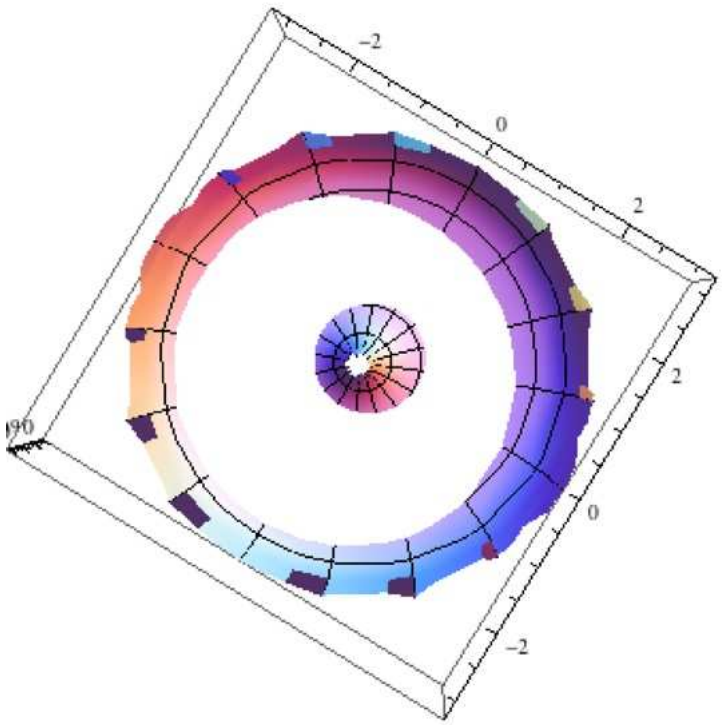
$$\text{Solve}\left[\frac{4\pi r^2}{c^2+r^2} == 2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right),\ r\right]$$

$$\left\{\left\{r \to -\frac{1.\sqrt{-2.82352\times10^{17}-2.82352\times10^{17}\sqrt{1.-1.\,\text{Sin}[\beta]^2}}}{\sqrt{-3.14159+3.14159\sqrt{1.-1.\,\text{Sin}[\beta]^2}}}\right\},\right.$$

$$\left.\left\{r \to \frac{\sqrt{-2.82352\times10^{17}-2.82352\times10^{17}\sqrt{1.-1.\,\text{Sin}[\beta]^2}}}{\sqrt{-3.14159+3.14159\sqrt{1.-1.\,\text{Sin}[\beta]^2}}}\right\}\right\}$$

```
RevolutionPlot3D[(√(-2.8235226668953677`*^17 - 2.8235226668953677`*^17 √(1.` - 1.` Sin[β]^2))) /
  (√(-3.141592653589793` + 3.141592653589793` √(1.` - 1.` Sin[β]^2))), {β, -π, π}]
```



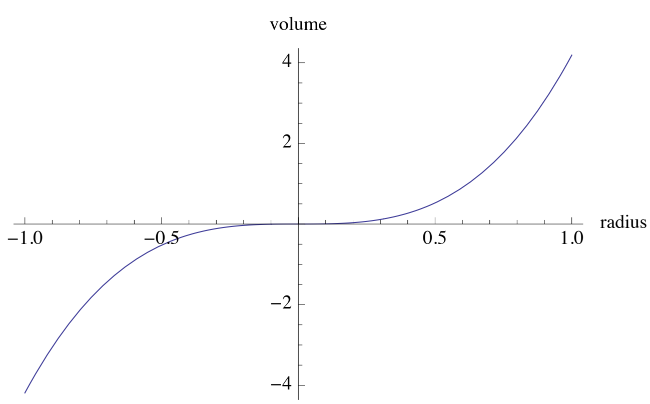


VII. Difference in Volumes of a Two Spheres from the Transformation of a Circle into a Cone

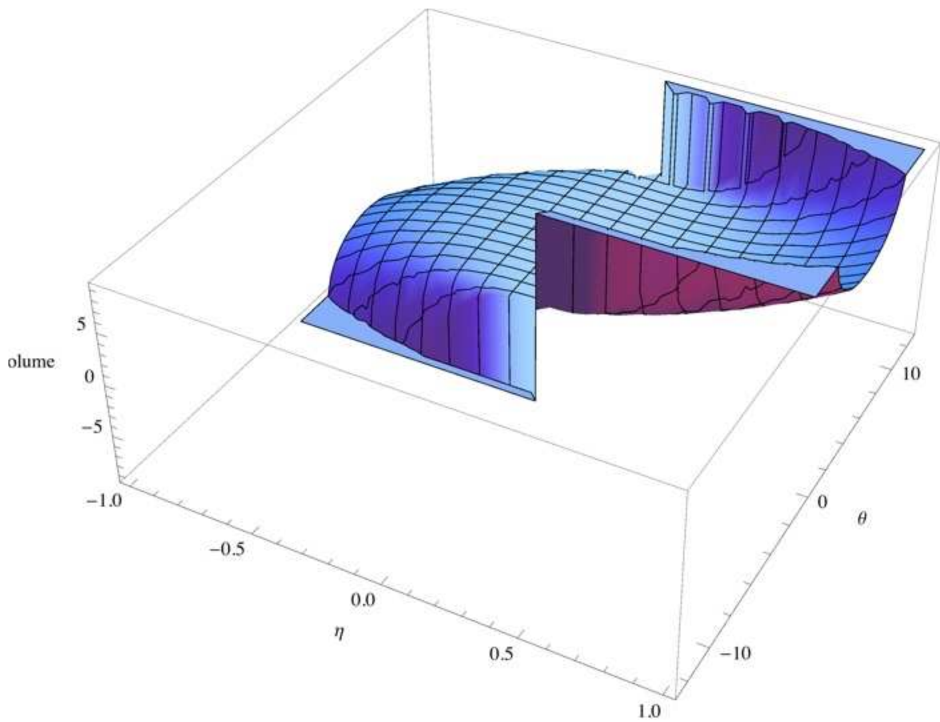
The purpose of this section is to show how extensive the implications of this theory are and to outline the framework from which more substitutions to the difference in volumes of a sphere could be made in the future. By no means will I attempt to visualize all the possible substitutions, for that would possibly be an infinite number, instead I will display just a handful that I find useful for describing certain specific visual contours.

$$(4/3)\,\pi\,(r)^3 = \text{Volume of Sphere} \tag{19}$$

```
Plot[(4 / 3) π (r)^3, {r, -1, 1}, AxesLabel → {radius, volume}]
```



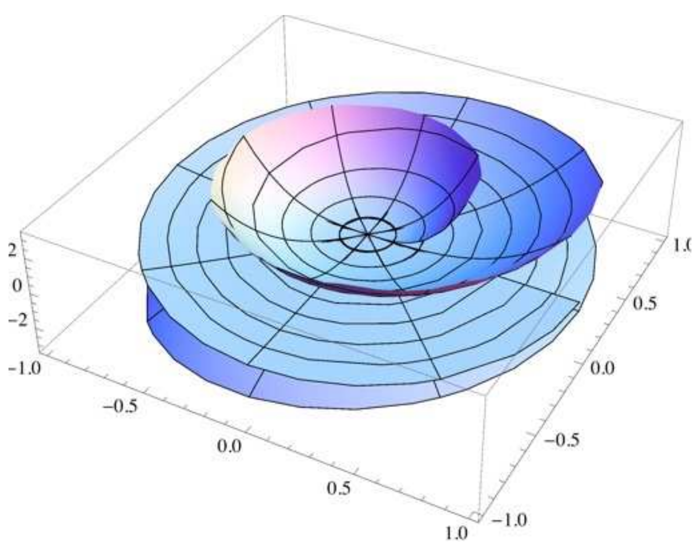
```
Plot3D[(4 / 3) π (2 π η / Sqrt[4 π θ - θ^2])^3, {η, -1, 1}, {θ, -4 π, 4 π}, AxesLabel → {η, θ, volume}]
```



$(4 / 3)\ \pi\ (r_1)\ \hat{}\ 3 = \text{Volume of Sphere of } r_1$ (20)

$r_1 = \frac{2 \pi r - r \theta}{2 \pi}$

```
RevolutionPlot3D[(4 / 3) π ((2 π r - r θ) / (2 π))^3, {r, -1, 1}, {θ, -2 π, 2 π}]
```

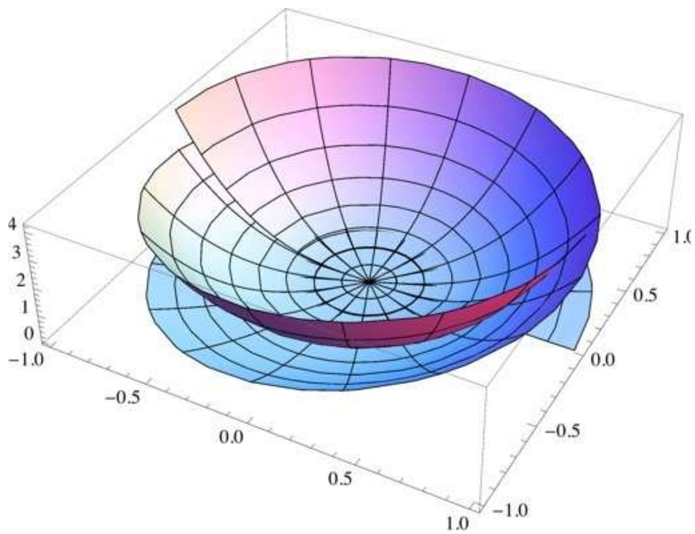


$(4 / 3)\ \pi\ (\eta)\ \hat{}\ 3 = \text{Volume of Sphere of } \eta$ (21)

$$\eta = \frac{\sqrt{4\,\pi\,r^2\,\Theta - r^2\,\Theta^2}}{2\,\pi} = r\,\mathrm{Sin}[\beta]$$

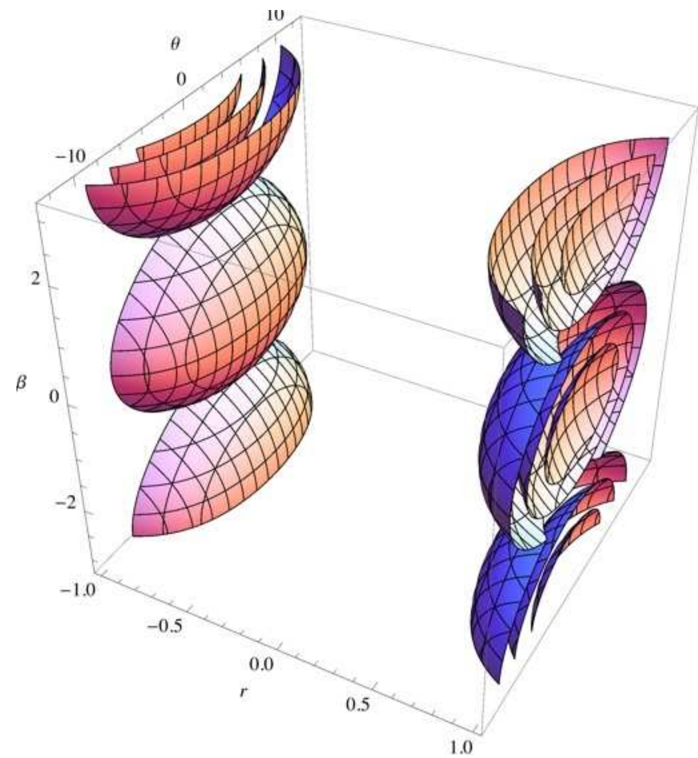
$$\Theta = 2\left(\pi + \sqrt{\pi^2 - \pi^2\,\mathrm{Sin}[\beta]^2}\right) = \frac{2\,\pi\left(r^2 + \sqrt{r^4 - r^2\,\eta^2}\right)}{r^2}$$

$$\mathbf{RevolutionPlot3D}\left[(4/3)\,\pi\left(\frac{\sqrt{4\,\pi\,r^2\,\Theta - r^2\,\Theta^2}}{2\,\pi}\right)^{\wedge}3,\ \{r, -1, 1\},\ \{\Theta, -2\,\pi, 2\,\pi\}\right]$$

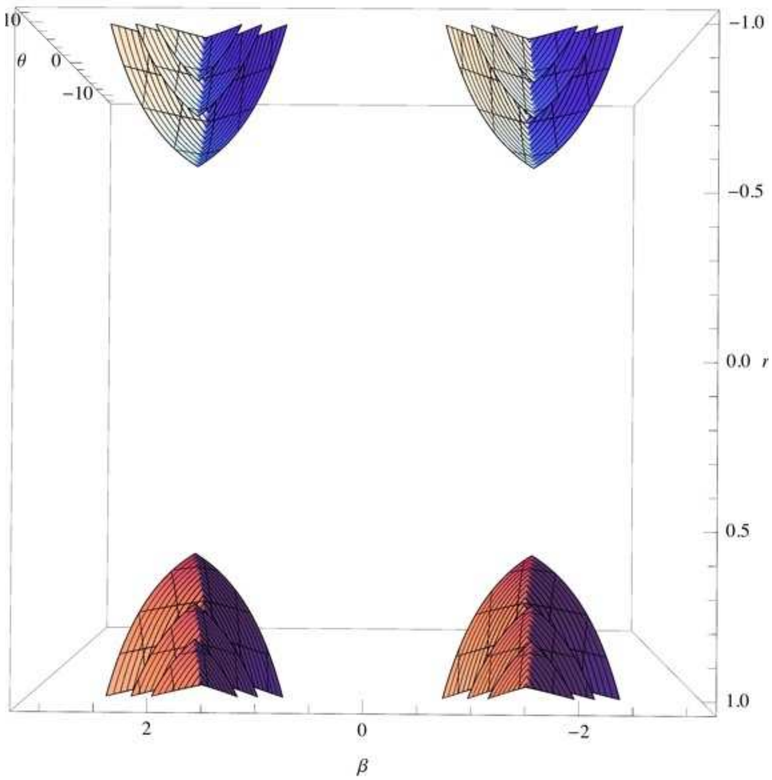


```
ContourPlot3D[(4 / 3) π (Sqrt[4 π r^2 2 (π + Sqrt[π^2 - π^2 Sin[β]^2]) - r^2 θ^2] / (2 π))^3,
 {r, -1, 1}, {θ, -4 π, 4 π}, {β, -π, π}, AxesLabel → Automatic]
```

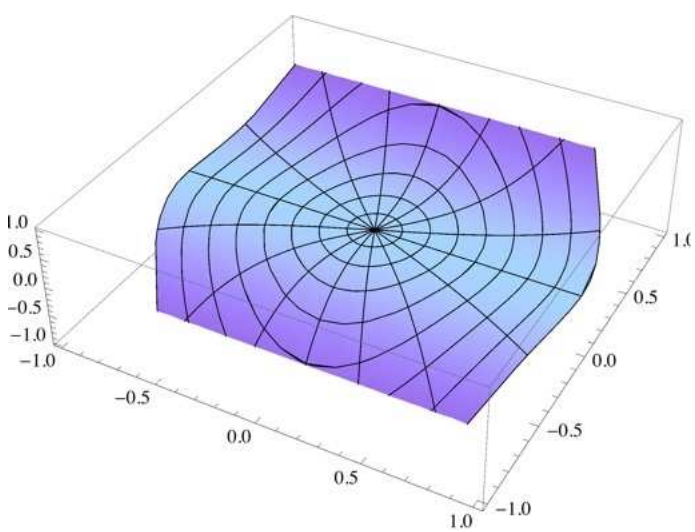
$$\text{ContourPlot3D}\left[(4/3)\,\pi\left(\frac{\sqrt{4\,\pi\,r^2\,2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)-r^2\,\theta^2}}{2\,\pi}\right)^3,\ \{r,-1,1\},\{\theta,-4\pi,4\pi\},\{\beta,-\pi,\pi\},\ \text{AxesLabel}\to\text{Automatic}\right]$$



```
ContourPlot3D[(4 / 3) π (Sqrt[4 π r^2 θ - r^2 (2 (π + Sqrt[π^2 - π^2 Sin[β]^2]))^2] / (2 π))^3,
  {r, -1, 1}, {θ, -4 π, 4 π}, {β, -π, π}, AxesLabel → Automatic]
```

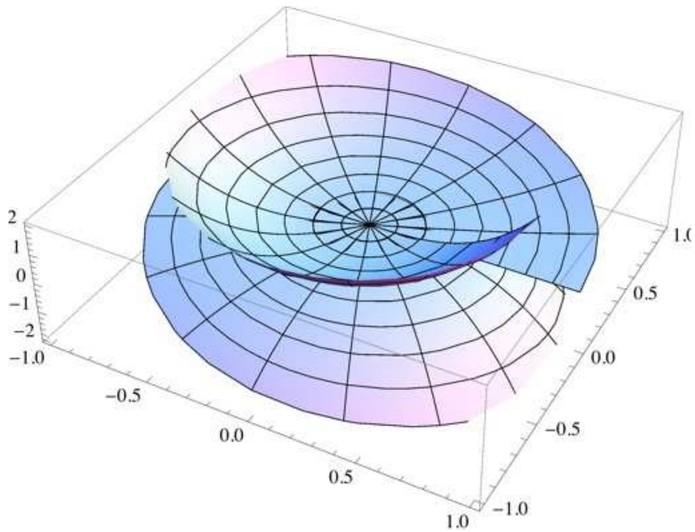



```
RevolutionPlot3D[(4 / 3) π (r Sin[β])^3, {r, -1, 1}, {β, -π, π}]
```



```
RevolutionPlot3D[(4 / 3) π (r Sin[ArcSin[Sqrt[(4 π - θ) θ] / (2 π)]])^3, {r, -1, 1}, {θ, -2 π, 2 π}]
```

$$\text{RevolutionPlot3D}\left[(4/3)\,\pi\left(r\,\text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\,\theta}}{2\pi}\right]\right]\right)\text{^}3,\ \{r,-1,1\},\ \{\theta,-2\pi,2\pi\}\right]$$



This section of the paper outlines the expressions for four differences in volumes of a sphere. These are:

$(4/3)\,\pi\,(r)\text{^}3 - (4/3)\,\pi\,(r_1)\text{^}3$ = Difference 1 in Volumes of Two Spheres

$(4/3)\,\pi\,(r)\text{^}3 - (4/3)\,\pi\,(\eta)\text{^}3$ = Difference 2 in Volumes of Two Spheres

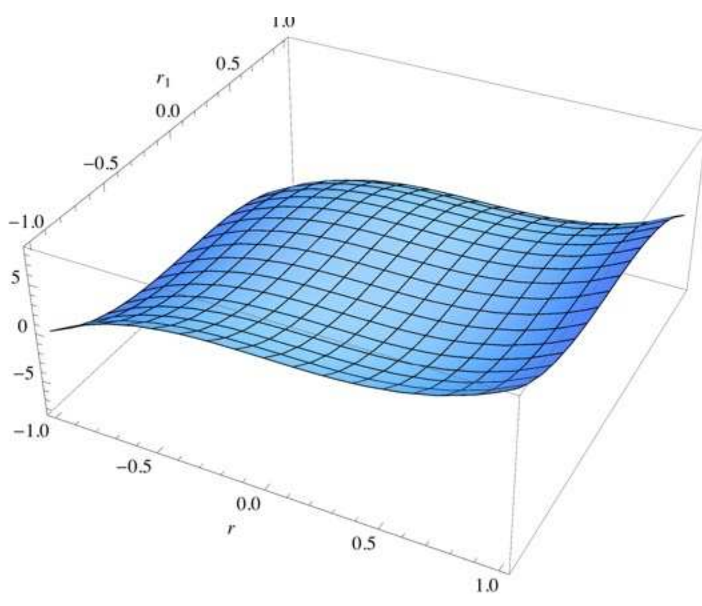
$(4/3)\,\pi\,(\eta)\text{^}3 - (4/3)\,\pi\,(r_1)\text{^}3$ = Difference 3 in Volumes of Two Spheres

$(4/3)\,\pi\,(r_1)\text{^}3 - (4/3)\,\pi\,(\eta)\text{^}3$ = Difference 4 in Volumes of Two Spheres

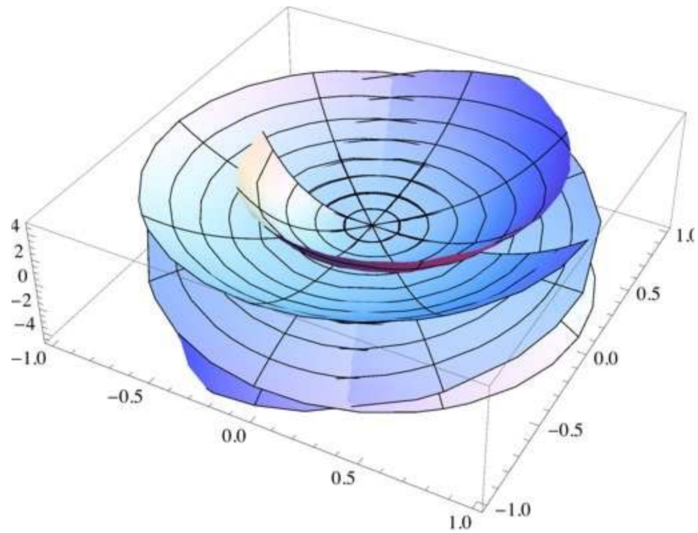
For each, I will show a few different expressions for them found from substitutions and the visual contours that are found upon graphing them.

$$(4/3)\,\pi\,(r)\text{^}3 - (4/3)\,\pi\,(r_1)\text{^}3 = \text{Difference 1 in Volumes of Two Spheres} \qquad (22)$$

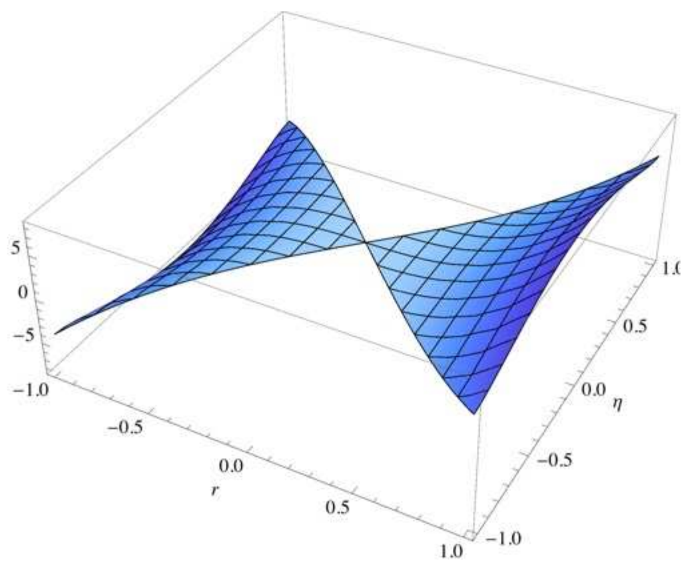
```
Plot3D[(4 / 3) π (r)^3 - (4 / 3) π (r1)^3, {r, -1, 1}, {r1, -1, 1}, AxesLabel → Automatic]
```



```
RevolutionPlot3D[(4 / 3) π (r) ^ 3 - (4 / 3) π ((2 π r - r θ)/(2 π)) ^ 3, {r, -1, 1}, {θ, -2 π, 2 π}]
```

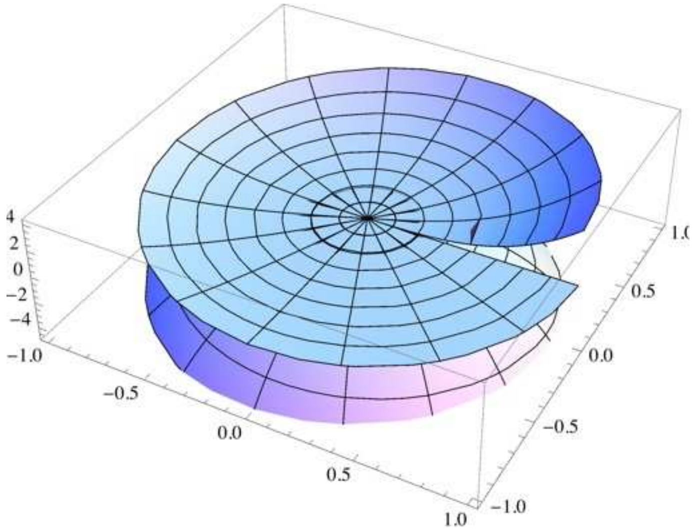



```
Plot3D[(4 / 3) π (r) ^ 3 - (4 / 3) π ((2 π r - r (2 π (r^2 + Sqrt[r^4 - r^2 η^2]))/r^2)/(2 π)) ^ 3,
 {r, -1, 1}, {η, -1, 1}, AxesLabel → Automatic]
```

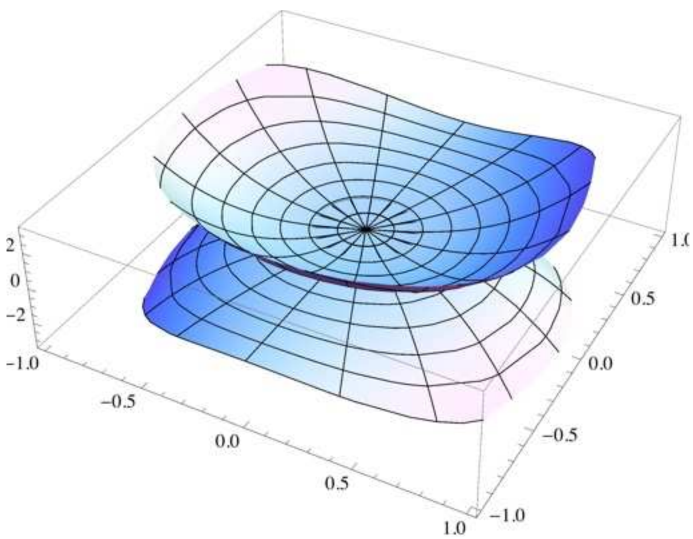


(4 / 3) π (r) ^ 3 - (4 / 3) π (η) ^ 3 = Difference 2 in Volumes of Two Spheres (23)

$$\text{RevolutionPlot3D}\left[(4/3)\,\pi\,(r)^{\wedge}3 - (4/3)\,\pi\left(\frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi}\right)^{\wedge}3,\ \{r, -1, 1\},\ \{\theta, -2\,\pi, 2\,\pi\}\right]$$

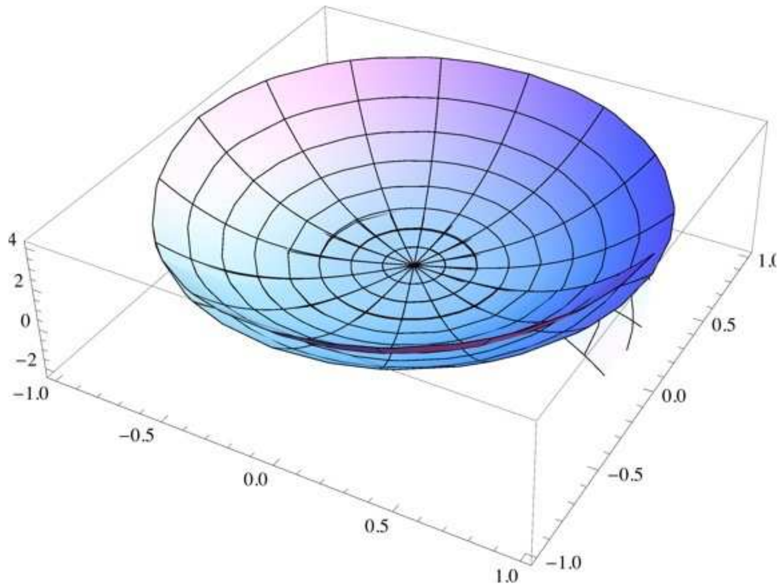


$$\text{RevolutionPlot3D}[(4/3)\,\pi\,(r)^{\wedge}3 - (4/3)\,\pi\,(r\,\text{Sin}[\beta])^{\wedge}3,\ \{r, -1, 1\},\ \{\beta, -\pi, \pi\}]$$



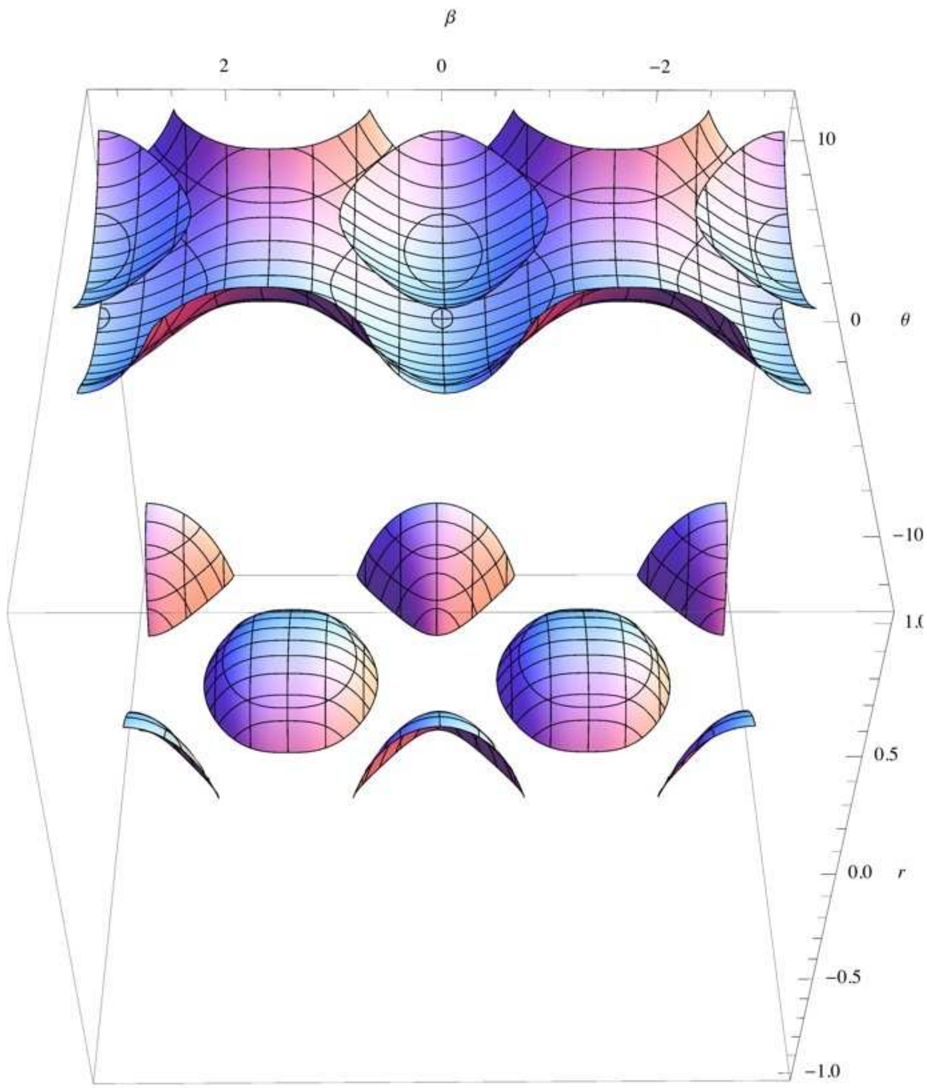
$(4/3)\,\pi\,(\eta)^{\wedge}3 - (4/3)\,\pi\,(r_1)^{\wedge}3$ = Difference 3 in Volumes of Two Spheres (24)

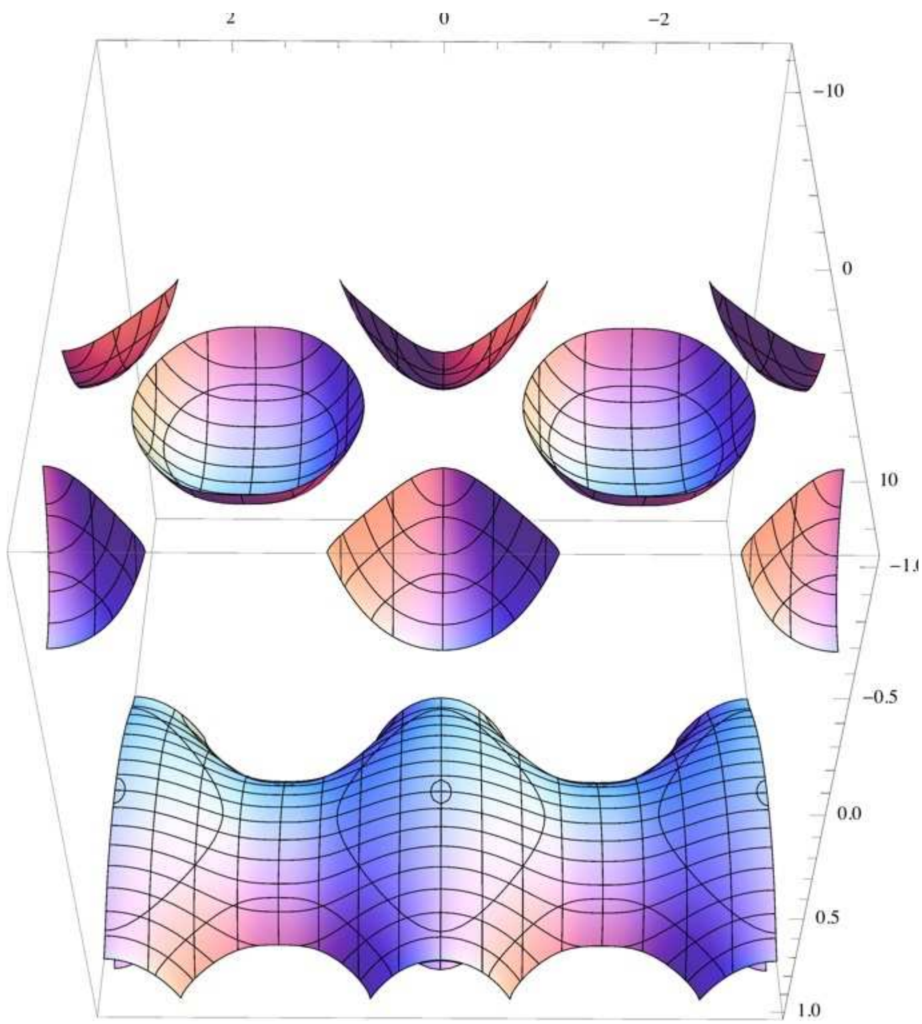
```
RevolutionPlot3D[
```
$$(4/3)\,\pi\left(\frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi}\right)\hat{}3 - (4/3)\,\pi\left(\frac{2\,\pi\,r - r\,\theta}{2\,\pi}\right)\hat{}3,\ \{r, -1, 1\},\ \{\theta, -2\,\pi, 2\,\pi\}\Big]$$



$$\text{ContourPlot3D}\left[(4/3)\,\pi\left(\frac{\sqrt{4\,\pi\,r^2\,(\theta)-r^2\,(\theta)^2}}{2\,\pi}\right)^{\wedge}3-(4/3)\,\pi\left(\frac{2\,\pi\,r-r\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right)}{2\,\pi}\right)^{\wedge}3,\right.$$

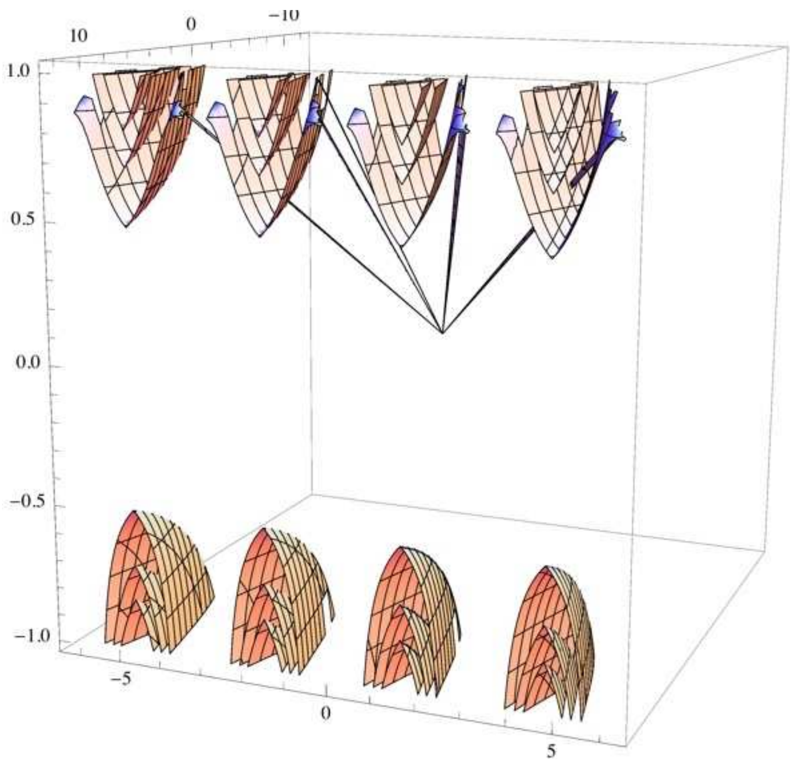
$$\left.\{r,-1,1\},\{\theta,-4\,\pi,4\,\pi\},\{\beta,-\pi,\pi\},\text{AxesLabel}\to\text{Automatic}\right]$$




2
0
−2
−10
0
10
−1.0
−0.5
0.0
0.5
1.0

$$\text{ContourPlot3D}\left[(4/3)\,\pi\left(\frac{\sqrt{4\,\pi\,r^2\,(\theta)-r^2\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right)^2}}{2\,\pi}\right)\text{^}3-\right.$$

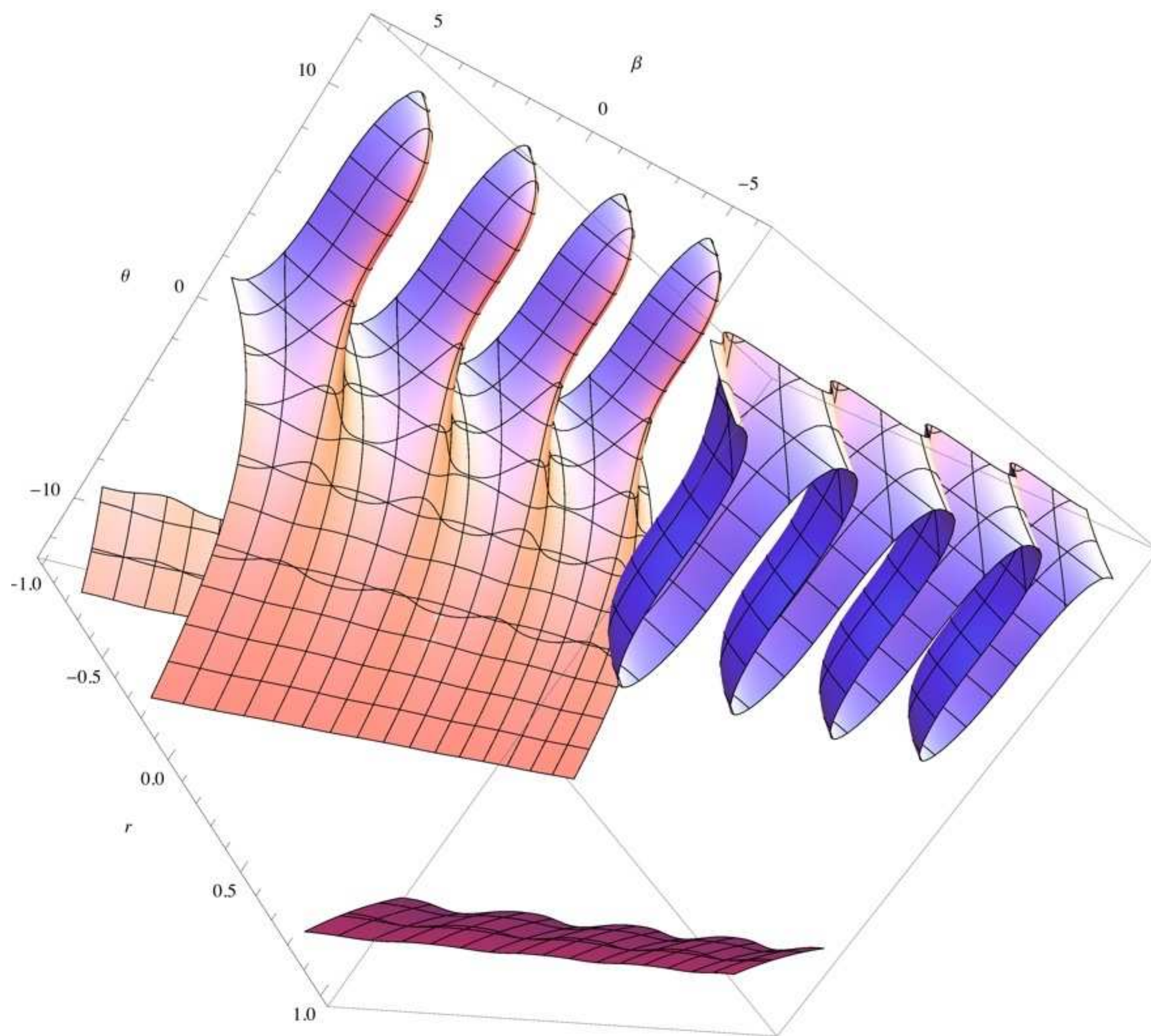
$$\left.(4/3)\,\pi\left(\frac{2\,\pi\,r-r\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right)}{2\,\pi}\right)\text{^}3,\ \{r,-1,1\},\ \{\theta,-4\,\pi,4\,\pi\},\ \{\beta,-2\,\pi,2\,\pi\}\right]$$



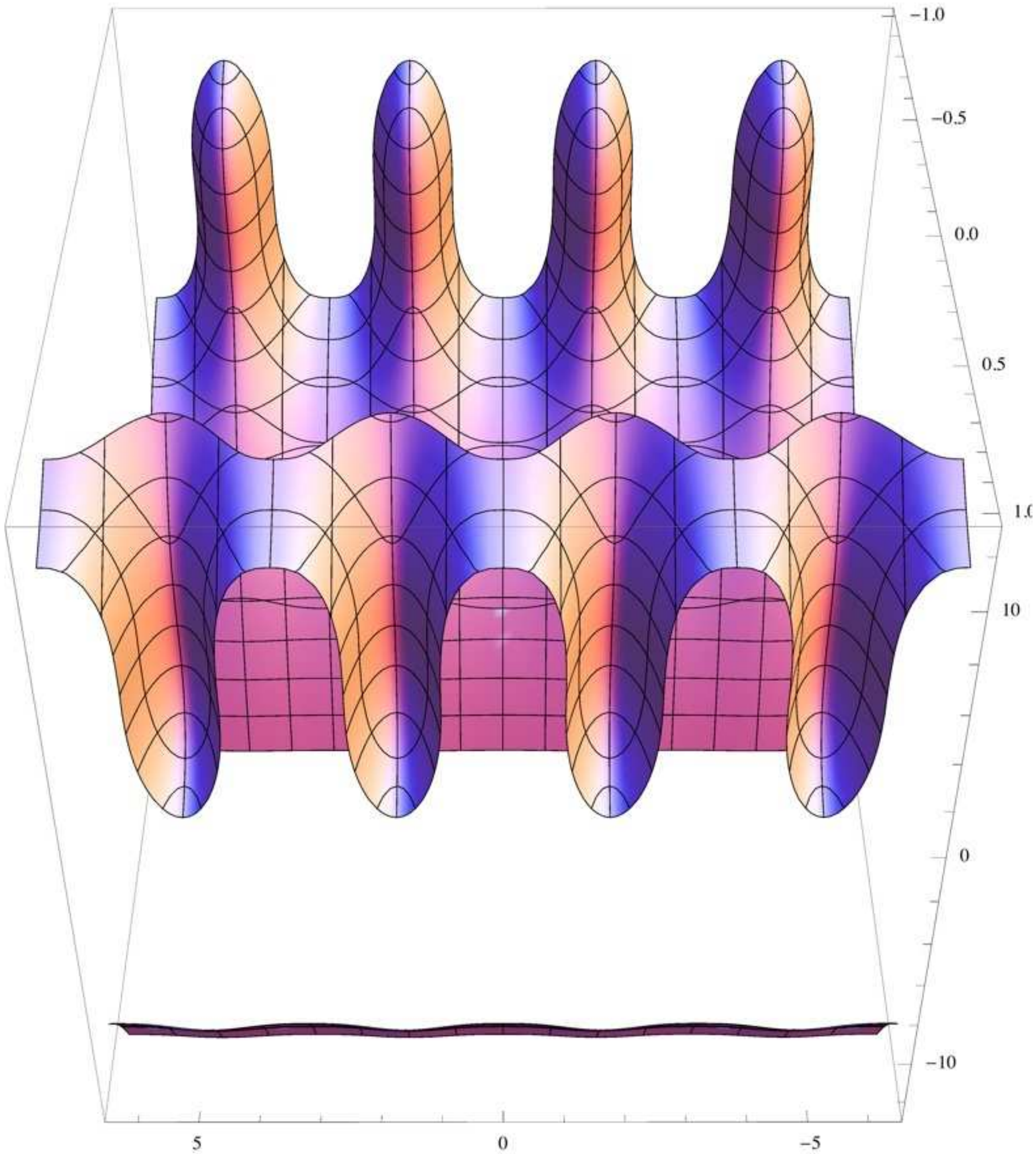
$$\text{ContourPlot3D}\left[(4/3)\,\pi\left(\frac{\sqrt{4\,\pi\,r^2\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right)-r^2\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right)^2}}{2\,\pi}\right)\text{^}3-\right.$$

$$\left.(4/3)\,\pi\left(\frac{2\,\pi\,r-r\,\theta}{2\,\pi}\right)\text{^}3,\ \{r,-1,1\},\ \{\theta,-4\,\pi,4\,\pi\},\ \{\beta,-2\,\pi,2\,\pi\},\ \text{AxesLabel}\to\text{Automatic}\right]$$

Adumbration One - "Michaelangelo's Formula"



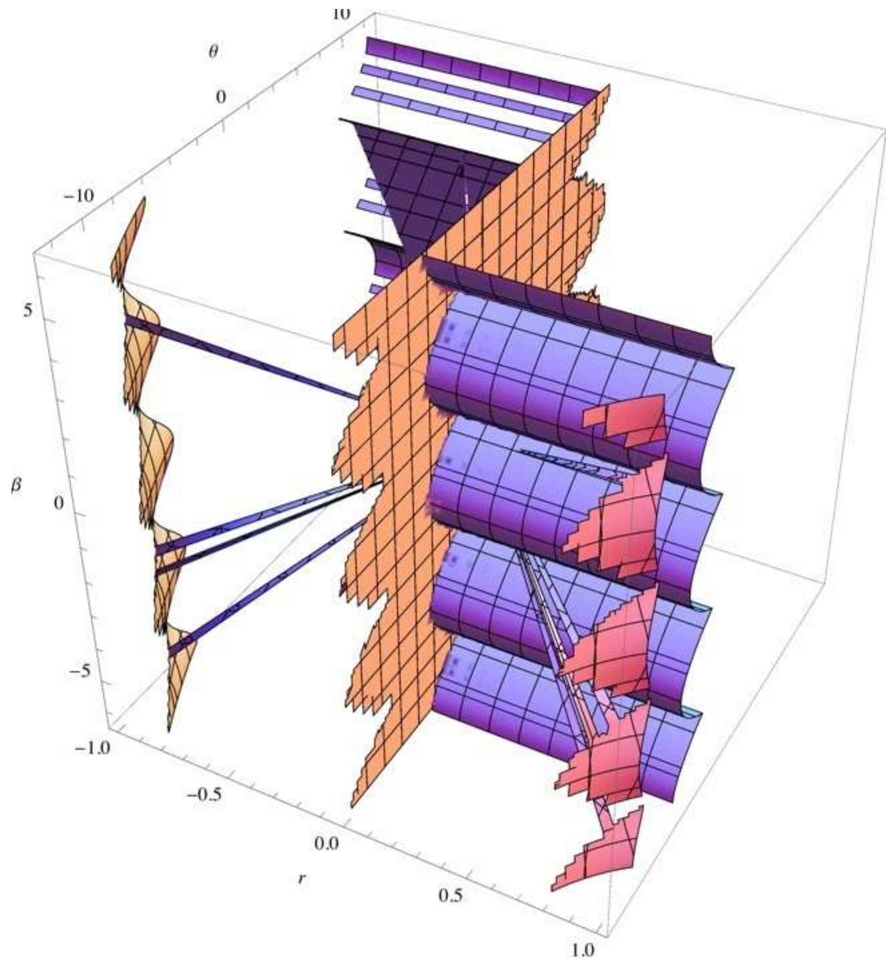
Adumbration Two



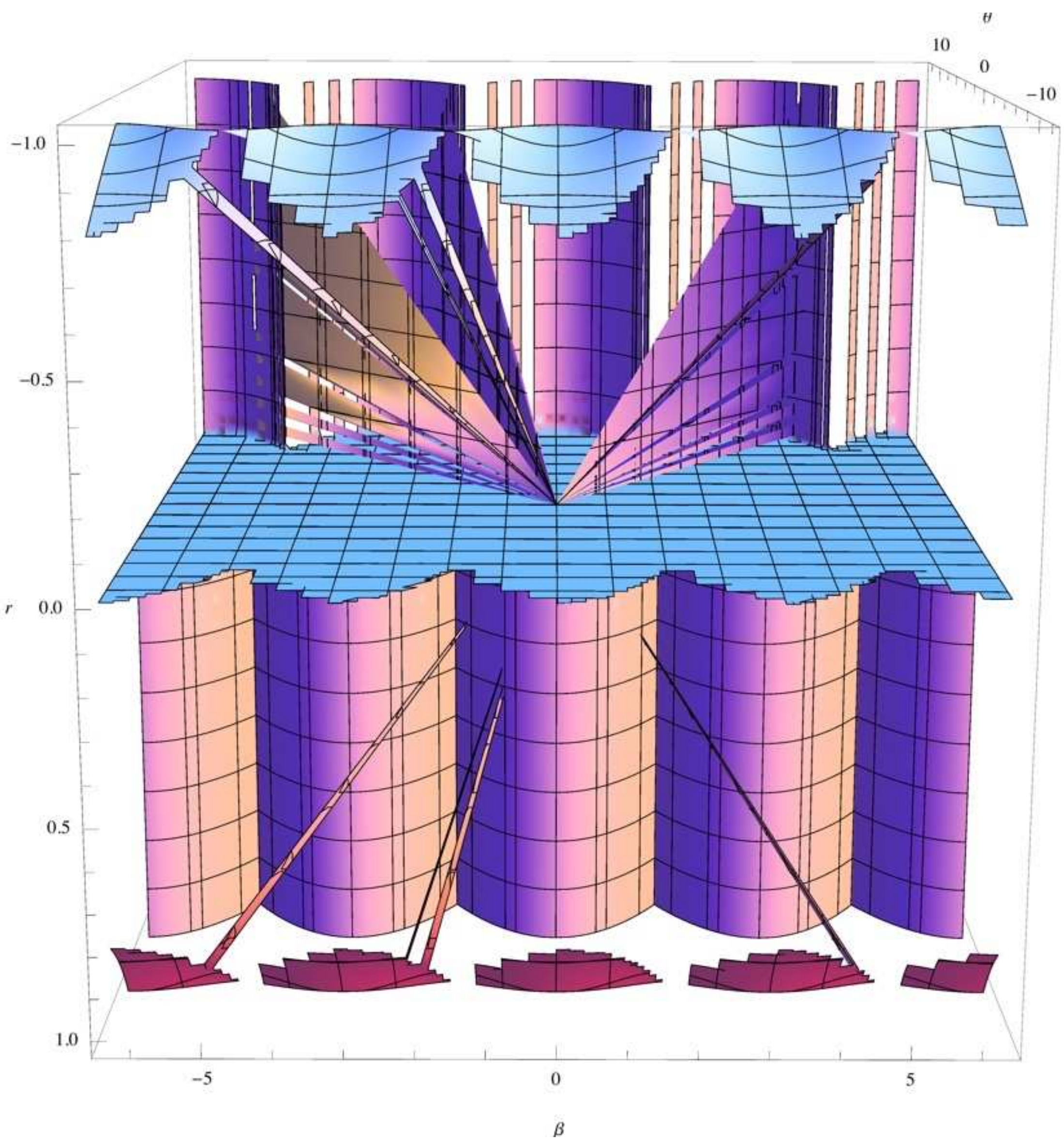
$$\mathbf{ContourPlot3D}\left[(4/3)\,\pi\left(\frac{\sqrt{4\,\pi\,r^2\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)\right)-r^2\,(\theta)^2}}{2\,\pi}\right)\text{^}3-(4/3)\,\pi\left(\frac{2\,\pi\,r-r\,\theta}{2\,\pi}\right)\text{^}3,\right.$$

$$\left.\{r,-1,1\},\{\theta,-4\,\pi,4\,\pi\},\{\beta,-2\,\pi,2\,\pi\},\mathbf{AxesLabel}\to\mathbf{Automatic}\right]$$

Adumbration One

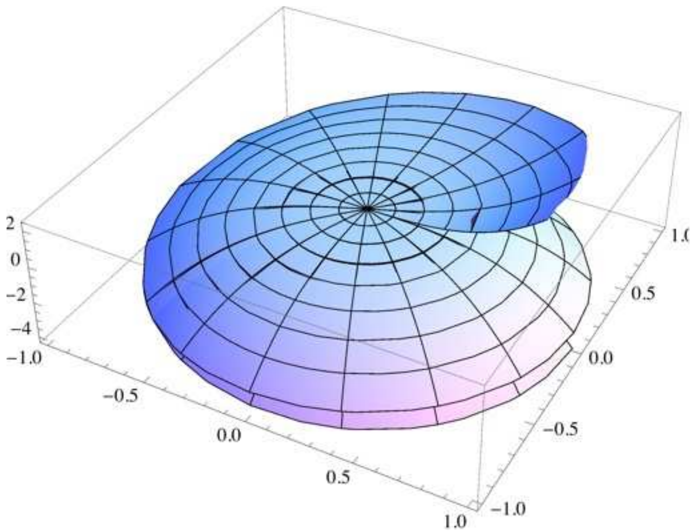


Adumbration Two

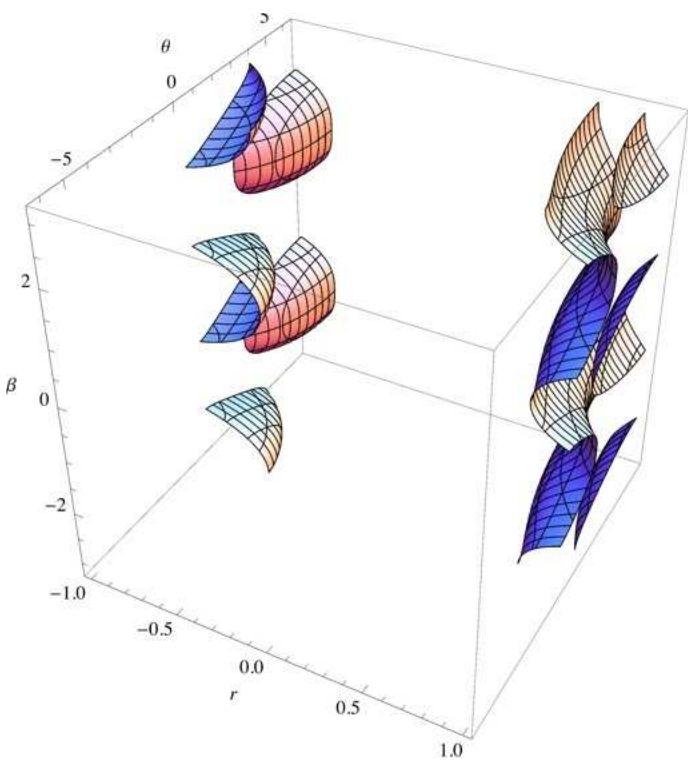


$(4/3)\,\pi\,(r_1)\,\hat{}\,3 - (4/3)\,\pi\,(\eta)\,\hat{}\,3$ = Difference 4 in Volumes of Two Spheres (25)

```
RevolutionPlot3D[
 (4 / 3) π ((2 π r - r θ)/(2 π))^3 - (4 / 3) π (Sqrt[4 π r^2 θ - r^2 θ^2]/(2 π))^3, {r, -1, 1}, {θ, -2 π, 2 π}]
```




```
ContourPlot3D[(4 / 3) π ((2 π r - r (2 (π + Sqrt[π^2 - π^2 Sin[β]^2])))/(2 π))^3 - (4 / 3) π (Sqrt[4 π r^2 θ - r^2 θ^2]/(2 π))^3,
 {r, -1, 1}, {θ, -2 π, 2 π}, {β, -π, π}, AxesLabel → Automatic]
```



VIII. Euler's Identity Applied through The "Cone Transformation"

A discussion of the origins of the meaning of complex analysis: Emptiness, substitution, and being.

■ Introducing the idea of emptiness within complex analysis.

The cone transformation refers to that transformation described through mathematics within the document : The Geometric Pattern of Perception by Parker Emmerson 2009 - 2010. In that document, it is shown from pure geometry and rigorous algebra that :

Lemma 4 *The height of the cone can be calculated in terms of only r and* θ, thus β is a function of θ alone.

Proof. Since we have shown that θ r = 2 π r - 2 π r_1 and $r_1 \to \sqrt{r^2 - \eta^2}$, we can substitute the expression for r_1, calculated from the Pythagorean theorem in terms of the height of the cone and the initial radius of the circle, into the expression for θ r in terms of the change in circumference of the initial circle to the circle that is the base of the cone into which the circle was transformed.

θ r = 2 π r - 2 π $\sqrt{(r\text{^}2 - \eta\text{^}2)}$, thus, $\eta = \frac{\sqrt{4 \pi r^2 \theta - r^2 \theta^2}}{2 \pi}$ = (r Sin[β]). From $\frac{2 \pi \eta}{\sqrt{4 \pi \theta - \theta^2}}$ = r, we note that: r = $\frac{2 \pi r \text{Sin}[\beta]}{\sqrt{4 \pi \theta - \theta^2}}$. So we solve the equation,

$$\text{Solve}\left[r == \frac{2 \pi r \text{Sin}[\beta]}{\sqrt{4 \pi \theta - \theta^2}}, \beta\right]$$

$$\left\{\left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{(4 \pi - \theta)\, \theta}}{2 \pi}\right]\right\}\right\}$$

From this, we see that $1 = \frac{2 \pi \text{Sin}[\beta]}{\sqrt{4 \pi \theta - \theta^2}}$, because $r = \frac{2 \pi r \text{Sin}[\beta]}{\sqrt{4 \pi \theta - \theta^2}}$

$$\text{Therefore, } \sqrt{-1} = i = \sqrt{-\frac{2 \pi \text{Sin}[\beta]}{\sqrt{4 \pi \theta - \theta^2}}} \qquad (26)$$

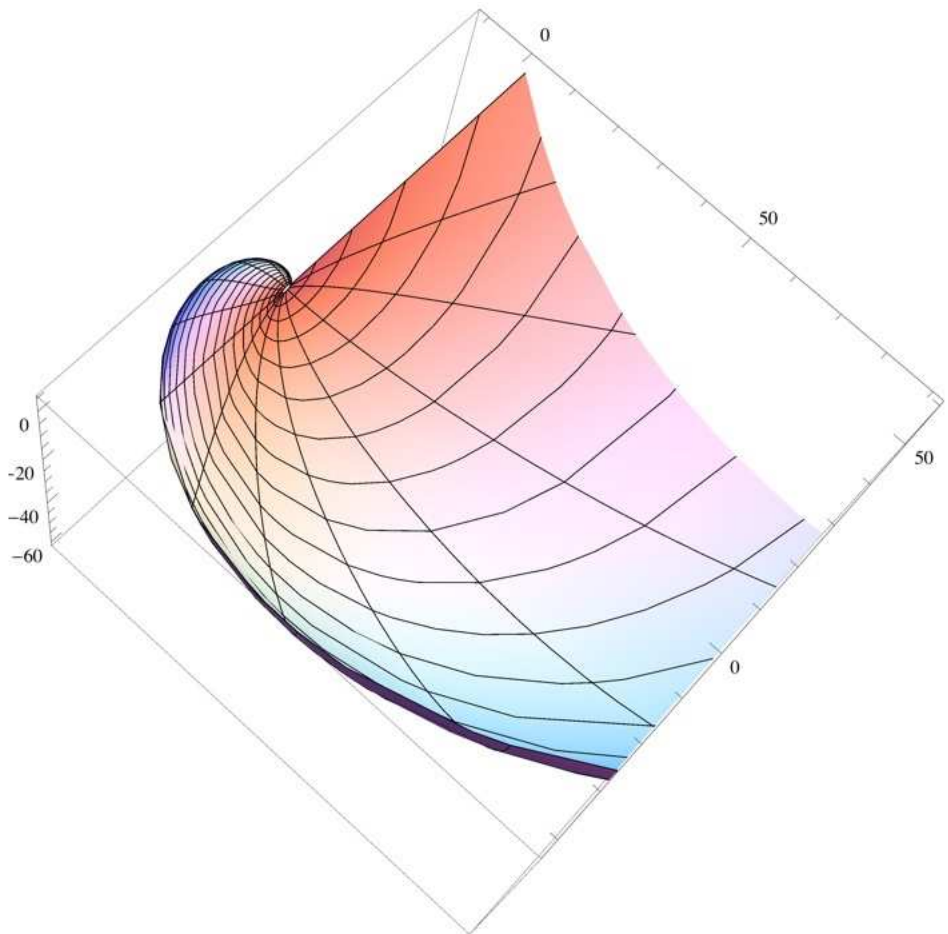
We can make a plot of $e^{i \theta}$, showing that i is a specific,

geometric function equal to $i = \sqrt{-\frac{2 \pi \text{Sin}[\beta]}{\sqrt{4 \pi \theta - \theta^2}}}$.

The function we have is :

$e^{\sqrt{-\frac{2 \pi \text{Sin}[\beta]}{\sqrt{4 \pi \theta - \theta^2}}}\, \theta}$, which we will graph.

$$\texttt{SphericalPlot3D}\left[e^{\sqrt{-\frac{2\,\pi\,\mathrm{Sin}[\beta]}{\sqrt{4\,\pi\,\theta-\theta^2}}\,\theta}},\ \{\theta,\ -2\,\pi,\ 2\,\pi\},\ \{\beta,\ -\pi/2,\ \pi/2\}\right]$$



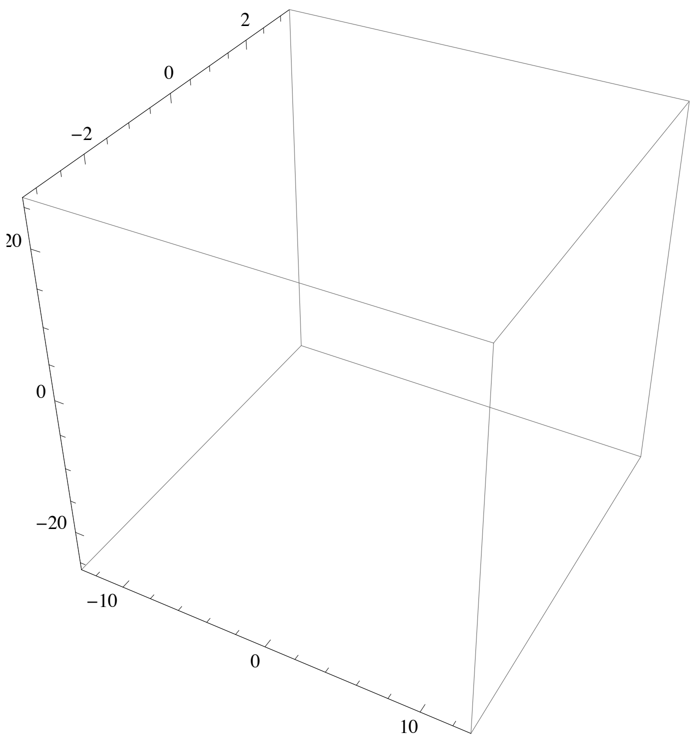
This is beautiful, a spiral, in the range at which we perceive regular distances. The range is 50 centimeters or 50 meters, scalable upon the units that we decide.

However, there is an intricate pattern within this expression. For example,

we can also say : $e^{\sqrt{-\frac{2\,\pi\,\mathrm{Sin}[\beta]}{\sqrt{4\,\pi\,\theta-\theta^2}}\,\alpha}}$, where $e^{\sqrt{-\frac{2\,\pi\,\mathrm{Sin}[\beta]}{\sqrt{4\,\pi\,2\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)-\theta^2}}\,\theta}} \neq e^{\sqrt{-\frac{2\,\pi\,\mathrm{Sin}[\beta]}{\sqrt{4\,\pi\,\theta-\theta^2}}\,\alpha}}$

Thus, we must also graph $e^{i\,\alpha} = e^{\sqrt{-\frac{2\,\pi\,\mathrm{Sin}[\beta]}{\sqrt{4\,\pi\,\theta-\theta^2}}\,\alpha}}$;
a graph with three variables is also possible.

ContourPlot3D$\left[e^{\sqrt{-\frac{2\pi \text{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}}}\alpha}, \{\theta, -4\pi, 4\pi\}, \{\beta, -\pi, \pi\}, \{\alpha, -8\pi, 8\pi\}\right]$

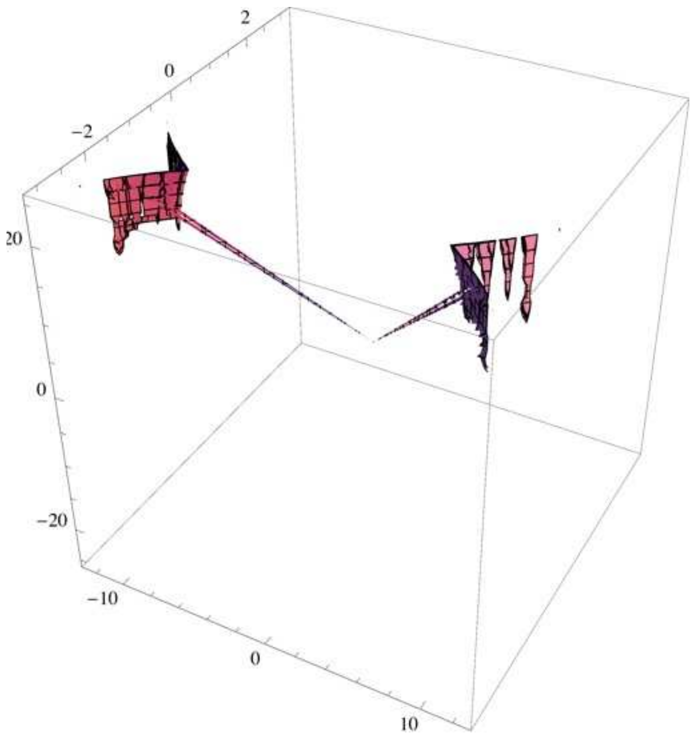


The emptiness of which is very interesting. This emptiness is especially interesting after further substitution, because it is not that all valid expressions in this manner are empty plots. We can contemplate why it is empty before and after applying a valid substitution, and why it is not empty after applying the valid substitution.

The valid substitution is : $\theta = 2\left(\pi + \sqrt{\pi^2 - \pi^2 \text{Sin}[\beta]^2}\right)$,

which we substitute into the position of θ within the square root in the denominator.

```
ContourPlot3D[
```
$$\text{ContourPlot3D}\left[e^{\sqrt{-\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{4\pi\,2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)-\theta^2}}\,\alpha}},\ \{\theta, -4\pi, 4\pi\},\ \{\beta, -\pi, \pi\},\ \{\alpha, -8\pi, 8\pi\}\right]$$

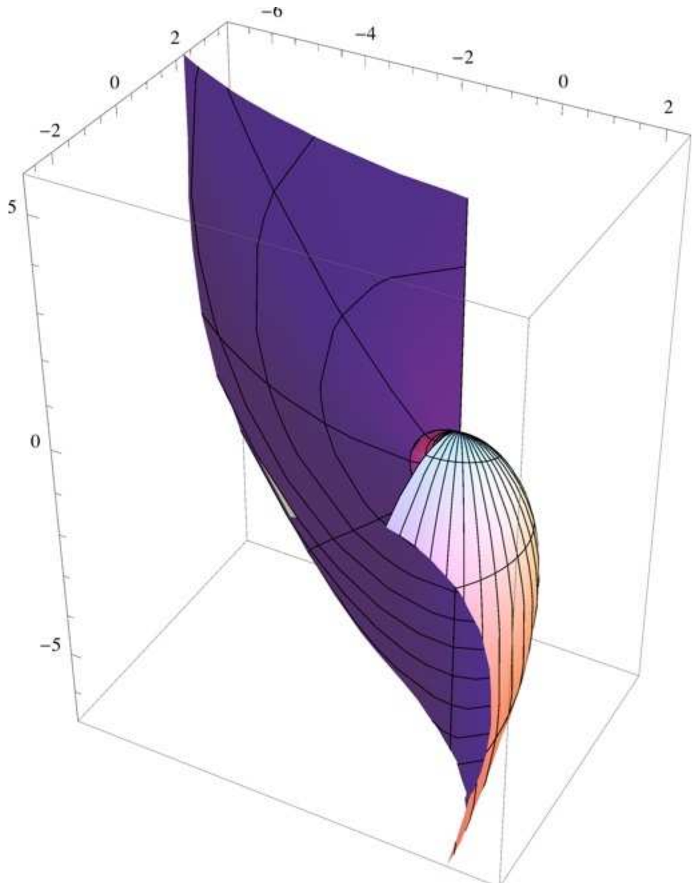


Where once we had a graph that was empty of form, we, after substitution from valid geometric and algebraic lemmas, we now have a box that contains a form within it.

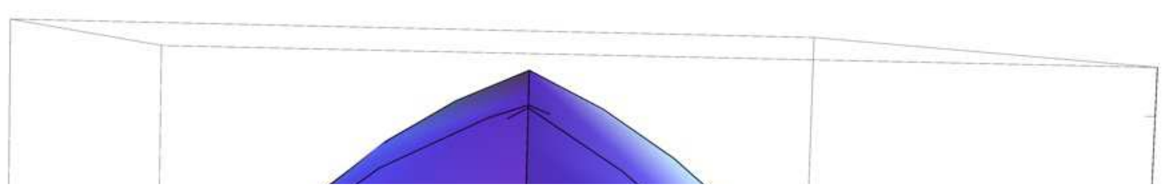
■ Configurational subsitutions

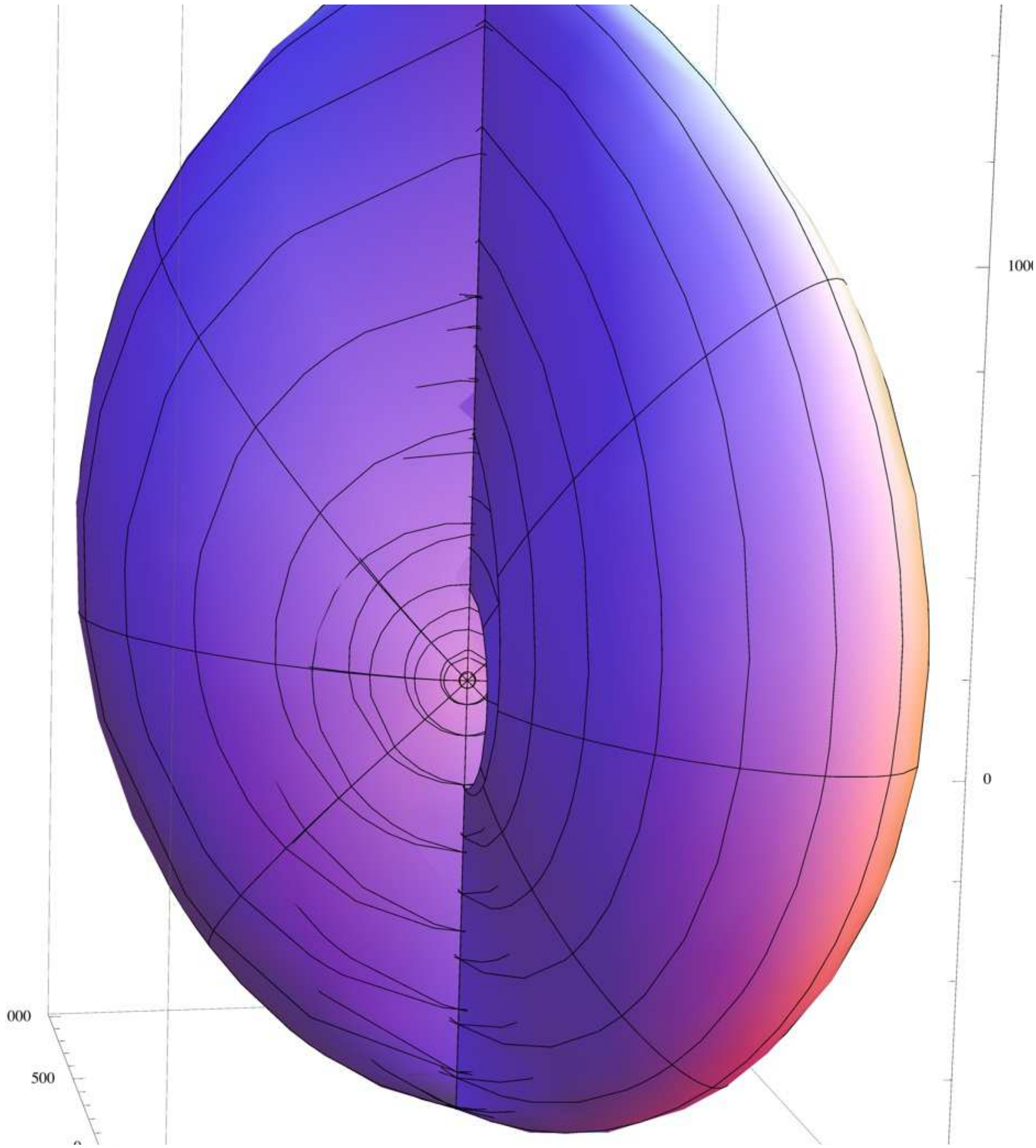
However, we can also,
from our initial Lemmas found through pure geometry and rigorous algebra,
graph other valid expressions of the equation $e^{\sqrt{-\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}}\,\theta}} = e^{i\,\theta}$. (27)

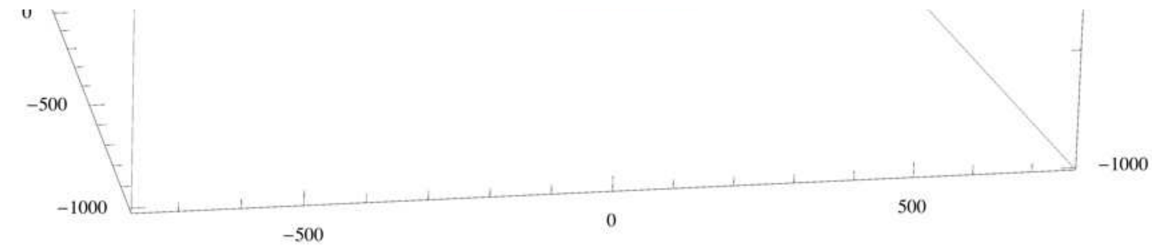
$$\texttt{SphericalPlot3D}\left[e^{\sqrt{-\frac{2\,\pi\,\texttt{Sin}[\beta]}{\sqrt{4\,\pi\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\texttt{Sin}[\beta]^2}\right)\right)-\theta^2}}}\;\theta},\ \{\theta,\,-2\,\pi,\,2\,\pi\},\ \{\beta,\,-\pi/2,\,\pi/2\}\right]$$



$$\texttt{SphericalPlot3D}\left[e^{\sqrt{-\frac{2\,\pi\,\texttt{Sin}[\beta]}{\sqrt{4\,\pi\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\texttt{Sin}[\beta]^2}\right)\right)-\theta^2}}}\;\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\texttt{Sin}[\beta]^2}\right)\right)},\ \{\theta,\,-2\,\pi,\,2\,\pi\},\ \{\beta,\,-\pi/2,\,\pi/2\}\right]$$



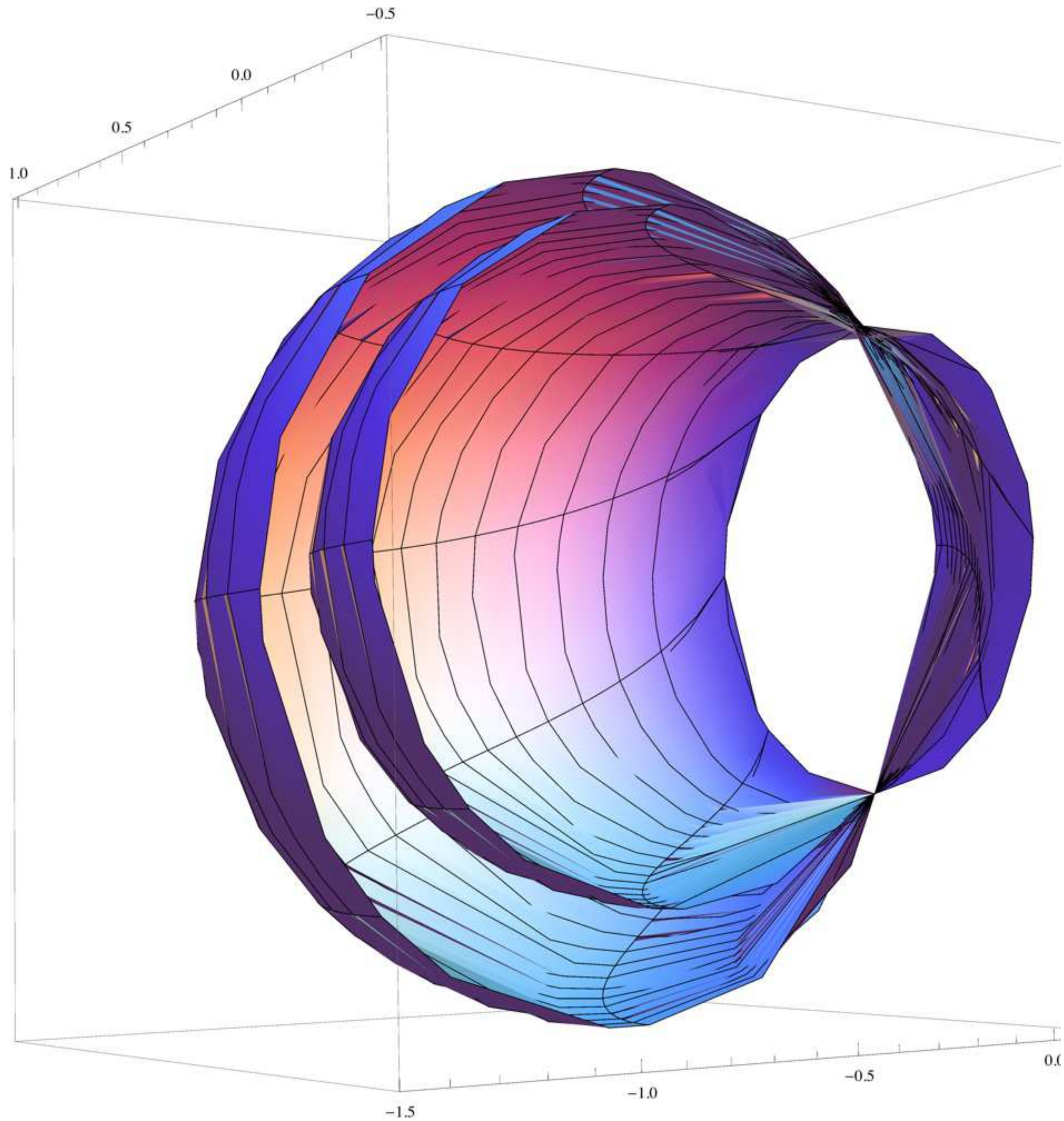

1000
0
000
500



Furthermore, we also have,

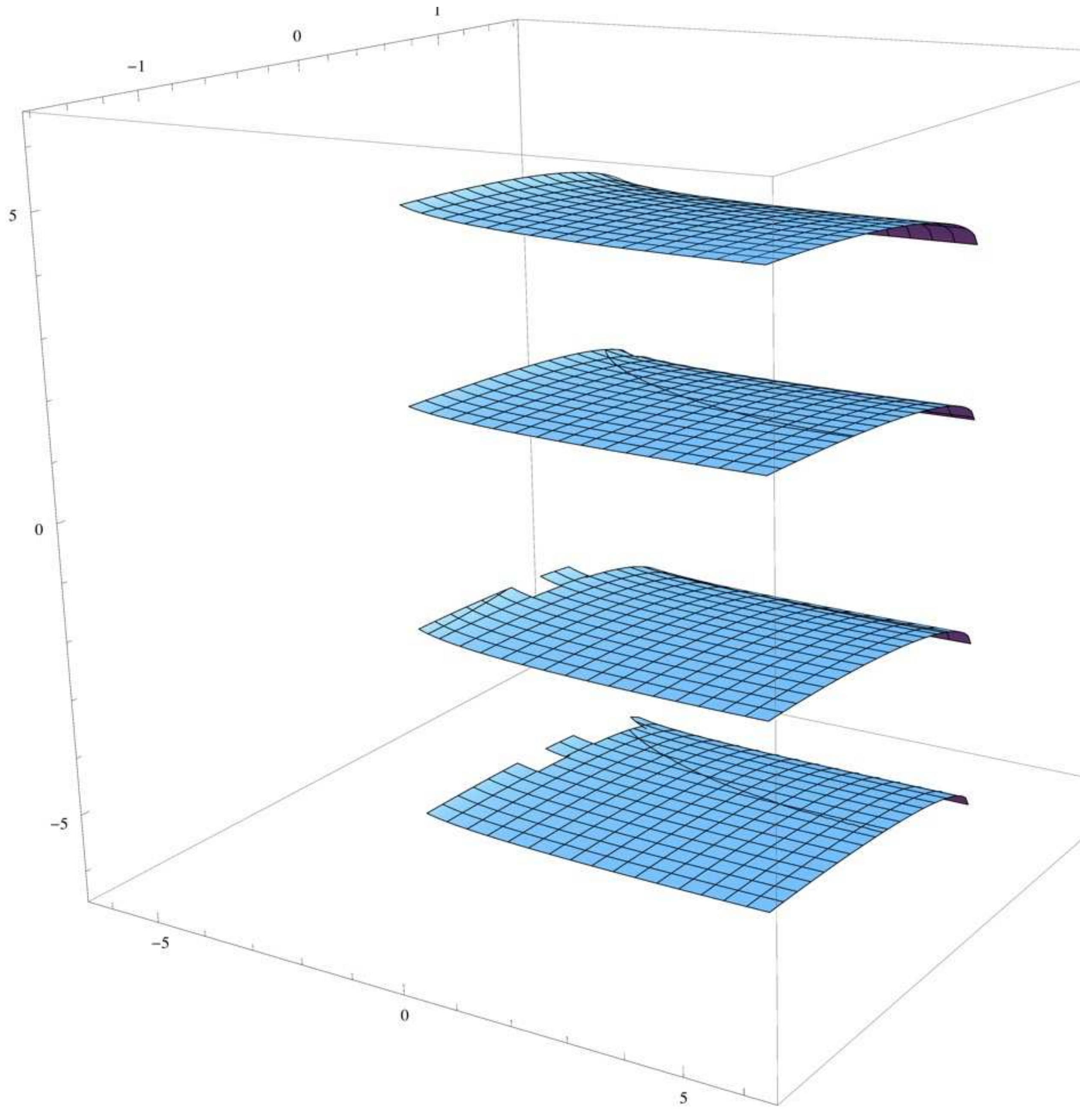
$$e^{\sqrt{-\frac{2\,\pi\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\theta-\theta^2}}}\;\theta} = i\,\text{Sin}[\theta] + \text{Cos}[\theta] = \sqrt{-\frac{2\,\pi\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\theta-\theta^2}}}\;\text{Sin}[\theta] + \text{Cos}[\theta]$$

$$\text{SphericalPlot3D}\left[\sqrt{-\frac{2\,\pi\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\theta-\theta^2}}}\ \text{Sin}[\theta]+\text{Cos}[\theta],\ \{\theta,\,-2\,\pi,\,2\,\pi\},\ \{\beta,\,-\pi/2,\,\pi/2\}\right]$$



$$e^{i\,\alpha}\,e^{\sqrt{-\frac{2\,\pi\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\theta-\theta^2}}}\,\alpha}=i\,\text{Sin}[\alpha]+\text{Cos}[\alpha]=\sqrt{-\frac{2\,\pi\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\theta-\theta^2}}}\ \text{Sin}[\alpha]+\text{Cos}[\alpha]$$

```
ContourPlot3D[Sqrt[-(2 π Sin[β])/Sqrt[4 π θ - θ^2]] Sin[α] + Cos[α], {θ, -2 π, 2 π}, {β, -π / 2, π / 2}, {α, -2 π, 2 π}]
```

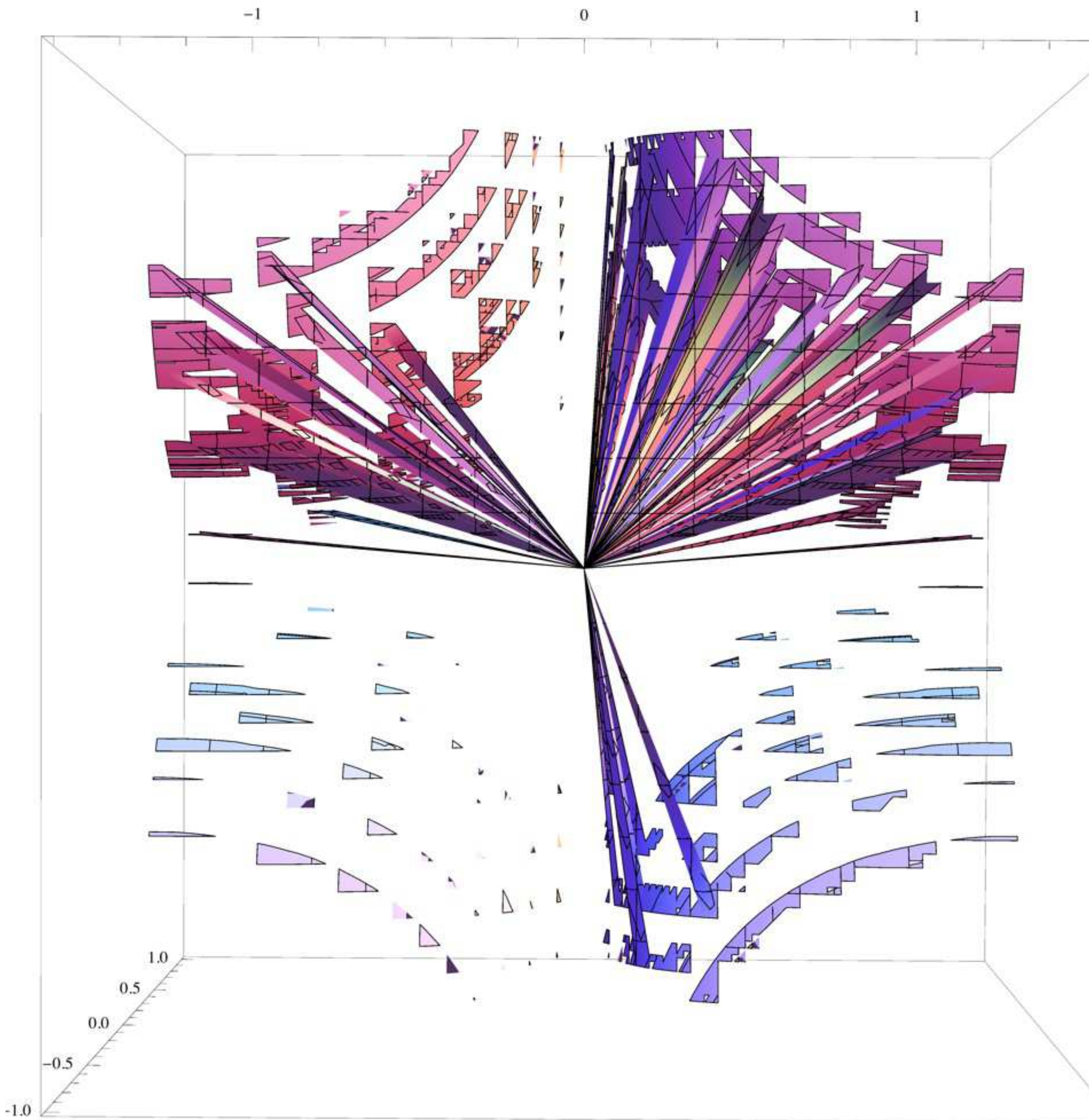


$$\text{Solve}\left[e^{\sqrt{-\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}}}\,\alpha}==x,\theta\right]$$

$$\left\{\left\{\theta\to\frac{2\left(\pi\,\text{Log}[x]^4-\sqrt{\pi^2\,\text{Log}[x]^8-\pi^2\,\alpha^4\,\text{Log}[x]^4\,\text{Sin}[\beta]^2}\right)}{\text{Log}[x]^4}\right\},\right.$$

$$\left.\left\{\theta\to\frac{2\left(\pi\,\text{Log}[x]^4+\sqrt{\pi^2\,\text{Log}[x]^8-\pi^2\,\alpha^4\,\text{Log}[x]^4\,\text{Sin}[\beta]^2}\right)}{\text{Log}[x]^4}\right\}\right\}$$

$$\text{ContourPlot3D}\left[\frac{2\left(\pi\,\text{Log}[x]^4+\sqrt{\pi^2\,\text{Log}[x]^8-\pi^2\,\alpha^4\,\text{Log}[x]^4\,\text{Sin}[\beta]^2}\right)}{\text{Log}[x]^4},\ \{x,-1,1\},\{\alpha,-\pi,\pi\},\{\beta,-\pi/2,\pi/2\}\right]$$



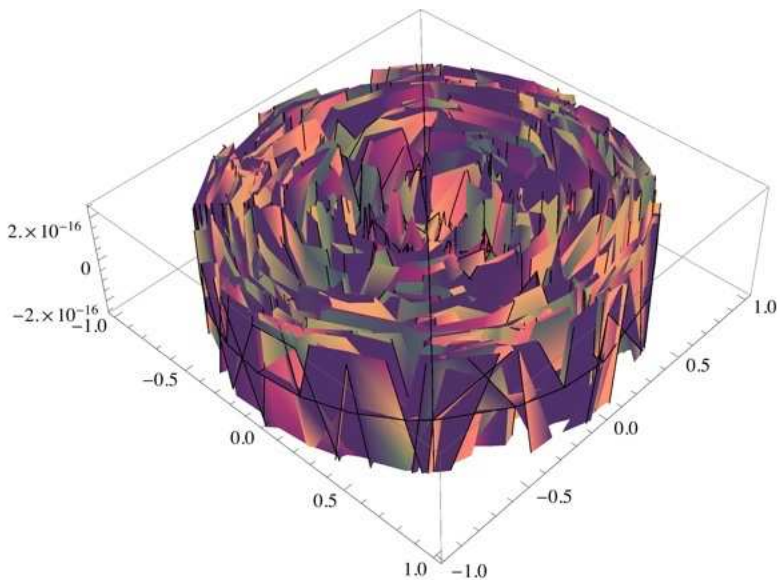
IX. Exploring Complex Infinity within the Cone Transformation

From Lemma 5, $\frac{2\,\pi\,\mathrm{Sin}[\beta]}{\sqrt{4\,\pi\,\theta-\theta^2}} = 1$ (28)

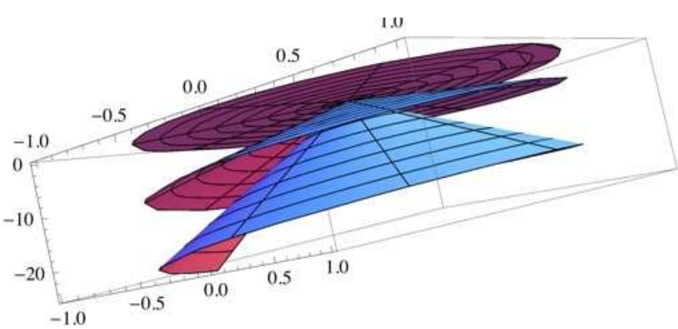
From Theorem 1, $0 = 2\,\pi\,r - 2\,\pi\,\frac{2\,\pi\,r - r\,\theta}{2\,\pi} - \theta\,r =$

$$2\,\pi\,r - 2\,\pi\sqrt{(r\hat{}2-\eta\hat{}2)} - \theta\,r = 2\,\pi\,r - 2\,\pi\sqrt{\left(r\hat{}2-\left(\frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi}\right)\hat{}2\right)} - \theta\,r \quad (29)$$

RevolutionPlot3D$\left[2\,\pi\,r - 2\,\pi\,\frac{2\,\pi\,r - r\,\theta}{2\,\pi} - \theta\,r,\ \{r, -1, 1\},\ \{\theta, -2\,\pi, 2\,\pi\}\right]$



RevolutionPlot3D$\left[2\,\pi\,r - 2\,\pi\sqrt{\left(r\hat{}2-\left(\frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi}\right)\hat{}2\right)} - \theta\,r,\ \{r, -1, 1\},\ \{\theta, -2\,\pi, 2\,\pi\}\right]$



$$\text{ComplexInfinity} = \frac{1}{0} = \frac{1}{\left(2\pi r - 2\pi \frac{2\pi r - r\theta}{2\pi} - \theta r\right)} =$$

$$\frac{\frac{2\pi \,\text{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}}}{\left(2\pi r - 2\pi \frac{2\pi r - r\theta}{2\pi} - \theta r\right)} = \frac{\frac{2\pi \,\text{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}}}{\left(2\pi r - r\theta - 2\pi\sqrt{r^2 - \frac{4\pi r^2\theta - r^2\theta^2}{4\pi^2}}\right)}$$

RevolutionPlot3D$\left[\frac{1}{\left(2\pi r - 2\pi\left(\frac{2\pi r - r\theta}{2\pi}\right) - \theta r\right)}, \{r, -1, 1\}, \{\theta, -2\pi, 2\pi\}\right]$

Power::infy : Infinite expression $\frac{1}{0.}$ encountered. ≫

∞::indet : Indeterminate expression 0 ComplexInfinity encountered. ≫

∞::indet : Indeterminate expression 0 ComplexInfinity encountered. ≫

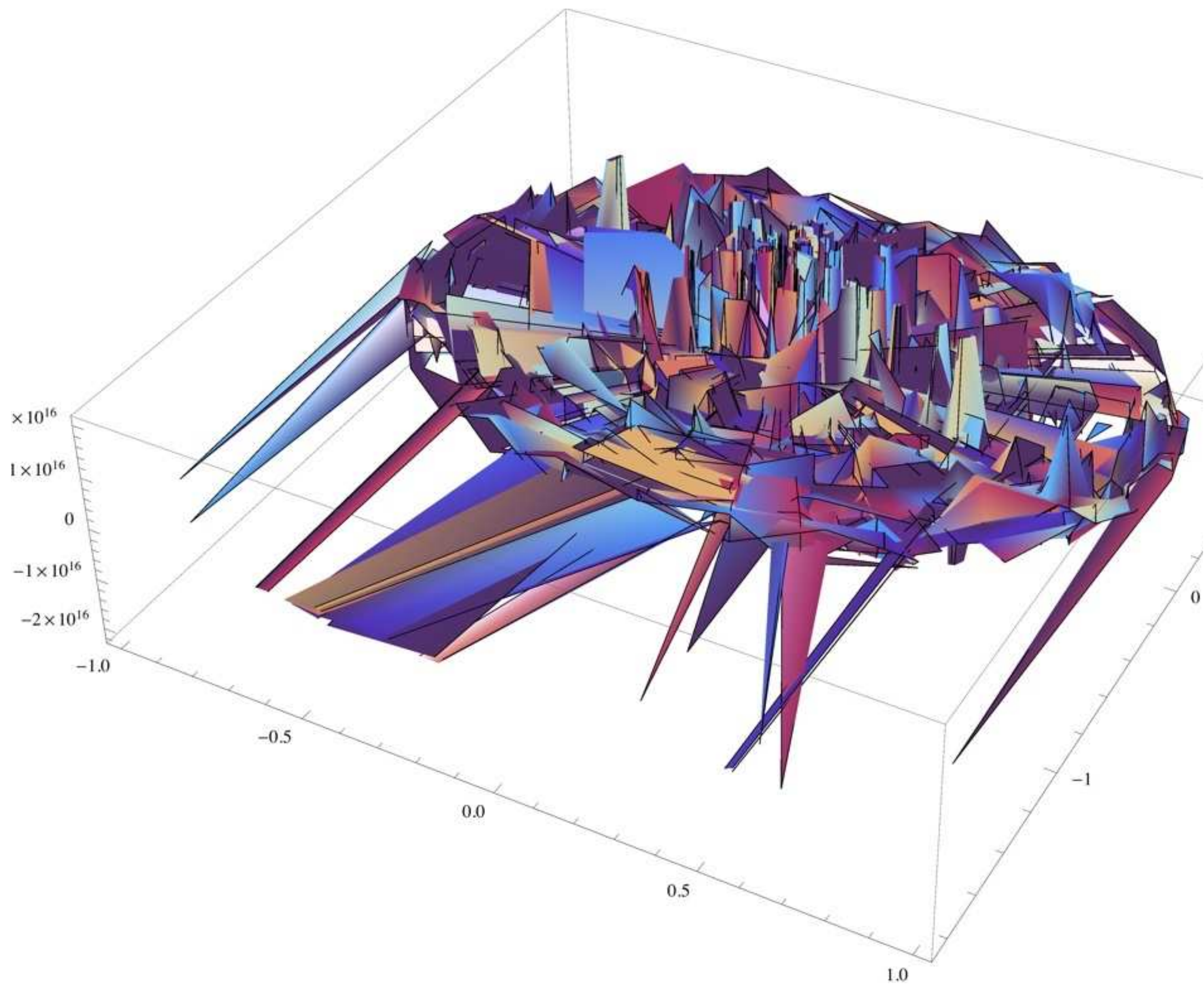
Power::infy : Infinite expression $\frac{1}{0.}$ encountered. ≫

∞::indet : Indeterminate expression 0 ComplexInfinity encountered. ≫

General::stop : Further output of ∞::indet will be suppressed during this calculation. ≫

Power::infy : Infinite expression $\frac{1}{0.}$ encountered. ≫

General::stop : Further output of Power::infy will be suppressed during this calculation. ≫


×10^16
1×10^16
0
−1×10^16
−2×10^16
−1.0
−0.5
0.0
0.5
1.0
0
−1

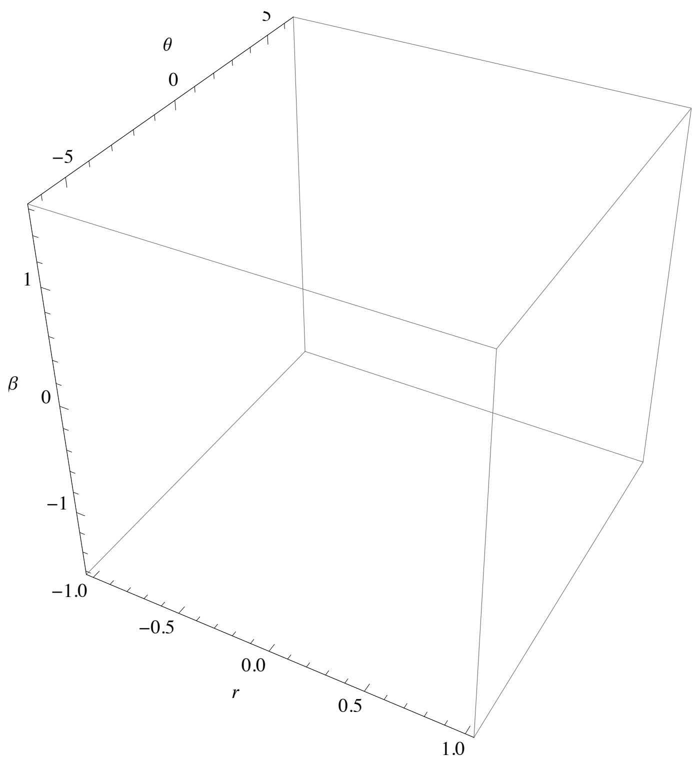
```
ContourPlot3D[(2 π Sin[β]/Sqrt[4 π θ - θ^2]) / (2 π r - 2 π ((2 π r - r θ)/(2 π)) - θ r), {r, -1, 1},
 {θ, -2 π, 2 π}, {β, -π / 2, π / 2}, AxesLabel → Automatic]
```

Power::infy : Infinite expression $\frac{1}{0.}$ encountered. ≫

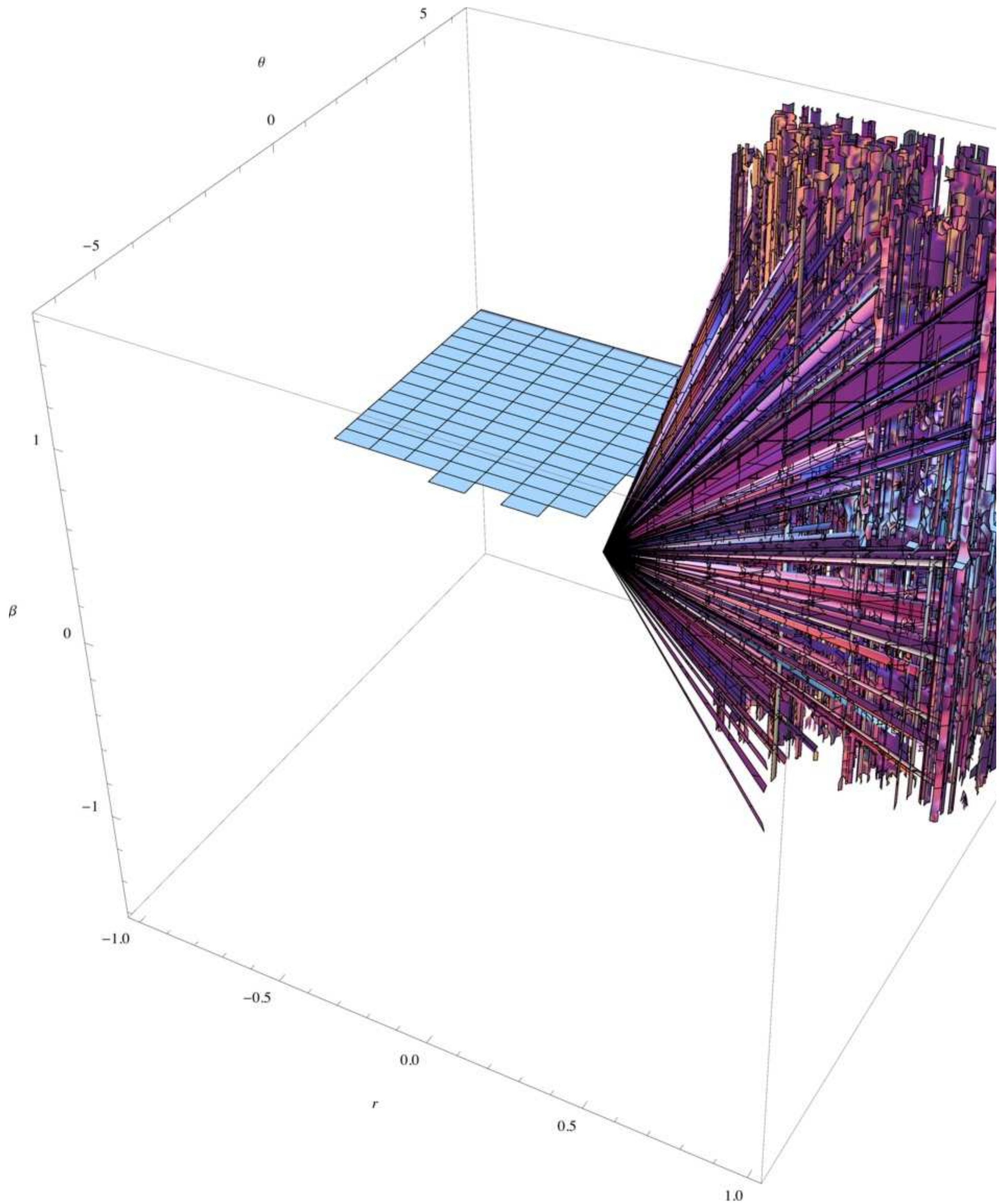
Power::infy : Infinite expression $\frac{1}{0.}$ encountered. ≫

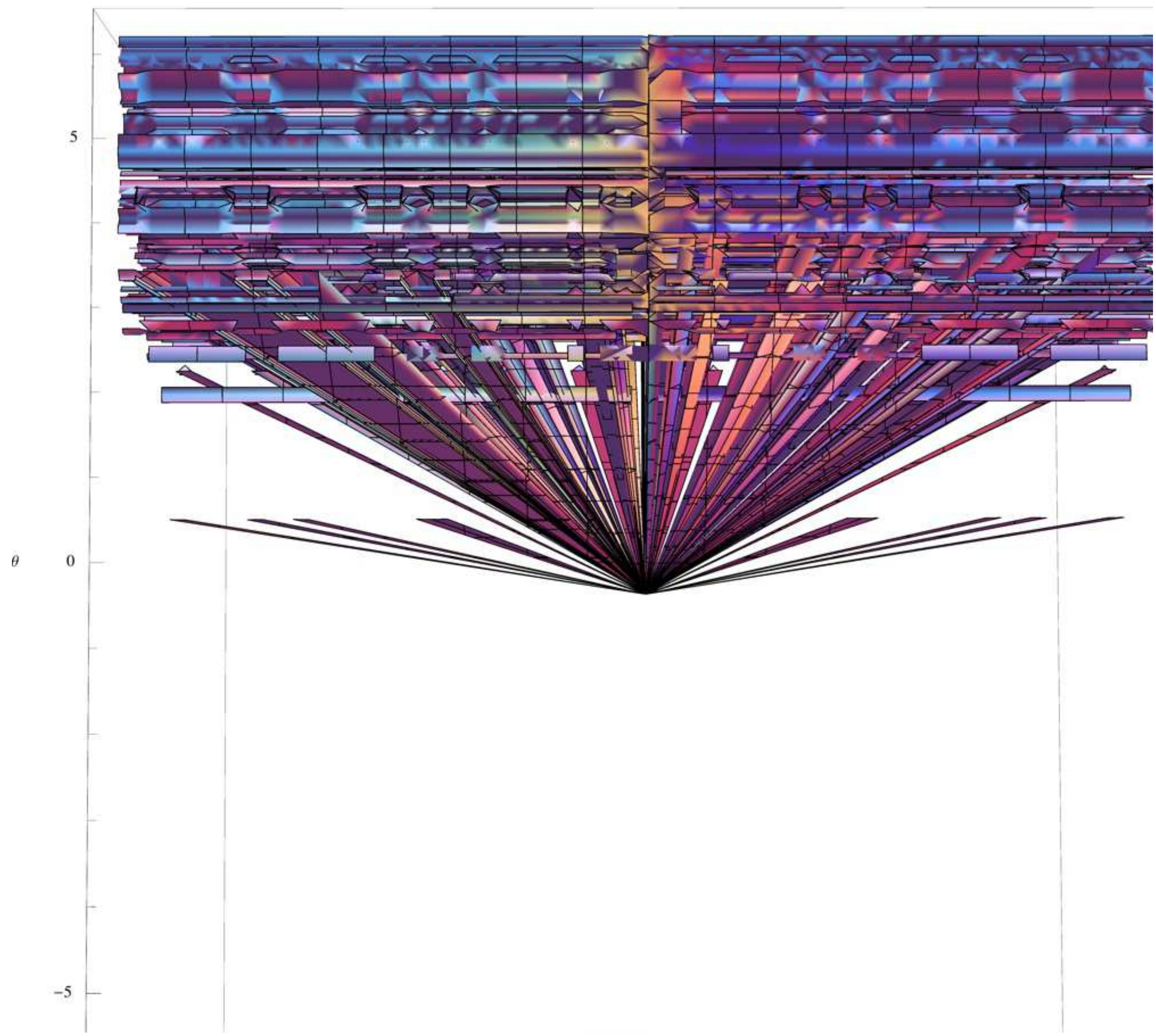
Power::infy : Infinite expression $\frac{1}{0.}$ encountered. ≫

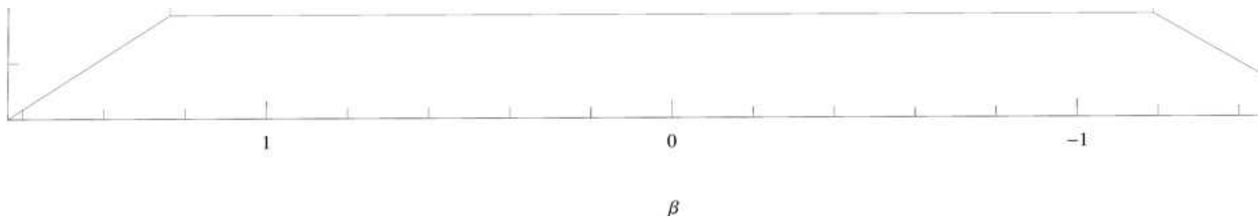
General::stop : Further output of Power::infy will be suppressed during this calculation. ≫



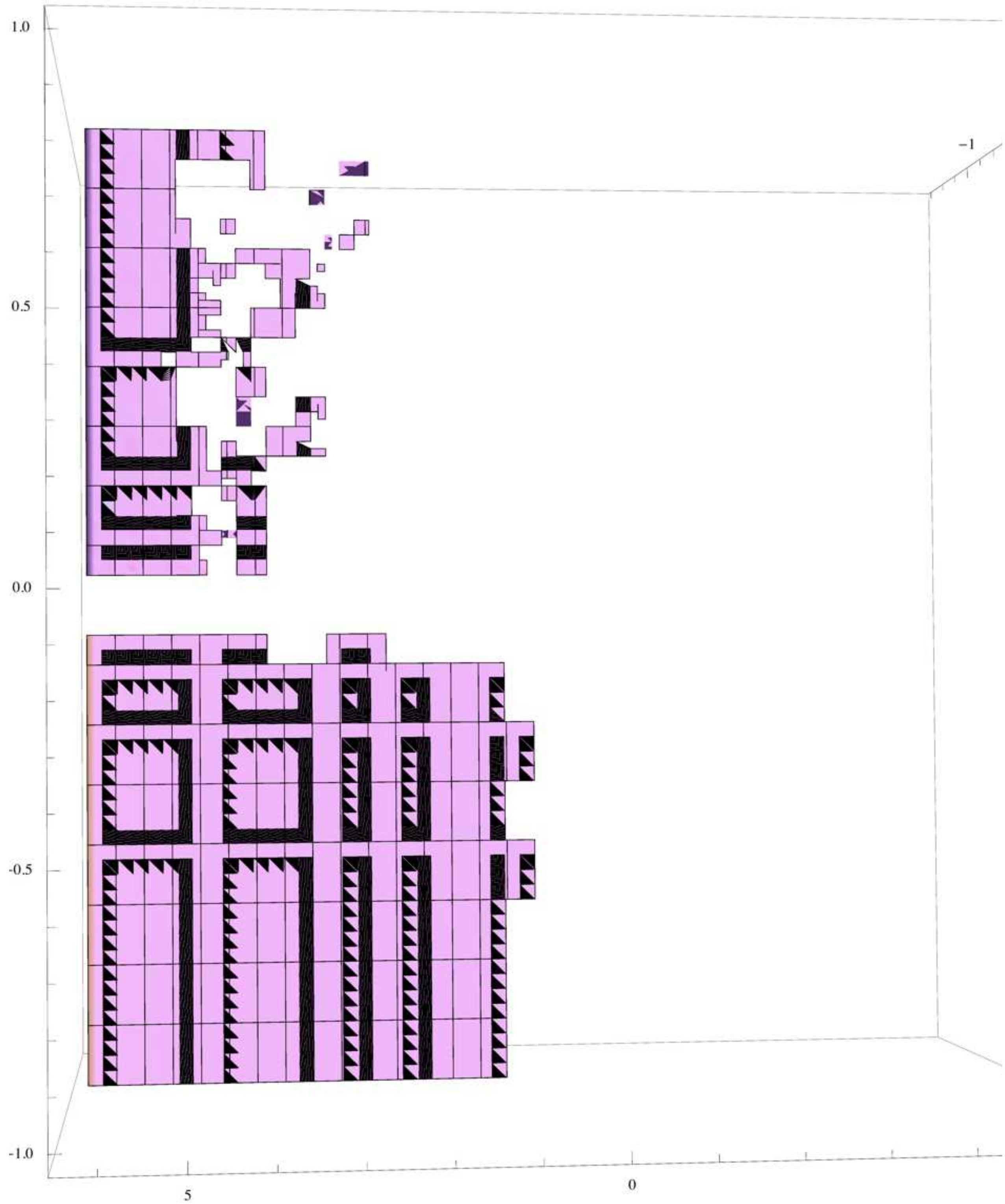

```
ContourPlot3D[(2 π Sin[β]/Sqrt[4 π θ - θ^2]) / (2 π r - r θ - 2 π Sqrt[r^2 - (4 π r^2 θ - r^2 θ^2)/(4 π^2)]),
 {r, -1, 1}, {θ, -2 π, 2 π}, {β, -π / 2, π / 2}, AxesLabel → Automatic]
```


5
θ
0
−5
1
β
0
−1
−1.0
−0.5
0.0
r
0.5
1.0


5
θ
0
−5



$$\text{ContourPlot3D}\left[\frac{\frac{2\,\pi\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\theta-\theta^2}}}{\left(2\,\pi\,r-r\,\theta-2\,\pi\,\sqrt{r^2-\frac{4\,\pi\,r^2\,\theta-r^2\,\theta^2}{4\,\pi^2}}\right)^2},\ \{r,\,-1,\,1\},\ \{\theta,\,-2\,\pi,\,2\,\pi\},\ \{\beta,\,-\pi\,/\,2,\,\pi\,/\,2\}\right]$$


1.0
0.5
0.0
-0.5
-1.0
5
0
-1

A notable effect in this function is the flickering on some of the sections and the intricately woven pattern of lines that runs through the surface.

X. Within the "Limits" of the Speed of Light

A short visualization by Parker Emmerson © 2009-2010

It is well known that normal matter does not travel at or faster than the speed of light. This paper will examine what this means for solutions to the velocity within the Lorentz coefficient ($\sqrt{1-\frac{v^2}{c^2}}$) found from the application of the Lorentz transformation to the equation for the height of the cone, which is found from the system of a difference in circumferences of two circles equaling an arc length applied to the Pythagorean theorem, in such a way that the Lorentz transformation should cancel out with itself as well as the velocity variable within it.

1. Introduction

In A Geometric Pattern of Perception (Emmerson, 2009), it was shown that:

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{(v)^2}{c^2}}}\sqrt{\frac{\theta}{\sqrt{1-\frac{(v)^2}{c^2}}}}\sqrt{4\pi r - r\theta}}{2\pi} == r\,\text{Sin}[\beta],\ v\right]$$

$$\left\{\left\{v\to-\left(1.\sqrt{\left(-1.12941\times10^{18}\,\theta+8.98755\times10^{16}\,\theta^2+3.54814\times10^{18}\,\text{Sin}[\beta]^2\right)}\right)\Big/\left(\sqrt{-12.5664\,\theta+\theta^2+39.4784\,\text{Sin}[\beta]^2}\right)\right\},\left\{v\to\left(\sqrt{\left(-1.12941\times10^{18}\,\theta+8.98755\times10^{16}\,\theta^2+3.54814\times10^{18}\,\text{Sin}[\beta]^2\right)}\right)\Big/\left(\sqrt{-12.5664\,\theta+\theta^2+39.4784\,\text{Sin}[\beta]^2}\right)\right\}\right\} \quad (31)$$

Velocity is less than speed of light for regular matter. The speed of light is greater than the purely algebraicized geometric solution to velocity (phenomenological velocity).

$$v < c > \left(\sqrt{\left(-1.12941\times10^{18}\,\theta+8.98755\times10^{16}\,\theta^2+3.54814\times10^{18}\,\text{Sin}[\beta]^2\right)}\right)\Big/\left(\sqrt{-12.5664\,\theta+\theta^2+39.4784\,\text{Sin}[\beta]^2}\right) \quad (32)$$

We can make the substitution,

$\theta = 2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)$ from A Geometric Pattern of Perception Theorems (Emmerson, 2009) and graph the inequality:

2. Graphics

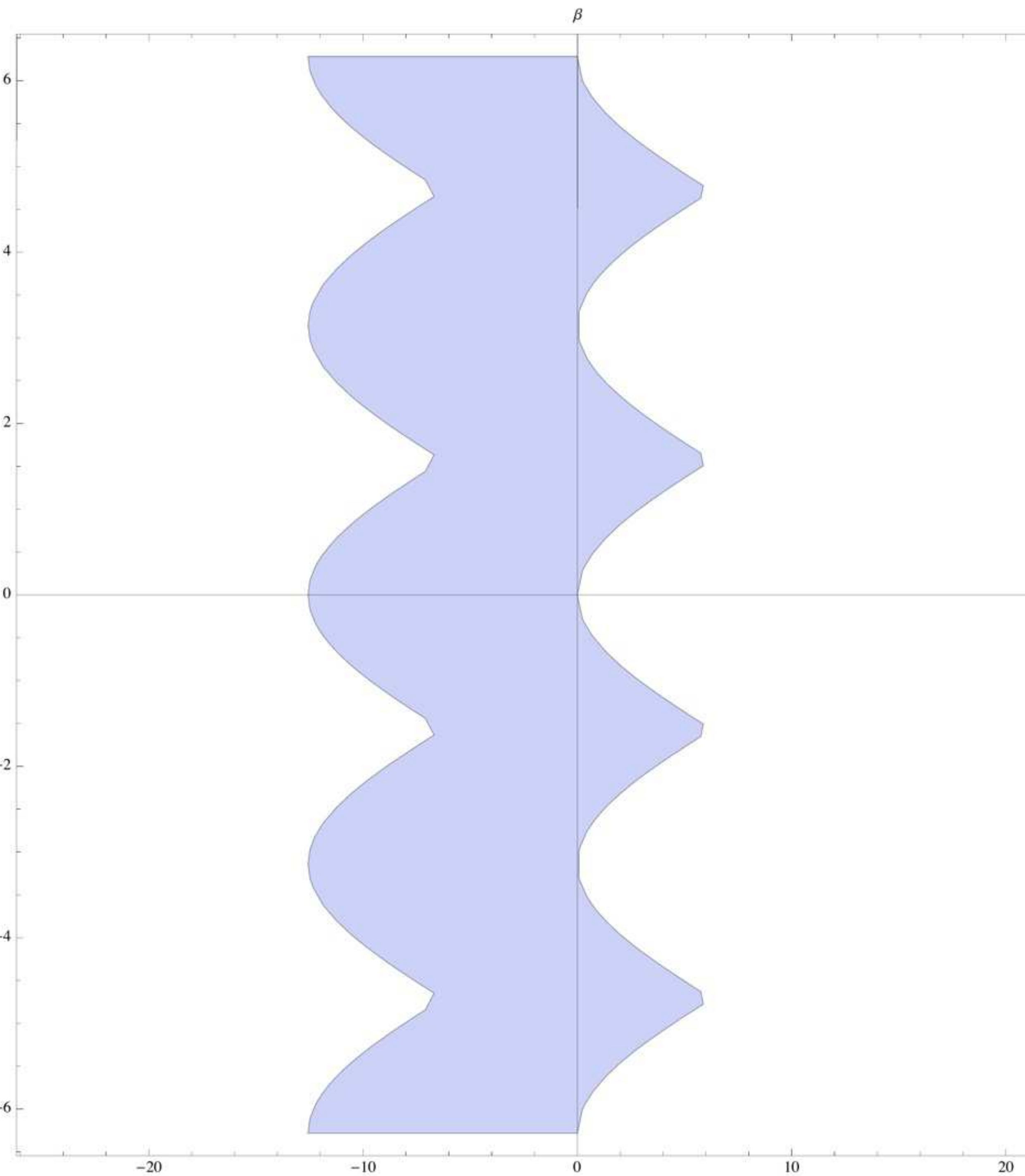
$$v = \left(\sqrt{\left(-1.1294090667581471`*^18\,\theta + 8.987551787368176`*^16\,\theta^2 + 3.5481432270250993`*^18\,\text{Sin}[\beta]^2\right)}\right) \Big/ \left(\sqrt{\left(-12.566370614359172`\,\theta + \left(2\left(\pi + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)\right)^2 + 39.47841760435743`\,\text{Sin}[\beta]^2\right)}\right) < c$$

$$c = 2.99792458\,\left(10^8\right)\ (\text{meters}\,/\,\text{second})$$

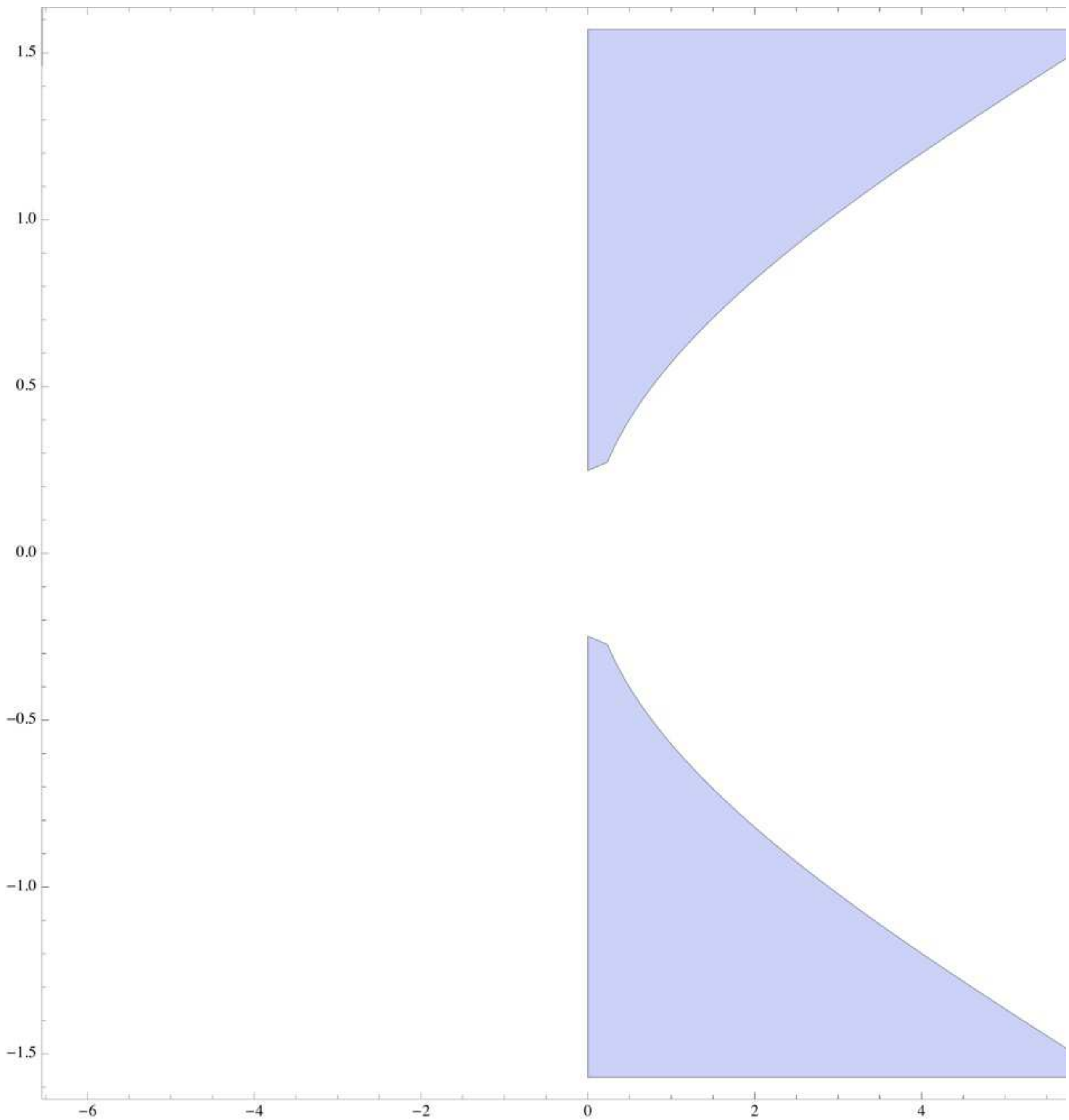
$$c := 2.99792458\,\left(10^8\right)$$

$$\text{RegionPlot}\Big[\left(\sqrt{\left(-1.1294090667581471`*^18\,\theta + 8.987551787368176`*^16\,\theta^2 + 3.5481432270250993`*^18\,\text{Sin}[\beta]^2\right)}\right) \Big/ \left(\sqrt{\left(-12.566370614359172`\,\theta + \left(2\left(\pi + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)\right)^2 + 39.47841760435743`\,\text{Sin}[\beta]^2\right)}\right) < c,\ \{\theta, -8\pi, 8\pi\}, \{\beta, -2\pi, 2\pi\}, \text{AxesLabel} \rightarrow \text{Automatic}\Big]$$

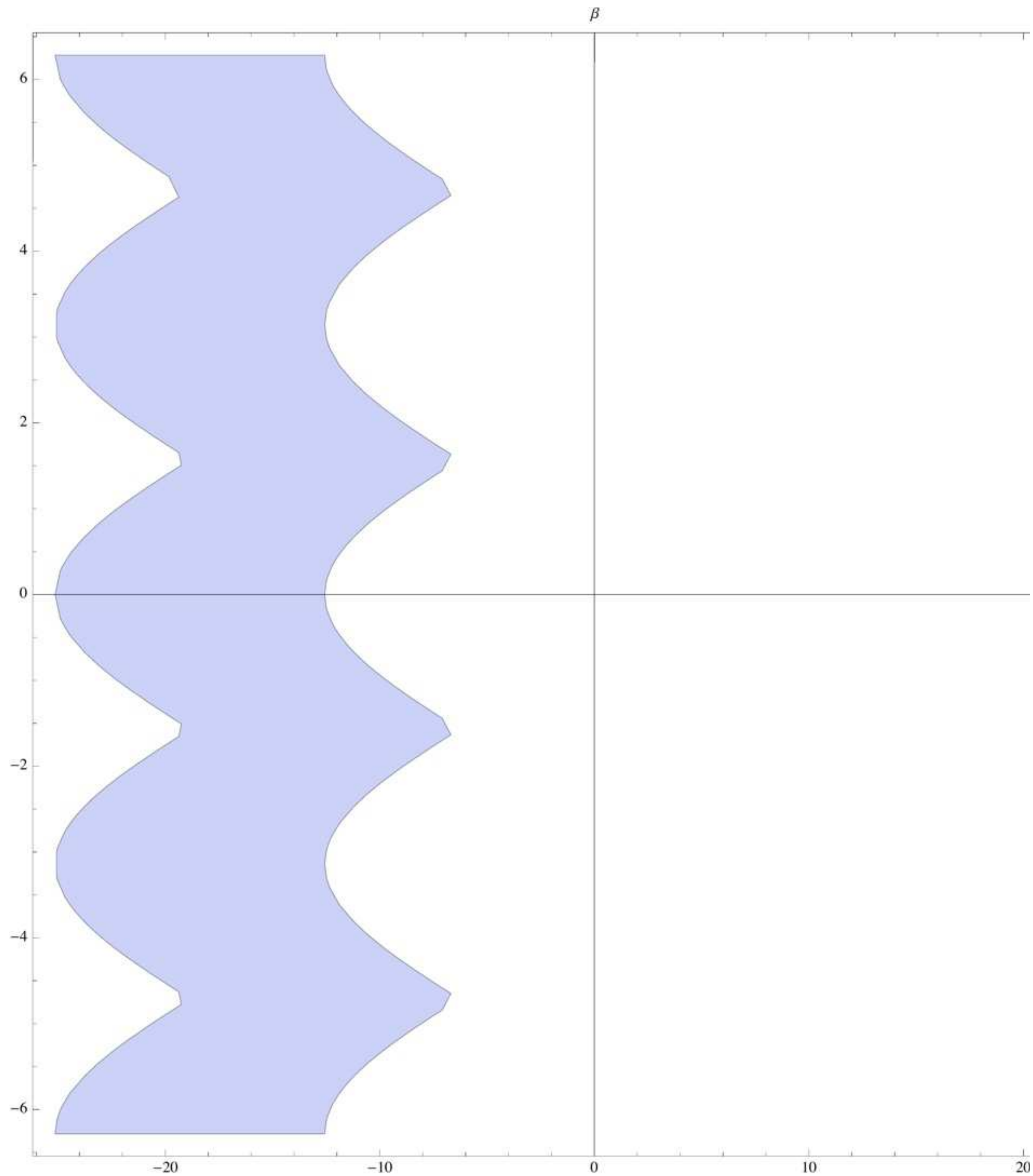
```
RegionPlot[(√(-1.1294090667581471`*^18 θ +
      8.987551787368176`*^16 θ^2 + 3.5481432270250993`*^18 Sin[β]^2)) /
  (√(-12.566370614359172` θ + (2 (π + √(π^2 - π^2 Sin[β]^2)))^2 + 39.47841760435743` Sin[β]^2)) < c,
 {θ, -8 π, 8 π}, {β, -2 π, 2 π}, AxesLabel → Automatic]
```



```
RegionPlot[(√(-1.1294090667581471`*^18 θ +
      8.987551787368176`*^16 θ^2 + 3.5481432270250993`*^18 Sin[β]^2)) /
   (√(-12.566370614359172` θ + (2 (π + √(π^2 - π^2 Sin[β]^2)))^2 + 39.47841760435743`
      Sin[ArcSin[√((4 π - θ) θ)/(2 π)]]^2)) < c, {θ, -2 π, 2 π}, {β, -π/2, π/2}]
```



```
RegionPlot[(√(-1.1294090667581471`*^18 2 (π + √(π^2 - π^2 Sin[β]^2)) +
      8.987551787368176`*^16 θ^2 + 3.5481432270250993`*^18 Sin[β]^2))/
    (√(-12.566370614359172` θ + (2 (π + √(π^2 - π^2 Sin[β]^2)))^2 + 39.47841760435743` Sin[β]^2)) < c,
  {θ, -8 π, 8 π}, {β, -2 π, 2 π}, AxesLabel → Automatic]
```


β
6
4
2
0
−2
−4
−6
−20
−10
0
10
20

3. Interpretation

What we see is the characteristic of oscillation. Regular matter will only oscillate within certain angular regions within a "conically univocal" system. It is also evident that switching the substitution pattern within the equation changes or limits the region of oscillation. Depending on the pattern of substitution selected, the oscillation of the matter – energy band shifts.

XI. The Wavelength with Constant Value

In this section, I will make a few considerations on the meanings of derivatives within the system and the proper application of them. Because the system described is complete (its parameters are all tied together and dependent on each other in some way or another), there are certain hypothetical inconsistencies.

$$2\pi r - 2\pi r_1 = \theta r \tag{33}$$

$$\frac{(2\pi r - 2\pi r_1)}{\theta} = r \tag{34}$$

velocity =

$$\text{wavelength of frequency} = v = \lambda f = r f = r / \tau = r\,(1 / (\theta / (2\pi))) = \frac{2\pi r}{\theta} =$$

$$\frac{2\pi s}{\theta^2} = \frac{2\pi \frac{(2\pi r - 2\pi r_1)}{\theta}}{\theta} = \frac{2\pi \frac{\left(2\pi r - 2\pi\sqrt{r^2-\eta^2}\right)}{\theta}}{\theta} = \frac{2\pi\left(2\pi r - 2\pi\sqrt{r^2-\eta^2}\right)}{\theta^2} =$$

$$\frac{2\pi\left(2\pi r - 2\pi\sqrt{r^2 - \left(\frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}\right)^2}\right)}{\theta^2} = \tag{35}$$

$$(1 / (1 / (2\pi)))\, D\left[\frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}, \theta\right] = \frac{4\pi r^2 - 2 r^2\theta}{2\sqrt{4\pi r^2\theta - r^2\theta^2}}$$

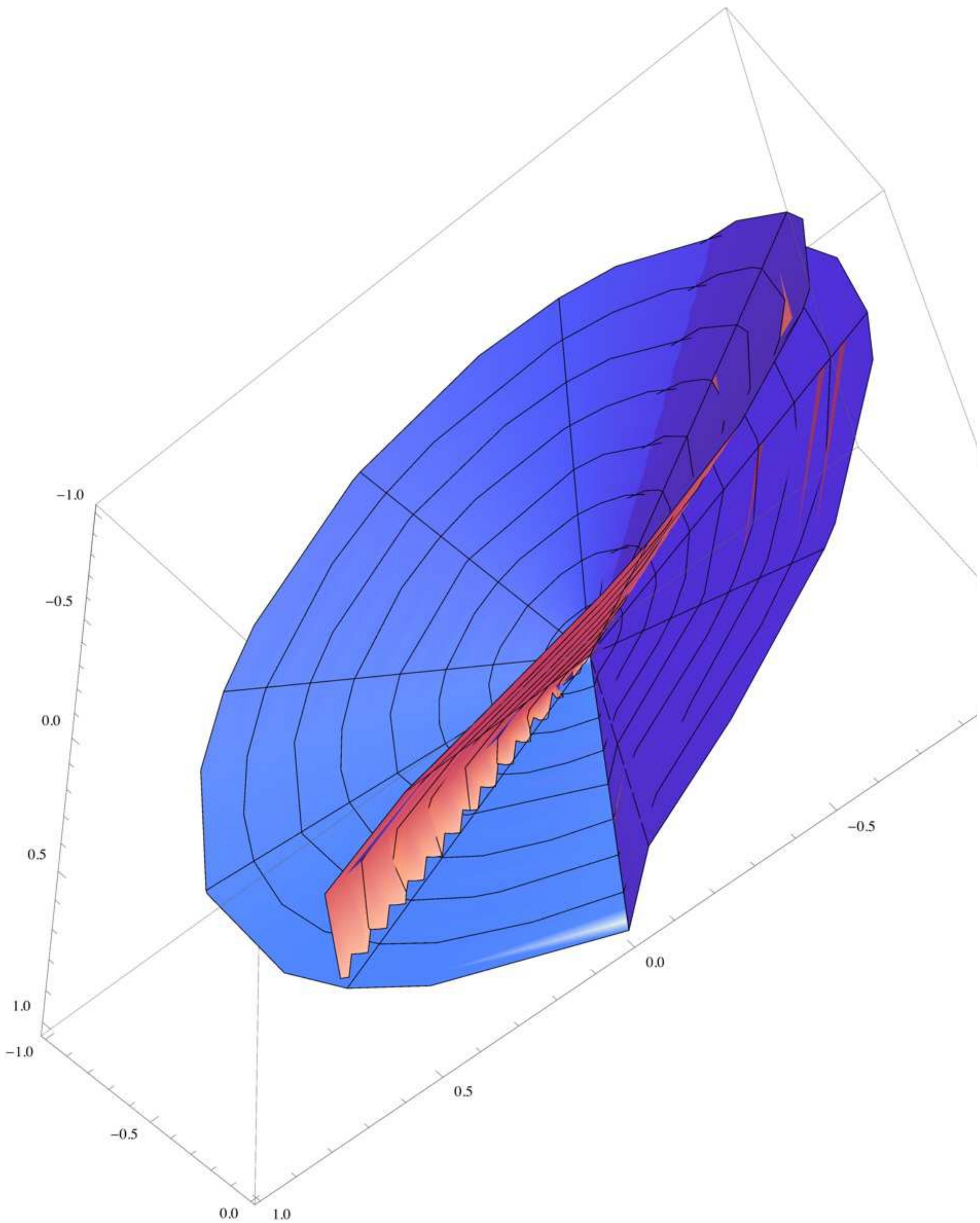
- η

$$\text{Solve}\left[\frac{2\pi\left(2\pi r - 2\pi\sqrt{r^2-(\eta)^2}\right)}{\theta^2} == \frac{4\pi r^2 - 2 r^2\theta}{2\sqrt{4\pi r^2\theta - r^2\theta^2}}, \eta\right]$$

$$\left\{\left\{\eta \to -\frac{1}{4\pi^2}\left(\sqrt{\left(64\pi^4 r^2 - \frac{256\pi^5 r^2}{4\pi-\theta} + 16\pi^3 r^2\theta + 4\pi^2 r^2\theta^2 + r^2\theta^4 + 16\pi^3 r\sqrt{r^2(4\pi-\theta)\theta} - \frac{64\pi^4 r\sqrt{r^2(4\pi-\theta)\theta}}{4\pi-\theta} + 8\pi^2 r\theta\sqrt{r^2(4\pi-\theta)\theta}\right)}\right)\right\},\right.$$

$$\left.\left\{\eta \to \frac{1}{4\pi^2}\left(\sqrt{\left(64\pi^4 r^2 - \frac{256\pi^5 r^2}{4\pi-\theta} + 16\pi^3 r^2\theta + 4\pi^2 r^2\theta^2 + r^2\theta^4 + 16\pi^3 r\sqrt{r^2(4\pi-\theta)\theta} - \frac{64\pi^4 r\sqrt{r^2(4\pi-\theta)\theta}}{4\pi-\theta} + 8\pi^2 r\theta\sqrt{r^2(4\pi-\theta)\theta}\right)}\right)\right\}\right\}$$

$$\text{RevolutionPlot3D}\left[-\frac{1}{4\pi^2}\left(\sqrt{\left(64\pi^4 r^2 - \frac{256\pi^5 r^2}{4\pi-\theta} + 16\pi^3 r^2\theta + 4\pi^2 r^2\theta^2 + r^2\theta^4 + 16\pi^3 r\sqrt{r^2(4\pi-\theta)\theta} - \frac{64\pi^4 r\sqrt{r^2(4\pi-\theta)\theta}}{4\pi-\theta} + 8\pi^2 r\theta\sqrt{r^2(4\pi-\theta)\theta}\right)}\right), \{r, -1, 1\}, \{\theta, -4\pi, 4\pi\}\right]$$


-1.0
-0.5
0.0
0.5
1.0
-1.0
-0.5
0.0
1.0
0.5
0.0
-0.5

- **r**

$$\text{Solve}\left[\frac{2\pi\left(2\pi r - 2\pi\sqrt{r^2-(\eta)^2}\right)}{\theta^2} == \frac{4\pi r^2 - 2r^2\theta}{2\sqrt{4\pi r^2\theta - r^2\theta^2}}, r\right]$$

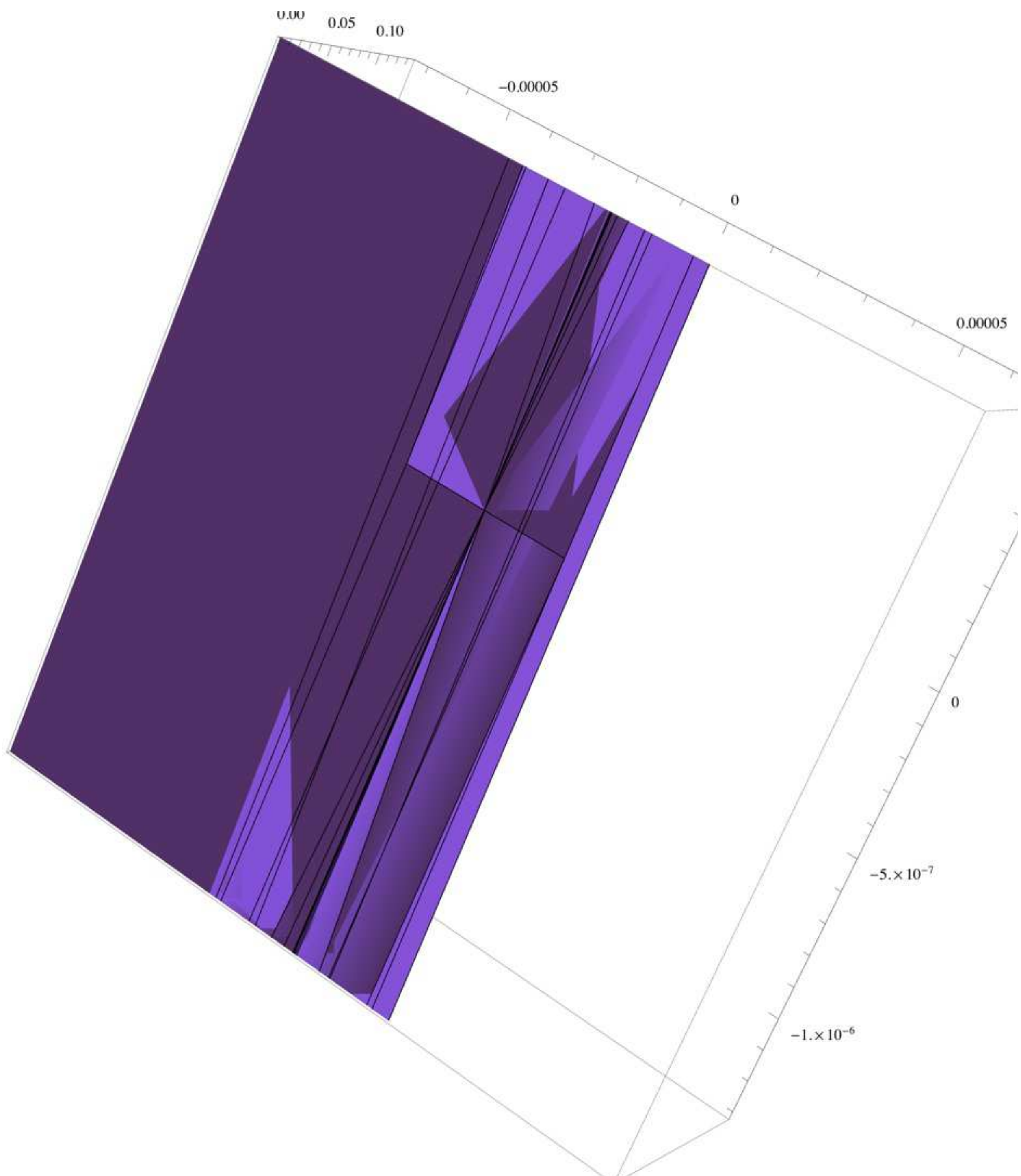
$$\{\{r \to -\sqrt{(-(256\pi^7\eta^2\theta^3)/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}) + (320\pi^6\eta^2\theta^4)/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}) - (128\pi^5\eta^2\theta^5)/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}) + (16\pi^4\eta^2\theta^6)/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}) - (128\pi^6\sqrt{(256\pi^5\eta^4\theta^3-448\pi^4\eta^4\theta^4+304\pi^3\eta^4\theta^5-100\pi^2\eta^4\theta^6+16\pi\eta^4\theta^7-\eta^4\theta^8)})/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}))}\},$$

$$\{r \to \sqrt{(-(256\pi^7\eta^2\theta^3)/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}) + (320\pi^6\eta^2\theta^4)/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}) - (128\pi^5\eta^2\theta^5)/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}) + (16\pi^4\eta^2\theta^6)/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}) - (128\pi^6\sqrt{(256\pi^5\eta^4\theta^3-448\pi^4\eta^4\theta^4+304\pi^3\eta^4\theta^5-100\pi^2\eta^4\theta^6+16\pi\eta^4\theta^7-\eta^4\theta^8)})/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}))}\},$$

$$\{r \to -\sqrt{(-(256\pi^7\eta^2\theta^3)/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}) + (320\pi^6\eta^2\theta^4)/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}) - (128\pi^5\eta^2\theta^5)/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}) + (16\pi^4\eta^2\theta^6)/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}) + (128\pi^6\sqrt{(256\pi^5\eta^4\theta^3-448\pi^4\eta^4\theta^4+304\pi^3\eta^4\theta^5-100\pi^2\eta^4\theta^6+16\pi\eta^4\theta^7-\eta^4\theta^8)})/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}))}\},$$

$$\{r \to \sqrt{(-(256\pi^7\eta^2\theta^3)/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}) + (320\pi^6\eta^2\theta^4)/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}) - (128\pi^5\eta^2\theta^5)/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}) + (16\pi^4\eta^2\theta^6)/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}) + (128\pi^6\sqrt{(256\pi^5\eta^4\theta^3-448\pi^4\eta^4\theta^4+304\pi^3\eta^4\theta^5-100\pi^2\eta^4\theta^6+16\pi\eta^4\theta^7-\eta^4\theta^8)})/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}))}\}\}$$

$$\text{RevolutionPlot3D}[\sqrt{(-(256\pi^7\eta^2\theta^3)/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}) + (320\pi^6\eta^2\theta^4)/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}) - (128\pi^5\eta^2\theta^5)/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}) + (16\pi^4\eta^2\theta^6)/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}) + (128\pi^6\sqrt{(256\pi^5\eta^4\theta^3-448\pi^4\eta^4\theta^4+304\pi^3\eta^4\theta^5-100\pi^2\eta^4\theta^6+16\pi\eta^4\theta^7-\eta^4\theta^8)})/(-1024\pi^7\theta^3+1280\pi^6\theta^4-512\pi^5\theta^5+80\pi^4\theta^6-32\pi^3\theta^7+24\pi^2\theta^8-8\pi\theta^9+\theta^{10}))}, \{\eta, -1, 1\}, \{\theta, -2\pi, 2\pi\}]$$


0.00
0.05
0.10
−0.00005
0
0.00005
0
−5.×10^−7
−1.×10^−6

- θ

$$\text{Solve}\left[\frac{2\pi\left(2\pi r - 2\pi\sqrt{r^2-(\eta)^2}\right)}{\theta^2} == \frac{4\pi r^2 - 2 r^2 \theta}{2\sqrt{4\pi r^2\theta - r^2\theta^2}}, \theta\right]$$

$$\Big\{\Big\{\theta \to \text{Root}\Big[-128\pi^5 r^2 + 64\pi^5\eta^2 + 128\pi^5 r\sqrt{r^2-\eta^2} + \left(32\pi^4 r^2 - 16\pi^4\eta^2 - 32\pi^4 r\sqrt{r^2-\eta^2}\right)\#1 + 4\pi^2 r^2\#1^3 - 4\pi r^2\#1^4 + r^2\#1^5\ \&, 1\Big]\Big\},$$

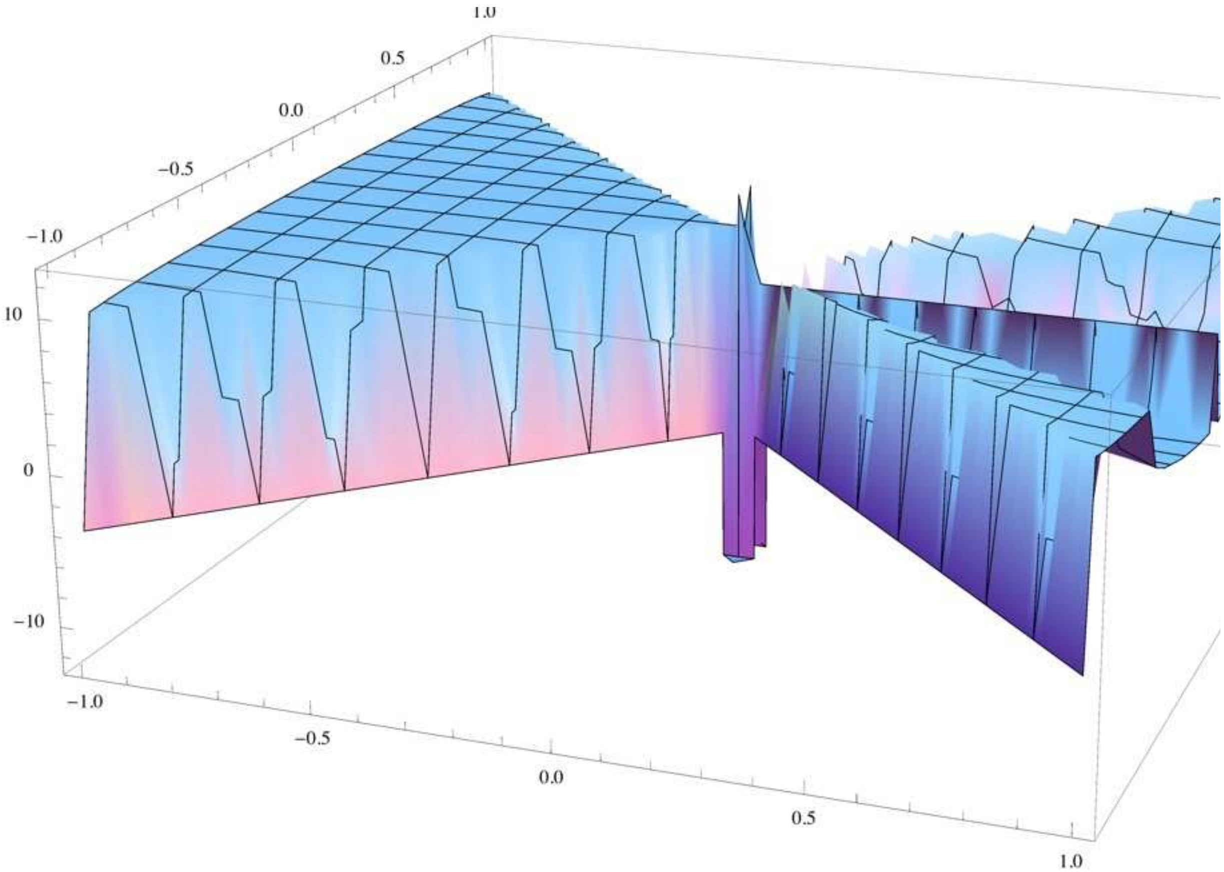
$$\Big\{\theta \to \text{Root}\Big[-128\pi^5 r^2 + 64\pi^5\eta^2 + 128\pi^5 r\sqrt{r^2-\eta^2} + \left(32\pi^4 r^2 - 16\pi^4\eta^2 - 32\pi^4 r\sqrt{r^2-\eta^2}\right)\#1 + 4\pi^2 r^2\#1^3 - 4\pi r^2\#1^4 + r^2\#1^5\ \&, 2\Big]\Big\},$$

$$\Big\{\theta \to \text{Root}\Big[-128\pi^5 r^2 + 64\pi^5\eta^2 + 128\pi^5 r\sqrt{r^2-\eta^2} + \left(32\pi^4 r^2 - 16\pi^4\eta^2 - 32\pi^4 r\sqrt{r^2-\eta^2}\right)\#1 + 4\pi^2 r^2\#1^3 - 4\pi r^2\#1^4 + r^2\#1^5\ \&, 3\Big]\Big\},$$

$$\Big\{\theta \to \text{Root}\Big[-128\pi^5 r^2 + 64\pi^5\eta^2 + 128\pi^5 r\sqrt{r^2-\eta^2} + \left(32\pi^4 r^2 - 16\pi^4\eta^2 - 32\pi^4 r\sqrt{r^2-\eta^2}\right)\#1 + 4\pi^2 r^2\#1^3 - 4\pi r^2\#1^4 + r^2\#1^5\ \&, 4\Big]\Big\},$$

$$\Big\{\theta \to \text{Root}\Big[-128\pi^5 r^2 + 64\pi^5\eta^2 + 128\pi^5 r\sqrt{r^2-\eta^2} + \left(32\pi^4 r^2 - 16\pi^4\eta^2 - 32\pi^4 r\sqrt{r^2-\eta^2}\right)\#1 + 4\pi^2 r^2\#1^3 - 4\pi r^2\#1^4 + r^2\#1^5\ \&, 5\Big]\Big\}\Big\}$$

```
Plot3D[Root[-128 π^5 r^2 + 64 π^5 η^2 + 128 π^5 r Sqrt[r^2 - η^2] + (32 π^4 r^2 - 16 π^4 η^2 - 32 π^4 r Sqrt[r^2 - η^2]) #1 +
   4 π^2 r^2 #1^3 - 4 π r^2 #1^4 + r^2 #1^5 &, 1], {r, -1, 1}, {η, -1, 1}]
```



$$\text{Solve}\left[\frac{2\pi\left(2\pi r-2\pi\sqrt{r^2-(r\,\text{Sin}[\beta])^2}\right)}{\theta^2}==\frac{4\pi r^2-2r^2\theta}{2\sqrt{4\pi r^2\theta-r^2\theta^2}},\theta\right]$$

$$\Big\{\Big\{\theta\to\text{Root}\Big[-128\pi^5 r+64\pi^5 r\,\text{Sin}[\beta]^2+128\pi^5\sqrt{-r^2\left(-1+\text{Sin}[\beta]^2\right)}+\left(32\pi^4 r-16\pi^4 r\,\text{Sin}[\beta]^2-32\pi^4\sqrt{-r^2\left(-1+\text{Sin}[\beta]^2\right)}\right)\#1+4\pi^2 r\#1^3-4\pi r\#1^4+r\#1^5\,\&,1\Big]\Big\},$$

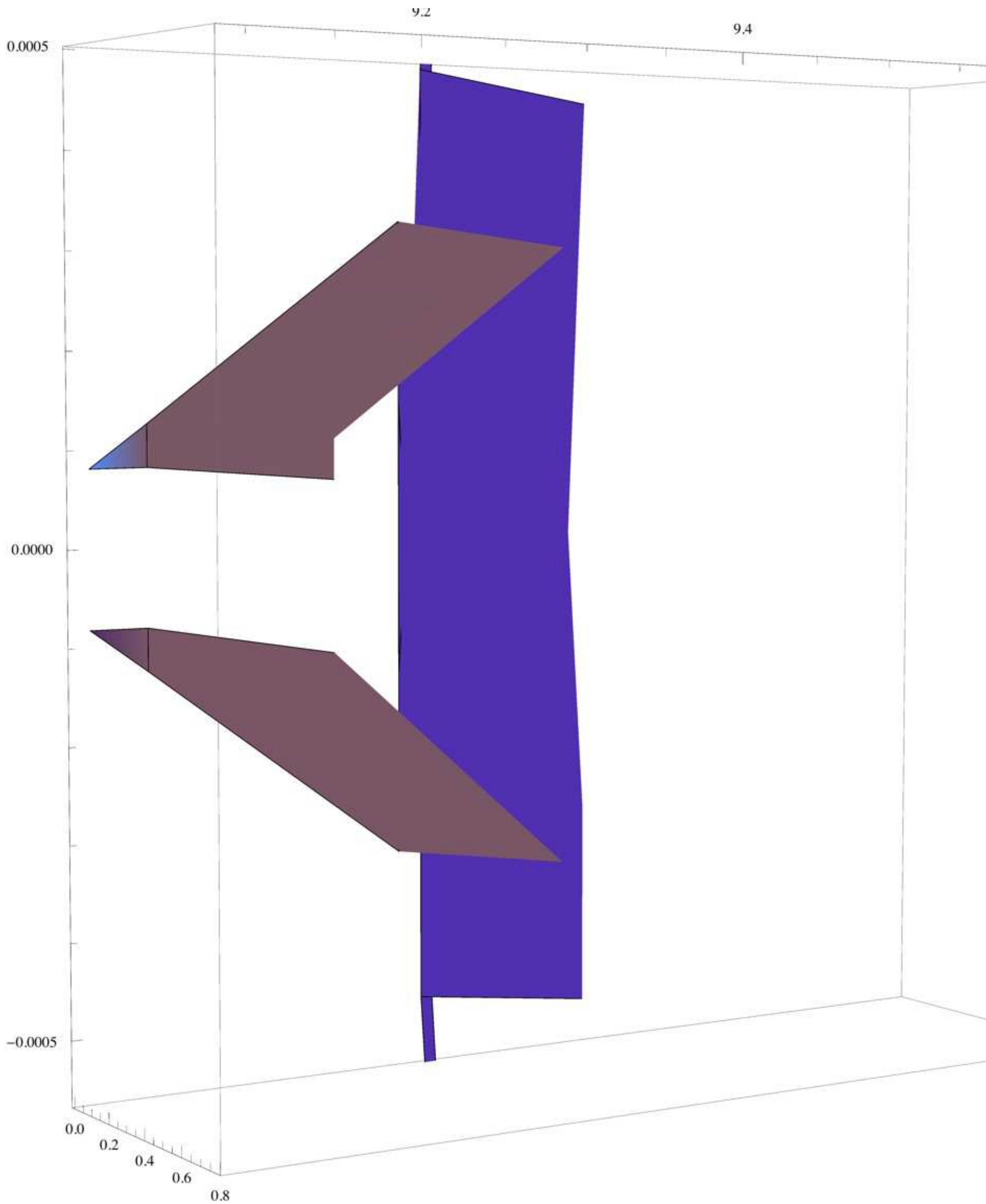
$$\Big\{\theta\to\text{Root}\Big[-128\pi^5 r+64\pi^5 r\,\text{Sin}[\beta]^2+128\pi^5\sqrt{-r^2\left(-1+\text{Sin}[\beta]^2\right)}+\left(32\pi^4 r-16\pi^4 r\,\text{Sin}[\beta]^2-32\pi^4\sqrt{-r^2\left(-1+\text{Sin}[\beta]^2\right)}\right)\#1+4\pi^2 r\#1^3-4\pi r\#1^4+r\#1^5\,\&,2\Big]\Big\},$$

$$\Big\{\theta\to\text{Root}\Big[-128\pi^5 r+64\pi^5 r\,\text{Sin}[\beta]^2+128\pi^5\sqrt{-r^2\left(-1+\text{Sin}[\beta]^2\right)}+\left(32\pi^4 r-16\pi^4 r\,\text{Sin}[\beta]^2-32\pi^4\sqrt{-r^2\left(-1+\text{Sin}[\beta]^2\right)}\right)\#1+4\pi^2 r\#1^3-4\pi r\#1^4+r\#1^5\,\&,3\Big]\Big\},$$

$$\Big\{\theta\to\text{Root}\Big[-128\pi^5 r+64\pi^5 r\,\text{Sin}[\beta]^2+128\pi^5\sqrt{-r^2\left(-1+\text{Sin}[\beta]^2\right)}+\left(32\pi^4 r-16\pi^4 r\,\text{Sin}[\beta]^2-32\pi^4\sqrt{-r^2\left(-1+\text{Sin}[\beta]^2\right)}\right)\#1+4\pi^2 r\#1^3-4\pi r\#1^4+r\#1^5\,\&,4\Big]\Big\},$$

$$\Big\{\theta\to\text{Root}\Big[-128\pi^5 r+64\pi^5 r\,\text{Sin}[\beta]^2+128\pi^5\sqrt{-r^2\left(-1+\text{Sin}[\beta]^2\right)}+\left(32\pi^4 r-16\pi^4 r\,\text{Sin}[\beta]^2-32\pi^4\sqrt{-r^2\left(-1+\text{Sin}[\beta]^2\right)}\right)\#1+4\pi^2 r\#1^3-4\pi r\#1^4+r\#1^5\,\&,5\Big]\Big\}\Big\}$$

```
RevolutionPlot3D[Root[-128 π^5 r + 64 π^5 r Sin[β]^2 +
    128 π^5 Sqrt[-r^2 (-1 + Sin[β]^2)] + (32 π^4 r - 16 π^4 r Sin[β]^2 - 32 π^4 Sqrt[-r^2 (-1 + Sin[β]^2)]) #1 +
    4 π^2 r #1^3 - 4 π r #1^4 + r #1^5 &, 5], {r, -10, 10}, {β, -20 π, 20 π}]
```



$$\textbf{Solve}\left[\frac{2\pi\left(2\pi r-2\pi\sqrt{r^2-\left(\frac{\sqrt{4\pi r^2\theta-r^2\theta^2}}{2\pi}\right)^2}\right)}{\theta^2}==\frac{4\pi r^2-2r^2\theta}{2\sqrt{4\pi r^2\theta-r^2\theta^2}},\theta\right]$$

$$\left\{\left\{\theta\to\frac{4\pi}{3}+\frac{4\pi}{3\left(-17+3\sqrt{33}\right)^{1/3}}-\frac{2}{3}\left(-17+3\sqrt{33}\right)^{1/3}\pi\right\},\right.$$

$$\left\{\theta\to\frac{4\pi}{3}-\frac{2\left(1+i\sqrt{3}\right)\pi}{3\left(-17+3\sqrt{33}\right)^{1/3}}+\frac{1}{3}\left(1-i\sqrt{3}\right)\left(-17+3\sqrt{33}\right)^{1/3}\pi\right\},$$

$$\left\{\theta\to\frac{4\pi}{3}-\frac{2\left(1-i\sqrt{3}\right)\pi}{3\left(-17+3\sqrt{33}\right)^{1/3}}+\frac{1}{3}\left(1+i\sqrt{3}\right)\left(-17+3\sqrt{33}\right)^{1/3}\pi\right\},$$

$\{\theta\to\pi\,\text{Root}[-256+192\,\#1-48\,\#1^2+8\,\#1^3-4\,\#1^4+\#1^5\,\&,1]\}$,
$\{\theta\to\pi\,\text{Root}[-256+192\,\#1-48\,\#1^2+8\,\#1^3-4\,\#1^4+\#1^5\,\&,2]\}$,
$\{\theta\to\pi\,\text{Root}[-256+192\,\#1-48\,\#1^2+8\,\#1^3-4\,\#1^4+\#1^5\,\&,3]\}$,
$\{\theta\to\pi\,\text{Root}[-256+192\,\#1-48\,\#1^2+8\,\#1^3-4\,\#1^4+\#1^5\,\&,4]\}$,
$\{\theta\to\pi\,\text{Root}[-256+192\,\#1-48\,\#1^2+8\,\#1^3-4\,\#1^4+\#1^5\,\&,5]\}\}$

$$\frac{2\pi r}{\theta}=\left(s_\eta/\theta\right)=r_\theta;\ r_\theta=\text{light year}\neq\text{velocity}\tag{36}$$

$$\frac{2\pi}{\theta}=1,\ \theta=2\pi\tag{37}$$

$$\frac{2\pi r}{\theta}=\frac{2\pi r_1}{\theta}+r=\frac{4\pi r^2-2r^2\theta}{2\sqrt{4\pi r^2\theta-r^2\theta^2}}\tag{38}$$

$$\textbf{Solve}\left[\frac{2\pi r_1}{\theta}+r==\frac{4\pi r^2-2r^2\theta}{2\sqrt{4\pi r^2\theta-r^2\theta^2}},r_1\right]$$

$$\left\{\left\{r_1\to-\frac{\theta\left(-2\pi r^2+r^2\theta+r\sqrt{r^2\,(4\pi-\theta)\,\theta}\right)}{2\pi\sqrt{r^2\,(4\pi-\theta)\,\theta}}\right\}\right\}\tag{39}$$

$$\left\{\left\{r_1\to\frac{2\pi r-r\theta}{2\pi}\right\}\right\}$$

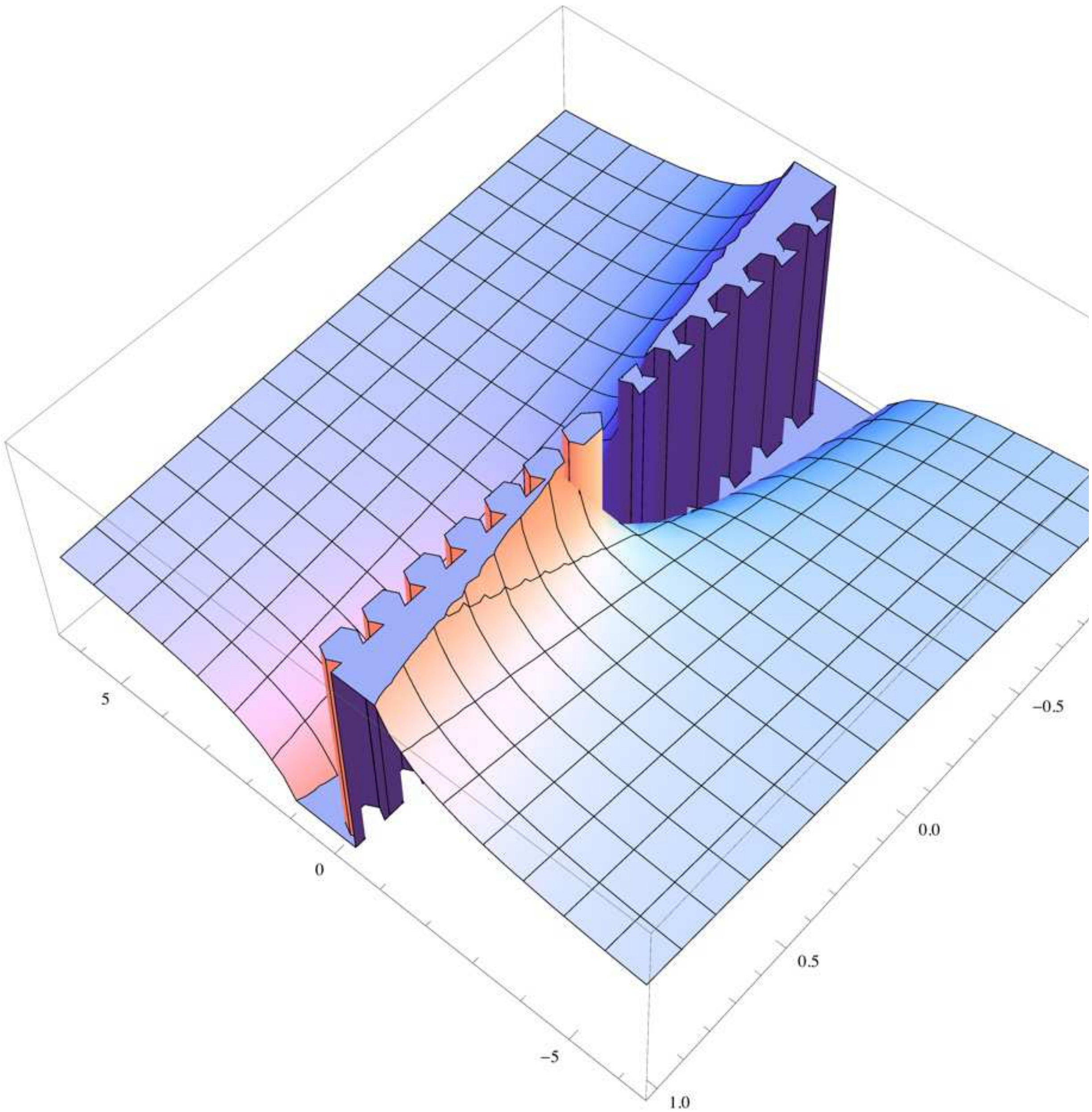
$$\text{Solve}\left[\frac{2\pi r - r\theta}{2\pi} == \frac{\theta\left(-2\pi r^2 + r^2\theta + r\sqrt{r^2(4\pi-\theta)\theta}\right)}{2\pi\sqrt{r^2(4\pi-\theta)\theta}}, \theta\right]$$

$$\left\{\left\{\theta \to \frac{28\pi}{15} + \frac{92\pi}{15\left(269 + 15\, i\sqrt{111}\right)^{1/3}} + \frac{2}{15}\left(269 + 15\, i\sqrt{111}\right)^{1/3}\pi\right\},\right.$$

$$\left\{\theta \to \frac{28\pi}{15} - \frac{46\left(1 + i\sqrt{3}\right)\pi}{15\left(269 + 15\, i\sqrt{111}\right)^{1/3}} - \frac{1}{15}\left(1 - i\sqrt{3}\right)\left(269 + 15\, i\sqrt{111}\right)^{1/3}\pi\right\},$$

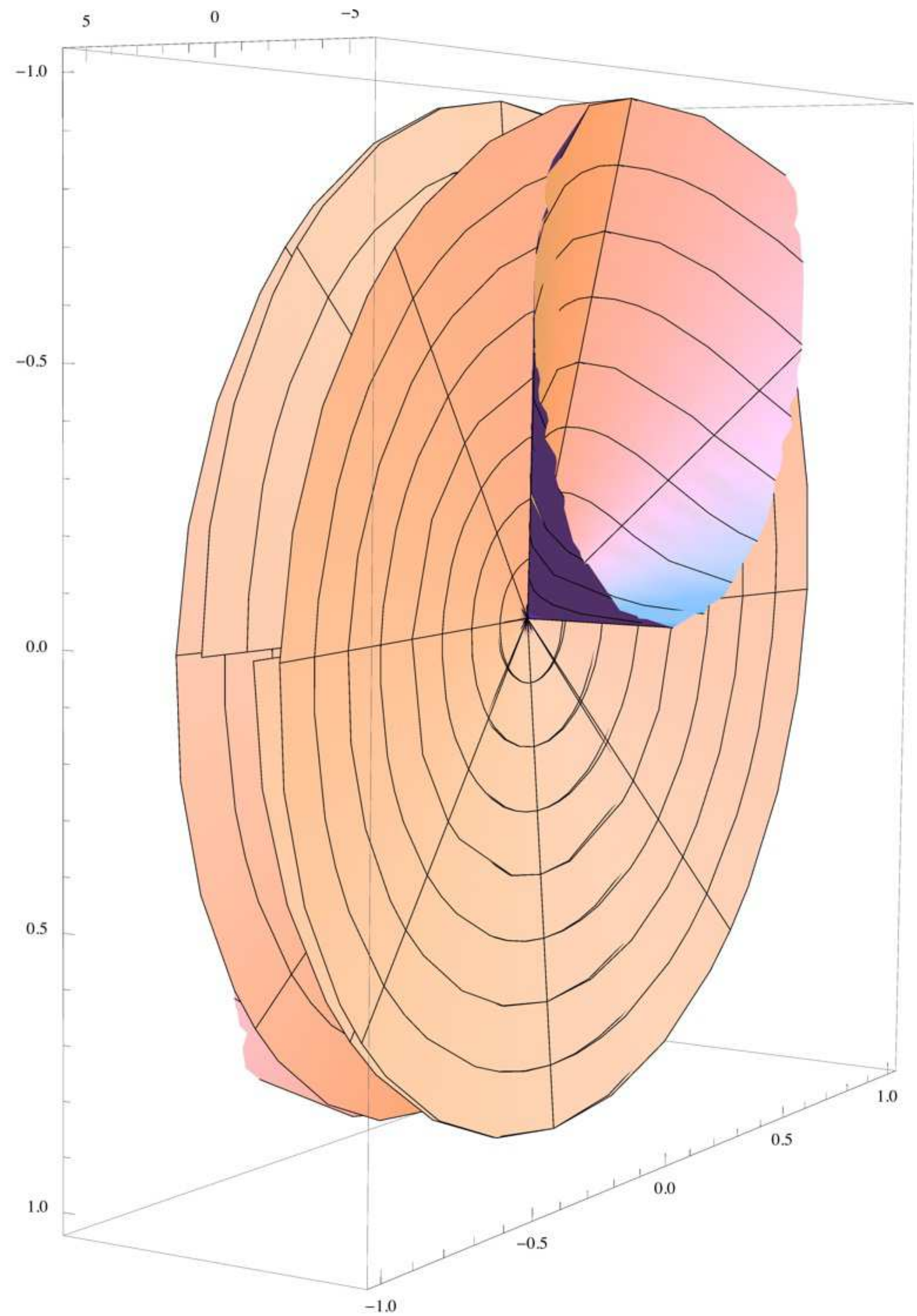
$$\left.\left\{\theta \to \frac{28\pi}{15} - \frac{46\left(1 - i\sqrt{3}\right)\pi}{15\left(269 + 15\, i\sqrt{111}\right)^{1/3}} - \frac{1}{15}\left(1 + i\sqrt{3}\right)\left(269 + 15\, i\sqrt{111}\right)^{1/3}\pi\right\}\right\}$$

```
Plot3D[2 π r / θ, {r, -1, 1}, {θ, -2 π, 2 π}]
```



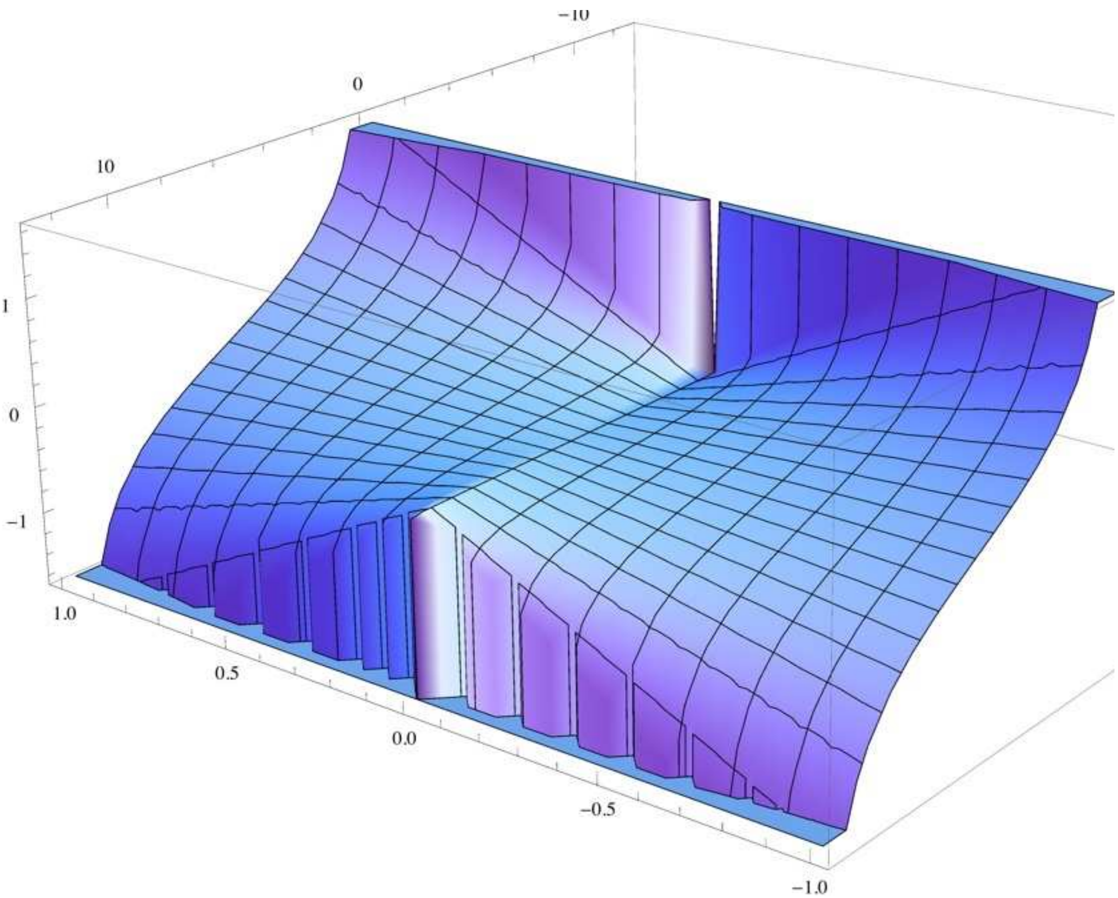
Now we can visually see how velocities would be superpositional. They can fit right on top of each other in this lego - like system while continuing along the same contour.

```
RevolutionPlot3D[2 π r/θ, {r, -1, 1}, {θ, -2 π, 2 π}]
```

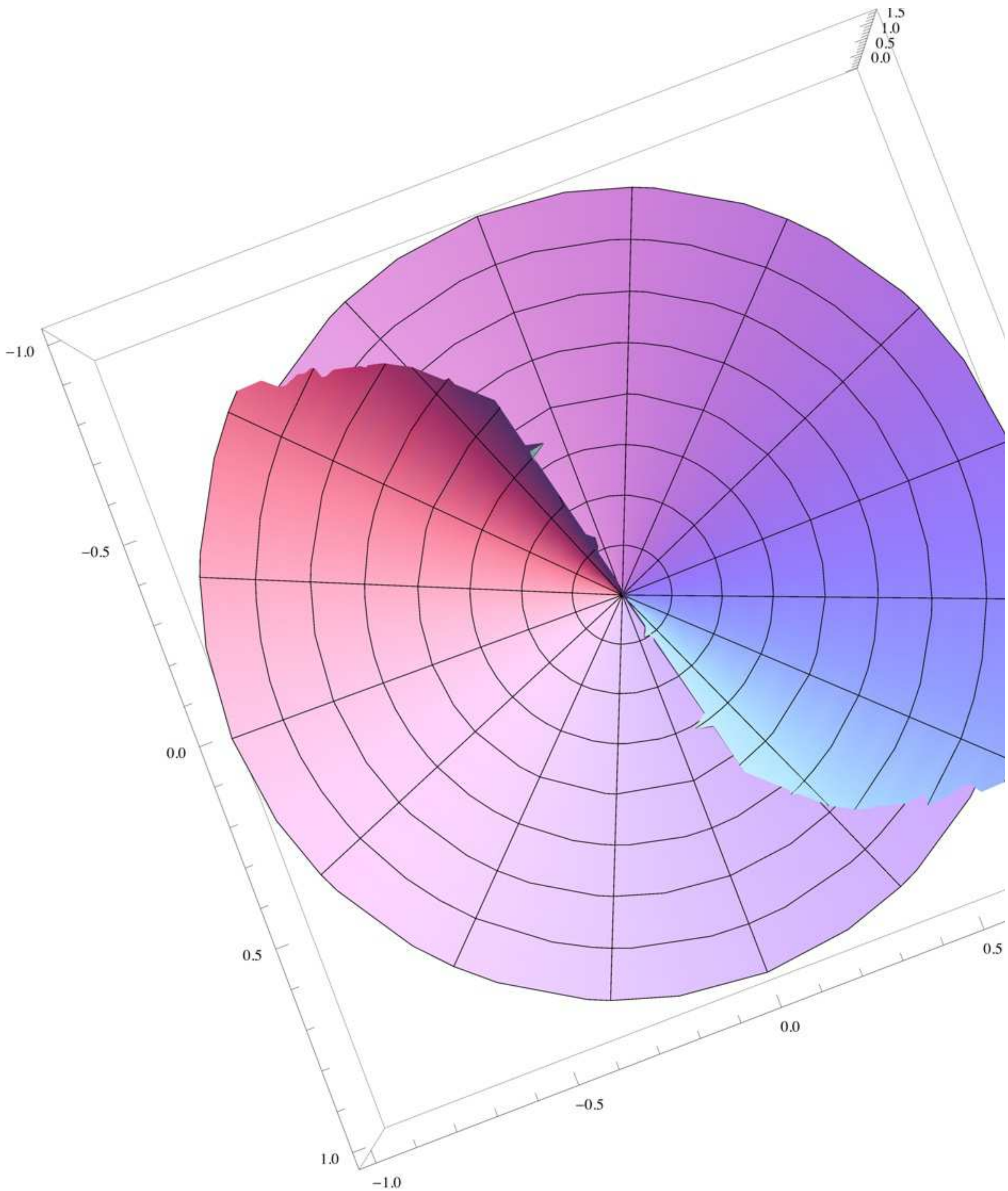


$$\text{Velocity when r is constant} = \frac{4\,\pi\,r^2 - 2\,r^2\,\theta}{2\,\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}} \tag{40}$$

$$\texttt{Plot3D}\left[\frac{4\,\pi\,r^2 - 2\,r^2\,\theta}{2\,\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}},\ \{r,\ -1,\ 1\},\ \{\theta,\ -4\,\pi,\ 4\,\pi\}\right]$$



$$\text{RevolutionPlot3D}\left[\frac{4\pi r^2 - 2r^2\theta}{2\sqrt{4\pi r^2\theta - r^2\theta^2}}, \{r, -1, 1\}, \{\theta, -2\pi, 2\pi\}\right]$$



We also have access to the equation, $\frac{4\pi r^2 - 2 r^2 \theta}{2\sqrt{4\pi r^2 \theta - r^2 \theta^2}} == r / (\theta / (2\pi))$,

which winds up having the r variable cancel out,
and delivers three exact solutions for theta.

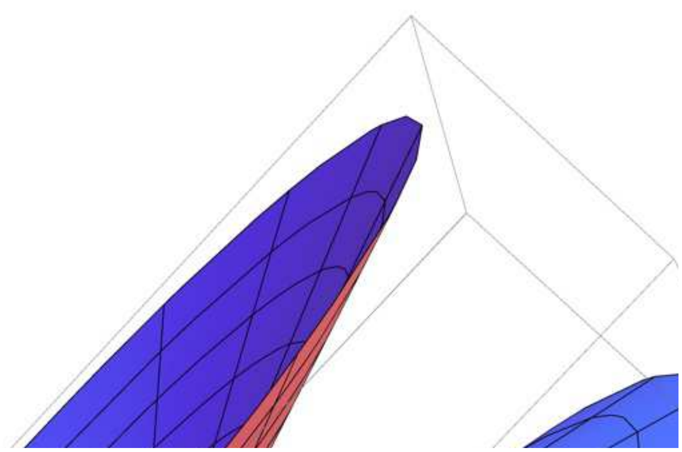
```
Solve[(4 π r^2 - 2 r^2 θ)/(2 Sqrt[4 π r^2 θ - r^2 θ^2]) == (2 π r)/θ, θ]
```

$$\left\{\left\{\theta \to \frac{2}{3}\left(2\pi - \frac{2\pi}{\left(17+3\sqrt{33}\right)^{1/3}} + \left(17+3\sqrt{33}\right)^{1/3}\pi\right)\right\},\right.$$

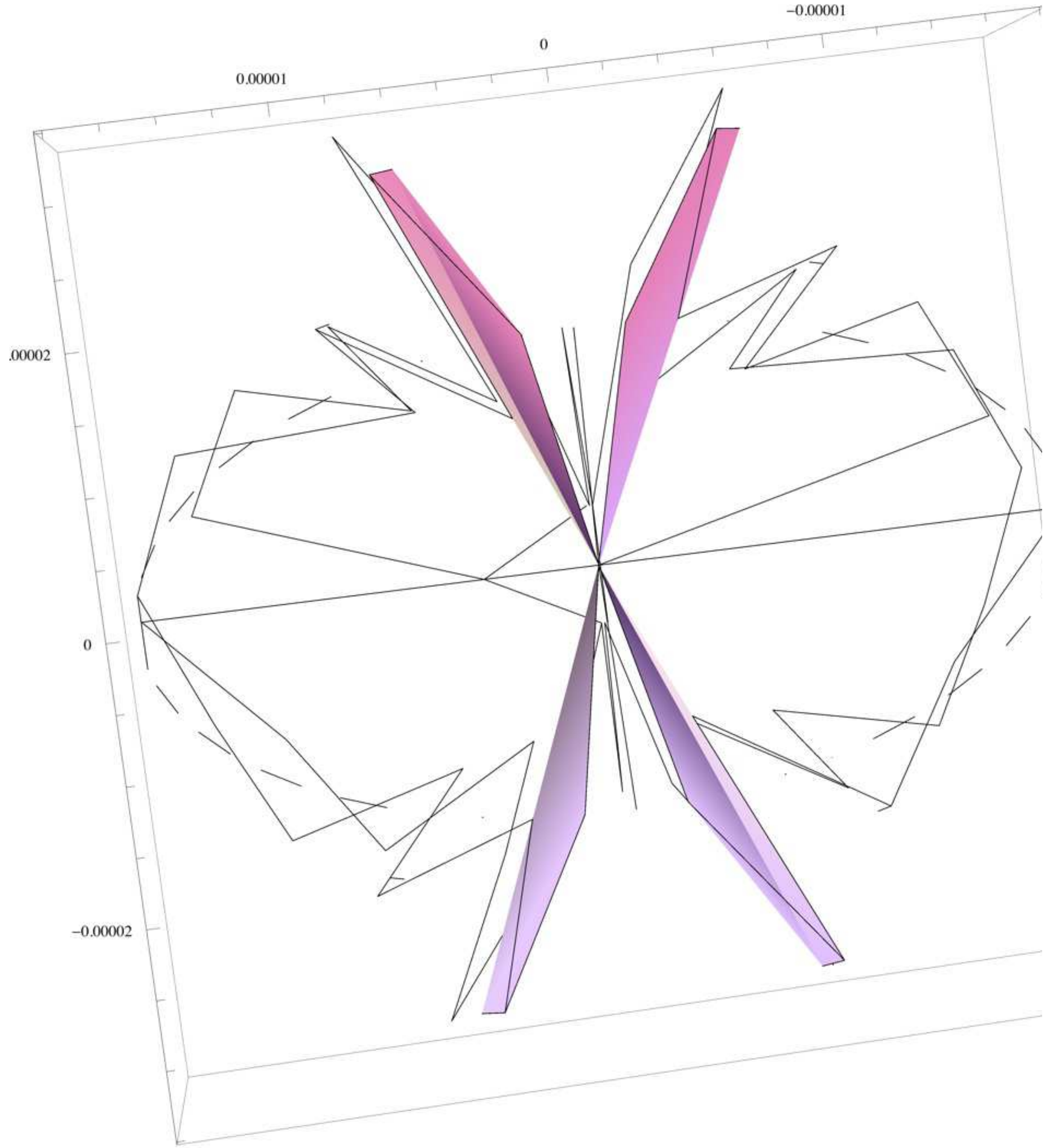
$$\left\{\theta \to \frac{4\pi}{3} + \frac{2\left(1+i\sqrt{3}\right)\pi}{3\left(17+3\sqrt{33}\right)^{1/3}} - \frac{1}{3}\left(1-i\sqrt{3}\right)\left(17+3\sqrt{33}\right)^{1/3}\pi\right\}, \tag{42}$$

$$\left.\left\{\theta \to \frac{4\pi}{3} + \frac{2\left(1-i\sqrt{3}\right)\pi}{3\left(17+3\sqrt{33}\right)^{1/3}} - \frac{1}{3}\left(1+i\sqrt{3}\right)\left(17+3\sqrt{33}\right)^{1/3}\pi\right\}\right\}$$

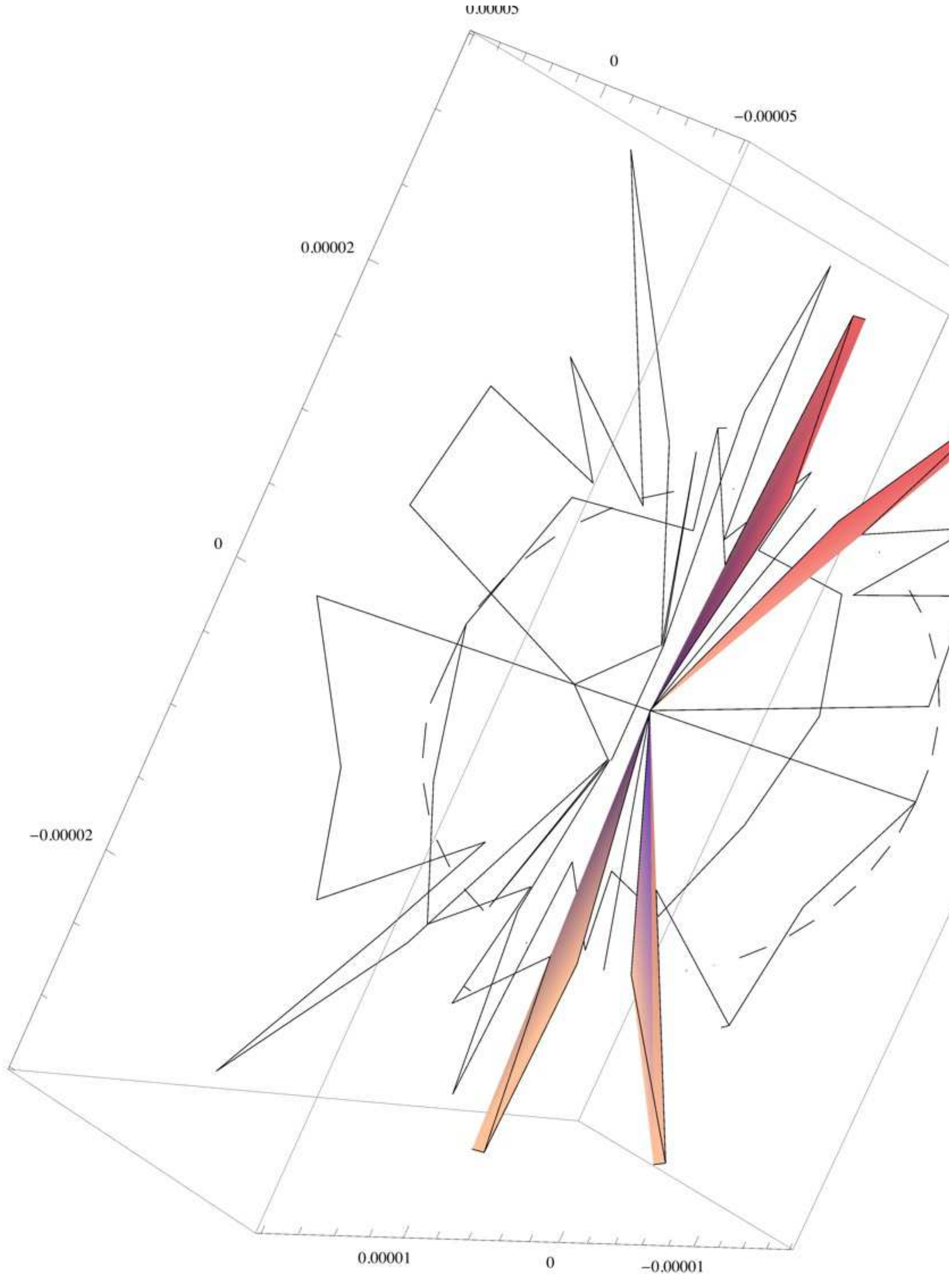
```
RevolutionPlot3D[r (2/3 (2 π - (2 π)/(17 + 3 Sqrt[33])^(1/3) + (17 + 3 Sqrt[33])^(1/3) π)), {r, -1, 1}]
```



$$\text{RevolutionPlot3D}\left[r\left(\frac{4\pi}{3}+\frac{2\left(1+i\sqrt{3}\right)\pi}{3\left(17+3\sqrt{33}\right)^{1/3}}-\frac{1}{3}\left(1-i\sqrt{3}\right)\left(17+3\sqrt{33}\right)^{1/3}\pi\right),\{r,-1,1\}\right]$$


-0.00001
0
0.00001
.00002
0
-0.00002

```
RevolutionPlot3D[r (4 π/3 + 2 (1 - ⅈ √3) π/(3 (17 + 3 √33)^(1/3)) - 1/3 (1 + ⅈ √3) (17 + 3 √33)^(1/3) π), {r, -1, 1}]
```



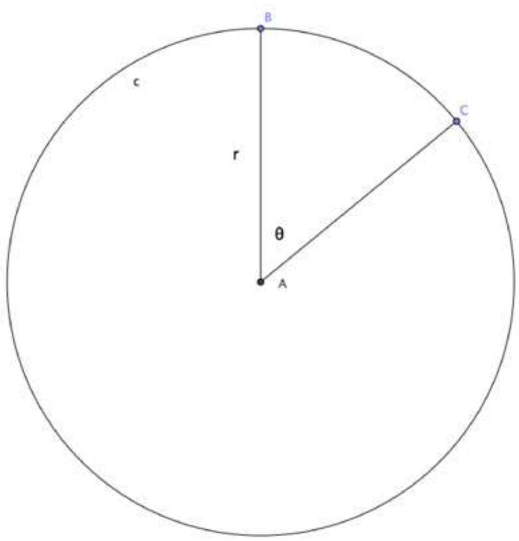
XII. A Paper on Leonard Euler' s Formula in Natural Scientific Thought:

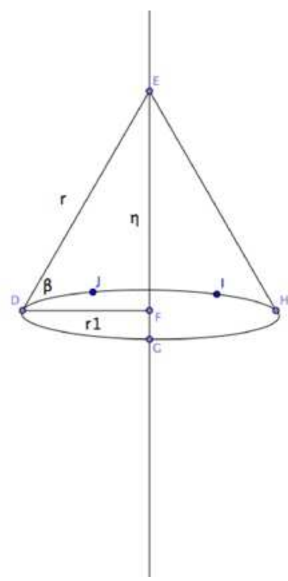
■ Euler's Identity With Substitutions from the Difference in Circumferences of Two Circles Applied to the Pythagorean Theorem

This paper uses Euler ' s Equation to deliver more expressions for an angular section of a circle. This first attempt to organize the multitude of solutions will begin with only the first solution to theta from a difference in circumferences of two circles applied to Pythagorean Theorem. The theorem, $2\pi r - 2\pi r_1 = \Theta r$ is provable and delivers the expression for theta

$\Theta = 2\left(\pi \pm \sqrt{\pi^2 - \pi^2 \text{Sin}[\beta]^2}\right)$.

$$e^{\wedge}(i\,\Theta) = i\,\text{Sin}[\Theta] + \text{Cos}[\Theta]$$





$$\Theta = 2\left(\pi + \sqrt{\pi^2 - \pi^2 \text{Sin}[\beta]^2}\right)$$

■ The Forms

$$e^{\wedge}\left(i\,2\left(\pi + \sqrt{\pi^2 - \pi^2 \text{Sin}[\beta]^2}\right)\right) = i\,\text{Sin}[\Theta] + \text{Cos}[\Theta]$$

$$e^{\wedge}\left(i\,2\left(\pi + \sqrt{\pi^2 - \pi^2 \text{Sin}[\beta]^2}\right)\right) = i\,\text{Sin}\left[2\left(\pi + \sqrt{\pi^2 - \pi^2 \text{Sin}[\beta]^2}\right)\right] + \text{Cos}[\Theta]$$

$$e^{\wedge}\left(i\,2\left(\pi + \sqrt{\pi^2 - \pi^2 \text{Sin}[\beta]^2}\right)\right) = i\,\text{Sin}[\Theta] + \text{Cos}\left[2\left(\pi + \sqrt{\pi^2 - \pi^2 \text{Sin}[\beta]^2}\right)\right]$$

$$e^{\wedge}(i\,\theta) == i\,\mathrm{Sin}\left[2\left(\pi + \sqrt{\pi^2 - \pi^2\,\mathrm{Sin}[\beta]^2}\right)\right] + \mathrm{Cos}[\theta]$$

$$e^{\wedge}(i\,\theta) == i\,\mathrm{Sin}[\theta] + \mathrm{Cos}\left[2\left(\pi + \sqrt{\pi^2 - \pi^2\,\mathrm{Sin}[\beta]^2}\right)\right]$$

$$e^{\wedge}(i\,\theta) == i\,\mathrm{Sin}\left[2\left(\pi + \sqrt{\pi^2 - \pi^2\,\mathrm{Sin}[\beta]^2}\right)\right] + \mathrm{Cos}\left[2\left(\pi + \sqrt{\pi^2 - \pi^2\,\mathrm{Sin}[\beta]^2}\right)\right]$$

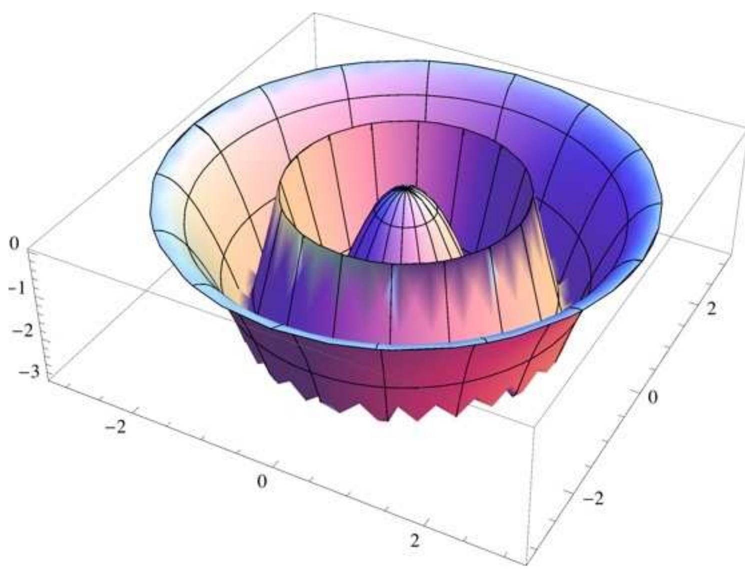
The Solutions

$$\mathrm{Solve}\left[e^{\wedge}\left(i\left(2\left(\pi + \sqrt{\pi^2 - \pi^2\,\mathrm{Sin}[\beta]^2}\right)\right)\right) == i\,\mathrm{Sin}[\theta] + \mathrm{Cos}[\theta],\ \theta\right]$$

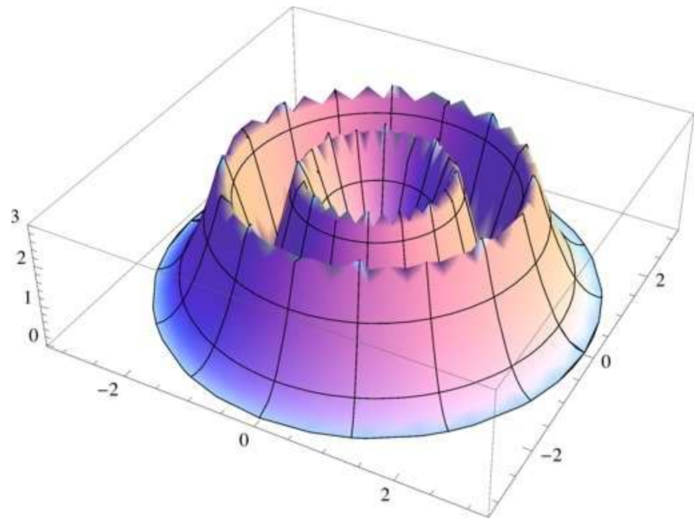
Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

$$\left\{\left\{\theta \to -\mathrm{ArcCos}\left[\frac{1}{2}\,e^{-2\,i\,\pi\sqrt{1-\mathrm{Sin}[\beta]^2}}\left(1 + e^{4\,i\,\pi\sqrt{1-\mathrm{Sin}[\beta]^2}}\right)\right]\right\},\right.$$
$$\left.\left\{\theta \to \mathrm{ArcCos}\left[\frac{1}{2}\,e^{-2\,i\,\pi\sqrt{1-\mathrm{Sin}[\beta]^2}}\left(1 + e^{4\,i\,\pi\sqrt{1-\mathrm{Sin}[\beta]^2}}\right)\right]\right\}\right\}$$

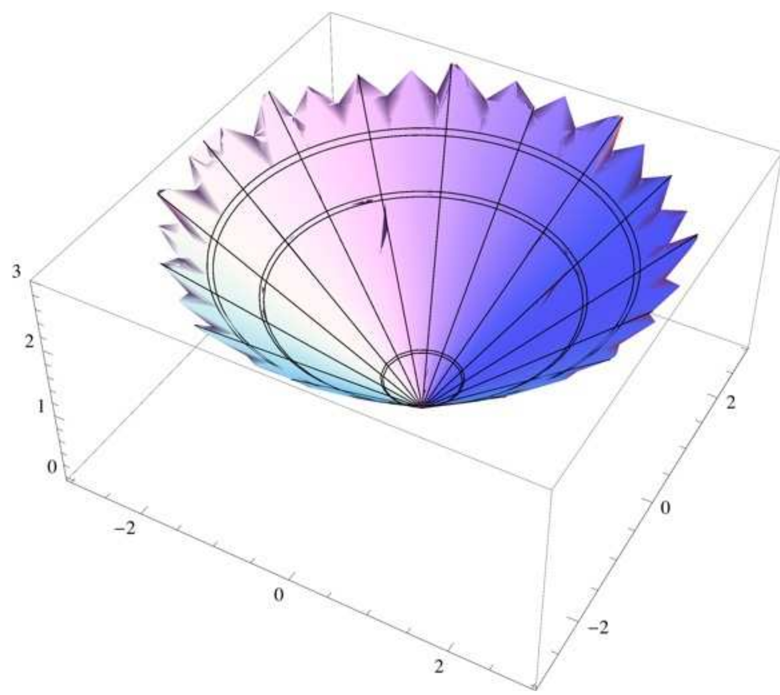
$$\mathrm{RevolutionPlot3D}\left[-\mathrm{ArcCos}\left[\frac{1}{2}\,e^{-2\,i\,\pi\sqrt{1-\mathrm{Sin}[\beta]^2}}\left(1 + e^{4\,i\,\pi\sqrt{1-\mathrm{Sin}[\beta]^2}}\right)\right],\ \{\beta, -\pi, \pi\}\right]$$



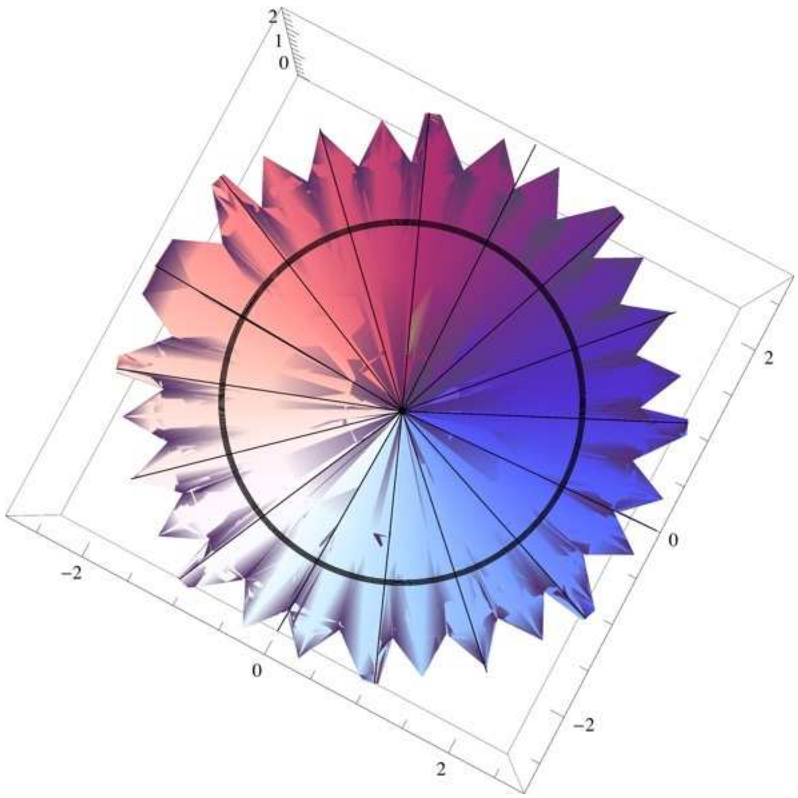
```
RevolutionPlot3D[ArcCos[1/2 e^(-2 i π Sqrt[1-Sin[β]^2]) (1 + e^(4 i π Sqrt[1-Sin[β]^2]))], {β, -π, π}]
```




```
RevolutionPlot3D[{-ArcCos[1/2 e^(-2 i π Sqrt[1-Sin[β]^2]) (1 + e^(4 i π Sqrt[1-Sin[β]^2]))],
  ArcCos[1/2 e^(-2 i π Sqrt[1-Sin[β]^2]) (1 + e^(4 i π Sqrt[1-Sin[β]^2]))]}, {β, -π, π}]
```



$$\text{RevolutionPlot3D}\left[\left\{-\text{ArcCos}\left[\frac{1}{2}\, e^{-2\, i\, \pi \sqrt{1-\text{Sin}[\beta]^2}} \left(1 + e^{4\, i\, \pi \sqrt{1-\text{Sin}[\beta]^2}}\right)\right],\ \text{ArcCos}\left[\frac{1}{2}\, e^{-2\, i\, \pi \sqrt{1-\text{Sin}[\beta]^2}} \left(1 + e^{4\, i\, \pi \sqrt{1-\text{Sin}[\beta]^2}}\right)\right]\right\}, \{\beta, -2\,\pi, 2\,\pi\}\right]$$

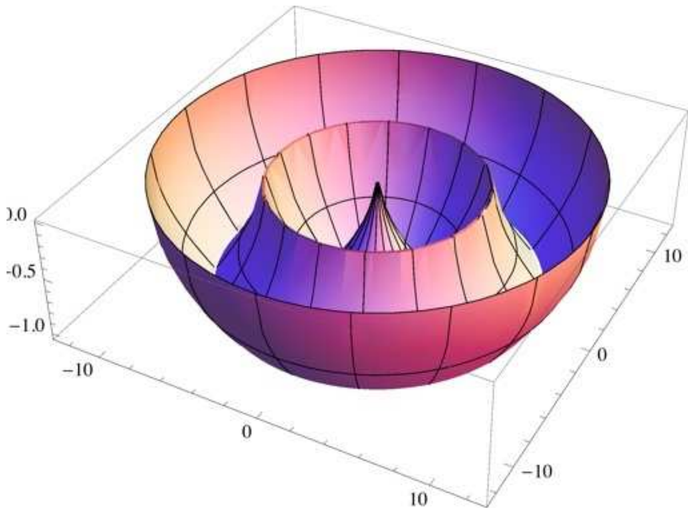


$$\text{Solve}\left[e \,\hat{}\, \left(i \left(2 \left(\pi + \sqrt{\pi^2 - \pi^2\, \text{Sin}[\beta]^2}\right)\right)\right) == i\, \text{Sin}[\theta] + \text{Cos}[\theta],\ \beta\right]$$

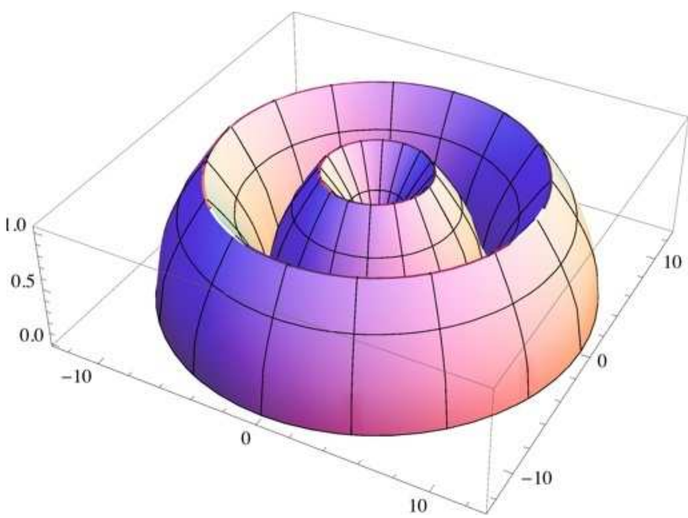
Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

$$\left\{\left\{\beta \to -\text{ArcSin}\left[\frac{\sqrt{\pi^2 + \text{Log}\left[-\sqrt{\text{Cos}[\theta] + i\,\text{Sin}[\theta]}\right]^2}}{\pi}\right]\right\},\ \left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{\pi^2 + \text{Log}\left[-\sqrt{\text{Cos}[\theta] + i\,\text{Sin}[\theta]}\right]^2}}{\pi}\right]\right\},\ \left\{\beta \to -\text{ArcSin}\left[\frac{\sqrt{4\,\pi^2 + \text{Log}[\text{Cos}[\theta] + i\,\text{Sin}[\theta]]^2}}{2\,\pi}\right]\right\},\ \left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{4\,\pi^2 + \text{Log}[\text{Cos}[\theta] + i\,\text{Sin}[\theta]]^2}}{2\,\pi}\right]\right\}\right\}$$

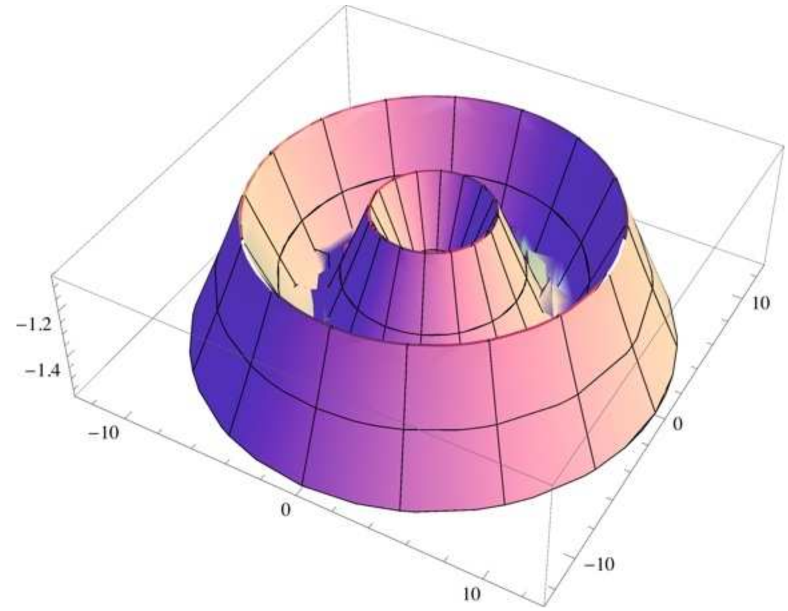
$$\text{RevolutionPlot3D}\left[-\text{ArcSin}\left[\frac{\sqrt{\pi^2 + \text{Log}\left[-\sqrt{\text{Cos}[\theta] + i\,\text{Sin}[\theta]}\right]^2}}{\pi}\right], \{\theta, -4\pi, 4\pi\}\right]$$



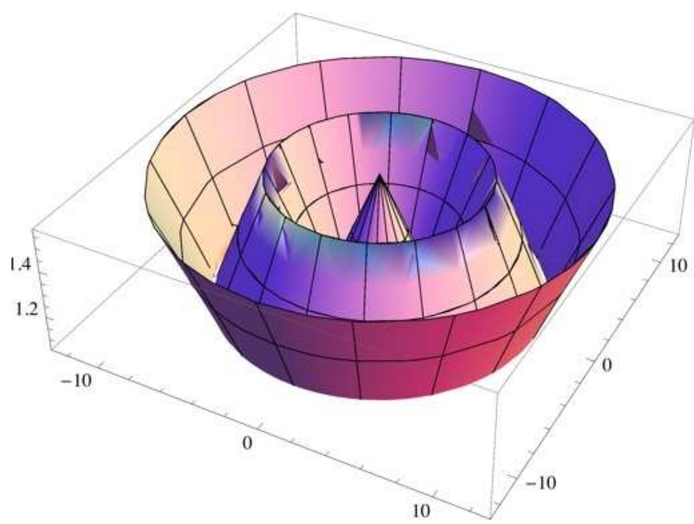
$$\text{RevolutionPlot3D}\left[\text{ArcSin}\left[\frac{\sqrt{\pi^2 + \text{Log}\left[-\sqrt{\text{Cos}[\theta] + i\,\text{Sin}[\theta]}\right]^2}}{\pi}\right], \{\theta, -4\pi, 4\pi\}\right]$$



$$\text{RevolutionPlot3D}\left[-\text{ArcSin}\left[\frac{\sqrt{4\,\pi^2 + \text{Log}[\text{Cos}[\theta] + i\,\text{Sin}[\theta]]^2}}{2\,\pi}\right], \{\theta, -4\,\pi, 4\,\pi\}\right]$$



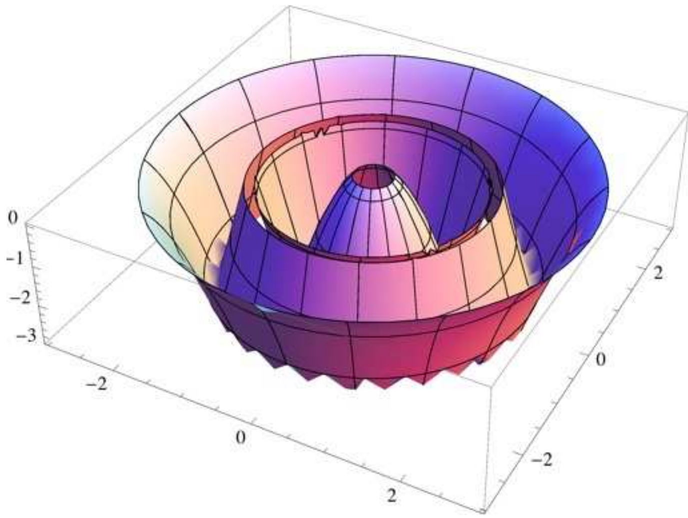
$$\text{RevolutionPlot3D}\left[\text{ArcSin}\left[\frac{\sqrt{4\,\pi^2 + \text{Log}[\text{Cos}[\theta] + i\,\text{Sin}[\theta]]^2}}{2\,\pi}\right], \{\theta, -4\,\pi, 4\,\pi\}\right]$$



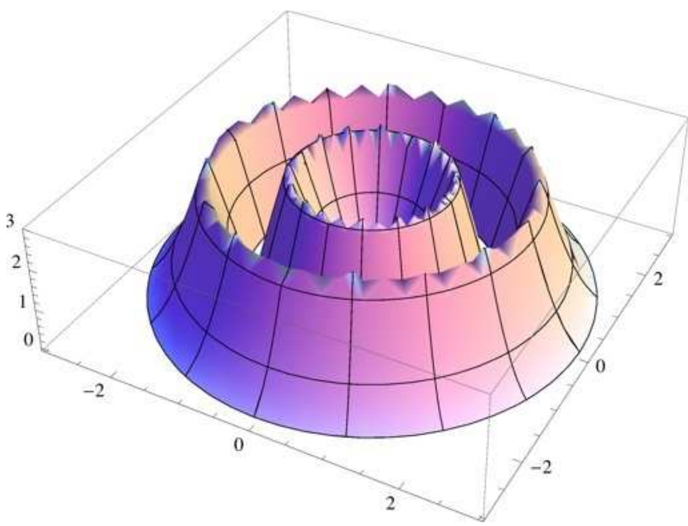
$$\text{Solve}\left[e^{\wedge}\left(i\,2\left(\pi + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)\right) == i\,\text{Sin}\left[2\left(\pi + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)\right] + \text{Cos}[\theta], \theta\right]$$

$$\left\{\left\{\theta \to -\text{ArcCos}\left[e^{2\,i\,\pi\sqrt{1-\text{Sin}[\beta]^2}} - i\,\text{Sin}\left[2\left(\pi + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)\right]\right]\right\},\right.$$
$$\left.\left\{\theta \to \text{ArcCos}\left[e^{2\,i\,\pi\sqrt{1-\text{Sin}[\beta]^2}} - i\,\text{Sin}\left[2\left(\pi + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)\right]\right]\right\}\right\}$$

RevolutionPlot3D$\left[-\text{ArcCos}\left[e^{2\,i\,\pi\sqrt{1-\text{Sin}[\beta]^2}} - i\,\text{Sin}\left[2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right]\right], \{\beta, -\pi, \pi\}\right]$



RevolutionPlot3D$\left[\text{ArcCos}\left[e^{2\,i\,\pi\sqrt{1-\text{Sin}[\beta]^2}} - i\,\text{Sin}\left[2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right]\right], \{\beta, -\pi, \pi\}\right]$



Solve$\left[e \wedge \left(i\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right)\right) == i\,\text{Sin}\left[2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right] + \text{Cos}[\theta], \beta\right]$

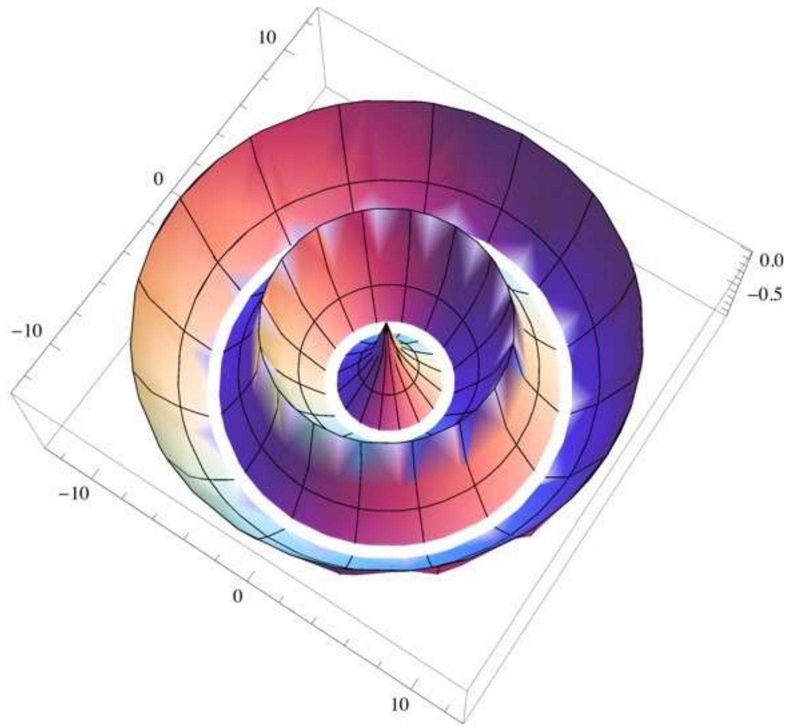
Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

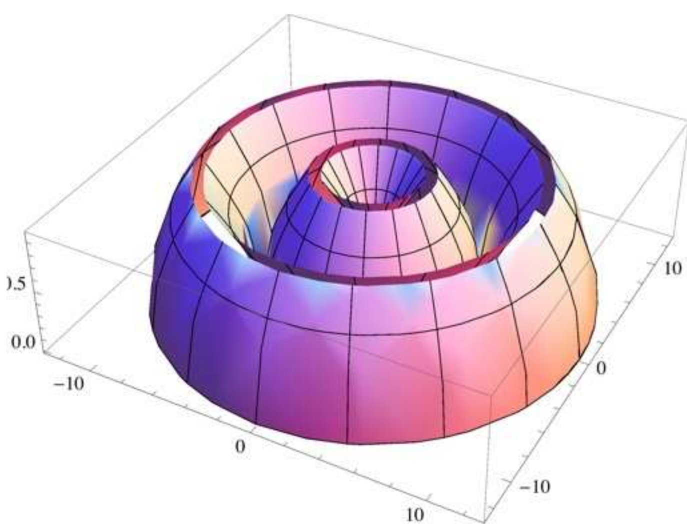
$$\left\{\left\{\beta\to-\text{ArcSin}\left[\sqrt{1-\frac{\text{ArcCos}\left[-\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}{\pi^2}}\right]\right\}, \left\{\beta\to\text{ArcSin}\left[\sqrt{1-\frac{\text{ArcCos}\left[-\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}{\pi^2}}\right]\right\},\right.$$

$$\left.\left\{\beta\to-\text{ArcSin}\left[\sqrt{1-\frac{\text{ArcCos}\left[\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}{\pi^2}}\right]\right\}, \left\{\beta\to\text{ArcSin}\left[\sqrt{1-\frac{\text{ArcCos}\left[\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}{\pi^2}}\right]\right\}\right\}$$

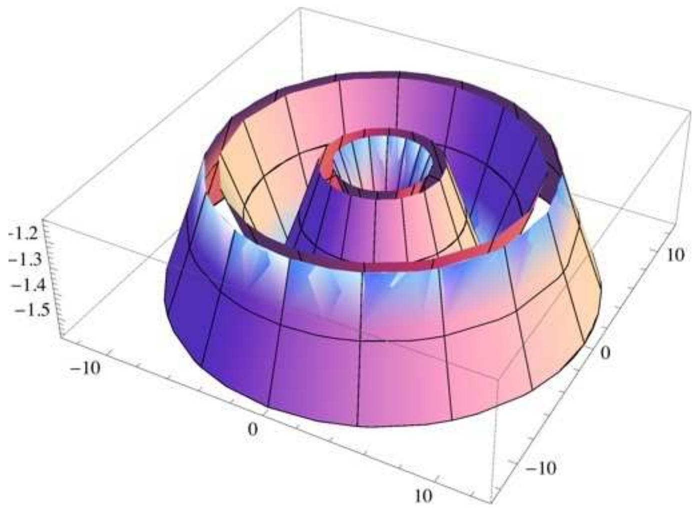
$$\texttt{RevolutionPlot3D}\left[-\texttt{ArcSin}\left[\sqrt{1-\frac{\texttt{ArcCos}\left[-\frac{\sqrt{1+\texttt{Cos}[\theta]}}{\sqrt{2}}\right]^2}{\pi^2}}\,\right],\ \{\theta,\ -4\,\pi,\ 4\,\pi\}\right]$$



$$\texttt{RevolutionPlot3D}\left[\texttt{ArcSin}\left[\sqrt{1-\frac{\texttt{ArcCos}\left[-\frac{\sqrt{1+\texttt{Cos}[\theta]}}{\sqrt{2}}\right]^2}{\pi^2}}\,\right],\ \{\theta,\ -4\,\pi,\ 4\,\pi\}\right]$$



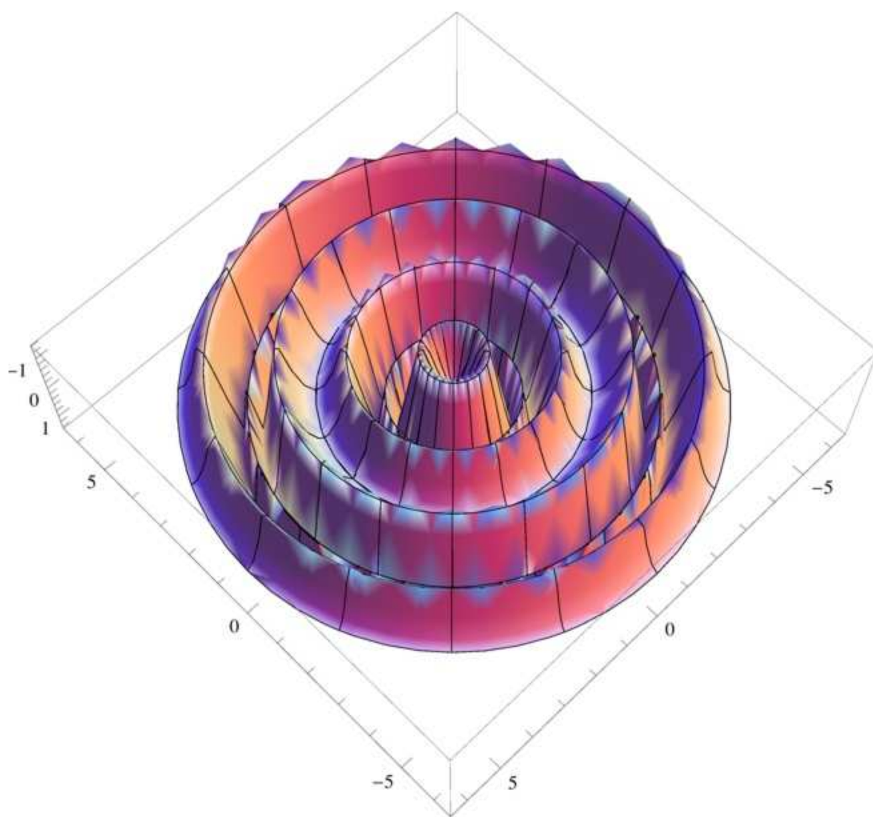
RevolutionPlot3D$\left[-\text{ArcSin}\left[\sqrt{1-\frac{\text{ArcCos}\left[\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}{\pi^2}}\right], \{\theta, -4\pi, 4\pi\}\right]$



Solve$\left[e^{\wedge}\left(i\, 2\left(\pi+\sqrt{\pi^2-\pi^2\, \text{Sin}[\beta]^2}\right)\right) == i\, \text{Sin}[\theta] + \text{Cos}\left[2\left(\pi+\sqrt{\pi^2-\pi^2\, \text{Sin}[\beta]^2}\right)\right], \theta\right]$

$\left\{\left\{\theta \to -i\, \text{ArcSinh}\left[e^{2\, i\, \pi\sqrt{1-\text{Sin}[\beta]^2}} - \text{Cos}\left[2\left(\pi+\sqrt{\pi^2-\pi^2\, \text{Sin}[\beta]^2}\right)\right]\right]\right\}\right\}$

RevolutionPlot3D$\left[-i\, \text{ArcSinh}\left[e^{2\, i\, \pi\sqrt{1-\text{Sin}[\beta]^2}} - \text{Cos}\left[2\left(\pi+\sqrt{\pi^2-\pi^2\, \text{Sin}[\beta]^2}\right)\right]\right], \{\beta, -2\pi, 2\pi\}\right]$



$$\textbf{Solve}\left[e^{\wedge}\left(i\,2\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)\right)==i\,\mathrm{Sin}[\theta]+\mathrm{Cos}\left[2\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)\right],\ \beta\right]$$

$$\left\{\left\{\beta\to-\mathrm{ArcSin}\left[\sqrt{1-\frac{\mathrm{ArcCos}\left[\mathrm{Csc}[\theta]\left(-\frac{\sqrt{1-\sqrt{1-\mathrm{Sin}[\theta]^2}}}{\sqrt{2}}-\frac{\sqrt{1-\mathrm{Sin}[\theta]^2}\sqrt{1-\sqrt{1-\mathrm{Sin}[\theta]^2}}}{\sqrt{2}}\right)\right]^2}{\pi^2}}\right]\right\},\right.$$

$$\left\{\beta\to\mathrm{ArcSin}\left[\sqrt{1-\frac{\mathrm{ArcCos}\left[\mathrm{Csc}[\theta]\left(-\frac{\sqrt{1-\sqrt{1-\mathrm{Sin}[\theta]^2}}}{\sqrt{2}}-\frac{\sqrt{1-\mathrm{Sin}[\theta]^2}\sqrt{1-\sqrt{1-\mathrm{Sin}[\theta]^2}}}{\sqrt{2}}\right)\right]^2}{\pi^2}}\right]\right\},$$

$$\left\{\beta\to-\mathrm{ArcSin}\left[\sqrt{1-\frac{\mathrm{ArcCos}\left[\mathrm{Csc}[\theta]\left(\frac{\sqrt{1-\sqrt{1-\mathrm{Sin}[\theta]^2}}}{\sqrt{2}}+\frac{\sqrt{1-\mathrm{Sin}[\theta]^2}\sqrt{1-\sqrt{1-\mathrm{Sin}[\theta]^2}}}{\sqrt{2}}\right)\right]^2}{\pi^2}}\right]\right\},$$

$$\left\{\beta\to\mathrm{ArcSin}\left[\sqrt{1-\frac{\mathrm{ArcCos}\left[\mathrm{Csc}[\theta]\left(\frac{\sqrt{1-\sqrt{1-\mathrm{Sin}[\theta]^2}}}{\sqrt{2}}+\frac{\sqrt{1-\mathrm{Sin}[\theta]^2}\sqrt{1-\sqrt{1-\mathrm{Sin}[\theta]^2}}}{\sqrt{2}}\right)\right]^2}{\pi^2}}\right]\right\},$$

$$\left\{\beta\to-\mathrm{ArcSin}\left[\sqrt{\left(1-\frac{1}{\pi^2}\mathrm{ArcCos}\left[\mathrm{Csc}[\theta]\left(\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\mathrm{Sin}[\theta]^2}}-\sqrt{1-\mathrm{Sin}[\theta]^2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\mathrm{Sin}[\theta]^2}}\right)\right]^2\right)}\right]\right\},$$

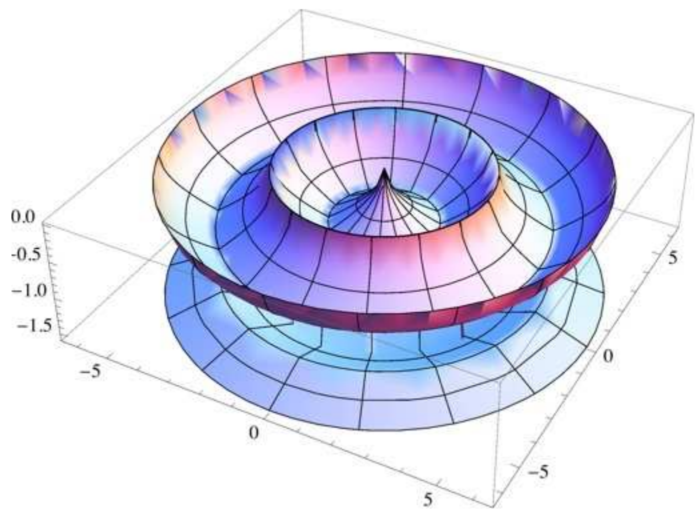
$$\left\{\beta\to\mathrm{ArcSin}\left[\sqrt{\left(1-\frac{1}{\pi^2}\mathrm{ArcCos}\left[\mathrm{Csc}[\theta]\left(\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\mathrm{Sin}[\theta]^2}}-\sqrt{1-\mathrm{Sin}[\theta]^2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\mathrm{Sin}[\theta]^2}}\right)\right]^2\right)}\right]\right\},$$

$$\left\{\beta\to-\mathrm{ArcSin}\left[\sqrt{\left(1-\frac{1}{\pi^2}\mathrm{ArcCos}\left[\mathrm{Csc}[\theta]\left(-\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\mathrm{Sin}[\theta]^2}}+\sqrt{1-\mathrm{Sin}[\theta]^2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\mathrm{Sin}[\theta]^2}}\right)\right]^2\right)}\right]\right\},$$

$$\left.\left\{\beta\to\mathrm{ArcSin}\left[\sqrt{\left(1-\frac{1}{\pi^2}\mathrm{ArcCos}\left[\mathrm{Csc}[\theta]\left(-\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\mathrm{Sin}[\theta]^2}}+\sqrt{1-\mathrm{Sin}[\theta]^2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\mathrm{Sin}[\theta]^2}}\right)\right]^2\right)}\right]\right\}\right\}$$

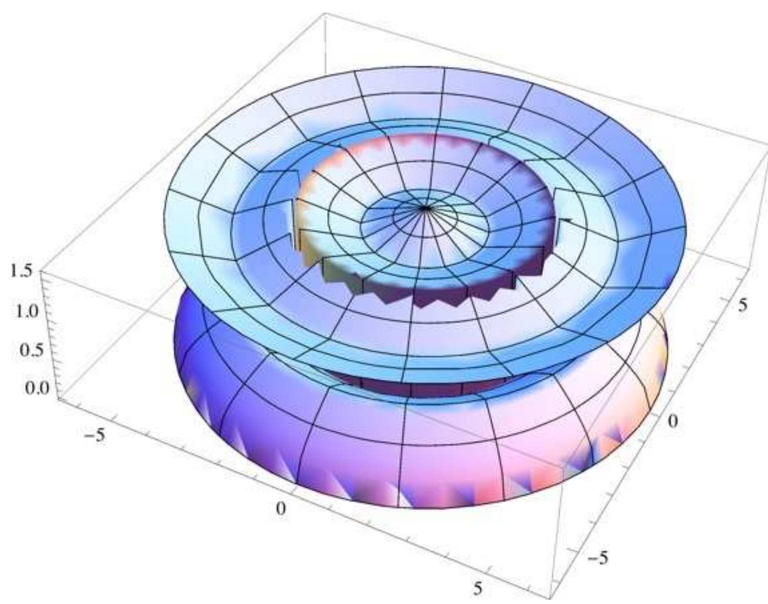
RevolutionPlot3D

$$\texttt{RevolutionPlot3D}\left[-\texttt{ArcSin}\left[\sqrt{1-\frac{\texttt{ArcCos}\left[\texttt{Csc}[\theta]\left(-\frac{\sqrt{1-\sqrt{1-\texttt{Sin}[\theta]^2}}}{\sqrt{2}}-\frac{\sqrt{1-\texttt{Sin}[\theta]^2}\,\sqrt{1-\sqrt{1-\texttt{Sin}[\theta]^2}}}{\sqrt{2}}\right)\right]^2}{\pi^2}}\right],\ \{\theta,\ -2\,\pi,\ 2\,\pi\}\right]$$



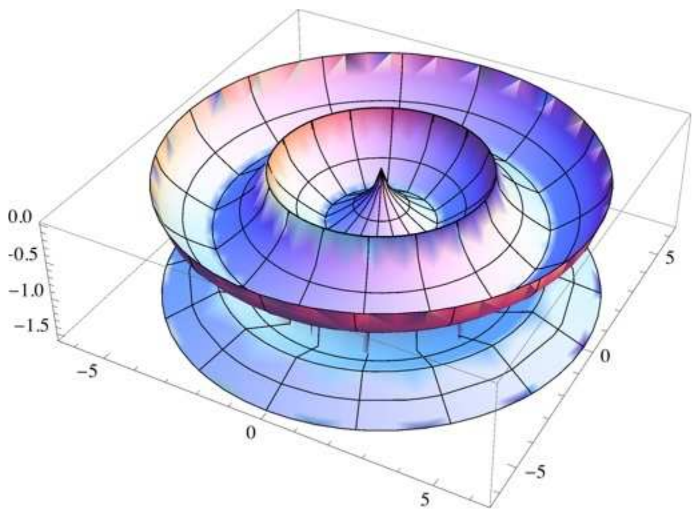
RevolutionPlot3D

$$\texttt{RevolutionPlot3D}\left[\texttt{ArcSin}\left[\sqrt{1-\frac{\texttt{ArcCos}\left[\texttt{Csc}[\theta]\left(-\frac{\sqrt{1-\sqrt{1-\texttt{Sin}[\theta]^2}}}{\sqrt{2}}-\frac{\sqrt{1-\texttt{Sin}[\theta]^2}\,\sqrt{1-\sqrt{1-\texttt{Sin}[\theta]^2}}}{\sqrt{2}}\right)\right]^2}{\pi^2}}\right],\ \{\theta,\ -2\,\pi,\ 2\,\pi\}\right]$$



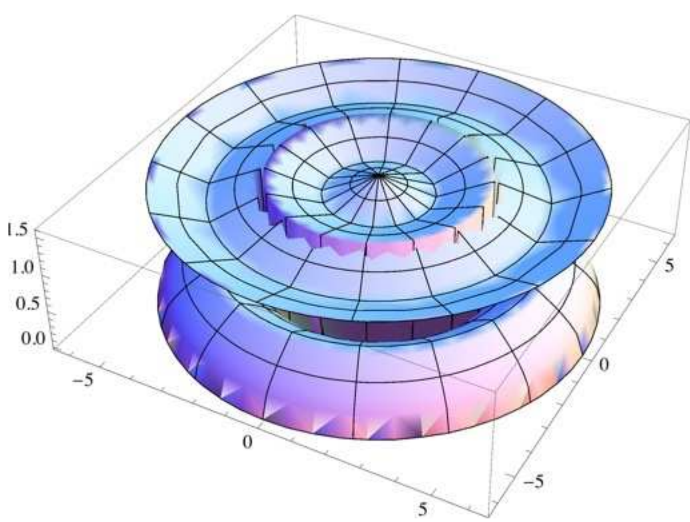
```
RevolutionPlot3D[-ArcSin[Sqrt[1 - ArcCos[Csc[θ] (Sqrt[1-Sqrt[1-Sin[θ]^2]]/Sqrt[2] + Sqrt[1-Sin[θ]^2] Sqrt[1-Sqrt[1-Sin[θ]^2]]/Sqrt[2])]^2/π^2]], {θ, -2 π, 2 π}]
```

$$-\mathrm{ArcSin}\left[\sqrt{1-\frac{\mathrm{ArcCos}\left[\mathrm{Csc}[\theta]\left(\frac{\sqrt{1-\sqrt{1-\mathrm{Sin}[\theta]^2}}}{\sqrt{2}}+\frac{\sqrt{1-\mathrm{Sin}[\theta]^2}\sqrt{1-\sqrt{1-\mathrm{Sin}[\theta]^2}}}{\sqrt{2}}\right)\right]^2}{\pi^2}}\right], \{\theta, -2\pi, 2\pi\}$$

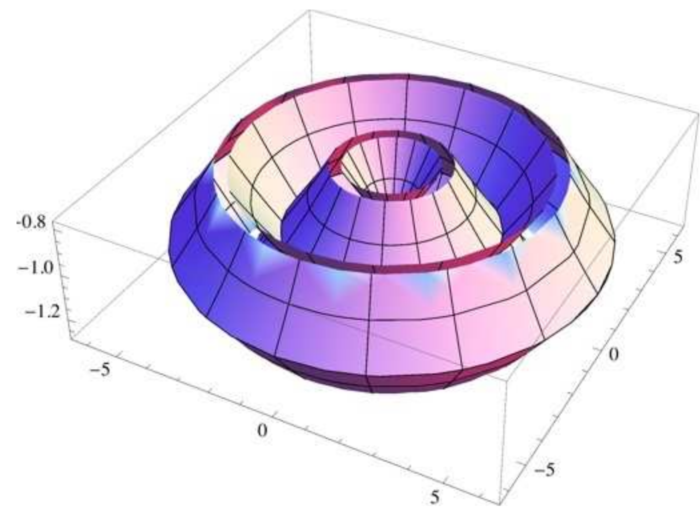



```
RevolutionPlot3D[ArcSin[Sqrt[1 - ArcCos[Csc[θ] (Sqrt[1-Sqrt[1-Sin[θ]^2]]/Sqrt[2] + Sqrt[1-Sin[θ]^2] Sqrt[1-Sqrt[1-Sin[θ]^2]]/Sqrt[2])]^2/π^2]], {θ, -2 π, 2 π}]
```

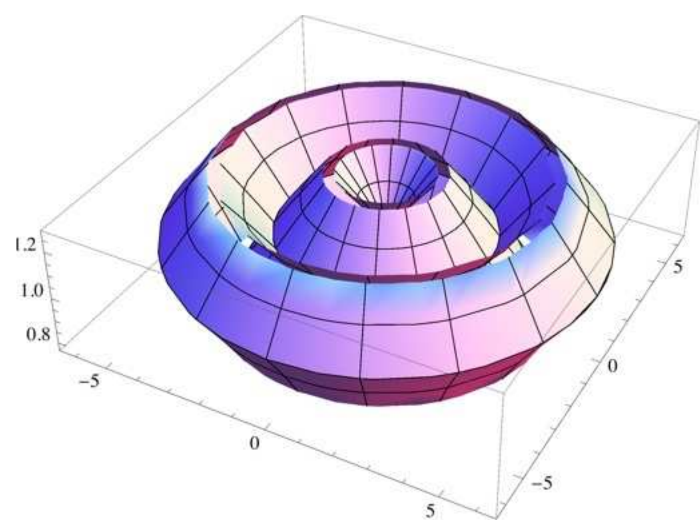
$$\mathrm{ArcSin}\left[\sqrt{1-\frac{\mathrm{ArcCos}\left[\mathrm{Csc}[\theta]\left(\frac{\sqrt{1-\sqrt{1-\mathrm{Sin}[\theta]^2}}}{\sqrt{2}}+\frac{\sqrt{1-\mathrm{Sin}[\theta]^2}\sqrt{1-\sqrt{1-\mathrm{Sin}[\theta]^2}}}{\sqrt{2}}\right)\right]^2}{\pi^2}}\right], \{\theta, -2\pi, 2\pi\}$$



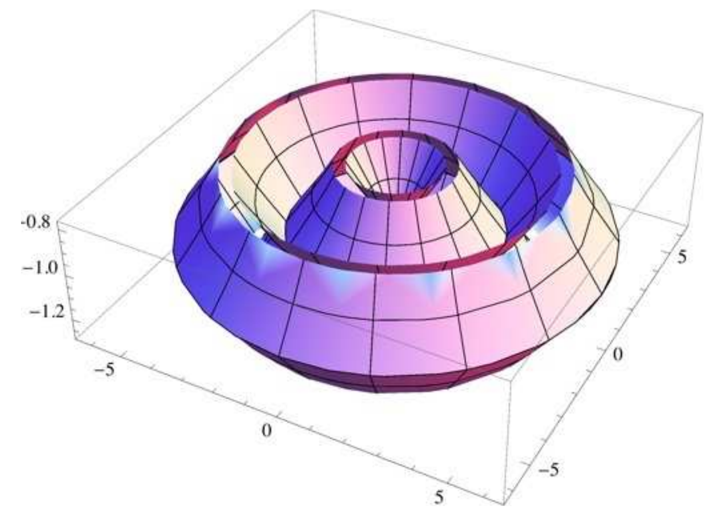
$$\text{RevolutionPlot3D}\left[-\text{ArcSin}\left[\sqrt{\left(1-\frac{1}{\pi^2}\text{ArcCos}\left[\text{Csc}[\theta]\left(\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}-\sqrt{1-\text{Sin}[\theta]^2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}\right)\right]^2\right)}\right],\{\theta,-2\pi,2\pi\}\right]$$



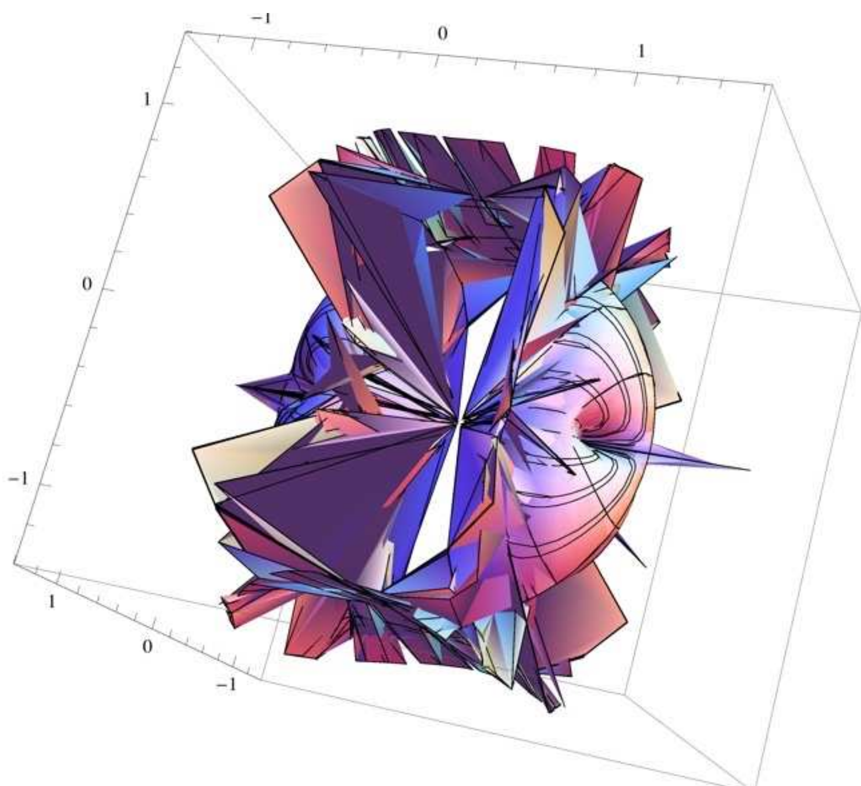
$$\text{RevolutionPlot3D}\left[\text{ArcSin}\left[\sqrt{\left(1-\frac{1}{\pi^2}\text{ArcCos}\left[\text{Csc}[\theta]\left(\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}-\sqrt{1-\text{Sin}[\theta]^2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}\right)\right]^2\right)}\right],\{\theta,-2\pi,2\pi\}\right]$$



```
RevolutionPlot3D[-ArcSin[
   Sqrt[(1 - 1/π^2 ArcCos[Csc[θ] (-Sqrt[1/2 + 1/2 Sqrt[1 - Sin[θ]^2]] + Sqrt[1 - Sin[θ]^2] Sqrt[1/2 + 1/2 Sqrt[1 - Sin[θ]^2]])])^2]],
  {θ, -2 π, 2 π}]
```



$$\text{SphericalPlot3D}\Bigg[\text{ArcSin}\Bigg[\sqrt{\left(1-\frac{1}{\pi^2}\text{ArcCos}\left[\text{Csc}\left[\frac{4\pi}{3}-\left(-4\pi^2+12\pi^2\,\text{Sin}[\beta]^2\right)\Big/\left(6\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}\right)+\frac{2}{3}\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}\right]\left(-\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}+\sqrt{1-\text{Sin}[\theta]^2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}\right)\right]\right)^2}\Bigg],\ \{\beta,-\pi,\pi\},\ \{\theta,-4\pi,4\pi\}\Bigg]$$



$$\text{ArcSin}\Big[\sqrt{\Big(\Big(1-\frac{1}{\pi^2}\text{ArcCos}\Big[\text{Csc}\Big[\frac{4\pi}{3}-\big(-4\pi^2+12\pi^2\text{Sin}[\beta]^2\big)\Big/\Big(6\big(-\pi^3+18\pi^3\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\text{Sin}[\beta]^2+11\pi^6\text{Sin}[\beta]^4+\pi^6\text{Sin}[\beta]^6}\big)^{1/3}\Big)+\frac{2}{3}\big(-\pi^3+18\pi^3\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\text{Sin}[\beta]^2+11\pi^6\text{Sin}[\beta]^4+\pi^6\text{Sin}[\beta]^6}\big)^{1/3}\Big]\Big(-\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}+\sqrt{1-\text{Sin}[\theta]^2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}\Big)\Big]\Big)^2\Big)}\Big]$$

meta - awareness of a pin popping a balloon. (tachyon light travel information is emulsed onto the the surface of the balloon).

$$\text{SphericalPlot3D}\Big[\Big\{\text{ArcSin}\Big[\sqrt{\Big(\Big(1-\frac{1}{\pi^2}\text{ArcCos}\Big[\text{Csc}\Big[\frac{4\pi}{3}-\big(-4\pi^2+12\pi^2\text{Sin}[\beta]^2\big)\Big/\Big(6\big(-\pi^3+18\pi^3\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\text{Sin}[\beta]^2+11\pi^6\text{Sin}[\beta]^4+\pi^6\text{Sin}[\beta]^6}\big)^{1/3}\Big)+\frac{2}{3}\big(-\pi^3+18\pi^3\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\text{Sin}[\beta]^2+11\pi^6\text{Sin}[\beta]^4+\pi^6\text{Sin}[\beta]^6}\big)^{1/3}\Big]\Big(-\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}+\sqrt{1-\text{Sin}[\theta]^2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}\Big)\Big]\Big)^2\Big)}\Big],$$

$$\Big(-13.16979643063896\text{`}\Big/\Big(-1.8378770664093453\text{`}-1.\text{`}\,\text{Log}\Big[\Big(\theta\,\text{Csc}[\beta]\sqrt{\big(-1.1294090667581471\text{`*^18}\,\theta+8.987551787368176\text{`*^16}\,\theta^2+3.5481432270250993\text{`*^18}\,\text{Sin}[\beta]^2\big)}\Big)\Big/\Big(\sqrt{-12.566370614359172\text{`}\,\theta+\theta^2+39.47841760435743\text{`}\,\text{Sin}[\beta]^2}\Big)\Big]\Big)\Big)\Big\},$$
$$\{\theta,-1.5\pi,1.5\pi\},\{\beta,-.75\pi,.75\pi\}\Big]$$

$$\theta=\frac{4\pi}{3}-\frac{-4\pi^2+12\pi^2\text{Sin}[\beta]^2}{6\big(-\pi^3+18\pi^3\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\text{Sin}[\beta]^2+11\pi^6\text{Sin}[\beta]^4+\pi^6\text{Sin}[\beta]^6}\big)^{1/3}}+\frac{2}{3}\big(-\pi^3+18\pi^3\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\text{Sin}[\beta]^2+11\pi^6\text{Sin}[\beta]^4+\pi^6\text{Sin}[\beta]^6}\big)^{1/3},$$

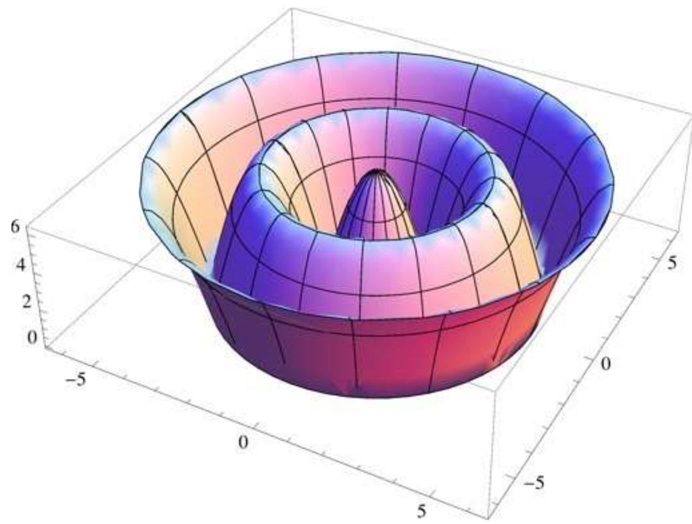
so substitutions of all kinds may be made.

$$\text{Solve}\Big[e^{\wedge}(i\,\theta)==i\,\text{Sin}\Big[2\Big(\pi+\sqrt{\pi^2-\pi^2\text{Sin}[\beta]^2}\Big)\Big]+\text{Cos}[\theta],\theta\Big]$$

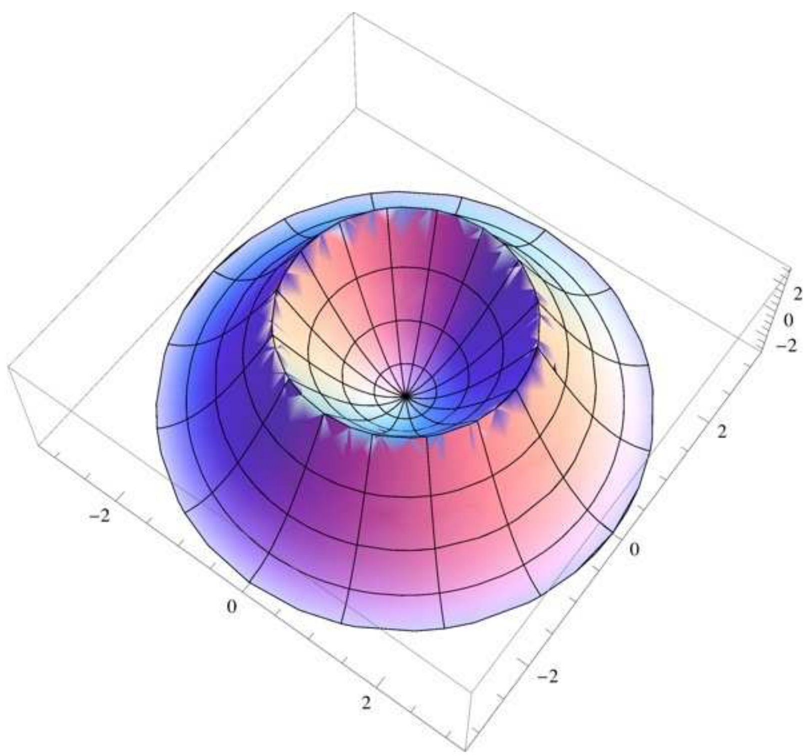
$$\Big\{\Big\{\theta\to\sqrt{e^{-2i\beta}\big(1+e^{2i\beta}\big)^2}\,\pi\Big\},\Big\{\theta\to -i\,\text{Log}\Big[-e^{-i\sqrt{e^{-2i\beta}(1+e^{2i\beta})^2}\,\pi}\Big]\Big\}\Big\}$$

```
RevolutionPlot3D[Sqrt[e^(-2 i β) (1 + e^(2 i β))^2] π, {β, -2 π, 2 π}]
```

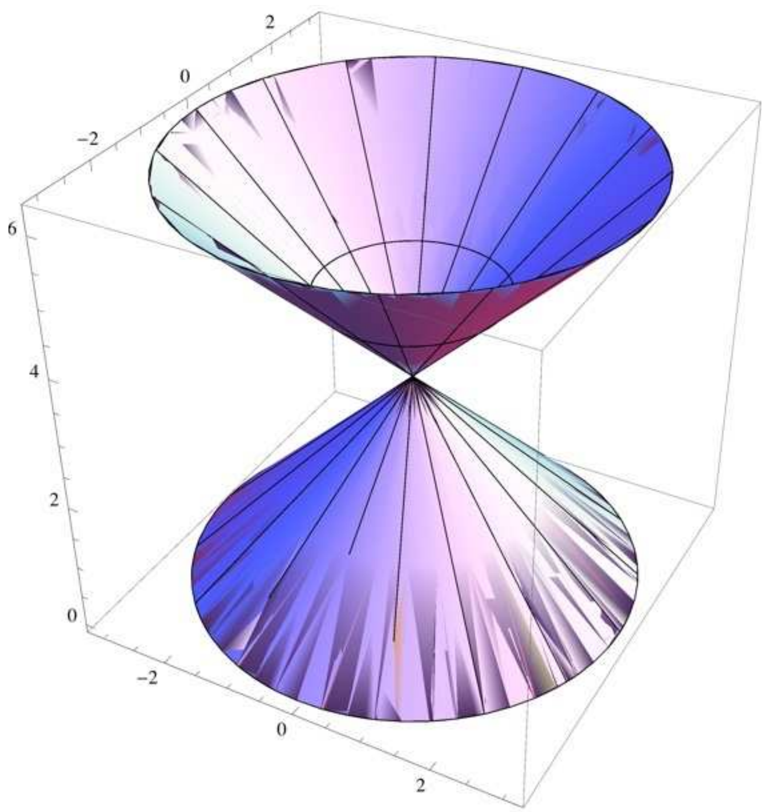
$$\text{RevolutionPlot3D}\left[\sqrt{e^{-2 i \beta}\left(1+e^{2 i \beta}\right)^2}\ \pi,\ \{\beta, -2\pi, 2\pi\}\right]$$



$$\text{RevolutionPlot3D}\left[-i\,\text{Log}\left[-e^{-i\sqrt{e^{-2 i \beta}\left(1+e^{2 i \beta}\right)^2}\ \pi}\right],\ \{\beta, -\pi, \pi\}\right]$$



$$\texttt{RevolutionPlot3D}\left[\left\{-\mathrm{i}\,\texttt{Log}\left[-\mathrm{e}^{-\mathrm{i}\sqrt{\mathrm{e}^{-2\,\mathrm{i}\,\beta}\left(1+\mathrm{e}^{2\,\mathrm{i}\,\beta}\right)^{2}\,\pi}}\right],\ \sqrt{\mathrm{e}^{-2\,\mathrm{i}\,\beta}\left(1+\mathrm{e}^{2\,\mathrm{i}\,\beta}\right)^{2}\,\pi}\right\},\ \{\beta,\,-2\,\pi,\,2\,\pi\}\right]$$



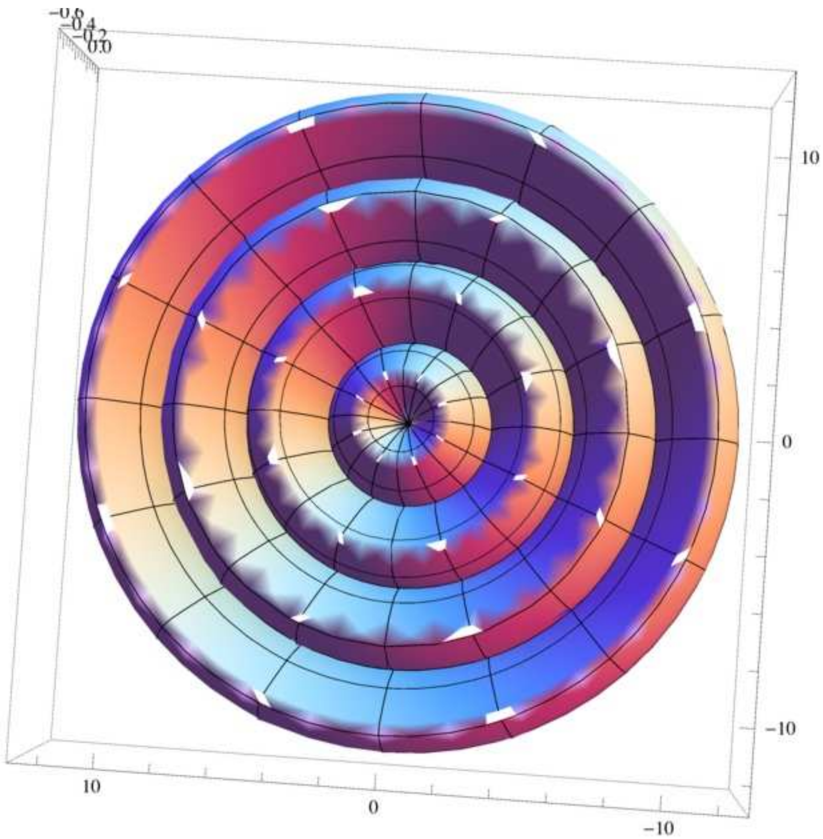
$$\texttt{Solve}\left[\mathrm{e}\,\hat{}\,(\mathrm{i}\,\theta) == \mathrm{i}\,\texttt{Sin}\left[2\left(\pi+\sqrt{\pi^{2}-\pi^{2}\,\texttt{Sin}[\beta]^{2}}\right)\right]+\texttt{Cos}[\theta],\ \beta\right]$$

$$\left\{\left\{\beta\to-\texttt{ArcSin}\left[\frac{1}{2}\sqrt{-\frac{4\,\mathrm{i}\,\texttt{ArcSinh}\left[\mathrm{e}^{\mathrm{i}\,\theta}-\texttt{Cos}[\theta]\right]}{\pi}+\frac{\texttt{ArcSinh}\left[\mathrm{e}^{\mathrm{i}\,\theta}-\texttt{Cos}[\theta]\right]^{2}}{\pi^{2}}}\right]\right\},\right.$$

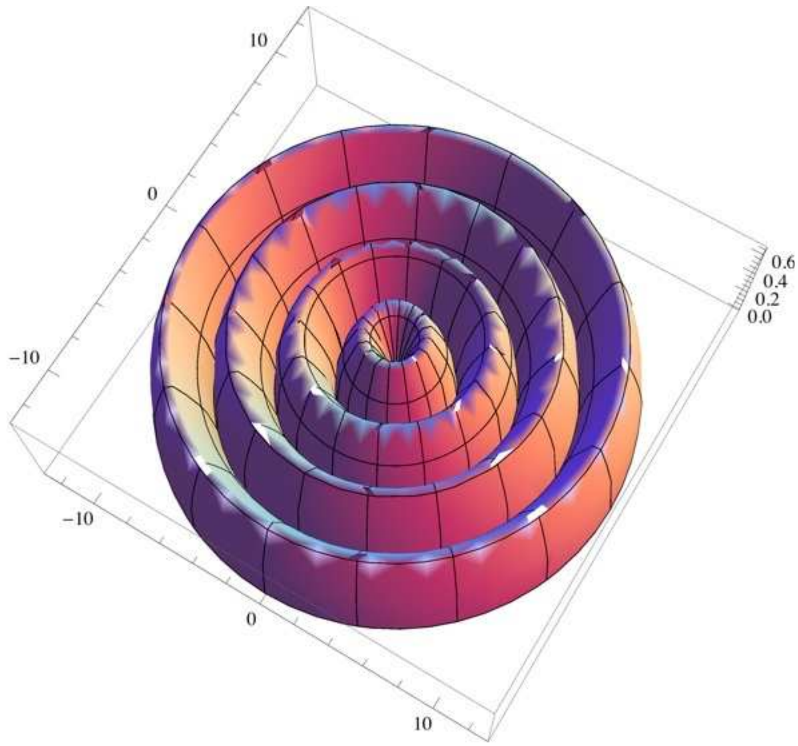
$$\left.\left\{\beta\to\texttt{ArcSin}\left[\frac{1}{2}\sqrt{-\frac{4\,\mathrm{i}\,\texttt{ArcSinh}\left[\mathrm{e}^{\mathrm{i}\,\theta}-\texttt{Cos}[\theta]\right]}{\pi}+\frac{\texttt{ArcSinh}\left[\mathrm{e}^{\mathrm{i}\,\theta}-\texttt{Cos}[\theta]\right]^{2}}{\pi^{2}}}\right]\right\}\right\}$$

```
RevolutionPlot3D[
```

$$-\text{ArcSin}\left[\frac{1}{2}\sqrt{-\frac{4\,i\,\text{ArcSinh}\left[e^{i\theta}-\text{Cos}[\theta]\right]}{\pi}+\frac{\text{ArcSinh}\left[e^{i\theta}-\text{Cos}[\theta]\right]^2}{\pi^2}}\right],\ \{\theta,\ -4\pi,\ 4\pi\}\Big]$$



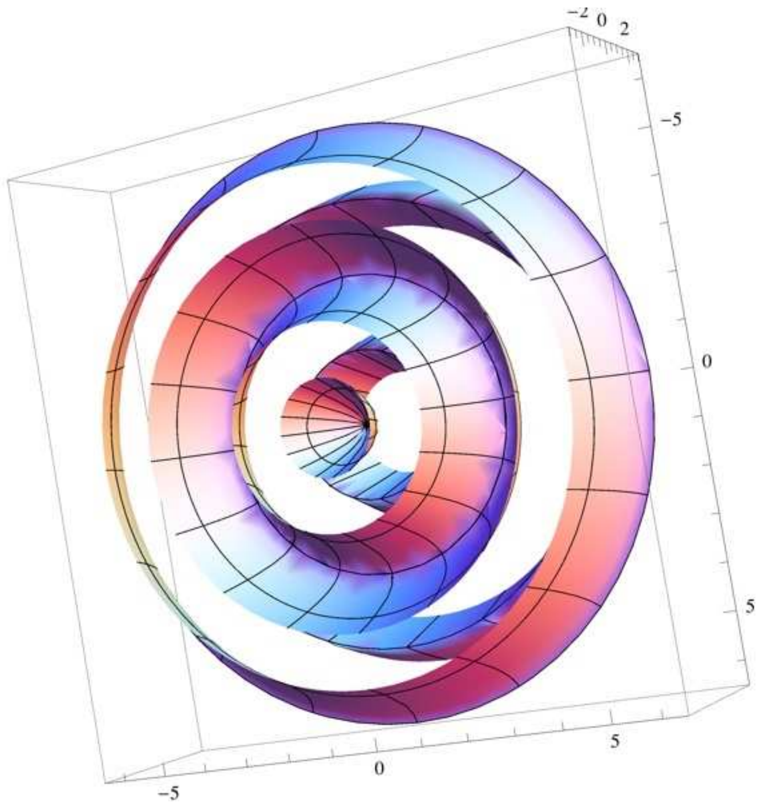
$$\text{RevolutionPlot3D}\left[\text{ArcSin}\left[\frac{1}{2}\sqrt{-\frac{4\,\mathbb{i}\,\text{ArcSinh}\left[e^{\mathbb{i}\,\theta}-\text{Cos}[\theta]\right]}{\pi}+\frac{\text{ArcSinh}\left[e^{\mathbb{i}\,\theta}-\text{Cos}[\theta]\right]^2}{\pi^2}}\,\right],\ \{\theta,-4\,\pi,4\,\pi\}\right]$$



$$\text{Solve}\left[e\,\hat{}\,(\mathbb{i}\,\theta) == \mathbb{i}\,\text{Sin}\left[2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right]+\text{Cos}\left[2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right],\ \theta\right]$$

$$\left\{\left\{\theta\rightarrow-\mathbb{i}\,\text{Log}\left[\text{Cos}\left[2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right]+\mathbb{i}\,\text{Sin}\left[2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right]\right]\right\}\right\}$$

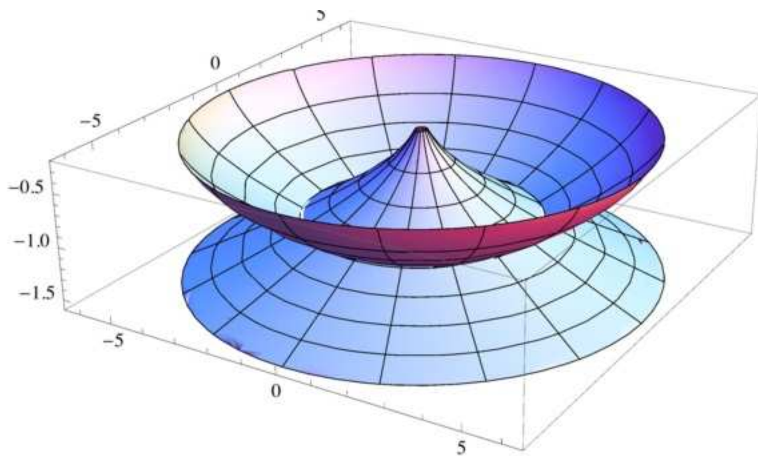
```
RevolutionPlot3D[
 -ⅈ Log[Cos[2 (π + Sqrt[π^2 - π^2 Sin[β]^2])] + ⅈ Sin[2 (π + Sqrt[π^2 - π^2 Sin[β]^2])]], {β, -2 π, 2 π}]
```



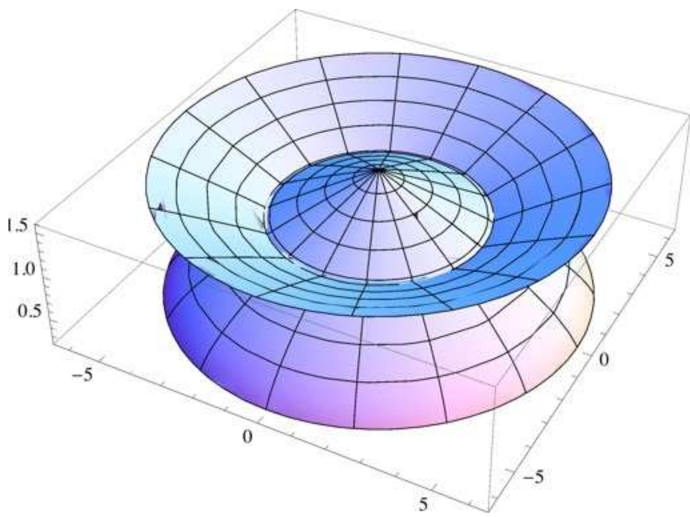
```
Solve[ⅇ^(ⅈ θ) == ⅈ Sin[2 (π + Sqrt[π^2 - π^2 Sin[β]^2])] + Cos[2 (π + Sqrt[π^2 - π^2 Sin[β]^2])], β]
```

```
{{β → -ArcSin[
    Sqrt[(1 - 1/π^2 ArcCos[((ⅈ Sqrt[(-1 + 2 Cos[θ] - Cos[θ]^2 + 2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] + Sin[θ]^2)]) /
          (2 Sqrt[Cos[θ] + ⅈ Sin[θ]]) + (ⅈ Cos[θ] Sqrt[(-1 + 2 Cos[θ] - Cos[θ]^2 + 2 ⅈ Sin[θ] -
               2 ⅈ Cos[θ] Sin[θ] + Sin[θ]^2)]) / (2 Sqrt[Cos[θ] + ⅈ Sin[θ]]) -
          (Sin[θ] Sqrt[(-1 + 2 Cos[θ] - Cos[θ]^2 + 2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] + Sin[θ]^2)]) /
          (2 Sqrt[Cos[θ] + ⅈ Sin[θ]])) / (-1 + Cos[θ] + ⅈ Sin[θ])]^2)]]},
  {β → ArcSin[Sqrt[(1 - 1/π^2 ArcCos[((ⅈ Sqrt[(-1 + 2 Cos[θ] - Cos[θ]^2 + 2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] +
               Sin[θ]^2)]) / (2 Sqrt[Cos[θ] + ⅈ Sin[θ]]) + (ⅈ Cos[θ] Sqrt[(-1 + 2 Cos[θ] - Cos[θ]^2 +
               2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] + Sin[θ]^2)]) / (2 Sqrt[Cos[θ] + ⅈ Sin[θ]]) -
          (Sin[θ] Sqrt[(-1 + 2 Cos[θ] - Cos[θ]^2 + 2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] + Sin[θ]^2)]) /
          (2 Sqrt[Cos[θ] + ⅈ Sin[θ]])) / (-1 + Cos[θ] + ⅈ Sin[θ])]^2)]]},
  {β → -ArcSin[Sqrt[(1 - 1/π^2 ArcCos[(-(ⅈ Sqrt[(-1 + 2 Cos[θ] - Cos[θ]^2 + 2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] +
               Sin[θ]^2)]) / (2 Sqrt[Cos[θ] + ⅈ Sin[θ]]) - (ⅈ Cos[θ] Sqrt[(-1 + 2 Cos[θ] - Cos[θ]^2 +
               2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] + Sin[θ]^2)]) / (2 Sqrt[Cos[θ] + ⅈ Sin[θ]]) +
          (Sin[θ] Sqrt[(-1 + 2 Cos[θ] - Cos[θ]^2 + 2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] + Sin[θ]^2)]) /
          (2 Sqrt[Cos[θ] + ⅈ Sin[θ]])) / (-1 + Cos[θ] + ⅈ Sin[θ])]^2)]]},
  {β → ArcSin[Sqrt[(1 - 1/π^2 ArcCos[(-(ⅈ Sqrt[(-1 + 2 Cos[θ] - Cos[θ]^2 + 2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] +
               Sin[θ]^2)]) / (2 Sqrt[Cos[θ] + ⅈ Sin[θ]]) - (ⅈ Cos[θ] Sqrt[(-1 + 2 Cos[θ] - Cos[θ]^2 +
               2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] + Sin[θ]^2)]) / (2 Sqrt[Cos[θ] + ⅈ Sin[θ]]) +
          (Sin[θ] Sqrt[(-1 + 2 Cos[θ] - Cos[θ]^2 + 2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] + Sin[θ]^2)]) /
          (2 Sqrt[Cos[θ] + ⅈ Sin[θ]])) / (-1 + Cos[θ] + ⅈ Sin[θ])]^2)]]}}
```

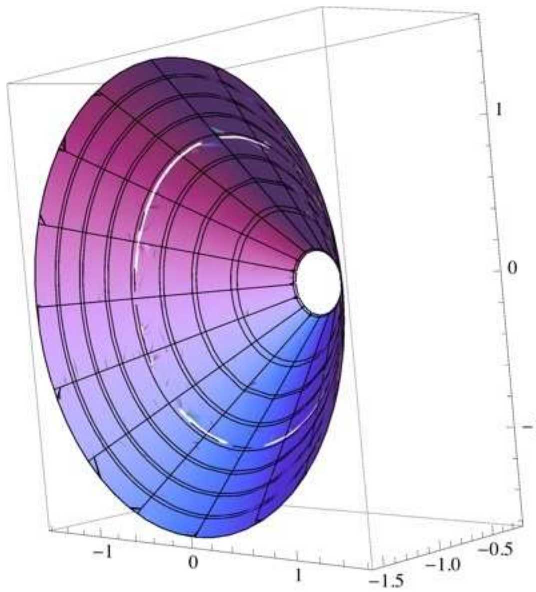
```
RevolutionPlot3D[
 -ArcSin[√(1 - 1/π^2 ArcCos[((ⅈ √(-1 + 2 Cos[θ] - Cos[θ]^2 + 2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] + Sin[θ]^2)) /
        (2 √(Cos[θ] + ⅈ Sin[θ])) + (ⅈ Cos[θ] √(-1 + 2 Cos[θ] - Cos[θ]^2 + 2 ⅈ Sin[θ] -
            2 ⅈ Cos[θ] Sin[θ] + Sin[θ]^2)) / (2 √(Cos[θ] + ⅈ Sin[θ])) -
        (Sin[θ] √(-1 + 2 Cos[θ] - Cos[θ]^2 + 2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] + Sin[θ]^2)) /
        (2 √(Cos[θ] + ⅈ Sin[θ]))) / (-1 + Cos[θ] + ⅈ Sin[θ])]^2)], {θ, -2 π, 2 π}]
```

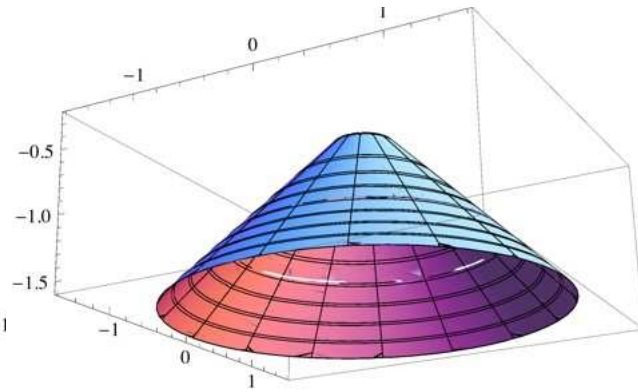


```
RevolutionPlot3D[
 ArcSin[√(1 - 1/π² ArcCos[((ⅈ √(-1 + 2 Cos[θ] - Cos[θ]² + 2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] + Sin[θ]²)) /
       (2 √(Cos[θ] + ⅈ Sin[θ])) + (ⅈ Cos[θ] √(-1 + 2 Cos[θ] - Cos[θ]² +
            2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] + Sin[θ]²)) / (2 √(Cos[θ] + ⅈ Sin[θ])) -
       (Sin[θ] √(-1 + 2 Cos[θ] - Cos[θ]² + 2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] + Sin[θ]²)) /
       (2 √(Cos[θ] + ⅈ Sin[θ]))) / (-1 + Cos[θ] + ⅈ Sin[θ])]²)], {θ, -2 π, 2 π}]
```




```
RevolutionPlot3D[{ArcSin[
   √(1 - 1/π² ArcCos[(-(ⅈ √(-1 + 2 Cos[θ] - Cos[θ]² + 2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] + Sin[θ]²)) /
          (2 √(Cos[θ] + ⅈ Sin[θ])) - (ⅈ Cos[θ] √(-1 + 2 Cos[θ] - Cos[θ]² +
               2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] + Sin[θ]²)) / (2 √(Cos[θ] + ⅈ Sin[θ])) +
          (Sin[θ] √(-1 + 2 Cos[θ] - Cos[θ]² + 2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] + Sin[θ]²)) /
          (2 √(Cos[θ] + ⅈ Sin[θ]))) / (-1 + Cos[θ] + ⅈ Sin[θ])]²)],
  -ArcSin[√(1 - 1/π² ArcCos[(-(ⅈ √(-1 + 2 Cos[θ] - Cos[θ]² + 2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] +
                  Sin[θ]²)) / (2 √(Cos[θ] + ⅈ Sin[θ])) - (ⅈ Cos[θ] √(-1 + 2 Cos[θ] - Cos[θ]² +
                 2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] + Sin[θ]²)) / (2 √(Cos[θ] + ⅈ Sin[θ])) +
            (Sin[θ] √(-1 + 2 Cos[θ] - Cos[θ]² + 2 ⅈ Sin[θ] - 2 ⅈ Cos[θ] Sin[θ] + Sin[θ]²)) /
            (2 √(Cos[θ] + ⅈ Sin[θ]))) / (-1 + Cos[θ] + ⅈ Sin[θ])]²)]}, {θ, -2 π, 2 π}]
```





$$\theta = \frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}$$

- **The Forms**

$$e\text{^}\left(i\,\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right) = i\,\text{Sin}[\theta] + \text{Cos}[\theta]$$

$$e\text{^}\left(i\,\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right) = i\,\text{Sin}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right] + \text{Cos}[\theta]$$

$$e\text{^}\left(i\,\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right) = i\,\text{Sin}[\theta] + \text{Cos}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right]$$

$$e\text{^}(i\,\theta) == i\,\text{Sin}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right] + \text{Cos}[\theta]$$

$$e\text{^}(i\,\theta) == i\,\text{Sin}[\theta] + \text{Cos}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right]$$

$$e\text{^}(i\,\theta) == i\,\text{Sin}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right] + \text{Cos}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right]$$

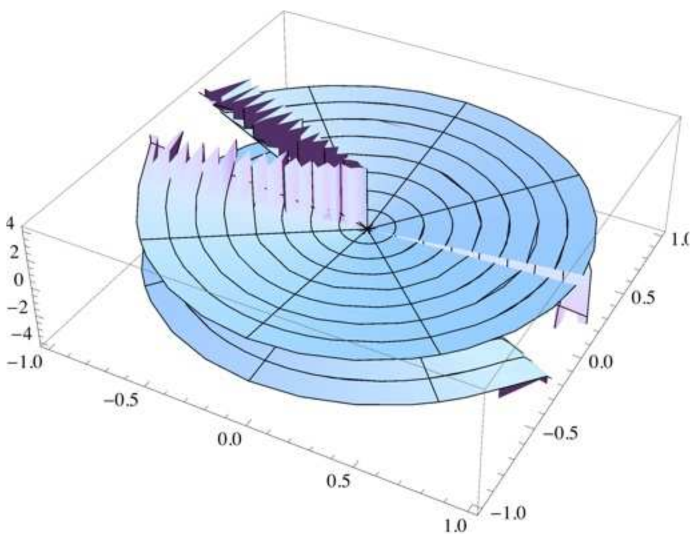
The Solutions

$$\textbf{Solve}\left[e\,\hat{}\left(i\,\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right) == i\,\text{Sin}[\theta]+\text{Cos}[\theta],\ r\right]$$

$$\left\{\left\{r\to-\frac{\pi\eta}{\sqrt{\pi^2+\text{Log}\left[-\sqrt{\text{Cos}[\theta]+i\,\text{Sin}[\theta]}\right]^2}}\right\},\left\{r\to\frac{\pi\eta}{\sqrt{\pi^2+\text{Log}\left[-\sqrt{\text{Cos}[\theta]+i\,\text{Sin}[\theta]}\right]^2}}\right\},$$

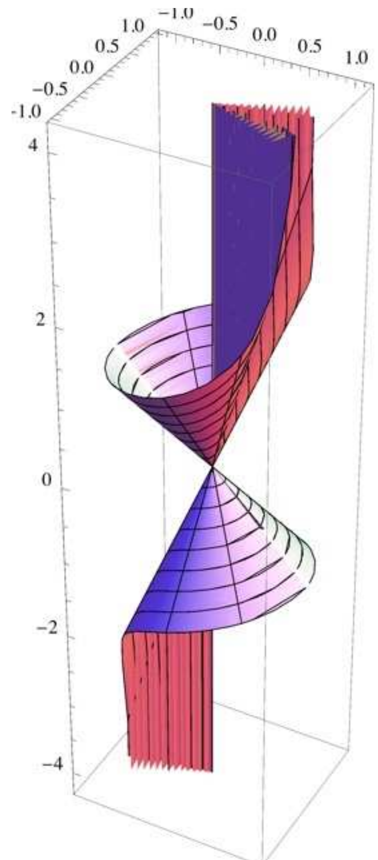
$$\left\{r\to-\frac{2\pi\eta}{\sqrt{4\pi^2+\text{Log}[\text{Cos}[\theta]+i\,\text{Sin}[\theta]]^2}}\right\},\left\{r\to\frac{2\pi\eta}{\sqrt{4\pi^2+\text{Log}[\text{Cos}[\theta]+i\,\text{Sin}[\theta]]^2}}\right\}\right\}$$

$$\textbf{RevolutionPlot3D}\left[-\frac{\pi\eta}{\sqrt{\pi^2+\text{Log}\left[-\sqrt{\text{Cos}[\theta]+i\,\text{Sin}[\theta]}\right]^2}},\ \{\eta,-1,1\},\ \{\theta,-2\pi,2\pi\}\right]$$



$$\textbf{RevolutionPlot3D}\left[\left\{-\frac{2\pi\eta}{\sqrt{4\pi^2+\text{Log}[\text{Cos}[\theta]+i\,\text{Sin}[\theta]]^2}},\ -\frac{\pi\eta}{\sqrt{\pi^2+\text{Log}\left[-\sqrt{\text{Cos}[\theta]+i\,\text{Sin}[\theta]}\right]^2}}\right\},\right.$$

$$\left.\{\eta,-1,1\},\ \{\theta,-2\pi,2\pi\}\right]$$



$$\text{RevolutionPlot3D}\left[-\frac{2\pi\eta}{\sqrt{4\pi^2+\text{Log}[\text{Cos}[\theta]+\mathbb{i}\,\text{Sin}[\theta]]^2}},\ \{\eta,-1,1\},\ \{\theta,-2\pi,2\pi\}\right]$$

$$\text{Solve}\left[e^{\wedge}\left(\mathbb{i}\,\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right)==\mathbb{i}\,\text{Sin}[\theta]+\text{Cos}[\theta],\ \eta\right]$$

$$\left\{\left\{\eta\to-\frac{\mathbb{i}\sqrt{-\pi^2 r^2-r^2\,\text{Log}\left[-\sqrt{\text{Cos}[\theta]+\mathbb{i}\,\text{Sin}[\theta]}\right]^2}}{\pi}\right\},\right.$$

$$\left\{\eta\to\frac{\mathbb{i}\sqrt{-\pi^2 r^2-r^2\,\text{Log}\left[-\sqrt{\text{Cos}[\theta]+\mathbb{i}\,\text{Sin}[\theta]}\right]^2}}{\pi}\right\},$$

$$\left.\left\{\eta\to-\frac{\mathbb{i}\sqrt{-4\pi^2 r^2-r^2\,\text{Log}[\text{Cos}[\theta]+\mathbb{i}\,\text{Sin}[\theta]]^2}}{2\pi}\right\},\ \left\{\eta\to\frac{\mathbb{i}\sqrt{-4\pi^2 r^2-r^2\,\text{Log}[\text{Cos}[\theta]+\mathbb{i}\,\text{Sin}[\theta]]^2}}{2\pi}\right\}\right\}$$

$$\text{Solve}\left[e^{\wedge}\left(i\,\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right)==i\,\text{Sin}[\theta]+\text{Cos}[\theta],\,\theta\right]$$

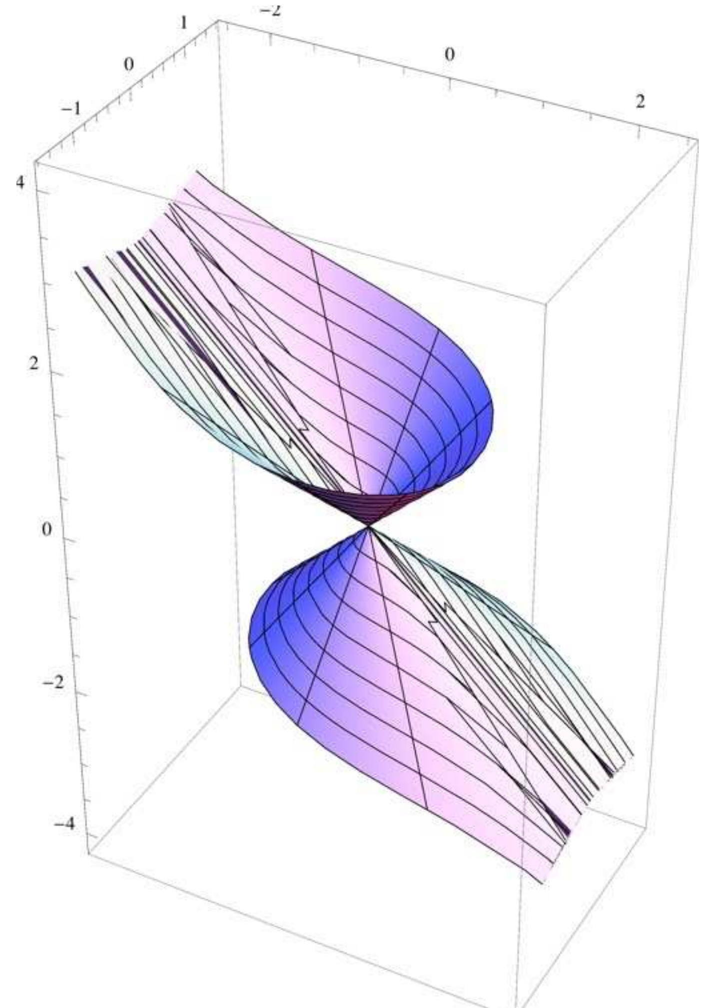
$$\left\{\left\{\theta\to-\text{ArcCos}\left[\frac{1}{2}\,e^{-\frac{2\,i\,\pi\sqrt{r^2\left(r^2-\eta^2\right)}}{r^2}}\left(1+e^{\frac{4\,i\,\pi\sqrt{r^2\left(r^2-\eta^2\right)}}{r^2}}\right)\right]\right\},\left\{\theta\to\text{ArcCos}\left[\frac{1}{2}\,e^{-\frac{2\,i\,\pi\sqrt{r^2\left(r^2-\eta^2\right)}}{r^2}}\left(1+e^{\frac{4\,i\,\pi\sqrt{r^2\left(r^2-\eta^2\right)}}{r^2}}\right)\right]\right\}\right\}$$

$$\text{Solve}\left[e^{\wedge}\left(i\,\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right)==i\,\text{Sin}\left[\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right]+\text{Cos}[\theta],\,r\right]$$

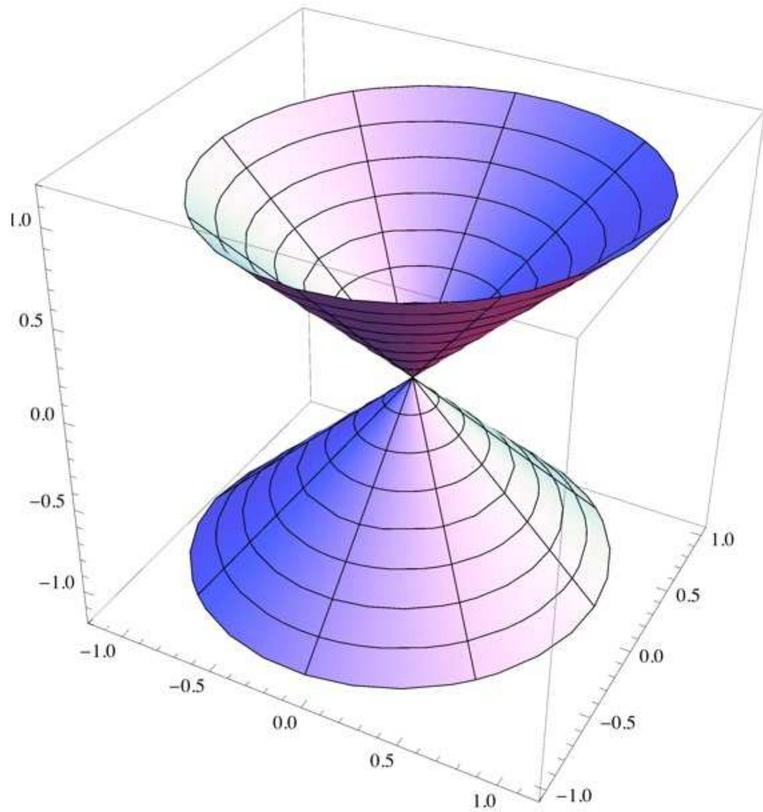
$$\left\{\left\{r\to-\frac{\pi\,\eta}{\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}}\right\},\left\{r\to\frac{\pi\,\eta}{\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}}\right\},\right.$$

$$\left.\left\{r\to-\frac{\pi\,\eta}{\sqrt{\pi^2-\text{ArcCos}\left[\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}}\right\},\left\{r\to\frac{\pi\,\eta}{\sqrt{\pi^2-\text{ArcCos}\left[\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}}\right\}\right\}$$

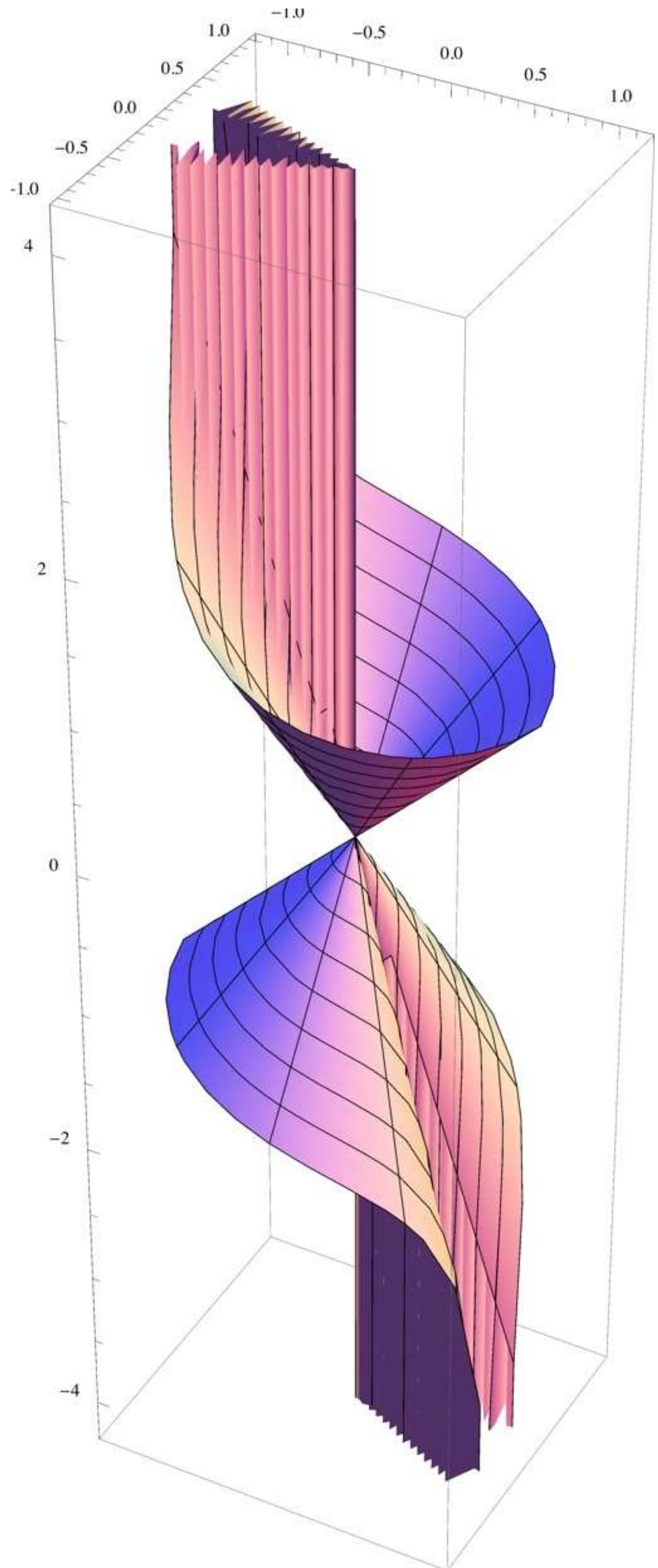
RevolutionPlot3D$\left[\left\{-\frac{\pi\,\eta}{\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}},\ \frac{\pi\,\eta}{\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}}\right\},\ \{\eta,-1,1\},\ \{\theta,-2\pi,2\pi\}\right]$



$$\text{RevolutionPlot3D}\left[\left\{-\frac{\pi\,\eta}{\sqrt{\pi^2-\text{ArcCos}\left[\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}},\ \frac{\pi\,\eta}{\sqrt{\pi^2-\text{ArcCos}\left[\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}}\right\},\ \{\eta,-1,1\},\ \{\theta,-2\pi,2\pi\}\right]$$



$$\text{RevolutionPlot3D}\left[\left\{-\frac{\pi\,\eta}{\sqrt{\pi^2-\text{ArcCos}\left[\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}},\ \frac{\pi\,\eta}{\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}}\right\},\ \{\eta,-1,1\},\ \{\theta,-2\pi,2\pi\}\right]$$

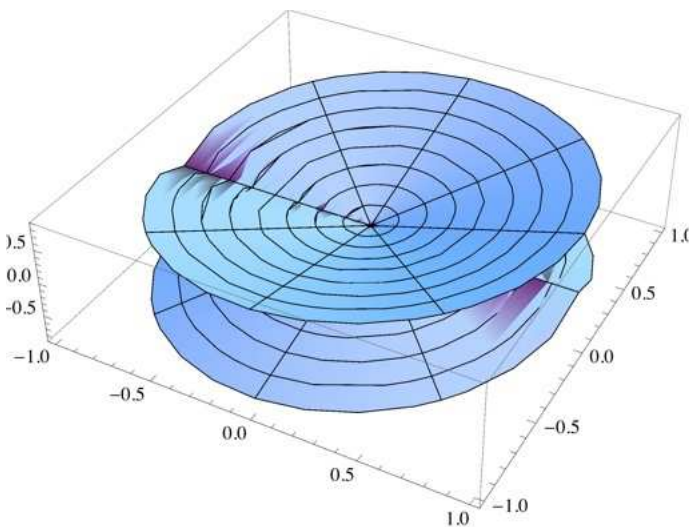

-1.0
-0.5
0.0
0.5
1.0
1.0
0.5
0.0
-0.5
-1.0
4
2
0
-2
-4

$$\textbf{Solve}\left[e^{\wedge}\left(i\,\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right) == i\,\textbf{Sin}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right]+\textbf{Cos}[\theta],\ \eta\right]$$

$$\left\{\left\{\eta\to-\frac{r\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}}{\pi}\right\},\ \left\{\eta\to\frac{r\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}}{\pi}\right\},\right.$$

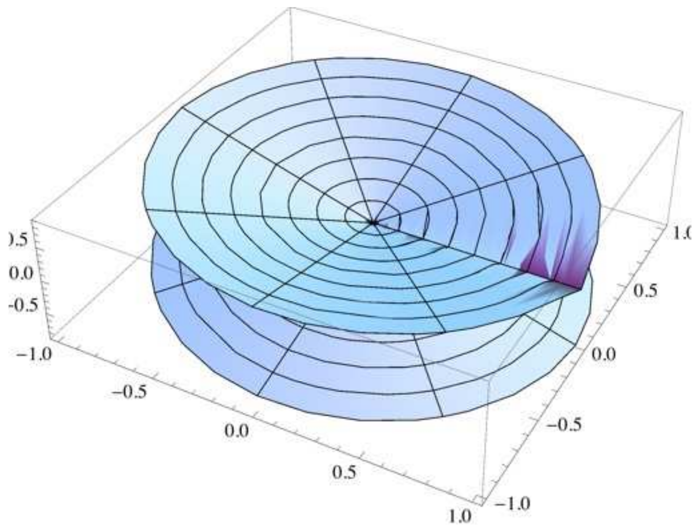
$$\left.\left\{\eta\to-\frac{r\sqrt{\pi^2-\text{ArcCos}\left[\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}}{\pi}\right\},\ \left\{\eta\to\frac{r\sqrt{\pi^2-\text{ArcCos}\left[\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}}{\pi}\right\}\right\}$$

$$\textbf{RevolutionPlot3D}\left[-\frac{r\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}}{\pi},\ \{r,-1,1\},\ \{\theta,-2\pi,2\pi\}\right]$$

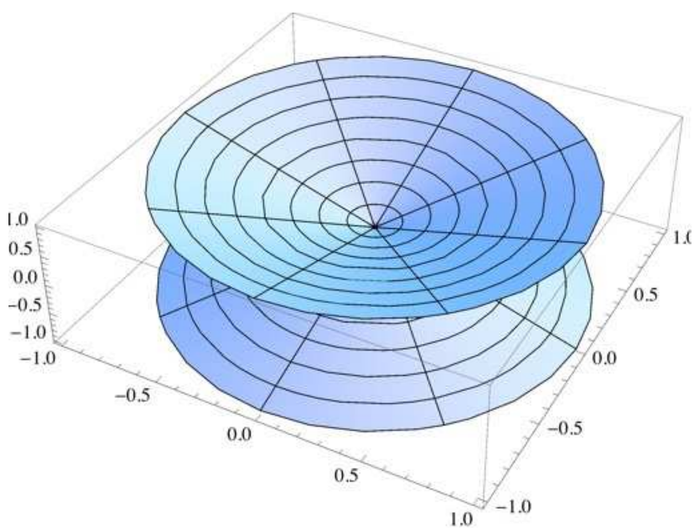


```
RevolutionPlot3D[ r Sqrt[π^2 - ArcCos[-Sqrt[1+Cos[θ]]/Sqrt[2]]^2] / π , {r, -1, 1}, {θ, -2 π, 2 π}]
```

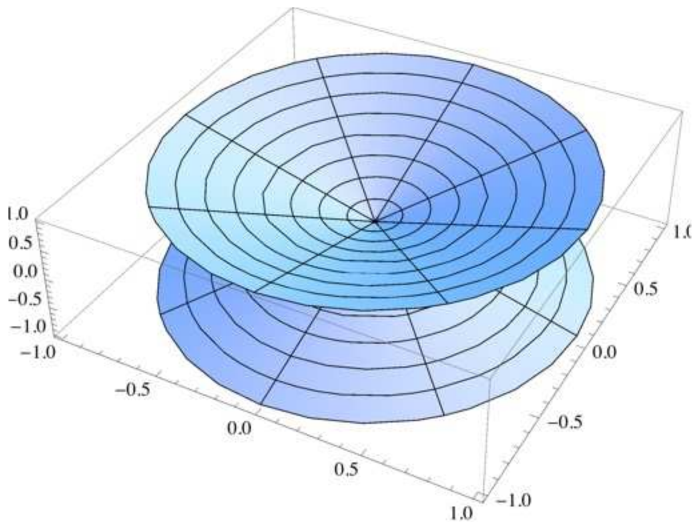
$$\text{RevolutionPlot3D}\left[\frac{r\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}}{\pi}, \{r,-1,1\}, \{\theta,-2\pi,2\pi\}\right]$$



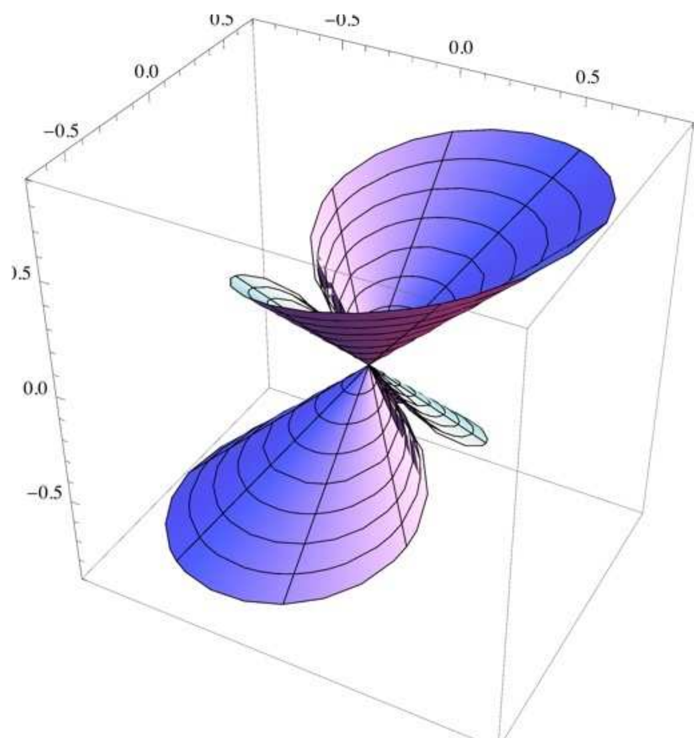
$$\text{RevolutionPlot3D}\left[-\frac{r\sqrt{\pi^2-\text{ArcCos}\left[\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}}{\pi}, \{r,-1,1\}, \{\theta,-2\pi,2\pi\}\right]$$



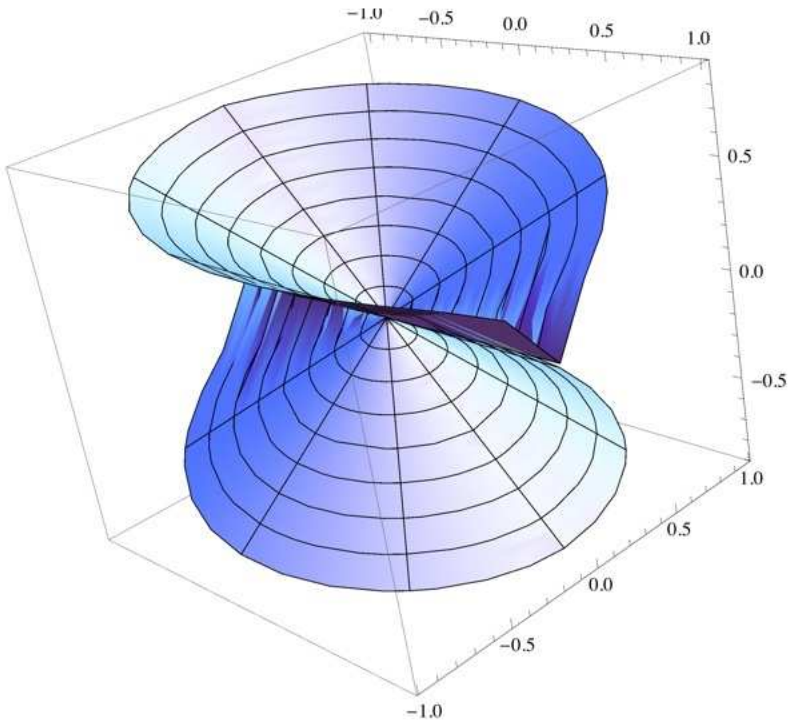
$$\text{RevolutionPlot3D}\left[\frac{r\sqrt{\pi^2-\text{ArcCos}\left[\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}}{\pi},\{r,-1,1\},\{\theta,-2\pi,2\pi\}\right]$$



$$\text{RevolutionPlot3D}\left[\left\{-\frac{r\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}}{\pi},\frac{r\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}}{\pi}\right\},\{r,-1,1\},\{\theta,-2\pi,2\pi\}\right]$$



$$\text{RevolutionPlot3D}\left[\left\{\frac{r\sqrt{\pi^2-\text{ArcCos}\left[\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}}{\pi},\ \frac{r\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1+\text{Cos}[\theta]}}{\sqrt{2}}\right]^2}}{\pi}\right\},\ \{r,-1,1\},\ \{\theta,-2\pi,2\pi\}\right]$$

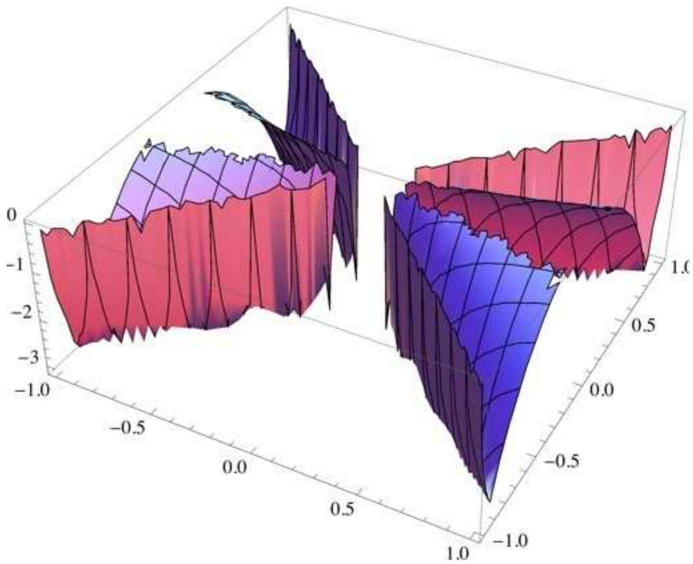


$$\text{Solve}\left[e^{\wedge}\left(i\,\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right) == i\,\text{Sin}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right]+\text{Cos}[\theta],\ \theta\right]$$

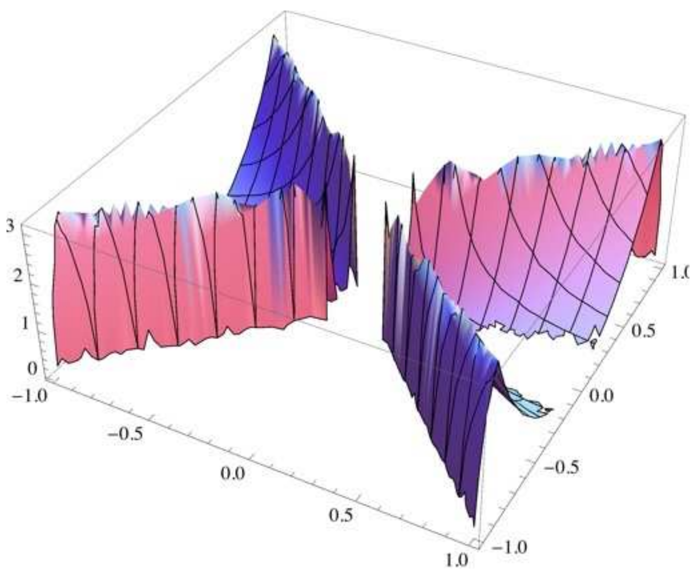
$$\left\{\left\{\theta\to-\text{ArcCos}\left[e^{\frac{2 i\pi\sqrt{r^2\left(r^2-\eta^2\right)}}{r^2}}-i\,\text{Sin}\left[\frac{2\pi\left(r^2+\sqrt{r^2\left(r^2-\eta^2\right)}\right)}{r^2}\right]\right]\right\},\right.$$

$$\left.\left\{\theta\to\text{ArcCos}\left[e^{\frac{2 i\pi\sqrt{r^2\left(r^2-\eta^2\right)}}{r^2}}-i\,\text{Sin}\left[\frac{2\pi\left(r^2+\sqrt{r^2\left(r^2-\eta^2\right)}\right)}{r^2}\right]\right]\right\}\right\}$$

$$\text{Plot3D}\left[-\text{ArcCos}\left[e^{\frac{2\,i\,\pi\sqrt{r^2\left(r^2-\eta^2\right)}}{r^2}}-i\,\text{Sin}\left[\frac{2\,\pi\left(r^2+\sqrt{r^2\left(r^2-\eta^2\right)}\right)}{r^2}\right]\right],\{r,-1,1\},\{\eta,-1,1\}\right]$$

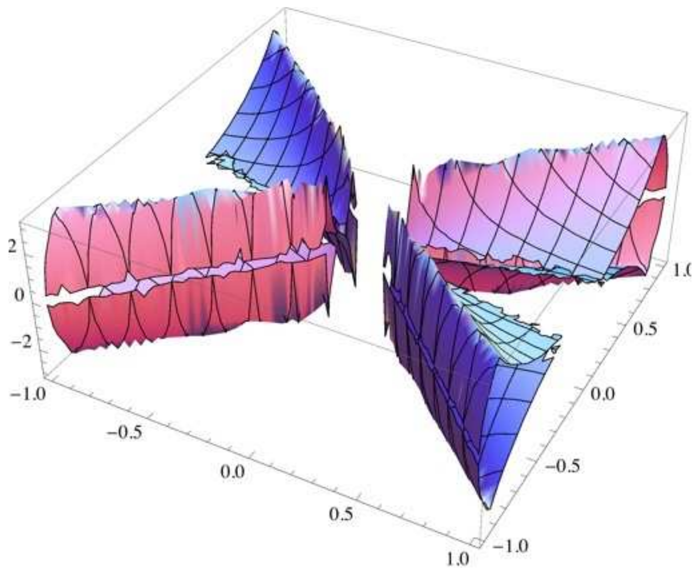


$$\text{Plot3D}\left[\text{ArcCos}\left[e^{\frac{2\,i\,\pi\sqrt{r^2\left(r^2-\eta^2\right)}}{r^2}}-i\,\text{Sin}\left[\frac{2\,\pi\left(r^2+\sqrt{r^2\left(r^2-\eta^2\right)}\right)}{r^2}\right]\right],\{r,-1,1\},\{\eta,-1,1\}\right]$$



$$\text{Plot3D}\left[\left\{-\text{ArcCos}\left[e^{\frac{2\,i\,\pi\sqrt{r^2\left(r^2-\eta^2\right)}}{r^2}}-i\,\text{Sin}\left[\frac{2\,\pi\left(r^2+\sqrt{r^2\left(r^2-\eta^2\right)}\right)}{r^2}\right]\right],\right.\right.$$

$$\left.\left.\text{ArcCos}\left[e^{\frac{2\,i\,\pi\sqrt{r^2\left(r^2-\eta^2\right)}}{r^2}}-i\,\text{Sin}\left[\frac{2\,\pi\left(r^2+\sqrt{r^2\left(r^2-\eta^2\right)}\right)}{r^2}\right]\right]\right\},\ \{r,-1,1\},\ \{\eta,-1,1\}\right]$$



$$\text{Solve}\left[e\text{^}\left(i\,\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right)==i\,\text{Sin}[\theta]+\text{Cos}\left[\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right],\ r\right]$$

$$\left\{\left\{r\to-\frac{\pi\,\eta}{\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1-\sqrt{1-\text{Sin}[\theta]^2}}}{\sqrt{2}}\right]^2}}\right\},\ \left\{r\to\frac{\pi\,\eta}{\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1-\sqrt{1-\text{Sin}[\theta]^2}}}{\sqrt{2}}\right]^2}}\right\},\right.$$

$$\left\{r\to-\frac{\pi\,\eta}{\sqrt{\pi^2-\text{ArcCos}\left[\frac{\sqrt{1-\sqrt{1-\text{Sin}[\theta]^2}}}{\sqrt{2}}\right]^2}}\right\},\ \left\{r\to\frac{\pi\,\eta}{\sqrt{\pi^2-\text{ArcCos}\left[\frac{\sqrt{1-\sqrt{1-\text{Sin}[\theta]^2}}}{\sqrt{2}}\right]^2}}\right\},$$

$$\left\{r\to-\frac{\pi\,\eta}{\sqrt{\pi^2-\text{ArcCos}\left[-\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}\right]^2}}\right\},\ \left\{r\to\frac{\pi\,\eta}{\sqrt{\pi^2-\text{ArcCos}\left[-\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}\right]^2}}\right\},$$

$$\left.\left\{r\to-\frac{\pi\,\eta}{\sqrt{\pi^2-\text{ArcCos}\left[\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}\right]^2}}\right\},\ \left\{r\to\frac{\pi\,\eta}{\sqrt{\pi^2-\text{ArcCos}\left[\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}\right]^2}}\right\}\right\}$$

$$\textbf{Solve}\left[e^{\wedge}\left(i\,\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right) == i\,\textbf{Sin}[\theta]+\textbf{Cos}\left[\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right],\ \eta\right]$$

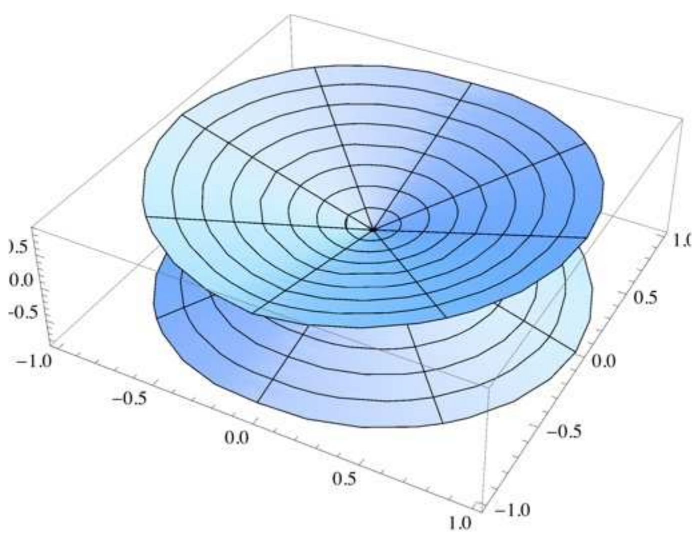
$$\left\{\left\{\eta\to-\frac{r\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1-\sqrt{1-\text{Sin}[\theta]^2}}}{\sqrt{2}}\right]^2}}{\pi}\right\},\ \left\{\eta\to\frac{r\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1-\sqrt{1-\text{Sin}[\theta]^2}}}{\sqrt{2}}\right]^2}}{\pi}\right\},\right.$$

$$\left\{\eta\to-\frac{r\sqrt{\pi^2-\text{ArcCos}\left[\frac{\sqrt{1-\sqrt{1-\text{Sin}[\theta]^2}}}{\sqrt{2}}\right]^2}}{\pi}\right\},\ \left\{\eta\to\frac{r\sqrt{\pi^2-\text{ArcCos}\left[\frac{\sqrt{1-\sqrt{1-\text{Sin}[\theta]^2}}}{\sqrt{2}}\right]^2}}{\pi}\right\},$$

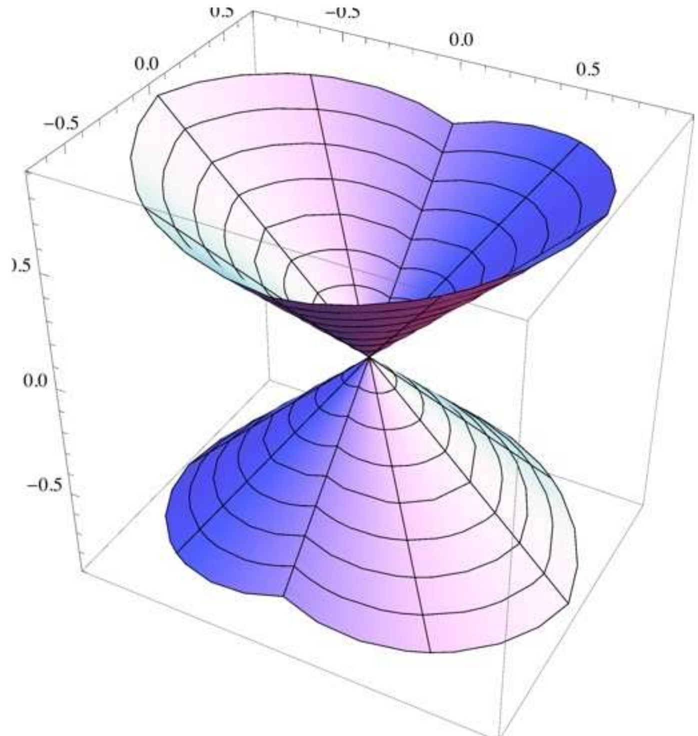
$$\left\{\eta\to-\frac{r\sqrt{\pi^2-\text{ArcCos}\left[-\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}\right]^2}}{\pi}\right\},\ \left\{\eta\to\frac{r\sqrt{\pi^2-\text{ArcCos}\left[-\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}\right]^2}}{\pi}\right\},$$

$$\left.\left\{\eta\to-\frac{r\sqrt{\pi^2-\text{ArcCos}\left[\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}\right]^2}}{\pi}\right\},\ \left\{\eta\to\frac{r\sqrt{\pi^2-\text{ArcCos}\left[\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}\right]^2}}{\pi}\right\}\right\}$$

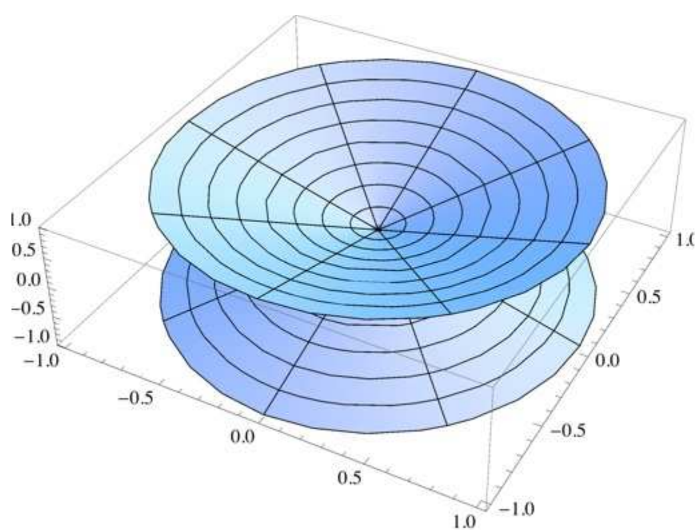
$$\textbf{RevolutionPlot3D}\left[-\frac{r\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1-\sqrt{1-\text{Sin}[\theta]^2}}}{\sqrt{2}}\right]^2}}{\pi},\ \{r,-1,1\},\ \{\theta,-2\,\pi,2\,\pi\}\right]$$



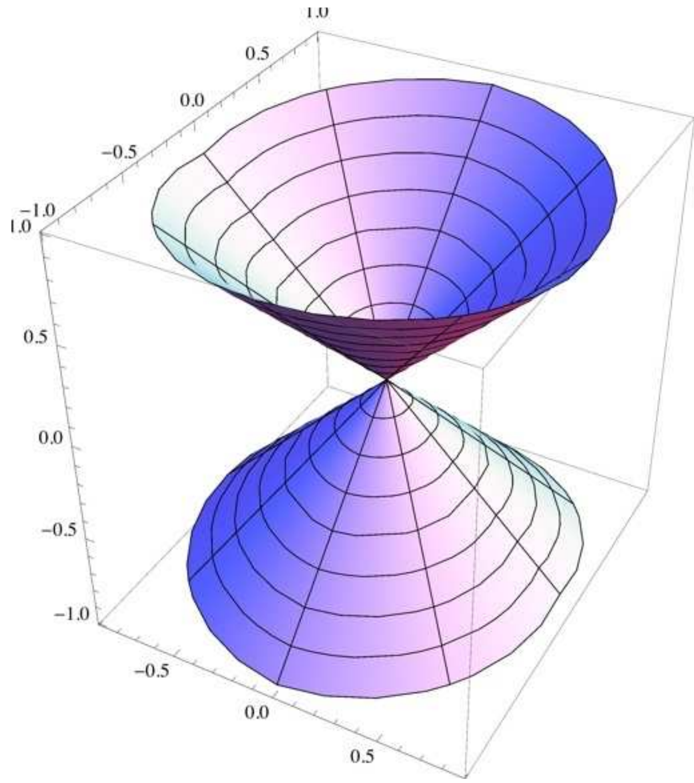
```
RevolutionPlot3D[{-(r Sqrt[π^2 - ArcCos[-Sqrt[1 - Sqrt[1 - Sin[θ]^2]]/Sqrt[2]]^2])/π, (r Sqrt[π^2 - ArcCos[-Sqrt[1 - Sqrt[1 - Sin[θ]^2]]/Sqrt[2]]^2])/π},
 {r, -1, 1}, {θ, -2 π, 2 π}]
```



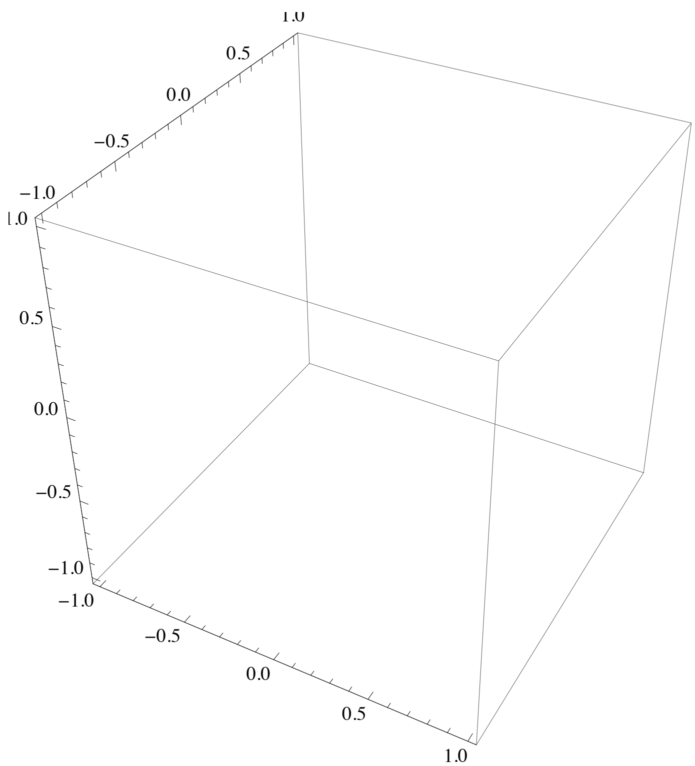

```
RevolutionPlot3D[-(r Sqrt[π^2 - ArcCos[Sqrt[1 - Sqrt[1 - Sin[θ]^2]]/Sqrt[2]]^2])/π, {r, -1, 1}, {θ, -2 π, 2 π}]
```



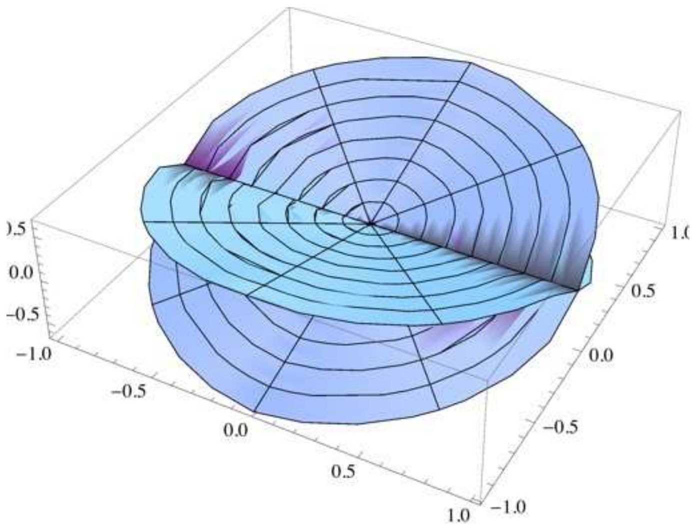
$$\text{RevolutionPlot3D}\left[\left\{-\frac{r\sqrt{\pi^2 - \text{ArcCos}\left[\frac{\sqrt{1-\sqrt{1-\text{Sin}[\theta]^2}}}{\sqrt{2}}\right]^2}}{\pi}, \frac{r\sqrt{\pi^2 - \text{ArcCos}\left[\frac{\sqrt{1-\sqrt{1-\text{Sin}[\theta]^2}}}{\sqrt{2}}\right]^2}}{\pi}\right\}, \{r, -1, 1\}, \{\theta, -2\pi, 2\pi\}\right]$$



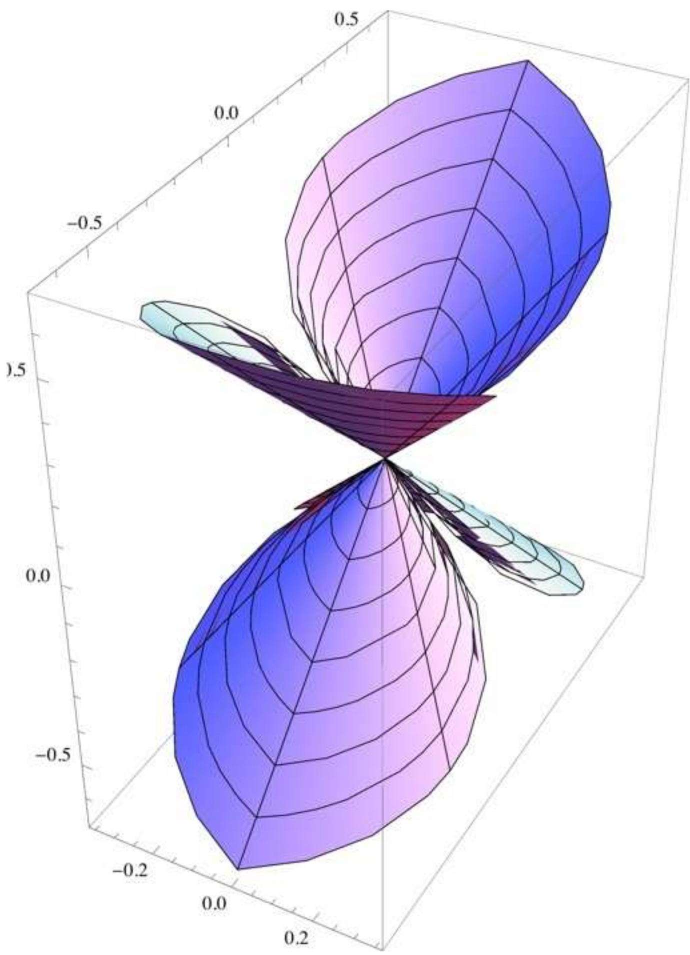
$$\text{RevolutionPlot3D}\left[\left\{-\frac{r\sqrt{\pi^2-\text{ArcCos}\left[\frac{\sqrt{1-\sqrt{1-\text{Sin}[\theta]^2}}}{\sqrt{2}}\right]^2}}{\pi}, \frac{r\sqrt{\pi^2-\text{ArcCos}\left[\frac{\sqrt{1-\sqrt{1-\text{Sin}[\theta]^2}}}{\sqrt{2}}\right]^2}}{\pi}, -\frac{r\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1-\sqrt{1-\text{Sin}[\theta]^2}}}{\sqrt{2}}\right]^2}}{\pi}, \frac{r\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1-\sqrt{1-\text{Sin}[\theta]^2}}}{\sqrt{2}}\right]^2}}{\pi}\right\}, \{r, -1, 1\}, \{\theta, -2\pi, 2\pi\}\right]$$



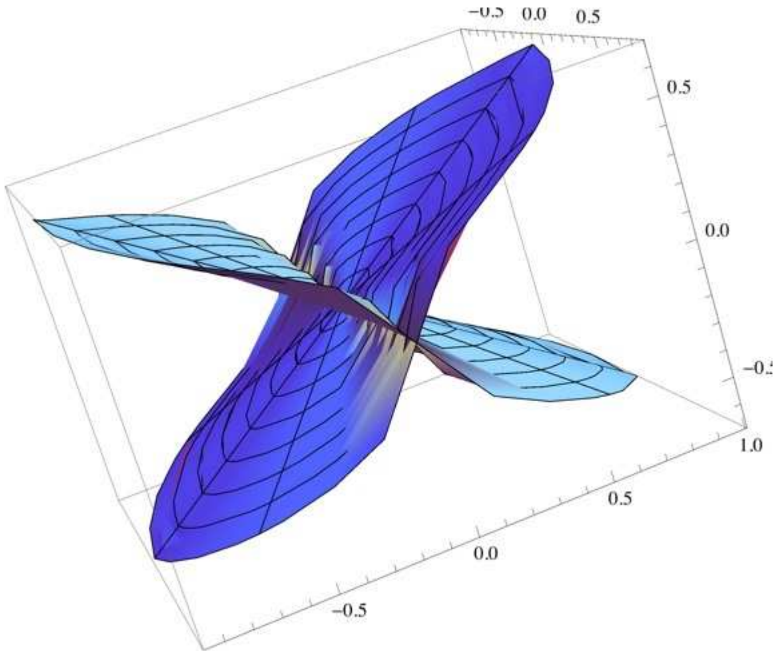
$$\text{RevolutionPlot3D}\left[-\frac{r\sqrt{\pi^2-\text{ArcCos}\left[-\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}\right]^2}}{\pi},\ \{r,-1,1\},\ \{\theta,-2\pi,2\pi\}\right]$$



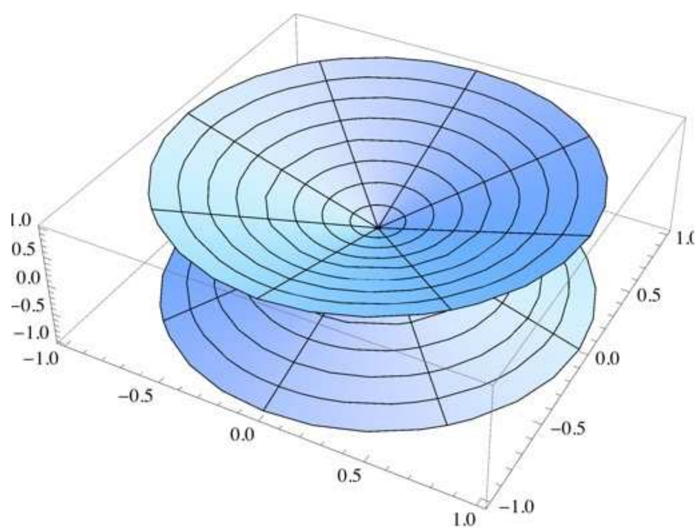
$$\text{RevolutionPlot3D}\left[\left\{\frac{r\sqrt{\pi^2 - \text{ArcCos}\left[-\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}\right]^2}}{\pi},\ -\frac{r\sqrt{\pi^2 - \text{ArcCos}\left[-\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}\right]^2}}{\pi}\right\}, \{r, -1, 1\}, \{\theta, -2\pi, 2\pi\}\right]$$



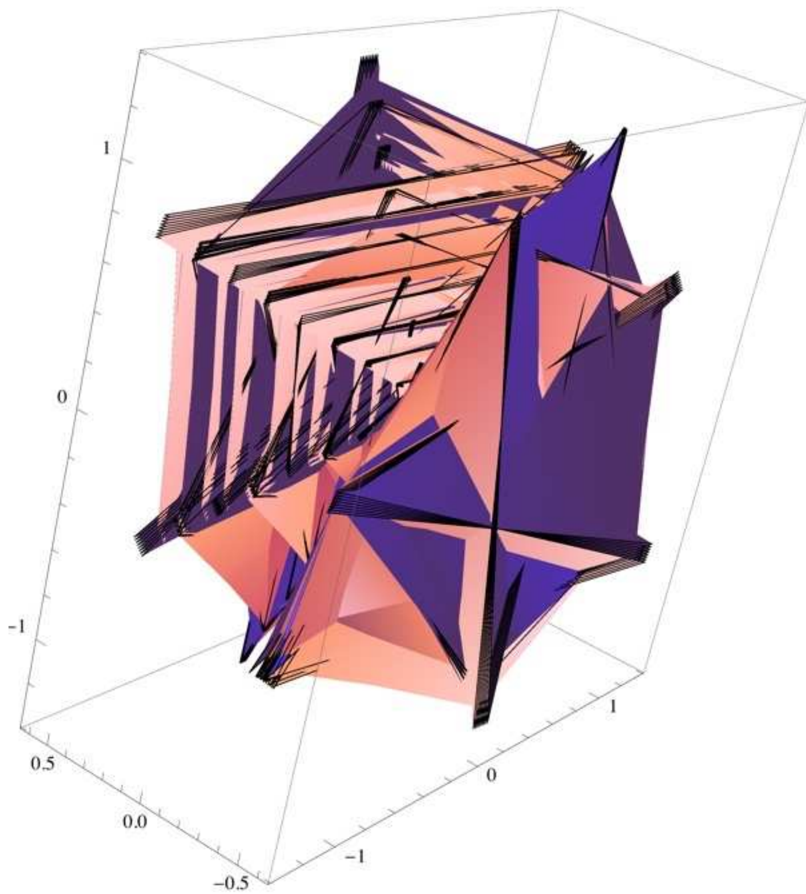
$$\text{RevolutionPlot3D}\left[\left\{\frac{r\sqrt{\pi^2-\text{ArcCos}\left[\frac{\sqrt{1-\sqrt{1-\text{Sin}[\theta]^2}}}{\sqrt{2}}\right]^2}}{\pi},\ \frac{r\sqrt{\pi^2-\text{ArcCos}\left[-\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}\right]^2}}{\pi}\right\},\right.$$
$$\left.\{r,-1,1\},\ \{\theta,-2\pi,2\pi\}\right]$$



$$\text{RevolutionPlot3D}\left[\frac{r\sqrt{\pi^2-\text{ArcCos}\left[\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}\right]^2}}{\pi},\ \{r,-1,1\},\ \{\theta,-2\pi,2\pi\}\right]$$



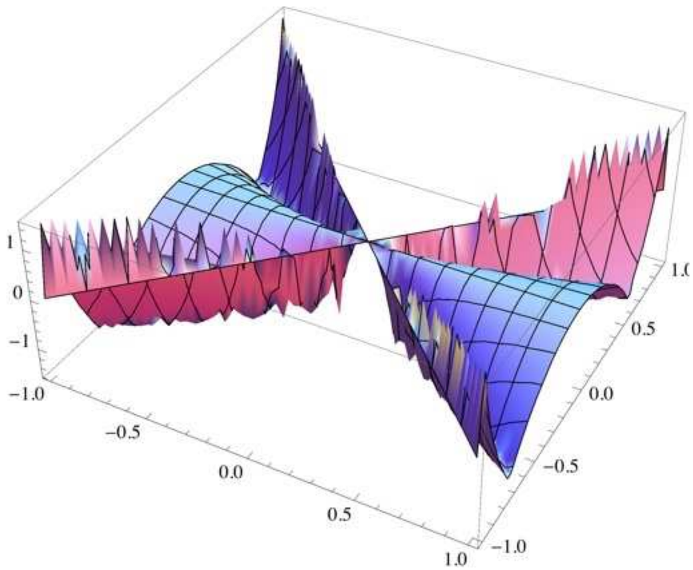
$$\text{RevolutionPlot3D}\left[\left\{\frac{r\sqrt{\pi^2 - \text{ArcCos}\left[\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}\right]^2}}{\pi}, \frac{r\sqrt{\pi^2 - \text{ArcCos}\left[\frac{\sqrt{1-\sqrt{1-\text{Sin}[\theta]^2}}}{\sqrt{2}}\right]^2}}{\pi}, \frac{r\sqrt{\pi^2 - \text{ArcCos}\left[-\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{1-\text{Sin}[\theta]^2}}\right]^2}}{\pi}\right\}, \{r, -1, 1\}, \{\theta, -80\,000\,\pi, 80\,000\,\pi\}\right]$$



$$\text{Solve}\left[e\wedge\left(i\,\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right) == i\,\text{Sin}[\theta] + \text{Cos}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right], \theta\right]$$

$$\left\{\left\{\theta \to -i\,\text{ArcSinh}\left[e^{\frac{2 i \pi\sqrt{r^2\left(r^2-\eta^2\right)}}{r^2}} - \text{Cos}\left[\frac{2\pi\left(r^2+\sqrt{r^2\left(r^2-\eta^2\right)}\right)}{r^2}\right]\right]\right\}\right\}$$

$$\text{Plot3D}\left[-i\,\text{ArcSinh}\left[e^{\frac{2 i \pi \sqrt{r^2 (r^2-\eta^2)}}{r^2}} - \text{Cos}\left[\frac{2\pi\left(r^2+\sqrt{r^2\left(r^2-\eta^2\right)}\right)}{r^2}\right]\right], \{r, -1, 1\}, \{\eta, -1, 1\}\right]$$



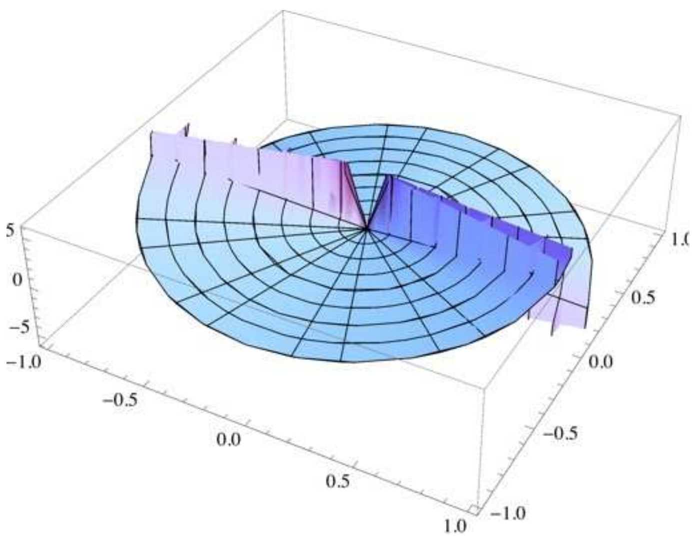
$$\text{Solve}\left[e^{\wedge}(i\,\theta) == i\,\text{Sin}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right] + \text{Cos}[\theta], r\right]$$

$$\left\{\left\{r \to -\frac{2\pi\eta}{\sqrt{-4 i \pi\,\text{ArcSinh}\left[e^{i\theta}-\text{Cos}[\theta]\right]+\text{ArcSinh}\left[e^{i\theta}-\text{Cos}[\theta]\right]^2}}\right\},\right.$$

$$\left.\left\{r \to \frac{2\pi\eta}{\sqrt{-4 i \pi\,\text{ArcSinh}\left[e^{i\theta}-\text{Cos}[\theta]\right]+\text{ArcSinh}\left[e^{i\theta}-\text{Cos}[\theta]\right]^2}}\right\}\right\}$$

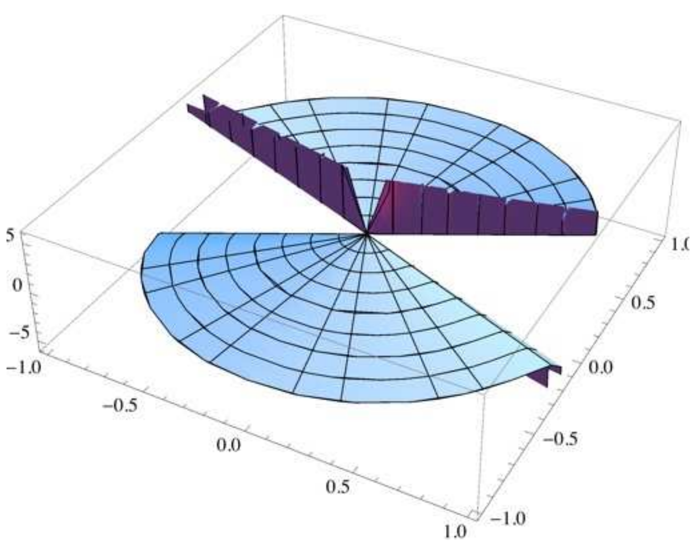
RevolutionPlot3D$\Big[$

$$-\frac{2\,\pi\,\eta}{\sqrt{-4\,\mathrm{i}\,\pi\,\mathrm{ArcSinh}\left[e^{\mathrm{i}\,\theta}-\mathrm{Cos}[\theta]\right]+\mathrm{ArcSinh}\left[e^{\mathrm{i}\,\theta}-\mathrm{Cos}[\theta]\right]^{2}}},\ \{\eta,\,-1,\,1\},\ \{\theta,\,-2\,\pi,\,2\,\pi\}\Big]$$

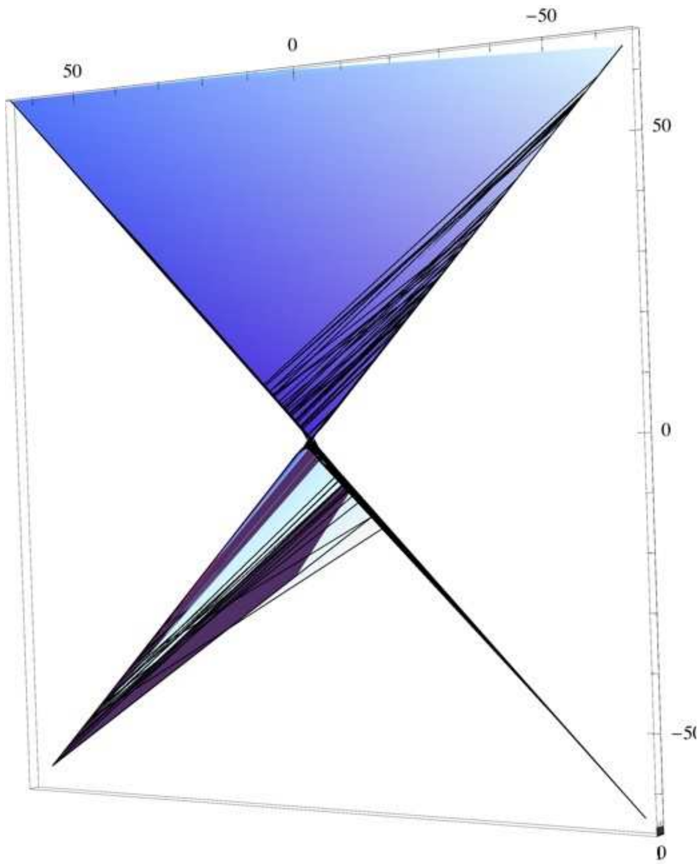


RevolutionPlot3D$\Big[$

$$\frac{2\,\pi\,\eta}{\sqrt{-4\,\mathrm{i}\,\pi\,\mathrm{ArcSinh}\left[e^{\mathrm{i}\,\theta}-\mathrm{Cos}[\theta]\right]+\mathrm{ArcSinh}\left[e^{\mathrm{i}\,\theta}-\mathrm{Cos}[\theta]\right]^{2}}},\ \{\eta,\,-1,\,1\},\ \{\theta,\,-2\,\pi,\,2\,\pi\}\Big]$$



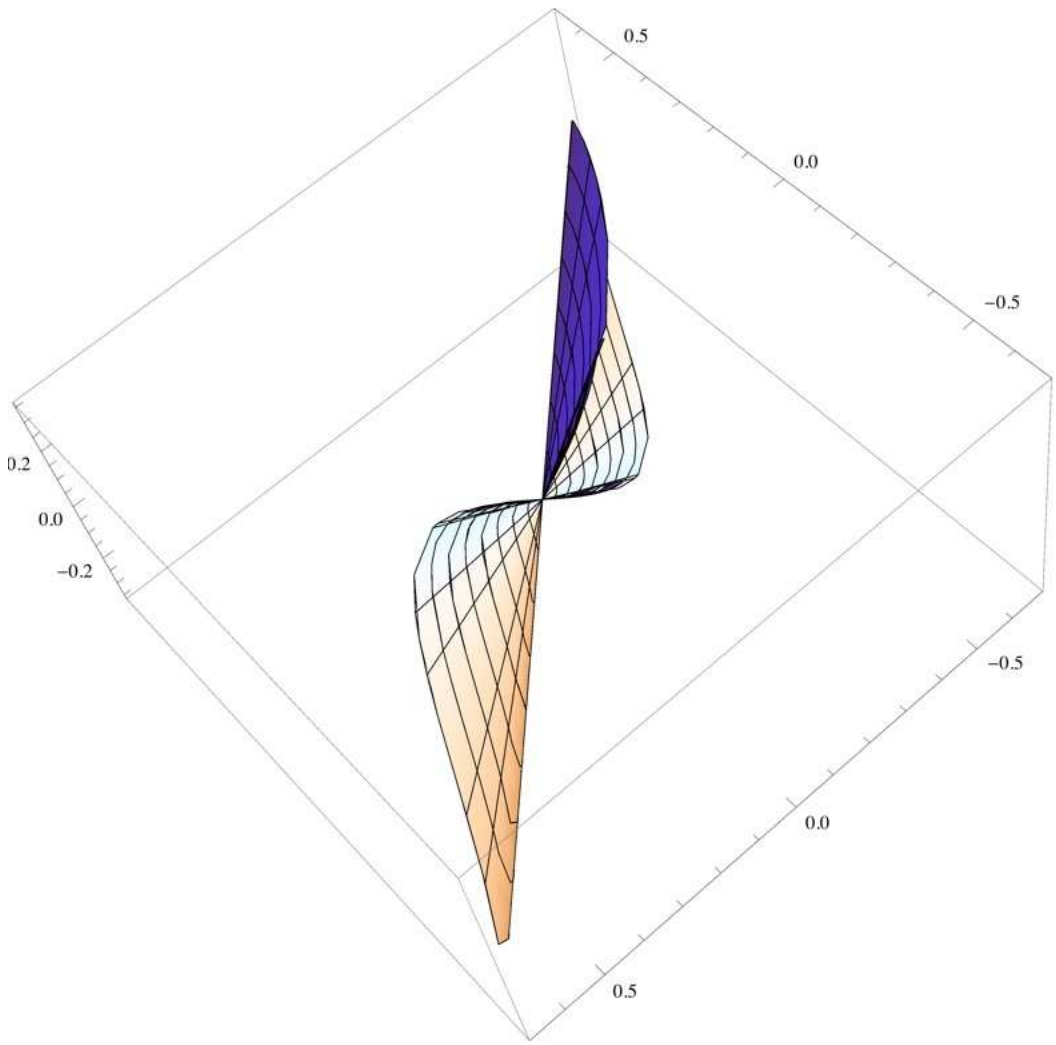
$$\text{RevolutionPlot3D}\left[\left\{-\frac{2\,\pi\,\eta}{\sqrt{-4\,i\,\pi\,\text{ArcSinh}\left[e^{i\,\theta}-\text{Cos}[\theta]\right]+\text{ArcSinh}\left[e^{i\,\theta}-\text{Cos}[\theta]\right]^2}},\ \frac{2\,\pi\,\eta}{\sqrt{-4\,i\,\pi\,\text{ArcSinh}\left[e^{i\,\theta}-\text{Cos}[\theta]\right]+\text{ArcSinh}\left[e^{i\,\theta}-\text{Cos}[\theta]\right]^2}}\right\},\ \{\eta,-1,1\},\ \{\theta,-2\,\pi,2\,\pi\}\right]$$



$$\text{Solve}\left[e\,\hat{}\,(i\,\theta) == i\,\text{Sin}\left[\frac{2\,\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right]+\text{Cos}[\theta],\ \eta\right]$$

$$\left\{\left\{\eta\to-\frac{r\,\sqrt{4\,\pi+i\,\text{ArcSinh}\left[e^{i\,\theta}-\text{Cos}[\theta]\right]}\,\sqrt{-i\,\text{ArcSinh}\left[e^{i\,\theta}-\text{Cos}[\theta]\right]}}{2\,\pi}\right\},\ \left\{\eta\to\frac{r\,\sqrt{4\,\pi+i\,\text{ArcSinh}\left[e^{i\,\theta}-\text{Cos}[\theta]\right]}\,\sqrt{-i\,\text{ArcSinh}\left[e^{i\,\theta}-\text{Cos}[\theta]\right]}}{2\,\pi}\right\}\right\}$$

$$\text{RevolutionPlot3D}\left[\left\{-\frac{r\,\sqrt{4\,\pi+i\,\text{ArcSinh}\left[e^{i\,\theta}-\text{Cos}[\theta]\right]}\,\sqrt{-i\,\text{ArcSinh}\left[e^{i\,\theta}-\text{Cos}[\theta]\right]}}{2\,\pi},\ \frac{r\,\sqrt{4\,\pi+i\,\text{ArcSinh}\left[e^{i\,\theta}-\text{Cos}[\theta]\right]}\,\sqrt{-i\,\text{ArcSinh}\left[e^{i\,\theta}-\text{Cos}[\theta]\right]}}{2\,\pi}\right\},\ \{r,-1,1\},\ \{\theta,-2\,\pi,2\,\pi\}\right]$$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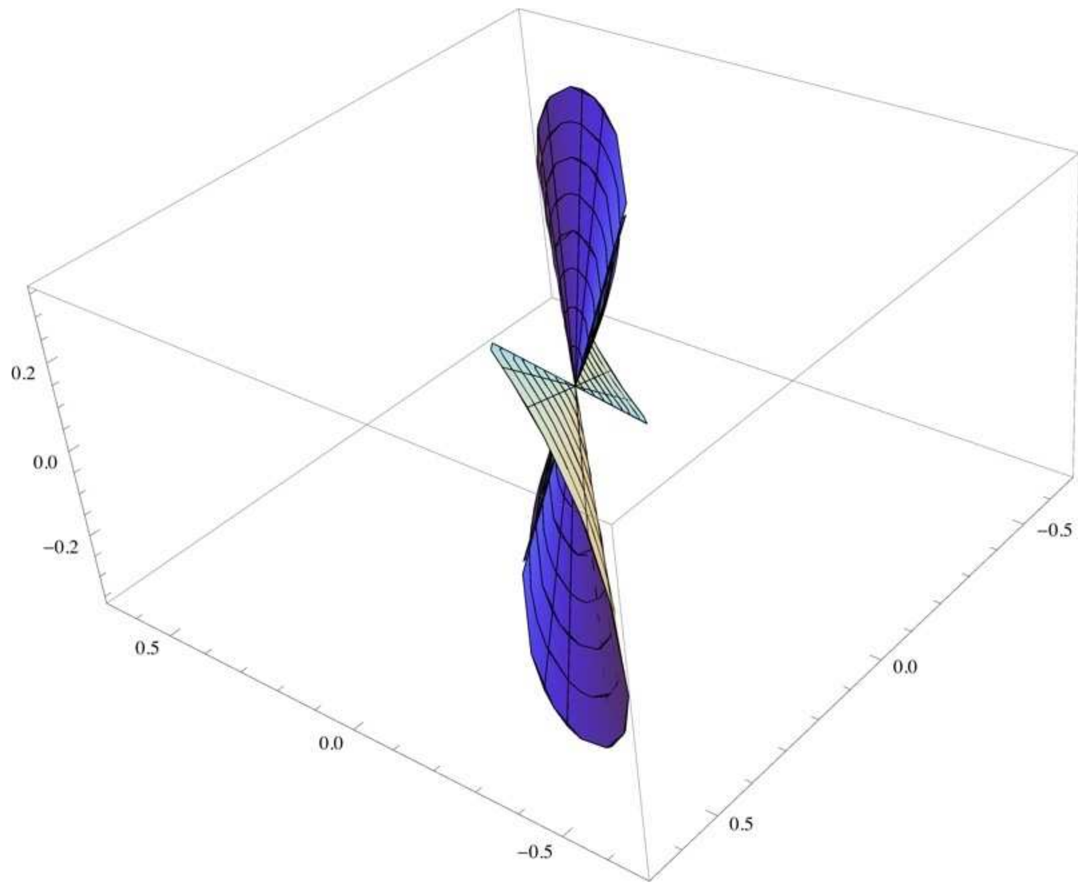
0.0
0.5


0.2
0.0
−0.2
0.5
0.0
−0.5
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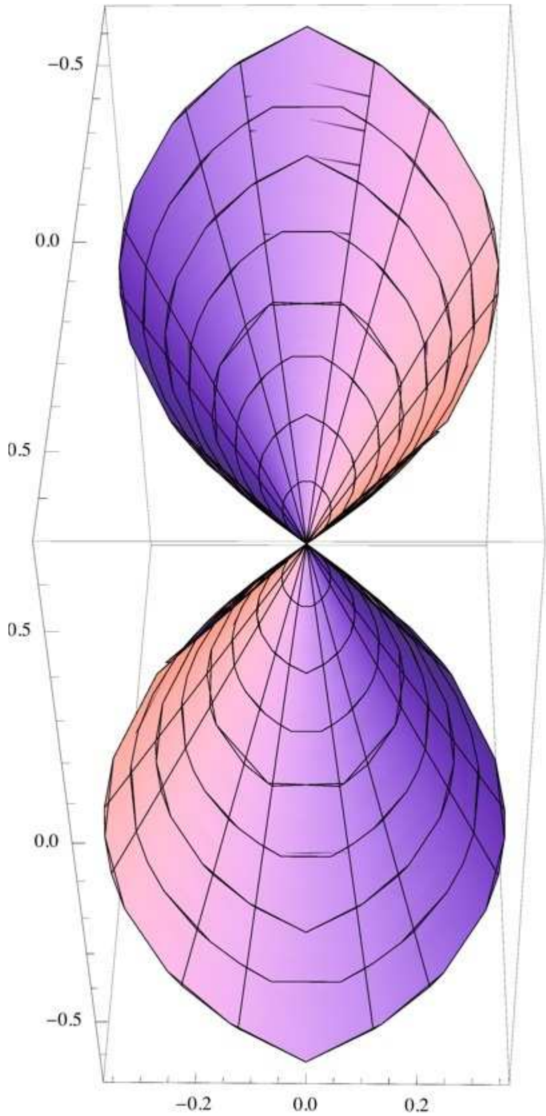
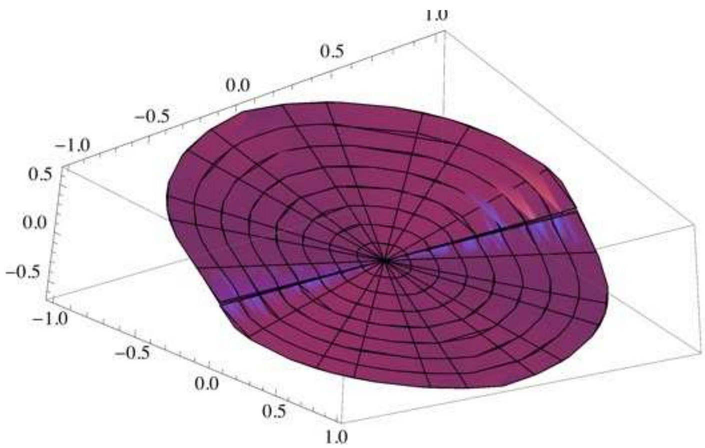
0.0
0.5


−0.5
0.0
0.5
0.5
0.0
−0.5
−0.2
0.0
0.2

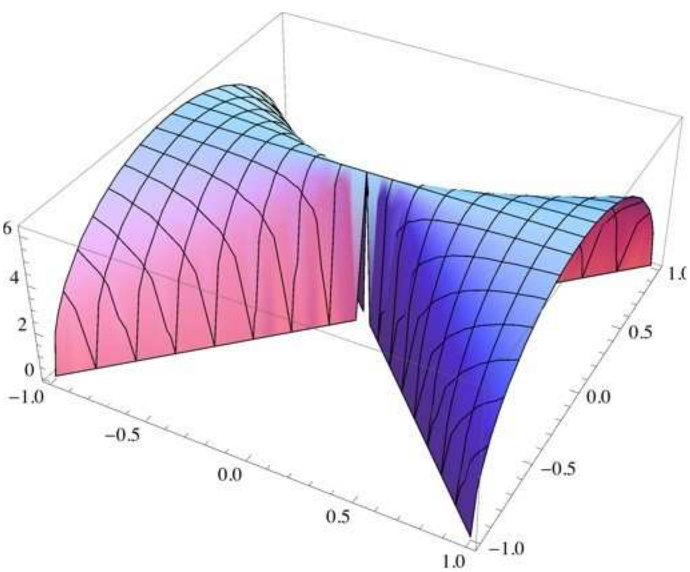
RevolutionPlot3D$\left[-\frac{r\sqrt{4\pi + i\,\text{ArcSinh}\left[e^{i\theta} - \text{Cos}[\theta]\right]}\ \sqrt{-i\,\text{ArcSinh}\left[e^{i\theta} - \text{Cos}[\theta]\right]}}{2\pi}, \{r, -1, 1\}, \{\theta, -2\pi, 2\pi\}\right]$



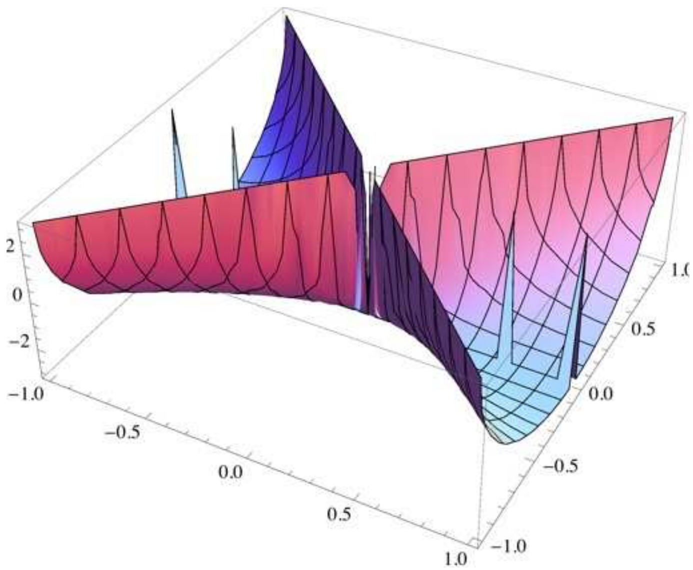
Solve$\left[e\text{^}(i\,\theta) == i\,\text{Sin}\left[\frac{2\pi\left(r^2 + \sqrt{r^4 - r^2\eta^2}\right)}{r^2}\right] + \text{Cos}[\theta], \theta\right]$

$$\left\{\left\{\theta \to \frac{2\pi\sqrt{r^2\left(r^2-\eta^2\right)}}{r^2}\right\}, \left\{\theta \to -i\,\text{Log}\left[-e^{-\frac{2 i \pi\sqrt{r^2\left(r^2-\eta^2\right)}}{r^2}}\right]\right\}\right\}$$

Plot3D$\left[\frac{2\pi\sqrt{r^2\left(r^2-\eta^2\right)}}{r^2}, \{r, -1, 1\}, \{\eta, -1, 1\}\right]$



```
Plot3D[-ⅈ Log[-ⅇ^(-(2 ⅈ π Sqrt[r^2 (r^2-η^2)])/r^2)], {r, -1, 1}, {η, -1, 1}]
```




```
Solve[ⅇ^(ⅈ θ) == ⅈ Sin[θ] + Cos[(2 π (r^2 + Sqrt[r^4 - r^2 η^2]))/r^2], r]
```

$$\left\{\left\{r \to -\frac{2\pi\eta}{\sqrt{-4\pi \text{ArcCos}\left[e^{i\theta} - i \text{Sin}[\theta]\right] - \text{ArcCos}\left[e^{i\theta} - i \text{Sin}[\theta]\right]^2}}\right\},\right.$$

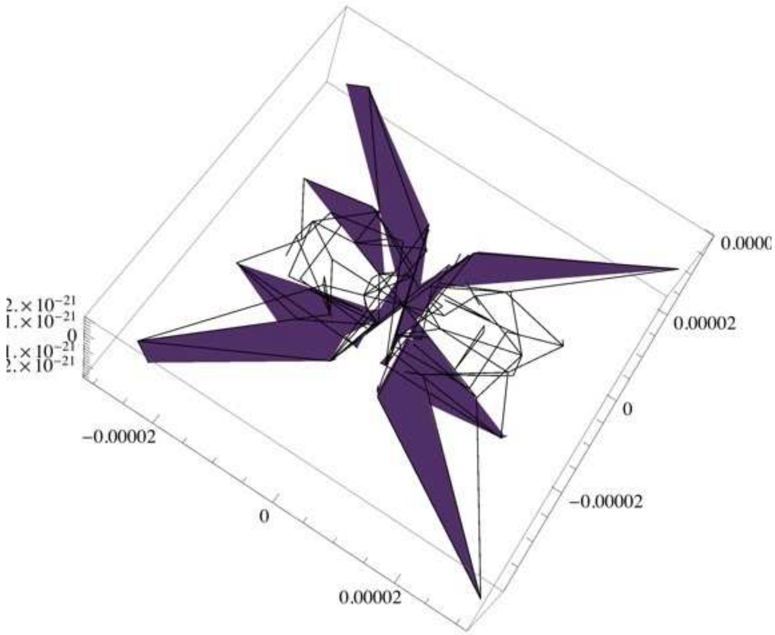
$$\left\{r \to \frac{2\pi\eta}{\sqrt{-4\pi \text{ArcCos}\left[e^{i\theta} - i \text{Sin}[\theta]\right] - \text{ArcCos}\left[e^{i\theta} - i \text{Sin}[\theta]\right]^2}}\right\},$$

$$\left\{r \to -\frac{2\pi\eta}{\sqrt{4\pi \text{ArcCos}\left[e^{i\theta} - i \text{Sin}[\theta]\right] - \text{ArcCos}\left[e^{i\theta} - i \text{Sin}[\theta]\right]^2}}\right\},$$

$$\left.\left\{r \to \frac{2\pi\eta}{\sqrt{4\pi \text{ArcCos}\left[e^{i\theta} - i \text{Sin}[\theta]\right] - \text{ArcCos}\left[e^{i\theta} - i \text{Sin}[\theta]\right]^2}}\right\}\right\}$$

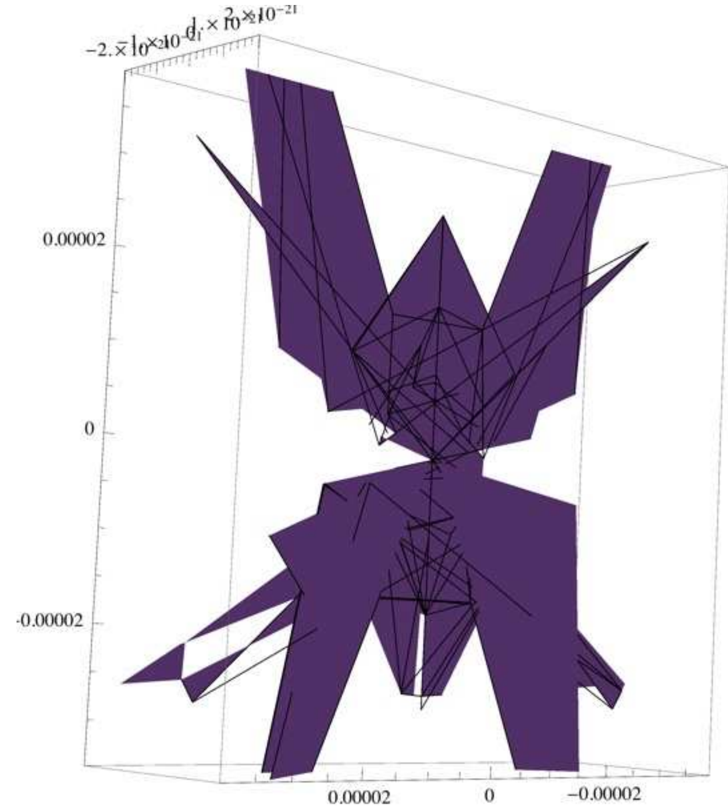
```
RevolutionPlot3D[
```

$$\texttt{RevolutionPlot3D}\left[-\frac{2\pi\eta}{\sqrt{-4\pi\,\mathrm{ArcCos}\left[e^{i\theta}-i\,\mathrm{Sin}[\theta]\right]-\mathrm{ArcCos}\left[e^{i\theta}-i\,\mathrm{Sin}[\theta]\right]^2}},\ \{\eta,-1,1\},\ \{\theta,-4\pi,4\pi\}\right]$$

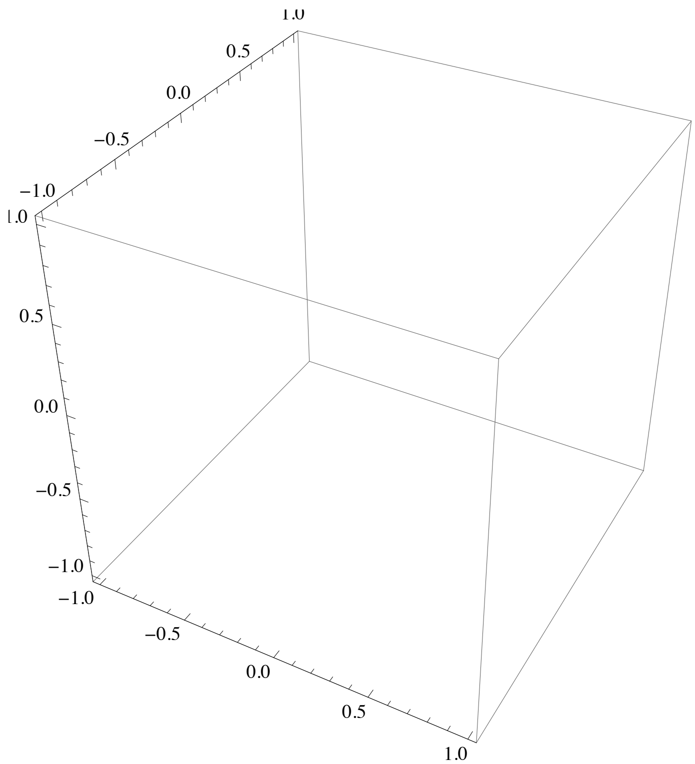


This is a piece of little known focus - textural lore that has crept up through the focused practice of buddhist teaching.

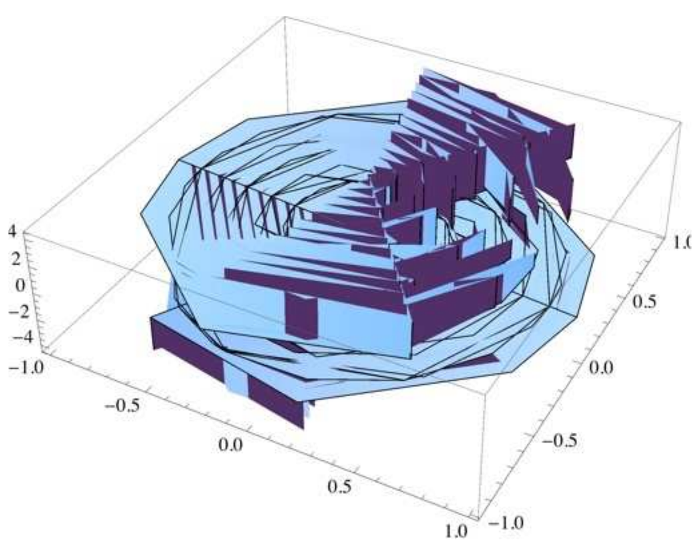
$$\text{RevolutionPlot3D}\left[\frac{2\,\pi\,\eta}{\sqrt{-4\,\pi\,\text{ArcCos}\left[e^{i\,\theta}-i\,\text{Sin}[\theta]\right]-\text{ArcCos}\left[e^{i\,\theta}-i\,\text{Sin}[\theta]\right]^{2}}},\ \{\eta,\ -1,\ 1\},\ \{\theta,\ -8\,\pi,\ 8\,\pi\}\right]$$



$$\text{RevolutionPlot3D}\left[\left\{-\frac{2\,\pi\,\eta}{\sqrt{-4\,\pi\,\text{ArcCos}\left[e^{i\,\theta}-i\,\text{Sin}[\theta]\right]-\text{ArcCos}\left[e^{i\,\theta}-i\,\text{Sin}[\theta]\right]^2}},\ \frac{2\,\pi\,\eta}{\sqrt{-4\,\pi\,\text{ArcCos}\left[e^{i\,\theta}-i\,\text{Sin}[\theta]\right]-\text{ArcCos}\left[e^{i\,\theta}-i\,\text{Sin}[\theta]\right]^2}}\right\},\ \{\eta,-1,1\},\ \{\theta,-2\,\pi,2\,\pi\}\right]$$

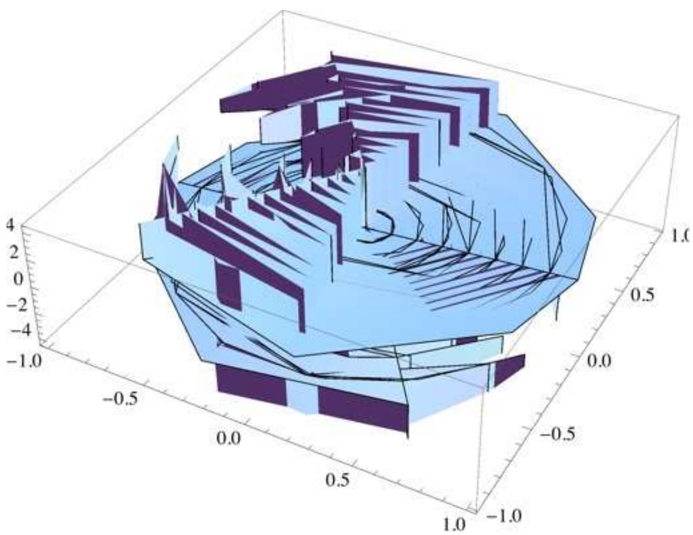


$$\text{RevolutionPlot3D}\left[\frac{2\,\pi\,\eta}{\sqrt{4\,\pi\,\text{ArcCos}\left[e^{i\,\theta}-i\,\text{Sin}[\theta]\right]-\text{ArcCos}\left[e^{i\,\theta}-i\,\text{Sin}[\theta]\right]^2}},\ \{\eta,-1,1\},\ \{\theta,-8\,\pi,8\,\pi\}\right]$$



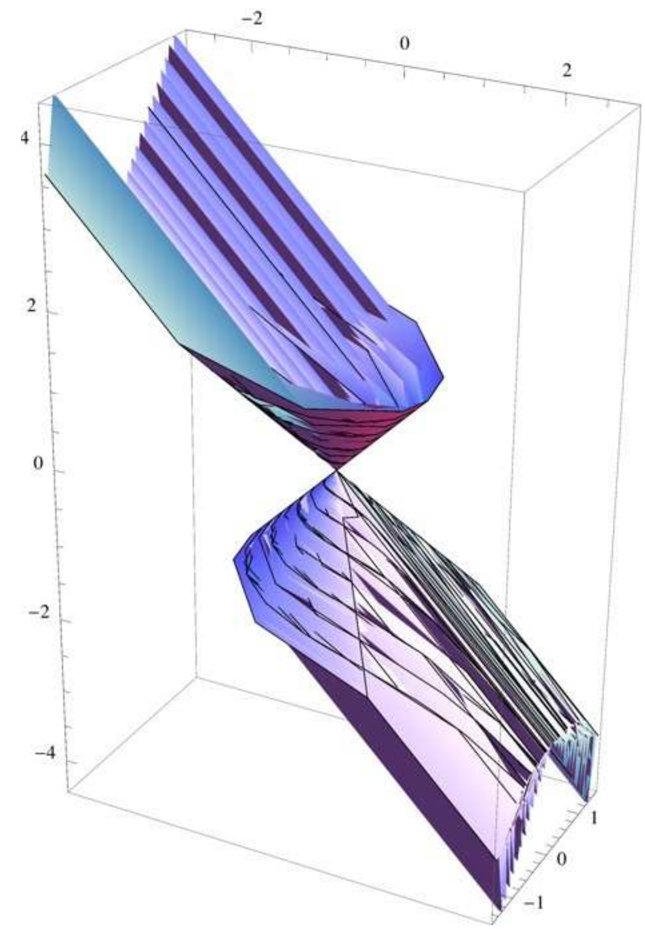
```
RevolutionPlot3D[
```

$$-\frac{2\pi\eta}{\sqrt{4\pi\,\text{ArcCos}\left[e^{i\theta}-i\,\text{Sin}[\theta]\right]-\text{ArcCos}\left[e^{i\theta}-i\,\text{Sin}[\theta]\right]^2}},\ \{\eta,-1,1\},\ \{\theta,-8\pi,8\pi\}\Big]$$



$$\text{RevolutionPlot3D}\left[\left\{\frac{2\,\pi\,\eta}{\sqrt{4\,\pi\,\text{ArcCos}\left[e^{i\,\theta}-i\,\text{Sin}[\theta]\right]-\text{ArcCos}\left[e^{i\,\theta}-i\,\text{Sin}[\theta]\right]^{2}}},\right.\right.$$

$$\left.\left.-\frac{2\,\pi\,\eta}{\sqrt{4\,\pi\,\text{ArcCos}\left[e^{i\,\theta}-i\,\text{Sin}[\theta]\right]-\text{ArcCos}\left[e^{i\,\theta}-i\,\text{Sin}[\theta]\right]^{2}}}\right\},\ \{\eta,\ -1,\ 1\},\ \{\theta,\ -8\,\pi,\ 8\,\pi\}\right]$$



```
Solve[e^(i θ) == i Sin[θ] + Cos[(2 π (r^2 + Sqrt[r^4 - r^2 η^2]))/r^2], η]
```

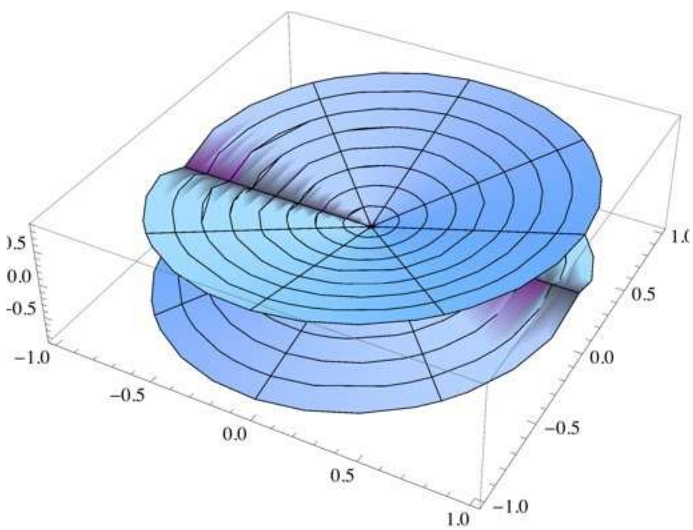
$$\left\{\left\{\eta \to -\frac{r \sqrt{4\pi - \text{ArcCos}\left[e^{i\theta} - i\,\text{Sin}[\theta]\right]}\ \sqrt{\text{ArcCos}\left[e^{i\theta} - i\,\text{Sin}[\theta]\right]}}{2\pi}\right\},\right.$$

$$\left\{\eta \to \frac{r \sqrt{4\pi - \text{ArcCos}\left[e^{i\theta} - i\,\text{Sin}[\theta]\right]}\ \sqrt{\text{ArcCos}\left[e^{i\theta} - i\,\text{Sin}[\theta]\right]}}{2\pi}\right\},$$

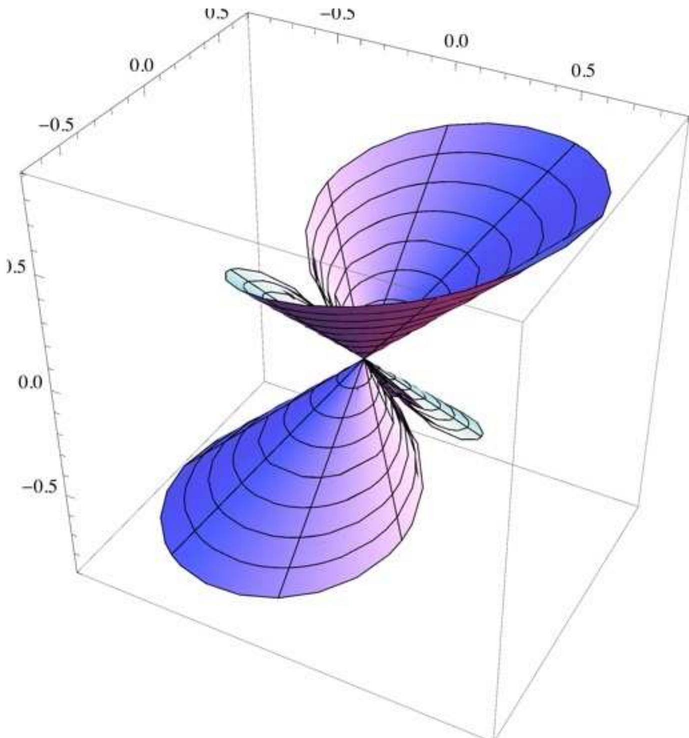
$$\left\{\eta \to -\frac{r \sqrt{-\text{ArcCos}\left[e^{i\theta} - i\,\text{Sin}[\theta]\right]}\ \sqrt{4\pi + \text{ArcCos}\left[e^{i\theta} - i\,\text{Sin}[\theta]\right]}}{2\pi}\right\},$$

$$\left.\left\{\eta \to \frac{r \sqrt{-\text{ArcCos}\left[e^{i\theta} - i\,\text{Sin}[\theta]\right]}\ \sqrt{4\pi + \text{ArcCos}\left[e^{i\theta} - i\,\text{Sin}[\theta]\right]}}{2\pi}\right\}\right\}$$

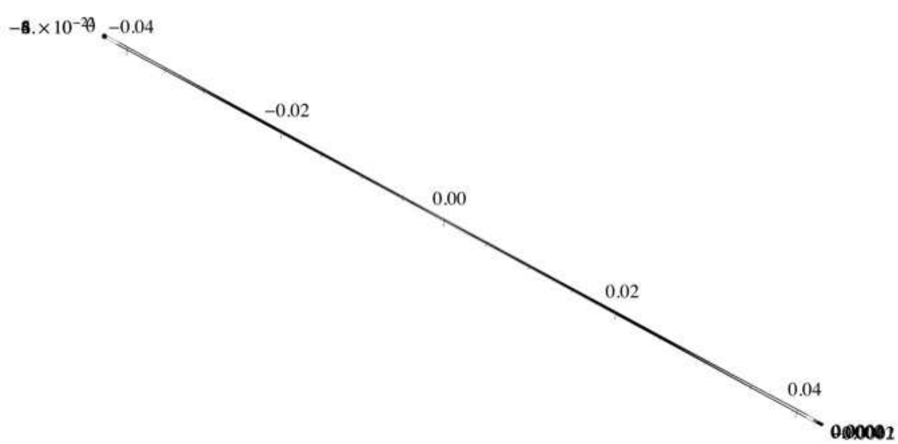
```
RevolutionPlot3D[
 -(r Sqrt[4 π - ArcCos[e^(i θ) - i Sin[θ]]] Sqrt[ArcCos[e^(i θ) - i Sin[θ]]])/(2 π), {r, -1, 1}, {θ, -2 π, 2 π}]
```



$$\text{RevolutionPlot3D}\left[\left\{\frac{r\sqrt{4\,\pi-\text{ArcCos}\left[e^{i\,\theta}-i\,\text{Sin}[\theta]\right]}\ \sqrt{\text{ArcCos}\left[e^{i\,\theta}-i\,\text{Sin}[\theta]\right]}}{2\,\pi},\ -\frac{r\sqrt{4\,\pi-\text{ArcCos}\left[e^{i\,\theta}-i\,\text{Sin}[\theta]\right]}\ \sqrt{\text{ArcCos}\left[e^{i\,\theta}-i\,\text{Sin}[\theta]\right]}}{2\,\pi}\right\},\ \{r,-1,1\},\ \{\theta,-2\,\pi,2\,\pi\}\right]$$

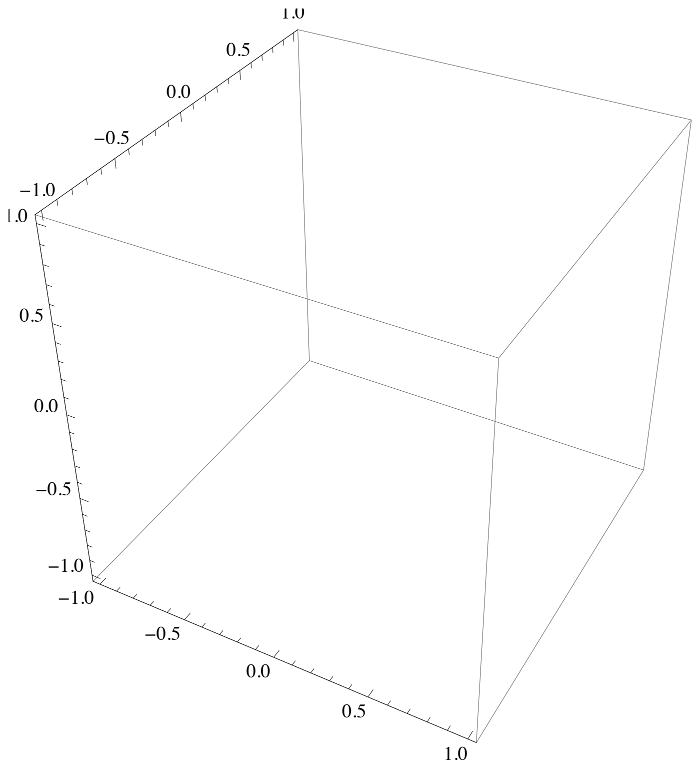


$$\text{RevolutionPlot3D}\left[\left\{\frac{r\sqrt{4\,\pi-\text{ArcCos}\left[e^{i\,\theta}-i\,\text{Sin}[\theta]\right]}\ \sqrt{\text{ArcCos}\left[e^{i\,\theta}-i\,\text{Sin}[\theta]\right]}}{2\,\pi},\ \frac{r\sqrt{-\text{ArcCos}\left[e^{i\,\theta}-i\,\text{Sin}[\theta]\right]}\ \sqrt{4\,\pi+\text{ArcCos}\left[e^{i\,\theta}-i\,\text{Sin}[\theta]\right]}}{2\,\pi}\right\},\ \{r,-1,1\},\ \{\theta,-2\,\pi,2\,\pi\}\right]$$



```
RevolutionPlot3D[{-(r Sqrt[-ArcCos[E^(I θ) - I Sin[θ]]] Sqrt[4 π + ArcCos[E^(I θ) - I Sin[θ]]])/(2 π),
  (r Sqrt[-ArcCos[E^(I θ) - I Sin[θ]]] Sqrt[4 π + ArcCos[E^(I θ) - I Sin[θ]]])/(2 π)}, {r, -1, 1}, {θ, -2 π, 2 π}]
```

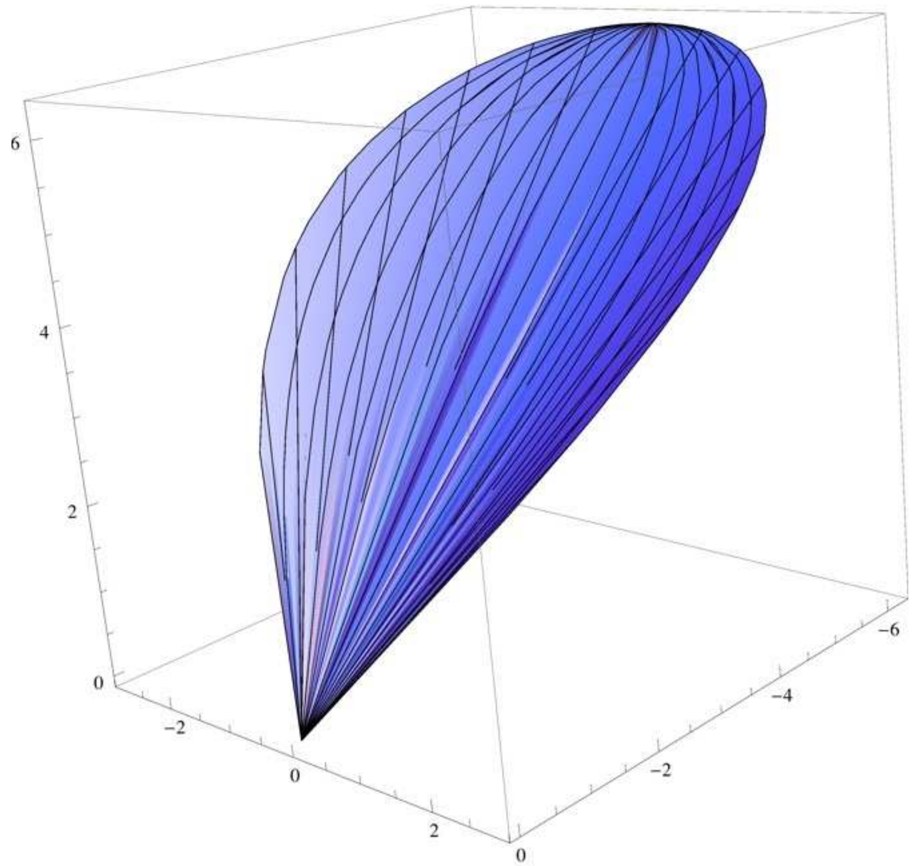
$$\text{RevolutionPlot3D}\left[\left\{-\frac{r\sqrt{-\text{ArcCos}\left[e^{i\theta}-i\,\text{Sin}[\theta]\right]}\sqrt{4\pi+\text{ArcCos}\left[e^{i\theta}-i\,\text{Sin}[\theta]\right]}}{2\pi},\frac{r\sqrt{-\text{ArcCos}\left[e^{i\theta}-i\,\text{Sin}[\theta]\right]}\sqrt{4\pi+\text{ArcCos}\left[e^{i\theta}-i\,\text{Sin}[\theta]\right]}}{2\pi}\right\},\{r,-1,1\},\{\theta,-2\pi,2\pi\}\right]$$



$$\text{Solve}\left[e\,\hat{}\,(i\,\theta) == i\,\text{Sin}[\theta] + \text{Cos}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\,\eta^2}\right)}{r^2}\right],\theta\right]$$

$$\left\{\left\{\theta\to-\frac{2\pi\sqrt{r^2\left(r^2-\eta^2\right)}}{r^2}\right\},\left\{\theta\to\frac{2\pi\sqrt{r^2\left(r^2-\eta^2\right)}}{r^2}\right\}\right\}$$

$$\text{RevolutionPlot3D}\left[\left\{-\frac{2\,\pi\,\sqrt{r^2\,\left(r^2-\eta^2\right)}}{r^2},\ \frac{2\,\pi\,\sqrt{r^2\,\left(r^2-\eta^2\right)}}{r^2}\right\},\ \{r,\,-1,\,1\},\ \{\eta,\,-1,\,1\}\right]$$



Solve$\left[e^{\wedge}(i\,\theta) == i\,\mathrm{Sin}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right] + \mathrm{Cos}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right], r\right]$

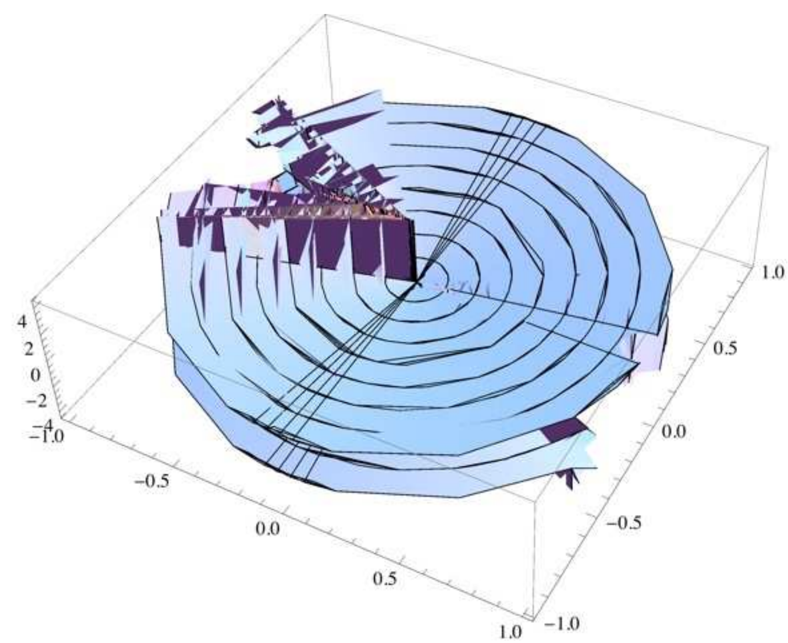
$$\left\{\left\{r \to -\frac{\pi\,\eta}{\sqrt{\pi^2 - \mathrm{ArcCos}\left[-\frac{\sqrt{1+2\,\mathrm{Cos}[\theta]+\mathrm{Cos}[\theta]^2+2\,i\,\mathrm{Sin}[\theta]+2\,i\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]-\mathrm{Sin}[\theta]^2}}{\sqrt{4\,\mathrm{Cos}[\theta]+4\,i\,\mathrm{Sin}[\theta]}}\right]^2}}\right\},\right.$$

$$\left\{r \to \frac{\pi\,\eta}{\sqrt{\pi^2 - \mathrm{ArcCos}\left[-\frac{\sqrt{1+2\,\mathrm{Cos}[\theta]+\mathrm{Cos}[\theta]^2+2\,i\,\mathrm{Sin}[\theta]+2\,i\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]-\mathrm{Sin}[\theta]^2}}{\sqrt{4\,\mathrm{Cos}[\theta]+4\,i\,\mathrm{Sin}[\theta]}}\right]^2}}\right\},$$

$$\left\{r \to -\frac{\pi\,\eta}{\sqrt{\pi^2 - \mathrm{ArcCos}\left[\frac{\sqrt{1+2\,\mathrm{Cos}[\theta]+\mathrm{Cos}[\theta]^2+2\,i\,\mathrm{Sin}[\theta]+2\,i\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]-\mathrm{Sin}[\theta]^2}}{\sqrt{4\,\mathrm{Cos}[\theta]+4\,i\,\mathrm{Sin}[\theta]}}\right]^2}}\right\},$$

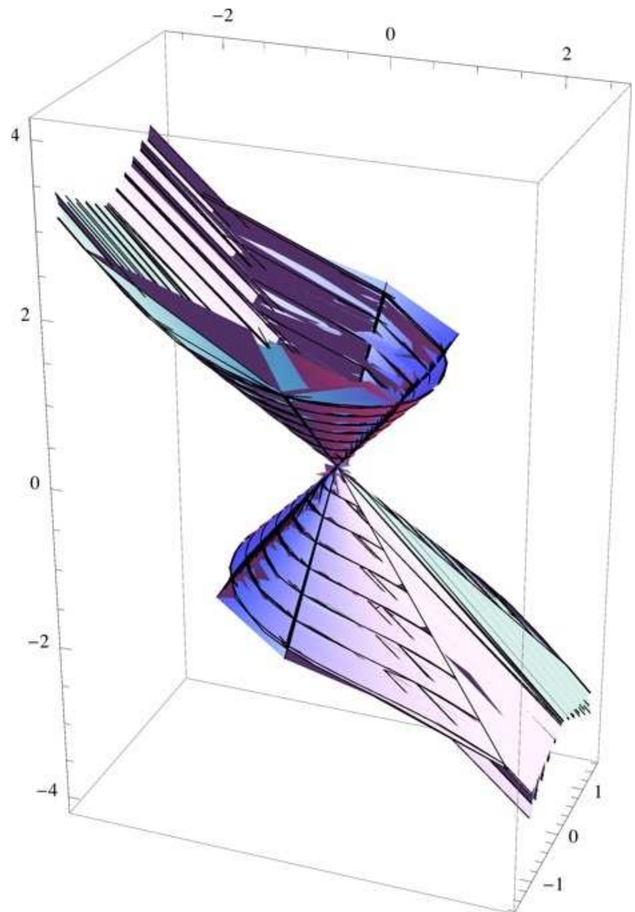
$$\left.\left\{r \to \frac{\pi\,\eta}{\sqrt{\pi^2 - \mathrm{ArcCos}\left[\frac{\sqrt{1+2\,\mathrm{Cos}[\theta]+\mathrm{Cos}[\theta]^2+2\,i\,\mathrm{Sin}[\theta]+2\,i\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]-\mathrm{Sin}[\theta]^2}}{\sqrt{4\,\mathrm{Cos}[\theta]+4\,i\,\mathrm{Sin}[\theta]}}\right]^2}}\right\}\right\}$$

RevolutionPlot3D$\left[-\frac{\pi\,\eta}{\sqrt{\pi^2 - \mathrm{ArcCos}\left[-\frac{\sqrt{1+2\,\mathrm{Cos}[\theta]+\mathrm{Cos}[\theta]^2+2\,i\,\mathrm{Sin}[\theta]+2\,i\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]-\mathrm{Sin}[\theta]^2}}{\sqrt{4\,\mathrm{Cos}[\theta]+4\,i\,\mathrm{Sin}[\theta]}}\right]^2}}, \{\eta, -1, 1\}, \{\theta, -4\pi, 4\pi\}\right]$

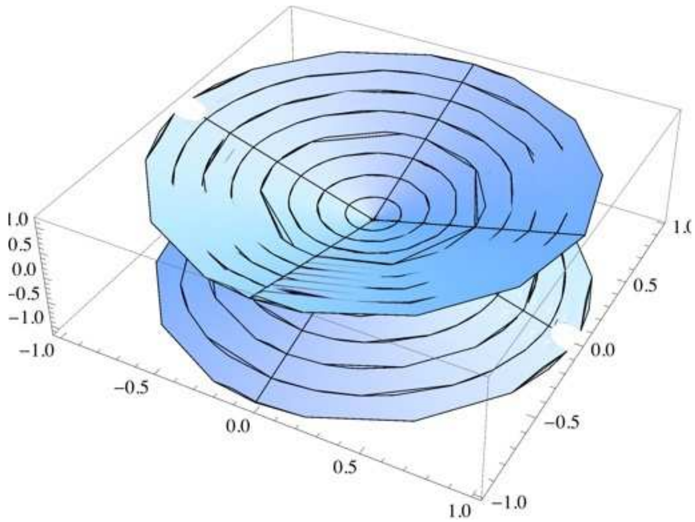


$$\text{RevolutionPlot3D}\left[\left\{-\frac{\pi\,\eta}{\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1+2\,\text{Cos}[\theta]+\text{Cos}[\theta]^2+2\,i\,\text{Sin}[\theta]+2\,i\,\text{Cos}[\theta]\,\text{Sin}[\theta]-\text{Sin}[\theta]^2}}{\sqrt{4\,\text{Cos}[\theta]+4\,i\,\text{Sin}[\theta]}}\right]^2}},\right.\right.$$

$$\left.\left.\frac{\pi\,\eta}{\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1+2\,\text{Cos}[\theta]+\text{Cos}[\theta]^2+2\,i\,\text{Sin}[\theta]+2\,i\,\text{Cos}[\theta]\,\text{Sin}[\theta]-\text{Sin}[\theta]^2}}{\sqrt{4\,\text{Cos}[\theta]+4\,i\,\text{Sin}[\theta]}}\right]^2}}\right\},\ \{\eta,-1,1\},\ \{\theta,-4\,\pi,4\,\pi\}\right]$$

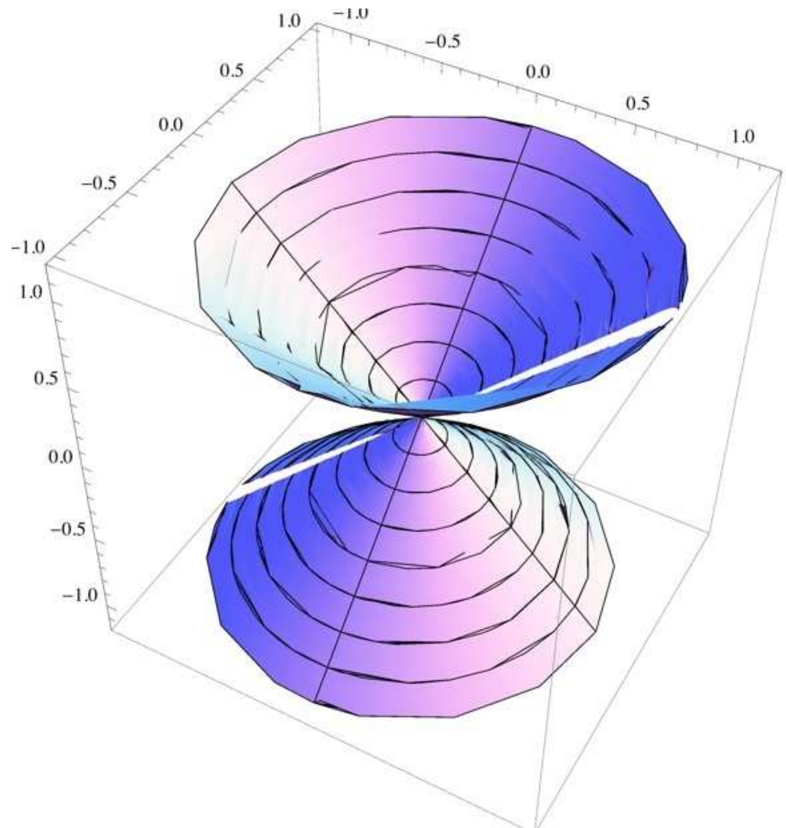


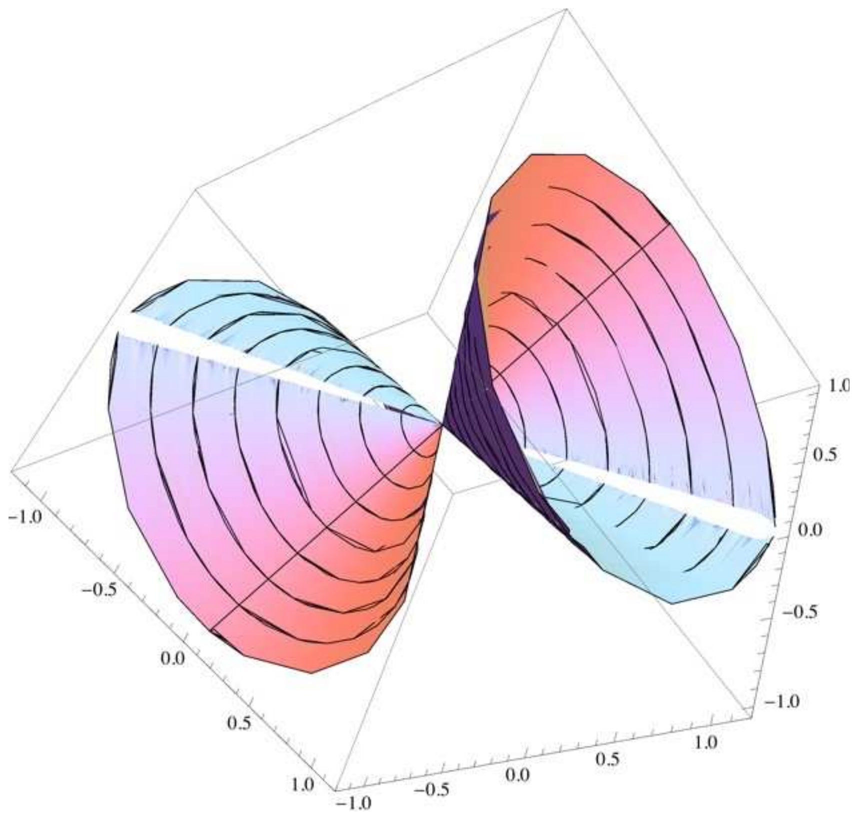
RevolutionPlot3D$\left[\dfrac{\pi\,\eta}{\sqrt{\pi^2 - \text{ArcCos}\left[\dfrac{\sqrt{1+2\,\text{Cos}[\theta]+\text{Cos}[\theta]^2+2\,\mathrm{i}\,\text{Sin}[\theta]+2\,\mathrm{i}\,\text{Cos}[\theta]\,\text{Sin}[\theta]-\text{Sin}[\theta]^2}}{\sqrt{4\,\text{Cos}[\theta]+4\,\mathrm{i}\,\text{Sin}[\theta]}}\right]^2}}, \{\eta, -1, 1\}, \{\theta, -4\pi, 4\pi\}\right]$



$$\textbf{RevolutionPlot3D}\left[\left\{-\frac{\pi\,\eta}{\sqrt{\pi^2-\textbf{ArcCos}\left[\frac{\sqrt{1+2\,\textbf{Cos}[\theta]+\textbf{Cos}[\theta]^2+2\,\mathbb{i}\,\textbf{Sin}[\theta]+2\,\mathbb{i}\,\textbf{Cos}[\theta]\,\textbf{Sin}[\theta]-\textbf{Sin}[\theta]^2}}{\sqrt{4\,\textbf{Cos}[\theta]+4\,\mathbb{i}\,\textbf{Sin}[\theta]}}\right]^2}},\right.\right.$$

$$\left.\left.\frac{\pi\,\eta}{\sqrt{\pi^2-\textbf{ArcCos}\left[\frac{\sqrt{1+2\,\textbf{Cos}[\theta]+\textbf{Cos}[\theta]^2+2\,\mathbb{i}\,\textbf{Sin}[\theta]+2\,\mathbb{i}\,\textbf{Cos}[\theta]\,\textbf{Sin}[\theta]-\textbf{Sin}[\theta]^2}}{\sqrt{4\,\textbf{Cos}[\theta]+4\,\mathbb{i}\,\textbf{Sin}[\theta]}}\right]^2}}\right\},\ \{\eta,\ -1,\ 1\},\ \{\theta,\ -4\,\pi,\ 4\,\pi\}\right]$$





$$\textbf{Solve}\left[e^{\wedge}(i\,\theta) == i\,\textbf{Sin}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right] + \textbf{Cos}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right],\ \eta\right]$$

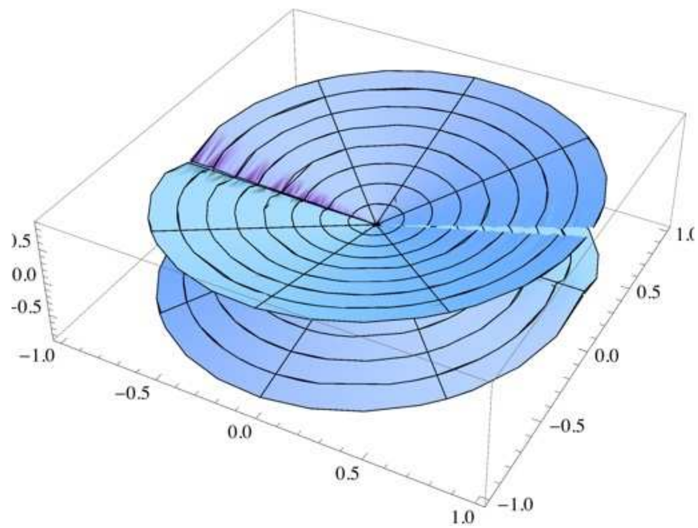
$$\left\{\left\{\eta \to -\frac{r\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1+2\,\text{Cos}[\theta]+\text{Cos}[\theta]^2+2\,i\,\text{Sin}[\theta]+2\,i\,\text{Cos}[\theta]\,\text{Sin}[\theta]-\text{Sin}[\theta]^2}}{\sqrt{4\,\text{Cos}[\theta]+4\,i\,\text{Sin}[\theta]}}\right]^2}}{\pi}\right\},\right.$$

$$\left\{\eta \to \frac{r\sqrt{\pi^2-\text{ArcCos}\left[-\frac{\sqrt{1+2\,\text{Cos}[\theta]+\text{Cos}[\theta]^2+2\,i\,\text{Sin}[\theta]+2\,i\,\text{Cos}[\theta]\,\text{Sin}[\theta]-\text{Sin}[\theta]^2}}{\sqrt{4\,\text{Cos}[\theta]+4\,i\,\text{Sin}[\theta]}}\right]^2}}{\pi}\right\},$$

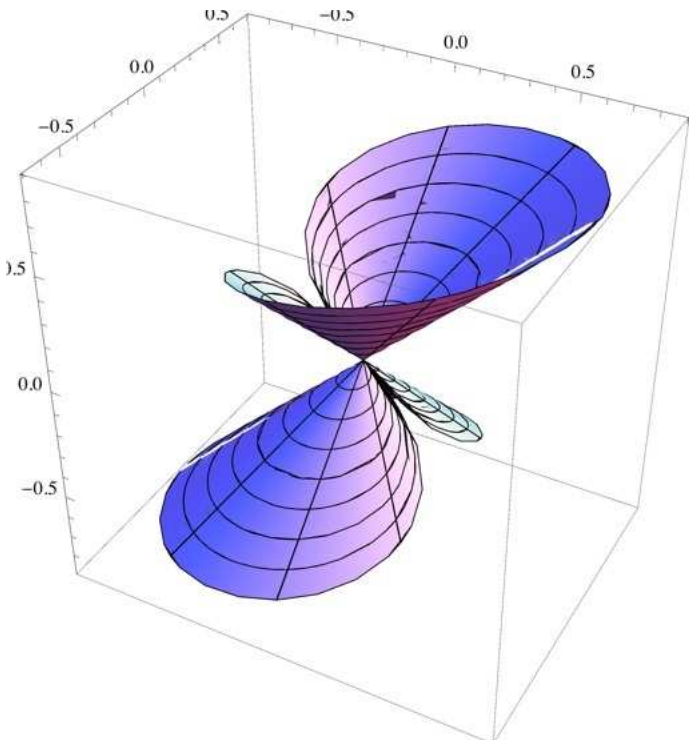
$$\left\{\eta \to -\frac{r\sqrt{\pi^2-\text{ArcCos}\left[\frac{\sqrt{1+2\,\text{Cos}[\theta]+\text{Cos}[\theta]^2+2\,i\,\text{Sin}[\theta]+2\,i\,\text{Cos}[\theta]\,\text{Sin}[\theta]-\text{Sin}[\theta]^2}}{\sqrt{4\,\text{Cos}[\theta]+4\,i\,\text{Sin}[\theta]}}\right]^2}}{\pi}\right\},$$

$$\left.\left\{\eta \to \frac{r\sqrt{\pi^2-\text{ArcCos}\left[\frac{\sqrt{1+2\,\text{Cos}[\theta]+\text{Cos}[\theta]^2+2\,i\,\text{Sin}[\theta]+2\,i\,\text{Cos}[\theta]\,\text{Sin}[\theta]-\text{Sin}[\theta]^2}}{\sqrt{4\,\text{Cos}[\theta]+4\,i\,\text{Sin}[\theta]}}\right]^2}}{\pi}\right\}\right\}$$

$$\texttt{RevolutionPlot3D}\left[-\frac{r\sqrt{\pi^2-\texttt{ArcCos}\left[-\frac{\sqrt{1+2\,\texttt{Cos}[\theta]+\texttt{Cos}[\theta]^2+2\,i\,\texttt{Sin}[\theta]+2\,i\,\texttt{Cos}[\theta]\,\texttt{Sin}[\theta]-\texttt{Sin}[\theta]^2}}{\sqrt{4\,\texttt{Cos}[\theta]+4\,i\,\texttt{Sin}[\theta]}}\right]^2}}{\pi},\ \{r,-1,1\},\ \{\theta,-2\pi,2\pi\}\right]$$

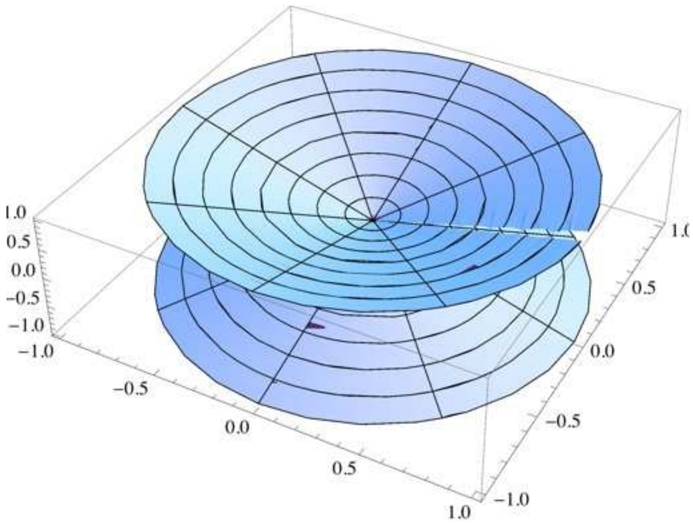


```
RevolutionPlot3D[{ (r Sqrt[π^2 - ArcCos[-(Sqrt[1+2 Cos[θ]+Cos[θ]^2+2 i Sin[θ]+2 i Cos[θ] Sin[θ]-Sin[θ]^2] / Sqrt[4 Cos[θ]+4 i Sin[θ]])]^2]) / π,
  -(r Sqrt[π^2 - ArcCos[-(Sqrt[1+2 Cos[θ]+Cos[θ]^2+2 i Sin[θ]+2 i Cos[θ] Sin[θ]-Sin[θ]^2] / Sqrt[4 Cos[θ]+4 i Sin[θ]])]^2]) / π}, {r, -1, 1}, {θ, -2 π, 2 π}]
```



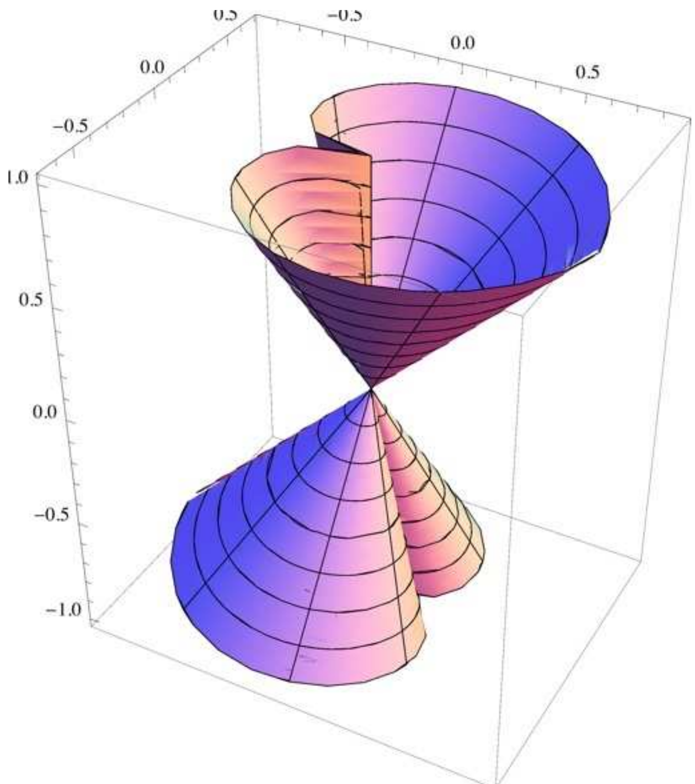
RevolutionPlot3D[

$$-\frac{r\sqrt{\pi^2-\text{ArcCos}\left[\frac{\sqrt{1+2\,\text{Cos}[\theta]+\text{Cos}[\theta]^2+2\,i\,\text{Sin}[\theta]+2\,i\,\text{Cos}[\theta]\,\text{Sin}[\theta]-\text{Sin}[\theta]^2}}{\sqrt{4\,\text{Cos}[\theta]+4\,i\,\text{Sin}[\theta]}}\right]^2}}{\pi}, \{r,-1,1\}, \{\theta,-2\pi,2\pi\}\Big]$$



$$\texttt{RevolutionPlot3D}\left[\left\{\frac{r\sqrt{\pi^2-\texttt{ArcCos}\left[-\frac{\sqrt{1+2\,\texttt{Cos}[\theta]+\texttt{Cos}[\theta]^2+2\,i\,\texttt{Sin}[\theta]+2\,i\,\texttt{Cos}[\theta]\,\texttt{Sin}[\theta]-\texttt{Sin}[\theta]^2}}{\sqrt{4\,\texttt{Cos}[\theta]+4\,i\,\texttt{Sin}[\theta]}}\right]^2}}{\pi},\right.\right.$$

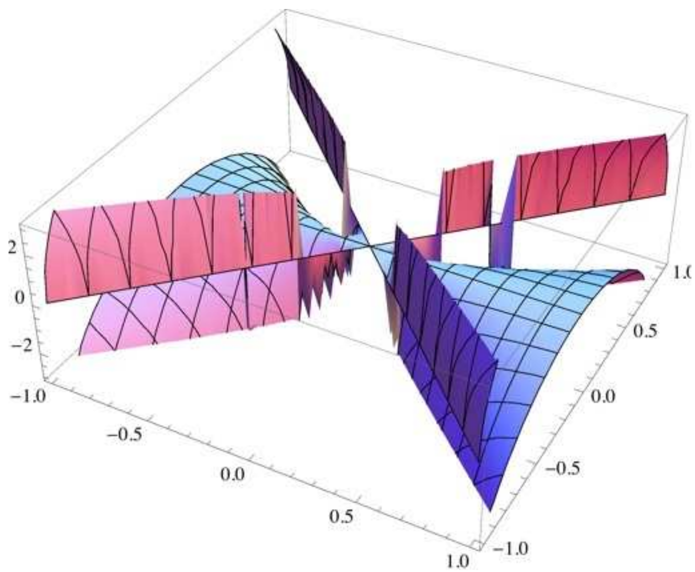
$$\left.\left.-\frac{r\sqrt{\pi^2-\texttt{ArcCos}\left[\frac{\sqrt{1+2\,\texttt{Cos}[\theta]+\texttt{Cos}[\theta]^2+2\,i\,\texttt{Sin}[\theta]+2\,i\,\texttt{Cos}[\theta]\,\texttt{Sin}[\theta]-\texttt{Sin}[\theta]^2}}{\sqrt{4\,\texttt{Cos}[\theta]+4\,i\,\texttt{Sin}[\theta]}}\right]^2}}{\pi}\right\},\ \{r,-1,1\},\ \{\theta,-2\pi,2\pi\}\right]$$



$$\texttt{Solve}\left[e\,\hat{}\,(i\,\theta) == i\,\texttt{Sin}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right]+\texttt{Cos}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right],\ \theta\right]$$

$$\left\{\left\{\theta\to -i\,\texttt{Log}\left[\texttt{Cos}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right]+i\,\texttt{Sin}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right]\right]\right\}\right\}$$

$$\text{Plot3D}\left[-i\,\text{Log}\left[\text{Cos}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right]+i\,\text{Sin}\left[\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right]\right],\{r,-1,1\},\{\eta,-1,1\}\right]$$



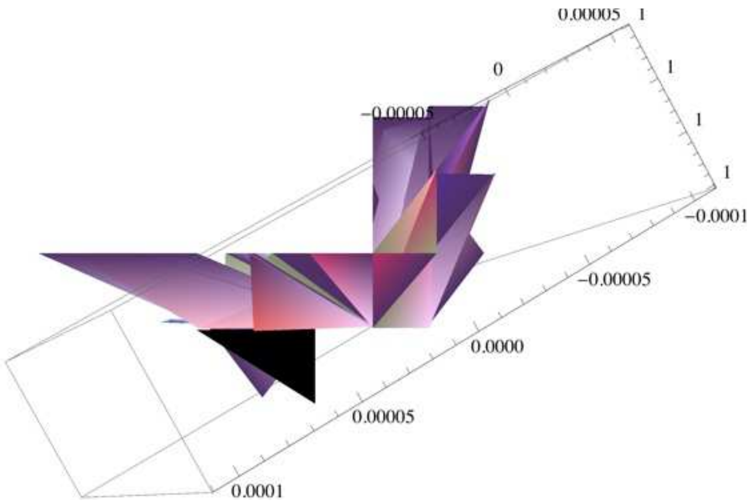
$$e\text{^}(i\,\beta) = \text{Cos}[\beta] + i\,\text{Sin}[\beta]$$

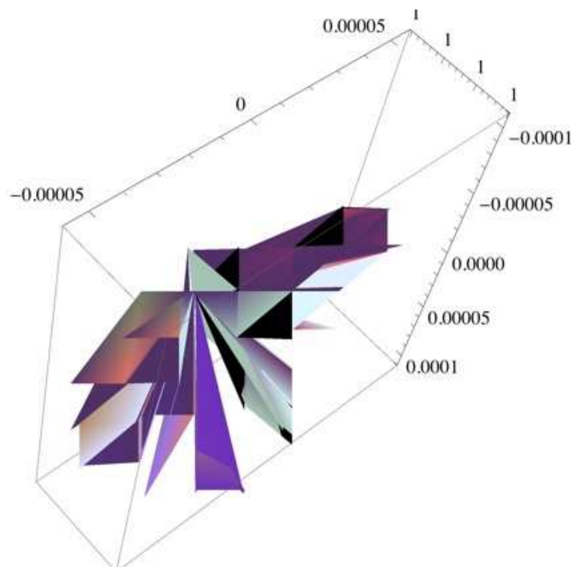
$$e\text{^}(i\,\theta) = \text{Cos}[\theta] + i\,\text{Sin}[\theta]$$

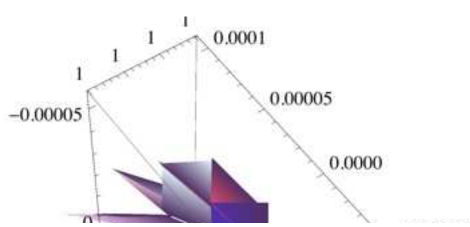
The great thing about scientific theory is that it is a practically endless exploration into varying expressions.

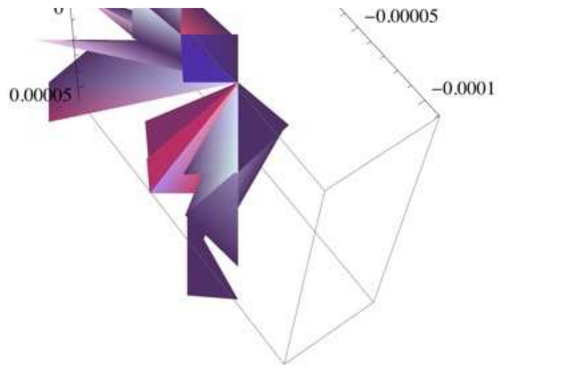
Example of expanding the theorem to $e^{i\theta_1}e^{i\theta_2}$, where $\theta\ _1 = \theta_{1+i\sqrt{3}\ ,1-i\sqrt{3}}$ and $\theta_2 = \theta_{1-i\sqrt{3}\ ,1+i\sqrt{3}}$

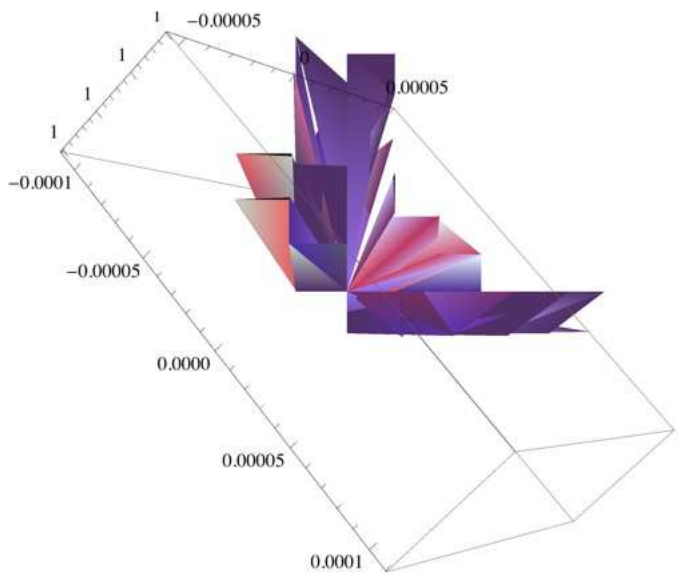
$$\text{RevolutionPlot3D}\Bigg[e\text{^}\Bigg(i\Bigg(\frac{4\pi}{3}+\left(\left(1+i\sqrt{3}\right)\left(-4\pi^2+12\pi^2\,\text{Sin}[\beta]^2\right)\right)\Big/$$
$$\left(12\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}\right)-$$
$$\frac{1}{3}\left(1-i\sqrt{3}\right)\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}\Bigg)\Bigg)$$
$$e\text{^}\Bigg(i\Bigg(\frac{4\pi}{3}+\left(\left(1-i\sqrt{3}\right)\left(-4\pi^2+12\pi^2\,\text{Sin}[\beta]^2\right)\right)\Big/$$
$$\left(12\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}\right)-\frac{1}{3}\left(1+i\sqrt{3}\right)$$
$$\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}\Bigg)\Bigg),\{\beta,-\pi,\pi\}\Bigg]$$


0.00005
1
0
1
−0.00005
1
1
−0.0001
−0.00005
0.0000
0.00005
0.0001


0.00005
1
1
1
0
1
−0.0001
−0.00005
−0.00005
0.0000
0.00005
0.0001

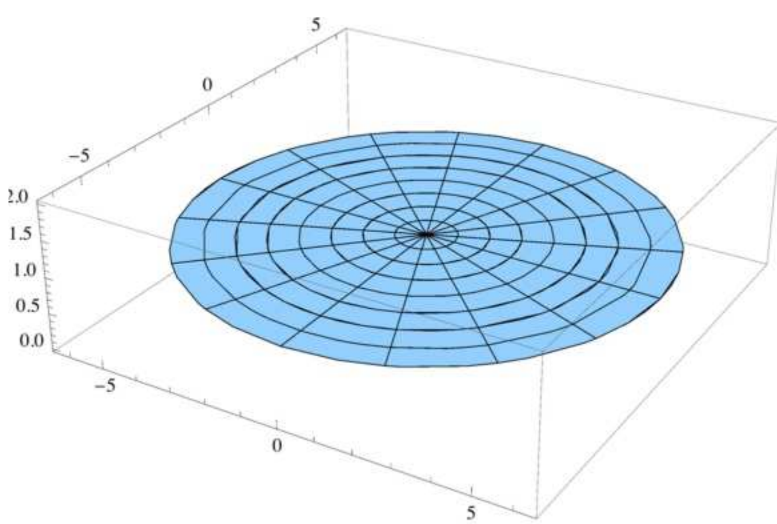

1
1
1
1
0.0001
−0.00005
0.00005
0.0000





RevolutionPlot3D[

$$\mathrm{e}^{\wedge}\left(\mathrm{i}\left(\mathbf{ArcSin}\left[\frac{\sqrt{-4\,\pi^2\,\theta+4\,\pi\,\theta^2-\theta^3}}{2\,\pi\,\sqrt{-4\,\pi+\theta}}\right]\right)\right)\,\mathrm{e}^{\wedge}\left(\mathrm{i}\left(-\mathbf{ArcSin}\left[\frac{\sqrt{-4\,\pi^2\,\theta+4\,\pi\,\theta^2-\theta^3}}{2\,\pi\,\sqrt{-4\,\pi+\theta}}\right]\right)\right),\ \{\theta,\ -2\,\pi,\ 2\,\pi\}\Big]$$



```
ⅇ^(ⅈ β) = Cos[β] + ⅈ Sin[β]

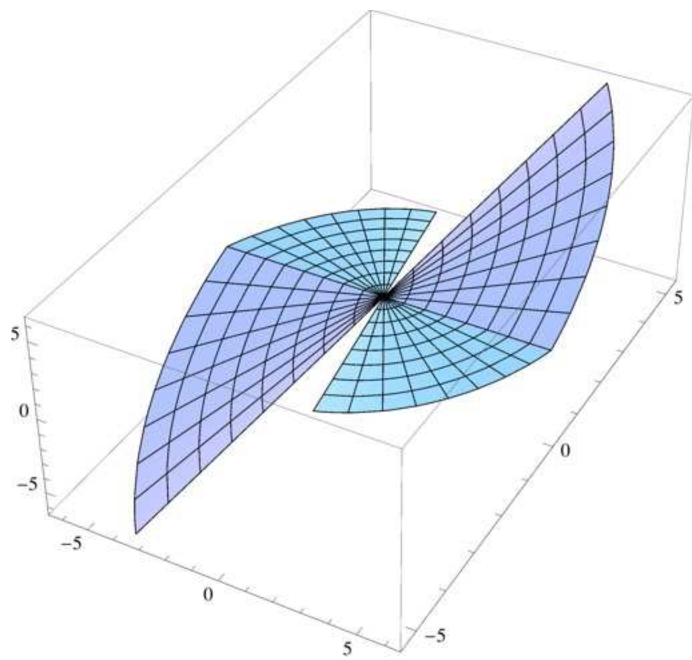
ⅇ^(ⅈ θ) = Cos[θ] + ⅈ Sin[θ]
```

XIII. Complex Analysis Applied to the Height of the Cone

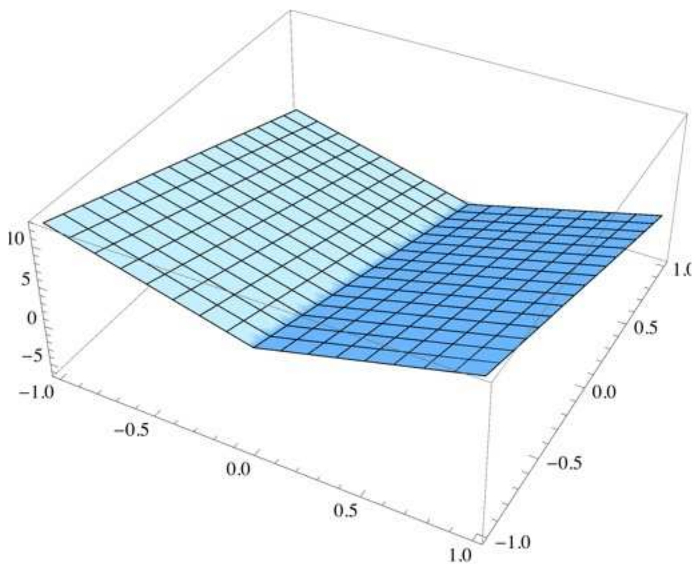
In complex analysis, a radius, **r = Abs[z] = Abs[x + ⅈ y] =** $\sqrt{x^2 + y^2}$, and, in our cone, x is translated to be the base of the cone, and y is translated to be the height of the cone. In this section, I will build up a few descriptions through visualization of the expressions that our new set of expressions for the basic parameters of the system gives us and apply a pattern of substitution in terms of the initial lemmas of transformation of a circle into a cone to the expressions for the height of the cone.

$$\theta r = \theta \text{Abs}[z] = 2\pi \text{Abs}[z] - 2\pi x \tag{43}$$

```
RevolutionPlot3D[θ Abs[z], {θ, -2 π, 2 π}, {z, -1, 1}]
```



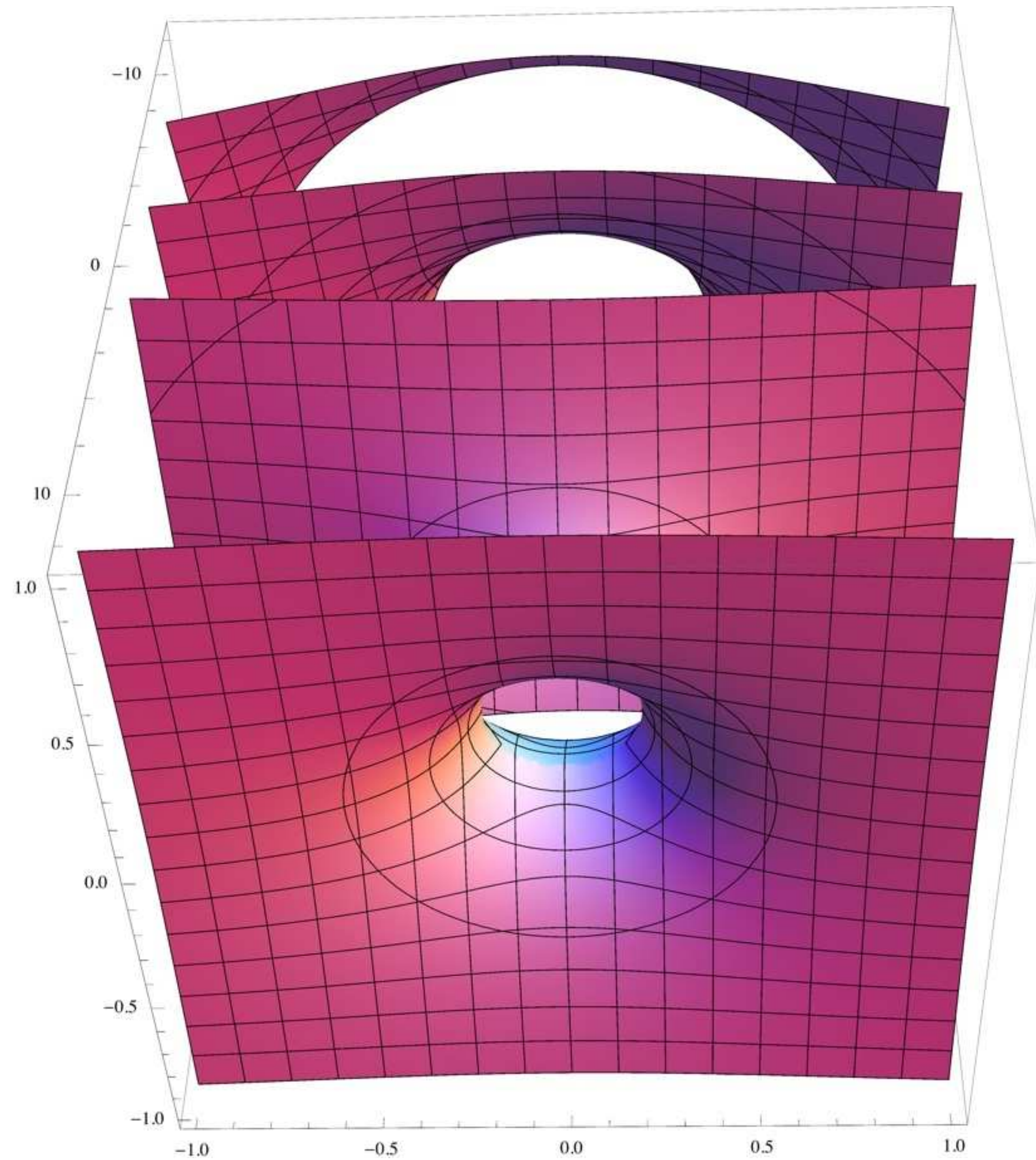
```
Plot3D[2 π Abs[z] - 2 π x, {z, -1, 1}, {x, -1, 1}]
```

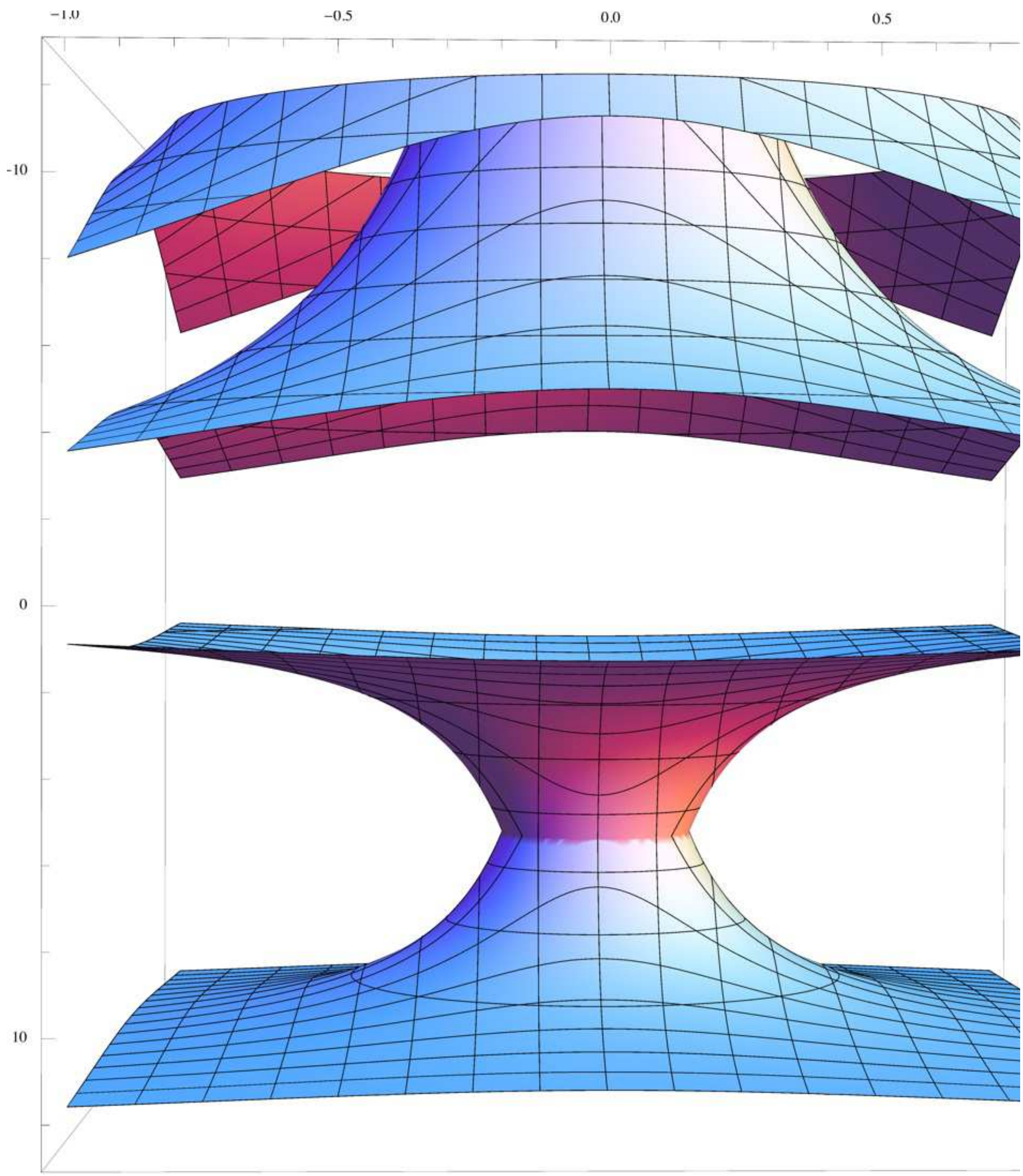


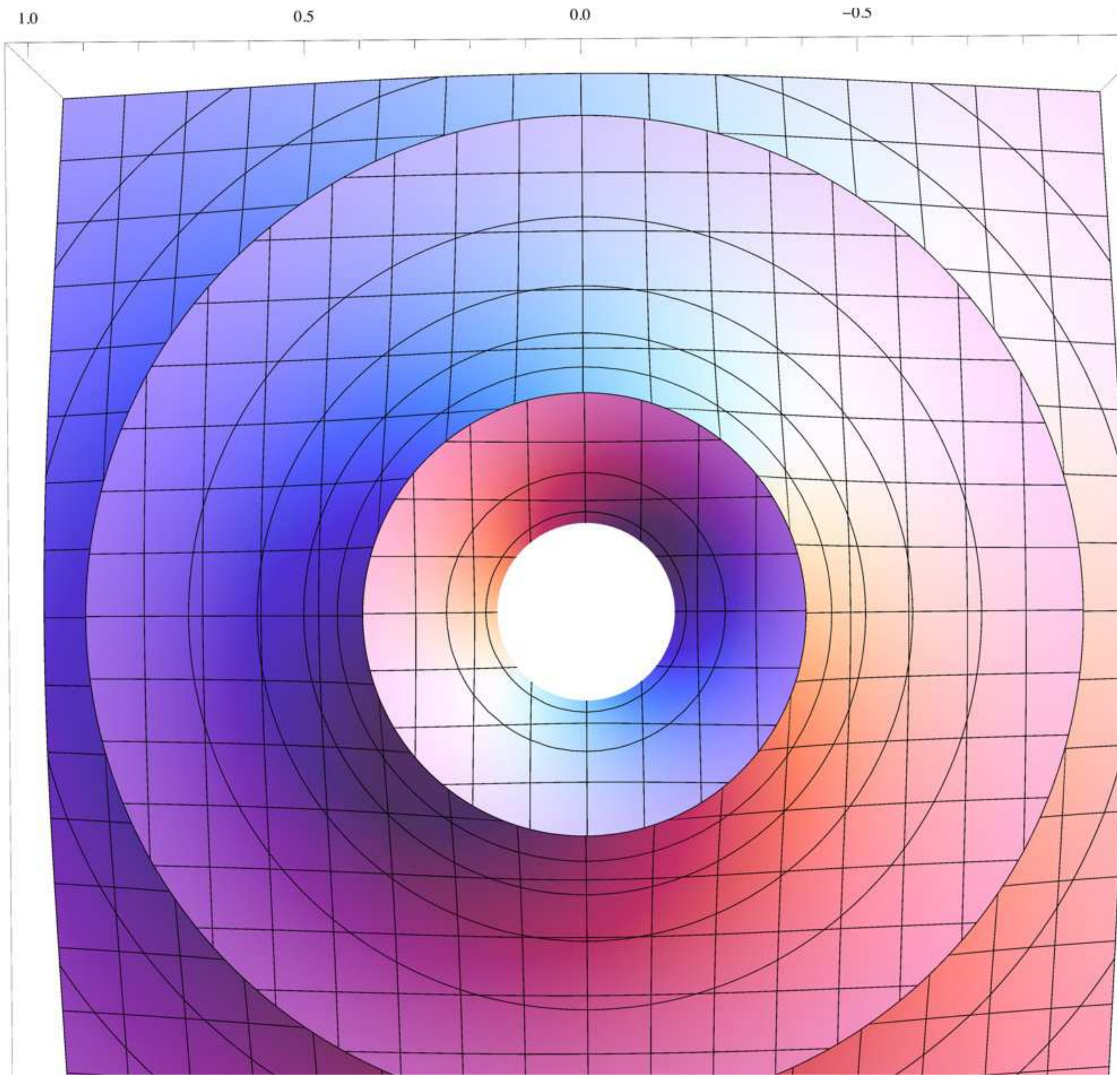
$$\theta \, \mathrm{Abs}[z] = 2\pi \, \mathrm{Abs}[z] - 2\pi \sqrt{((\mathrm{Abs}[z])^2 - y^2)} \quad (44)$$

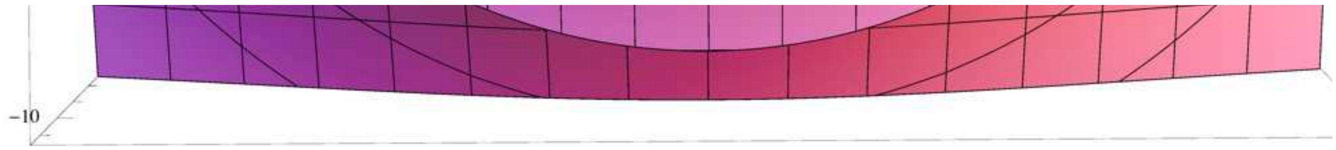
z = x+i y

```
ContourPlot3D[2 π Abs[x + ⅈ y] - 2 π Sqrt[(Abs[x + ⅈ y]^2 - ((Sqrt[4 π - θ] Sqrt[θ] Abs[x + ⅈ y])/(2 π))^2)],
 {x, -1, 1}, {y, -1, 1}, {θ, -4 π, 4 π}]
```




-1.0
-0.5
0.0
0.5
-10
0
10

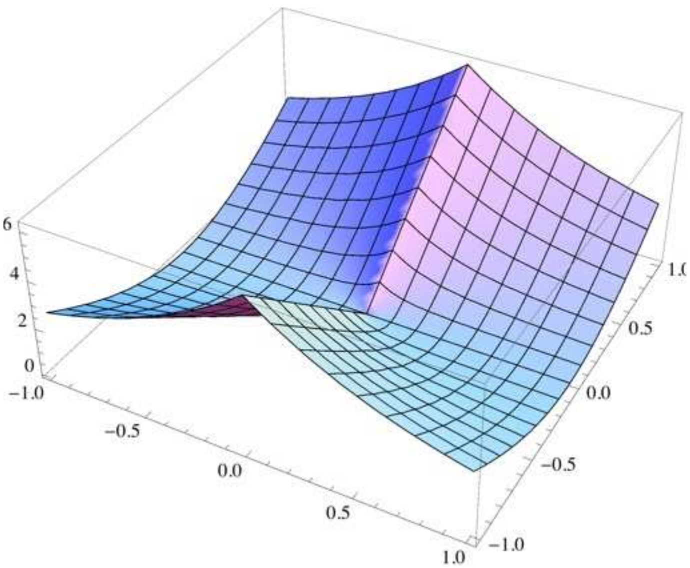

1.0
0.5
0.0
−0.5



$r = \text{Abs}[z] = \text{Abs}[x + i\, y] = \sqrt{x^2 + y^2}$

$$\theta\, r = 2\pi\, \text{Abs}[x + i\, y] - 2\pi \sqrt{(\text{Abs}[x + i\, y]^{\wedge}2 - y^{\wedge}2)} \quad (45)$$

```
Plot3D[2 π Abs[x + ⅈ y] - 2 π Sqrt[(Abs[x + ⅈ y]^2 - y^2)], {x, -1, 1}, {y, -1, 1}]
```

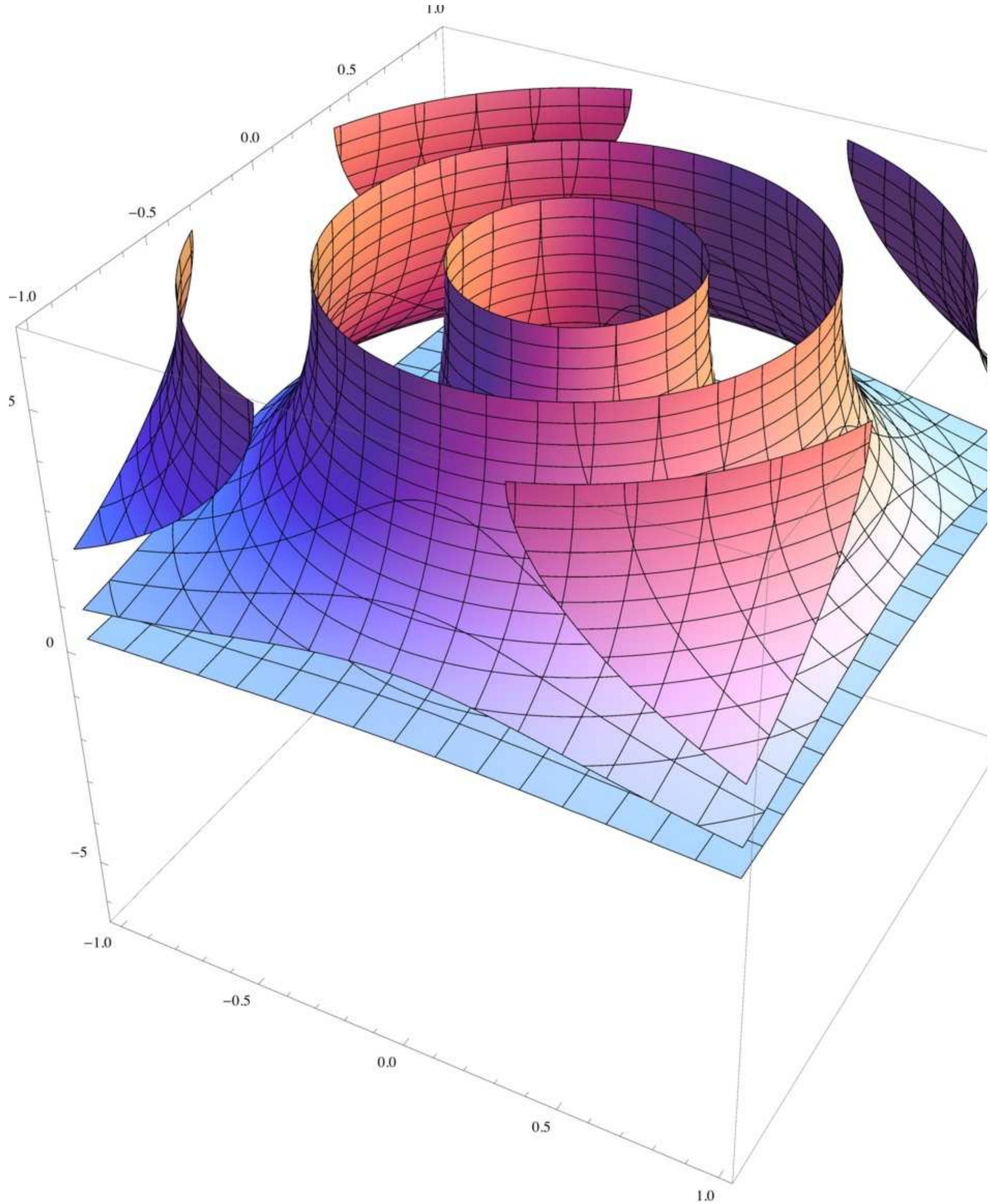



```
Solve[θ Abs[z] == 2 π Abs[z] - 2 π Sqrt[(Abs[z]^2 - y^2)], y]
```

$$\left\{\left\{y \to -\frac{\sqrt{4\pi - \theta}\ \sqrt{\theta}\ \text{Abs}[z]}{2\pi}\right\}, \left\{y \to \frac{\sqrt{4\pi - \theta}\ \sqrt{\theta}\ \text{Abs}[z]}{2\pi}\right\}\right\}$$

$$y = \text{Abs}[z]\, \text{Sin}[\beta] = \sqrt{-x^2 + \text{Abs}[z]^2} = \frac{\sqrt{4\pi - \theta}\ \sqrt{\theta}\ \text{Abs}[z]}{2\pi} = \frac{\sqrt{4\pi - \theta}\ \sqrt{\theta}\ \text{Abs}[x + i\, y]}{2\pi} \quad (46)$$

```
ContourPlot3D[Sqrt[4 π - θ] Sqrt[θ] Abs[x + ⅈ y] / (2 π), {x, -1, 1}, {y, -1, 1}, {θ, -2 π, 2 π}]
```



$$\text{Solve}\left[y == \frac{\sqrt{4\pi - \theta}\ \sqrt{\theta}\ \text{Abs}[z]}{2\pi}, \theta\right]$$

$$\left\{\left\{\theta \to \frac{2\pi\left(\text{Abs}[z]^2 - \sqrt{-y^2\,\text{Abs}[z]^2 + \text{Abs}[z]^4}\right)}{\text{Abs}[z]^2}\right\}, \left\{\theta \to \frac{2\pi\left(\text{Abs}[z]^2 + \sqrt{-y^2\,\text{Abs}[z]^2 + \text{Abs}[z]^4}\right)}{\text{Abs}[z]^2}\right\}\right\}$$

Now I can visualize the height of the cone in terms of complex analysis.

$$y \to \frac{\sqrt{4\pi - \theta}\ \sqrt{\theta}\ \text{Abs}[z]}{2\pi} =$$

$$\frac{\sqrt{4\pi - \frac{2\pi\left(\text{Abs}[z]^2+\sqrt{-y^2\,\text{Abs}[z]^2+\text{Abs}[z]^4}\right)}{\text{Abs}[z]^2}}\ \sqrt{\frac{2\pi\left(\text{Abs}[z]^2+\sqrt{-y^2\,\text{Abs}[z]^2+\text{Abs}[z]^4}\right)}{\text{Abs}[z]^2}}\ \text{Abs}[z]}{2\pi} =$$

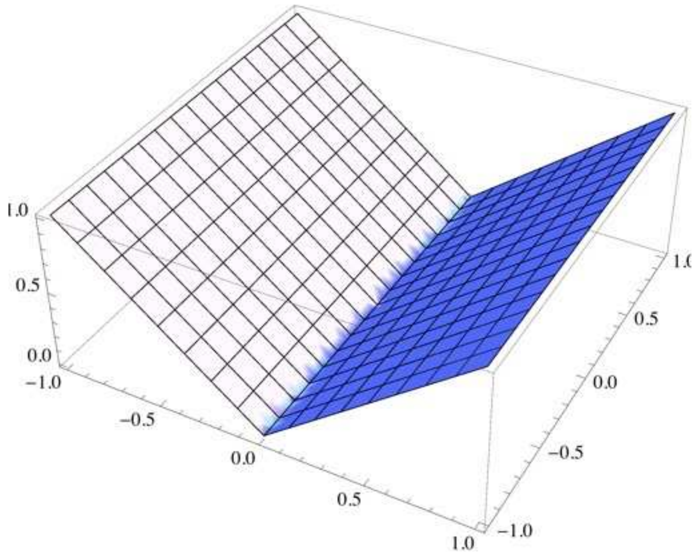
$$\frac{1}{2\pi}\sqrt{4\pi - \theta}\ \sqrt{\frac{2\pi\left(\text{Abs}[z]^2 + \sqrt{-y^2\,\text{Abs}[z]^2 + \text{Abs}[z]^4}\right)}{\text{Abs}[z]^2}}\ \text{Abs}[z] = \tag{47}$$

$$\frac{1}{2\pi}\sqrt{4\pi - \frac{2\pi\left(\text{Abs}[z]^2 + \sqrt{-y^2\,\text{Abs}[z]^2 + \text{Abs}[z]^4}\right)}{\text{Abs}[z]^2}}\ \sqrt{\theta}\ \text{Abs}[z] =$$

$$\frac{\sqrt{4\pi - \frac{2\pi\left(\text{Abs}[x+i\,y]^2+\sqrt{-y^2\,\text{Abs}[x+i\,y]^2+\text{Abs}[x+i\,y]^4}\right)}{\text{Abs}[x+i\,y]^2}}\ \sqrt{\theta}\ \text{Abs}[x+i\,y]}{2\pi} = \sqrt{4\pi - \theta}$$

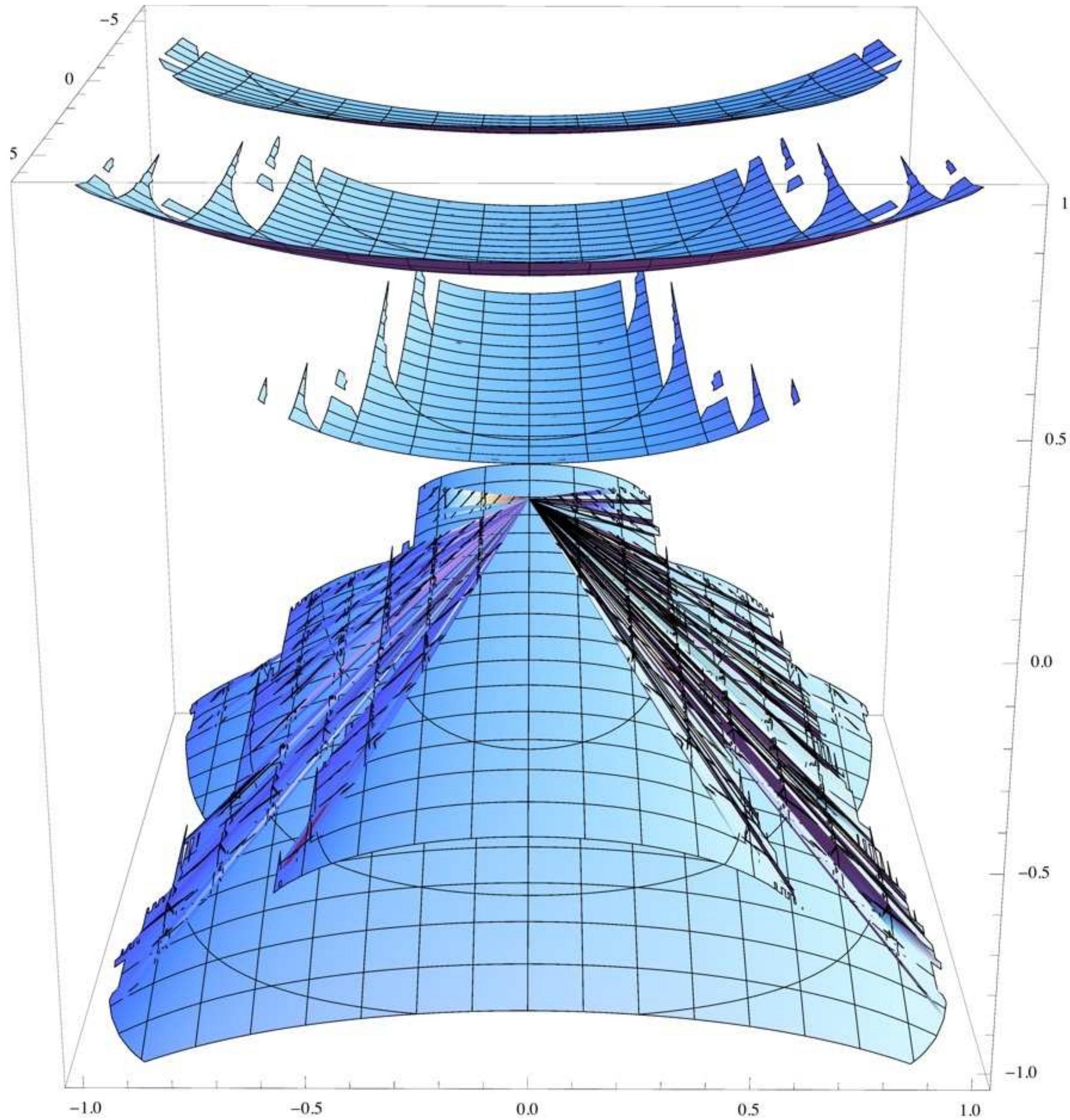
$$\sqrt{\frac{2\pi\left(\text{Abs}[x+i\,y]^2 + \sqrt{-y^2\,\text{Abs}[x+i\,y]^2 + \text{Abs}[x+i\,y]^4}\right)}{\text{Abs}[x+i\,y]^2}}\ \text{Abs}[x+i\,y]$$

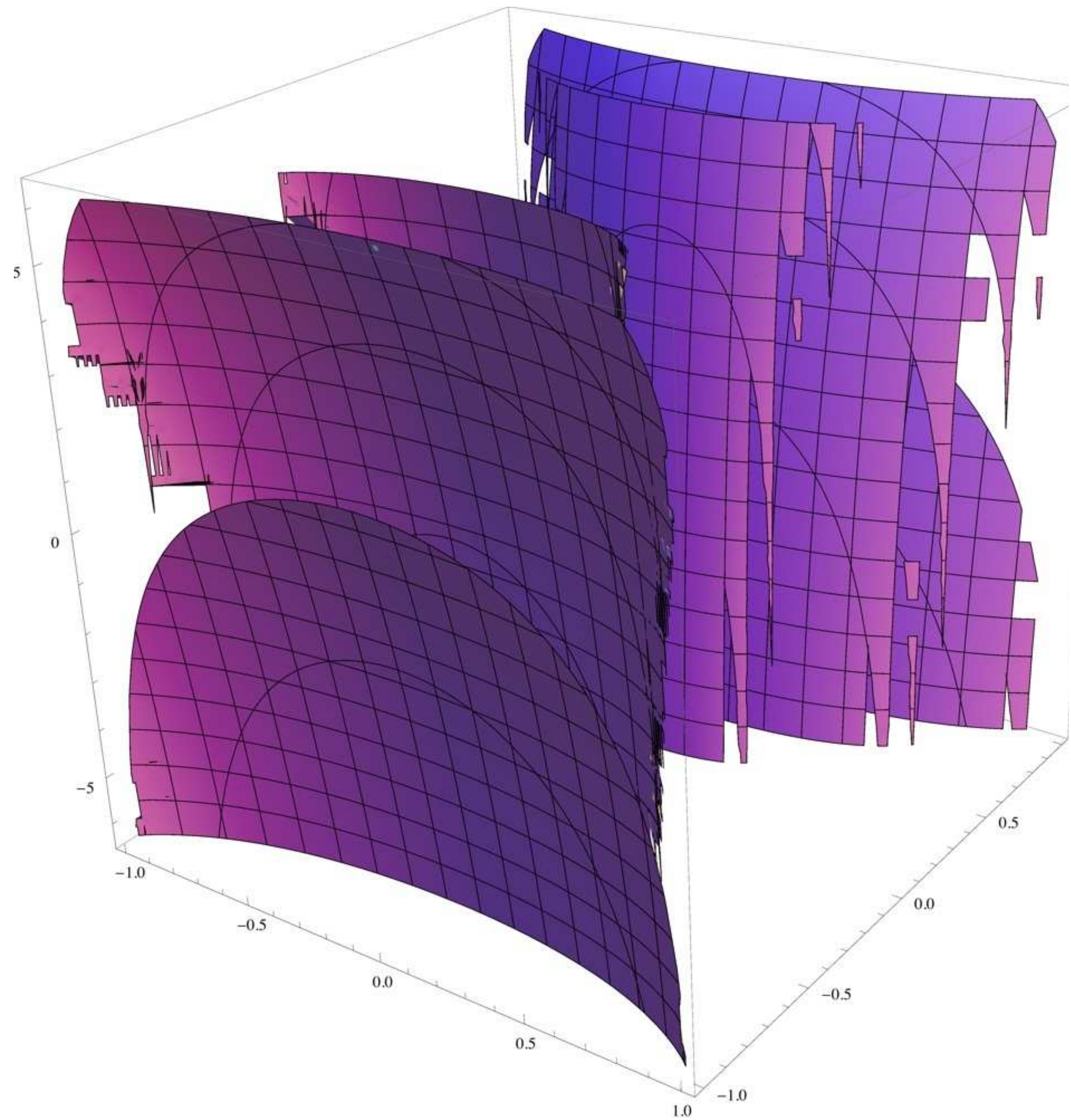
$$\texttt{Plot3D}\left[\frac{1}{2\,\pi}\sqrt{4\,\pi-\frac{2\,\pi\left(\texttt{Abs}[z]^2+\sqrt{-y^2\,\texttt{Abs}[z]^2+\texttt{Abs}[z]^4}\right)}{\texttt{Abs}[z]^2}}\sqrt{\frac{2\,\pi\left(\texttt{Abs}[z]^2+\sqrt{-y^2\,\texttt{Abs}[z]^2+\texttt{Abs}[z]^4}\right)}{\texttt{Abs}[z]^2}}\ \texttt{Abs}[z],\ \{y,-1,1\},\ \{z,-1,1\}\right]$$



$$\text{ContourPlot3D}\left[\frac{1}{2\pi}\sqrt{4\pi-\theta}\sqrt{\frac{2\pi\left(\text{Abs}[z]^2+\sqrt{-y^2\,\text{Abs}[z]^2+\text{Abs}[z]^4}\right)}{\text{Abs}[z]^2}}\ \text{Abs}[z],\right.$$

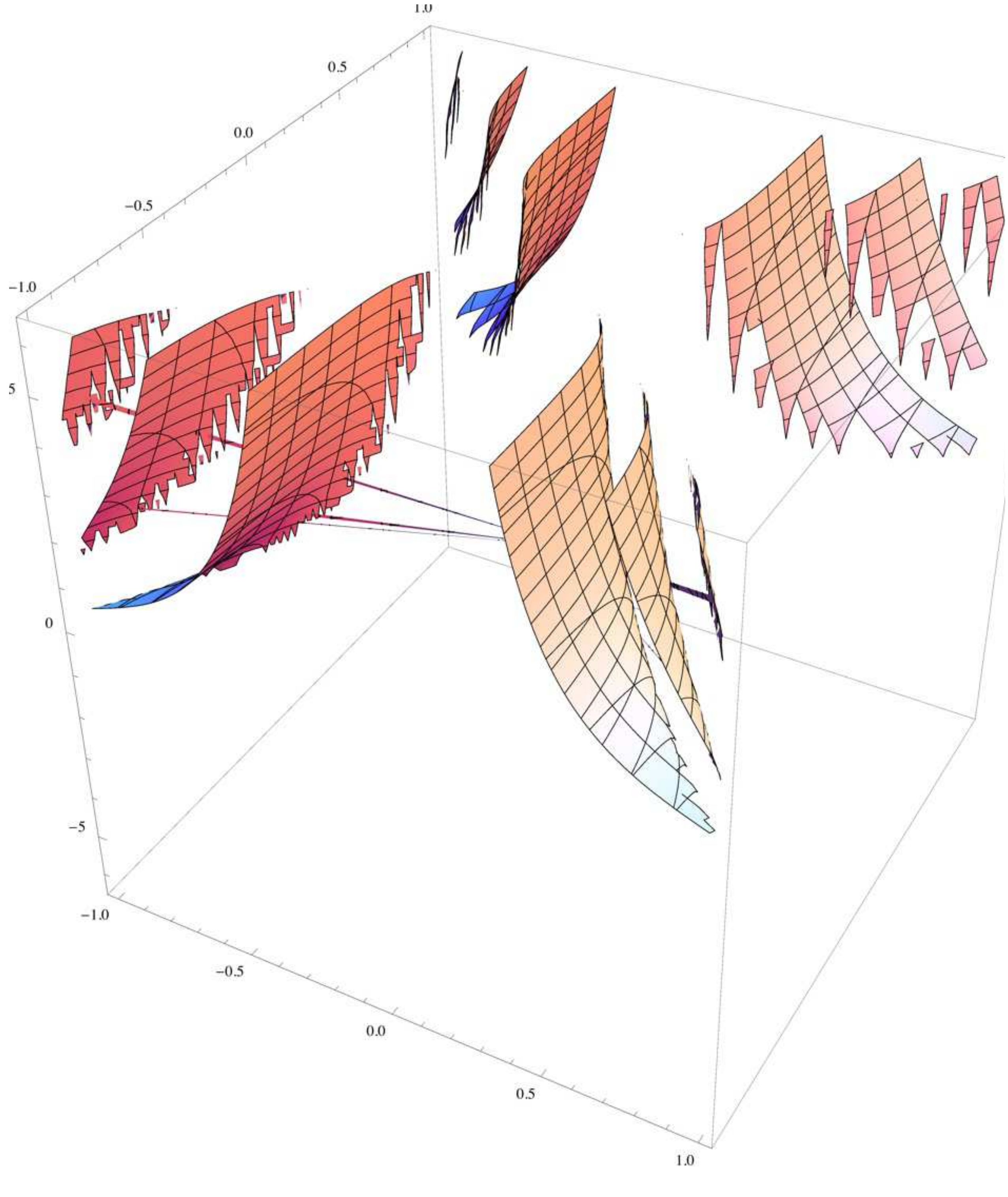
$$\left.\{y,-1,1\},\{z,-1,1\},\{\theta,-2\pi,2\pi\}\right]$$




5
0
−5
−1.0
−0.5
0.0
0.5
1.0
−1.0
−0.5
0.0
0.5

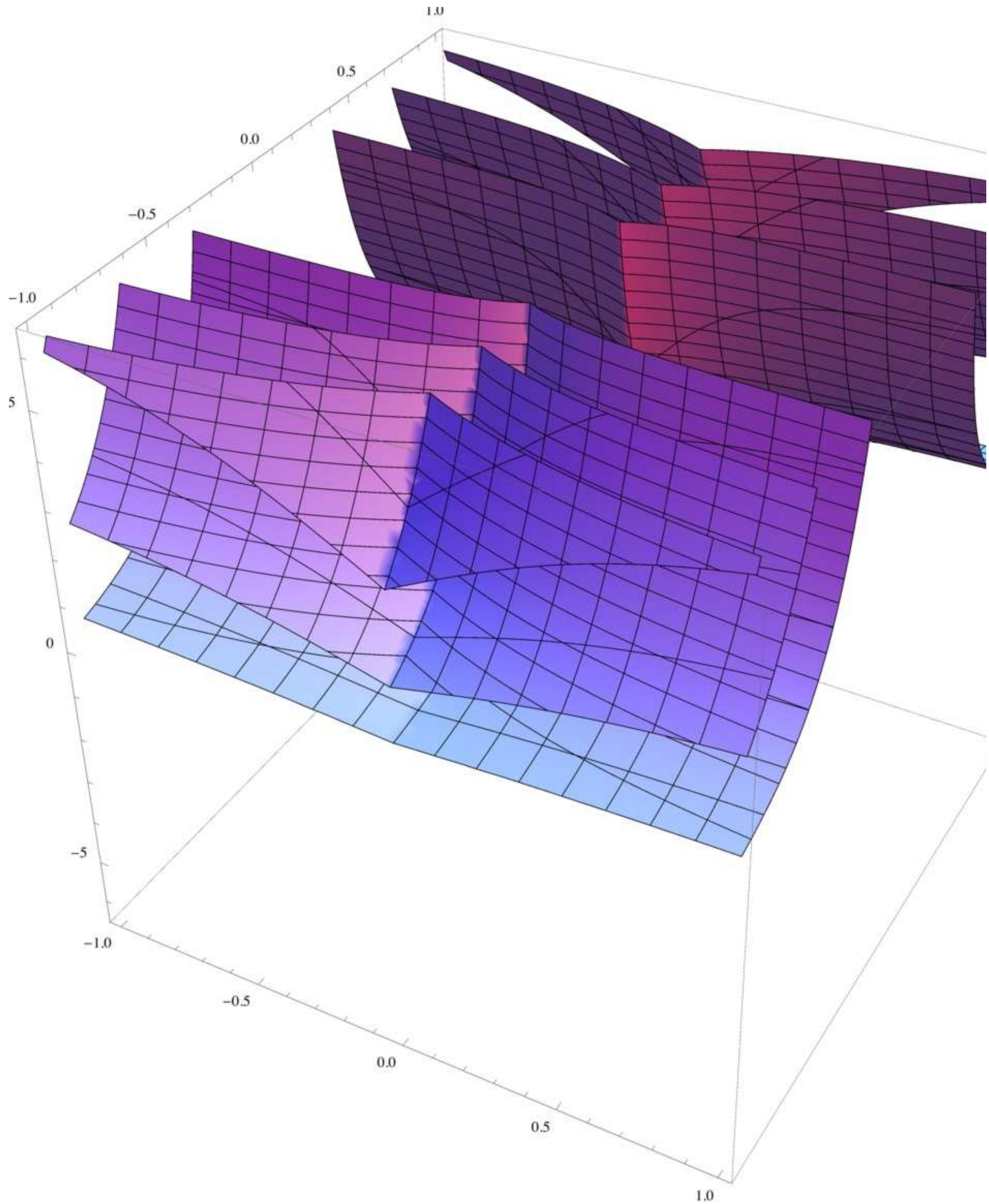
$$\text{ContourPlot3D}\left[\frac{1}{2\pi}\sqrt{4\pi - \frac{2\pi\left(\text{Abs}[z]^2 + \sqrt{-y^2\,\text{Abs}[z]^2 + \text{Abs}[z]^4}\right)}{\text{Abs}[z]^2}}\ \sqrt{\theta}\ \text{Abs}[z],\right.$$

$$\left.\{y, -1, 1\}, \{z, -1, 1\}, \{\theta, -2\pi, 2\pi\}\right]$$

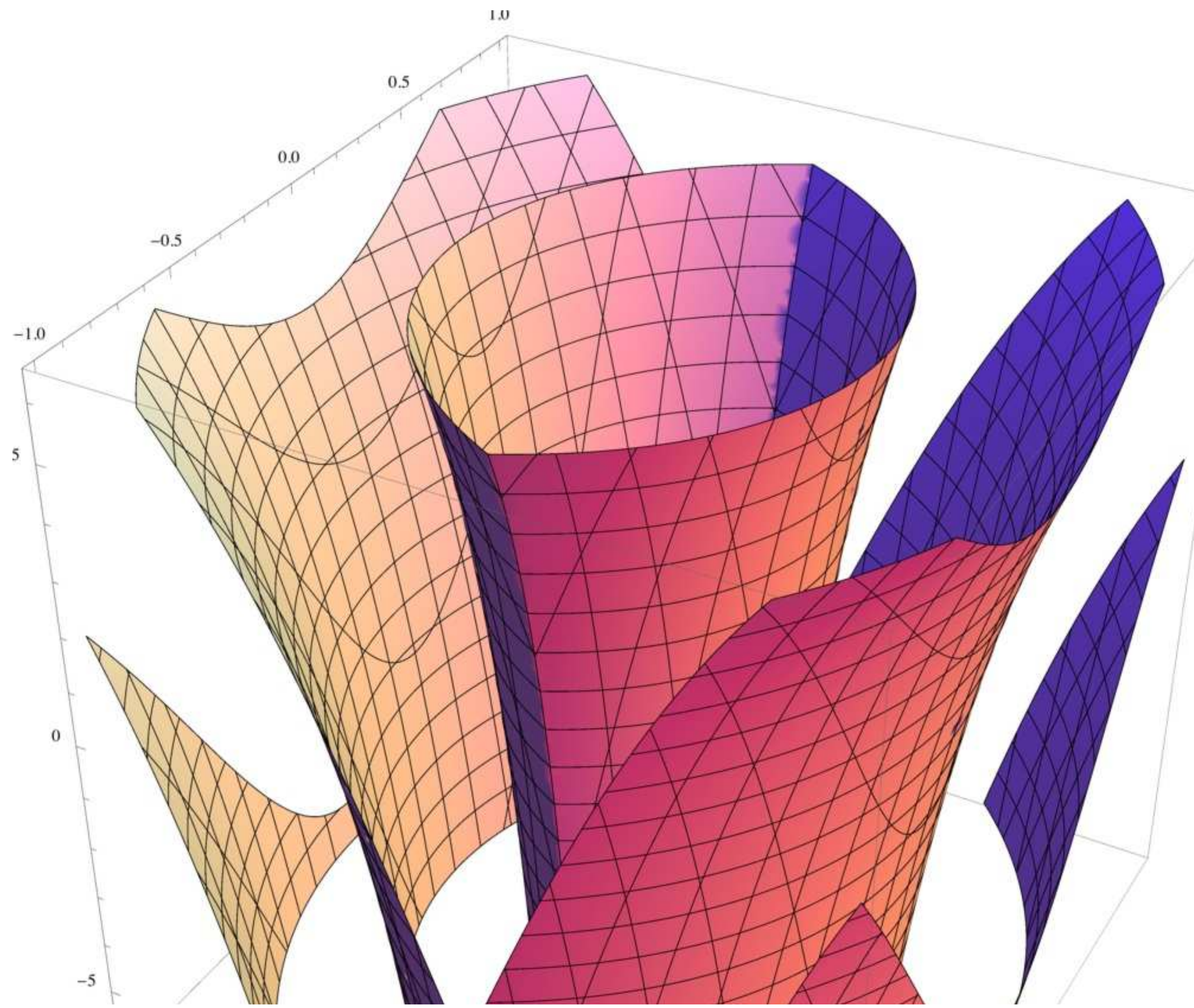


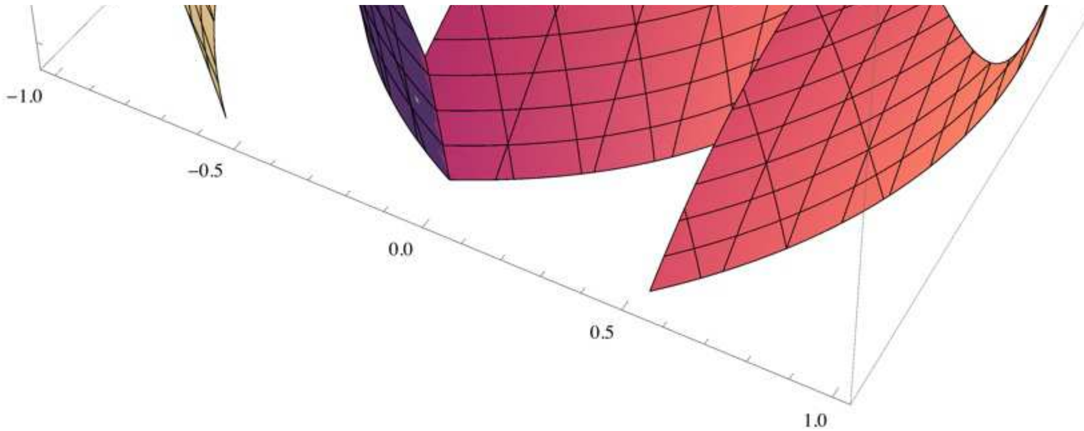
$$\texttt{ContourPlot3D}\left[\frac{\sqrt{4\pi - \frac{2\pi\left(\texttt{Abs}[x+i\,y]^2+\sqrt{-y^2\,\texttt{Abs}[x+i\,y]^2+\texttt{Abs}[x+i\,y]^4}\right)}{\texttt{Abs}[x+i\,y]^2}}\;\sqrt{\theta}\;\texttt{Abs}[x+i\,y]}{2\pi},\right.$$

$$\left.\{x, -1, 1\}, \{y, -1, 1\}, \{\theta, -2\pi, 2\pi\}\right]$$


1.0
0.5
0.0
−0.5
−1.0
5
0
−5
−1.0
−0.5
0.0
0.5
1.0

```
ContourPlot3D[1/(2 π) Sqrt[4 π - θ] Sqrt[(2 π (Abs[x + ⅈ y]^2 + Sqrt[-y^2 Abs[x + ⅈ y]^2 + Abs[x + ⅈ y]^4]))/Abs[x + ⅈ y]^2] Abs[x + ⅈ y],
 {x, -1, 1}, {y, -1, 1}, {θ, -2 π, 2 π}]
```





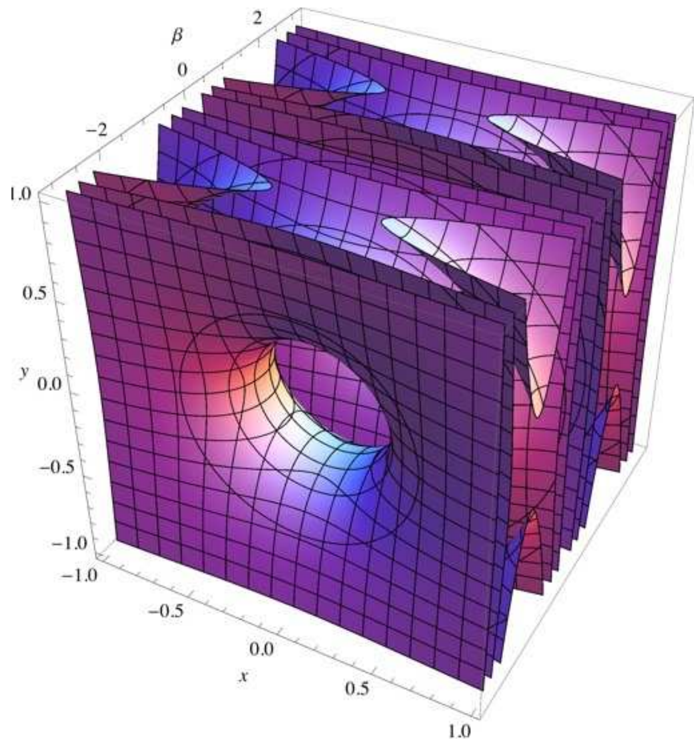
$$y \to \frac{\sqrt{4\pi-\Theta}\ \sqrt{\Theta}\ \text{Abs}[z]}{2\pi} =$$

$$\frac{\sqrt{4\pi-2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)}\ \sqrt{2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)}\ \text{Abs}[x+\mathbb{i}\,y]}{2\pi} =$$

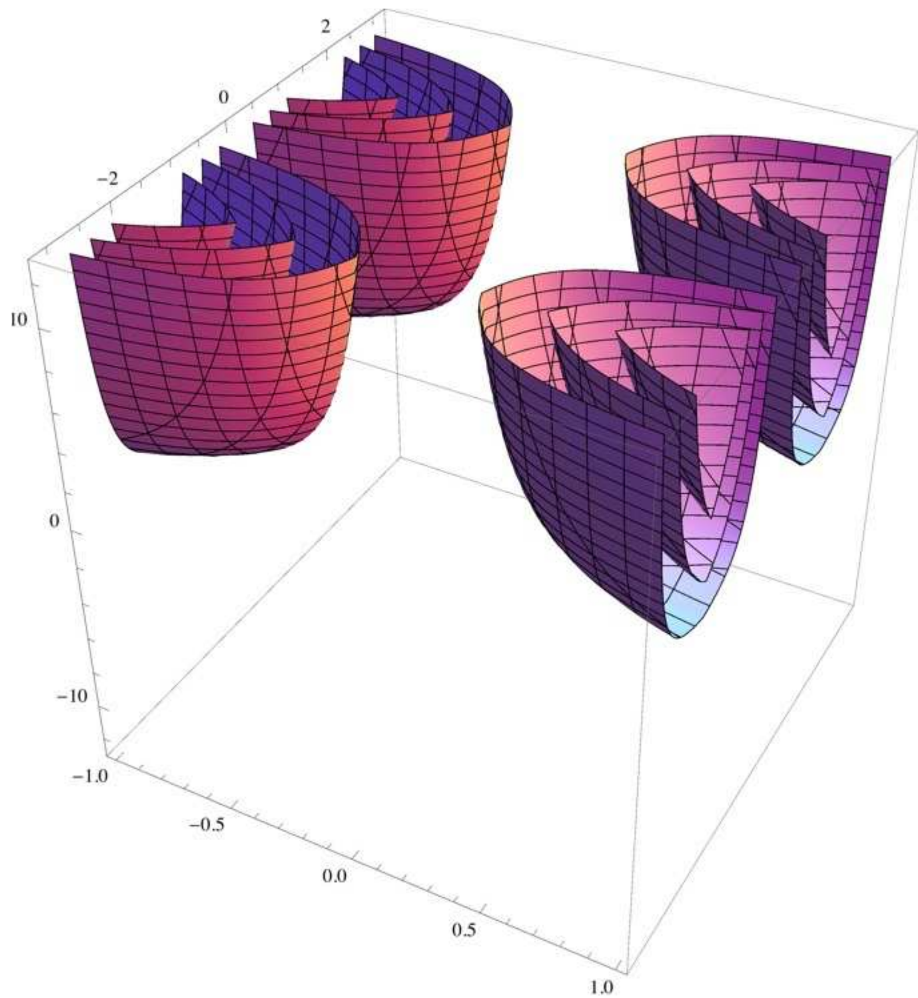
$$\frac{\sqrt{4\pi-2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)}\ \sqrt{\Theta}\ \text{Abs}[z]}{2\pi} = \qquad (48)$$

$$\frac{\sqrt{4\pi-\Theta}\ \sqrt{2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)}\ \text{Abs}[z]}{2\pi}$$

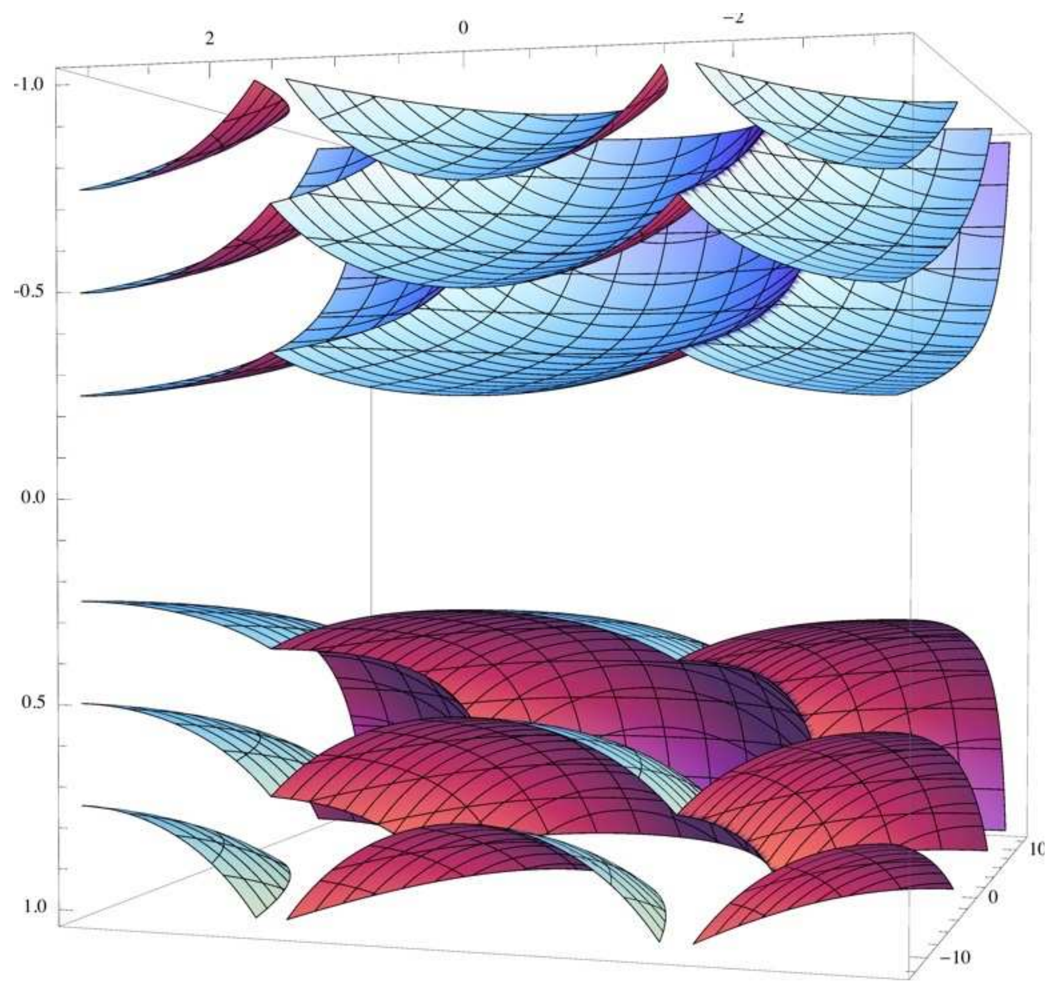
```
ContourPlot3D[(Sqrt[4 π - 2 (π + Sqrt[π^2 - π^2 Sin[β]^2])] Sqrt[2 (π + Sqrt[π^2 - π^2 Sin[β]^2])] Abs[x + ⅈ y])/(2 π),
 {x, -1, 1}, {β, -π, π}, {y, -1, 1}, AxesLabel → Automatic]
```



$$\text{ContourPlot3D}\left[\frac{\sqrt{4\pi - 2\left(\pi + \sqrt{\pi^2 - \pi^2 \,\text{Sin}[\beta]^2}\right)}\ \sqrt{\theta}\ \text{Abs}[z]}{2\pi}, \{z, -1, 1\}, \{\beta, -\pi, \pi\}, \{\theta, -4\pi, 4\pi\}\right]$$



$$\text{ContourPlot3D}\left[\frac{\sqrt{4\pi - \theta}\ \sqrt{2\left(\pi + \sqrt{\pi^2 - \pi^2 \,\text{Sin}[\beta]^2}\right)}\ \text{Abs}[z]}{2\pi}, \{z, -1, 1\}, \{\beta, -\pi, \pi\}, \{\theta, -4\pi, 4\pi\}\right]$$



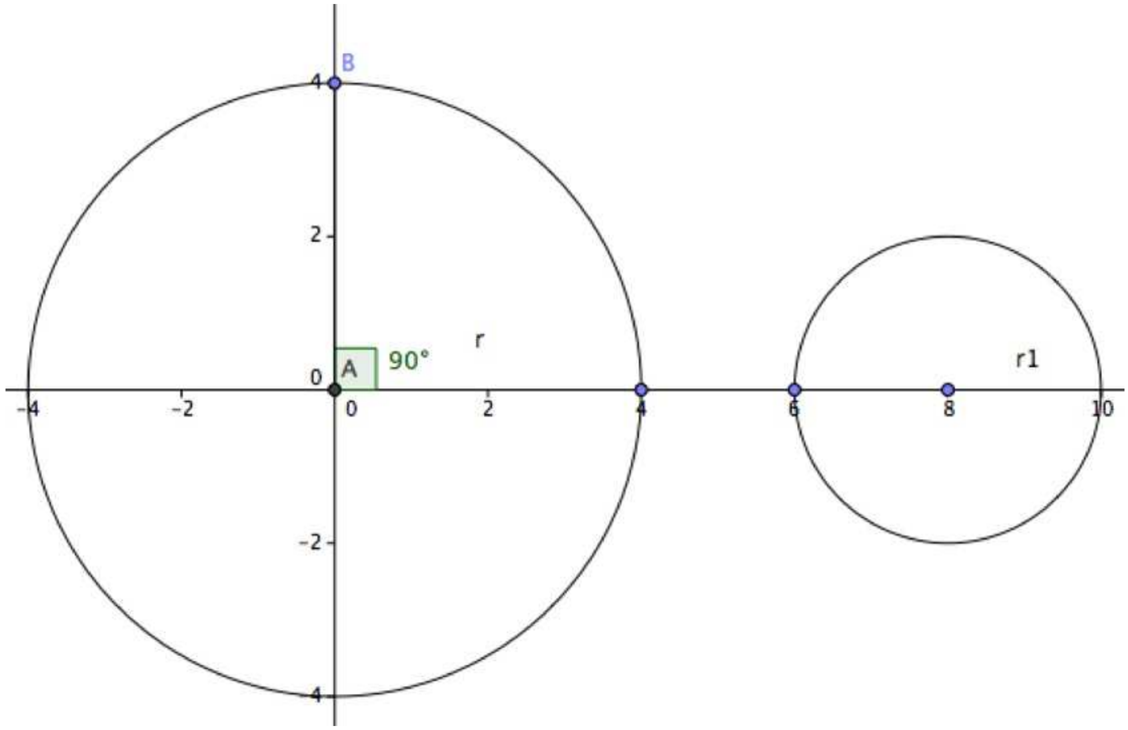
XIV. A Proof for Non-Comprehensiveness of Euclidean Geometry (Due to its Consistency)

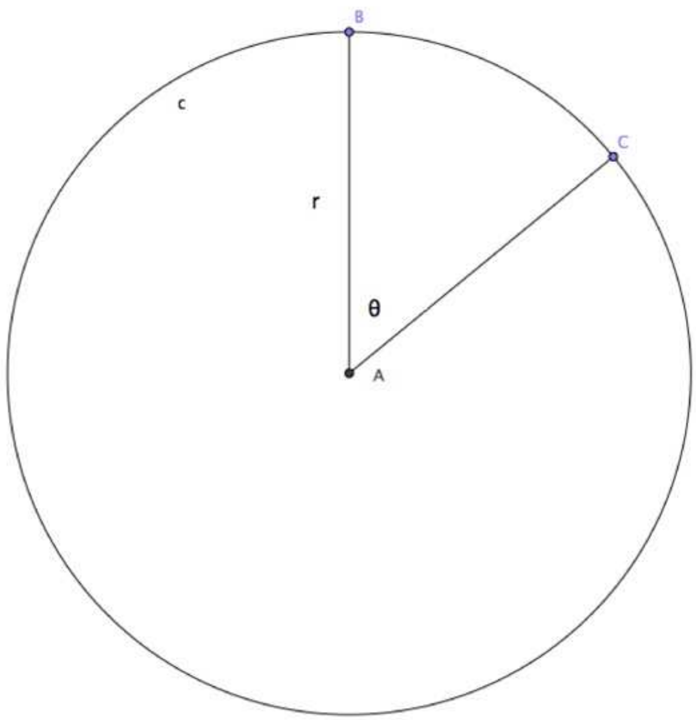
- **Postulates**

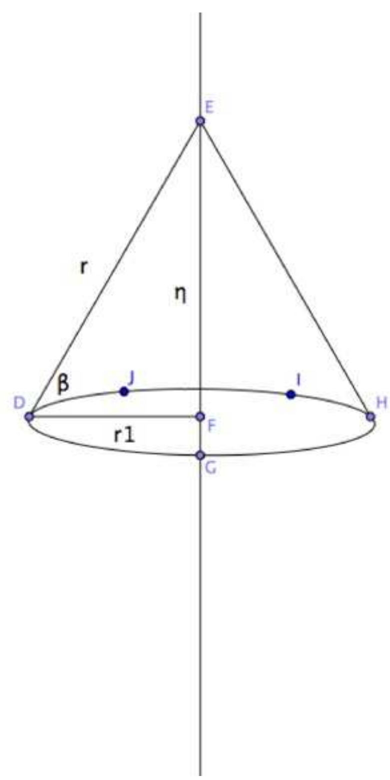
I. Math for Transforming a Circle into a Cone

by Parker Emmerson

Parameters not drawn to scale.







When a sector of a circle is collapsed (removed), we may "fold up" the resulting shape into a cone. The parameters are related by the following theorem :

Theorem 1 *When a sector of angle θ is removed from a circle of radius r and the resulting shape is folded into a cone, then the base of the cone has radius r_1 given by $r_1 = r - \frac{(r\,\theta)}{2\pi}$; and height η, given by* $\eta = \sqrt{r^2 - r_1{}^2} = \text{r Sin}[\beta]$

Proof. The circumference of the initial circle is 2π r and the wedge removed has an arc length r θ. Therefore, the remaining circumference is of length r $(2\pi - \theta)$, and after the fold, this is the circumference of the base of the cone.

Establishing the circumference of the base of the cone, from the equation, $\theta r = 2\pi r - 2\pi r_1$, we calculate that its radius r_1 is $\frac{2\pi r - r\theta}{2\pi}$, which simplifies to $r - \frac{r\theta}{2\pi}$. Thus, we have proved the first part of the theorem.

To find the height of the cone, η, we apply the Pythagorean theorem to a right triangle formed between the apex of the cone, the center of the base, and a point on the circumference of the base. This gives $\eta = \sqrt{r^2 - r_1^2} = r \,\mathrm{Sin}[\beta]$, where β is the angle formed by the slant of the cone and the base of the cone. The initial radius is always equal to the slant of the cone, and the height of the cone is always orthogonal to the center of the base of the cone.

Lemma 1 The height of the cone can be caluclated in terms of r and θ.

Proof.

$$\theta r = 2\pi r - 2\pi r_1$$

$$\eta = \sqrt{r^2 - r_1^2}$$

$$\theta r = 2\pi r - 2\pi \sqrt{(r\hat{\ }2 - \eta\hat{\ }2)}$$

Solving this equation we find that,

$$\left\{\left\{\eta \to -\frac{\sqrt{4\pi r^2 \theta - r^2 \theta^2}}{2\pi}\right\}, \left\{\eta \to \frac{\sqrt{4\pi r^2 \theta - r^2 \theta^2}}{2\pi}\right\}\right\}$$

Lemma 2 The angle θ can be calculated in terms of r and η.

Proof

$$\mathbf{Solve}\left[\eta == \frac{\sqrt{4\pi r^2 \theta - r^2 \theta^2}}{2\pi}, \theta\right]$$

$$\left\{\left\{\theta \to \frac{2\pi\left(r^2 - \sqrt{r^4 - r^2\eta^2}\right)}{r^2}\right\}, \left\{\theta \to \frac{2\pi\left(r^2 + \sqrt{r^4 - r^2\eta^2}\right)}{r^2}\right\}\right\}$$

Lemma 3 The initial radius is a function of θ and η.

$$\mathrm{Solve}\left[\frac{\sqrt{4\pi r^2 \theta - r^2 \theta^2}}{2\pi} == \eta, r\right]$$

$$\left\{\left\{r \to -\frac{2\pi\eta}{\sqrt{4\pi\theta - \theta^2}}\right\}, \left\{r \to \frac{2\pi\eta}{\sqrt{4\pi\theta - \theta^2}}\right\}\right\}$$

Lemma 4 *The height of the cone can be calculated in terms of only r and* θ, thus β is a function of θ alone.

Proof. Since we have shown that $\theta r = 2\pi r - 2\pi r_1$ and $r_1 \to \sqrt{r^2 - \eta^2}$, we can substitute the expression for r_1, calculated from the Pythagorean theorem in terms of the height of the cone and the initial radius of the circle, into the expression for θ r in terms of the change in circumference of the initial circle to the circle that is the base of the cone into which the circle was transformed.

$\theta r = 2\pi r - 2\pi\sqrt{(r\hat{\ }2 - \eta\hat{\ }2)}$, thus, $\eta = \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}$ = (r Sin[β]). From $\frac{2\pi\eta}{\sqrt{4\pi\theta - \theta^2}}$ = r, we note that: r = $\frac{2\pi r\,\mathrm{Sin}[\beta]}{\sqrt{4\pi\theta - \theta^2}}$. So we solve the equation,

$$\text{Solve}\left[r == \frac{2\,\pi\,r\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\Theta - \Theta^2}},\ \beta\right]$$

$$\left\{\left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{(4\,\pi - \Theta)\,\Theta}}{2\,\pi}\right]\right\}\right\}$$

Lemma 5 *The height of the cone can be calculated in terms of only r and θ*, thus θ is a function of β alone.

Proof. Since we have shown that θ r = 2 π r - 2 π r_1 and $r_1 \to \sqrt{r^2 - \eta^2}$, we can substitute the expression for r_1, calculated from the Pythagorean theorem in terms of the height of the cone and the initial radius of the circle, into the expression for θ r in terms of the change in circumference of the initial circle to the circle that is the base of the cone into which the circle was transformed.

θ r = 2 π r - 2 π $\sqrt{(r\hat{}2 - \eta\hat{}2)}$, thus, $\eta = \frac{\sqrt{4\,\pi\,r^2\,\Theta - r^2\,\Theta^2}}{2\,\pi}$ = (r Sin[β]). From $\frac{2\,\pi\,\eta}{\sqrt{4\,\pi\,\Theta - \Theta^2}}$= r, we note that: r = $\frac{2\,\pi\,r\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\Theta - \Theta^2}}$. So we solve the equation,

$$\text{Solve}\left[r == \frac{2\,\pi\,r\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\Theta - \Theta^2}},\ \Theta\right]$$

$$\left\{\left\{\Theta \to 2\left(\pi - \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)\right\},\ \left\{\Theta \to 2\left(\pi + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)\right\}\right\}$$

■ The Statement that Concludes in Paradox (Evidence of Euclidean Geometry's not Being Comprehensive of All Geometries)

Theorem 4 The initial radius of the circle, r, is a function of only the angle taken out of the initial circle, θ; i.e. r=f(θ). This cannot be proven with Euclidean geometry, thus, this is evidence that Euclidean geometry is consistent, because, any consistent system is not comprehensive, as shown by Gödel in his Incompleteness Theorems.

Proof.

Lemma 6 From Lemma 4, it can be shown that $1=\frac{2\,\pi\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\Theta - \Theta^2}}$

Proof.

$$r == \frac{2\,\pi\,r\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\Theta - \Theta^2}}\text{; r cancels on both sides, therefore } 1 == \frac{2\,\pi\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\Theta - \Theta^2}}$$

Lemma 7 From Theorem 1 and lemma 1, it can be shown that:

$$0 = \Theta\,r - \left(2\,\pi\,r\ -\ 2\,\pi\,\sqrt{\left(r\hat{}2 - \frac{\sqrt{4\,\pi\,r^2\,\Theta - r^2\,\Theta^2}}{2\,\pi}\hat{}2\right)}\right)$$

Proof.

$\theta\, r = 2\,\pi\, r - 2\,\pi \sqrt{(r\hat{}2 - \eta\hat{}2)}$

Subtract θ*r from both sides.

$$0 = \theta\, r - \left(2\,\pi\, r - 2\,\pi \sqrt{\left(r\hat{}2 - \frac{\sqrt{4\,\pi\, r^2\,\theta - r^2\,\theta^2}}{2\,\pi}\hat{}2\right)}\right)$$

Lemma 8 One can calculate the radius as a function of θ and β from constructing the equation 1-1=0

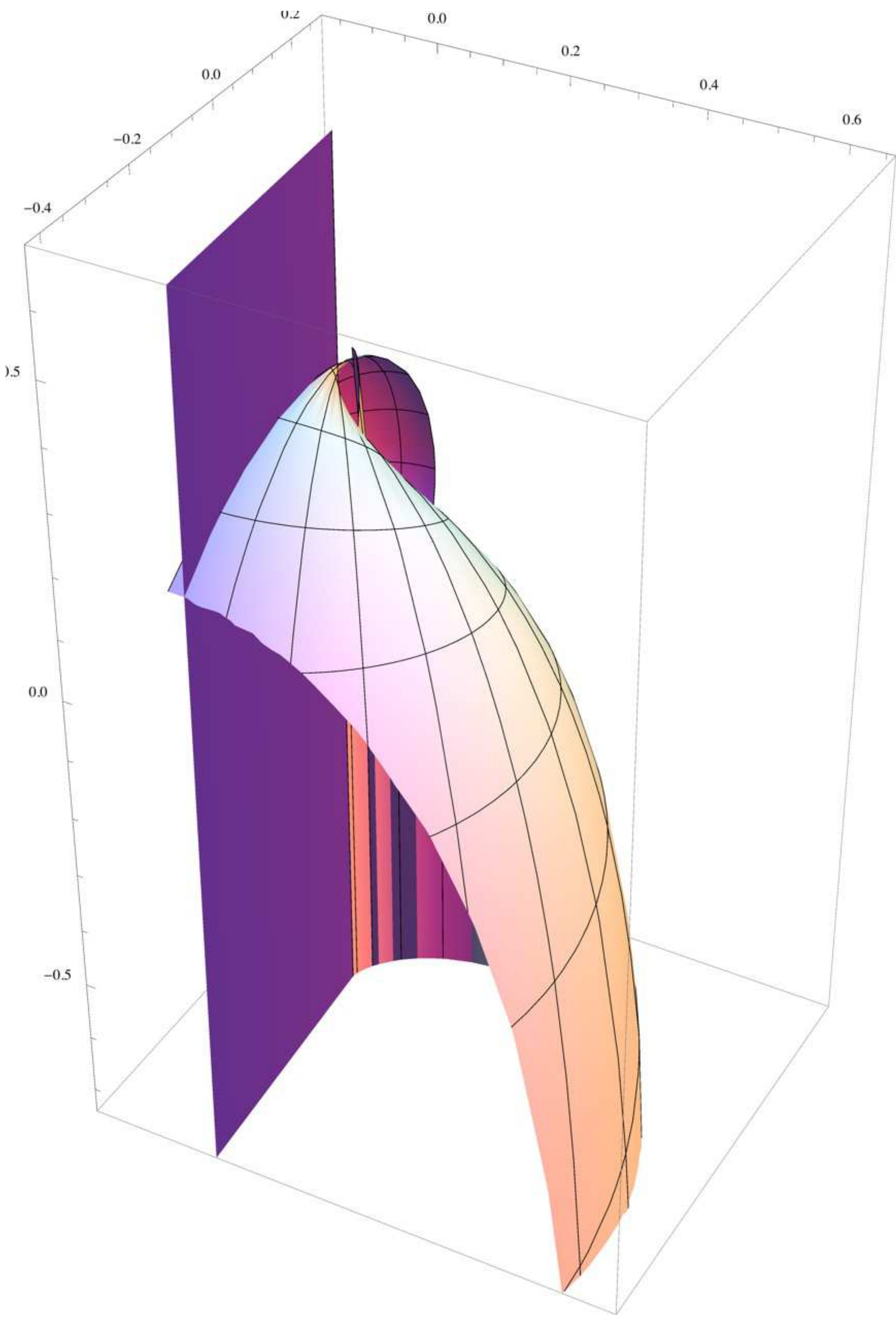
as $1 - \frac{2\,\pi\, \mathrm{Sin}[\beta]}{\sqrt{4\,\pi\,\theta - \theta^2}} == \theta\, r - \left(2\,\pi\, r - 2\,\pi \sqrt{\left(r\hat{}2 - \frac{\sqrt{4\,\pi\, r^2\,\theta - r^2\,\theta^2}}{2\,\pi}\hat{}2\right)}\right)$.

Proof.

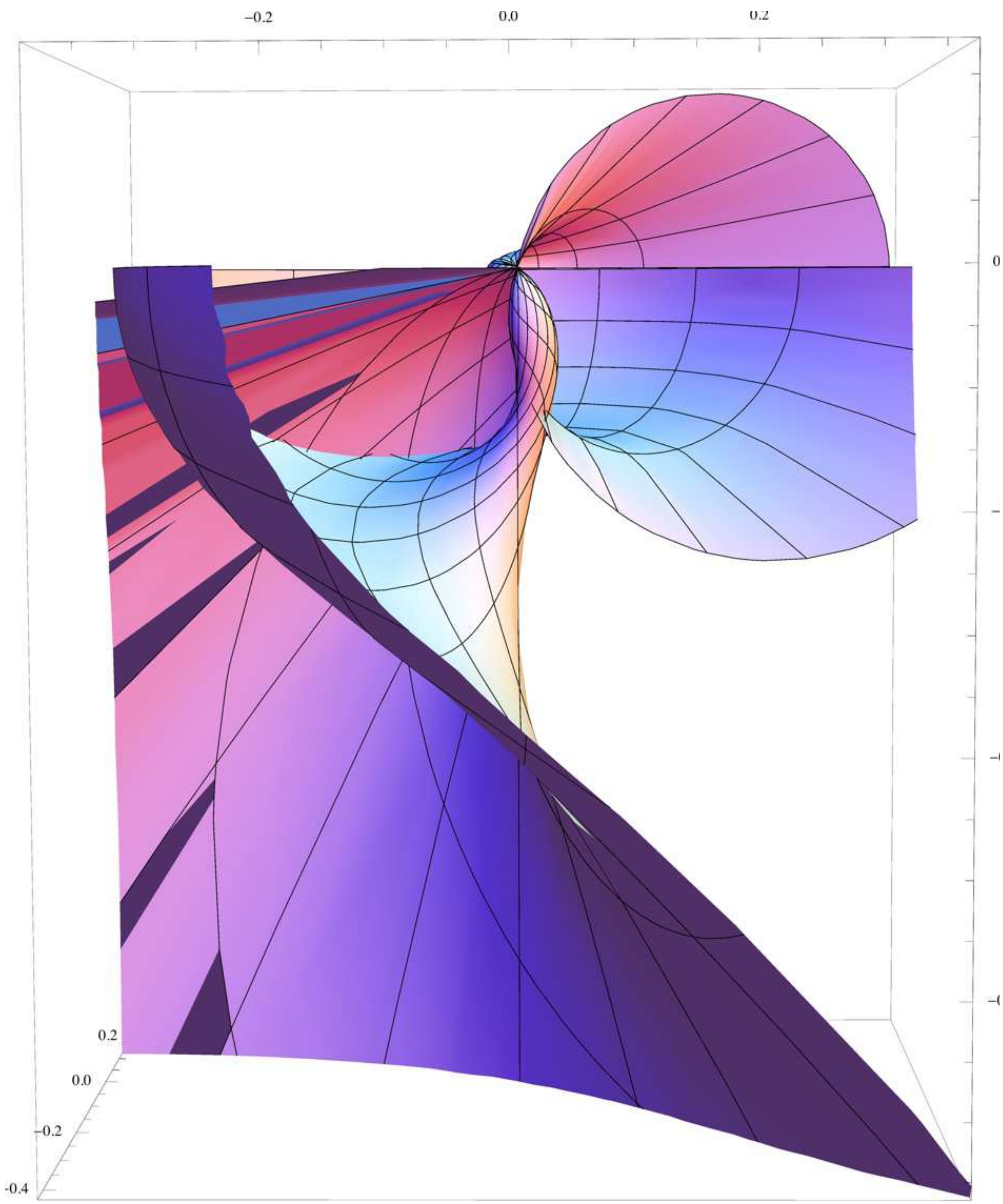
$$\mathbf{Solve}\left[1 - \frac{2\,\pi\,\mathrm{Sin}[\beta]}{\sqrt{4\,\pi\,\theta - \theta^2}} == \theta\, r - \left(2\,\pi\, r - 2\,\pi \sqrt{\left(r\hat{}2 - \frac{\sqrt{4\,\pi\, r^2\,\theta - r^2\,\theta^2}}{2\,\pi}\hat{}2\right)}\right),\ r\right]$$

$$\left\{\left\{r \to \left(-4\,\pi\,\theta + \theta^2 + 2\,\pi\sqrt{(4\,\pi - \theta)\,\theta}\ \mathrm{Sin}[\beta] + 4\,\pi^2\,\mathrm{Sin}[\beta]^2 - \frac{2\,\pi^2\sqrt{(4\,\pi - \theta)\,\theta}\ \mathrm{Sin}[\beta]^3}{4\,\pi - \theta} - \frac{2\,\pi^2\sqrt{(4\,\pi - \theta)\,\theta}\ \mathrm{Sin}[\beta]^3}{\theta}\right) \Big/ \left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)\right\}\right\}$$

$$\mathbf{SphericalPlot3D}\left[\left(-4\,\pi\,\theta + \theta^2 + 2\,\pi\sqrt{(4\,\pi - \theta)\,\theta}\ \mathrm{Sin}[\beta] + 4\,\pi^2\,\mathrm{Sin}[\beta]^2 - \frac{2\,\pi^2\sqrt{(4\,\pi - \theta)\,\theta}\ \mathrm{Sin}[\beta]^3}{4\,\pi - \theta} - \frac{2\,\pi^2\sqrt{(4\,\pi - \theta)\,\theta}\ \mathrm{Sin}[\beta]^3}{\theta}\right) \Big/ \left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right),\ \{\theta, -2\,\pi, 2\,\pi\},\ \{\beta, -\pi/2, \pi/2\}\right]$$


0.2
0.0
−0.2
−0.4
0.0
0.2
0.4
0.6
0.5
0.0
−0.5

$$\text{SphericalPlot3D}\Big[\Big(-4\,\pi\,\theta+\theta^2+2\,\pi\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]+4\,\pi^2\,\text{Sin}[\beta]^2-\frac{2\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{4\,\pi-\theta}-\frac{2\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\Big)\Big/\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right),\ \{\beta,\,-\pi/2,\,\pi/2\},\ \{\theta,\,-2\,\pi,\,2\,\pi\}\Big]$$

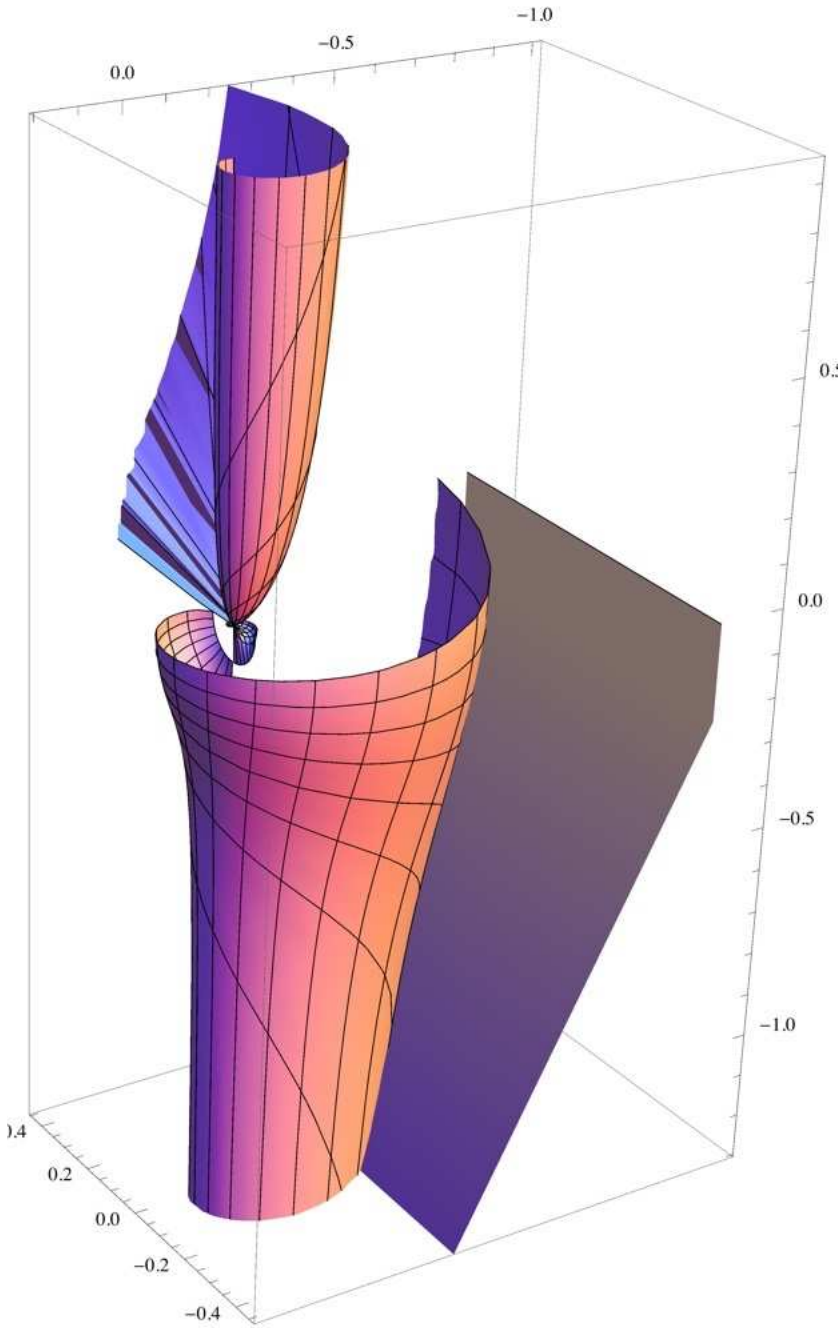


−0.2
0.0
0.2
0
0.2
0.0
-0.2
-0.4

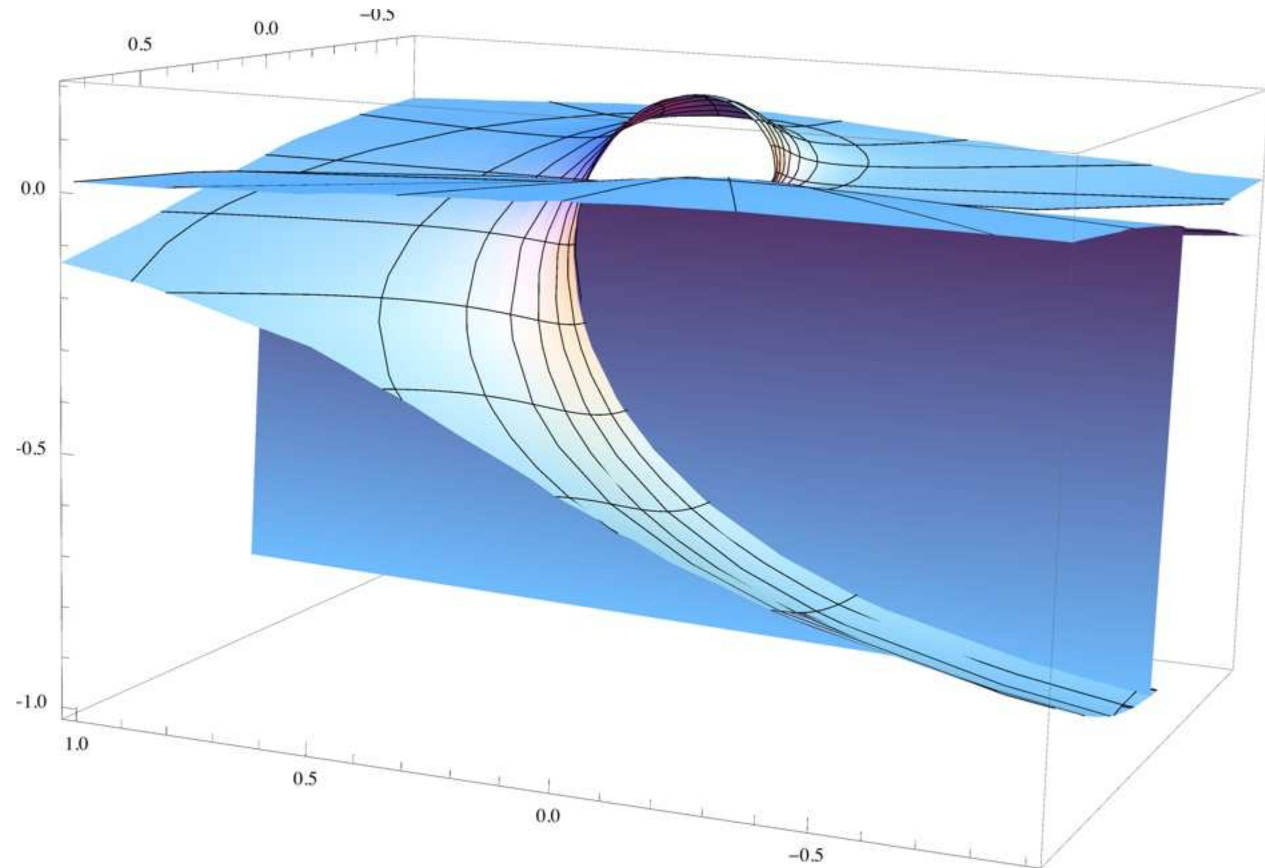
$$\text{Solve}\left[1-\frac{\sqrt{4\pi\theta-\theta^2}}{2\pi\,\text{Sin}[\beta]}==\theta\,r-\left(2\pi r-2\pi\sqrt{\left(r\wedge 2-\frac{\sqrt{4\pi r^2\theta-r^2\theta^2}}{2\pi}\wedge 2\right)}\right),\ r\right]$$

$$\left\{\left\{r\to\left(-8\pi^3+4\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Csc}[\beta]+8\pi^2\theta\,\text{Csc}[\beta]^2-2\pi\theta^2\,\text{Csc}[\beta]^2-4\pi\theta\sqrt{(4\pi-\theta)\,\theta}\,\text{Csc}[\beta]^3+\theta^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Csc}[\beta]^3\right)\Big/\left(4\left(8\pi^4-4\pi^3\theta-8\pi^3\theta\,\text{Csc}[\beta]^2+6\pi^2\theta^2\,\text{Csc}[\beta]^2-\pi\theta^3\,\text{Csc}[\beta]^2\right)\right)\right\}\right\}$$

```
SphericalPlot3D[(-8 π^3 + 4 π^2 Sqrt[(4 π - θ) θ] Csc[β] + 8 π^2 θ Csc[β]^2 -
    2 π θ^2 Csc[β]^2 - 4 π θ Sqrt[(4 π - θ) θ] Csc[β]^3 + θ^2 Sqrt[(4 π - θ) θ] Csc[β]^3) /
  (4 (8 π^4 - 4 π^3 θ - 8 π^3 θ Csc[β]^2 + 6 π^2 θ^2 Csc[β]^2 - π θ^3 Csc[β]^2)), {β, -π / 2, π / 2}, {θ, -2 π, 2 π}]
```



```
SphericalPlot3D[(-8 π^3 + 4 π^2 √((4 π - θ) θ) Csc[β] + 8 π^2 θ Csc[β]^2 -
    2 π θ^2 Csc[β]^2 - 4 π θ √((4 π - θ) θ) Csc[β]^3 + θ^2 √((4 π - θ) θ) Csc[β]^3) /
  (4 (8 π^4 - 4 π^3 θ - 8 π^3 θ Csc[β]^2 + 6 π^2 θ^2 Csc[β]^2 - π θ^3 Csc[β]^2)), {θ, -2 π, 2 π}, {β, -π / 2, π / 2}]
```

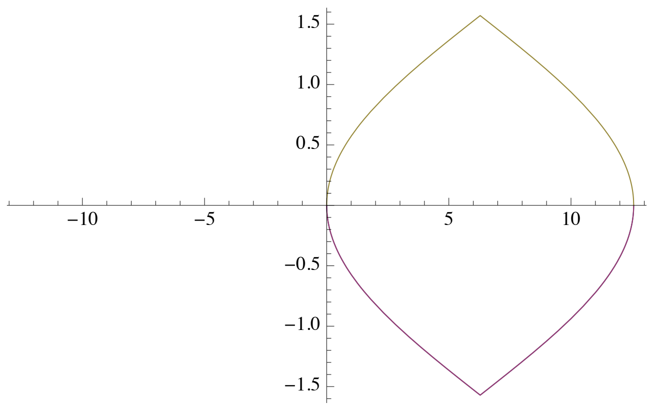



```
Solve[(-4 π θ + θ^2 + 2 π √((4 π - θ) θ) Sin[β] + 4 π^2 Sin[β]^2 - (2 π^2 √((4 π - θ) θ) Sin[β]^3)/(4 π - θ) -
    (2 π^2 √((4 π - θ) θ) Sin[β]^3)/θ) / (16 π^2 θ - 12 π θ^2 + 2 θ^3 - 16 π^3 Sin[β]^2 + 8 π^2 θ Sin[β]^2) ==
  (-8 π^3 + 4 π^2 √((4 π - θ) θ) Csc[β] + 8 π^2 θ Csc[β]^2 - 2 π θ^2 Csc[β]^2 - 4 π θ √((4 π - θ) θ) Csc[β]^3 +
    θ^2 √((4 π - θ) θ) Csc[β]^3) / (4 (8 π^4 - 4 π^3 θ - 8 π^3 θ Csc[β]^2 + 6 π^2 θ^2 Csc[β]^2 - π θ^3 Csc[β]^2)), β]
```

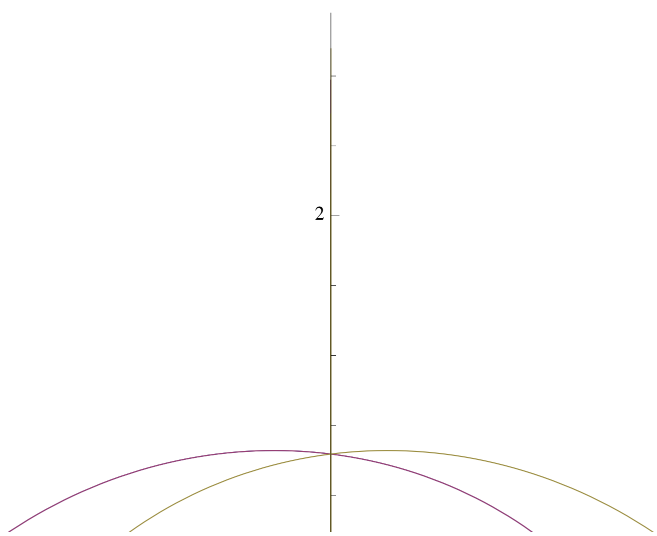
$$\left\{\left\{\beta \to \text{ArcCsc}\left[\frac{1}{2}\left(-\frac{\sqrt{(4\pi-\theta)\,\theta}}{4\pi-\theta}-\frac{\sqrt{(4\pi-\theta)\,\theta}}{\theta}\right)\right]\right\},\right.$$

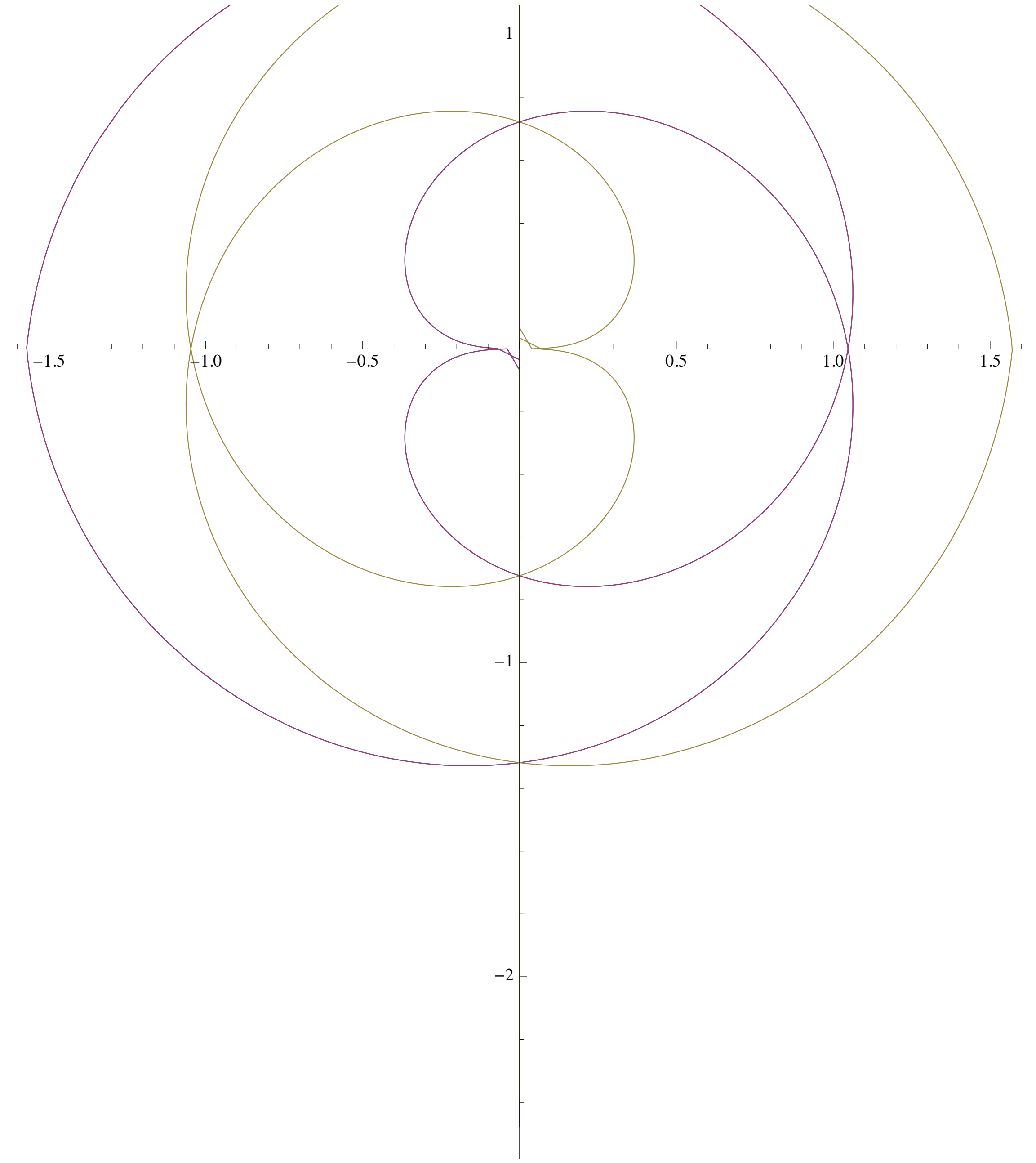
$$\left.\left\{\beta \to -\text{ArcCsc}\left[\frac{1}{2}\left(\frac{\sqrt{(4\pi-\theta)\,\theta}}{4\pi-\theta}+\frac{\sqrt{(4\pi-\theta)\,\theta}}{\theta}\right)\right]\right\},\ \left\{\beta \to \text{ArcCsc}\left[\frac{\sqrt{4\pi\theta-\theta^2}}{2\,(4\pi-\theta)}+\frac{\sqrt{4\pi\theta-\theta^2}}{2\,\theta}\right]\right\}\right\}$$

$$\text{Plot}\left[\left\{-\text{ArcCsc}\left[\frac{1}{2}\left(\frac{\sqrt{(4\pi-\theta)\,\theta}}{4\pi-\theta}+\frac{\sqrt{(4\pi-\theta)\,\theta}}{\theta}\right)\right],\right.\right.$$
$$\left.\left.-\text{ArcCsc}\left[\frac{1}{2}\left(\frac{\sqrt{(4\pi-\theta)\,\theta}}{4\pi-\theta}+\frac{\sqrt{(4\pi-\theta)\,\theta}}{\theta}\right)\right],\ \text{ArcCsc}\left[\frac{\sqrt{4\pi\theta-\theta^2}}{2\,(4\pi-\theta)}+\frac{\sqrt{4\pi\theta-\theta^2}}{2\,\theta}\right]\right\},\ \{\theta,\ -4\pi,\ 4\pi\}\right]$$



$$\text{PolarPlot}\left[\left\{\text{ArcCsc}\left[\frac{1}{2}\left(-\frac{\sqrt{(4\pi-\theta)\,\theta}}{4\pi-\theta}-\frac{\sqrt{(4\pi-\theta)\,\theta}}{\theta}\right)\right],\right.\right.$$
$$\left.\left.-\text{ArcCsc}\left[\frac{1}{2}\left(\frac{\sqrt{(4\pi-\theta)\,\theta}}{4\pi-\theta}+\frac{\sqrt{(4\pi-\theta)\,\theta}}{\theta}\right)\right],\ \text{ArcCsc}\left[\frac{\sqrt{4\pi\theta-\theta^2}}{2\,(4\pi-\theta)}+\frac{\sqrt{4\pi\theta-\theta^2}}{2\,\theta}\right]\right\},\ \{\theta,\ -10\pi,\ 10\pi\}\right]$$

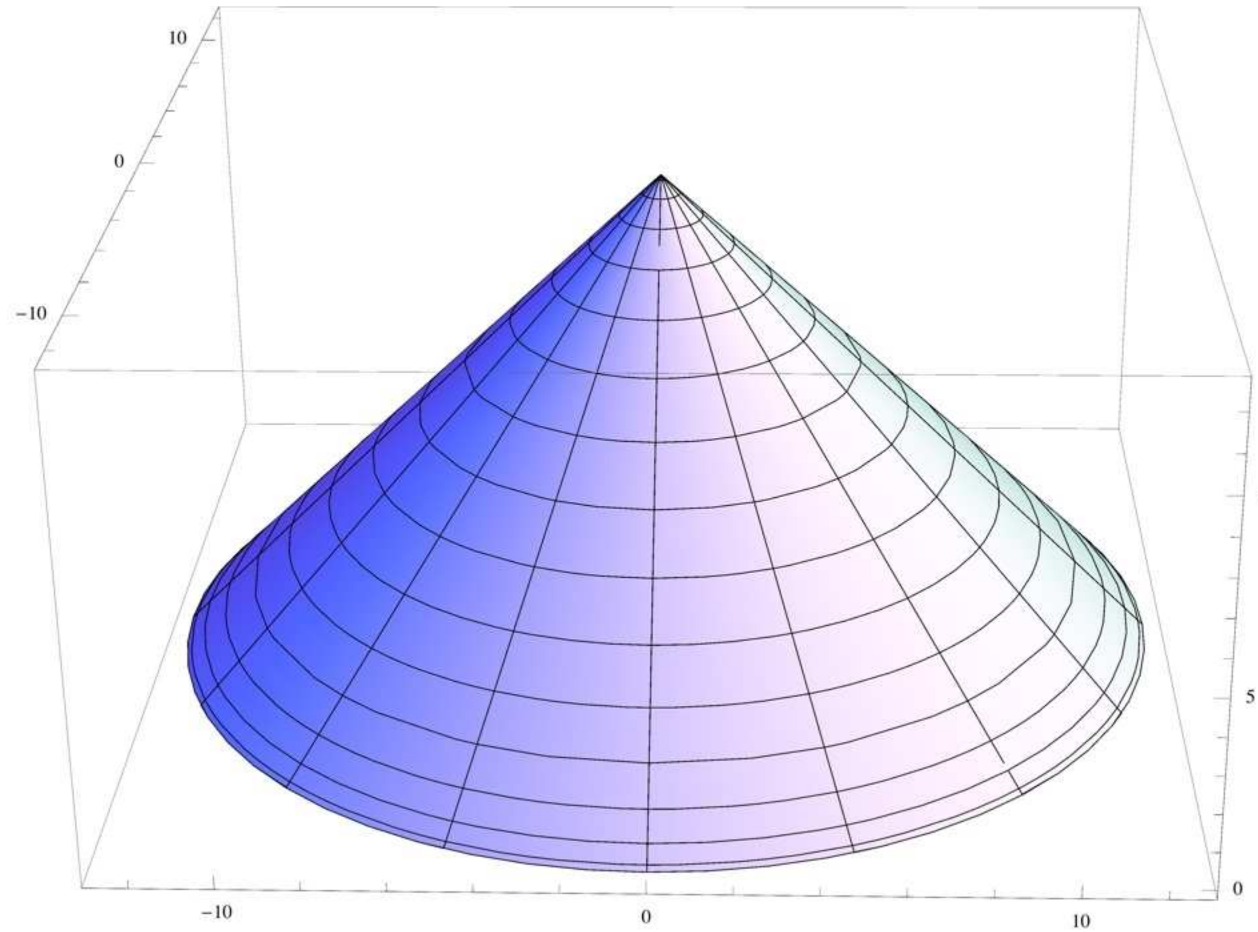




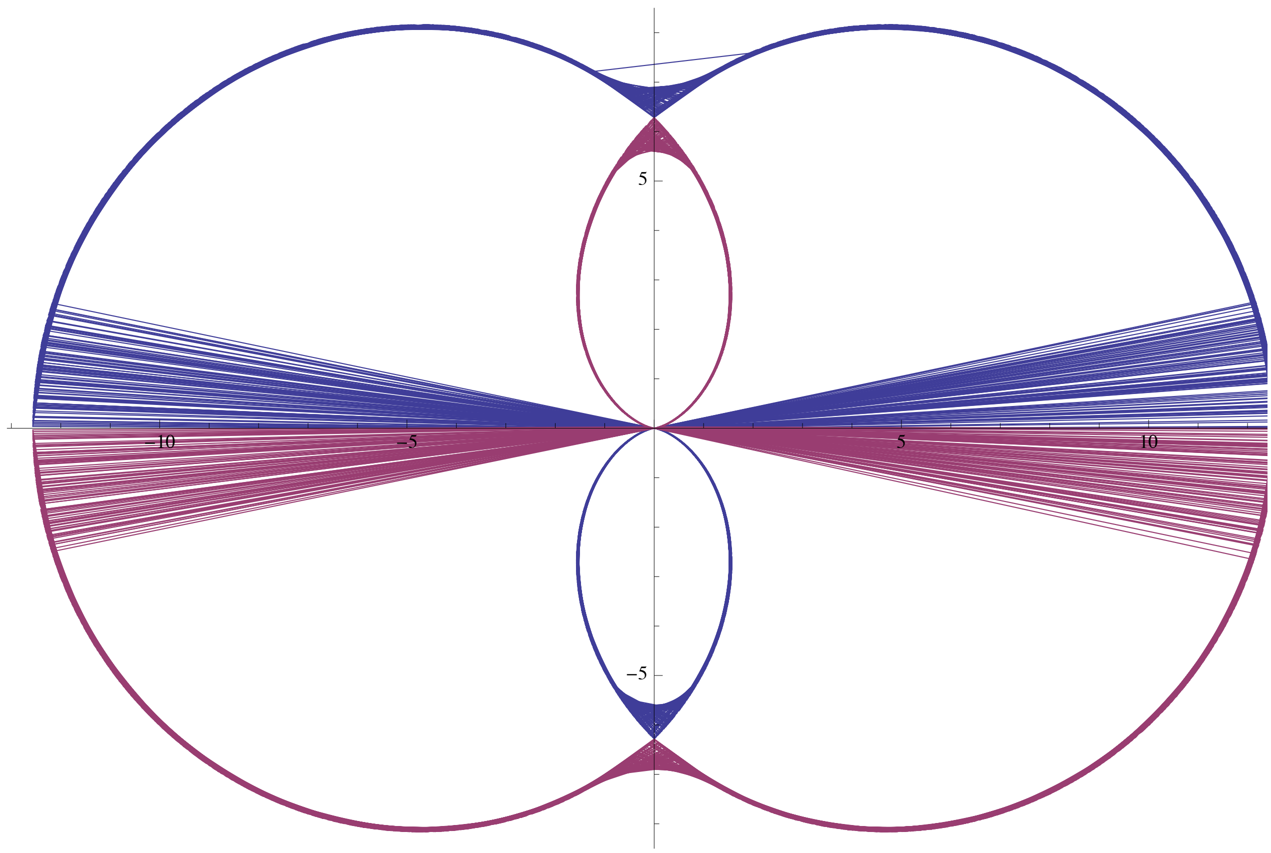
$$\text{Solve}\left[\left(-4\,\pi\,\theta+\theta^2+2\,\pi\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]+4\,\pi^2\,\text{Sin}[\beta]^2-\frac{2\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{4\,\pi-\theta}-\frac{2\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)==\left(-8\,\pi^3+4\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Csc}[\beta]+8\,\pi^2\,\theta\,\text{Csc}[\beta]^2-2\,\pi\,\theta^2\,\text{Csc}[\beta]^2-4\,\pi\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Csc}[\beta]^3+\theta^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Csc}[\beta]^3\right)\Big/\left(4\left(8\,\pi^4-4\,\pi^3\,\theta-8\,\pi^3\,\theta\,\text{Csc}[\beta]^2+6\,\pi^2\,\theta^2\,\text{Csc}[\beta]^2-\pi\,\theta^3\,\text{Csc}[\beta]^2\right)\right),\,\theta\right]$$

$$\left\{\left\{\theta\to 2\left(\pi-\sqrt{-\pi^2+\pi^2\,\text{Csc}[\beta]^2}\;\text{Sin}[\beta]\right)\right\},\left\{\theta\to 2\left(\pi+\sqrt{-\pi^2+\pi^2\,\text{Csc}[\beta]^2}\;\text{Sin}[\beta]\right)\right\}\right\}$$

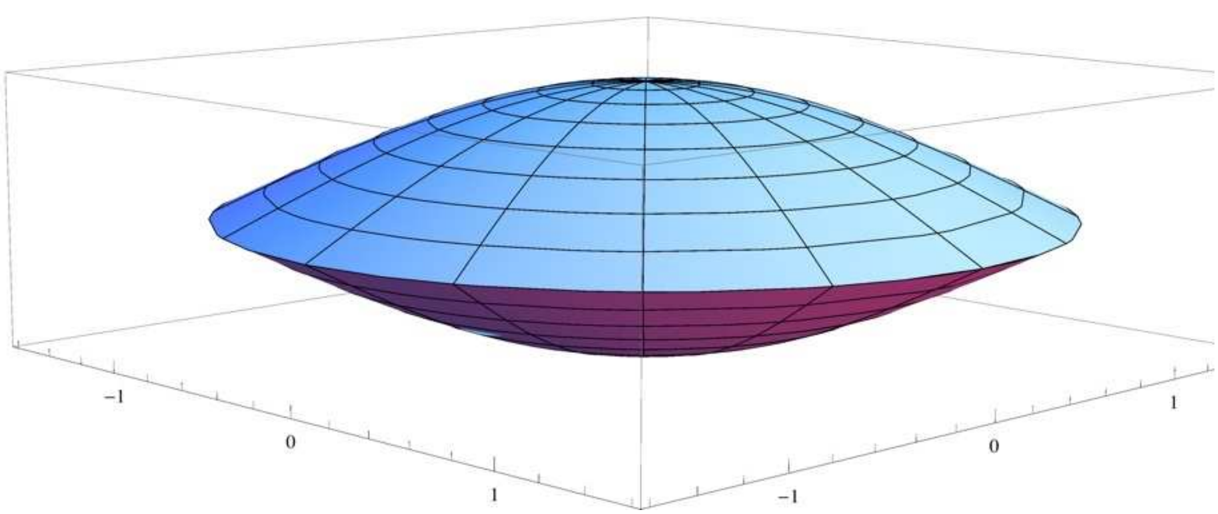
```
RevolutionPlot3D[
 {2 (π + Sqrt[-π^2 + π^2 Csc[β]^2] Sin[β]), 2 (π - Sqrt[-π^2 + π^2 Csc[β]^2] Sin[β])}, {β, -π/2, π/2}]
```



```
PolarPlot[{2 (π + Sqrt[-π^2 + π^2 Csc[β]^2] Sin[β]), 2 (π - Sqrt[-π^2 + π^2 Csc[β]^2] Sin[β])},
 {β, -10^2 π, 10^2 π}]
```




```
RevolutionPlot3D[2 (π - Sqrt[-π^2 + π^2 Csc[β]^2] Sin[β]), {β, -π/2, π/2}]
```



Lemma 9 From Lemma 5, θ is a function of β. From Lemma 4, β is a function of θ. Therefore, r is a function of theta or beta. This cannot be proven with Euclidean geometry, because one can draw a circle of any size and take any angle (arc length) from it.

Euclidean geometry is consistent, but not comprehensive of all geometries, including the system related to difference in circumferences of two circles equaling an arc length (applied to the Pythagorean theorem to form *a* cone).

```
RegionPlot3D[(-4 π θ + θ^2 + 2 π Sqrt[(4 π - θ) θ] Sin[β] + 4 π^2 Sin[β]^2 - (2 π^2 Sqrt[(4 π - θ) θ] Sin[β]^3)/(4 π - θ) -
    (2 π^2 Sqrt[(4 π - θ) θ] Sin[β]^3)/θ) / (16 π^2 θ - 12 π θ^2 + 2 θ^3 - 16 π^3 Sin[β]^2 + 8 π^2 θ Sin[β]^2) ==
  (-8 π^3 + 4 π^2 Sqrt[(4 π - θ) θ] Csc[β] + 8 π^2 θ Csc[β]^2 - 2 π θ^2 Csc[β]^2 -
     4 π θ Sqrt[(4 π - θ) θ] Csc[β]^3 + θ^2 Sqrt[(4 π - θ) θ] Csc[β]^3) /
   (4 (8 π^4 - 4 π^3 θ - 8 π^3 θ Csc[β]^2 + 6 π^2 θ^2 Csc[β]^2 - π θ^3 Csc[β]^2)), {θ, -2 π, 2 π}, {β, -π / 2, π / 2}]
```

XV. Woven Velocity Patterns

We can formulate the expression for the height of the cone in terms of the speed of light of non – relativistically transformed time :

$$\frac{\sqrt{r \sqrt{1-\frac{v^2}{c^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}}\ \sqrt{4 \pi r - r \theta}}{2 \pi} == c\ (\theta / (2 \pi)) = rate * time = distance \qquad (49)$$

This will give us introspection into the meaning of relativity' s relation of the consciously perceptual system to the visual system and if the visual system has evolved with a response to relativistic principles. We will be able to visualize the ambient optic array and the inhomogeneities within it.

```
ContourPlot3D[(Sqrt[r Sqrt[1 - v^2/c^2]] Sqrt[θ/Sqrt[1 - v^2/c^2]] Sqrt[4 π r - r θ])/(2 π), {r, -1, 1}, {θ, -2 π, 2 π}, {v, -c, c}]
```

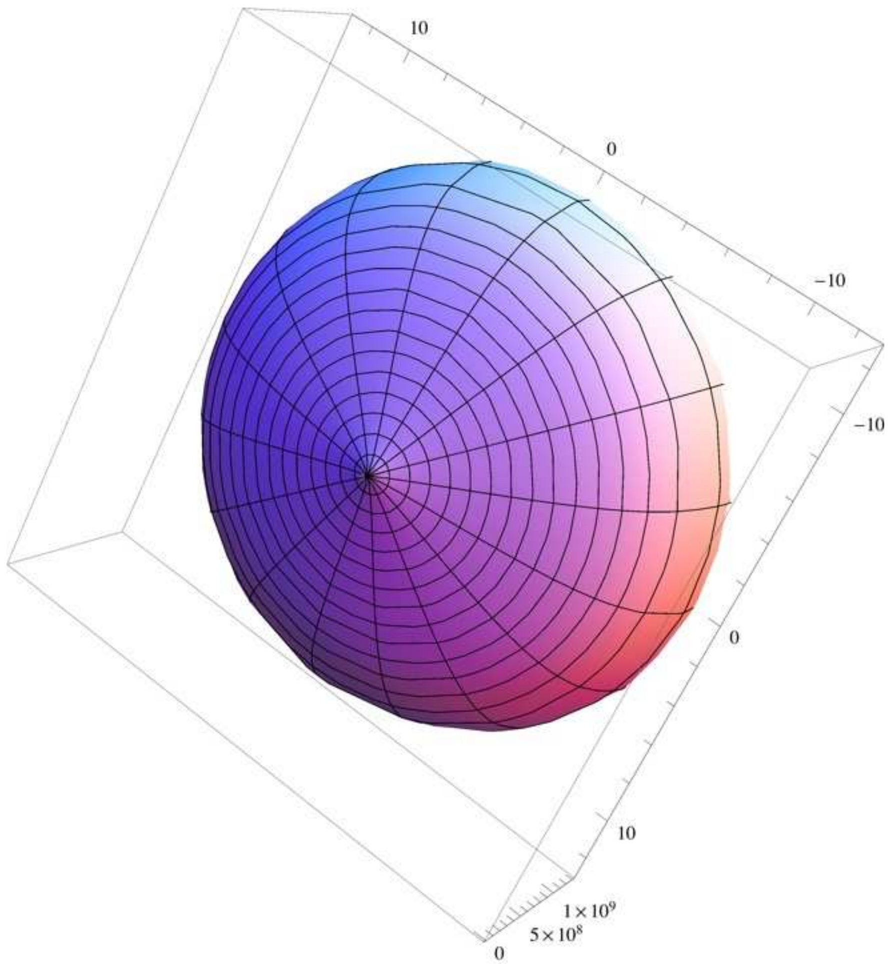
```
Solve[(Sqrt[r Sqrt[1 - v^2/c^2]] Sqrt[θ/Sqrt[1 - v^2/c^2]] Sqrt[4 π r - r θ])/(2 π) == c (θ / (2 π)), v]
```

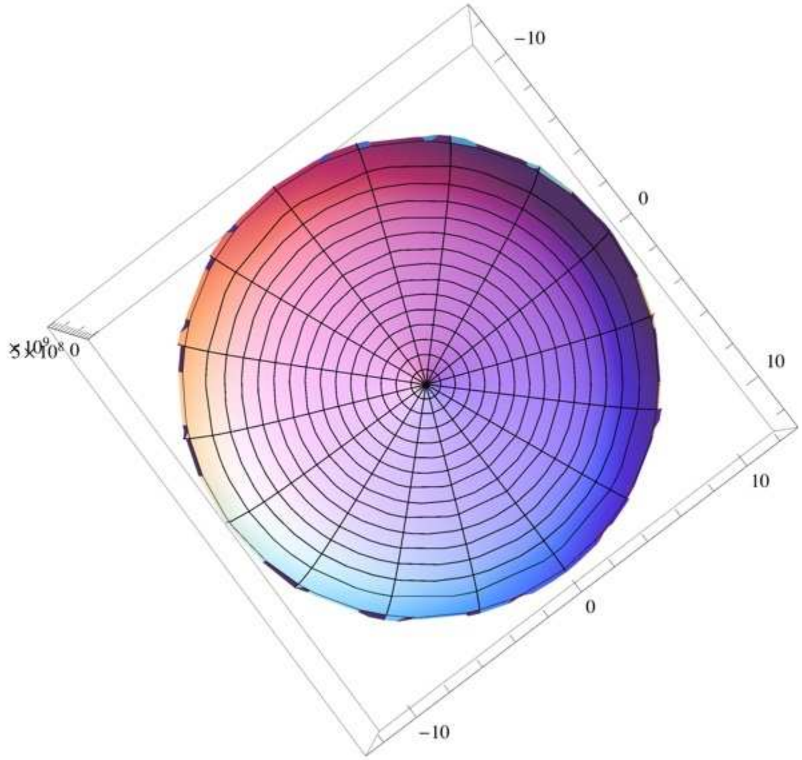
{}

```
Solve[(Sqrt[r Sqrt[1 - v^2/c^2]] Sqrt[θ/Sqrt[1 - v^2/c^2]] Sqrt[4 π r - r θ])/(2 π) == c (θ / (2 π)), r]
```

$$\left\{\left\{r \to -\frac{c\sqrt{\theta}}{\sqrt{4\pi-\theta}}\right\}, \left\{r \to \frac{c\sqrt{\theta}}{\sqrt{4\pi-\theta}}\right\}\right\}$$

$$\text{RevolutionPlot3D}\left[\frac{c\sqrt{\theta}}{\sqrt{4\pi-\theta}}, \{\theta, -4\pi, 4\pi\}\right]$$

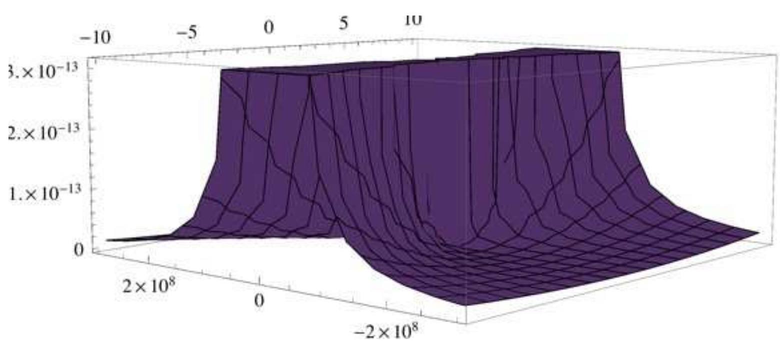




$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{c^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}}\ \sqrt{4\pi r - r\theta}}{2\pi} == c\ (\theta / (2\pi)),\ \theta\right]$$

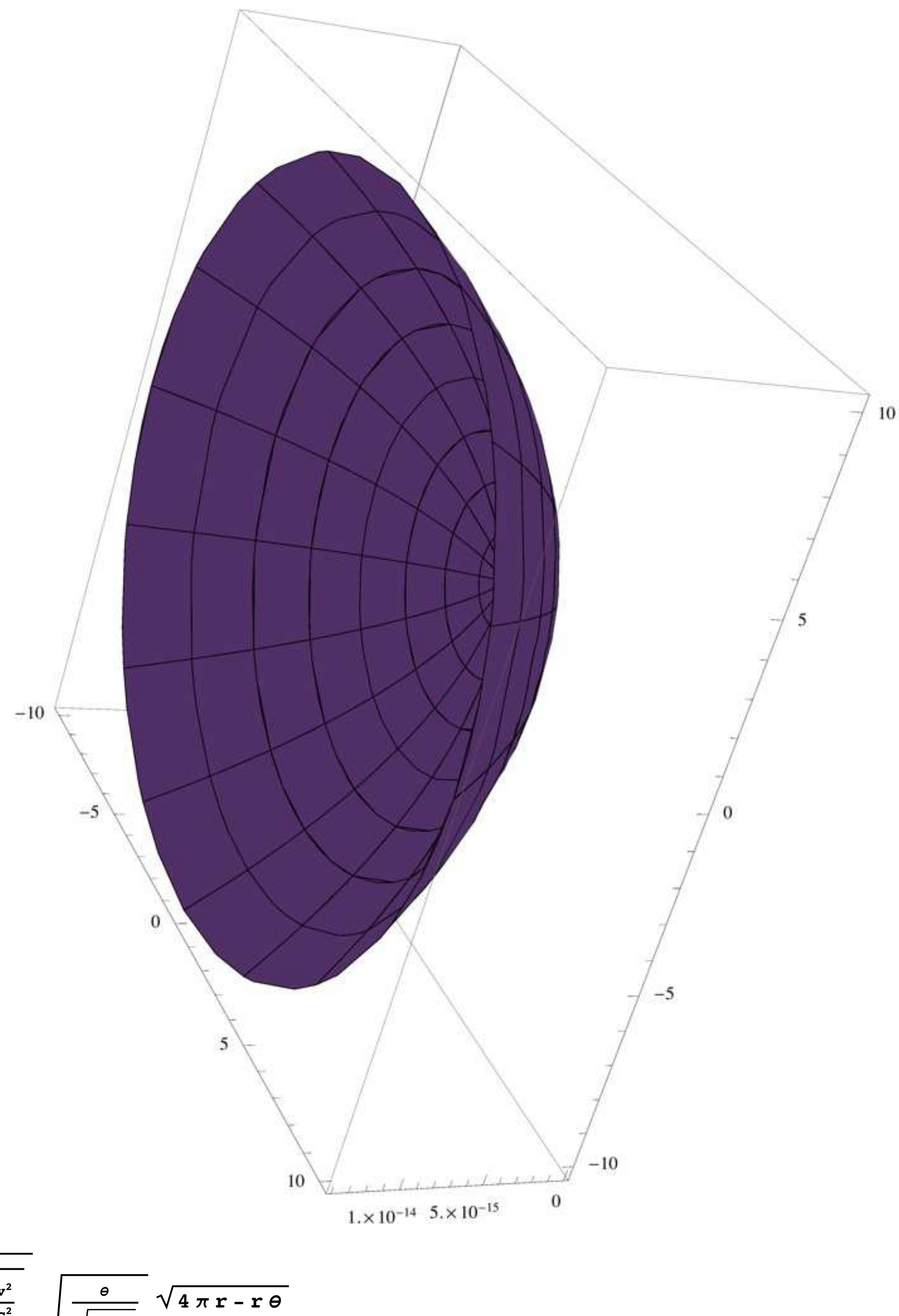
$$\left\{\{\theta \to 0\},\ \left\{\theta \to \frac{4\pi r^2}{c^2 + r^2}\right\}\right\}$$

$$\text{Plot3D}\left[\frac{4\pi r^2}{c^2 + r^2},\ \{r, -10, 10\},\ \{c, -(2.99792458 * 10\text{^}8),\ (2.99792458 * 10\text{^}8)\}\right]$$




```
c := (2.99792458 * 10^8)
```

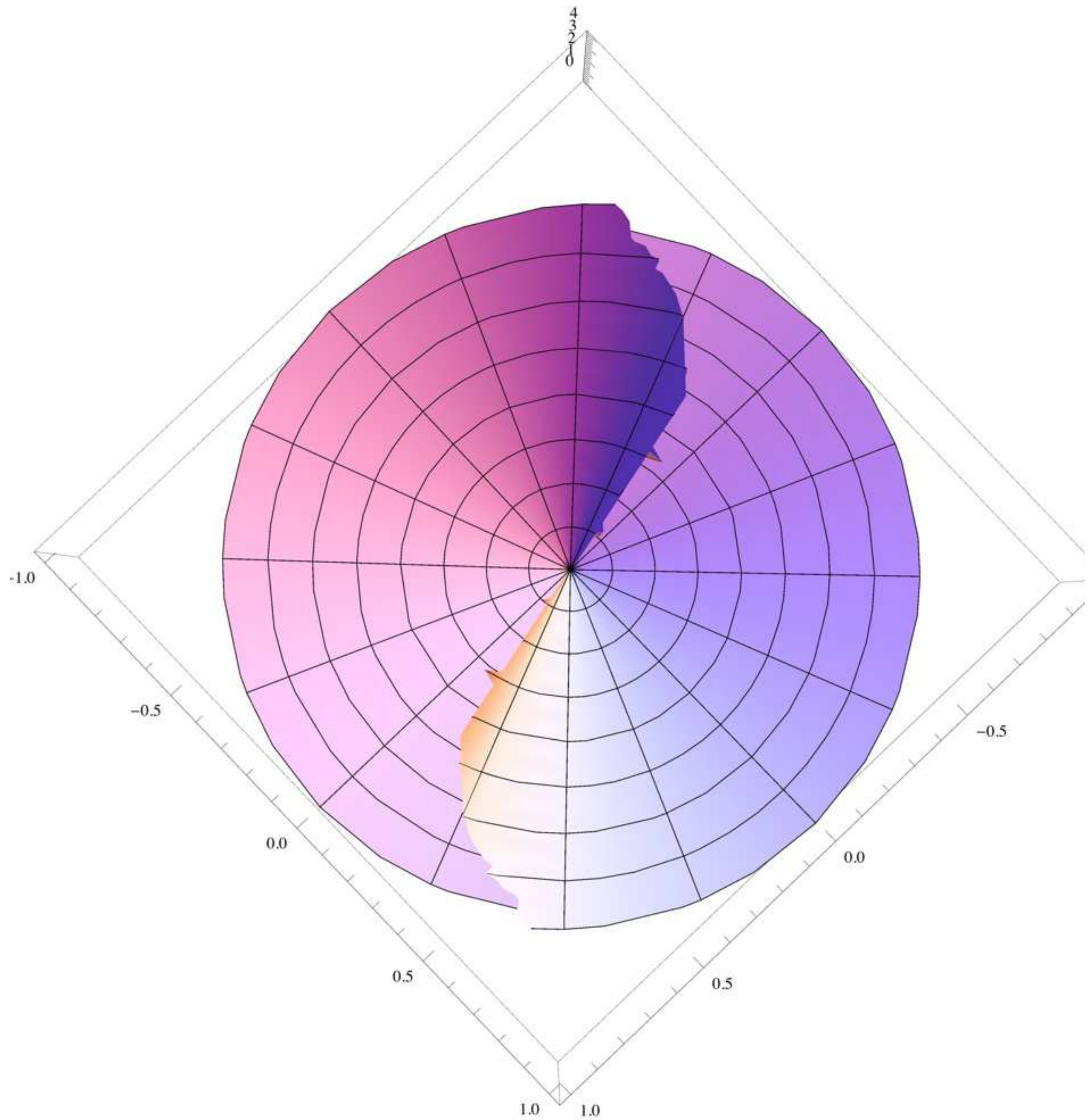
$$\texttt{RevolutionPlot3D}\left[\frac{4\,\pi\,r^2}{c^2+r^2},\ \{r,\ -10,\ 10\}\right]$$



$$\texttt{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{c^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}}\ \sqrt{4\,\pi\,r-r\,\theta}}{2\,\pi} == c\ (\theta\ /\ (2\,\pi)),\ c\right]$$

$$\left\{\left\{c\to-\frac{\sqrt{4\,\pi\,r^2-r^2\,\theta}}{\sqrt{\theta}}\right\},\ \left\{c\to\frac{\sqrt{4\,\pi\,r^2-r^2\,\theta}}{\sqrt{\theta}}\right\}\right\}$$

$$\text{RevolutionPlot3D}\left[\left\{\frac{\sqrt{4\,\pi\,r^2 - r^2\,\theta}}{\sqrt{\theta}}\right\},\ \{r,\ -1,\ 1\},\ \{\theta,\ -2\,\pi,\ 2\,\pi\}\right]$$



We can formulate the relativistic equation for the height of the cone in terms of the speed of light of relativistic time :

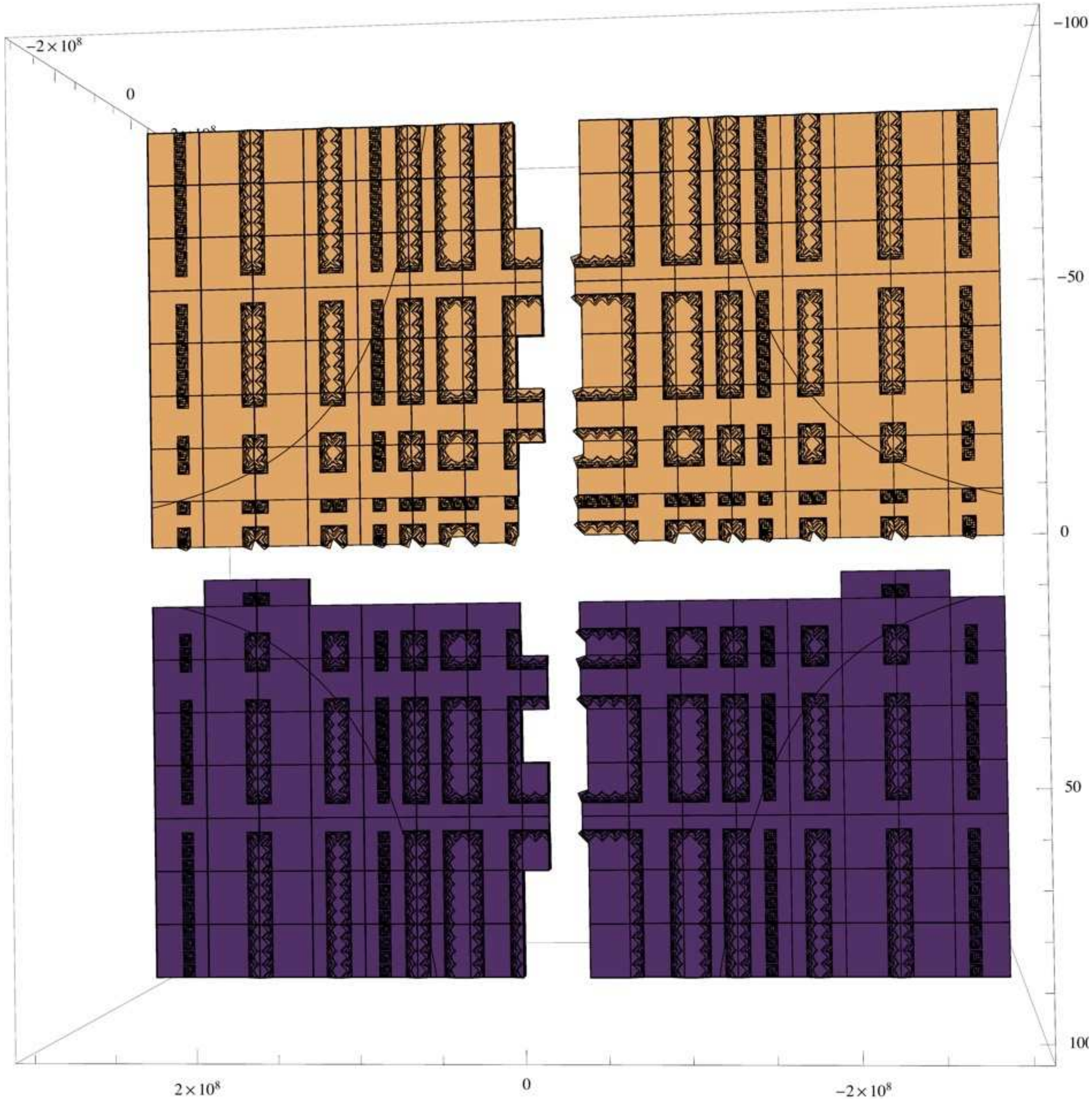
$$\frac{\sqrt{r\sqrt{1-\frac{v^2}{c^2}}}\sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}}\sqrt{4\pi r-r\theta}}{2\pi}\sqrt{1-\frac{v^2}{c^2}} == c\left((\theta/(2\pi))\Big/\sqrt{1-\frac{v^2}{c^2}}\right)$$

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{c^2}}}\sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}}\sqrt{4\pi r-r\theta}}{2\pi}\sqrt{1-\frac{v^2}{c^2}} == c\left((\theta/(2\pi))\Big/\sqrt{1-\frac{v^2}{c^2}}\right), \theta\right]$$

$$\left\{\{\theta\to 0\}, \left\{\theta\to\frac{4\pi\left(c^4 r^2-2c^2r^2v^2+r^2v^4\right)}{c^6+c^4r^2-2c^2r^2v^2+r^2v^4}\right\}\right\}$$

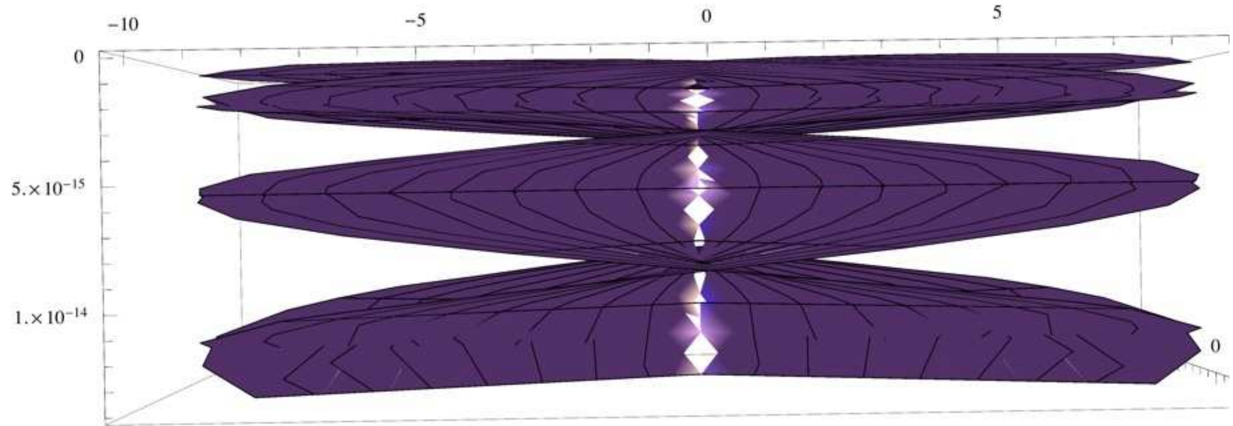
Time is visualized from contour to be a gradient of texture. We can also visualize the layout when we know that light travels at a fairly constant speed through substances like air. However, when we consider it to be a variable, which it truly is, we can see how there is information contained within time' s relation to the velocity of a perceiver, the speed of light being perceived, and the initial radius of the circle transforming into a cone, which is the wavelength of light.

```
ContourPlot3D[4 π (c^4 r^2 - 2 c^2 r^2 v^2 + r^2 v^4)/(c^6 + c^4 r^2 - 2 c^2 r^2 v^2 + r^2 v^4), {r, -100, 100},
 {v, -(2.99792458 * 10^8), (2.99792458 * 10^8)}, {c, -(2.99792458 * 10^8), (2.99792458 * 10^8)}]
```

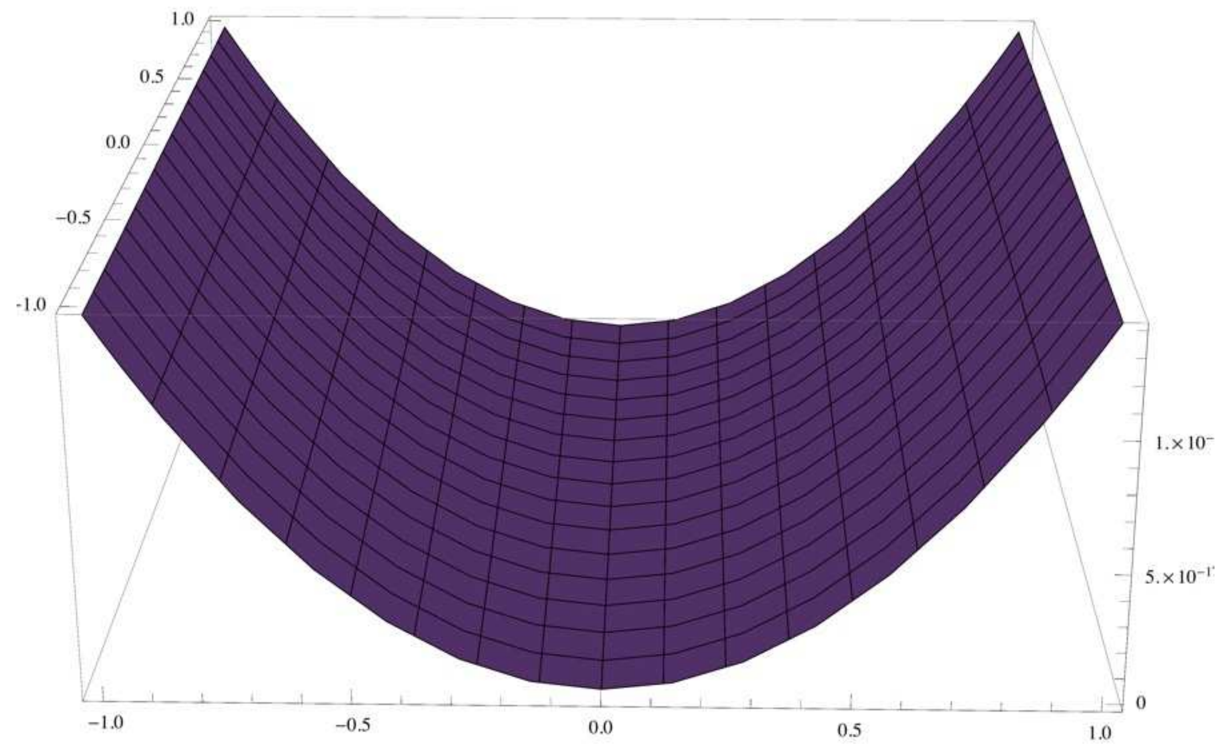


```
RevolutionPlot3D[4 π (c^4 r^2 - 2 c^2 r^2 v^2 + r^2 v^4) / (c^6 + c^4 r^2 - 2 c^2 r^2 v^2 + r^2 v^4), {v, -10, 10}, {r, -10, 10}]
```

$$\text{RevolutionPlot3D}\left[\frac{4\pi\left(c^4 r^2 - 2c^2 r^2 v^2 + r^2 v^4\right)}{c^6 + c^4 r^2 - 2c^2 r^2 v^2 + r^2 v^4}, \{v, -10, 10\}, \{r, -10, 10\}\right]$$



$$\text{Plot3D}\left[\frac{4\pi\left(c^4 r^2 - 2c^2 r^2 v^2 + r^2 v^4\right)}{c^6 + c^4 r^2 - 2c^2 r^2 v^2 + r^2 v^4}, \{r, -1, 1\}, \{v, -1, 1\}\right]$$



$$\textbf{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{c^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}}\ \sqrt{4\pi r - r\theta}}{2\pi}\sqrt{1-\frac{v^2}{c^2}} == c\left((\theta\,/\,(2\pi))\Big/\sqrt{1-\frac{v^2}{c^2}}\right),\ v\right]$$

$$\left\{\left\{v\to-\sqrt{-\frac{4c^2\pi r^2}{-4\pi r^2+r^2\theta}+\frac{c^2r^2\theta}{-4\pi r^2+r^2\theta}-\frac{\sqrt{c^6r^2(4\pi-\theta)\theta}}{-4\pi r^2+r^2\theta}}\right\},\right.$$

$$\left\{v\to\sqrt{-\frac{4c^2\pi r^2}{-4\pi r^2+r^2\theta}+\frac{c^2r^2\theta}{-4\pi r^2+r^2\theta}-\frac{\sqrt{c^6r^2(4\pi-\theta)\theta}}{-4\pi r^2+r^2\theta}}\right\},$$

$$\left\{v\to-\sqrt{-\frac{4c^2\pi r^2}{-4\pi r^2+r^2\theta}+\frac{c^2r^2\theta}{-4\pi r^2+r^2\theta}+\frac{\sqrt{4c^6\pi r^2\theta-c^6r^2\theta^2}}{-4\pi r^2+r^2\theta}}\right\},$$

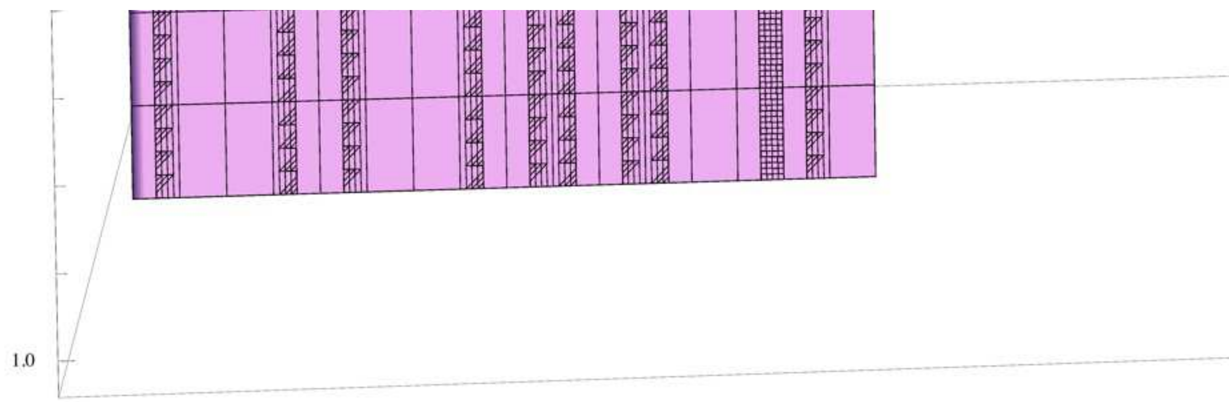
$$\left.\left\{v\to\sqrt{-\frac{4c^2\pi r^2}{-4\pi r^2+r^2\theta}+\frac{c^2r^2\theta}{-4\pi r^2+r^2\theta}+\frac{\sqrt{4c^6\pi r^2\theta-c^6r^2\theta^2}}{-4\pi r^2+r^2\theta}}\right\}\right\}$$

$$\textbf{ContourPlot3D}\left[\left\{\sqrt{-\frac{4c^2\pi r^2}{-4\pi r^2+r^2\theta}+\frac{c^2r^2\theta}{-4\pi r^2+r^2\theta}-\frac{\sqrt{c^6r^2(4\pi-\theta)\theta}}{-4\pi r^2+r^2\theta}},\right.\right.$$

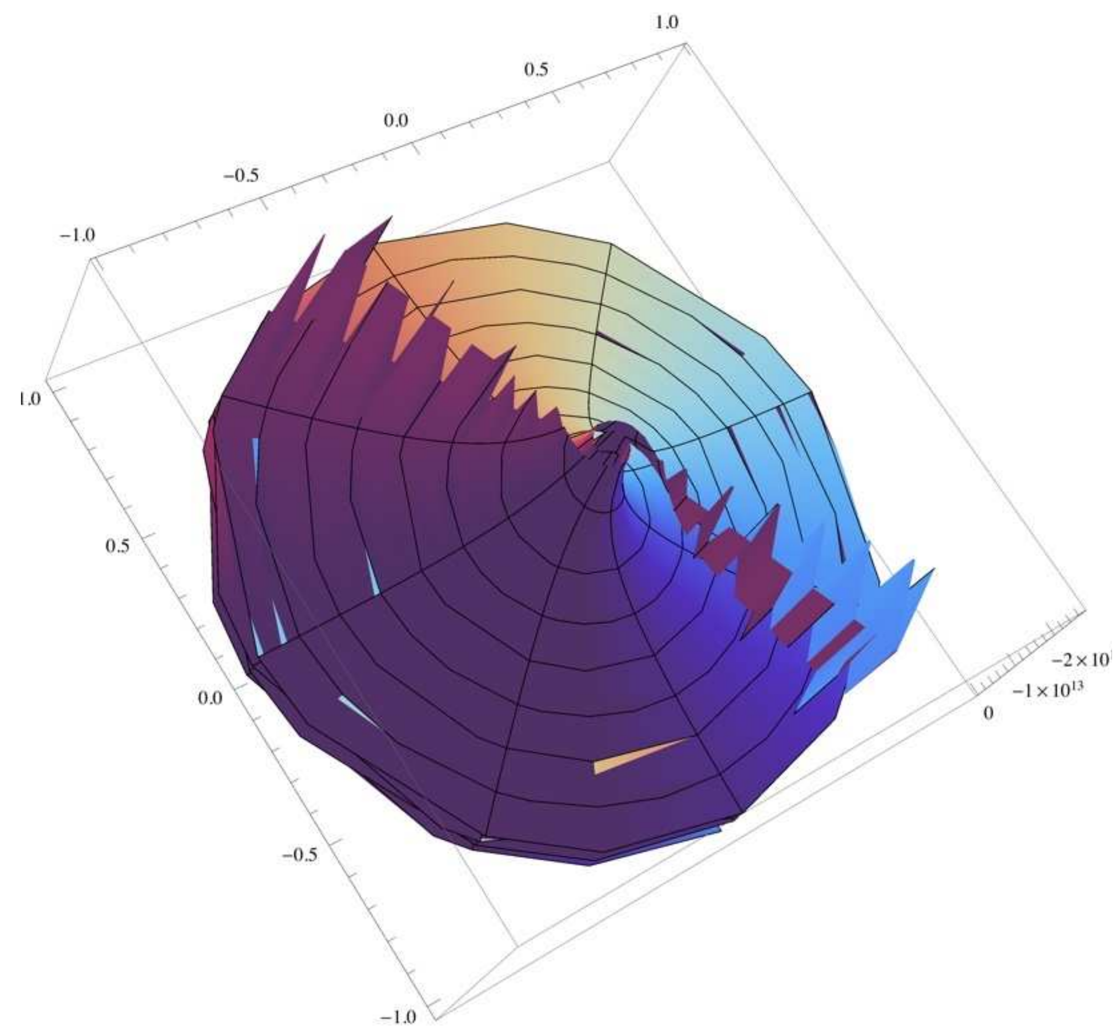
$$\left.-\sqrt{-\frac{4c^2\pi r^2}{-4\pi r^2+r^2\theta}+\frac{c^2r^2\theta}{-4\pi r^2+r^2\theta}-\frac{\sqrt{c^6r^2(4\pi-\theta)\theta}}{-4\pi r^2+r^2\theta}}\right\},\ \{r,-1,1\},$$

$$\left.\{\theta,-4\pi,4\pi\},\ \{c,-(2.99792458*10\verb|^|8),\ (2.99792458*10\verb|^|8)\}\right]$$

−0.5
0.0
0.5



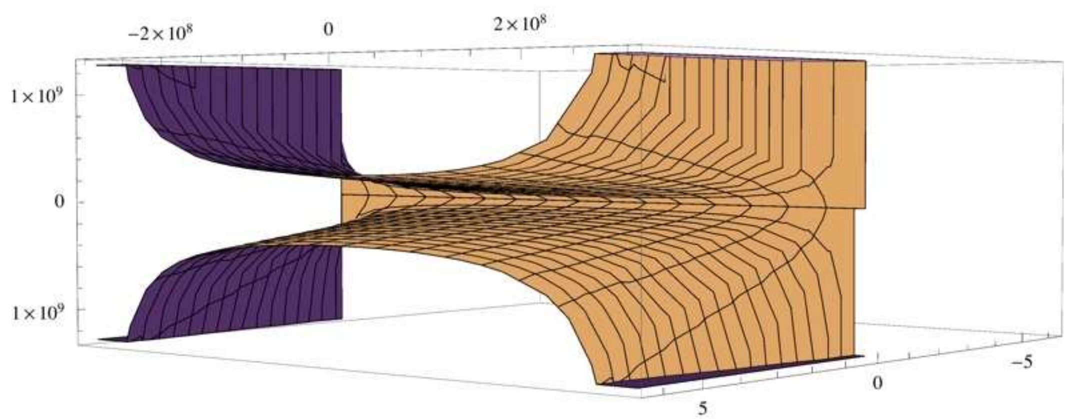
$$\text{RevolutionPlot3D}\left[-\sqrt{-\frac{4\,c^2\,\pi\,r^2}{-4\,\pi\,r^2+r^2\,\theta}+\frac{c^2\,r^2\,\theta}{-4\,\pi\,r^2+r^2\,\theta}-\frac{\sqrt{c^6\,r^2\,(4\,\pi-\theta)\,\theta}}{-4\,\pi\,r^2+r^2\,\theta}},\ \{r,-1,1\},\ \{\theta,-4\,\pi,4\,\pi\}\right]$$



$$\textbf{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{c^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}}\ \sqrt{4\pi r - r\theta}}{2\pi}\sqrt{1-\frac{v^2}{c^2}} == c\left((\theta/(2\pi))\Big/\sqrt{1-\frac{v^2}{c^2}}\right),\ r\right]$$

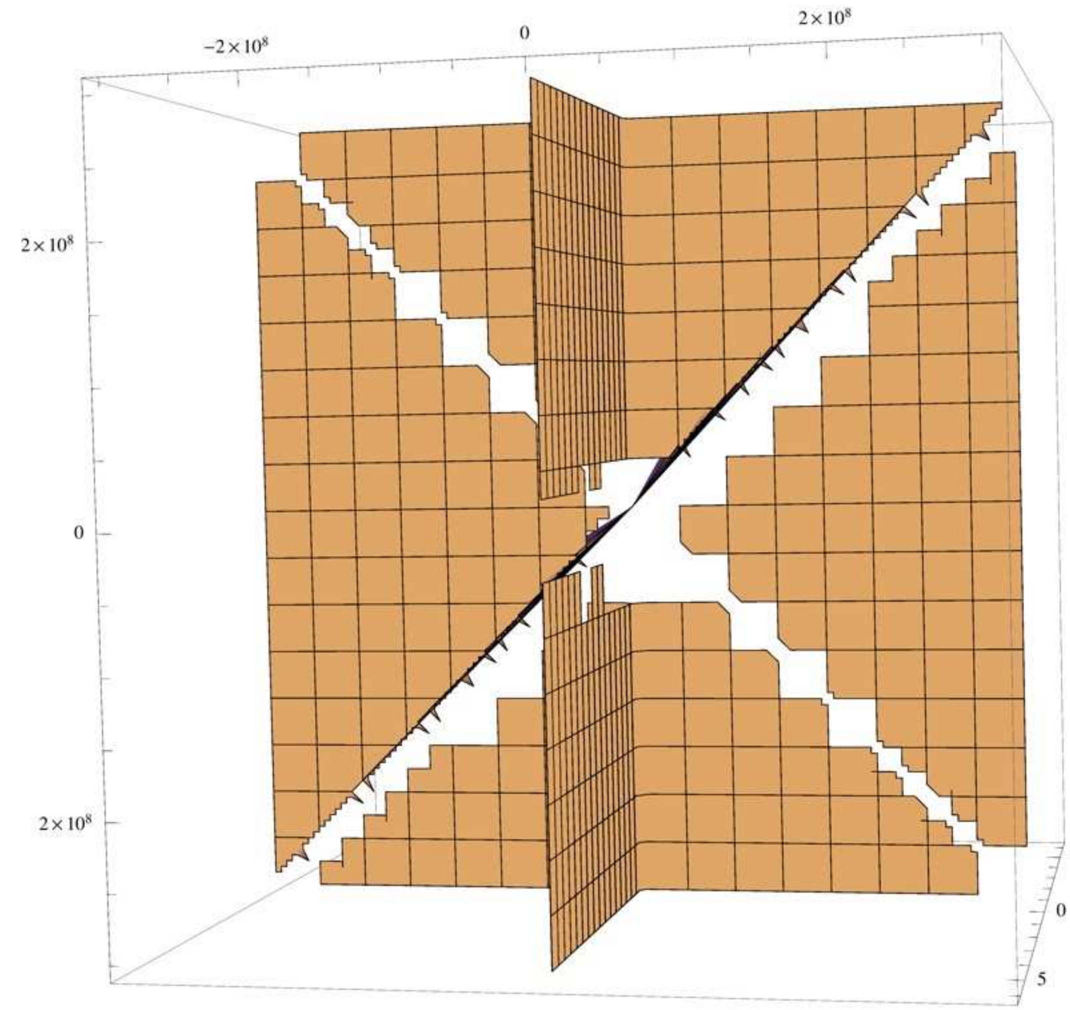
$$\left\{\left\{r \to -\frac{i\, c^3 \sqrt{\theta}}{(c-v)\ (c+v)\ \sqrt{-4\pi+\theta}}\right\},\ \left\{r \to \frac{i\, c^3 \sqrt{\theta}}{(c-v)\ (c+v)\ \sqrt{-4\pi+\theta}}\right\}\right\}$$

$$\textbf{Plot3D}\left[\left\{\frac{i\,(2.99792458 * 10^{\wedge}8)^3 \sqrt{\theta}}{((2.99792458 * 10^{\wedge}8) - v)\ ((2.99792458 * 10^{\wedge}8) + v)\ \sqrt{-4\pi+\theta}},\right.\right.$$
$$\left.-\frac{i\,(2.99792458 * 10^{\wedge}8)^3 \sqrt{\theta}}{((2.99792458 * 10^{\wedge}8) - v)\ ((2.99792458 * 10^{\wedge}8) + v)\ \sqrt{-4\pi+\theta}}\right\},$$
$$\left.\{\theta, -2\pi, 2\pi\},\ \{v, -(2.99792458 * 10^{\wedge}8),\ (2.99792458 * 10^{\wedge}8)\}\right]$$



$$\text{ContourPlot3D}\left[\left\{-\frac{\mathbb{i}\, c^3 \sqrt{\theta}}{(c-v)\ (c+v)\ \sqrt{-4\,\pi+\theta}},\ \frac{\mathbb{i}\, c^3 \sqrt{\theta}}{(c-v)\ (c+v)\ \sqrt{-4\,\pi+\theta}}\right\},\right.$$

```
{θ, -2 π, 2 π}, {v, -((2.99792458 * 10^8)), ((2.99792458 * 10^8))},
{c, -(2.99792458 * 10^8), (2.99792458 * 10^8)}]
```



During this section of the paper' s main body, we will discuss the gradient found when we make a contour plot of the radial distance perceived as a wavelength, or the changed distance perceived to be an actual distance.

I am proposing with the relativisitic interpretations of the functions of the height of the cone, that layout and contour are contained within the information available in ambient light through the inherency of relativity to the pickup of the invariant, r, of the visual system by the mind.

We can solve this equation for the speed of light in terms of theta, velocity, and radius (wavelength). However, the solutions are very long, and the visualization methods somewhat difficult. I will therefore refer the reader to addendum no. 7 of this work.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{c^2}}}\;\sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}}\;\sqrt{4\pi r-r\theta}}{2\pi}\sqrt{1-\frac{v^2}{c^2}}==c\left((\theta/(2\pi))\Big/\sqrt{1-\frac{v^2}{c^2}}\right),\,c\right]$$

$\{\{c\to$

$$-\sqrt{\Big(-\frac{r^2}{3}+\frac{4\pi r^2}{3\theta}-(8\,2^{1/3}\pi r^4)\Big/\Big(3\,(128\pi^3 r^6-96\pi^2 r^6\theta-288\pi^2 r^4 v^2\theta+24\pi r^6\theta^2+144\pi r^4 v^2\theta^2+}$$

$$108\pi r^2 v^4\theta^2-2r^6\theta^3-18r^4v^2\theta^3-27r^2v^4\theta^3+3\sqrt{3}\sqrt{(-256\pi^3 r^6 v^6\theta^3+192\pi^2 r^6 v^6}$$

$$\theta^4+432\pi^2 r^4 v^8\theta^4-48\pi r^6 v^6\theta^5-216\pi r^4 v^8\theta^5+4r^6v^6\theta^6+27r^4v^8\theta^6))^{1/3}\Big)-$$

$$(8\,2^{1/3}\pi r^2 v^2)\Big/\Big(128\pi^3 r^6-96\pi^2 r^6\theta-288\pi^2 r^4 v^2\theta+24\pi r^6\theta^2+144\pi r^4 v^2\theta^2+$$

$$108\pi r^2 v^4\theta^2-2r^6\theta^3-18r^4v^2\theta^3-27r^2v^4\theta^3+3\sqrt{3}\sqrt{(-256\pi^3 r^6 v^6\theta^3+192\pi^2 r^6 v^6\theta^4+}$$

$$432\pi^2 r^4 v^8\theta^4-48\pi r^6 v^6\theta^5-216\pi r^4 v^8\theta^5+4r^6v^6\theta^6+27r^4v^8\theta^6)\Big)^{1/3}+$$

$$(16\,2^{1/3}\pi^2 r^4)\Big/\Big(3\theta\,(128\pi^3 r^6-96\pi^2 r^6\theta-288\pi^2 r^4 v^2\theta+24\pi r^6\theta^2+144\pi r^4 v^2\theta^2+$$

$$108\pi r^2 v^4\theta^2-2r^6\theta^3-18r^4v^2\theta^3-27r^2v^4\theta^3+3\sqrt{3}\sqrt{(-256\pi^3 r^6 v^6\theta^3+192\pi^2 r^6 v^6}$$

$$\theta^4+432\pi^2 r^4 v^8\theta^4-48\pi r^6 v^6\theta^5-216\pi r^4 v^8\theta^5+4r^6v^6\theta^6+27r^4v^8\theta^6))^{1/3}\Big)+$$

$$(2^{1/3} r^4\theta)\Big/\Big(3\,(128\pi^3 r^6-96\pi^2 r^6\theta-288\pi^2 r^4 v^2\theta+24\pi r^6\theta^2+144\pi r^4 v^2\theta^2+$$

$$108\pi r^2 v^4\theta^2-2r^6\theta^3-18r^4v^2\theta^3-27r^2v^4\theta^3+3\sqrt{3}\sqrt{(-256\pi^3 r^6 v^6\theta^3+192\pi^2 r^6 v^6}$$

$$\theta^4+432\pi^2 r^4 v^8\theta^4-48\pi r^6 v^6\theta^5-216\pi r^4 v^8\theta^5+4r^6v^6\theta^6+27r^4v^8\theta^6))^{1/3}\Big)+$$

$$(2\,2^{1/3} r^2 v^2\theta)\Big/\Big(128\pi^3 r^6-96\pi^2 r^6\theta-288\pi^2 r^4 v^2\theta+24\pi r^6\theta^2+144\pi r^4 v^2\theta^2+$$

$$108\pi r^2 v^4\theta^2-2r^6\theta^3-18r^4v^2\theta^3-27r^2v^4\theta^3+3\sqrt{3}\sqrt{(-256\pi^3 r^6 v^6\theta^3+192\pi^2 r^6 v^6\theta^4+}$$

$$432\pi^2 r^4 v^8\theta^4-48\pi r^6 v^6\theta^5-216\pi r^4 v^8\theta^5+4r^6v^6\theta^6+27r^4v^8\theta^6)\Big)^{1/3}+$$

$$\frac{1}{3\,2^{1/3}\theta}\Big(128\pi^3 r^6-96\pi^2 r^6\theta-288\pi^2 r^4 v^2\theta+24\pi r^6\theta^2+144\pi r^4 v^2\theta^2+108\pi r^2 v^4\theta^2-$$

$$2r^6\theta^3-18r^4v^2\theta^3-27r^2v^4\theta^3+3\sqrt{3}\sqrt{(-256\pi^3 r^6 v^6\theta^3+192\pi^2 r^6 v^6\theta^4+}$$

$$432\,\pi^2\,r^4\,v^8\,\theta^4-48\,\pi\,r^6\,v^6\,\theta^5-216\,\pi\,r^4\,v^8\,\theta^5+4\,r^6\,v^6\,\theta^6+27\,r^4\,v^8\,\theta^6\Big)\Big)^{1/3}\Bigg)\Bigg\},$$

$$\Big\{c\to\sqrt{\Bigg(-\frac{r^2}{3}+\frac{4\,\pi\,r^2}{3\,\theta}-\left(8\,2^{1/3}\,\pi\,r^4\right)\Big/\Big(3\,\Big(128\,\pi^3\,r^6-96\,\pi^2\,r^6\,\theta-288\,\pi^2\,r^4\,v^2\,\theta+24\,\pi\,r^6\,\theta^2+144\,\pi\,r^4\,v^2\,\theta^2+}$$

$$108\,\pi\,r^2\,v^4\,\theta^2-2\,r^6\,\theta^3-18\,r^4\,v^2\,\theta^3-27\,r^2\,v^4\,\theta^3+3\,\sqrt{3}\,\sqrt{\Big(-256\,\pi^3\,r^6\,v^6\,\theta^3+192\,\pi^2\,r^6\,v^6\,\theta^4+}$$

$$432\,\pi^2\,r^4\,v^8\,\theta^4-48\,\pi\,r^6\,v^6\,\theta^5-216\,\pi\,r^4\,v^8\,\theta^5+4\,r^6\,v^6\,\theta^6+27\,r^4\,v^8\,\theta^6\Big)}\Big)^{1/3}\Big)-$$

$$\left(8\,2^{1/3}\,\pi\,r^2\,v^2\right)\Big/\Big(128\,\pi^3\,r^6-96\,\pi^2\,r^6\,\theta-288\,\pi^2\,r^4\,v^2\,\theta+24\,\pi\,r^6\,\theta^2+144\,\pi\,r^4\,v^2\,\theta^2+$$

$$108\,\pi\,r^2\,v^4\,\theta^2-2\,r^6\,\theta^3-18\,r^4\,v^2\,\theta^3-27\,r^2\,v^4\,\theta^3+3\,\sqrt{3}\,\sqrt{\Big(-256\,\pi^3\,r^6\,v^6\,\theta^3+192\,\pi^2\,r^6\,v^6}$$

$$\theta^4+432\,\pi^2\,r^4\,v^8\,\theta^4-48\,\pi\,r^6\,v^6\,\theta^5-216\,\pi\,r^4\,v^8\,\theta^5+4\,r^6\,v^6\,\theta^6+27\,r^4\,v^8\,\theta^6\Big)\Big)^{1/3}+$$

$$\left(16\,2^{1/3}\,\pi^2\,r^4\right)\Big/\Big(3\,\theta\,\Big(128\,\pi^3\,r^6-96\,\pi^2\,r^6\,\theta-288\,\pi^2\,r^4\,v^2\,\theta+24\,\pi\,r^6\,\theta^2+144\,\pi\,r^4\,v^2\,\theta^2+$$

$$108\,\pi\,r^2\,v^4\,\theta^2-2\,r^6\,\theta^3-18\,r^4\,v^2\,\theta^3-27\,r^2\,v^4\,\theta^3+3\,\sqrt{3}\,\sqrt{\Big(-256\,\pi^3\,r^6\,v^6\,\theta^3+192\,\pi^2\,r^6\,v^6\,\theta^4+}$$

$$432\,\pi^2\,r^4\,v^8\,\theta^4-48\,\pi\,r^6\,v^6\,\theta^5-216\,\pi\,r^4\,v^8\,\theta^5+4\,r^6\,v^6\,\theta^6+27\,r^4\,v^8\,\theta^6\Big)\Big)^{1/3}\Big)+$$

$$\left(2^{1/3}\,r^4\,\theta\right)\Big/\Big(3\,\Big(128\,\pi^3\,r^6-96\,\pi^2\,r^6\,\theta-288\,\pi^2\,r^4\,v^2\,\theta+24\,\pi\,r^6\,\theta^2+144\,\pi\,r^4\,v^2\,\theta^2+$$

$$108\,\pi\,r^2\,v^4\,\theta^2-2\,r^6\,\theta^3-18\,r^4\,v^2\,\theta^3-27\,r^2\,v^4\,\theta^3+3\,\sqrt{3}\,\sqrt{\Big(-256\,\pi^3\,r^6\,v^6\,\theta^3+192\,\pi^2\,r^6\,v^6\,\theta^4+}$$

$$432\,\pi^2\,r^4\,v^8\,\theta^4-48\,\pi\,r^6\,v^6\,\theta^5-216\,\pi\,r^4\,v^8\,\theta^5+4\,r^6\,v^6\,\theta^6+27\,r^4\,v^8\,\theta^6\Big)\Big)^{1/3}\Big)+$$

$$\left(2\,2^{1/3}\,r^2\,v^2\,\theta\right)\Big/\Big(128\,\pi^3\,r^6-96\,\pi^2\,r^6\,\theta-288\,\pi^2\,r^4\,v^2\,\theta+24\,\pi\,r^6\,\theta^2+144\,\pi\,r^4\,v^2\,\theta^2+$$

$$108\,\pi\,r^2\,v^4\,\theta^2-2\,r^6\,\theta^3-18\,r^4\,v^2\,\theta^3-27\,r^2\,v^4\,\theta^3+3\,\sqrt{3}\,\sqrt{\Big(-256\,\pi^3\,r^6\,v^6\,\theta^3+192\,\pi^2\,r^6\,v^6}$$

$$\theta^4+432\,\pi^2\,r^4\,v^8\,\theta^4-48\,\pi\,r^6\,v^6\,\theta^5-216\,\pi\,r^4\,v^8\,\theta^5+4\,r^6\,v^6\,\theta^6+27\,r^4\,v^8\,\theta^6\Big)\Big)^{1/3}+$$

$$\frac{1}{3\,2^{1/3}\,\theta}\Big(128\,\pi^3\,r^6-96\,\pi^2\,r^6\,\theta-288\,\pi^2\,r^4\,v^2\,\theta+24\,\pi\,r^6\,\theta^2+144\,\pi\,r^4\,v^2\,\theta^2+108\,\pi\,r^2\,v^4\,\theta^2-$$

$$2\,r^6\,\theta^3-18\,r^4\,v^2\,\theta^3-27\,r^2\,v^4\,\theta^3+3\,\sqrt{3}\,\sqrt{\Big(-256\,\pi^3\,r^6\,v^6\,\theta^3+192\,\pi^2\,r^6\,v^6\,\theta^4+432}$$

$$\pi^2\,r^4\,v^8\,\theta^4-48\,\pi\,r^6\,v^6\,\theta^5-216\,\pi\,r^4\,v^8\,\theta^5+4\,r^6\,v^6\,\theta^6+27\,r^4\,v^8\,\theta^6\Big)\Big)^{1/3}\Bigg)\Big\},$$

$$\Big\{c\to-\sqrt{\Bigg(-\frac{r^2}{3}+\frac{4\,\pi\,r^2}{3\,\theta}+\left(4\,2^{1/3}\,\pi\,r^4\right)\Big/\Big(3\,\Big(128\,\pi^3\,r^6-96\,\pi^2\,r^6\,\theta-288\,\pi^2\,r^4\,v^2\,\theta+24\,\pi\,r^6\,\theta^2+144\,\pi\,r^4\,v^2}$$

$$\theta^2 + 108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{\left(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 + 432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\right)}\Big)^{1/3}\Bigg) -$$

$$\left(4\,i\,2^{1/3}\,\pi\,r^4\right) \Big/ \Bigg(\sqrt{3}\,\Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 + 108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{\left(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 + 432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\right)}\Big)^{1/3}\Bigg) +$$

$$\left(4\,2^{1/3}\,\pi\,r^2\,v^2\right) \Big/ \Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 + 108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{\left(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 + 432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\right)}\Big)^{1/3} -$$

$$\left(4\,i\,2^{1/3}\,\sqrt{3}\,\pi\,r^2\,v^2\right) \Big/ \Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 + 108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{\left(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 + 432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\right)}\Big)^{1/3} -$$

$$\left(8\,2^{1/3}\,\pi^2\,r^4\right) \Big/ \Bigg(3\,\theta\,\Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 + 108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{\left(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 + 432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\right)}\Big)^{1/3}\Bigg) +$$

$$\left(8\,i\,2^{1/3}\,\pi^2\,r^4\right) \Big/ \Bigg(\sqrt{3}\,\theta\,\Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 + 108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{\left(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 + 432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\right)}\Big)^{1/3}\Bigg) -$$

$$\left(r^4\,\theta\right) \Big/ \Bigg(3\,2^{2/3}\,\Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 + 108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{\left(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 + 432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\right)}\Big)^{1/3}\Bigg) +$$

$$\left(i\,r^4\,\theta\right) \Big/ \Bigg(2^{2/3}\,\sqrt{3}\,\Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{\left(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\right.}$$

$$\left.\left.\left.\theta^4 + 432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\right)\right)^{1/3}\right) -$$

$$\left(2^{1/3}\,r^2\,v^2\,\theta\right) \Big/ \left(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +\right.$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{\left(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 +\right.}$$

$$\left.\left.432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\right)\right)^{1/3} +$$

$$\left(i\,2^{1/3}\,\sqrt{3}\,r^2\,v^2\,\theta\right) \Big/ \left(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +\right.$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{\left(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 +\right.}$$

$$\left.\left.432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\right)\right)^{1/3} -$$

$$\frac{1}{6\,2^{1/3}\,\theta}\left(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 + 108\,\pi\,r^2\,v^4\,\theta^2 -\right.$$

$$2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{\left(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 +\right.}$$

$$\left.\left.432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\right)\right)^{1/3} -$$

$$\frac{1}{2\,2^{1/3}\,\sqrt{3}\,\theta}\,i\left(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +\right.$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{\left(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 +\right.}$$

$$\left.\left.\left.432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\right)\right)^{1/3}\right)\Big\},$$

$$\Big\{c \to \sqrt{\left(-\frac{r^2}{3} + \frac{4\,\pi\,r^2}{3\,\theta} + \left(4\,2^{1/3}\,\pi\,r^4\right) \Big/ \left(3\left(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +\right.\right.\right.}$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{\left(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 +\right.}$$

$$\left.\left.\left.432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\right)\right)^{1/3}\right) -$$

$$\left(4\,i\,2^{1/3}\,\pi\,r^4\right) \Big/ \left(\sqrt{3}\left(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +\right.\right.$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{\left(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 +\right.}$$

$$\left.\left.\left.432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\right)\right)^{1/3}\right) +$$

$$\left(4\,2^{1/3}\,\pi\,r^2\,v^2\right) \Big/ \left(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +\right.$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{}\,(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6$$

$$\theta^4 + 432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6))^{1/3} -$$

$$(4\,i\,2^{1/3}\,\sqrt{3}\,\pi\,r^2\,v^2) \Big/ (128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{}\,(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6$$

$$\theta^4 + 432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6))^{1/3} -$$

$$(8\,2^{1/3}\,\pi^2\,r^4) \Big/ (3\,\theta\,(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{}\,(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 +$$

$$432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6))^{1/3}) +$$

$$(8\,i\,2^{1/3}\,\pi^2\,r^4) \Big/ (\sqrt{3}\,\theta\,(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{}\,(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 +$$

$$432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6))^{1/3}) -$$

$$(r^4\,\theta) \Big/ (3\,2^{2/3}\,(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{}\,(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 +$$

$$432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6))^{1/3}) +$$

$$(i\,r^4\,\theta) \Big/ (2^{2/3}\,\sqrt{3}\,(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{}\,(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 +$$

$$432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6))^{1/3}) -$$

$$(2^{1/3}\,r^2\,v^2\,\theta) \Big/ (128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{}\,(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6$$

$$\theta^4 + 432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6))^{1/3} +$$

$$(i\,2^{1/3}\,\sqrt{3}\,r^2\,v^2\,\theta) \Big/ (128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{}\,(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6$$

$$\theta^4 + 432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\Big)\Big)^{1/3} -$$

$$\frac{1}{6\,2^{1/3}\,\theta}\Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 + 108\,\pi\,r^2\,v^4\,\theta^2 -$$

$$2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\sqrt{3}\,\sqrt{\Big(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 + 432}$$

$$\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\Big)\Big)^{1/3} -$$

$$\frac{1}{2\,2^{1/3}\,\sqrt{3}\,\theta}\,i\,\Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\sqrt{3}\,\sqrt{\Big(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 +}$$

$$432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\Big)\Big)^{1/3}\Bigg)\Big\},$$

$$\Big\{c \to -\sqrt{\Bigg(-\frac{r^2}{3} + \frac{4\,\pi\,r^2}{3\,\theta} + \left(4\,2^{1/3}\,\pi\,r^4\right)\Big/\Big(3\,\Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2}$$

$$\theta^2 + 108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\sqrt{3}\,\sqrt{\Big(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6}$$

$$v^6\,\theta^4 + 432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\Big)\Big)^{1/3}\Big) +$$

$$\left(4\,i\,2^{1/3}\,\pi\,r^4\right)\Big/\Big(\sqrt{3}\,\Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\sqrt{3}\,\sqrt{\Big(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6}$$

$$\theta^4 + 432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\Big)\Big)^{1/3}\Big) +$$

$$\left(4\,2^{1/3}\,\pi\,r^2\,v^2\right)\Big/\Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\sqrt{3}\,\sqrt{\Big(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 +}$$

$$432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\Big)\Big)^{1/3} +$$

$$\left(4\,i\,2^{1/3}\,\sqrt{3}\,\pi\,r^2\,v^2\right)\Big/\Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\sqrt{3}\,\sqrt{\Big(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 +}$$

$$432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6\Big)\Big)^{1/3} -$$

$$\left(8\,2^{1/3}\,\pi^2\,r^4\right)\Big/\Big(3\,\theta\,\Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\sqrt{3}\,\sqrt{\Big(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6}$$

$$\theta^4 + 432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6)\Big)^{1/3}\Big) -$$

$$\left(8\,i\,2^{1/3}\,\pi^2\,r^4\right) \Big/ \Big(\sqrt{3}\,\theta\,\Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{}\,(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6$$

$$\theta^4 + 432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6)\Big)^{1/3}\Big) -$$

$$\left(r^4\,\theta\right) \Big/ \Big(3\,2^{2/3}\,\Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{}\,(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6$$

$$\theta^4 + 432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6)\Big)^{1/3}\Big) -$$

$$\left(i\,r^4\,\theta\right) \Big/ \Big(2^{2/3}\,\sqrt{3}\,\Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{}\,(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6$$

$$\theta^4 + 432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6)\Big)^{1/3}\Big) -$$

$$\left(2^{1/3}\,r^2\,v^2\,\theta\right) \Big/ \Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{}\,(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 +$$

$$432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6)\Big)^{1/3} -$$

$$\left(i\,2^{1/3}\,\sqrt{3}\,r^2\,v^2\,\theta\right) \Big/ \Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{}\,(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 +$$

$$432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6)\Big)^{1/3} -$$

$$\frac{1}{6\,2^{1/3}\,\theta}\,\Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 + 108\,\pi\,r^2\,v^4\,\theta^2 -$$

$$2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{}\,(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 +$$

$$432\,\pi^2\,r^4\,v^8\,\theta^4 - 48\,\pi\,r^6\,v^6\,\theta^5 - 216\,\pi\,r^4\,v^8\,\theta^5 + 4\,r^6\,v^6\,\theta^6 + 27\,r^4\,v^8\,\theta^6)\Big)^{1/3} +$$

$$\frac{1}{2\,2^{1/3}\,\sqrt{3}\,\theta}\,i\,\Big(128\,\pi^3\,r^6 - 96\,\pi^2\,r^6\,\theta - 288\,\pi^2\,r^4\,v^2\,\theta + 24\,\pi\,r^6\,\theta^2 + 144\,\pi\,r^4\,v^2\,\theta^2 +$$

$$108\,\pi\,r^2\,v^4\,\theta^2 - 2\,r^6\,\theta^3 - 18\,r^4\,v^2\,\theta^3 - 27\,r^2\,v^4\,\theta^3 + 3\,\sqrt{3}\,\sqrt{}\,(-256\,\pi^3\,r^6\,v^6\,\theta^3 + 192\,\pi^2\,r^6\,v^6\,\theta^4 +$$

$$432\,\pi^2\,r^4\,v^8\,\theta^4-48\,\pi\,r^6\,v^6\,\theta^5-216\,\pi\,r^4\,v^8\,\theta^5+4\,r^6\,v^6\,\theta^6+27\,r^4\,v^8\,\theta^6\Big)\Big)^{1/3}\Big)\Big\},$$

$$\Big\{c\to\sqrt{\Big(-\frac{r^2}{3}+\frac{4\,\pi\,r^2}{3\,\theta}+\left(4\,2^{1/3}\,\pi\,r^4\right)\Big/\Big(3\,\Big(128\,\pi^3\,r^6-96\,\pi^2\,r^6\,\theta-288\,\pi^2\,r^4\,v^2\,\theta+24\,\pi\,r^6\,\theta^2+144\,\pi\,r^4\,v^2\,\theta^2+}$$

$$108\,\pi\,r^2\,v^4\,\theta^2-2\,r^6\,\theta^3-18\,r^4\,v^2\,\theta^3-27\,r^2\,v^4\,\theta^3+3\,\sqrt{3}\,\sqrt{\Big(-256\,\pi^3\,r^6\,v^6\,\theta^3+192\,\pi^2\,r^6\,v^6\,\theta^4+}$$

$$432\,\pi^2\,r^4\,v^8\,\theta^4-48\,\pi\,r^6\,v^6\,\theta^5-216\,\pi\,r^4\,v^8\,\theta^5+4\,r^6\,v^6\,\theta^6+27\,r^4\,v^8\,\theta^6\Big)\Big)^{1/3}\Big)+$$

$$\left(4\,i\,2^{1/3}\,\pi\,r^4\right)\Big/\Big(\sqrt{3}\,\Big(128\,\pi^3\,r^6-96\,\pi^2\,r^6\,\theta-288\,\pi^2\,r^4\,v^2\,\theta+24\,\pi\,r^6\,\theta^2+144\,\pi\,r^4\,v^2\,\theta^2+$$

$$108\,\pi\,r^2\,v^4\,\theta^2-2\,r^6\,\theta^3-18\,r^4\,v^2\,\theta^3-27\,r^2\,v^4\,\theta^3+3\,\sqrt{3}\,\sqrt{\Big(-256\,\pi^3\,r^6\,v^6\,\theta^3+192\,\pi^2\,r^6\,v^6\,\theta^4+}$$

$$432\,\pi^2\,r^4\,v^8\,\theta^4-48\,\pi\,r^6\,v^6\,\theta^5-216\,\pi\,r^4\,v^8\,\theta^5+4\,r^6\,v^6\,\theta^6+27\,r^4\,v^8\,\theta^6\Big)\Big)^{1/3}\Big)+$$

$$\left(4\,2^{1/3}\,\pi\,r^2\,v^2\right)\Big/\Big(128\,\pi^3\,r^6-96\,\pi^2\,r^6\,\theta-288\,\pi^2\,r^4\,v^2\,\theta+24\,\pi\,r^6\,\theta^2+144\,\pi\,r^4\,v^2\,\theta^2+$$

$$108\,\pi\,r^2\,v^4\,\theta^2-2\,r^6\,\theta^3-18\,r^4\,v^2\,\theta^3-27\,r^2\,v^4\,\theta^3+3\,\sqrt{3}\,\sqrt{\Big(-256\,\pi^3\,r^6\,v^6\,\theta^3+192\,\pi^2\,r^6\,v^6}$$

$$\theta^4+432\,\pi^2\,r^4\,v^8\,\theta^4-48\,\pi\,r^6\,v^6\,\theta^5-216\,\pi\,r^4\,v^8\,\theta^5+4\,r^6\,v^6\,\theta^6+27\,r^4\,v^8\,\theta^6\Big)\Big)^{1/3}+$$

$$\left(4\,i\,2^{1/3}\,\sqrt{3}\,\pi\,r^2\,v^2\right)\Big/\Big(128\,\pi^3\,r^6-96\,\pi^2\,r^6\,\theta-288\,\pi^2\,r^4\,v^2\,\theta+24\,\pi\,r^6\,\theta^2+144\,\pi\,r^4\,v^2\,\theta^2+$$

$$108\,\pi\,r^2\,v^4\,\theta^2-2\,r^6\,\theta^3-18\,r^4\,v^2\,\theta^3-27\,r^2\,v^4\,\theta^3+3\,\sqrt{3}\,\sqrt{\Big(-256\,\pi^3\,r^6\,v^6\,\theta^3+192\,\pi^2\,r^6\,v^6}$$

$$\theta^4+432\,\pi^2\,r^4\,v^8\,\theta^4-48\,\pi\,r^6\,v^6\,\theta^5-216\,\pi\,r^4\,v^8\,\theta^5+4\,r^6\,v^6\,\theta^6+27\,r^4\,v^8\,\theta^6\Big)\Big)^{1/3}-$$

$$\left(8\,2^{1/3}\,\pi^2\,r^4\right)\Big/\Big(3\,\theta\,\Big(128\,\pi^3\,r^6-96\,\pi^2\,r^6\,\theta-288\,\pi^2\,r^4\,v^2\,\theta+24\,\pi\,r^6\,\theta^2+144\,\pi\,r^4\,v^2\,\theta^2+$$

$$108\,\pi\,r^2\,v^4\,\theta^2-2\,r^6\,\theta^3-18\,r^4\,v^2\,\theta^3-27\,r^2\,v^4\,\theta^3+3\,\sqrt{3}\,\sqrt{\Big(-256\,\pi^3\,r^6\,v^6\,\theta^3+192\,\pi^2\,r^6\,v^6\,\theta^4+}$$

$$432\,\pi^2\,r^4\,v^8\,\theta^4-48\,\pi\,r^6\,v^6\,\theta^5-216\,\pi\,r^4\,v^8\,\theta^5+4\,r^6\,v^6\,\theta^6+27\,r^4\,v^8\,\theta^6\Big)\Big)^{1/3}\Big)-$$

$$\left(8\,i\,2^{1/3}\,\pi^2\,r^4\right)\Big/\Big(\sqrt{3}\,\theta\,\Big(128\,\pi^3\,r^6-96\,\pi^2\,r^6\,\theta-288\,\pi^2\,r^4\,v^2\,\theta+24\,\pi\,r^6\,\theta^2+144\,\pi\,r^4\,v^2\,\theta^2+$$

$$108\,\pi\,r^2\,v^4\,\theta^2-2\,r^6\,\theta^3-18\,r^4\,v^2\,\theta^3-27\,r^2\,v^4\,\theta^3+3\,\sqrt{3}\,\sqrt{\Big(-256\,\pi^3\,r^6\,v^6\,\theta^3+192\,\pi^2\,r^6\,v^6\,\theta^4+}$$

$$432\,\pi^2\,r^4\,v^8\,\theta^4-48\,\pi\,r^6\,v^6\,\theta^5-216\,\pi\,r^4\,v^8\,\theta^5+4\,r^6\,v^6\,\theta^6+27\,r^4\,v^8\,\theta^6\Big)\Big)^{1/3}\Big)-$$

$$\left(r^4\,\theta\right)\Big/\Big(3\,2^{2/3}\,\Big(128\,\pi^3\,r^6-96\,\pi^2\,r^6\,\theta-288\,\pi^2\,r^4\,v^2\,\theta+24\,\pi\,r^6\,\theta^2+144\,\pi\,r^4\,v^2\,\theta^2+$$

$$108\,\pi\,r^2\,v^4\,\theta^2-2\,r^6\,\theta^3-18\,r^4\,v^2\,\theta^3-27\,r^2\,v^4\,\theta^3+3\,\sqrt{3}\,\sqrt{\Big(-256\,\pi^3\,r^6\,v^6\,\theta^3+192\,\pi^2\,r^6\,v^6\,\theta^4+}$$

$$432\,\pi^2\,r^4\,v^8\,\theta^4-48\,\pi\,r^6\,v^6\,\theta^5-216\,\pi\,r^4\,v^8\,\theta^5+4\,r^6\,v^6\,\theta^6+27\,r^4\,v^8\,\theta^6\big)\Big)^{1/3}\Big)-$$

$$(\mathbb{i}\,r^4\,\theta)\Big/\Big(2^{2/3}\sqrt{3}\,\Big(128\,\pi^3\,r^6-96\,\pi^2\,r^6\,\theta-288\,\pi^2\,r^4\,v^2\,\theta+24\,\pi\,r^6\,\theta^2+144\,\pi\,r^4\,v^2\,\theta^2+$$
$$108\,\pi\,r^2\,v^4\,\theta^2-2\,r^6\,\theta^3-18\,r^4\,v^2\,\theta^3-27\,r^2\,v^4\,\theta^3+3\sqrt{3}\,\sqrt{\big(-256\,\pi^3\,r^6\,v^6\,\theta^3+192\,\pi^2\,r^6\,v^6\,\theta^4+}$$
$$432\,\pi^2\,r^4\,v^8\,\theta^4-48\,\pi\,r^6\,v^6\,\theta^5-216\,\pi\,r^4\,v^8\,\theta^5+4\,r^6\,v^6\,\theta^6+27\,r^4\,v^8\,\theta^6\big)\Big)^{1/3}\Big)-$$

$$(2^{1/3}\,r^2\,v^2\,\theta)\Big/\Big(128\,\pi^3\,r^6-96\,\pi^2\,r^6\,\theta-288\,\pi^2\,r^4\,v^2\,\theta+24\,\pi\,r^6\,\theta^2+144\,\pi\,r^4\,v^2\,\theta^2+$$
$$108\,\pi\,r^2\,v^4\,\theta^2-2\,r^6\,\theta^3-18\,r^4\,v^2\,\theta^3-27\,r^2\,v^4\,\theta^3+3\sqrt{3}\,\sqrt{\big(-256\,\pi^3\,r^6\,v^6\,\theta^3+192\,\pi^2\,r^6\,v^6}$$
$$\theta^4+432\,\pi^2\,r^4\,v^8\,\theta^4-48\,\pi\,r^6\,v^6\,\theta^5-216\,\pi\,r^4\,v^8\,\theta^5+4\,r^6\,v^6\,\theta^6+27\,r^4\,v^8\,\theta^6\big)\Big)^{1/3}-$$

$$\big(\mathbb{i}\,2^{1/3}\sqrt{3}\,r^2\,v^2\,\theta\big)\Big/\Big(128\,\pi^3\,r^6-96\,\pi^2\,r^6\,\theta-288\,\pi^2\,r^4\,v^2\,\theta+24\,\pi\,r^6\,\theta^2+144\,\pi\,r^4\,v^2\,\theta^2+$$
$$108\,\pi\,r^2\,v^4\,\theta^2-2\,r^6\,\theta^3-18\,r^4\,v^2\,\theta^3-27\,r^2\,v^4\,\theta^3+3\sqrt{3}\,\sqrt{\big(-256\,\pi^3\,r^6\,v^6\,\theta^3+192\,\pi^2\,r^6\,v^6}$$
$$\theta^4+432\,\pi^2\,r^4\,v^8\,\theta^4-48\,\pi\,r^6\,v^6\,\theta^5-216\,\pi\,r^4\,v^8\,\theta^5+4\,r^6\,v^6\,\theta^6+27\,r^4\,v^8\,\theta^6\big)\Big)^{1/3}-$$

$$\frac{1}{6\,2^{1/3}\,\theta}\Big(128\,\pi^3\,r^6-96\,\pi^2\,r^6\,\theta-288\,\pi^2\,r^4\,v^2\,\theta+24\,\pi\,r^6\,\theta^2+144\,\pi\,r^4\,v^2\,\theta^2+108\,\pi\,r^2\,v^4\,\theta^2-$$
$$2\,r^6\,\theta^3-18\,r^4\,v^2\,\theta^3-27\,r^2\,v^4\,\theta^3+3\sqrt{3}\,\sqrt{\big(-256\,\pi^3\,r^6\,v^6\,\theta^3+192\,\pi^2\,r^6\,v^6\,\theta^4+432}$$
$$\pi^2\,r^4\,v^8\,\theta^4-48\,\pi\,r^6\,v^6\,\theta^5-216\,\pi\,r^4\,v^8\,\theta^5+4\,r^6\,v^6\,\theta^6+27\,r^4\,v^8\,\theta^6\big)\Big)^{1/3}+$$

$$\frac{1}{2\,2^{1/3}\sqrt{3}\,\theta}\,\mathbb{i}\,\Big(128\,\pi^3\,r^6-96\,\pi^2\,r^6\,\theta-288\,\pi^2\,r^4\,v^2\,\theta+24\,\pi\,r^6\,\theta^2+144\,\pi\,r^4\,v^2\,\theta^2+$$
$$108\,\pi\,r^2\,v^4\,\theta^2-2\,r^6\,\theta^3-18\,r^4\,v^2\,\theta^3-27\,r^2\,v^4\,\theta^3+3\sqrt{3}\,\sqrt{\big(-256\,\pi^3\,r^6\,v^6\,\theta^3+192\,\pi^2\,r^6\,v^6\,\theta^4+}$$
$$432\,\pi^2\,r^4\,v^8\,\theta^4-48\,\pi\,r^6\,v^6\,\theta^5-216\,\pi\,r^4\,v^8\,\theta^5+4\,r^6\,v^6\,\theta^6+27\,r^4\,v^8\,\theta^6\big)\Big)^{1/3}\Big)\}\}$$

XVI. Spatio - Temporal Configurations for Faster than Light Travel (v > c): Mathematics for a Warp Engine.

By Parker Emmerson © 2009-2010

There are spatio – temporal, "univocally conic," configurative positions at which it is possible that entities travel faster than light.

a) Computational Results from the Lorentz Transformation

Theorem 3 The "innate velocity," v, within the Lorentz transformation can be solved for in terms of the system of the circle transforming into a cone. If r is multiplied by the Lorenz transformation, then it measures the distance in the prime system, denoted by r '. If t ' equals $\frac{\left(\frac{\theta}{(2\pi)}\right)}{\sqrt{1-\left(\frac{v^2}{c^2}\right)}}$, then the quantity r $\theta = \theta$ ' r '. We are only dealing with algebraic forms and the solutions necessitated by them. Logical, algebraic, reasoning will be given why, when using the exact speed of light, 2.99792458 (10^8) meters per second, the units of the speed of light can be ignored for the purposes of calculation and computation (they canel out - they are equal to one). This theorem states that, although, normal algebra would require the speed of light as a quantity to cancel out, valid expressions for the solutions for the intrinsic velocity, v, can be found in terms of η, r, and θ, or θ and β, depending on the expression used for the height of the cone.

Proof.

$$c = 2.99792458\left(10^8\right) \text{ meters per second}$$

$$r' = r\sqrt{1-\frac{v^2}{c^2}}$$

$$t' = \frac{\left(\frac{\theta}{(2\pi)}\right)}{\sqrt{1-\left(\frac{v^2}{c^2}\right)}}$$

$$t' = \frac{\theta'}{2\pi}$$

$$2\pi t' = \theta'$$

$$\theta' = \frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}$$

Therefore,

$$r' * \theta' = \left(\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}\right)\left(r\sqrt{1-\frac{v^2}{c^2}}\right)$$

$$\left(\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}\right)\left(r\sqrt{1-\frac{v^2}{c^2}}\right) = r\,\theta$$

$$r' * \theta' = r\,\theta = 2\,\pi\,r - 2\,\pi\,r_1 = 2\,\pi\,r - 2\,\pi\sqrt{r\,\hat{}\,2 - \eta\,\hat{}\,2}$$

$$\text{Solve}\left[r\,\theta == 2\,\pi\,r - 2\,\pi\sqrt{r\,\hat{}\,2 - \eta\,\hat{}\,2}\,,\ \eta\right]$$

$$\left\{\left\{\eta \to -\frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi}\right\},\ \left\{\eta \to \frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi}\right\}\right\}$$

$$\textbf{Solve}\left[r'\,\theta' == 2\,\pi\,r - 2\,\pi\sqrt{r\,\hat{}\,2 - \eta\,\hat{}\,2}\,,\ \eta\right]$$

$$\left\{\left\{\eta \to -\frac{\sqrt{r'}\ \sqrt{\theta'}\ \sqrt{4\,\pi\,r - r'\,\theta'}}{2\,\pi}\right\},\ \left\{\eta \to \frac{\sqrt{r'}\ \sqrt{\theta'}\ \sqrt{4\,\pi\,r - r'\,\theta'}}{2\,\pi}\right\}\right\}$$

The argument follows modus ponens, saying that, through commutation, r ' θ ' = θ r, therefore η = $\frac{\sqrt{r'}\ \sqrt{\theta'}\ \sqrt{4\,\pi\,r - r'\,\theta'}}{2\,\pi} = \frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi}$

η =

$$\frac{\sqrt{r'}\ \sqrt{\theta'}\ \sqrt{4\,\pi\,r - r'\,\theta'}}{2\,\pi} = \frac{\sqrt{r\sqrt{1-\frac{v^2}{c^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}}\ \sqrt{4\,\pi\,r - r\sqrt{1-\frac{v^2}{c^2}}\ \frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}}}{2\,\pi} = \frac{\sqrt{r\,\theta}\ \sqrt{4\,\pi\,r - r\,\theta}}{2\,\pi}$$

where c is its numeric value of the speed of light, its units being shown to cancel out.

$$\frac{\sqrt{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}}{\sqrt{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}}$$

$$\textbf{Solve}\left[\sqrt{\frac{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}}\,,\ \text{meters}\right]$$

{{}}

$$\textbf{Solve}\left[\sqrt{\frac{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}}\,,\ \text{second}\right]$$

{{}}

$$\text{Solve}\left[\sqrt{\frac{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}}==\sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}},\ v\right]$$

{{}}

$$\text{Solve}\left[\sqrt{\frac{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}}==\sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}},\ c\right]$$

$$\text{Solve}\left[\text{True},\ 2.99792\times 10^{8}\right]$$

$$\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}}}\ \sqrt{4\pi r-r\theta}}{2\pi}=$$

$$(1)\quad \frac{\sqrt{r\sqrt{1-\frac{v^2}{(c)^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{(c)^2}}}}\ \sqrt{4\pi r-r\theta}}{2\pi}$$

Logically, the units of the speed of light will cancel out, therefore, their relationship will be set equal to one and taken out of the equation for the purposes of computational calculation.

$$\text{Solve}\left[r'\theta' == 2\pi r-2\pi\sqrt{r^{\wedge}2-\eta^{\wedge}2},\ \eta\right]$$

$$\left\{\left\{\eta\to-\frac{\sqrt{r'}\ \sqrt{\theta'}\ \sqrt{4\pi r-r'\theta'}}{2\pi}\right\},\ \left\{\eta\to\frac{\sqrt{r'}\ \sqrt{\theta'}\ \sqrt{4\pi r-r'\theta'}}{2\pi}\right\}\right\}$$

The argument follows modus ponens, saying that, through commutation, $r'\theta' = \theta r$, therefore $\eta =$ $\frac{\sqrt{r'}\ \sqrt{\theta'}\ \sqrt{4\pi r-r'\theta'}}{2\pi}=\frac{\sqrt{4\pi r^2\theta-r^2\theta^2}}{2\pi}$

$\eta =$

$$\frac{\sqrt{r'}\ \sqrt{\theta'}\ \sqrt{4\pi r-r'\theta'}}{2\pi}=\frac{\sqrt{r\sqrt{1-\frac{v^2}{c^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}}\ \sqrt{4\pi r-r\sqrt{1-\frac{v^2}{c^2}}\ \frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}}}{2\pi}=\frac{\sqrt{r\theta}\ \sqrt{4\pi r-r\theta}}{2\pi}$$

where c is its numeric value of the speed of light, its units being shown to cancel out.

$$\frac{\sqrt{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}}{\sqrt{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}}==\sqrt{\frac{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}}==\sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}}$$

$$\text{Solve}\left[\sqrt{\frac{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}}==\sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}},\ \text{meters}\right]$$

{{}}

$$\text{Solve}\left[\sqrt{\frac{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}}==\sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}},\ \text{second}\right]$$

{{}}

$$\text{Solve}\left[\sqrt{\frac{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}}==\sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}},\ v\right]$$

{{}}

$$\text{Solve}\left[\sqrt{\frac{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}}==\sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}},\ c\right]$$

$$\text{Solve}\left[\text{True},\ 2.99792\times 10^{8}\right]$$

$$\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}}}\ \sqrt{4\pi r-r\theta}}{2\pi}=$$

$$\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}\ \frac{\theta}{\sqrt{1-\frac{v^2}{(c\,(\text{meters/second}))^2}}}}\ \sqrt{4\pi r-r\theta}}{2\pi}=$$

$$\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c)^2}}\ \frac{\theta}{\sqrt{1-\frac{v^2}{(c)^2}}}}\ \sqrt{4\pi r-r\theta}}{2\pi}=(1)\ \frac{\sqrt{r\sqrt{1-\frac{v^2}{(c)^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{(c)^2}}}}\ \sqrt{4\pi r-r\theta}}{2\pi}$$

Logically, the units of the speed of light will cancel out, therefore, their relationship will be set equal to one and taken out of the equation for the purposes of computational calculation.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}\ \frac{\theta}{\sqrt{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}}}\ \sqrt{4\pi r-r\theta}}{2\pi}==\eta,\ \text{meters}\right]$$

{{}}

Meters cancel out.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}\ \frac{\theta}{\sqrt{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}}}\ \sqrt{4\pi r-r\theta}}{2\pi}==\eta,\ \text{second}\right]$$

{{}}

Seconds cancel out.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}\ \frac{\theta}{\sqrt{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}}}\ \sqrt{4\pi r-r\theta}}{2\pi}==\eta,\ c\right]$$

{{}}

The numeric c cancels out.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}}}\ \sqrt{4\pi r-r\theta}}{2\pi}==\eta,\ r\right]$$

$$\left\{\left\{r\to-\frac{2\pi\eta}{\sqrt{4\pi\theta-\theta^2}}\right\},\ \left\{r\to\frac{2\pi\eta}{\sqrt{4\pi\theta-\theta^2}}\right\}\right\}$$

Radius yields the result from Lemma 3.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c)^2}}\ \frac{\theta}{\sqrt{1-\frac{v^2}{(c)^2}}}}\ \sqrt{4\pi r-r\theta}}{2\pi}==\eta,\ v\right]$$

{{}}

Velocity cancels out.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c)^2}}\ \frac{\theta}{\sqrt{1-\frac{v^2}{(c)^2}}}}\ \sqrt{4\pi r - r\theta}}{2\pi} == \eta,\ v\right]$$

{{}}

Radius yields the result from Lemma 3.

{}

Velocity cancels out. Only when using the exact speed of light in scientific notation can solutions be found.

We set the speed of light equal to its numeric value for the purpose of making computations, dropping the units, because in the expression for the height of the cone, they would cancel out anyway. It should be noted that this is necessary for computing the function of the velocity and that the exact speed of light is to be used as well as that the numeric value of the speed of light has to be in the form of scientific notation in order to find results to this equation.

```
c := 2.99792458 (10^8)
```

Theorem 3 Continued From the expression of the height of the cone of Lemma 1, with the Lorentz transformations implicitly expressed, we can solve for the velocity within the Lorenz coefficient in terms of the height of the cone, the initial radius, and the angle, θ.

Proof.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{(v)^2}{c^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{(v)^2}{c^2}}}}\ \sqrt{4\pi r - r\theta}}{2\pi} == \eta,\ v\right]$$

$$\left\{\left\{v \to -\frac{1.\sqrt{3.54814\times 10^{18}\,\eta^2 - 1.12941\times 10^{18}\,r^2\,\theta + 8.98755\times 10^{16}\,r^2\,\theta^2}}{\sqrt{39.4784\,\eta^2 - 12.5664\,r^2\,\theta + r^2\,\theta^2}}\right\},\right.$$

$$\left.\left\{v \to \frac{\sqrt{3.54814\times 10^{18}\,\eta^2 - 1.12941\times 10^{18}\,r^2\,\theta + 8.98755\times 10^{16}\,r^2\,\theta^2}}{\sqrt{39.4784\,\eta^2 - 12.5664\,r^2\,\theta + r^2\,\theta^2}}\right\}\right\}$$

Theorem 3 Continued From the expression of the height of the cone, from Lemma 1 with the Lorentz transformations implicitly expressed, we can solve for the velocity within the Lorenz coefficient in terms of θ and β.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{(v)^2}{c^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{(v)^2}{c^2}}}}\ \sqrt{4\pi r - r\theta}}{2\pi} == r\,\text{Sin}[\beta],\ v\right]$$

$$\left\{\left\{v \to -\frac{1.\sqrt{-1.12941\times 10^{18}\,\theta + 8.98755\times 10^{16}\,\theta^2 + 3.54814\times 10^{18}\,\text{Sin}[\beta]^2}}{\sqrt{-12.5664\,\theta + \theta^2 + 39.4784\,\text{Sin}[\beta]^2}}\right\},\right.$$

$$\left.\left\{v \to \frac{\sqrt{-1.12941\times 10^{18}\,\theta + 8.98755\times 10^{16}\,\theta^2 + 3.54814\times 10^{18}\,\text{Sin}[\beta]^2}}{\sqrt{-12.5664\,\theta + \theta^2 + 39.4784\,\text{Sin}[\beta]^2}}\right\}\right\}$$

b) Now, Apply the Substitution, and Set the Result Equal to 99*(2.99792458 (10^8)) and Solve the Equation.

Lemma 2 The angle θ can be calculated in terms of r and η.

Proof

$$\text{Solve}\left[\eta == \frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi},\ \theta\right]$$

$$\left\{\left\{\theta \to \frac{2\,\pi\left(r^2 - \sqrt{r^4 - r^2\,\eta^2}\right)}{r^2}\right\},\ \left\{\theta \to \frac{2\,\pi\left(r^2 + \sqrt{r^4 - r^2\,\eta^2}\right)}{r^2}\right\}\right\}$$

Lemma 4 *The height of the cone can be calculated in terms of only r and* θ, thus β is a function of θ alone.

Proof. Since we have shown that θ r = 2 π r - 2 π r_1 and $r_1 \to \sqrt{r^2 - \eta^2}$, we can substitute the expression for r_1, calculated from the Pythagorean theorem in terms of the height of the cone and the initial radius of the circle, into the expression for θ r in terms of the change in circumference of the initial circle to the circle that is the base of the cone into which the circle was transformed.

θ r = 2 π r - 2 π $\sqrt{(r\hat{}2 - \eta\hat{}2)}$, thus, $\eta = \frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi}$ = (r Sin[β]). From $\frac{2\,\pi\,\eta}{\sqrt{4\,\pi\,\theta - \theta^2}}$ = r, we note that: r = $\frac{2\,\pi\,r\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\theta - \theta^2}}$. So we solve the equation,

$$\text{Solve}\left[r == \frac{2\,\pi\,r\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\theta - \theta^2}},\ \beta\right]$$

$$\left\{\left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{(4\,\pi - \theta)\,\theta}}{2\,\pi}\right]\right\}\right\}$$

$$v = \left(\sqrt{\left(-1.1294090667581471`*\hat{}18\,\theta + 8.987551787368176`*\hat{}16\,\theta^2 + 3.5481432270250993`*\hat{}18\,\text{Sin}[\beta]^2\right)}\right) \Big/ \left(\sqrt{-12.566370614359172`\,\theta + \theta^2 + 39.47841760435743`\,\text{Sin}[\beta]^2}\right) =$$

$$\left(\sqrt{\left(-1.1294090667581471`*\hat{}18\,\frac{2\,\pi\left(r^2 + \sqrt{r^4 - r^2\,\eta^2}\right)}{r^2} + 8.987551787368176`*\hat{}16\left(\frac{2\,\pi\left(r^2 + \sqrt{r^4 - r^2\,\eta^2}\right)}{r^2}\right)^2 + 3.5481432270250993`*\hat{}18\,\text{Sin}[\beta]^2\right)}\right) \Big/ \left(\sqrt{\left(-12.566370614359172`\,\frac{2\,\pi\left(r^2 + \sqrt{r^4 - r^2\,\eta^2}\right)}{r^2} + \left(\frac{2\,\pi\left(r^2 + \sqrt{r^4 - r^2\,\eta^2}\right)}{r^2}\right)^2 + 39.47841760435743`\,\text{Sin}[\beta]^2\right)}\right)$$

```
Solve[(√(-1.1294090667581471`*^18 (2 π (r^2 + √(r^4 - r^2 η^2)))/r^2 +
      8.987551787368176`*^16 ((2 π (r^2 + √(r^4 - r^2 η^2)))/r^2)^2 + 3.5481432270250993`*^18 Sin[β]^2)) /
   (√(-12.566370614359172` (2 π (r^2 + √(r^4 - r^2 η^2)))/r^2 + ((2 π (r^2 + √(r^4 - r^2 η^2)))/r^2)^2 +
      39.47841760435743` Sin[β]^2)) == (99.0 * (10^8)), r]
```

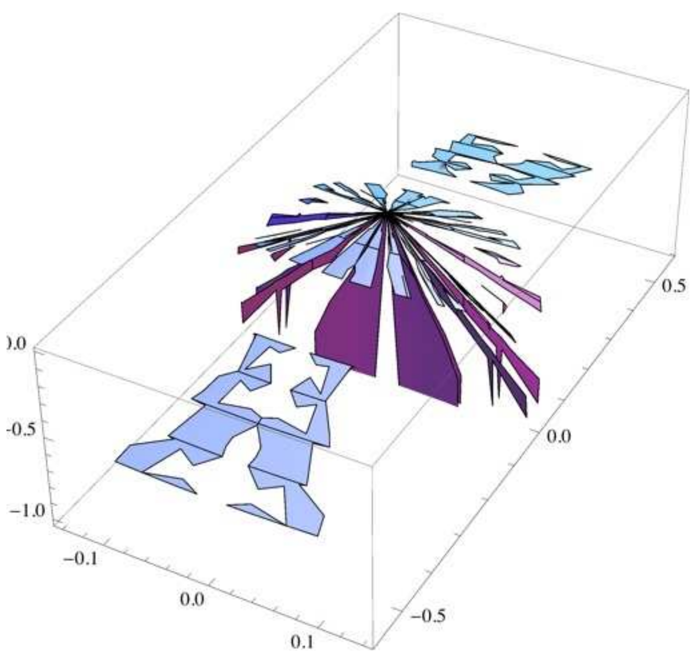
$$\left\{\left\{r\to -0.5\sqrt{\left(-\frac{2.\ \eta^2\left(1.65145\times10^{20}-1.35532\times10^{52}\,\mathrm{Sin}[\beta]^2\right)}{-1.65145\times10^{20}+6.77658\times10^{51}\,\mathrm{Sin}[\beta]^4}-2.\sqrt{\left(\frac{\eta^4\left(1.65145\times10^{20}-1.35532\times10^{52}\,\mathrm{Sin}[\beta]^2\right)^2}{\left(-1.65145\times10^{20}+6.77658\times10^{51}\,\mathrm{Sin}[\beta]^4\right)^2}-\frac{2.71063\times10^{52}\,\eta^4}{-1.65145\times10^{20}+6.77658\times10^{51}\,\mathrm{Sin}[\beta]^4}\right)}\right)}\right\},\right.$$

$$\left\{r\to 0.5\sqrt{\left(-\frac{2.\ \eta^2\left(1.65145\times10^{20}-1.35532\times10^{52}\,\mathrm{Sin}[\beta]^2\right)}{-1.65145\times10^{20}+6.77658\times10^{51}\,\mathrm{Sin}[\beta]^4}-2.\sqrt{\left(\frac{\eta^4\left(1.65145\times10^{20}-1.35532\times10^{52}\,\mathrm{Sin}[\beta]^2\right)^2}{\left(-1.65145\times10^{20}+6.77658\times10^{51}\,\mathrm{Sin}[\beta]^4\right)^2}-\frac{2.71063\times10^{52}\,\eta^4}{-1.65145\times10^{20}+6.77658\times10^{51}\,\mathrm{Sin}[\beta]^4}\right)}\right)}\right\},$$

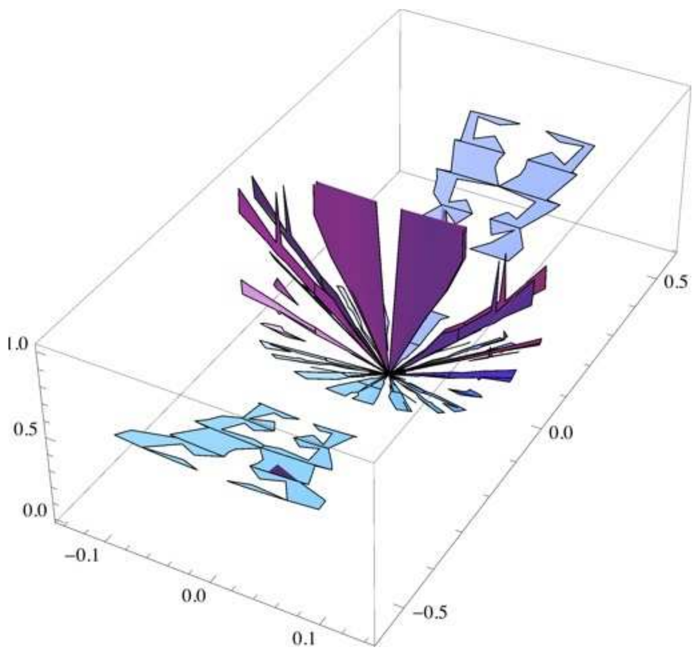
$$\left\{r\to -0.5\sqrt{\left(-\frac{2.\ \eta^2\left(1.65145\times10^{20}-1.35532\times10^{52}\,\mathrm{Sin}[\beta]^2\right)}{-1.65145\times10^{20}+6.77658\times10^{51}\,\mathrm{Sin}[\beta]^4}+2.\sqrt{\left(\frac{\eta^4\left(1.65145\times10^{20}-1.35532\times10^{52}\,\mathrm{Sin}[\beta]^2\right)^2}{\left(-1.65145\times10^{20}+6.77658\times10^{51}\,\mathrm{Sin}[\beta]^4\right)^2}-\frac{2.71063\times10^{52}\,\eta^4}{-1.65145\times10^{20}+6.77658\times10^{51}\,\mathrm{Sin}[\beta]^4}\right)}\right)}\right\},$$

$$\left.\left\{r\to 0.5\sqrt{\left(-\frac{2.\ \eta^2\left(1.65145\times10^{20}-1.35532\times10^{52}\,\mathrm{Sin}[\beta]^2\right)}{-1.65145\times10^{20}+6.77658\times10^{51}\,\mathrm{Sin}[\beta]^4}+2.\sqrt{\left(\frac{\eta^4\left(1.65145\times10^{20}-1.35532\times10^{52}\,\mathrm{Sin}[\beta]^2\right)^2}{\left(-1.65145\times10^{20}+6.77658\times10^{51}\,\mathrm{Sin}[\beta]^4\right)^2}-\frac{2.71063\times10^{52}\,\eta^4}{-1.65145\times10^{20}+6.77658\times10^{51}\,\mathrm{Sin}[\beta]^4}\right)}\right)}\right\}\right\}$$

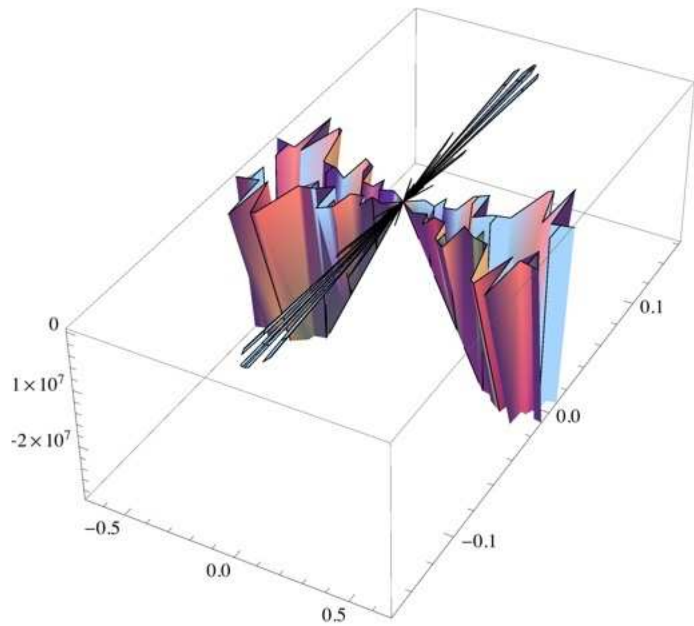
```
RevolutionPlot3D[
 -0.5` Sqrt[(-(2.` η^2 (1.6514528686265244`*^20 - 1.3553156905595964`*^52 Sin[β]^2)) /
     (-1.6514528686265244`*^20 + 6.776578452797982`*^51 Sin[β]^4) -
    2.` Sqrt[(η^4 (1.6514528686265244`*^20 - 1.3553156905595964`*^52 Sin[β]^2)^2 /
        (-1.6514528686265244`*^20 + 6.776578452797982`*^51 Sin[β]^4)^2 -
       2.7106313811191934`*^52 η^4 /
        (-1.6514528686265244`*^20 + 6.776578452797982`*^51 Sin[β]^4))])],
 {η, -1, 1}, {β, -π / 2, π / 2}]
```



```
RevolutionPlot3D[0.5` Sqrt[(-(2.` η^2 (1.6514528686265244`*^20 - 1.3553156905595964`*^52 Sin[β]^2)) /
      (-1.6514528686265244`*^20 + 6.776578452797982`*^51 Sin[β]^4) -
     2.` Sqrt[(η^4 (1.6514528686265244`*^20 - 1.3553156905595964`*^52 Sin[β]^2)^2 /
          (-1.6514528686265244`*^20 + 6.776578452797982`*^51 Sin[β]^4)^2 -
         2.7106313811191934`*^52 η^4 /
          (-1.6514528686265244`*^20 + 6.776578452797982`*^51 Sin[β]^4))])],
  {η, -1, 1}, {β, -π / 2, π / 2}]
```

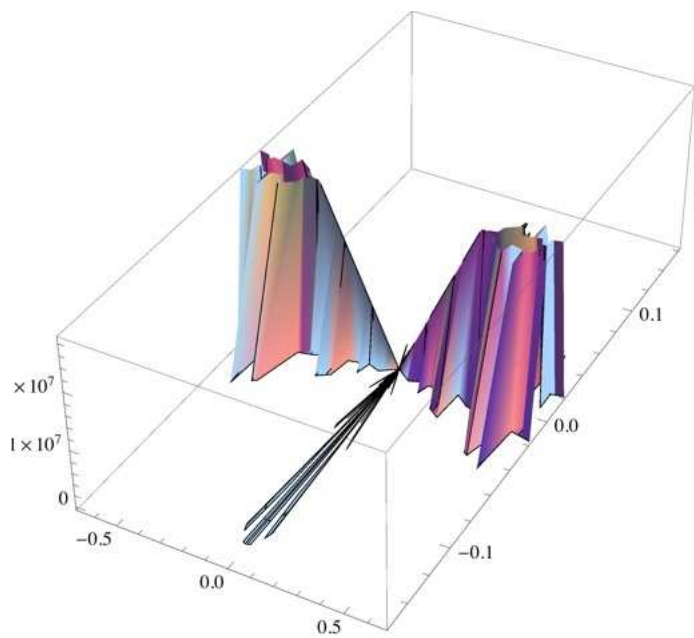


```
RevolutionPlot3D[
 -0.5` Sqrt[(-(2.` η^2 (1.6514528686265244`*^20 - 1.3553156905595964`*^52 Sin[β]^2)) /
      (-1.6514528686265244`*^20 + 6.776578452797982`*^51 Sin[β]^4) +
     2.` Sqrt[(η^4 (1.6514528686265244`*^20 - 1.3553156905595964`*^52 Sin[β]^2)^2 /
          (-1.6514528686265244`*^20 + 6.776578452797982`*^51 Sin[β]^4)^2 -
         2.7106313811191934`*^52 η^4 /
          (-1.6514528686265244`*^20 + 6.776578452797982`*^51 Sin[β]^4))])],
 {η, -1, 1}, {β, -π / 2, π / 2}]
```



$$\text{RevolutionPlot3D}\Big[0.5` \sqrt{\Big(-\big(2.`\ \eta^2\ (1.6514528686265244`*^20 - 1.3553156905595964`*^52\ \text{Sin}[\beta]^2)\big) \Big/ (-1.6514528686265244`*^20 + 6.776578452797982`*^51\ \text{Sin}[\beta]^4) + 2.` \sqrt{\left(\frac{\eta^4\ (1.6514528686265244`*^20 - 1.3553156905595964`*^52\ \text{Sin}[\beta]^2)^2}{(-1.6514528686265244`*^20 + 6.776578452797982`*^51\ \text{Sin}[\beta]^4)^2} - \frac{2.7106313811191934`*^52\ \eta^4}{-1.6514528686265244`*^20 + 6.776578452797982`*^51\ \text{Sin}[\beta]^4}\right)}\Big)},$$

$$\{\eta, -1, 1\}, \{\beta, -\pi/2, \pi/2\}\Big]$$



■ c) You Can Also Solve for the Height of the Cone, η.

1. Select a solution to the radius at faster than light travel from a proper solution from having solved the innate velocity within the Lorentz transformation. This can be done in several ways when using the exact speed of light in scientific notation. (51)

2. You will probably get multiple solutions to the initial radius. Select one of them and then solve for the height of the cone. (52)

3. Example Below : (53)

```
Solve[r == 0.5` √( (-(2.` η^2 (1.6514528686265244`*^20 - 1.3553156905595964`*^52 Sin[β]^2)) /
      (-1.6514528686265244`*^20 + 6.776578452797982`*^51 Sin[β]^4) +
     2.` √( (η^4 (1.6514528686265244`*^20 - 1.3553156905595964`*^52 Sin[β]^2)^2) /
          (-1.6514528686265244`*^20 + 6.776578452797982`*^51 Sin[β]^4)^2 -
          2.7106313811191934`*^52 η^4 / (-1.6514528686265244`*^20 + 6.776578452797982`*^51 Sin[β]^4) )), η]
```

```
{{η →
   -0.5 √( (-(2. r^2 (-1.08097×10^77 + 8.87129×10^108 Sin[β]^2 + 1.33069×10^109 Sin[β]^4 - 1.09207×10^141
            Sin[β]^6 - 5.46037×10^140 Sin[β]^8 + 4.48122×10^172 Sin[β]^10 +
           7.46871×10^171 Sin[β]^12 - 6.12942×10^203 Sin[β]^14)) /
       (-4.43565×10^108 - 5.07698×10^92 Sin[β]^2 + 5.46037×10^140 Sin[β]^4 + 3.88622×10^124 Sin[β]^6 -
         2.24061×10^172 Sin[β]^8 - 7.39814×10^155 Sin[β]^10 + 3.06471×10^203 Sin[β]^12) -
      2. √( ((r^4 (-1.08097×10^77 + 8.87129×10^108 Sin[β]^2 + 1.33069×10^109 Sin[β]^4 -
                1.09207×10^141 Sin[β]^6 - 5.46037×10^140 Sin[β]^8 + 4.48122×10^172 Sin[β]^10 +
                7.46871×10^171 Sin[β]^12 - 6.12942×10^203 Sin[β]^14)^2) / (-4.43565×10^108 -
              5.07698×10^92 Sin[β]^2 + 5.46037×10^140 Sin[β]^4 + 3.88622×10^124 Sin[β]^6 -
              2.24061×10^172 Sin[β]^8 - 7.39814×10^155 Sin[β]^10 + 3.06471×10^203 Sin[β]^12)^2 -
          (4. r^4 (1.08097×10^77 - 1.77426×10^109 Sin[β]^4 + 1.09207×10^141 Sin[β]^8 -
               2.98748×10^172 Sin[β]^12 + 3.06471×10^203 Sin[β]^16)) / (-4.43565×10^108 -
             5.07698×10^92 Sin[β]^2 + 5.46037×10^140 Sin[β]^4 + 3.88622×10^124 Sin[β]^6 -
             2.24061×10^172 Sin[β]^8 - 7.39814×10^155 Sin[β]^10 + 3.06471×10^203 Sin[β]^12))))},
 {η → 0.5 √( (-(2. r^2 (-1.08097×10^77 + 8.87129×10^108 Sin[β]^2 + 1.33069×10^109 Sin[β]^4 -
           1.09207×10^141 Sin[β]^6 - 5.46037×10^140 Sin[β]^8 + 4.48122×10^172 Sin[β]^10 +
           7.46871×10^171 Sin[β]^12 - 6.12942×10^203 Sin[β]^14)) /
       (-4.43565×10^108 - 5.07698×10^92 Sin[β]^2 + 5.46037×10^140 Sin[β]^4 + 3.88622×10^124 Sin[β]^6 -
         2.24061×10^172 Sin[β]^8 - 7.39814×10^155 Sin[β]^10 + 3.06471×10^203 Sin[β]^12) -
      2. √( ((r^4 (-1.08097×10^77 + 8.87129×10^108 Sin[β]^2 + 1.33069×10^109 Sin[β]^4 -
                1.09207×10^141 Sin[β]^6 - 5.46037×10^140 Sin[β]^8 + 4.48122×10^172 Sin[β]^10 +
                7.46871×10^171 Sin[β]^12 - 6.12942×10^203 Sin[β]^14)^2) / (-4.43565×10^108 -
              5.07698×10^92 Sin[β]^2 + 5.46037×10^140 Sin[β]^4 + 3.88622×10^124 Sin[β]^6 -
              2.24061×10^172 Sin[β]^8 - 7.39814×10^155 Sin[β]^10 + 3.06471×10^203 Sin[β]^12)^2 -
          (4. r^4 (1.08097×10^77 - 1.77426×10^109 Sin[β]^4 + 1.09207×10^141 Sin[β]^8 -
```

$$2.98748\times10^{172}\,\mathrm{Sin}[\beta]^{12}+3.06471\times10^{203}\,\mathrm{Sin}[\beta]^{16}))\,/\,(-4.43565\times10^{108}-$$
$$5.07698\times10^{92}\,\mathrm{Sin}[\beta]^{2}+5.46037\times10^{140}\,\mathrm{Sin}[\beta]^{4}+3.88622\times10^{124}\,\mathrm{Sin}[\beta]^{6}-$$
$$2.24061\times10^{172}\,\mathrm{Sin}[\beta]^{8}-7.39814\times10^{155}\,\mathrm{Sin}[\beta]^{10}+3.06471\times10^{203}\,\mathrm{Sin}[\beta]^{12})))\},$$

$$\{\eta\to-0.5\sqrt{(-(2.\,r^{2}\,(-1.08097\times10^{77}+8.87129\times10^{108}\,\mathrm{Sin}[\beta]^{2}+1.33069\times10^{109}\,\mathrm{Sin}[\beta]^{4}-}$$
$$1.09207\times10^{141}\,\mathrm{Sin}[\beta]^{6}-5.46037\times10^{140}\,\mathrm{Sin}[\beta]^{8}+4.48122\times10^{172}\,\mathrm{Sin}[\beta]^{10}+$$
$$7.46871\times10^{171}\,\mathrm{Sin}[\beta]^{12}-6.12942\times10^{203}\,\mathrm{Sin}[\beta]^{14}))\,/$$
$$(-4.43565\times10^{108}-5.07698\times10^{92}\,\mathrm{Sin}[\beta]^{2}+5.46037\times10^{140}\,\mathrm{Sin}[\beta]^{4}+3.88622\times10^{124}\,\mathrm{Sin}[\beta]^{6}-$$
$$2.24061\times10^{172}\,\mathrm{Sin}[\beta]^{8}-7.39814\times10^{155}\,\mathrm{Sin}[\beta]^{10}+3.06471\times10^{203}\,\mathrm{Sin}[\beta]^{12})+$$
$$2.\sqrt{((r^{4}\,(-1.08097\times10^{77}+8.87129\times10^{108}\,\mathrm{Sin}[\beta]^{2}+1.33069\times10^{109}\,\mathrm{Sin}[\beta]^{4}-}$$
$$1.09207\times10^{141}\,\mathrm{Sin}[\beta]^{6}-5.46037\times10^{140}\,\mathrm{Sin}[\beta]^{8}+4.48122\times10^{172}\,\mathrm{Sin}[\beta]^{10}+$$
$$7.46871\times10^{171}\,\mathrm{Sin}[\beta]^{12}-6.12942\times10^{203}\,\mathrm{Sin}[\beta]^{14})^{2})\,/\,(-4.43565\times10^{108}-$$
$$5.07698\times10^{92}\,\mathrm{Sin}[\beta]^{2}+5.46037\times10^{140}\,\mathrm{Sin}[\beta]^{4}+3.88622\times10^{124}\,\mathrm{Sin}[\beta]^{6}-$$
$$2.24061\times10^{172}\,\mathrm{Sin}[\beta]^{8}-7.39814\times10^{155}\,\mathrm{Sin}[\beta]^{10}+3.06471\times10^{203}\,\mathrm{Sin}[\beta]^{12})^{2}-$$
$$(4.\,r^{4}\,(1.08097\times10^{77}-1.77426\times10^{109}\,\mathrm{Sin}[\beta]^{4}+1.09207\times10^{141}\,\mathrm{Sin}[\beta]^{8}-$$
$$2.98748\times10^{172}\,\mathrm{Sin}[\beta]^{12}+3.06471\times10^{203}\,\mathrm{Sin}[\beta]^{16}))\,/\,(-4.43565\times10^{108}-$$
$$5.07698\times10^{92}\,\mathrm{Sin}[\beta]^{2}+5.46037\times10^{140}\,\mathrm{Sin}[\beta]^{4}+3.88622\times10^{124}\,\mathrm{Sin}[\beta]^{6}-$$
$$2.24061\times10^{172}\,\mathrm{Sin}[\beta]^{8}-7.39814\times10^{155}\,\mathrm{Sin}[\beta]^{10}+3.06471\times10^{203}\,\mathrm{Sin}[\beta]^{12})))\},$$

$$\{\eta\to0.5\sqrt{(-(2.\,r^{2}\,(-1.08097\times10^{77}+8.87129\times10^{108}\,\mathrm{Sin}[\beta]^{2}+1.33069\times10^{109}\,\mathrm{Sin}[\beta]^{4}-}$$
$$1.09207\times10^{141}\,\mathrm{Sin}[\beta]^{6}-5.46037\times10^{140}\,\mathrm{Sin}[\beta]^{8}+4.48122\times10^{172}\,\mathrm{Sin}[\beta]^{10}+$$
$$7.46871\times10^{171}\,\mathrm{Sin}[\beta]^{12}-6.12942\times10^{203}\,\mathrm{Sin}[\beta]^{14}))\,/$$
$$(-4.43565\times10^{108}-5.07698\times10^{92}\,\mathrm{Sin}[\beta]^{2}+5.46037\times10^{140}\,\mathrm{Sin}[\beta]^{4}+3.88622\times10^{124}\,\mathrm{Sin}[\beta]^{6}-$$
$$2.24061\times10^{172}\,\mathrm{Sin}[\beta]^{8}-7.39814\times10^{155}\,\mathrm{Sin}[\beta]^{10}+3.06471\times10^{203}\,\mathrm{Sin}[\beta]^{12})+$$
$$2.\sqrt{((r^{4}\,(-1.08097\times10^{77}+8.87129\times10^{108}\,\mathrm{Sin}[\beta]^{2}+1.33069\times10^{109}\,\mathrm{Sin}[\beta]^{4}-}$$
$$1.09207\times10^{141}\,\mathrm{Sin}[\beta]^{6}-5.46037\times10^{140}\,\mathrm{Sin}[\beta]^{8}+4.48122\times10^{172}\,\mathrm{Sin}[\beta]^{10}+$$
$$7.46871\times10^{171}\,\mathrm{Sin}[\beta]^{12}-6.12942\times10^{203}\,\mathrm{Sin}[\beta]^{14})^{2})\,/\,(-4.43565\times10^{108}-$$
$$5.07698\times10^{92}\,\mathrm{Sin}[\beta]^{2}+5.46037\times10^{140}\,\mathrm{Sin}[\beta]^{4}+3.88622\times10^{124}\,\mathrm{Sin}[\beta]^{6}-$$
$$2.24061\times10^{172}\,\mathrm{Sin}[\beta]^{8}-7.39814\times10^{155}\,\mathrm{Sin}[\beta]^{10}+3.06471\times10^{203}\,\mathrm{Sin}[\beta]^{12})^{2}-$$
$$(4.\,r^{4}\,(1.08097\times10^{77}-1.77426\times10^{109}\,\mathrm{Sin}[\beta]^{4}+1.09207\times10^{141}\,\mathrm{Sin}[\beta]^{8}-$$
$$2.98748\times10^{172}\,\mathrm{Sin}[\beta]^{12}+3.06471\times10^{203}\,\mathrm{Sin}[\beta]^{16}))\,/\,(-4.43565\times10^{108}-$$
$$5.07698\times10^{92}\,\mathrm{Sin}[\beta]^{2}+5.46037\times10^{140}\,\mathrm{Sin}[\beta]^{4}+3.88622\times10^{124}\,\mathrm{Sin}[\beta]^{6}-$$
$$2.24061\times10^{172}\,\mathrm{Sin}[\beta]^{8}-7.39814\times10^{155}\,\mathrm{Sin}[\beta]^{10}+3.06471\times10^{203}\,\mathrm{Sin}[\beta]^{12})))\}\}$$

```
Solve[r == 0.5` Sqrt[(-(2.` η^2 (1.6514528686265244`*^20 - 1.3553156905595964`*^52 Sin[β]^2)) /
    (-1.6514528686265244`*^20 + 6.776578452797982`*^51 Sin[β]^4) -
   2.` Sqrt[((η^4 (1.6514528686265244`*^20 - 1.3553156905595964`*^52 Sin[β]^2)^2) /
       (-1.6514528686265244`*^20 + 6.776578452797982`*^51 Sin[β]^4)^2 -
      (2.7106313811191934`*^52 η^4) /
       (-1.6514528686265244`*^20 + 6.776578452797982`*^51 Sin[β]^4))])], η]
```

$$\Big\{\Big\{\eta \to$$

$$-0.5\sqrt{\Big(-\Big(2.\,r^2\,\big(-1.08097\times10^{77}+8.87129\times10^{108}\,\text{Sin}[\beta]^2+1.33069\times10^{109}\,\text{Sin}[\beta]^4-1.09207\times10^{141}}$$

$$\text{Sin}[\beta]^6-5.46037\times10^{140}\,\text{Sin}[\beta]^8+4.48122\times10^{172}\,\text{Sin}[\beta]^{10}+$$

$$7.46871\times10^{171}\,\text{Sin}[\beta]^{12}-6.12942\times10^{203}\,\text{Sin}[\beta]^{14}\big)\Big)\Big/$$

$$\big(-4.43565\times10^{108}-5.07698\times10^{92}\,\text{Sin}[\beta]^2+5.46037\times10^{140}\,\text{Sin}[\beta]^4+3.88622\times10^{124}\,\text{Sin}[\beta]^6-$$

$$2.24061\times10^{172}\,\text{Sin}[\beta]^8-7.39814\times10^{155}\,\text{Sin}[\beta]^{10}+3.06471\times10^{203}\,\text{Sin}[\beta]^{12}\big)-$$

$$2.\sqrt{\Big(\Big(r^4\,\big(-1.08097\times10^{77}+8.87129\times10^{108}\,\text{Sin}[\beta]^2+1.33069\times10^{109}\,\text{Sin}[\beta]^4-}$$

$$1.09207\times10^{141}\,\text{Sin}[\beta]^6-5.46037\times10^{140}\,\text{Sin}[\beta]^8+4.48122\times10^{172}\,\text{Sin}[\beta]^{10}+$$

$$7.46871\times10^{171}\,\text{Sin}[\beta]^{12}-6.12942\times10^{203}\,\text{Sin}[\beta]^{14}\big)^2\Big)\Big/\big(-4.43565\times10^{108}-$$

$$5.07698\times10^{92}\,\text{Sin}[\beta]^2+5.46037\times10^{140}\,\text{Sin}[\beta]^4+3.88622\times10^{124}\,\text{Sin}[\beta]^6-$$

$$2.24061\times10^{172}\,\text{Sin}[\beta]^8-7.39814\times10^{155}\,\text{Sin}[\beta]^{10}+3.06471\times10^{203}\,\text{Sin}[\beta]^{12}\big)^2-$$

$$\Big(4.\,r^4\,\big(1.08097\times10^{77}-1.77426\times10^{109}\,\text{Sin}[\beta]^4+1.09207\times10^{141}\,\text{Sin}[\beta]^8-$$

$$2.98748\times10^{172}\,\text{Sin}[\beta]^{12}+3.06471\times10^{203}\,\text{Sin}[\beta]^{16}\big)\Big)\Big/\big(-4.43565\times10^{108}-$$

$$5.07698\times10^{92}\,\text{Sin}[\beta]^2+5.46037\times10^{140}\,\text{Sin}[\beta]^4+3.88622\times10^{124}\,\text{Sin}[\beta]^6-$$

$$2.24061\times10^{172}\,\text{Sin}[\beta]^8-7.39814\times10^{155}\,\text{Sin}[\beta]^{10}+3.06471\times10^{203}\,\text{Sin}[\beta]^{12}\big)\Big)\Big)\Big\},$$

$$\Big\{\eta \to 0.5\sqrt{\Big(-\Big(2.\,r^2\,\big(-1.08097\times10^{77}+8.87129\times10^{108}\,\text{Sin}[\beta]^2+1.33069\times10^{109}\,\text{Sin}[\beta]^4-}$$

$$1.09207\times10^{141}\,\text{Sin}[\beta]^6-5.46037\times10^{140}\,\text{Sin}[\beta]^8+4.48122\times10^{172}\,\text{Sin}[\beta]^{10}+$$

$$7.46871\times10^{171}\,\text{Sin}[\beta]^{12}-6.12942\times10^{203}\,\text{Sin}[\beta]^{14}\big)\Big)\Big/$$

$$\big(-4.43565\times10^{108}-5.07698\times10^{92}\,\text{Sin}[\beta]^2+5.46037\times10^{140}\,\text{Sin}[\beta]^4+3.88622\times10^{124}\,\text{Sin}[\beta]^6-$$

$$2.24061\times10^{172}\,\text{Sin}[\beta]^8-7.39814\times10^{155}\,\text{Sin}[\beta]^{10}+3.06471\times10^{203}\,\text{Sin}[\beta]^{12}\big)-$$

$$2.\sqrt{\Big(\Big(r^4\,\big(-1.08097\times10^{77}+8.87129\times10^{108}\,\text{Sin}[\beta]^2+1.33069\times10^{109}\,\text{Sin}[\beta]^4-}$$

$$1.09207\times10^{141}\,\text{Sin}[\beta]^6-5.46037\times10^{140}\,\text{Sin}[\beta]^8+4.48122\times10^{172}\,\text{Sin}[\beta]^{10}+$$

$$7.46871\times10^{171}\,\text{Sin}[\beta]^{12}-6.12942\times10^{203}\,\text{Sin}[\beta]^{14}\big)^2\Big)\Big/\big(-4.43565\times10^{108}-$$

$$5.07698\times10^{92}\,\text{Sin}[\beta]^2+5.46037\times10^{140}\,\text{Sin}[\beta]^4+3.88622\times10^{124}\,\text{Sin}[\beta]^6-$$

$$2.24061\times10^{172}\,\text{Sin}[\beta]^8-7.39814\times10^{155}\,\text{Sin}[\beta]^{10}+3.06471\times10^{203}\,\text{Sin}[\beta]^{12}\big)^2-$$

$$\Big(4.\,r^4\,\big(1.08097\times10^{77}-1.77426\times10^{109}\,\text{Sin}[\beta]^4+1.09207\times10^{141}\,\text{Sin}[\beta]^8-$$

$$2.98748\times10^{172}\,\text{Sin}[\beta]^{12}+3.06471\times10^{203}\,\text{Sin}[\beta]^{16}\big)\Big)\Big/\big(-4.43565\times10^{108}-$$

$$5.07698\times10^{92}\,\text{Sin}[\beta]^2+5.46037\times10^{140}\,\text{Sin}[\beta]^4+3.88622\times10^{124}\,\text{Sin}[\beta]^6-$$

$$2.24061\times10^{172}\,\text{Sin}[\beta]^8-7.39814\times10^{155}\,\text{Sin}[\beta]^{10}+3.06471\times10^{203}\,\text{Sin}[\beta]^{12}\big)\Big)\Big)\Big\},$$

$$\Big\{\eta \to -0.5\sqrt{\Big(-\Big(2.\,r^2\,\big(-1.08097\times10^{77}+8.87129\times10^{108}\,\text{Sin}[\beta]^2+1.33069\times10^{109}\,\text{Sin}[\beta]^4-}$$

$$1.09207\times10^{141}\,\text{Sin}[\beta]^6-5.46037\times10^{140}\,\text{Sin}[\beta]^8+4.48122\times10^{172}\,\text{Sin}[\beta]^{10}+$$

$$7.46871\times10^{171}\,\text{Sin}[\beta]^{12}-6.12942\times10^{203}\,\text{Sin}[\beta]^{14}\big)\Big)\Big/$$

$$\big(-4.43565\times10^{108}-5.07698\times10^{92}\,\text{Sin}[\beta]^2+5.46037\times10^{140}\,\text{Sin}[\beta]^4+3.88622\times10^{124}\,\text{Sin}[\beta]^6-$$

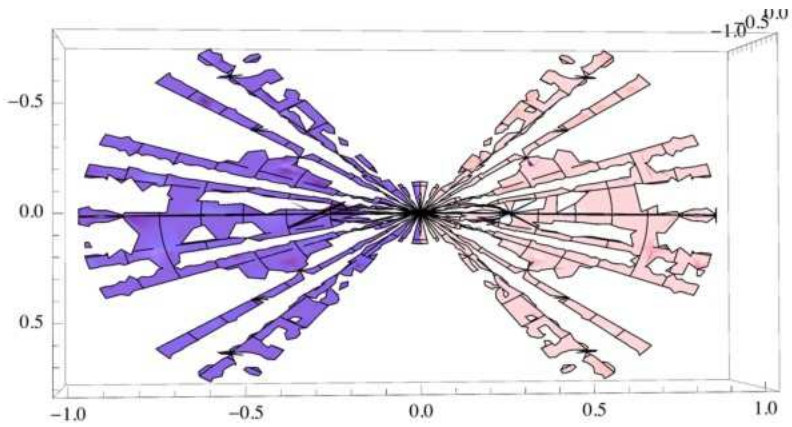
$$2.24061\times10^{172}\,\text{Sin}[\beta]^8-7.39814\times10^{155}\,\text{Sin}[\beta]^{10}+3.06471\times10^{203}\,\text{Sin}[\beta]^{12}\big)+$$

$$2.\sqrt{\Big(\Big(r^4\,\big(-1.08097\times10^{77}+8.87129\times10^{108}\,\text{Sin}[\beta]^2+1.33069\times10^{109}\,\text{Sin}[\beta]^4-}$$

$$1.09207\times10^{141}\,\mathrm{Sin}[\beta]^{6}-5.46037\times10^{140}\,\mathrm{Sin}[\beta]^{8}+4.48122\times10^{172}\,\mathrm{Sin}[\beta]^{10}+$$
$$7.46871\times10^{171}\,\mathrm{Sin}[\beta]^{12}-6.12942\times10^{203}\,\mathrm{Sin}[\beta]^{14})^{2})\Big/\big(-4.43565\times10^{108}-$$
$$5.07698\times10^{92}\,\mathrm{Sin}[\beta]^{2}+5.46037\times10^{140}\,\mathrm{Sin}[\beta]^{4}+3.88622\times10^{124}\,\mathrm{Sin}[\beta]^{6}-$$
$$2.24061\times10^{172}\,\mathrm{Sin}[\beta]^{8}-7.39814\times10^{155}\,\mathrm{Sin}[\beta]^{10}+3.06471\times10^{203}\,\mathrm{Sin}[\beta]^{12}\big)^{2}-$$
$$\big(4.\ r^{4}\ (1.08097\times10^{77}-1.77426\times10^{109}\,\mathrm{Sin}[\beta]^{4}+1.09207\times10^{141}\,\mathrm{Sin}[\beta]^{8}-$$
$$2.98748\times10^{172}\,\mathrm{Sin}[\beta]^{12}+3.06471\times10^{203}\,\mathrm{Sin}[\beta]^{16})\big)\big/\big(-4.43565\times10^{108}-$$
$$5.07698\times10^{92}\,\mathrm{Sin}[\beta]^{2}+5.46037\times10^{140}\,\mathrm{Sin}[\beta]^{4}+3.88622\times10^{124}\,\mathrm{Sin}[\beta]^{6}-$$
$$2.24061\times10^{172}\,\mathrm{Sin}[\beta]^{8}-7.39814\times10^{155}\,\mathrm{Sin}[\beta]^{10}+3.06471\times10^{203}\,\mathrm{Sin}[\beta]^{12})))\},$$

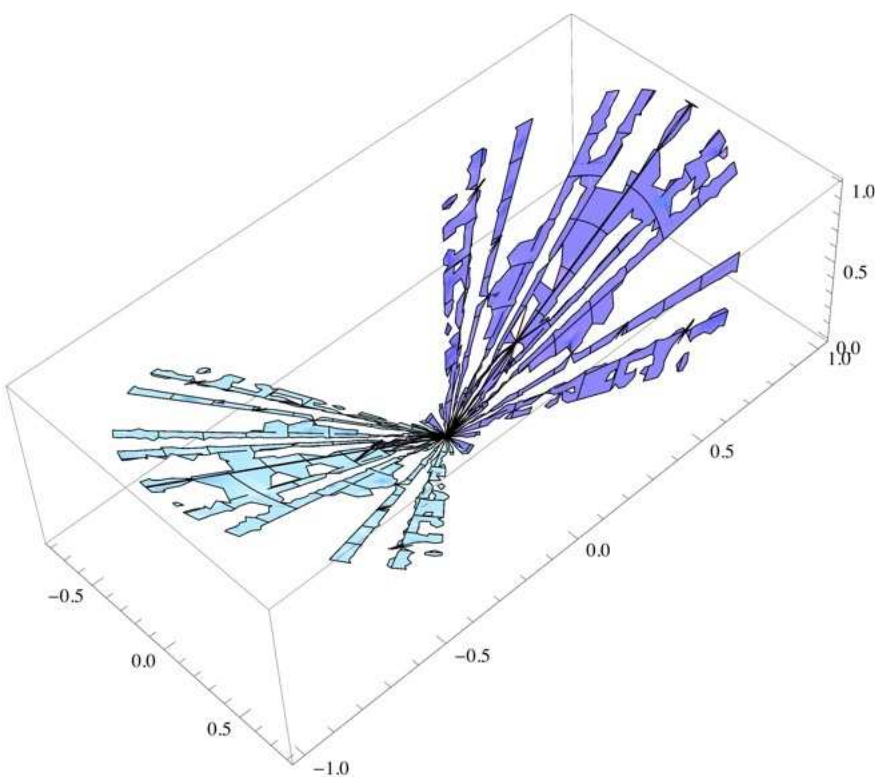
$$\{\eta\to0.5\sqrt{}\big(-\big(2.\ r^{2}\ (-1.08097\times10^{77}+8.87129\times10^{108}\,\mathrm{Sin}[\beta]^{2}+1.33069\times10^{109}\,\mathrm{Sin}[\beta]^{4}-$$
$$1.09207\times10^{141}\,\mathrm{Sin}[\beta]^{6}-5.46037\times10^{140}\,\mathrm{Sin}[\beta]^{8}+4.48122\times10^{172}\,\mathrm{Sin}[\beta]^{10}+$$
$$7.46871\times10^{171}\,\mathrm{Sin}[\beta]^{12}-6.12942\times10^{203}\,\mathrm{Sin}[\beta]^{14})\big)\big/$$
$$\big(-4.43565\times10^{108}-5.07698\times10^{92}\,\mathrm{Sin}[\beta]^{2}+5.46037\times10^{140}\,\mathrm{Sin}[\beta]^{4}+3.88622\times10^{124}\,\mathrm{Sin}[\beta]^{6}-$$
$$2.24061\times10^{172}\,\mathrm{Sin}[\beta]^{8}-7.39814\times10^{155}\,\mathrm{Sin}[\beta]^{10}+3.06471\times10^{203}\,\mathrm{Sin}[\beta]^{12}\big)+$$
$$2.\ \sqrt{}\big(\big(r^{4}\ (-1.08097\times10^{77}+8.87129\times10^{108}\,\mathrm{Sin}[\beta]^{2}+1.33069\times10^{109}\,\mathrm{Sin}[\beta]^{4}-$$
$$1.09207\times10^{141}\,\mathrm{Sin}[\beta]^{6}-5.46037\times10^{140}\,\mathrm{Sin}[\beta]^{8}+4.48122\times10^{172}\,\mathrm{Sin}[\beta]^{10}+$$
$$7.46871\times10^{171}\,\mathrm{Sin}[\beta]^{12}-6.12942\times10^{203}\,\mathrm{Sin}[\beta]^{14})^{2}\big)\Big/\big(-4.43565\times10^{108}-$$
$$5.07698\times10^{92}\,\mathrm{Sin}[\beta]^{2}+5.46037\times10^{140}\,\mathrm{Sin}[\beta]^{4}+3.88622\times10^{124}\,\mathrm{Sin}[\beta]^{6}-$$
$$2.24061\times10^{172}\,\mathrm{Sin}[\beta]^{8}-7.39814\times10^{155}\,\mathrm{Sin}[\beta]^{10}+3.06471\times10^{203}\,\mathrm{Sin}[\beta]^{12}\big)^{2}-$$
$$\big(4.\ r^{4}\ (1.08097\times10^{77}-1.77426\times10^{109}\,\mathrm{Sin}[\beta]^{4}+1.09207\times10^{141}\,\mathrm{Sin}[\beta]^{8}-$$
$$2.98748\times10^{172}\,\mathrm{Sin}[\beta]^{12}+3.06471\times10^{203}\,\mathrm{Sin}[\beta]^{16})\big)\big/\big(-4.43565\times10^{108}-$$
$$5.07698\times10^{92}\,\mathrm{Sin}[\beta]^{2}+5.46037\times10^{140}\,\mathrm{Sin}[\beta]^{4}+3.88622\times10^{124}\,\mathrm{Sin}[\beta]^{6}-$$
$$2.24061\times10^{172}\,\mathrm{Sin}[\beta]^{8}-7.39814\times10^{155}\,\mathrm{Sin}[\beta]^{10}+3.06471\times10^{203}\,\mathrm{Sin}[\beta]^{12})))\}\}$$

```
RevolutionPlot3D[-0.5` √(-(2.` r^2 (-1.0809673819336597`*^77 + 8.871291948744909`*^108 Sin[β]^2 +
      1.3306937923117365`*^109 Sin[β]^4 - 1.0920748695361118`*^141 Sin[β]^6 -
      5.46037434768056`*^140 Sin[β]^8 + 4.481224484411375`*^172 Sin[β]^10 +
      7.468707474018959`*^171 Sin[β]^12 - 6.129424956678444`*^203 Sin[β]^14)) /
   (-4.435645974372455`*^108 - 5.076976983244755`*^92 Sin[β]^2 + 5.46037434768056`*^140
      Sin[β]^4 + 3.886215303428729`*^124 Sin[β]^6 - 2.2406122422056876`*^172 Sin[β]^8 -
     7.398135940236569`*^155 Sin[β]^10 + 3.064712478339222`*^203 Sin[β]^12) -
  2.` √((r^4 (-1.0809673819336597`*^77 + 8.871291948744909`*^108 Sin[β]^2 +
          1.3306937923117365`*^109 Sin[β]^4 - 1.0920748695361118`*^141 Sin[β]^6 -
          5.46037434768056`*^140 Sin[β]^8 + 4.481224484411375`*^172 Sin[β]^10 +
          7.468707474018959`*^171 Sin[β]^12 - 6.129424956678444`*^203 Sin[β]^14)^2) /
      (-4.435645974372455`*^108 - 5.076976983244755`*^92 Sin[β]^2 + 5.46037434768056`*^140
          Sin[β]^4 + 3.886215303428729`*^124 Sin[β]^6 - 2.2406122422056876`*^172 Sin[β]^8 -
         7.398135940236569`*^155 Sin[β]^10 + 3.064712478339222`*^203 Sin[β]^12)^2 -
     (4.` r^4 (1.0809673819336597`*^77 - 1.774258389748982`*^109 Sin[β]^4 +
          1.092074869536112`*^141 Sin[β]^8 - 2.987482989607584`*^172 Sin[β]^12 +
          3.0647124783392222`*^203 Sin[β]^16)) /
      (-4.435645974372455`*^108 - 5.076976983244755`*^92 Sin[β]^2 +
        5.46037434768056`*^140 Sin[β]^4 + 3.886215303428729`*^124 Sin[β]^6 -
        2.2406122422056876`*^172 Sin[β]^8 - 7.398135940236569`*^155 Sin[β]^10 +
        3.064712478339222`*^203 Sin[β]^12))), {r, -1, 1}, {β, -π / 2, π / 2}]
```



−0.5
0.0
0.5
−1.0
−0.5
0.0
0.5
1.0

```
RevolutionPlot3D[0.5` √(-(2.` r^2 (-1.0809673819336597`*^77 + 8.871291948744909`*^108 Sin[β]^2 +
      1.3306937923117365`*^109 Sin[β]^4 - 1.0920748695361118`*^141 Sin[β]^6 -
      5.46037434768056`*^140 Sin[β]^8 + 4.481224484411375`*^172 Sin[β]^10 +
      7.468707474018959`*^171 Sin[β]^12 - 6.129424956678444`*^203 Sin[β]^14)) /
   (-4.435645974372455`*^108 - 5.076976983244755`*^92 Sin[β]^2 + 5.46037434768056`*^140
      Sin[β]^4 + 3.886215303428729`*^124 Sin[β]^6 - 2.2406122422056876`*^172 Sin[β]^8 -
     7.398135940236569`*^155 Sin[β]^10 + 3.064712478339222`*^203 Sin[β]^12) -
   2.` √(((r^4 (-1.0809673819336597`*^77 + 8.871291948744909`*^108 Sin[β]^2 +
           1.3306937923117365`*^109 Sin[β]^4 - 1.0920748695361118`*^141 Sin[β]^6 -
           5.46037434768056`*^140 Sin[β]^8 + 4.481224484411375`*^172 Sin[β]^10 +
           7.468707474018959`*^171 Sin[β]^12 - 6.129424956678444`*^203 Sin[β]^14)^2) /
       (-4.435645974372455`*^108 - 5.076976983244755`*^92 Sin[β]^2 + 5.46037434768056`*^140
           Sin[β]^4 + 3.886215303428729`*^124 Sin[β]^6 - 2.2406122422056876`*^172 Sin[β]^8 -
          7.398135940236569`*^155 Sin[β]^10 + 3.064712478339222`*^203 Sin[β]^12)^2 -
      (4.` r^4 (1.0809673819336597`*^77 - 1.774258389748982`*^109 Sin[β]^4 +
           1.092074869536112`*^141 Sin[β]^8 - 2.987482989607584`*^172 Sin[β]^12 +
           3.0647124783392222`*^203 Sin[β]^16)) /
       (-4.435645974372455`*^108 - 5.076976983244755`*^92 Sin[β]^2 +
         5.46037434768056`*^140 Sin[β]^4 + 3.886215303428729`*^124 Sin[β]^6 -
         2.2406122422056876`*^172 Sin[β]^8 - 7.398135940236569`*^155 Sin[β]^10 +
         3.064712478339222`*^203 Sin[β]^12))), {r, -1, 1}, {β, -π / 2, π / 2}]
```

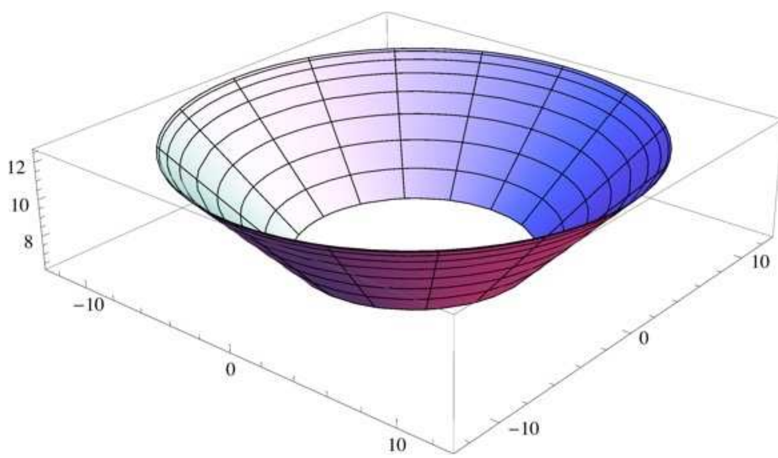


- **d) Apply the Substitution,** $\left\{\theta \to 2\left(\pi + \sqrt{\pi^2 - \pi^2 \text{Sin}[\beta]^2}\right)\right\}$

```
Solve[99 (2.99792458 (10^8)) == (√(-1.1294090667581471`*^18 θ +
      8.987551787368176`*^16 θ^2 + 3.5481432270250993`*^18 Sin[β]^2)) /
   (√(-12.566370614359172` (2 (π + √(π^2 - π^2 Sin[β]^2))) + θ^2 + 39.47841760435743` Sin[β]^2)), θ]
```

$$\left\{\left\{\theta \to 1.42883\times10^{-19}\left(-4.48719\times10^{15} - 1.\sqrt{\left(3.8679\times10^{39} - 1.93375\times10^{39}\,\text{Sin}[\beta]^2 + 3.8679\times10^{39}\sqrt{1. + 0.\,\text{Sin}[\beta] - 1.\,\text{Sin}[\beta]^2}\right)}\right)\right\},\right.$$
$$\left.\left\{\theta \to 1.42883\times10^{-19}\left(-4.48719\times10^{15} + \sqrt{\left(3.8679\times10^{39} - 1.93375\times10^{39}\,\text{Sin}[\beta]^2 + 3.8679\times10^{39}\sqrt{1. + 0.\,\text{Sin}[\beta] - 1.\,\text{Sin}[\beta]^2}\right)}\right)\right\}\right\}$$

```
RevolutionPlot3D[{1.4288257082550006`*^-19
   (-4.487190804107819`*^15 - 1.` √(3.8679026663078`*^39 - 1.933754001249597`*^39 Sin[β]^2 +
        3.8679026461729186`*^39 √(1.` + 0.` Sin[β] - 1.` Sin[β]^2))), 1.4288257082550006`*^-19
   (-4.487190804107819`*^15 + √(3.8679026663078`*^39 - 1.933754001249597`*^39 Sin[β]^2 +
        3.8679026461729186`*^39 √(1.` + 0.` Sin[β] - 1.` Sin[β]^2)))}, {β, -π / 2, π / 2}]
```

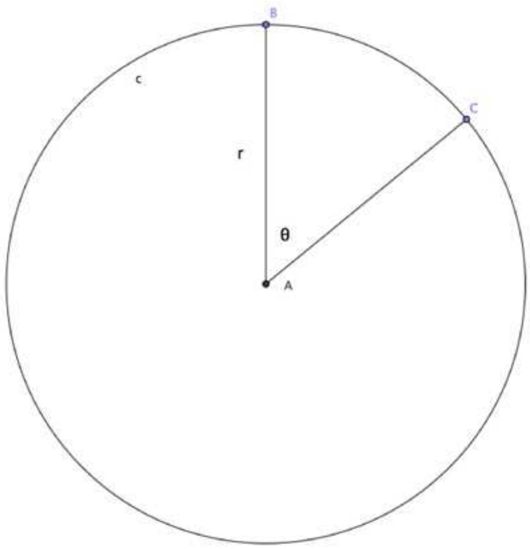


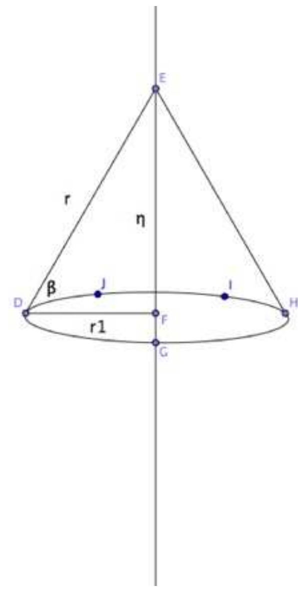
XVII. Volume of a Cone

by Parker Emmerson

Volume of Cone =

$$(1/3)\ \text{height} * \text{area of base} = (1/3)\ \frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi}\,\pi\left(r - \frac{r\,\theta}{2\,\pi}\right)\text{^}2 \qquad (54)$$





Theorem 1 *When a sector of angle θ is removed from a circle of radius r and the resulting shape is folded into a cone, then the base of the cone has radius r_1 given by $r_1 = r - \frac{(r\theta)}{2\pi}$; and height η, given by* $\eta = \sqrt{r^2 - r_1^2}$ =r Sin[β]

Proof. The circumference of the initial circle is 2π r and the wedge removed has an arc length r θ. Therefore, the remaining circumference is of length r $(2\pi - \theta)$, and after the fold, this is the circumference of the base of the cone.

Establishing the circumference of the base of the cone, from the equation, θ r = 2π r - $2\pi r_1$, we calculate that its radius r_1 is $\frac{2\pi r - r\theta}{2\pi}$, which simplifies to r - $\frac{r\theta}{2\pi}$. Thus, we have proved the first part of the theorem.

To find the height of the cone, η, we apply the Pythagorean theorem to a right triangle formed between the apex of the cone, the center of the base, and a point on the circumference of the base. This gives $\eta = \sqrt{r^2 - r_1^2}$ = r Sin[β], where β is the angle formed by the slant of the cone and the base of the cone. The initial radius is always equal to the slant of the cone, and the height of the cone is always orthogonal to the center of the base of the cone.

Lemma 1 The height of the cone can be caluclated in terms of r and θ.

Proof.

$\theta r = 2\pi r - 2\pi r_1$

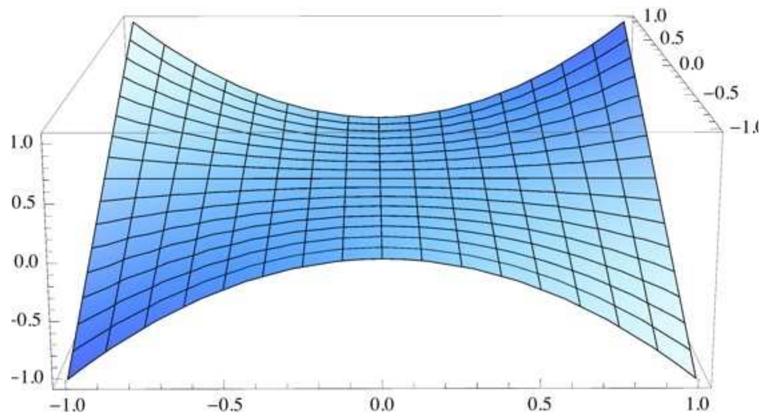
$\eta = \sqrt{r^2 - r_1^2}$

$\theta r = 2\pi r - 2\pi\sqrt{(r^2 - \eta^2)}$

$$\eta = \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}$$

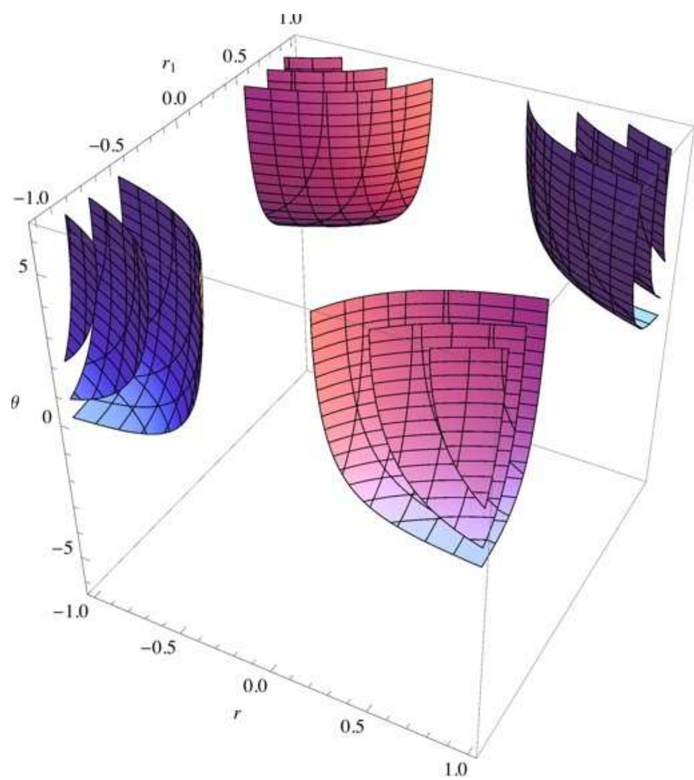
$$\text{Volume of Cone} = (1/3)\pi((\text{radius of base})^2) * \text{height} = (1/3)\pi(r_1^2) * \eta = (1/3)\frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}\pi\left(r - \frac{r\theta}{2\pi}\right)^2 \quad (55)$$

```
Plot3D[(1 / 3) π ( r1^2) * η, {r1, -1, 1}, {η, -1, 1}]
```



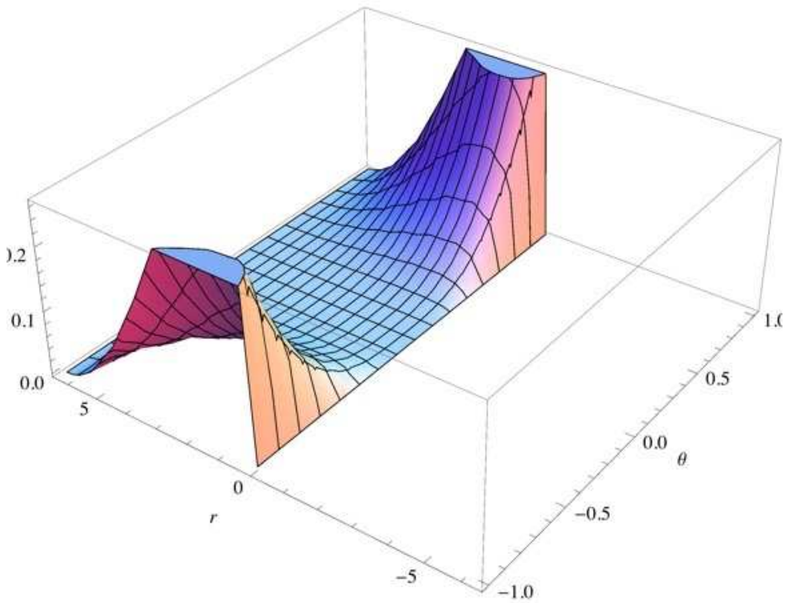
Therefore, we make a substitution for our expressions of the height and base of the cone into the formula for the of the volume of the cone.

$$\text{ContourPlot3D}\left[(1/3)\ \frac{\sqrt{4\pi r^2 \theta - r^2 \theta^2}}{2\pi}\ \pi\ (r_1)\text{^}2,\right.$$
$$\left.\{r, -1, 1\}, \{r_1, -1, 1\}, \{\theta, -2\pi, 2\pi\}, \{\text{AxesLabel} \to \text{Automatic}\}\right]$$



e

$$\texttt{Plot3D}\left[(1/3)\ \frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi}\ \pi\left(r - \frac{r\,\theta}{2\,\pi}\right)\texttt{\^{}2, \{r, -1, 1\}, \{}\theta\texttt{, -2}\,\pi\texttt{, 2}\,\pi\texttt{\}, \{AxesLabel} \to \{\theta\texttt{, r\}\}}\right]$$



Lemma 2 The angle θ can be calculated in terms of r and η.

Proof

$$\texttt{Solve}\left[\eta == \frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi},\ \theta\right]$$

$$\left\{\left\{\theta \to \frac{2\,\pi\left(r^2 - \sqrt{r^4 - r^2\,\eta^2}\right)}{r^2}\right\},\ \left\{\theta \to \frac{2\,\pi\left(r^2 + \sqrt{r^4 - r^2\,\eta^2}\right)}{r^2}\right\}\right\}$$

Lemma 5 *The height of the cone can be calculated in terms of only r and* θ, thus θ is a function of β alone.

Proof. Since we have shown that θ r = 2 π r - 2 π r_1 and $r_1 \to \sqrt{r^2 - \eta^2}$, we can substitute the expression for r_1, calculated from the Pythagorean theorem in terms of the height of the cone and the initial radius of the circle, into the expression for θ r in terms of the change in circumference of the initial circle to the circle that is the base of the cone into which the circle was transformed.

θ r = 2 π r - 2 π $\sqrt{(r\texttt{\^{}}2 - \eta\texttt{\^{}}2)}$, thus, $\eta = \frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi}$ = (r Sin[β]). From $\frac{2\,\pi\,\eta}{\sqrt{4\,\pi\,\theta - \theta^2}}$= r, we note that: r = $\frac{2\,\pi\,r\,\mathrm{Sin}[\beta]}{\sqrt{4\,\pi\,\theta - \theta^2}}$. So we solve the equation,

$$\texttt{Solve}\left[r == \frac{2\,\pi\,r\,\mathrm{Sin}[\beta]}{\sqrt{4\,\pi\,\theta - \theta^2}},\ \theta\right]$$

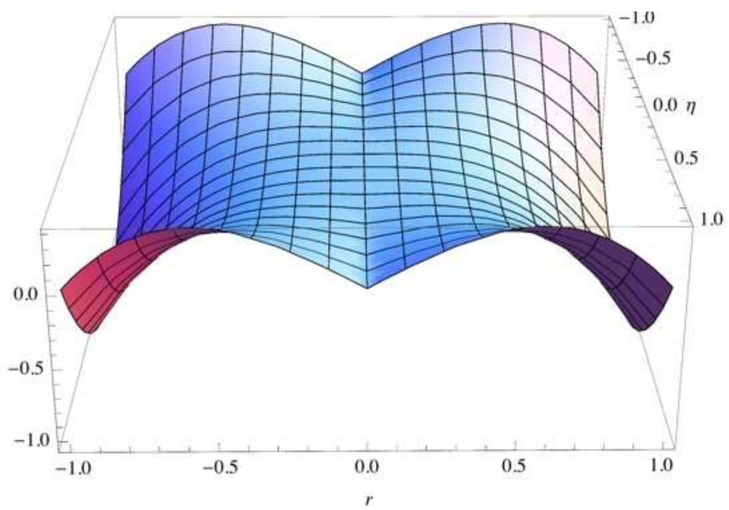
$$\left\{\left\{\theta \to 2\left(\pi - \sqrt{\pi^2 - \pi^2\,\mathrm{Sin}[\beta]^2}\right)\right\},\ \left\{\theta \to 2\left(\pi + \sqrt{\pi^2 - \pi^2\,\mathrm{Sin}[\beta]^2}\right)\right\}\right\}$$

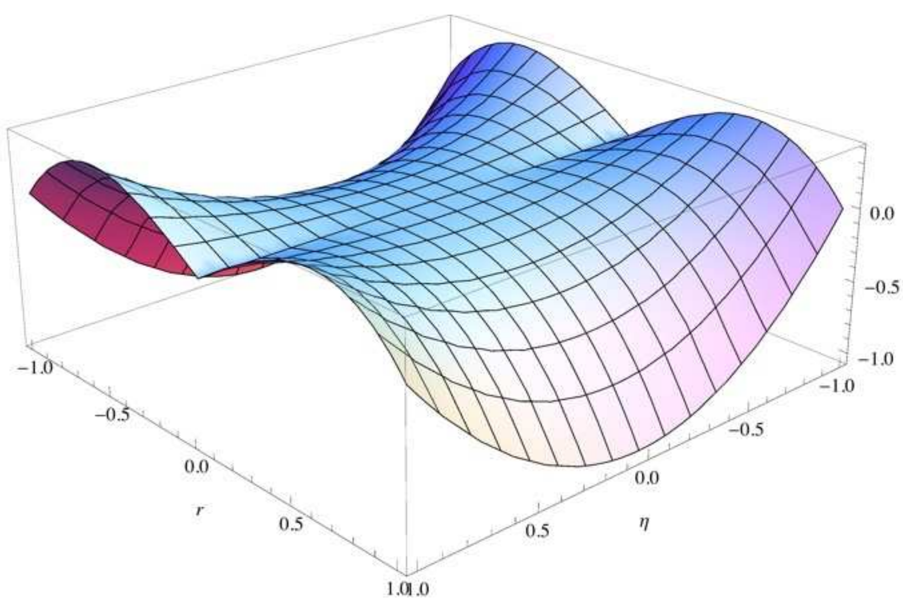
Volume of Cone =

$$(1/3)\ \frac{\sqrt{4\pi r^2\ \frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2} - r^2\left(\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right)^2}}{2\pi}\ \pi\left(r - \frac{r\ \frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}}{2\pi}\right)\hat{}\ 2$$

$$\textbf{Plot3D}\Big[(1/3)\ \frac{\sqrt{4\pi r^2\ \frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2} - r^2\left(\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right)^2}}{2\pi}\ \pi\left(r - \frac{r\ \frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}}{2\pi}\right)\hat{}\ 2,$$

{r, -1, 1}, {η, -1, 1}, {AxesLabel → {η, r}}]





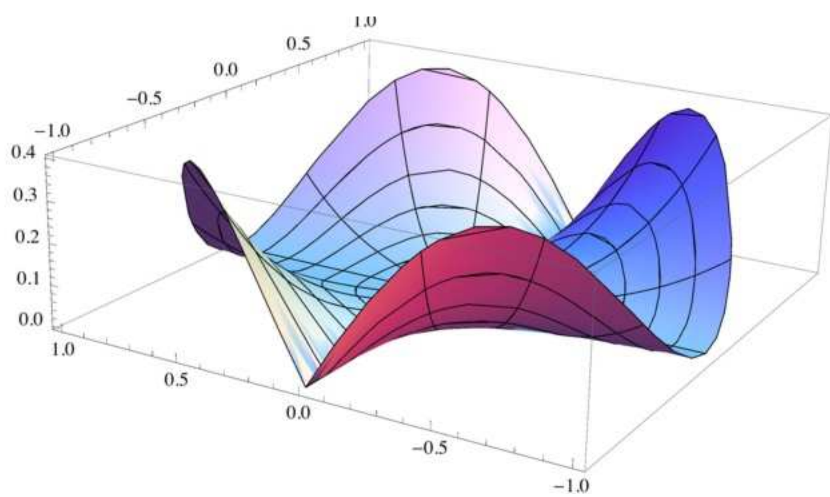
Volume of Cone =

$$(1/3)\ \frac{\sqrt{4\,\pi\, r^2\ 2\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)-r^2\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)\right)^2}}{2\,\pi}$$

$$\pi\left(r-\frac{r\ 2\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)}{2\,\pi}\right)\hat{}\,2$$

$$\mathbf{RevolutionPlot3D}\Big[(1/3)\ \frac{\sqrt{4\,\pi\, r^2\ 2\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)-r^2\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)\right)^2}}{2\,\pi}$$

$$\pi\left(r-\frac{r\ 2\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)}{2\,\pi}\right)\hat{}\,2,\ \{r,-1,1\},\ \{\beta,-\pi,\pi\}\Big]$$



XVIII. Frequency in Terms of Transcendental, Phenomenological Velocity from Energy of Electron = Energy of Photon: Descriptions of Mass

Relevant Equations, Facts, and Solutions

■ I. Commonplace Values and Equations

c = 2.99792458 (10^8) meters per second = speed of light in a vaccuum

m = mass of electron = 9.109382900000001`*^-31 kg

h = Planck ' s Constant = 6.62606896 * 10 ^ (- 34) J * s

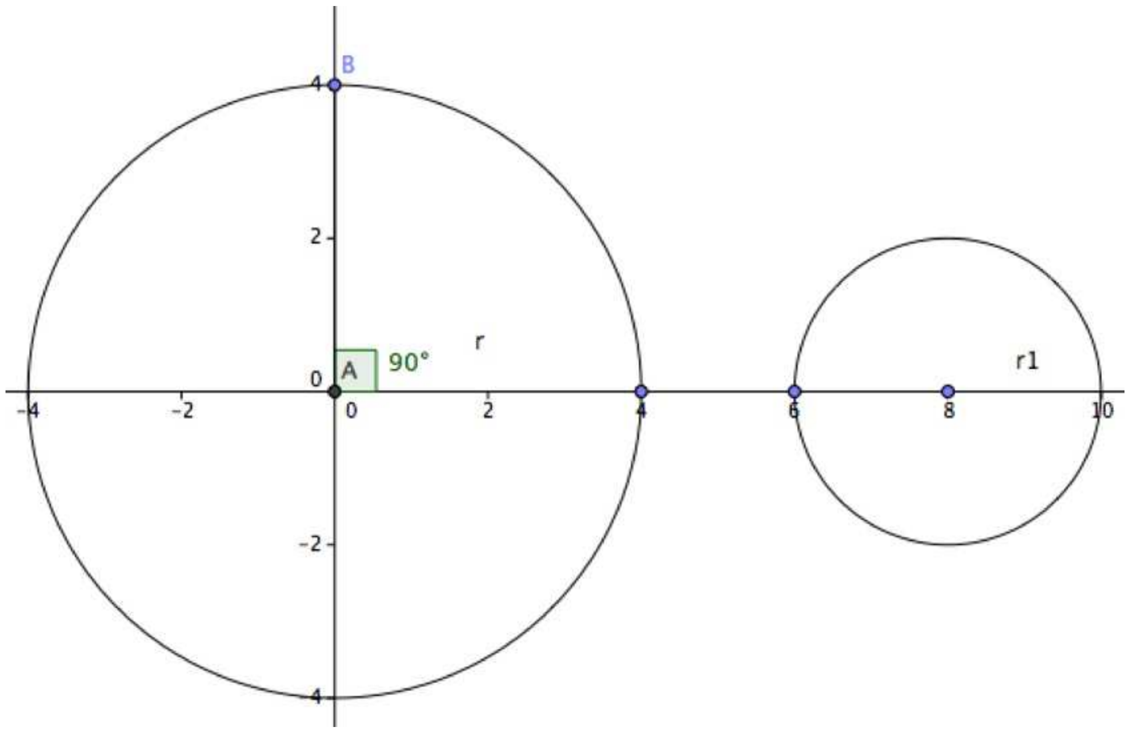
Energy = m c ^ 2

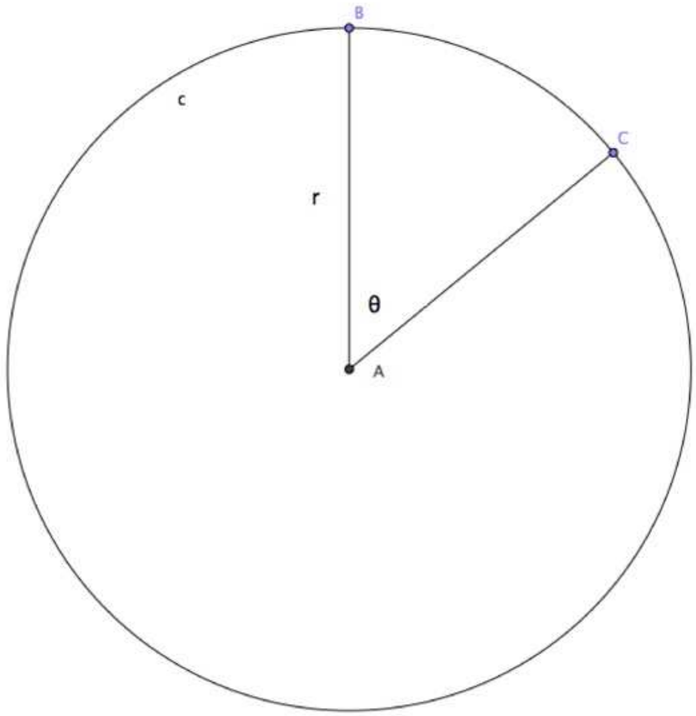
velocity = wavelength * frequency

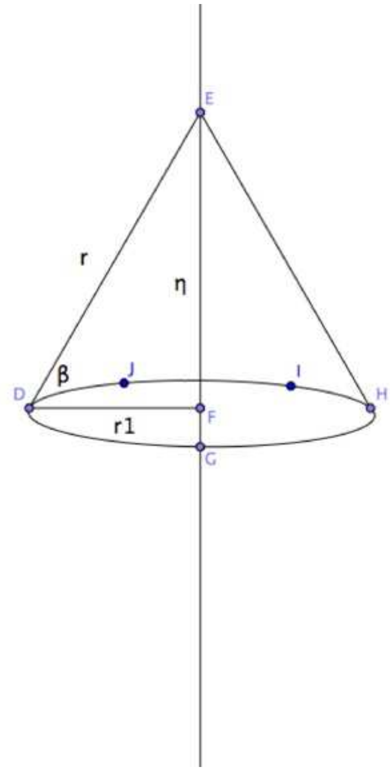
Energy = E = h ν = (Planck ' s Constant) * (Frequency)

■ II. Math for Transforming a Circle into a Cone

by Parker Emmerson







When a sector of a circle is collapsed (removed), we may "fold up" the resulting shape into a cone. The parameters are related by the following theorem :

Theorem 1 *When a sector of angle θ is removed from a circle of radius r and the resulting shape is folded into a cone, then the base of the cone has radius r_1 given by* $r_1 = r - \frac{(r\theta)}{2\pi}$; *and height η, given by* $\eta = \sqrt{r^2 - r_1^2} = \text{r Sin}[\beta]$

Proof. The circumference of the initial circle is 2π r and the wedge removed has an arc length r θ. Therefore, the remaining circumference is of length r $(2\pi - \theta)$, and after the fold, this is the circumference of the base of the cone.

Establishing the circumference of the base of the cone, from the equation, $\theta \text{ r} = 2\pi \text{ r} - 2\pi r_1$, we calculate that its radius r_1 is $\frac{2\pi \text{r} - \text{r}\theta}{2\pi}$, which simplifies to r - $\frac{\text{r}\theta}{2\pi}$. Thus, we have proved the first part of the theorem.

To find the height of the cone, η, we apply the Pythagorean theorem to a right triangle formed between the apex of the cone, the center of the base, and a point on the circumference of the base. This gives $\eta = \sqrt{r^2 - r_1^2} = \text{r Sin}[\beta]$, where β is the angle formed by the slant of the cone and the base of the cone. The initial radius is always equal to the slant of the cone, and the height of the cone is always orthogonal to the center of the base of the cone.

Lemma 1 The height of the cone can be caluclated in terms of r and θ.

Proof.

$\theta \text{ r} = 2\pi \text{ r} - 2\pi r_1$

$\eta = \sqrt{r^2 - r_1^2}$

$\theta \text{ r} = 2\pi \text{ r} - 2\pi \sqrt{(\text{r}^\wedge 2 - \eta^\wedge 2)}$

Solving this equation we find that,

$$\left\{\left\{\eta \to -\frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi}\right\},\ \left\{\eta \to \frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi}\right\}\right\}$$

```
Solve[Limit[(2 Pi) / Sqrt[4 Pi θ - θ^2], {θ → -Infinity, θ → Infinity}] ==
  {2 π r - 2 π x - θ r, 2 π r - 2 π x - θ r}, θ]
```

$$\left\{\left\{\theta \to \frac{2\,(\pi\,r - \pi\,x)}{r}\right\}\right\}$$

Lemma 2 The angle θ can be calculated in terms of r and η.

Proof

$$\text{Solve}\left[\eta == \frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi},\ \theta\right]$$

$$\left\{\left\{\theta \to \frac{2\,\pi\left(r^2 - \sqrt{r^4 - r^2\,\eta^2}\right)}{r^2}\right\},\ \left\{\theta \to \frac{2\,\pi\left(r^2 + \sqrt{r^4 - r^2\,\eta^2}\right)}{r^2}\right\}\right\}$$

Lemma 3 The initial radius is a function of θ and η.

$$\text{Solve}\left[\frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi} == \eta,\ r\right]$$

$$\left\{\left\{r \to -\frac{2\,\pi\,\eta}{\sqrt{4\,\pi\,\theta - \theta^2}}\right\},\ \left\{r \to \frac{2\,\pi\,\eta}{\sqrt{4\,\pi\,\theta - \theta^2}}\right\}\right\}$$

Lemma 4 *The height of the cone can be calculated in terms of only r and* θ, thus β is a function of θ alone.

Proof. Since we have shown that θ r = 2 π r - 2 π r_1 and $r_1 \to \sqrt{r^2 - \eta^2}$, we can substitute the expression for r_1, calculated from the Pythagorean theorem in terms of the height of the cone and the initial radius of the circle, into the expression for θ r in terms of the change in circumference of the initial circle to the circle that is the base of the cone into which the circle was transformed.

$\theta\,r = 2\,\pi\,r - 2\,\pi\sqrt{(r\hat{}2 - \eta\hat{}2)}$, thus, $\eta = \frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi}$ = (r Sin[β]). From $\frac{2\,\pi\,\eta}{\sqrt{4\,\pi\,\theta - \theta^2}}$ = r, we note that: r = $\frac{2\,\pi\,r\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\theta - \theta^2}}$. So we solve the equation,

$$\text{Solve}\left[r == \frac{2\,\pi\,r\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\theta - \theta^2}},\ \beta\right]$$

$$\left\{\left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{(4\,\pi - \theta)\,\theta}}{2\,\pi}\right]\right\}\right\}$$

Lemma 5 *The height of the cone can be calculated in terms of only r and* θ, thus θ is a function of β alone.

Proof. Since we have shown that θ r = 2 π r - 2 π r_1 and $r_1 \to \sqrt{r^2 - \eta^2}$, we can substitute the expression for r_1, calculated from the Pythagorean theorem in terms of the height of the cone and the initial radius of the circle, into the expression for θ r in terms of the change in circumference of the initial circle to the circle that is the base of the cone into which the circle was transformed.

θ r = 2 π r - 2 π $\sqrt{(r\text{^}2 - \eta\text{^}2)}$, thus, $\eta = \frac{\sqrt{4 \pi r^2 \theta - r^2 \theta^2}}{2 \pi}$ = (r Sin[β]). From $\frac{2 \pi \eta}{\sqrt{4 \pi \theta - \theta^2}}$ = r, we note that: r = $\frac{2 \pi r \text{Sin}[\beta]}{\sqrt{4 \pi \theta - \theta^2}}$. So we solve the equation,

$$\text{Solve}\left[r == \frac{2 \pi r \text{Sin}[\beta]}{\sqrt{4 \pi \theta - \theta^2}}, \theta\right]$$

$$\left\{\left\{\theta \to 2 \left(\pi - \sqrt{\pi^2 - \pi^2 \text{Sin}[\beta]^2}\right)\right\}, \left\{\theta \to 2 \left(\pi + \sqrt{\pi^2 - \pi^2 \text{Sin}[\beta]^2}\right)\right\}\right\}$$

- **A note about time passing like a clock.**

The elapse of one unit of time, t, can be expressed by a constant function of the angle θ. The simplest expression is t (seconds) = $\frac{\theta}{2\pi}$; θ = k t , where k is 2π, because one unit of time is equal to one revolution of θ through a circle.

Proof. θ r = 2 π r - 2 π (r - r t) yields $t = \frac{\theta}{2 \pi}$.

Theorem 2 When we designate that a single unit of time passes per revolution of the angle through the total number of radians in a circle, instantaneous velocity through the distance of the height of the cone can be found by taking the first derivative of the expression for that distance, which is in terms of r and θ, with respect to t = $\frac{\theta}{2 \pi}$. There is also a velocity through the height of the cone, which is equal to wavelength times frequency = $\lambda f = \frac{\eta}{\left(\frac{\theta}{(2 \pi)}\right)}$ considered the average velocity through the height of the cone. Under the condition that one unit of time passes with one revolution of the circle, these two velocities are equal to each other at the position where a 30-60-90 triangle is formed between the apex, center of the base of the cone, and point on the circumference of the circle of the base of the cone.

Proof.

To prove this, we can substitute r Sin[β] for the height of the cone in the expression of velocity = ((2 π η)/θ) and find a real and two complex solutions for theta in terms of β , thus from Lemma 4, we can solve for β exactly.

$$\text{Instantaneous Velocity} = \frac{d\eta}{dt} = \frac{d\eta}{d\frac{\theta}{(2\pi)}} = D\left[\frac{\sqrt{4 \pi r^2 \theta - r^2 \theta^2}}{2 \pi}, t\right] =$$

$$D\left[k \frac{\sqrt{4 \pi r^2 \theta - r^2 \theta^2}}{2 \pi}, \theta\right] = D\left[2 \pi \frac{\sqrt{4 \pi r^2 \theta - r^2 \theta^2}}{2 \pi}, \theta\right] = \frac{4 \pi r^2 - 2 r^2 \theta}{2 \sqrt{4 \pi r^2 \theta - r^2 \theta^2}}$$

$$\text{Average Velocity} = (\eta / (\theta / 2 \pi))$$

$$\text{Instantaneous Velocity} = \frac{4 \pi r^2 - 2 r^2 \theta}{2 \sqrt{4 \pi r^2 \theta - r^2 \theta^2}} = \text{Average Velocity} = \frac{2 \pi \eta}{\theta}$$

$$\text{Solve}\left[\frac{4 \pi r^2 - 2 r^2 \theta}{2 \sqrt{4 \pi r^2 \theta - r^2 \theta^2}} == \frac{2 \pi \eta}{\theta}, \theta\right]$$

$$\left\{\left\{\theta \to \frac{4\pi}{3} + \frac{-4\pi^2 r^4 + 12\pi^2 r^2 \eta^2}{6\pi r^2 \left(r^6 - 18 r^4 \eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11 r^8 \eta^4 + r^6 \eta^6}\right)^{1/3}} - \frac{2\pi \left(r^6 - 18 r^4 \eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11 r^8 \eta^4 + r^6 \eta^6}\right)^{1/3}}{3 r^2}\right\},\right.$$

$$\left\{\theta \to \frac{4\pi}{3} - \frac{\left(1 + i\sqrt{3}\right)\left(-4\pi^2 r^4 + 12\pi^2 r^2 \eta^2\right)}{12\pi r^2 \left(r^6 - 18 r^4 \eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11 r^8 \eta^4 + r^6 \eta^6}\right)^{1/3}} + \frac{\left(1 - i\sqrt{3}\right)\pi \left(r^6 - 18 r^4 \eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11 r^8 \eta^4 + r^6 \eta^6}\right)^{1/3}}{3 r^2}\right\},$$

$$\left.\left\{\theta \to \frac{4\pi}{3} - \frac{\left(1 - i\sqrt{3}\right)\left(-4\pi^2 r^4 + 12\pi^2 r^2 \eta^2\right)}{12\pi r^2 \left(r^6 - 18 r^4 \eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11 r^8 \eta^4 + r^6 \eta^6}\right)^{1/3}} + \frac{\left(1 + i\sqrt{3}\right)\pi \left(r^6 - 18 r^4 \eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11 r^8 \eta^4 + r^6 \eta^6}\right)^{1/3}}{3 r^2}\right\}\right\}$$

$$\text{Solve}\left[\frac{k\left(4\pi r^2 - 2 r^2 \theta\right)}{4\pi\sqrt{4\pi r^2 \theta - r^2 \theta^2}} == \frac{k\, r\, \text{Sin}[\beta]}{\theta}, \theta\right]$$

$$\left\{\left\{\theta \to \frac{4\pi}{3} - \frac{-4\pi^2 + 12\pi^2 \text{Sin}[\beta]^2}{6\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}} + \frac{2}{3}\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}\right\},\right.$$

$$\left\{\theta \to \frac{4\pi}{3} + \frac{\left(1 + i\sqrt{3}\right)\left(-4\pi^2 + 12\pi^2 \text{Sin}[\beta]^2\right)}{12\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}} - \frac{1}{3}\left(1 - i\sqrt{3}\right)\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}\right\},$$

$$\left.\left\{\theta \to \frac{4\pi}{3} + \frac{\left(1 - i\sqrt{3}\right)\left(-4\pi^2 + 12\pi^2 \text{Sin}[\beta]^2\right)}{12\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}} - \frac{1}{3}\left(1 + i\sqrt{3}\right)\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}\right\}\right\}$$

The real solution for θ, solved from equating the instantaneous velocity to the average velocity, can be equated with the real solution for the expression for θ from Lemma 4 to yield an exact solution for β that tells us that when these solutions for theta are equal, a 30-60-90 triangle is formed between the azimuth of the cone, the point on the base of the cone and the center of the base of the cone.

$$\text{Solve}\left[\frac{4\pi}{3} - \frac{-4\pi^2 + 12\pi^2 \text{Sin}[\beta]^2}{6\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}} + \frac{2}{3}\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3} == 2\left(\pi + \sqrt{\pi^2 - \pi^2 \text{Sin}[\beta]^2}\right), \beta\right]$$

$$\left\{\left\{\beta \to -\frac{\pi}{3}\right\}, \left\{\beta \to \frac{\pi}{3}\right\}\right\}$$

We know that the height of the cone is perpendicular to the center of the base of the cone, so this proves a 30-60-90 triangle, because the sum of the angles of the triangle must be 180 degrees or π radians.

Lemma 7 We can show that $\beta = \frac{\pi}{3}$, thus we can show that there are two solutions to θ at which this occurs.

Proof.

$$\text{ArcSin}\left[\frac{\sqrt{(4\pi - \theta)\,\theta}}{2\pi}\right] = \beta$$

$$\text{Solve}\left[\text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\,\theta}}{2\pi}\right] == \frac{\pi}{3}, \theta\right]$$

$$\{\{\theta \to \pi\}, \{\theta \to 3\pi\}\}$$

Lemma 8 We can show can show that the position at which instantaneous rate of change of the height of the cone with respect to theta equals average rate of change of the height of the cone, 'per theta measure,' at

$$\left\{\left\{\beta \to -\text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right]\right\}, \left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right]\right\}\right\}.$$

Proof.

$$\text{Solve}\left[\theta == \frac{4\pi}{3} - \frac{-4\pi^2 + 12\pi^2\,\text{Sin}[\beta]^2}{6\left(-\pi^3 + 18\pi^3\,\text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2 + 11\pi^6\,\text{Sin}[\beta]^4 + \pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}} + \frac{2}{3}\left(-\pi^3 + 18\pi^3\,\text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2 + 11\pi^6\,\text{Sin}[\beta]^4 + \pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}, \beta\right]$$

$$\left\{\left\{\beta \to -\text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right]\right\}, \left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right]\right\}\right\}$$

Theorem 3 The "innate velocity," v, within the Lorentz transformation can be solved for in terms of the system of the circle transforming into a cone. If r is multiplied by the Lorenz transformation, then it measures the distance in the prime system, denoted by r '. If t ' equals $\frac{\left(\frac{\theta}{(2\pi)}\right)}{\sqrt{1-\left(\frac{v^2}{c^2}\right)}}$, then the quantity r $\theta = \theta$ ' r '. We are only dealing with algebraic forms and the solutions necessitated by them. Logical, algebraic, reasoning will be given why, when using the exact speed of light, 2.99792458 (10^8) meters per second, the units of the speed of light can be ignored for the purposes of calculation and computation. This theorem states that, although, normal algebra would require the speed of light as a quantity to cancel out, valid expressions for the solutions for the intrinsic velocity, v, can be found in terms of η, r, and θ, or θ and β, depending on the expression used for the height of the cone.

Proof.

$$c = 2.99792458\,(10^8)\ \text{meters per second}$$

$$r' = r\sqrt{1 - \frac{v^2}{c^2}}$$

$$t' = \frac{\left(\frac{\theta}{(2\pi)}\right)}{\sqrt{1-\left(\frac{v^2}{c^2}\right)}}$$

$$t' = \frac{\theta'}{2\pi}$$

$$2\pi t' = \theta'$$

$$\theta' = \frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}$$

Therefore,

$$r' * \theta' = \left(\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}\right)\left(r\sqrt{1-\frac{v^2}{c^2}}\right)$$

$$\left(\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}\right)\left(r\sqrt{1-\frac{v^2}{c^2}}\right) = r\,\theta$$

$$r' * \theta' = r\,\theta = 2\pi r - 2\pi r_1 = 2\pi r - 2\pi\sqrt{r\text{^}2 - \eta\text{^}2}$$

$$\text{Solve}\left[r\,\theta == 2\pi r - 2\pi\sqrt{r\text{^}2-\eta\text{^}2}\,,\ \eta\right]$$

$$\left\{\left\{\eta \to -\frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}\right\},\ \left\{\eta \to \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}\right\}\right\}$$

$$\textbf{Solve}\left[r'\,\theta' == 2\pi r - 2\pi\sqrt{r\text{^}2-\eta\text{^}2}\,,\ \eta\right]$$

$$\left\{\left\{\eta \to -\frac{\sqrt{r'}\,\sqrt{\theta'}\,\sqrt{4\pi r - r'\theta'}}{2\pi}\right\},\ \left\{\eta \to \frac{\sqrt{r'}\,\sqrt{\theta'}\,\sqrt{4\pi r - r'\theta'}}{2\pi}\right\}\right\}$$

The argument follows modus ponens, saying that, through commutation, $r'\,\theta' = \theta\, r$, therefore $\eta = \frac{\sqrt{r'}\,\sqrt{\theta'}\,\sqrt{4\pi r - r'\theta'}}{2\pi} = \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}$

$\eta =$

$$\frac{\sqrt{r'}\ \sqrt{\theta'}\ \sqrt{4\,\pi\,r - r'\,\theta'}}{2\,\pi} = \frac{\sqrt{r\sqrt{1-\frac{v^2}{c^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}}\ \sqrt{4\,\pi\,r - r\sqrt{1-\frac{v^2}{c^2}}\ \frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}}}{2\,\pi} = \frac{\sqrt{r\,\theta}\ \sqrt{4\,\pi\,r - r\,\theta}}{2\,\pi}$$

where c is its numeric value of the speed of light, its units being shown to cancel out.

$$\frac{\sqrt{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}}{\sqrt{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}}$$

$$\textbf{Solve}\left[\sqrt{\frac{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}}\ ,\ \text{meters}\right]$$

{{}}

$$\textbf{Solve}\left[\sqrt{\frac{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}}\ ,\ \text{second}\right]$$

{{}}

$$\textbf{Solve}\left[\sqrt{\frac{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}}\ ,\ v\right]$$

{{}}

$$\textbf{Solve}\left[\sqrt{\frac{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}{1-\frac{v^2}{(c\ (\text{meters}/\text{second}))^2}}} == \sqrt{\frac{1-\frac{v^2}{(c)^2}}{1-\frac{v^2}{(c)^2}}}\ ,\ c\right]$$

$\text{Solve}\left[\text{True},\ 2.99792\times 10^{8}\right]$

$$\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}\;\sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}}}\;\sqrt{4\pi r-r\theta}}}{2\pi}=$$

$$\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}\;\frac{\theta}{\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}}\;\sqrt{4\pi r-r\theta}}}{2\pi}=$$

$$\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c)^2}}\;\frac{\theta}{\sqrt{1-\frac{v^2}{(c)^2}}}\;\sqrt{4\pi r-r\theta}}}{2\pi}=(1)\;\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c)^2}}\;\sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{(c)^2}}}}\;\sqrt{4\pi r-r\theta}}}{2\pi}$$

Logically, the units of the speed of light will cancel out, therefore, their relationship will be set equal to one and taken out of the equation for the purposes of computational calculation.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}\;\frac{\theta}{\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}}\;\sqrt{4\pi r-r\theta}}}{2\pi}==\eta,\ \text{meters}\right]$$

{{}}

Meters cancel out.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}\;\frac{\theta}{\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}}\;\sqrt{4\pi r-r\theta}}}{2\pi}==\eta,\ \text{second}\right]$$

{{}}

Seconds cancel out.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}\;\frac{\theta}{\sqrt{1-\frac{v^2}{(c\,(\text{meters}/\text{second}))^2}}}\;\sqrt{4\pi r-r\theta}}}{2\pi}==\eta,\ c\right]$$

{{}}

The numeric c cancels out.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c\ (\text{meters/second}))^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{(c\ (\text{meters/second}))^2}}}}\ \sqrt{4\pi r - r\theta}}{2\pi} == \eta,\ r\right]$$

$$\left\{\left\{r \to -\frac{2\pi\eta}{\sqrt{4\pi\theta-\theta^2}}\right\},\ \left\{r \to \frac{2\pi\eta}{\sqrt{4\pi\theta-\theta^2}}\right\}\right\}$$

Radius yields the result from Lemma 3.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c)^2}}\ \frac{\theta}{\sqrt{1-\frac{v^2}{(c)^2}}}}\ \sqrt{4\pi r - r\theta}}{2\pi} == \eta,\ v\right]$$

{{}}

Velocity cancels out.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c)^2}}\ \frac{\theta}{\sqrt{1-\frac{v^2}{(c)^2}}}}\ \sqrt{4\pi r - r\theta}}{2\pi} == \eta,\ v\right]$$

{{}}

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{(c)^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{(c)^2}}}}\ \sqrt{4\pi r - r\theta}}{2\pi} == \eta,\ v\right]$$

{}

Velocity cancels out. Everything *cancels out*. Only when using the exact speed of light, in scientific notation, can solutions to the innate velocity be found.

We set the speed of light equal to its numeric value for the purpose of making computations, dropping the units, because in the expression for the height of the cone, they would cancel out anyway. It should be noted that this is necessary for computing the function of the velocity and that the exact speed of light is to be used as well as that the numeric value of the speed of light has to be in the form of scientific notation in order to find results to this equation.

Theorem 3 Continued From the expression of the height of the cone of Lemma 1, with the Lorentz transformations implicitly expressed, we can solve for the velocity within the Lorenz coefficient in terms of the height of the cone, the initial radius, and the angle, θ when using the exact speed of light in scientific notation and only when it is its exact (or extremely closely approximated) value expressed in scientific notation.

Proof.

```
c := 2.99792458 * (10^8)
```

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{(v)^2}{c^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{(v)^2}{c^2}}}}\ \sqrt{4\pi r - r\theta}}{2\pi} == \eta,\ v\right]$$

$$\left\{\left\{v \to -\frac{1.\ \sqrt{3.54814\times10^{18}\,\eta^2 - 1.12941\times10^{18}\,r^2\,\theta + 8.98755\times10^{16}\,r^2\,\theta^2}}{\sqrt{39.4784\,\eta^2 - 12.5664\,r^2\,\theta + r^2\,\theta^2}}\right\},\right.$$
$$\left.\left\{v \to \frac{\sqrt{3.54814\times10^{18}\,\eta^2 - 1.12941\times10^{18}\,r^2\,\theta + 8.98755\times10^{16}\,r^2\,\theta^2}}{\sqrt{39.4784\,\eta^2 - 12.5664\,r^2\,\theta + r^2\,\theta^2}}\right\}\right\}$$

Theorem 3 Continued From the expression of the height of the cone, from Lemma 1 with the Lorentz transformations implicitly expressed, we can solve for the velocity within the Lorenz coefficient in terms of θ and β.

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{(v)^2}{c^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{(v)^2}{c^2}}}}\ \sqrt{4\pi r - r\theta}}{2\pi} == r\,\text{Sin}[\beta],\ v\right]$$

$$\left\{\left\{v \to -\frac{1.\ \sqrt{-1.12941\times10^{18}\,\theta + 8.98755\times10^{16}\,\theta^2 + 3.54814\times10^{18}\,\text{Sin}[\beta]^2}}{\sqrt{-12.5664\,\theta + \theta^2 + 39.4784\,\text{Sin}[\beta]^2}}\right\},\right.$$
$$\left.\left\{v \to \frac{\sqrt{-1.12941\times10^{18}\,\theta + 8.98755\times10^{16}\,\theta^2 + 3.54814\times10^{18}\,\text{Sin}[\beta]^2}}{\sqrt{-12.5664\,\theta + \theta^2 + 39.4784\,\text{Sin}[\beta]^2}}\right\}\right\}$$

■ The Statement that Concludes in Paradox (Evidence of Euclidean Geometry's not Being Comprehensive of All Geometries)

Theorem 4 The initial radius of the circle, r, is a function of only the angle taken out of the initial circle, θ; i.e. r=f(θ). This cannot be proven with Euclidean geometry, thus, this is evidence that Euclidean geometry is consistent, because, any consistent system is not comprehensive, as shown by Gödel in his Incompleteness Theorems.

Proof.

Lemma 6 From Lemma 4, it can be shown that $1=\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}}$

Proof.

$$r == \frac{2\pi\,r\,\text{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}};\ \text{r cancels on both sides, therefore } 1 == \frac{2\pi\,\text{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}}$$

Lemma 7 From Theorem 1 and lemma 1, it can be shown that:

$$0 = \theta\,r - \left(2\pi r - 2\pi\sqrt{\left(r\hat{}2 - \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}\hat{}2\right)}\right)$$

Proof.

$\theta\, r = 2\pi r - 2\pi\sqrt{(r\hat{}2 - \eta\hat{}2)}$

Subtract θ*r from both sides.

$$0 = \theta\, r - \left(2\pi r - 2\pi\sqrt{\left(r\hat{}2 - \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}\hat{}2\right)}\right)$$

Lemma 8 One can calculate the radius as a function of θ and β from constructing the equation 1-1=0

as $1 - \frac{2\pi\,\text{Sin}[\beta]}{\sqrt{4\pi\theta - \theta^2}} == \theta\, r - \left(2\pi r - 2\pi\sqrt{\left(r\hat{}2 - \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}\hat{}2\right)}\right)$.

Proof.

$$\textbf{Solve}\left[1 - \frac{2\pi\,\text{Sin}[\beta]}{\sqrt{4\pi\theta - \theta^2}} == \theta\, r - \left(2\pi r - 2\pi\sqrt{\left(r\hat{}2 - \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}\hat{}2\right)}\right),\ r\right]$$

$$\left\{\left\{r \to \left(-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi - \theta)\,\theta}\ \text{Sin}[\beta] + 4\pi^2\,\text{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi - \theta)\,\theta}\ \text{Sin}[\beta]^3}{4\pi - \theta} - \frac{2\pi^2\sqrt{(4\pi - \theta)\,\theta}\ \text{Sin}[\beta]^3}{\theta}\right) \Big/ \left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\text{Sin}[\beta]^2 + 8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right\}\right\}$$

Lemma 9 From Lemma 5, θ is a function of β. From Lemma 4, β is a function of θ. Therefore, r is a function of theta or beta. This cannot be proven with Euclidean geometry, because one can draw a circle of any size and take any angle (arc length) from it.

Euclidean geometry is consistent, but not comprehensive of all geometries,
including the system related to difference in circumferences of two
circles equaling an arc length (applied to the Pythagorean theorem to form *a* cone).

Application of These Facts, Equations, and Solutions to the Properties of an Electron whose Energy Equals that of a Photon

■ Process of Discovery

From v = $\left(\sqrt{\left(3.54814\times 10^{18}\,\eta^2 - 1.12941\times 10^{18}\,r^2\,\theta + 8.98755\times 10^{16}\,r^2\,\theta^2\right)}\right) \Big/ \left(\sqrt{39.4784\,\eta^2 - 12.5664\,r^2\,\theta + r^2\,\theta^2}\right)$,

we can say that velocity = (wavelength) * (frequency) =

$\lambda\,\nu = \mathbf{v} = \left(\sqrt{\left(3.54814\times 10^{18}\,\eta^2 - 1.12941\times 10^{18}\,r^2\,\theta + 8.98755\times 10^{16}\,r^2\,\theta^2\right)}\right) \Big/ \left(\sqrt{39.4784\,\eta^2 - 12.5664\,r^2\,\theta + r^2\,\theta^2}\right)$. We can then solve this equation for ν.

```
Solve[
 (√ (3.5481432270250993`*^18 η² - 1.1294090667581471`*^18 r² θ + 8.987551787368176`*^16 r² θ²)) /
  (√(39.47841760435743` η² - 12.566370614359172` r² θ + r² θ²)) == λ ν, ν]
```

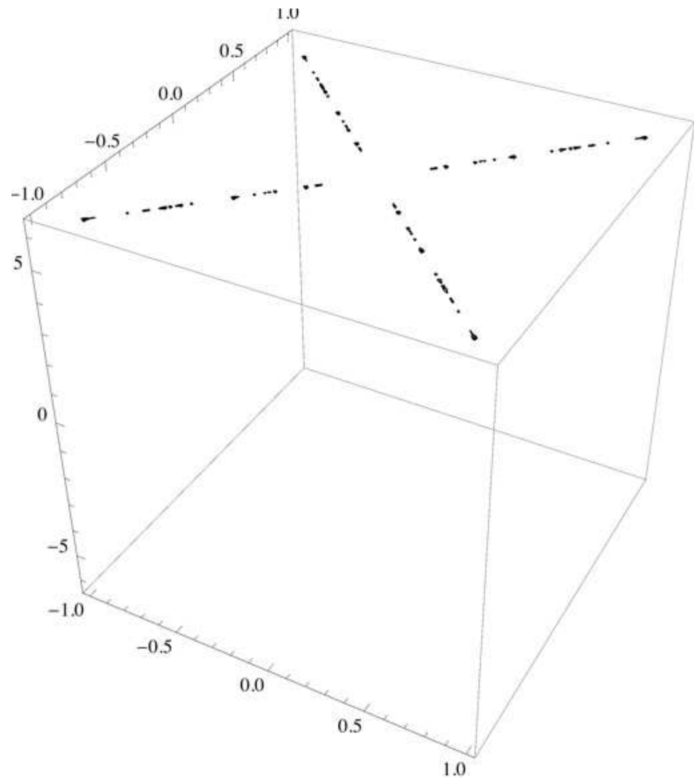
$$\left\{\left\{\nu \to \frac{1.\sqrt{3.54814\times 10^{18}\,\eta^2 - 1.12941\times 10^{18}\,r^2\,\theta + 8.98755\times 10^{16}\,r^2\,\theta^2}}{\sqrt{39.4784\,\eta^2 - 12.5664\,r^2\,\theta + r^2\,\theta^2}\;\lambda}\right\}\right\}$$

We then set h ν = Energy = m c^2, where m is the mass of an electron, c, the speed of light in a vaccuum, and h Planck' s Constant. We solve that equation for λ, which is wavelength.

```
Solve[
 h (1.` √ (3.5481432270250993`*^18 η² - 1.1294090667581471`*^18 r² θ + 8.987551787368176`*^16 r²
      θ²)) / (√(39.47841760435743` η² - 12.566370614359172` r² θ + r² θ²) λ) == m c^2, λ]
```

$$\left\{\left\{\lambda \to \frac{8.65133\times 10^{-13}\sqrt{1.10879\times 10^{17}\,\eta^2 - 3.5294\times 10^{16}\,r^2\,\theta + 2.80861\times 10^{15}\,r^2\,\theta^2}}{\sqrt{1.40969\times 10^{16}\,\eta^2 - 4.48719\times 10^{15}\,r^2\,\theta + 3.57079\times 10^{14}\,r^2\,\theta^2}}\right\}\right\}$$

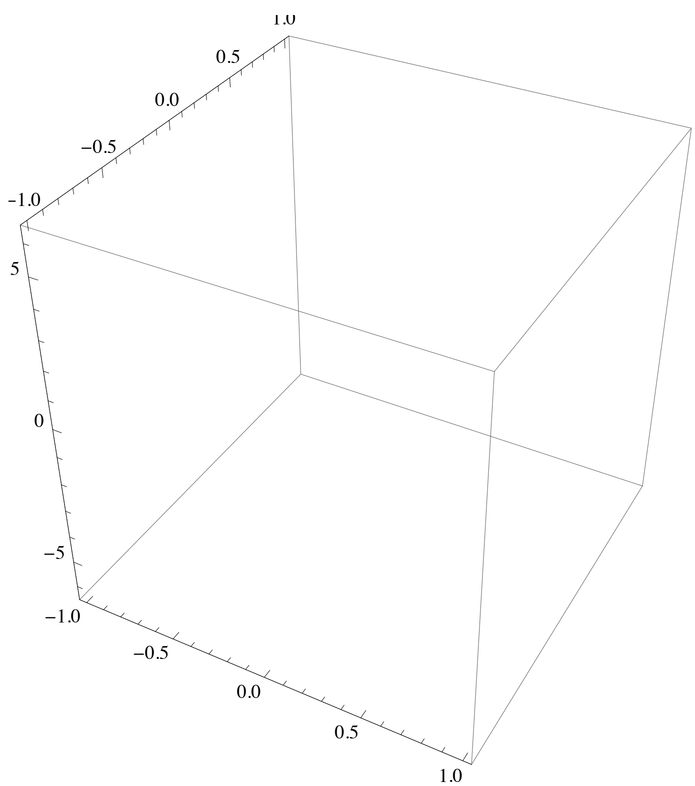
```
ContourPlot3D[
 (8.651330708334268`*^-13 √(1.1087947584453435`*^17 η^2 - 3.5294033336192096`*^16 r^2 θ +
        2.808609933552555`*^15 r^2 θ^2)) /
  (√(1.40969256654408`*^16 η^2 - 4.487190804107819`*^15 r^2 θ + 3.57079298535128`*^14 r^2 θ^2)),
 {r, -1, 1}, {η, -1, 1}, {θ, -2 π, 2 π}]
```



- ### Visualization of Introspections through Substitutions

- **Making the substitution,** $\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}$**, yields the following visualizations.**

$$\text{ContourPlot3D}\Big[\left(8.651330708334268`*^{-13}\sqrt{\left(1.1087947584453435`*^{17}\,\eta^2-3.5294033336192096`*^{16}\,r^2\,\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}+2.808609933552555`*^{15}\,r^2\,\theta^2\right)}\right)\Big/\left(\sqrt{\left(1.40969256654408`*^{16}\,\eta^2-4.487190804107819`*^{15}\,r^2\,\theta+3.57079298535128`*^{14}\,r^2\,\theta^2\right)}\right),\ \{r,-1,1\},\{\eta,-1,1\},\{\theta,-2\pi,2\pi\}\Big]$$

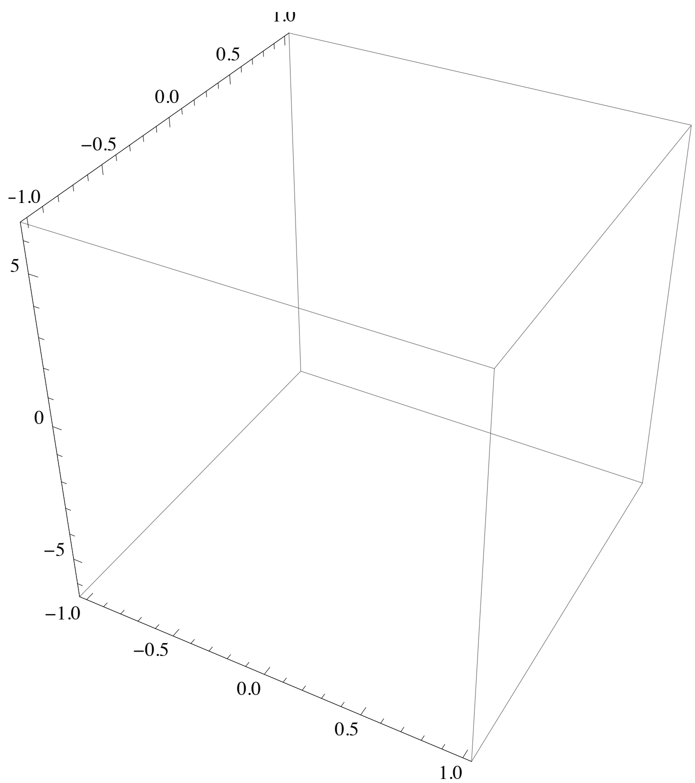


ContourPlot3D[

$$\left(8.651330708334268`*^{\wedge}-13\sqrt{\left(1.1087947584453435`*^{\wedge}17\,\eta^2 - 3.5294033336192096`*^{\wedge}16\,r^2\,\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2} + 2.808609933552555`*^{\wedge}15\,r^2\left(\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right)^2\right)}\right)\Big/$$

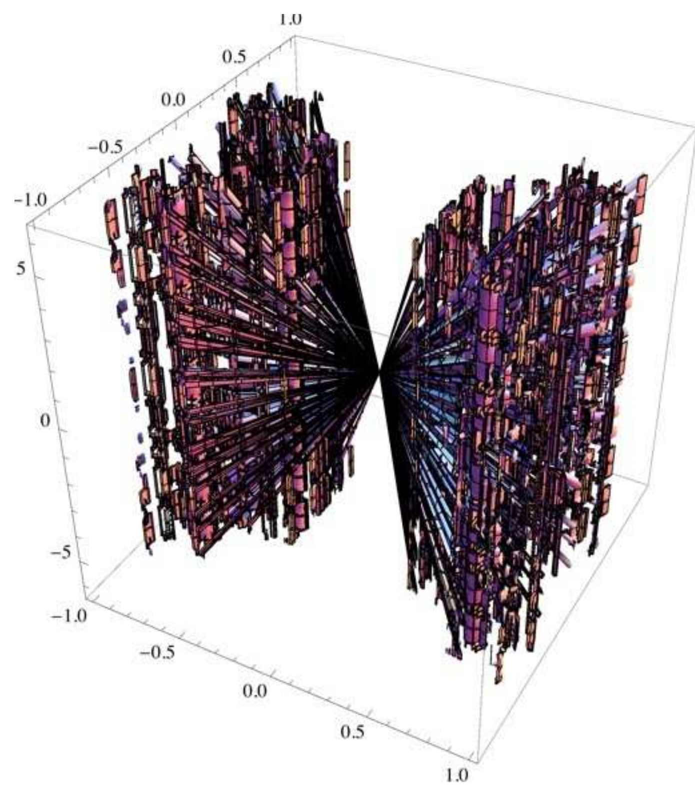
$$\left(\sqrt{\left(1.40969256654408`*^{\wedge}16\,\eta^2 - 4.487190804107819`*^{\wedge}15\,r^2\,\theta + 3.57079298535128`*^{\wedge}14\,r^2\,\theta^2\right)}\right),$$

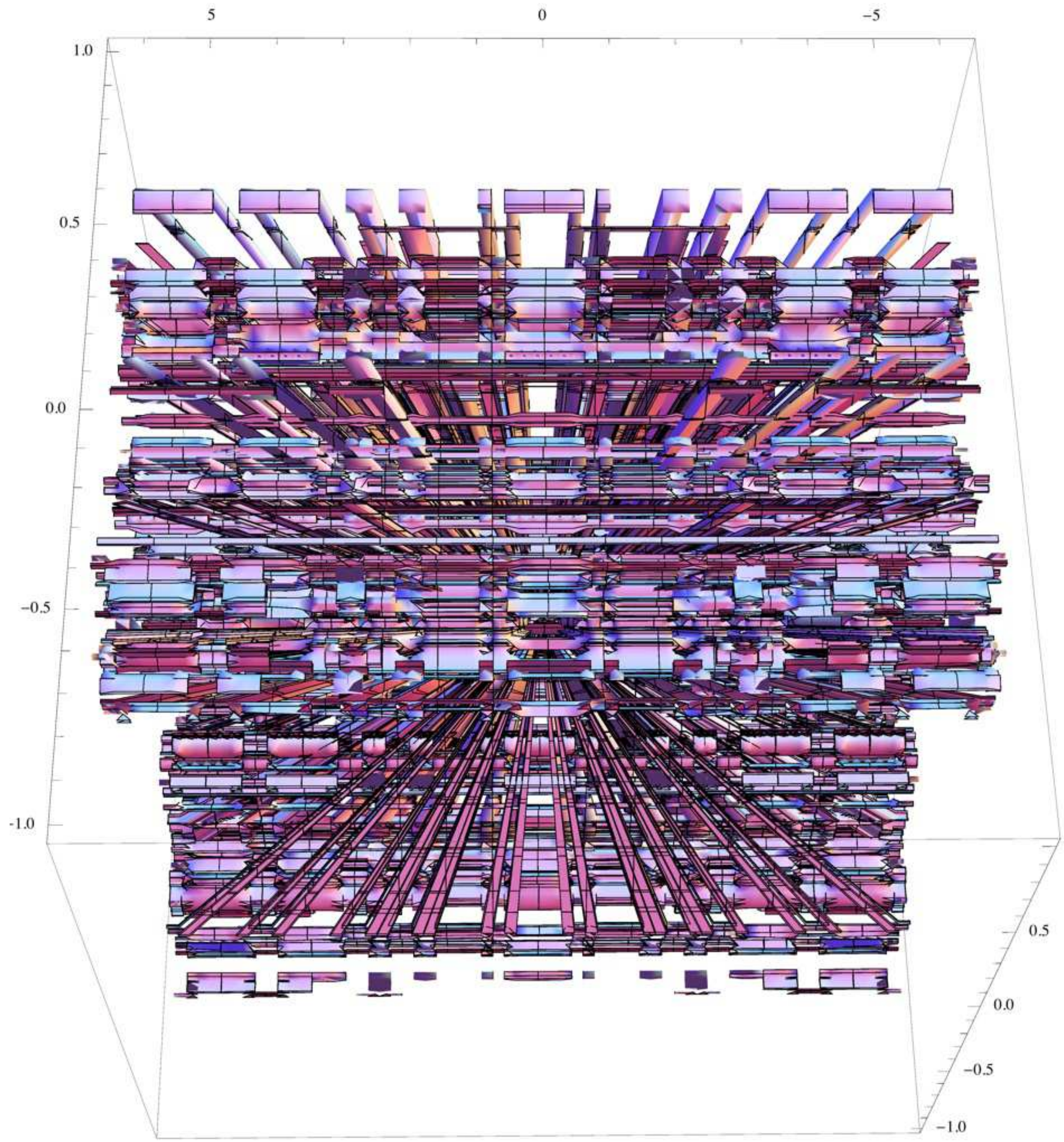
{r, -1, 1}, {η, -1, 1}, {θ, -2 π, 2 π}]



ContourPlot3D$\Bigg[$

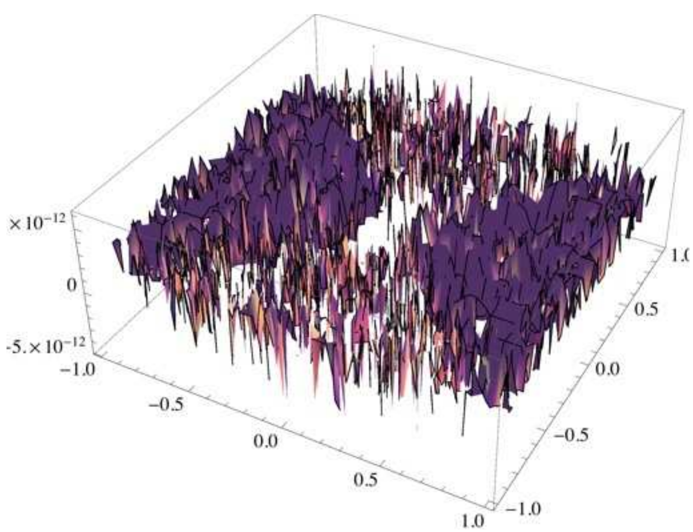
$$\left(8.651330708334268`*^{\wedge}-13\sqrt{\left(1.1087947584453435`*^{\wedge}17\,\eta^2 - 3.5294033336192096`*^{\wedge}16\,r^2\,\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2} + 2.808609933552555`*^{\wedge}15\,r^2\left(\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right)^2\right)}\right) \Big/ \left(\sqrt{\left(1.40969256654408`*^{\wedge}16\,\eta^2 - 4.487190804107819`*^{\wedge}15\,r^2\,\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2} + 3.57079298535128`*^{\wedge}14\,r^2\,\theta^2\right)}\right),\ \{r,-1,1\},\ \{\eta,-1,1\},\ \{\theta,-2\pi,2\pi\}\Bigg]$$




5
0
−5
1.0
0.5
0.0
−0.5
-1.0
0.5
0.0
−0.5
−1.0

```
Plot3D[(8.651330708334268`*^-13
  Sqrt[(1.1087947584453435`*^17 η^2 - 3.5294033336192096`*^16 r^2 (2 π (r^2 + Sqrt[r^4 - r^2 η^2]))/r^2 +
     2.808609933552555`*^15 r^2 ((2 π (r^2 + Sqrt[r^4 - r^2 η^2]))/r^2)^2)] /
  (Sqrt[(1.40969256654408`*^16 η^2 - 4.487190804107819`*^15 r^2 (2 π (r^2 + Sqrt[r^4 - r^2 η^2]))/r^2 +
     3.57079298535128`*^14 r^2 ((2 π (r^2 + Sqrt[r^4 - r^2 η^2]))/r^2)^2)]), {r, -1, 1}, {η, -1, 1}]
```

Power::infy : Infinite expression $\frac{1}{0.}$ encountered. ≫



- **If we substitute the univocal solution to the radius for the variable, r.**

$$r := \left(-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta] + 4\pi^2\,\mathrm{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]^3}{4\pi-\theta} - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]^3}{\theta}\right) \Big/ \left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\mathrm{Sin}[\beta]^2 + 8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)$$

$$\text{ContourPlot3D}\Big[\left(8.651330708334268`*^-13\sqrt{\left(1.1087947584453435`*^17\,\eta^2 - 3.5294033336192096`*^16\,r^2\,\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2} + 2.808609933552555`*^15\,r^2\left(\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right)^2\right)}\right)\Big/$$

$$\left(\sqrt{\left(1.40969256654408`*^16\,\eta^2 - 4.487190804107819`*^15\,r^2\,\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2} + 3.57079298535128`*^14\,r^2\left(\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right)^2\right)}\right),$$

$$\{\theta, -2\pi, 2\pi\}, \{\beta, -\pi/2, \pi/2\}, \{\eta, -1, 1\}\Big]$$

Solutions to Mass

$$\nu == \left(1.`\sqrt{}\left(3.5481432270250993`*^18\,\eta^2 - 1.1294090667581471`*^18\,r^2\,\theta + 8.987551787368176`*^16\,r^2\,\theta^2\right)\right)\Big/\left(\sqrt{39.47841760435743`\,\eta^2 - 12.566370614359172`\,r^2\,\theta + r^2\,\theta^2}\;\lambda\right)$$

$$h\,\nu = m\,c\text{^}2 = h\left(\left(1.`\sqrt{}\left(3.5481432270250993`*^18\,\eta^2 - 1.1294090667581471`*^18\,r^2\,\theta + 8.987551787368176`*^16\,r^2\,\theta^2\right)\right)\Big/\left(\sqrt{39.47841760435743`\,\eta^2 - 12.566370614359172`\,r^2\,\theta + r^2\,\theta^2}\;\lambda\right)\right)$$

$$\text{Solve}\Big[h\left(\left(1.`\sqrt{}\left(3.5481432270250993`*^18\,\eta^2 - 1.1294090667581471`*^18\,r^2\,\theta + 8.987551787368176`*^16\,r^2\,\theta^2\right)\right)\Big/\left(\sqrt{39.47841760435743`\,\eta^2 - 12.566370614359172`\,r^2\,\theta + r^2\,\theta^2}\;\lambda\right)\right) == m\,c\text{^}2,\ m\Big]$$

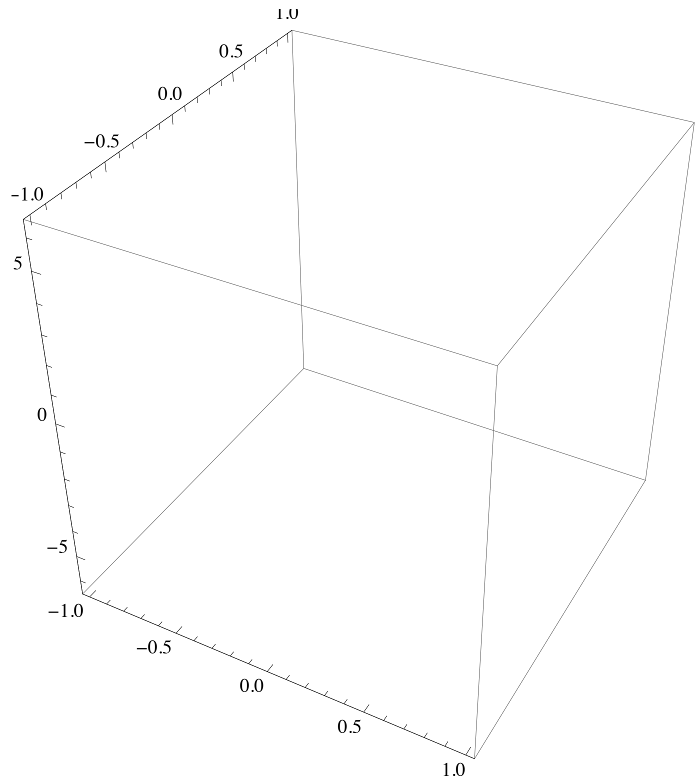
$$\left\{\left\{m \to \frac{1.\,h\sqrt{3.54814\times10^{18}\,\eta^2 - 1.12941\times10^{18}\,r^2\,\theta + 8.98755\times10^{16}\,r^2\,\theta^2}}{c^2\sqrt{39.4784\,\eta^2 - 12.5664\,r^2\,\theta + r^2\,\theta^2}\;\lambda}\right\}\right\}$$

$$h := 6.62606896 * 10\text{^}(-34)$$

$$c := 2.99792458\left(10^8\right)$$

- **If the wavelength, λ equals the initial radius of the circle, r.**

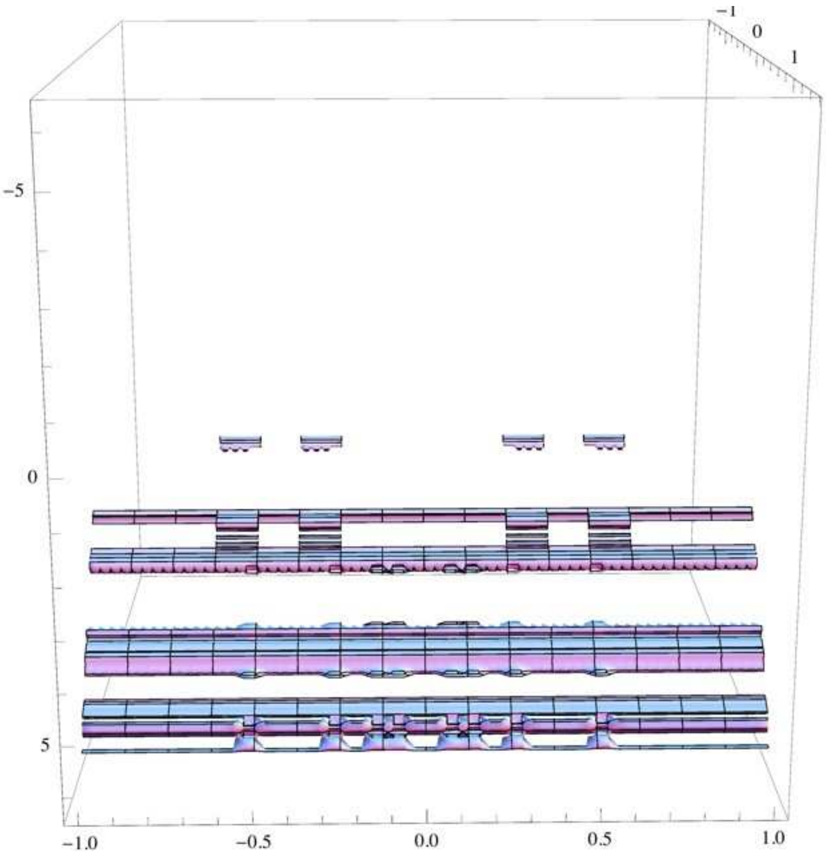
```
ContourPlot3D[(1.` h √(3.5481432270250993`*^18 η² -
      1.1294090667581471`*^18 r² θ + 8.987551787368176`*^16 r² θ²)) /
  (c² √(39.47841760435743` η² - 12.566370614359172` r² θ + r² θ²) r),
 {r, -1, 1}, {η, -1, 1}, {θ, -2 π, 2 π}]
```



- **If we substitute the univocal solution to the radius for the variable, r.**

$$r := \left(-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta] + 4\pi^2\,\text{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{4\pi-\theta} - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\right) \Big/ \left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\text{Sin}[\beta]^2 + 8\pi^2\theta\,\text{Sin}[\beta]^2\right)$$

```
ContourPlot3D[(1.` h √(3.5481432270250993`*^18 η^2 -
      1.1294090667581471`*^18 r^2 θ + 8.987551787368176`*^16 r^2 θ^2)) /
  (c^2 √(39.47841760435743` η^2 - 12.566370614359172` r^2 θ + r^2 θ^2) r),
 {θ, -2 π, 2 π}, {η, -1, 1}, {β, -π / 2, π / 2}]
```



- **If we substitute the univocal solution to the radius for the variable, r and we make the substitution $\eta = \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}$, the equation makes the computer stall and the spinning wheel of death.**

$$r := \left(-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta] + 4\pi^2\,\mathrm{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{4\pi-\theta} - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{\theta}\right) \Big/ \left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\mathrm{Sin}[\beta]^2 + 8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)$$

$$\eta := \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}$$

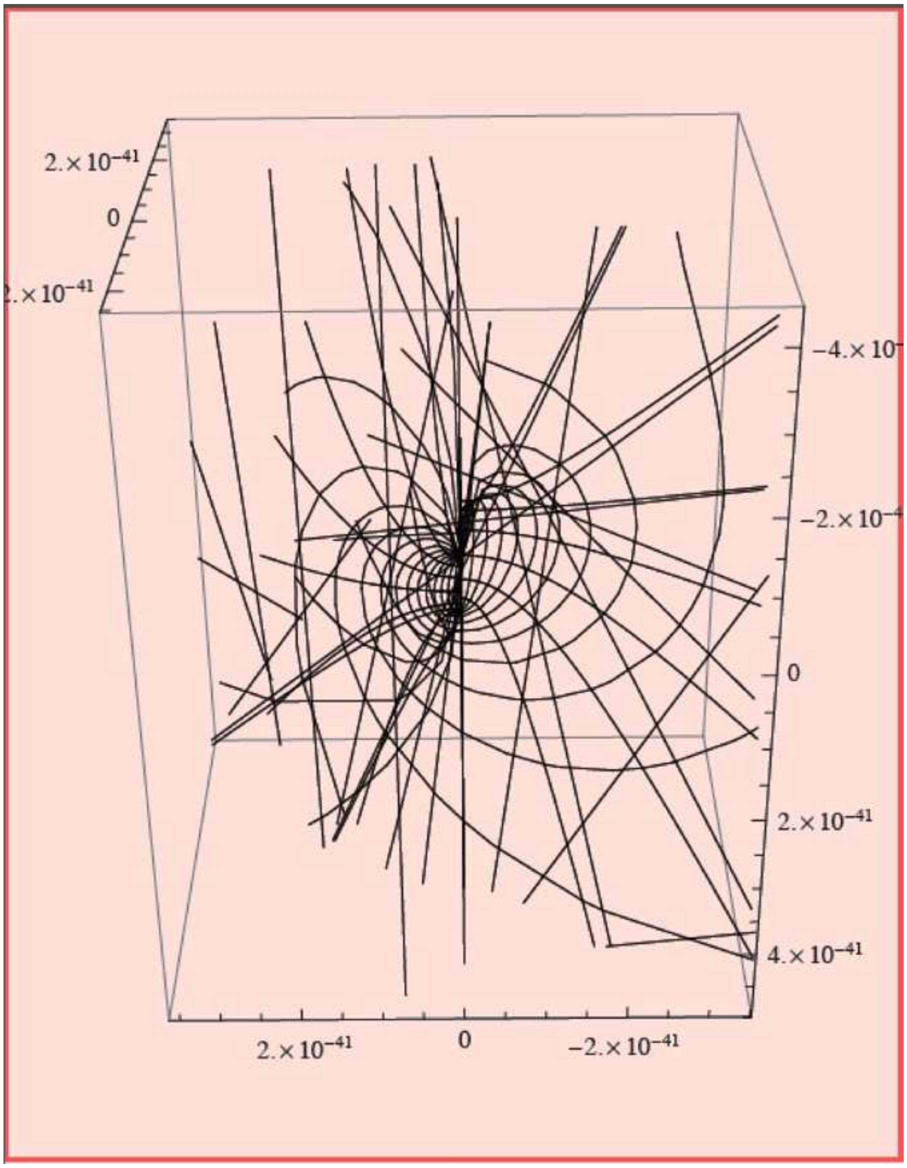
```
SphericalPlot3D[(1.` h √(3.5481432270250993`*^18 η^2 -
        1.1294090667581471`*^18 r^2 θ + 8.987551787368176`*^16 r^2 θ^2)) /
  (c^2 Sqrt[39.47841760435743` η^2 - 12.566370614359172` r^2 θ + r^2 θ^2] r), {θ,
  -2 π, 2 π}, {β, -π / 2, π / 2}]
```

- **If we substitute the univocal solution to the radius for the variable, r and we make the substitution** η = r Sin[β].

$$r := \left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]+4\pi^2\,\mathrm{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)$$

```
η := r Sin[β]
```

```
SphericalPlot3D[(1.` h √(3.5481432270250993`*^18 η^2 -
        1.1294090667581471`*^18 r^2 θ + 8.987551787368176`*^16 r^2 θ^2)) /
  (c^2 Sqrt[39.47841760435743` η^2 - 12.566370614359172` r^2 θ + r^2 θ^2] r), {θ,
  -2 π, 2 π}, {β, -π / 2, π / 2}]
```



Time Solutions Used for Describing Mass and Frequency

$$v = \left(\sqrt{\left(-1.1294090667581471`*^{\wedge}18\,\theta + 8.987551787368176`*^{\wedge}16\,\theta^2 + 3.5481432270250993`*^{\wedge}18\,\text{Sin}[\beta]^2\right)}\right) \Big/ \left(\sqrt{-12.566370614359172`\,\theta + \theta^2 + 39.47841760435743`\,\text{Sin}[\beta]^2}\right)$$

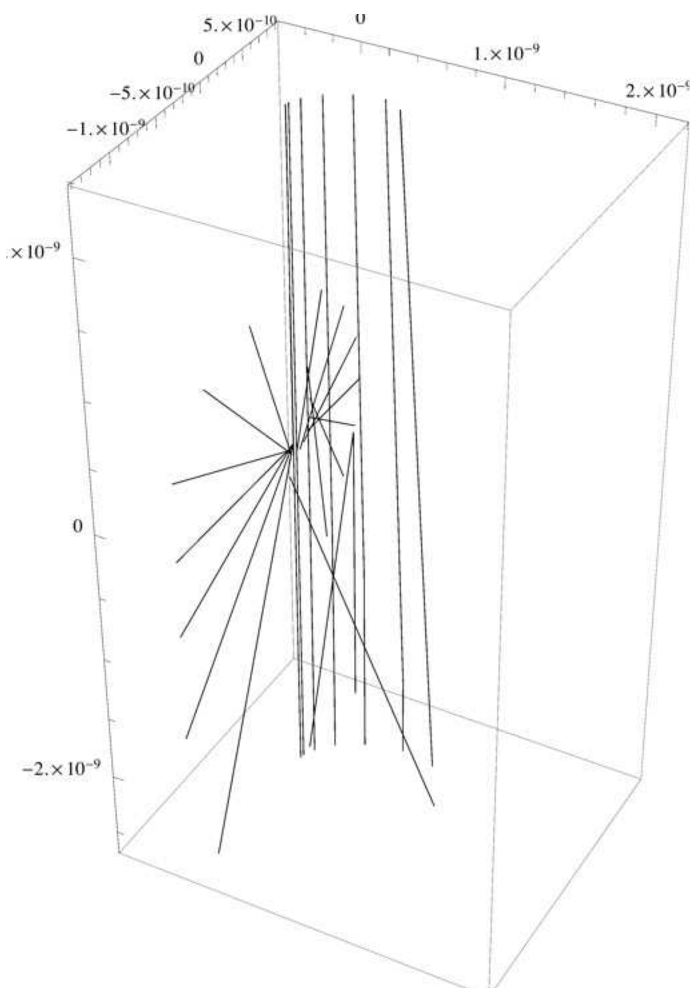
$$v = \lambda / t = r / t$$

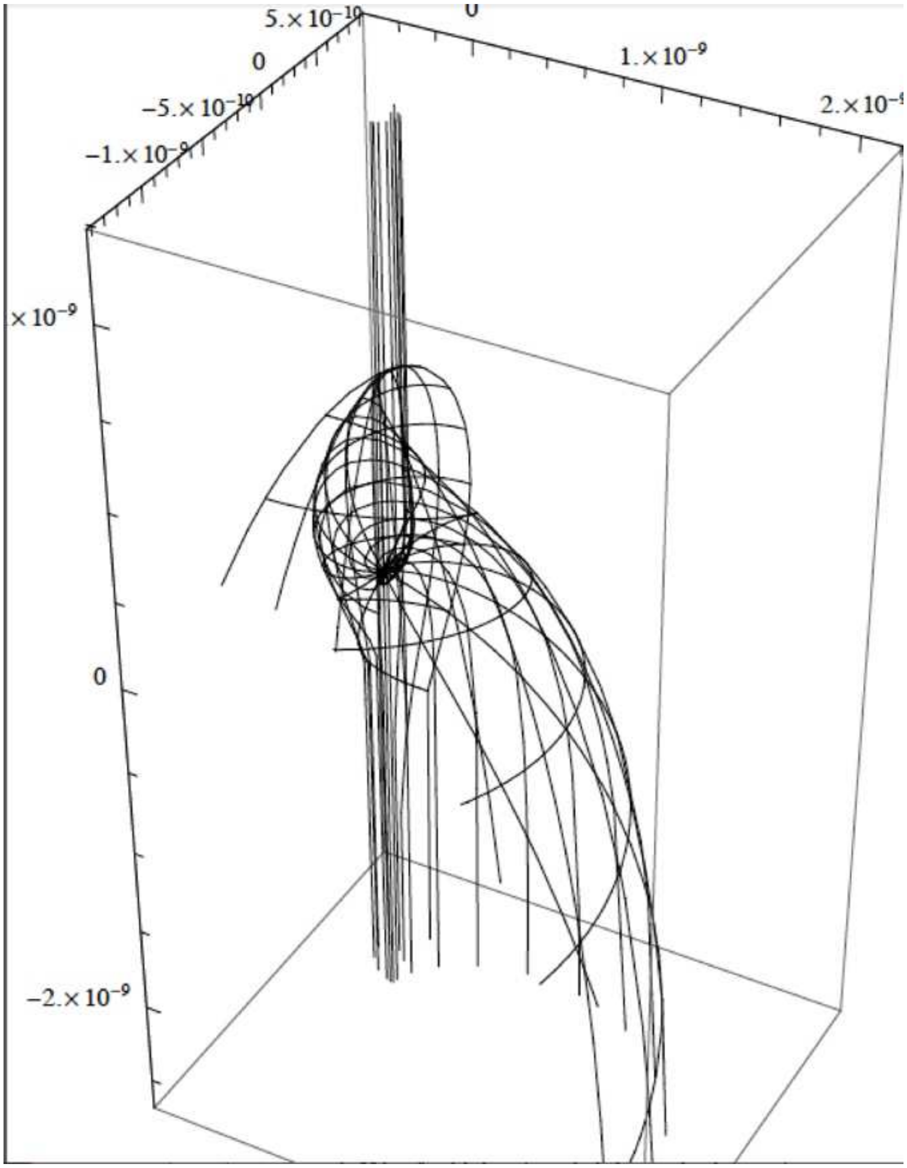
$$r := \left(-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi - \theta)\,\theta}\,\text{Sin}[\beta] + 4\pi^2\,\text{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{4\pi - \theta} - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta}\right) \Big/ \left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\text{Sin}[\beta]^2 + 8\pi^2\theta\,\text{Sin}[\beta]^2\right)$$

```
Solve[(√(-1.1294090667581471`*^18 θ +
      8.987551787368176`*^16 θ^2 + 3.5481432270250993`*^18 Sin[β]^2)) /
   (√(-12.566370614359172` θ + θ^2 + 39.47841760435743` Sin[β]^2)) == r / t, t]
```

$$\left\{\left\{t \to \left(-12.5664\,\theta\sqrt{-4.48719\times10^{15}\,\theta + 3.57079\times10^{14}\,\theta^2 + 1.40969\times10^{16}\,\mathrm{Sin}[\beta]^2} + 1.\,\theta^2\sqrt{-4.48719\times10^{15}\,\theta + 3.57079\times10^{14}\,\theta^2 + 1.40969\times10^{16}\,\mathrm{Sin}[\beta]^2} + 6.28319\sqrt{12.5664\,\theta - 1.\,\theta^2}\,\mathrm{Sin}[\beta]\sqrt{-4.48719\times10^{15}\,\theta + 3.57079\times10^{14}\,\theta^2 + 1.40969\times10^{16}\,\mathrm{Sin}[\beta]^2}\right) \Big/ \left(1.68802\times10^{10}\,\theta\sqrt{-3.5294\times10^{16}\,\theta + 2.80861\times10^{15}\,\theta^2 + 1.10879\times10^{17}\,\mathrm{Sin}[\beta]^2} - 4.02985\times10^{9}\,\theta^2\sqrt{-3.5294\times10^{16}\,\theta + 2.80861\times10^{15}\,\theta^2 + 1.10879\times10^{17}\,\mathrm{Sin}[\beta]^2} + 2.1379\times10^{8}\,\theta^3\sqrt{-3.5294\times10^{16}\,\theta + 2.80861\times10^{15}\,\theta^2 + 1.10879\times10^{17}\,\mathrm{Sin}[\beta]^2}\right)\right\}\right\}$$

```
SphericalPlot3D[
 (-12.566370614359172` θ √(-4.487190804107819`*^15 θ + 3.57079298535128`*^14 θ^2 +
       1.40969256654408`*^16 Sin[β]^2) + 1.` θ^2
    √(-4.487190804107819`*^15 θ + 3.57079298535128`*^14 θ^2 + 1.40969256654408`*^16 Sin[β]^2) +
   6.283185307179586` √(12.566370614359172` θ - 1.` θ^2) Sin[β]
    √(-4.487190804107819`*^15 θ + 3.57079298535128`*^14 θ^2 + 1.40969256654408`*^16 Sin[β]^2)) /
  (1.6880179234807384`*^10 θ √(-3.5294033336192096`*^16 θ +
       2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2) -
   4.029845948245144`*^9 θ^2 √(-3.5294033336192096`*^16 θ + 2.808609933552555`*^15 θ^2 +
       1.1087947584453435`*^17 Sin[β]^2) +
   2.1378996752068695`*^8 θ^3 √(-3.5294033336192096`*^16 θ + 2.808609933552555`*^15 θ^2 +
       1.1087947584453435`*^17 Sin[β]^2)), {θ, -2 π, 2 π}, {β, -π / 2, π / 2}]
```





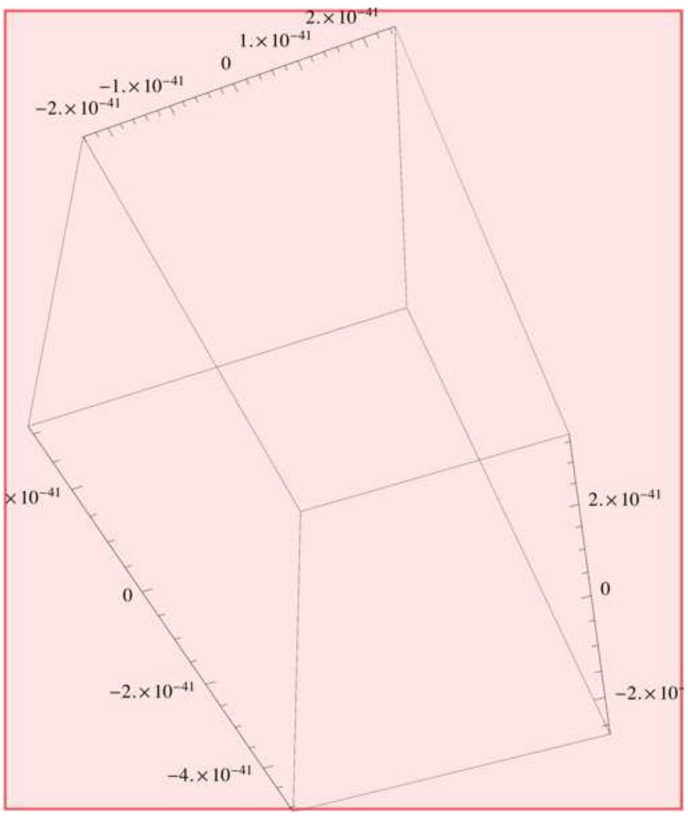
$$v = \frac{1}{t} = 1 \Big/ \Big(\big(-12.566370614359172` \, \theta \sqrt{\left(-4.487190804107819`*^{\wedge}15\, \theta + 3.57079298535128`*^{\wedge}14\, \theta^2 + 1.40969256654408`*^{\wedge}16\, \mathrm{Sin}[\beta]^2\right)} + 1.` \, \theta^2 \sqrt{\left(-4.487190804107819`*^{\wedge}15\, \theta + 3.57079298535128`*^{\wedge}14\, \theta^2 + 1.40969256654408`*^{\wedge}16\, \mathrm{Sin}[\beta]^2\right)} + 6.283185307179586` \sqrt{12.566370614359172` \, \theta - 1.` \, \theta^2} \, \mathrm{Sin}[\beta] \sqrt{\left(-4.487190804107819`*^{\wedge}15\, \theta + 3.57079298535128`*^{\wedge}14\, \theta^2 + 1.40969256654408`*^{\wedge}16\, \mathrm{Sin}[\beta]^2\right)} \big) \Big/ \left(1.6880179234807384`*^{\wedge}10\, \theta \sqrt{\left(-3.5294033336192096`*^{\wedge}16\, \theta + 2.808609933552555`*^{\wedge}15\, \theta^2 + 1.1087947584453435`*^{\wedge}17\, \mathrm{Sin}[\beta]^2\right)} - 4.029845948245144`*^{\wedge}9\, \theta^2 \sqrt{\left(-3.5294033336192096`*^{\wedge}16\, \theta + 2.808609933552555`*^{\wedge}15\, \theta^2 + 1.1087947584453435`*^{\wedge}17\, \mathrm{Sin}[\beta]^2\right)} + 2.1378996752068695`*^{\wedge}8\, \theta^3 \sqrt{\left(-3.5294033336192096`*^{\wedge}16\, \theta + 2.808609933552555`*^{\wedge}15\, \theta^2 + 1.1087947584453435`*^{\wedge}17\, \mathrm{Sin}[\beta]^2\right)}\right) \Big)$$

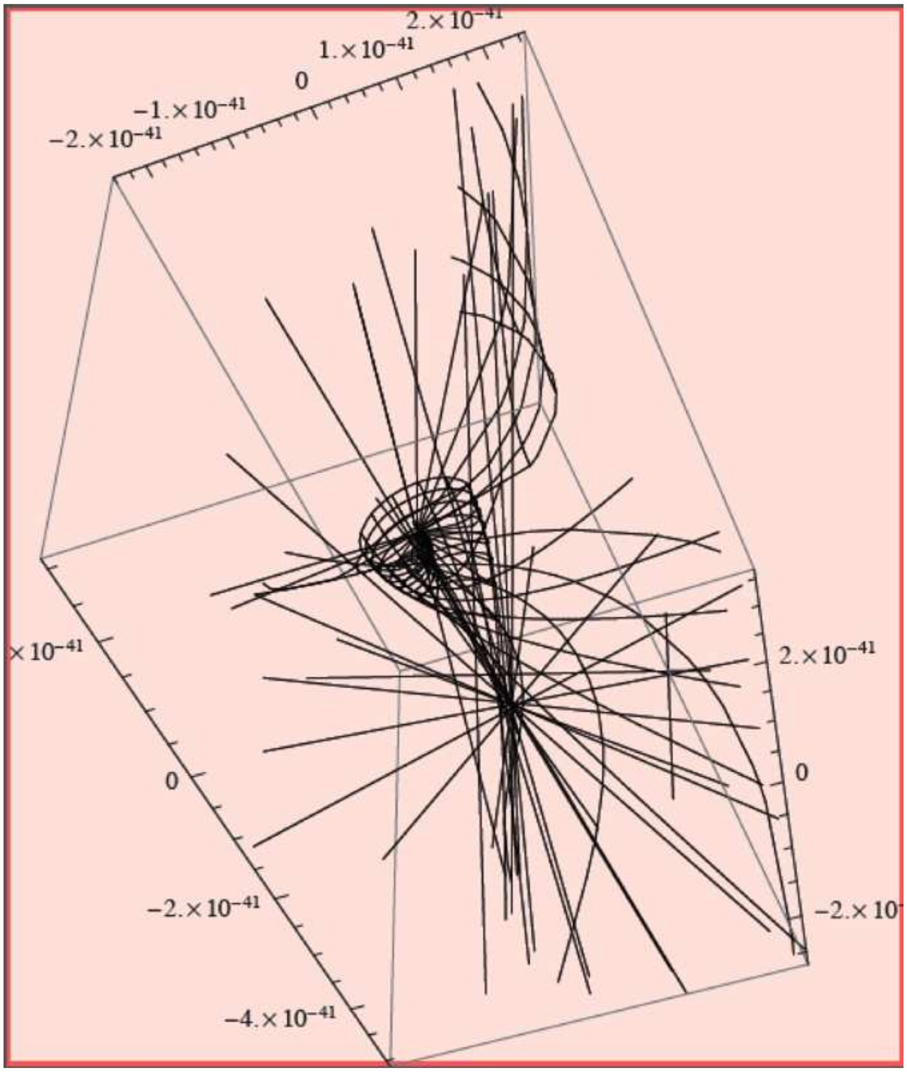
```
Energy =
 h ν = h (1/ ((-12.566370614359172` θ √(-4.487190804107819`*^15 θ + 3.57079298535128`*^14 θ^2 +
          1.40969256654408`*^16 Sin[β]^2) + 1.` θ^2 √(-4.487190804107819`*^15 θ +
          3.57079298535128`*^14 θ^2 + 1.40969256654408`*^16 Sin[β]^2) + 6.283185307179586`
        √(12.566370614359172` θ - 1.` θ^2) Sin[β] √(-4.487190804107819`*^15 θ +
          3.57079298535128`*^14 θ^2 + 1.40969256654408`*^16 Sin[β]^2)) /
      (1.6880179234807384`*^10 θ √(-3.5294033336192096`*^16 θ +
          2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2) -
       4.029845948245144`*^9 θ^2 √(-3.5294033336192096`*^16 θ +
          2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2) +
       2.1378996752068695`*^8 θ^3 √(-3.5294033336192096`*^16 θ +
          2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2)))) = m c^2
```

```
Solve[h (1/ ((-12.566370614359172` θ √(-4.487190804107819`*^15 θ + 3.57079298535128`*^14 θ^2 +
          1.40969256654408`*^16 Sin[β]^2) + 1.` θ^2 √(-4.487190804107819`*^15 θ +
          3.57079298535128`*^14 θ^2 + 1.40969256654408`*^16 Sin[β]^2) + 6.283185307179586`
        √(12.566370614359172` θ - 1.` θ^2) Sin[β] √(-4.487190804107819`*^15 θ +
          3.57079298535128`*^14 θ^2 + 1.40969256654408`*^16 Sin[β]^2)) /
      (1.6880179234807384`*^10 θ √(-3.5294033336192096`*^16 θ +
          2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2) -
       4.029845948245144`*^9 θ^2 √(-3.5294033336192096`*^16 θ +
          2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2) +
       2.1378996752068695`*^8 θ^3 √(-3.5294033336192096`*^16 θ +
          2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2)))) == m c^2, m]
```

$$\Big\{\Big\{m \to \Big(7.3725\times 10^{-51}\Big(1.68802\times 10^{10}\,\theta\sqrt{-3.5294\times 10^{16}\,\theta + 2.80861\times 10^{15}\,\theta^2 + 1.10879\times 10^{17}\,\text{Sin}[\beta]^2} - 4.02985\times 10^{9}\,\theta^2\sqrt{-3.5294\times 10^{16}\,\theta + 2.80861\times 10^{15}\,\theta^2 + 1.10879\times 10^{17}\,\text{Sin}[\beta]^2} + 2.1379\times 10^{8}\,\theta^3\sqrt{-3.5294\times 10^{16}\,\theta + 2.80861\times 10^{15}\,\theta^2 + 1.10879\times 10^{17}\,\text{Sin}[\beta]^2}\Big)\Big) \Big/ \Big(-12.5664\,\theta\sqrt{-4.48719\times 10^{15}\,\theta + 3.57079\times 10^{14}\,\theta^2 + 1.40969\times 10^{16}\,\text{Sin}[\beta]^2} + 1.\,\theta^2\sqrt{-4.48719\times 10^{15}\,\theta + 3.57079\times 10^{14}\,\theta^2 + 1.40969\times 10^{16}\,\text{Sin}[\beta]^2} + 6.28319\sqrt{12.5664\,\theta - 1.\,\theta^2}\,\text{Sin}[\beta]\sqrt{-4.48719\times 10^{15}\,\theta + 3.57079\times 10^{14}\,\theta^2 + 1.40969\times 10^{16}\,\text{Sin}[\beta]^2}\Big)\Big\}\Big\}$$

```
SphericalPlot3D[
 (7.372495999759142`*^-51 (1.6880179234807384`*^10 θ √(-3.5294033336192096`*^16 θ +
        2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2) -
     4.029845948245144`*^9 θ^2 √(-3.5294033336192096`*^16 θ +
        2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2) +
     2.1378996752068695`*^8 θ^3 √(-3.5294033336192096`*^16 θ +
        2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2))) /
  (-12.566370614359172` θ √(-4.487190804107819`*^15 θ + 3.57079298535128`*^14 θ^2 +
       1.40969256654408`*^16 Sin[β]^2) + 1.` θ^2
     √(-4.487190804107819`*^15 θ + 3.57079298535128`*^14 θ^2 + 1.40969256654408`*^16 Sin[β]^2) +
    6.283185307179586` √(12.566370614359172` θ - 1.` θ^2) Sin[β]
     √(-4.487190804107819`*^15 θ + 3.57079298535128`*^14 θ^2 + 1.40969256654408`*^16 Sin[β]^2)),
 {β, -π / 2, π / 2}, {θ, -2 π, 2 π}]
```

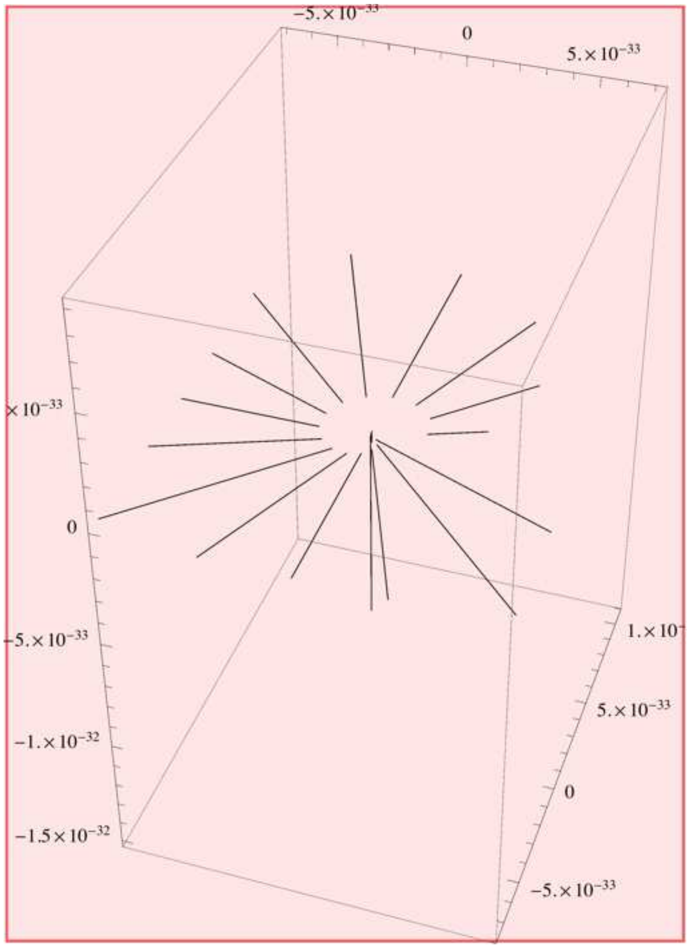




■ Momentum

```
m v = ((7.372495999759142`*^-51
    (1.6880179234807384`*^10 θ √(-3.5294033336192096`*^16 θ + 2.808609933552555`*^15 θ^2 +
         1.1087947584453435`*^17 Sin[β]^2) - 4.029845948245144`*^9 θ^2
       √(-3.5294033336192096`*^16 θ + 2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17
          Sin[β]^2) + 2.1378996752068695`*^8 θ^3 √(-3.5294033336192096`*^16 θ +
         2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2))) /
   (-12.566370614359172` θ √(-4.487190804107819`*^15 θ + 3.57079298535128`*^14 θ^2 +
         1.40969256654408`*^16 Sin[β]^2) + 1.` θ^2
      √(-4.487190804107819`*^15 θ + 3.57079298535128`*^14 θ^2 + 1.40969256654408`*^16 Sin[β]^2) +
     6.283185307179586` √(12.566370614359172` θ - 1.` θ^2) Sin[β]
      √(-4.487190804107819`*^15 θ + 3.57079298535128`*^14 θ^2 + 1.40969256654408`*^16 Sin[β]^2)))
```

```
SphericalPlot3D[
 ((7.372495999759142`*^-51 (1.6880179234807384`*^10 θ √(-3.5294033336192096`*^16 θ +
         2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2) -
      4.029845948245144`*^9 θ^2 √(-3.5294033336192096`*^16 θ +
         2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2) +
      2.1378996752068695`*^8 θ^3 √(-3.5294033336192096`*^16 θ +
         2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2))) /
   (-12.566370614359172` θ √(-4.487190804107819`*^15 θ + 3.57079298535128`*^14 θ^2 +
        1.40969256654408`*^16 Sin[β]^2) + 1.` θ^2 √(-4.487190804107819`*^15 θ +
        3.57079298535128`*^14 θ^2 + 1.40969256654408`*^16 Sin[β]^2) +
     6.283185307179586` √(12.566370614359172` θ - 1.` θ^2) Sin[β] √(-4.487190804107819`*^15 θ +
        3.57079298535128`*^14 θ^2 + 1.40969256654408`*^16 Sin[β]^2)))
  ((√(-1.1294090667581471`*^18 θ + 8.987551787368176`*^16 θ^2 +
        3.5481432270250993`*^18 Sin[β]^2)) /
    (√(-12.566370614359172` θ + θ^2 + 39.47841760435743` Sin[β]^2))), {β, -π /
  2, π / 2}, {θ, -2
  π, 2
  π}]
```

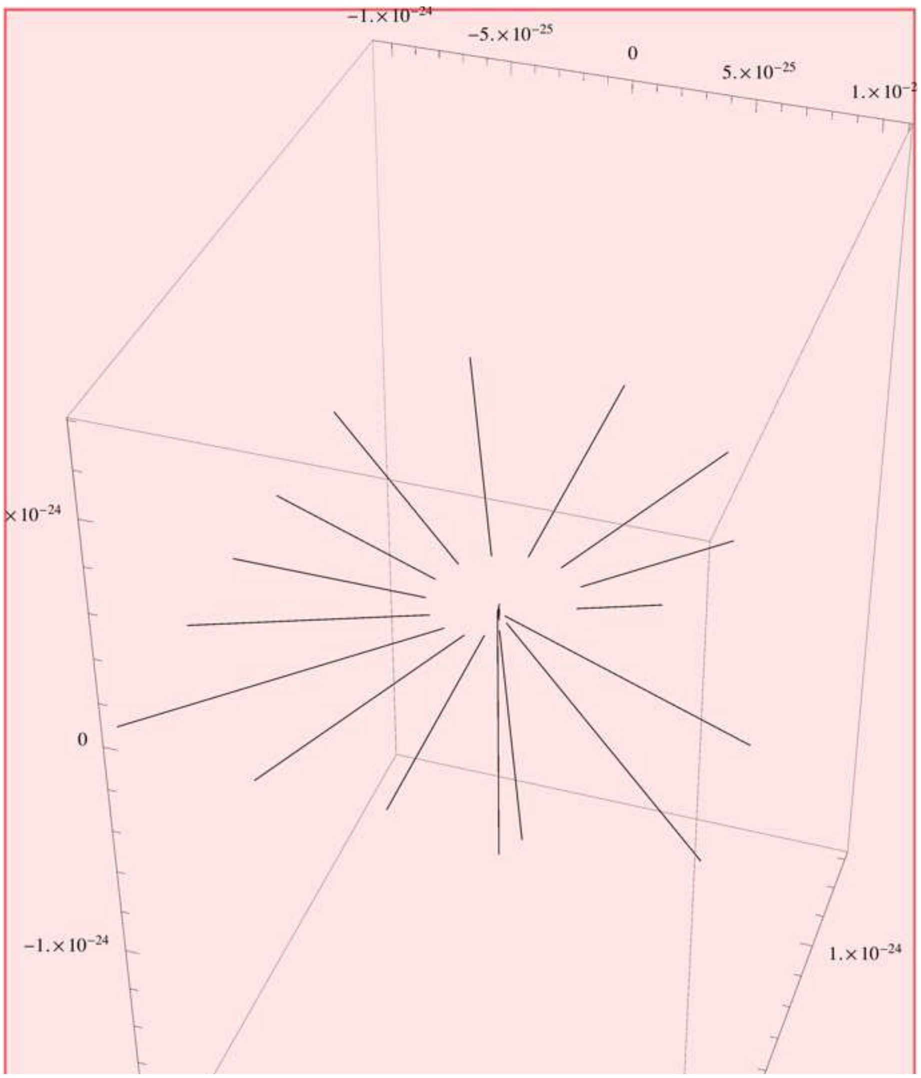


−5.× 10^−33
0
5.× 10^−33
× 10^−33
0
−5.× 10^−33
−1.× 10^−32
−1.5× 10^−32
1.× 10
5.× 10^−33
0
−5.× 10^−33

- **Kinetic Energy**

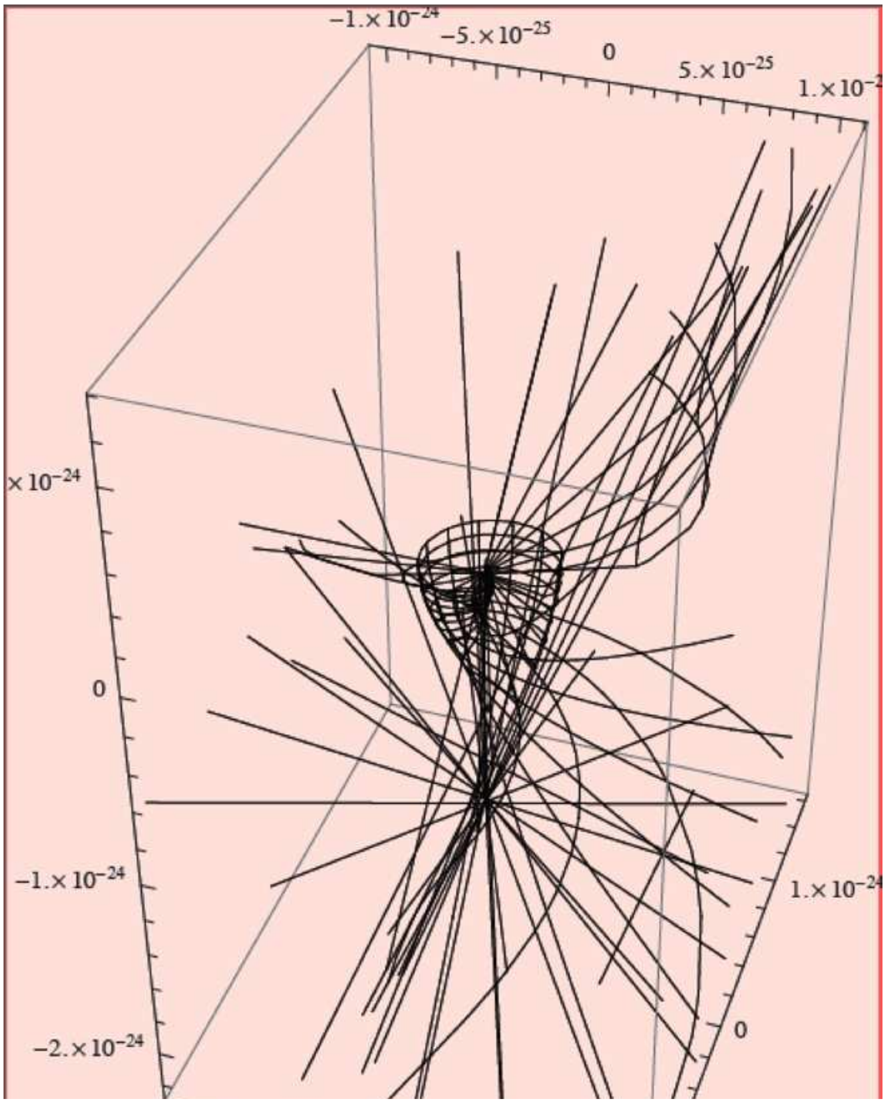
```
KineticEnergy = .5 m v^2 =
  .5 (((7.372495999759142`*^-51 (1.6880179234807384`*^10 θ √(-3.5294033336192096`*^16
                θ + 2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2) -
          4.029845948245144`*^9 θ^2 √(-3.5294033336192096`*^16 θ +
              2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2) +
          2.1378996752068695`*^8 θ^3 √(-3.5294033336192096`*^16 θ +
              2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2))) /
       (-12.566370614359172` θ √(-4.487190804107819`*^15 θ + 3.57079298535128`*^14 θ^2 +
              1.40969256654408`*^16 Sin[β]^2) + 1.` θ^2 √(-4.487190804107819`*^15 θ +
              3.57079298535128`*^14 θ^2 + 1.40969256654408`*^16 Sin[β]^2) + 6.283185307179586`
          √(12.566370614359172` θ - 1.` θ^2) Sin[β] √(-4.487190804107819`*^15 θ +
              3.57079298535128`*^14 θ^2 + 1.40969256654408`*^16 Sin[β]^2)))
     ((√(-1.1294090667581471`*^18 θ + 8.987551787368176`*^16 θ^2 +
              3.5481432270250993`*^18 Sin[β]^2)) /
        (√(-12.566370614359172` θ + θ^2 + 39.47841760435743` Sin[β]^2)))^2)

SphericalPlot3D[
  .5 (((7.372495999759142`*^-51 (1.6880179234807384`*^10 θ √(-3.5294033336192096`*^16 θ +
              2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2) -
          4.029845948245144`*^9 θ^2 √(-3.5294033336192096`*^16 θ +
              2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2) +
          2.1378996752068695`*^8 θ^3 √(-3.5294033336192096`*^16 θ +
              2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2))) /
       (-12.566370614359172` θ √(-4.487190804107819`*^15 θ + 3.57079298535128`*^14 θ^2 +
              1.40969256654408`*^16 Sin[β]^2) + 1.` θ^2 √(-4.487190804107819`*^15 θ +
              3.57079298535128`*^14 θ^2 + 1.40969256654408`*^16 Sin[β]^2) + 6.283185307179586`
          √(12.566370614359172` θ - 1.` θ^2) Sin[β] √(-4.487190804107819`*^15 θ +
              3.57079298535128`*^14 θ^2 + 1.40969256654408`*^16 Sin[β]^2)))
     ((√(-1.1294090667581471`*^18 θ + 8.987551787368176`*^16 θ^2 +
              3.5481432270250993`*^18 Sin[β]^2)) /
        (√(-12.566370614359172` θ + θ^2 + 39.47841760435743` Sin[β]^2)))^
     2), {β, -π / 2, π / 2}, {θ, -2 π, 2 π}]
```

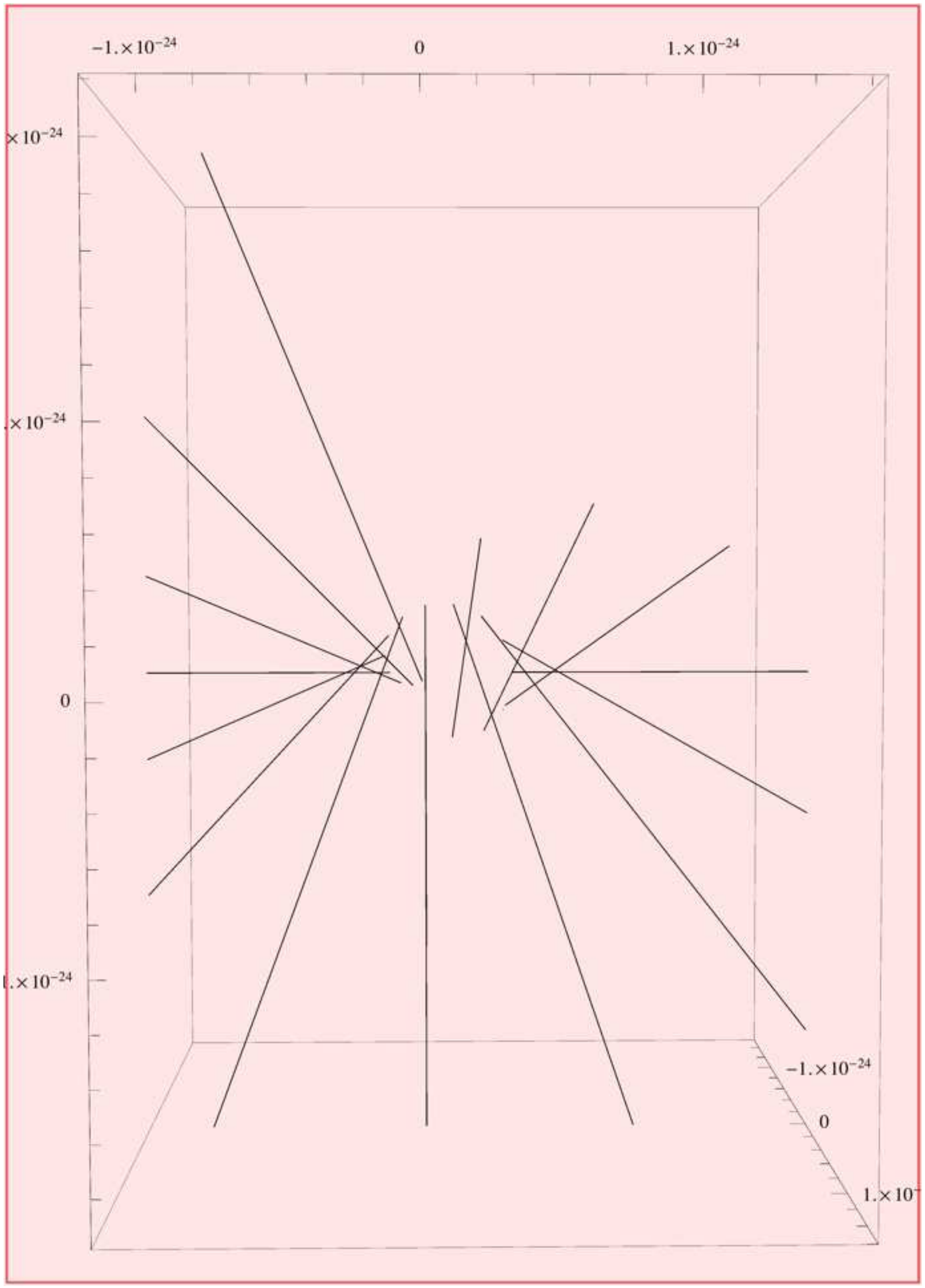


−1.×10^−24
−5.×10^−25
0
5.×10^−25
1.×10^−2
×10^−24
0
−1.×10^−24
1.×10^−24

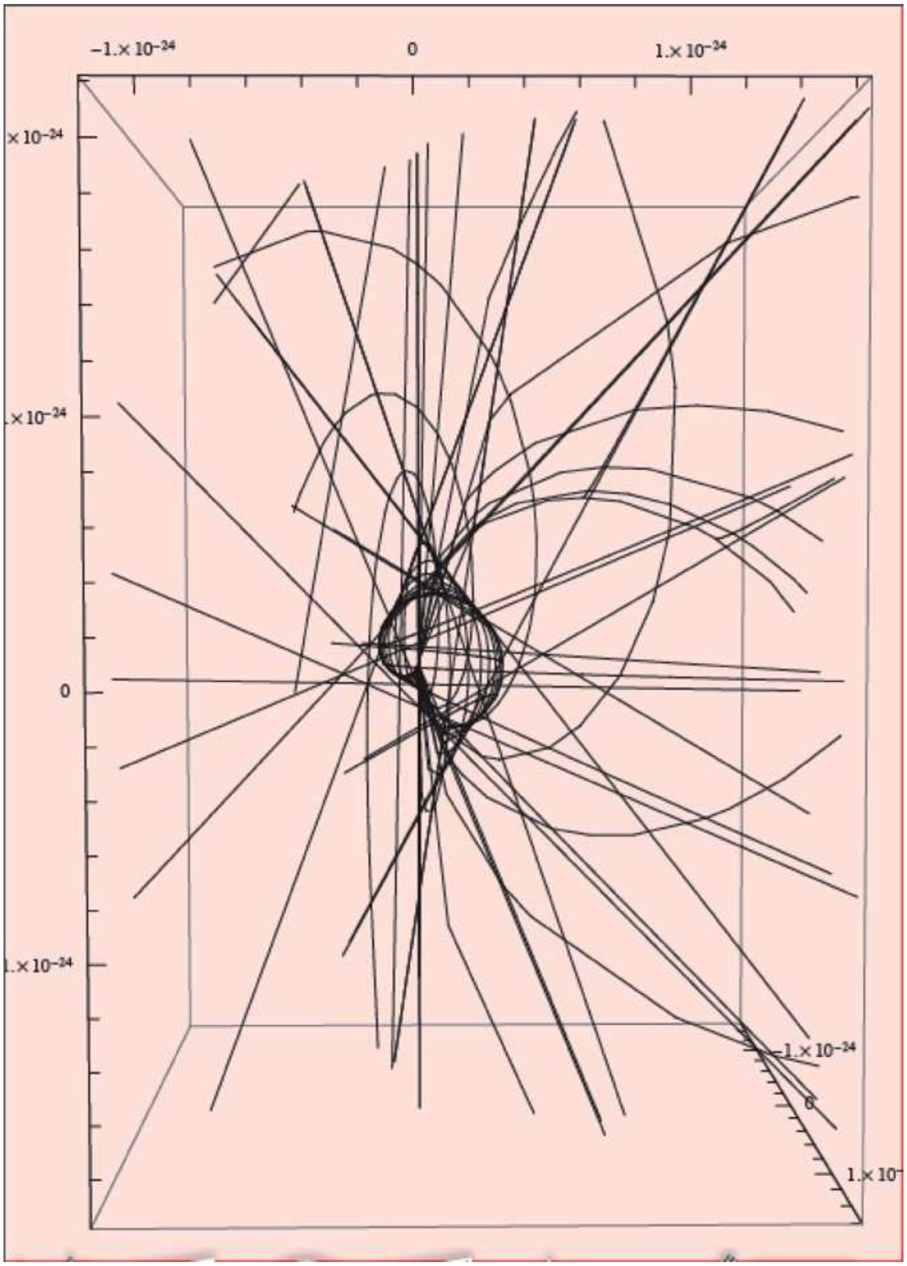

−2.×10^−24
0
−1.×10^−24


−1.×10^−24
−5.×10^−25
0
5.×10^−25
×10^−24
0
−1.×10^−24
−2.×10^−24
1.×10^−24
0


```
SphericalPlot3D[
 .5 (((7.372495999759142`*^-51 (1.6880179234807384`*^10 θ √(-3.5294033336192096`*^16 θ +
            2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2) -
         4.029845948245144`*^9 θ^2 √(-3.5294033336192096`*^16 θ +
            2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2) +
         2.1378996752068695`*^8 θ^3 √(-3.5294033336192096`*^16 θ +
            2.808609933552555`*^15 θ^2 + 1.1087947584453435`*^17 Sin[β]^2))) /
     (-12.566370614359172` θ √(-4.487190804107819`*^15 θ + 3.57079298535128`*^14 θ^2 +
           1.40969256654408`*^16 Sin[β]^2) + 1.` θ^2 √(-4.487190804107819`*^15 θ +
           3.57079298535128`*^14 θ^2 + 1.40969256654408`*^16 Sin[β]^2) + 6.283185307179586`
        √(12.566370614359172` θ - 1.` θ^2) Sin[β] √(-4.487190804107819`*^15 θ +
           3.57079298535128`*^14 θ^2 + 1.40969256654408`*^16 Sin[β]^2)))
    ((√(-1.1294090667581471`*^18 θ + 8.987551787368176`*^16 θ^2 +
          3.5481432270250993`*^18 Sin[β]^2)) /
      (√(-12.566370614359172` θ + θ^2 + 39.47841760435743` Sin[β]^2)))^
    2), {θ, -2 π, 2 π}, {β, -π / 2, π / 2}]
```



−1.×10^{-24}
0
1.×10^{-24}
×10^{-24}
.×10^{-24}
0
l.×10^{-24}
−1.×10^{-24}
0
1.×10



XIX. Ontology of Phenomenological Velocity

Why is the velocity considered phenomenological?

The velocity is considered phenomenological because it comes from manipulation of algebra as a descriptive language for geometry. Logic and perception (visualization) are interpenetrating ideas. This leads to a phenomenon that could normally go unseen. However, there is a normative manner in which the phenomenon is present to the conscious perceiver. Through logical manipulation of algebraic forms, a phenomenon of visualization is brought about in a substitutive adumbrative pattern. Thus, the studies are phenomenological. In the next paragraphs, I will describe how this is a valid interpretation of physical univocity of light particles.

From Section Theorem 3 Section 3 and section XIX, we can form the equation :

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{(v)^2}{c^2}}}\sqrt{\frac{\theta}{\sqrt{1-\frac{(v)^2}{c^2}}}}\sqrt{4\pi r - r\theta}}{2\pi} == r\,\text{Sin}[\beta],\ v\right]$$

$$\left\{\left\{v \to -\frac{1.\sqrt{-1.12941\times10^{18}\,\theta + 8.98755\times10^{16}\,\theta^2 + 3.54814\times10^{18}\,\text{Sin}[\beta]^2}}{\sqrt{-12.5664\,\theta + \theta^2 + 39.4784\,\text{Sin}[\beta]^2}}\right\},\right.$$

$$\left.\left\{v \to \frac{\sqrt{-1.12941\times10^{18}\,\theta + 8.98755\times10^{16}\,\theta^2 + 3.54814\times10^{18}\,\text{Sin}[\beta]^2}}{\sqrt{-12.5664\,\theta + \theta^2 + 39.4784\,\text{Sin}[\beta]^2}}\right\}\right\}$$

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{v^2}{c^2}}}\sqrt{\frac{\theta}{\sqrt{1-\frac{v^2}{c^2}}}}\sqrt{4\pi r - r\theta}}{2\pi}\sqrt{1-\frac{v^2}{c^2}} == c\left((\theta/(2\pi))\Big/\sqrt{1-\frac{v^2}{c^2}}\right),\ v\right]$$

$$\left\{\left\{v \to -\sqrt{-\frac{4c^2\pi r^2}{-4\pi r^2 + r^2\theta} + \frac{c^2 r^2\theta}{-4\pi r^2 + r^2\theta} - \frac{\sqrt{c^6 r^2(4\pi-\theta)\theta}}{-4\pi r^2 + r^2\theta}}\right\},\right.$$

$$\left\{v \to \sqrt{-\frac{4c^2\pi r^2}{-4\pi r^2 + r^2\theta} + \frac{c^2 r^2\theta}{-4\pi r^2 + r^2\theta} - \frac{\sqrt{c^6 r^2(4\pi-\theta)\theta}}{-4\pi r^2 + r^2\theta}}\right\},$$

$$\left\{v \to -\sqrt{-\frac{4c^2\pi r^2}{-4\pi r^2 + r^2\theta} + \frac{c^2 r^2\theta}{-4\pi r^2 + r^2\theta} + \frac{\sqrt{4c^6\pi r^2\theta - c^6 r^2\theta^2}}{-4\pi r^2 + r^2\theta}}\right\},$$

$$\left.\left\{v \to \sqrt{-\frac{4c^2\pi r^2}{-4\pi r^2 + r^2\theta} + \frac{c^2 r^2\theta}{-4\pi r^2 + r^2\theta} + \frac{\sqrt{4c^6\pi r^2\theta - c^6 r^2\theta^2}}{-4\pi r^2 + r^2\theta}}\right\}\right\}$$

$$\text{velocity} = \sqrt{-\frac{4c^2\pi r^2}{-4\pi r^2 + r^2\theta} + \frac{c^2 r^2\theta}{-4\pi r^2 + r^2\theta} + \frac{\sqrt{4c^6\pi r^2\theta - c^6 r^2\theta^2}}{-4\pi r^2 + r^2\theta}} ==$$

$$\left(\sqrt{\left(-1.12941\times10^{18}\,\theta + 8.98755\times10^{16}\,\theta^2 + 3.54814\times10^{18}\,\text{Sin}[\beta]^2\right)}\right)\Big/$$

$$\left(\sqrt{-12.5664\,\theta + \theta^2 + 39.4784\,\text{Sin}[\beta]^2}\right)$$

Every variable in the system can be placed in terms of the speed of light and one other variable. From this, when the speed of light is known, as in a vacuum state, we see that there is a statement that is true but cannot be proven using Euclidean geometry. That statement is the fact that initial radius is a function of the angle taken out of the initial circle alone when the speed of light is known. This is obviously not provable via Euclidean geometry, because one could draw a circle any size and take any wedge out of it by folding. Because one variable in the system is a function of one other variable alone, we find the statement that is true but cannot be proven, and we must rely on faith. This brings scientific inquiries and method to understanding of the spiritual journey. Univocity has several different interpretations. One interpretation is how the number one is expressed through pure geometry. Another meaning is how, when the height of the cone or any other variable in the system is fixed, every other variable in the system can be placed in terms of one other variable. In this interpretation of univocity, we see the relationship of each variable in terms of the speed of light and one other variable. For instance, it is shown that r = f(c, θ). It is also shown that r = f(c, β). It is also shown that, θ = f(c, r).

$$\text{Solve}\left[\sqrt{-\frac{4\,c^2\,\pi\,r^2}{-4\,\pi\,r^2+r^2\,\theta}+\frac{c^2\,r^2\,\theta}{-4\,\pi\,r^2+r^2\,\theta}+\frac{\sqrt{4\,c^6\,\pi\,r^2\,\theta-c^6\,r^2\,\theta^2}}{-4\,\pi\,r^2+r^2\,\theta}} == \left(\sqrt{\left(-1.1294090667581471\text{`*\^{}}18\,\theta+8.987551787368176\text{`*\^{}}16\,\theta^2+3.5481432270250993\text{`*\^{}}18\,\text{Sin}[\beta]^2\right)}\right)\Big/\left(\sqrt{-12.566370614359172\text{`}\,\theta+\theta^2+39.47841760435743\text{`}\,\text{Sin}[\beta]^2}\right),\ r\right]$$

{}

$$\text{Solve}\left[\sqrt{-\frac{4 c^2 \pi r^2}{-4 \pi r^2 + r^2 \theta} + \frac{c^2 r^2 \theta}{-4 \pi r^2 + r^2 \theta} + \frac{\sqrt{4 c^6 \pi r^2 \theta - c^6 r^2 \theta^2}}{-4 \pi r^2 + r^2 \theta}} == \left(\sqrt{\left(-1.1294090667581471\text{`*^18}\, \theta + 8.987551787368176\text{`*^16}\, \theta^2 + 3.5481432270250993\text{`*^18}\, \text{Sin}[\beta]^2\right)}\right) \Big/ \left(\sqrt{-12.566370614359172\text{`}\, \theta + \theta^2 + 39.47841760435743\text{`}\, \text{Sin}[\beta]^2}\right), r\right]$$

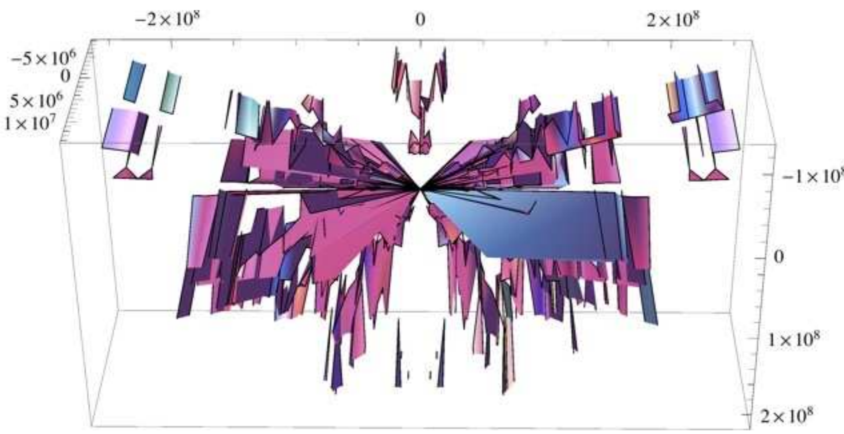
$$\Big\{\Big\{r \to -\Big(1.\, c^3 \sqrt{\theta} \sqrt{\left(1.62943\times10^{45}\theta^2 - 3.88997\times10^{44}\theta^3 + 3.09554\times10^{43}\theta^4 - 8.21118\times10^{41}\theta^5 - 1.0238\times10^{46}\theta\,\text{Sin}[\beta]^2 + 1.62943\times10^{45}\theta^2\,\text{Sin}[\beta]^2 - 6.48329\times10^{43}\theta^3\,\text{Sin}[\beta]^2 + 1.60818\times10^{46}\,\text{Sin}[\beta]^4 - 1.27975\times10^{45}\theta\,\text{Sin}[\beta]^4\right)}\Big) \Big/ \Big(\sqrt{\left(1.65397\times10^{80}\theta^2 - 3.68058\times10^{63}c^2\theta^2 + 2.0476\times10^{46}c^4\theta^2 - 5.26475\times10^{79}\theta^3 + 1.17157\times10^{63}c^2\theta^3 - 6.51771\times10^{45}c^4\theta^3 + 6.28433\times10^{78}\theta^4 - 1.39845\times10^{62}c^2\theta^4 + 7.77994\times10^{44}c^4\theta^4 - 3.33394\times10^{77}\theta^5 + 7.41902\times10^{60}c^2\theta^5 - 4.12739\times10^{43}c^4\theta^5 + 6.63267\times10^{75}\theta^6 - 1.47597\times10^{59}c^2\theta^6 + 8.21118\times10^{41}c^4\theta^6 - 1.03922\times10^{81}\theta\,\text{Sin}[\beta]^2 + 2.31258\times10^{64}c^2\theta\,\text{Sin}[\beta]^2 - 1.28654\times10^{47}c^4\theta\,\text{Sin}[\beta]^2 + 2.48096\times10^{80}\theta^2\,\text{Sin}[\beta]^2 - 5.52087\times10^{63}c^2\theta^2\,\text{Sin}[\beta]^2 + 3.0714\times10^{46}c^4\theta^2\,\text{Sin}[\beta]^2 - 1.97428\times10^{79}\theta^3\,\text{Sin}[\beta]^2 + 4.39337\times10^{62}c^2\theta^3\,\text{Sin}[\beta]^2 - 2.44414\times10^{45}c^4\theta^3\,\text{Sin}[\beta]^2 + 5.23694\times10^{77}\theta^4\,\text{Sin}[\beta]^2 - 1.16538\times10^{61}c^2\theta^4\,\text{Sin}[\beta]^2 + 6.48329\times10^{43}c^4\theta^4\,\text{Sin}[\beta]^2 + 1.6324\times10^{81}\,\text{Sin}[\beta]^4 - 3.63259\times10^{64}c^2\,\text{Sin}[\beta]^4 + 2.0209\times10^{47}c^4\,\text{Sin}[\beta]^4 - 2.59805\times10^{80}\theta\,\text{Sin}[\beta]^4 + 5.78144\times10^{63}c^2\theta\,\text{Sin}[\beta]^4 - 3.21636\times10^{46}c^4\theta\,\text{Sin}[\beta]^4 + 1.03373\times10^{79}\theta^2\,\text{Sin}[\beta]^4 - 2.30036\times10^{62}c^2\theta^2\,\text{Sin}[\beta]^4 + 1.27975\times10^{45}c^4\theta^2\,\text{Sin}[\beta]^4\right)}\Big)\Big\},$$

$$\Big\{r \to \Big(c^3 \sqrt{\theta} \sqrt{\left(1.62943\times10^{45}\theta^2 - 3.88997\times10^{44}\theta^3 + 3.09554\times10^{43}\theta^4 - 8.21118\times10^{41}\theta^5 - 1.0238\times10^{46}\theta\,\text{Sin}[\beta]^2 + 1.62943\times10^{45}\theta^2\,\text{Sin}[\beta]^2 - 6.48329\times10^{43}\theta^3\,\text{Sin}[\beta]^2 + 1.60818\times10^{46}\,\text{Sin}[\beta]^4 - 1.27975\times10^{45}\theta\,\text{Sin}[\beta]^4\right)}\Big) \Big/ \Big(\sqrt{\left(1.65397\times10^{80}\theta^2 - 3.68058\times10^{63}c^2\theta^2 + 2.0476\times10^{46}c^4\theta^2 - 5.26475\times10^{79}\theta^3 + 1.17157\times10^{63}c^2\theta^3 - 6.51771\times10^{45}c^4\theta^3 + 6.28433\times10^{78}\theta^4 - 1.39845\times10^{62}c^2\theta^4 + 7.77994\times10^{44}c^4\theta^4 - 3.33394\times10^{77}\theta^5 + 7.41902\times10^{60}c^2\theta^5 - 4.12739\times10^{43}c^4\theta^5 + 6.63267\times10^{75}\theta^6 - 1.47597\times10^{59}c^2\theta^6 + 8.21118\times10^{41}c^4\theta^6 - 1.03922\times10^{81}\theta\,\text{Sin}[\beta]^2 + 2.31258\times10^{64}c^2\theta\,\text{Sin}[\beta]^2 - 1.28654\times10^{47}c^4\theta\,\text{Sin}[\beta]^2 + 2.48096\times10^{80}\theta^2\,\text{Sin}[\beta]^2 - 5.52087\times10^{63}c^2\theta^2\,\text{Sin}[\beta]^2 + 3.0714\times10^{46}c^4\theta^2\,\text{Sin}[\beta]^2 - 1.97428\times10^{79}\theta^3\,\text{Sin}[\beta]^2 + 4.39337\times10^{62}c^2\theta^3\,\text{Sin}[\beta]^2 - 2.44414\times10^{45}c^4\theta^3\,\text{Sin}[\beta]^2 + 5.23694\times10^{77}\theta^4\,\text{Sin}[\beta]^2 - 1.16538\times10^{61}c^2\theta^4\,\text{Sin}[\beta]^2 + 6.48329\times10^{43}c^4\theta^4\,\text{Sin}[\beta]^2 + 1.6324\times10^{81}\,\text{Sin}[\beta]^4 - 3.63259\times10^{64}c^2\,\text{Sin}[\beta]^4 + 2.0209\times10^{47}c^4\,\text{Sin}[\beta]^4 - 2.59805\times10^{80}\theta\,\text{Sin}[\beta]^4 + 5.78144\times10^{63}c^2\theta\,\text{Sin}[\beta]^4 - 3.21636\times10^{46}c^4\theta\,\text{Sin}[\beta]^4 + 1.03373\times10^{79}\theta^2\,\text{Sin}[\beta]^4 - 2.30036\times10^{62}c^2\theta^2\,\text{Sin}[\beta]^4 + 1.27975\times10^{45}c^4\theta^2\,\text{Sin}[\beta]^4\right)}\Big)\Big\}\Big\}$$

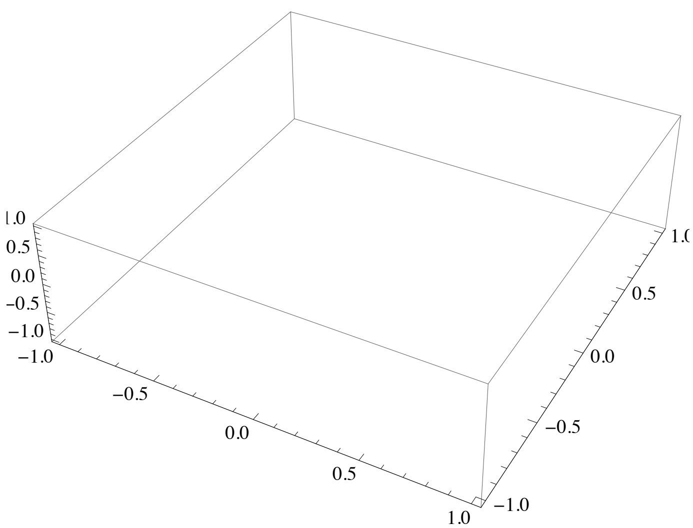
$$\theta := 2\left(\pi + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)$$

```
c := 2.99792458 * (10^8)
```

```
RevolutionPlot3D[
 (c^3 Sqrt[θ] Sqrt[(1.6294275228490475`*^45 θ^2 - 3.889971670071122`*^44 θ^3 + 3.0955410988962723`*^43 θ^4 -
      8.211177790557656`*^41 θ^5 - 1.0237995070679164`*^46 θ Sin[β]^2 +
      1.6294275228490475`*^45 θ^2 Sin[β]^2 - 6.483286116785203`*^43 θ^3 Sin[β]^2 +
      1.6081805050767087`*^46 Sin[β]^4 - 1.2797493838348955`*^45 θ Sin[β]^4)]) /
  (Sqrt[(1.6539703637441768`*^80 θ^2 - 3.680580435861964`*^63 c^2 θ^2 + 2.0475990141358328`*^46 c^4 θ^2 -
     5.264751182347718`*^79 θ^3 + 1.1715651396295084`*^63 c^2 θ^3 - 6.51771009139619`*^45 c^4 θ^3 +
     6.2843338111465474`*^78 θ^4 - 1.3984528734463713`*^62 c^2 θ^4 +
     7.779943340142244`*^44 c^4 θ^4 - 3.333942633611675`*^77 θ^5 + 7.419022916335157`*^60 c^2 θ^5 -
     4.1273881318616967`*^43 c^4 θ^5 + 6.632668126551373`*^75 θ^6 - 1.4759677125584868`*^59 c^2 θ^6 +
     8.211177790557656`*^41 c^4 θ^6 - 1.0392202287987887`*^81 θ Sin[β]^2 +
     2.312576891650053`*^64 c^2 θ Sin[β]^2 - 1.286544404061367`*^47 c^4 θ Sin[β]^2 +
     2.480955545616265`*^80 θ^2 Sin[β]^2 - 5.520870653792946`*^63 c^2 θ^2 Sin[β]^2 +
     3.071398521203749`*^46 c^4 θ^2 Sin[β]^2 - 1.974281693380394`*^79 θ^3 Sin[β]^2 +
     4.393369273610656`*^62 c^2 θ^3 Sin[β]^2 - 2.444141284273571`*^45 c^4 θ^3 Sin[β]^2 +
     5.236944842622124`*^77 θ^4 Sin[β]^2 - 1.1653773945386426`*^61 c^2 θ^4 Sin[β]^2 +
     6.483286116785203`*^43 c^4 θ^4 Sin[β]^2 + 1.6324033181280892`*^81 Sin[β]^4 -
     3.6325872868346625`*^64 c^2 Sin[β]^4 + 2.0208992241581245`*^47 c^4 Sin[β]^4 -
     2.5980505719969717`*^80 θ Sin[β]^4 + 5.7814422291251325`*^63 c^2 θ Sin[β]^4 -
     3.2163610101534174`*^46 c^4 θ Sin[β]^4 + 1.0337314773401105`*^79 θ^2 Sin[β]^4 -
     2.3003627724137275`*^62 c^2 θ^2 Sin[β]^4 + 1.2797493838348955`*^45 c^4 θ^2 Sin[β]^4)]),
 {c, -3 * 10^8, 3 * 10^8}, {β, -π / 2, π / 2}]
```

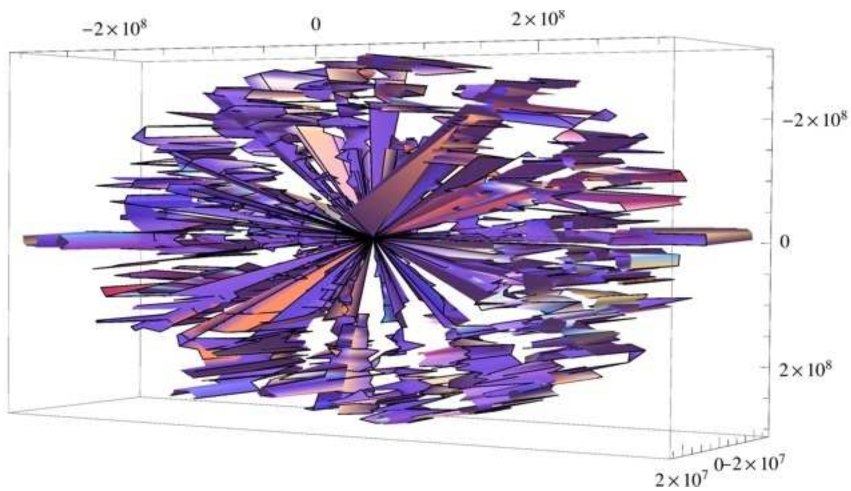


```
RevolutionPlot3D[
 (c^3 Sqrt[θ] Sqrt[(1.6294275228490475`*^45 θ^2 - 3.889971670071122`*^44 θ^3 + 3.0955410988962723`*^43 θ^4 -
      8.211177790557656`*^41 θ^5 - 1.0237995070679164`*^46 θ Sin[β]^2 +
      1.6294275228490475`*^45 θ^2 Sin[β]^2 - 6.483286116785203`*^43 θ^3 Sin[β]^2 +
      1.6081805050767087`*^46 Sin[β]^4 - 1.2797493838348955`*^45 θ Sin[β]^4)]) /
  (Sqrt[(1.6539703637441768`*^80 θ^2 - 3.680580435861964`*^63 c^2 θ^2 + 2.0475990141358328`*^46 c^4 θ^2 -
     5.264751182347718`*^79 θ^3 + 1.1715651396295084`*^63 c^2 θ^3 - 6.51771009139619`*^45 c^4 θ^3 +
     6.2843338111465474`*^78 θ^4 - 1.3984528734463713`*^62 c^2 θ^4 + 7.779943340142244`*^44 c^4 θ^4 -
     3.333942633611675`*^77 θ^5 + 7.419022916335157`*^60 c^2 θ^5 - 4.1273881318616967`*^43 c^4 θ^5 +
     6.632668126551373`*^75 θ^6 - 1.4759677125584868`*^59 c^2 θ^6 + 8.211177790557656`*^41 c^4 θ^6 -
     1.0392202287987887`*^81 θ Sin[β]^2 + 2.312576891650053`*^64 c^2 θ Sin[β]^2 -
     1.286544404061367`*^47 c^4 θ Sin[β]^2 + 2.480955545616265`*^80 θ^2 Sin[β]^2 -
     5.520870653792946`*^63 c^2 θ^2 Sin[β]^2 + 3.071398521203749`*^46 c^4 θ^2 Sin[β]^2 -
     1.974281693380394`*^79 θ^3 Sin[β]^2 + 4.393369273610656`*^62 c^2 θ^3 Sin[β]^2 -
     2.444141284273571`*^45 c^4 θ^3 Sin[β]^2 + 5.236944842622124`*^77 θ^4 Sin[β]^2 -
     1.1653773945386426`*^61 c^2 θ^4 Sin[β]^2 + 6.483286116785203`*^43 c^4 θ^4 Sin[β]^2 +
     1.6324033181280892`*^81 Sin[β]^4 - 3.6325872868346625`*^64 c^2 Sin[β]^4 +
     2.0208992241581245`*^47 c^4 Sin[β]^4 - 2.5980505719969717`*^80 θ Sin[β]^4 +
     5.7814422291251325`*^63 c^2 θ Sin[β]^4 - 3.2163610101534174`*^46 c^4 θ Sin[β]^4 +
     1.0337314773401105`*^79 θ^2 Sin[β]^4 - 2.3003627724137275`*^62 c^2 θ^2 Sin[β]^4 +
     1.2797493838348955`*^45 c^4 θ^2 Sin[β]^4)]), {β, -π / 2, π / 2}]
```

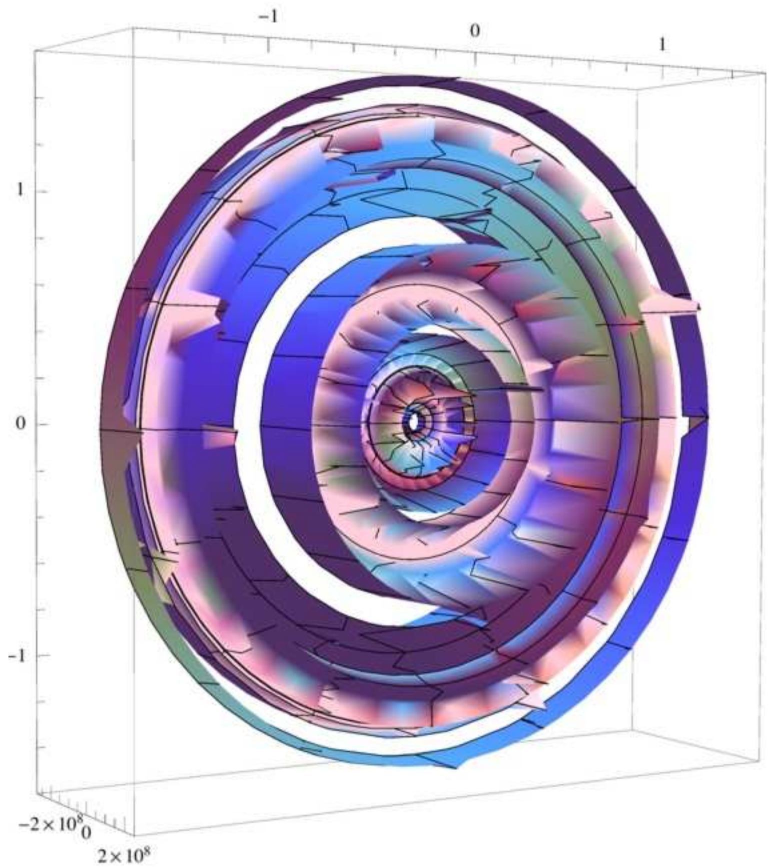


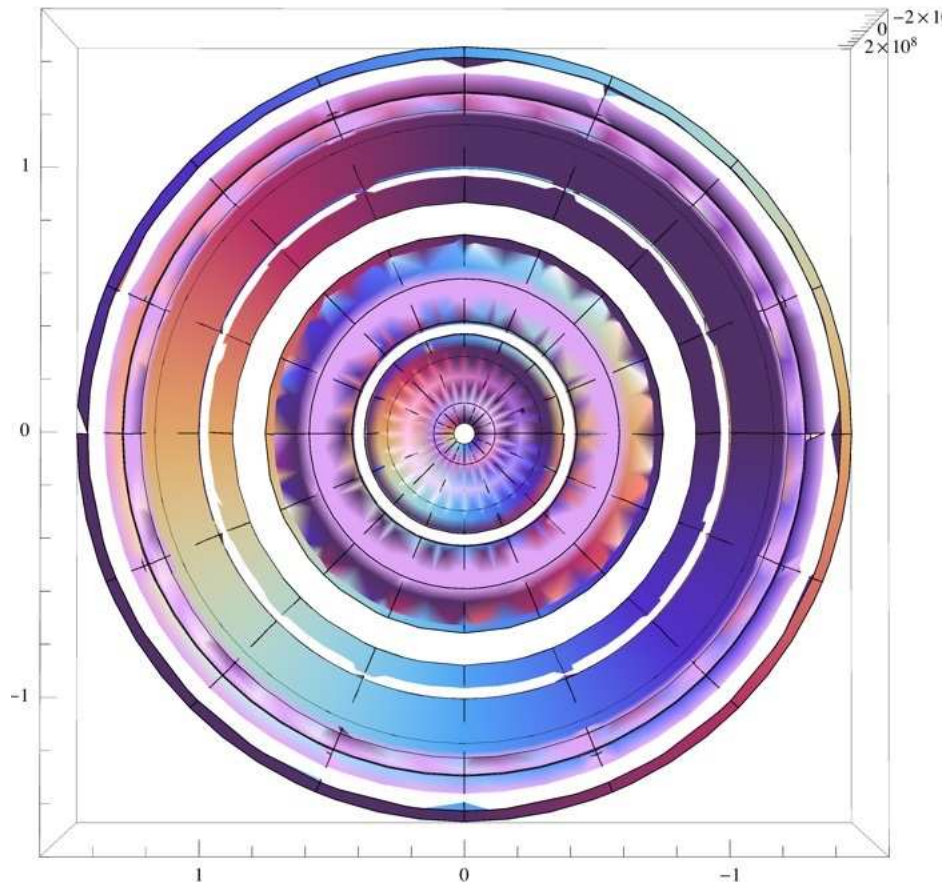

```
β := ArcSin[Sqrt[(4 π - θ) θ] / (2 π)]
```

```
RevolutionPlot3D[
 (c^3 Sqrt[θ] √(1.6294275228490475`*^45 θ^2 - 3.889971670071122`*^44 θ^3 + 3.0955410988962723`*^43 θ^4 -
      8.211177790557656`*^41 θ^5 - 1.0237995070679164`*^46 θ Sin[β]^2 +
      1.6294275228490475`*^45 θ^2 Sin[β]^2 - 6.483286116785203`*^43 θ^3 Sin[β]^2 +
      1.6081805050767087`*^46 Sin[β]^4 - 1.2797493838348955`*^45 θ Sin[β]^4)) /
  (√(1.6539703637441768`*^80 θ^2 - 3.680580435861964`*^63 c^2 θ^2 + 2.0475990141358328`*^46 c^4 θ^2 -
     5.264751182347718`*^79 θ^3 + 1.1715651396295084`*^63 c^2 θ^3 - 6.51771009139619`*^45 c^4 θ^3 +
     6.2843338111465474`*^78 θ^4 - 1.3984528734463713`*^62 c^2 θ^4 +
     7.779943340142244`*^44 c^4 θ^4 - 3.333942633611675`*^77 θ^5 + 7.419022916335157`*^60 c^2 θ^5 -
     4.1273881318616967`*^43 c^4 θ^5 + 6.632668126551373`*^75 θ^6 - 1.4759677125584868`*^59 c^2 θ^6 +
     8.211177790557656`*^41 c^4 θ^6 - 1.0392202287987887`*^81 θ Sin[β]^2 +
     2.312576891650053`*^64 c^2 θ Sin[β]^2 - 1.286544404061367`*^47 c^4 θ Sin[β]^2 +
     2.480955545616265`*^80 θ^2 Sin[β]^2 - 5.520870653792946`*^63 c^2 θ^2 Sin[β]^2 +
     3.071398521203749`*^46 c^4 θ^2 Sin[β]^2 - 1.974281693380394`*^79 θ^3 Sin[β]^2 +
     4.393369273610656`*^62 c^2 θ^3 Sin[β]^2 - 2.444141284273571`*^45 c^4 θ^3 Sin[β]^2 +
     5.236944842622124`*^77 θ^4 Sin[β]^2 - 1.1653773945386426`*^61 c^2 θ^4 Sin[β]^2 +
     6.483286116785203`*^43 c^4 θ^4 Sin[β]^2 + 1.6324033181280892`*^81 Sin[β]^4 -
     3.6325872868346625`*^64 c^2 Sin[β]^4 + 2.0208992241581245`*^47 c^4 Sin[β]^4 -
     2.5980505719969717`*^80 θ Sin[β]^4 + 5.7814422291251325`*^63 c^2 θ Sin[β]^4 -
     3.2163610101534174`*^46 c^4 θ Sin[β]^4 + 1.0337314773401105`*^79 θ^2 Sin[β]^4 -
     2.3003627724137275`*^62 c^2 θ^2 Sin[β]^4 + 1.2797493838348955`*^45 c^4 θ^2 Sin[β]^4)),
 {c, -3 * 10^8, 3 * 10^8}, {θ, -π / 2, π / 2}]
```



```
RevolutionPlot3D[
 (c^3 Sqrt[θ] Sqrt[(1.6294275228490475`*^45 θ^2 - 3.889971670071122`*^44 θ^3 + 3.0955410988962723`*^43 θ^4 -
       8.211177790557656`*^41 θ^5 - 1.0237995070679164`*^46 θ Sin[β]^2 +
       1.6294275228490475`*^45 θ^2 Sin[β]^2 - 6.483286116785203`*^43 θ^3 Sin[β]^2 +
       1.6081805050767087`*^46 Sin[β]^4 - 1.2797493838348955`*^45 θ Sin[β]^4)]) /
  (Sqrt[(1.6539703637441768`*^80 θ^2 - 3.680580435861964`*^63 c^2 θ^2 + 2.0475990141358328`*^46 c^4 θ^2 -
      5.264751182347718`*^79 θ^3 + 1.1715651396295084`*^63 c^2 θ^3 - 6.51771009139619`*^45 c^4 θ^3 +
      6.2843338111465474`*^78 θ^4 - 1.3984528734463713`*^62 c^2 θ^4 + 7.779943340142244`*^44 c^4 θ^4 -
      3.333942633611675`*^77 θ^5 + 7.419022916335157`*^60 c^2 θ^5 - 4.1273881318616967`*^43 c^4 θ^5 +
      6.632668126551373`*^75 θ^6 - 1.4759677125584868`*^59 c^2 θ^6 + 8.211177790557656`*^41 c^4 θ^6 -
      1.0392202287987887`*^81 θ Sin[β]^2 + 2.312576891650053`*^64 c^2 θ Sin[β]^2 -
      1.286544404061367`*^47 c^4 θ Sin[β]^2 + 2.480955545616265`*^80 θ^2 Sin[β]^2 -
      5.520870653792946`*^63 c^2 θ^2 Sin[β]^2 + 3.071398521203749`*^46 c^4 θ^2 Sin[β]^2 -
      1.974281693380394`*^79 θ^3 Sin[β]^2 + 4.393369273610656`*^62 c^2 θ^3 Sin[β]^2 -
      2.444141284273571`*^45 c^4 θ^3 Sin[β]^2 + 5.236944842622124`*^77 θ^4 Sin[β]^2 -
      1.1653773945386426`*^61 c^2 θ^4 Sin[β]^2 + 6.483286116785203`*^43 c^4 θ^4 Sin[β]^2 +
      1.6324033181280892`*^81 Sin[β]^4 - 3.6325872868346625`*^64 c^2 Sin[β]^4 +
      2.0208992241581245`*^47 c^4 Sin[β]^4 - 2.5980505719969717`*^80 θ Sin[β]^4 +
      5.7814422291251325`*^63 c^2 θ Sin[β]^4 - 3.2163610101534174`*^46 c^4 θ Sin[β]^4 +
      1.0337314773401105`*^79 θ^2 Sin[β]^4 - 2.3003627724137275`*^62 c^2 θ^2 Sin[β]^4 +
      1.2797493838348955`*^45 c^4 θ^2 Sin[β]^4)]), {θ, -π / 2, π / 2}]
```





$$\text{Solve}\left[\sqrt{-\frac{4\,c^2\,\pi\,r^2}{-4\,\pi\,r^2+r^2\,\theta}+\frac{c^2\,r^2\,\theta}{-4\,\pi\,r^2+r^2\,\theta}+\frac{\sqrt{4\,c^6\,\pi\,r^2\,\theta-c^6\,r^2\,\theta^2}}{-4\,\pi\,r^2+r^2\,\theta}} ==\right.$$

```
(√(-1.1294090667581471`*^18 θ + 8.987551787368176`*^16 θ^2 +
     3.5481432270250993`*^18 Sin[β]^2)) /
  (√(-12.566370614359172` θ + θ^2 + 39.47841760435743` Sin[β]^2)), θ]
```

$$\left\{\left\{\theta \to \frac{\left(1.01506\times10^{35}-2.25882\times10^{18}\,c^2+12.5664\,c^4\right)\,r^2}{c^6+8.07761\times10^{33}\,r^2-1.79751\times10^{17}\,c^2\,r^2+c^4\,r^2}\right\}\right\}$$

N[2.2588181335162944`*^18]

2.25882×10^{18}

2.2588181335162944`*^18

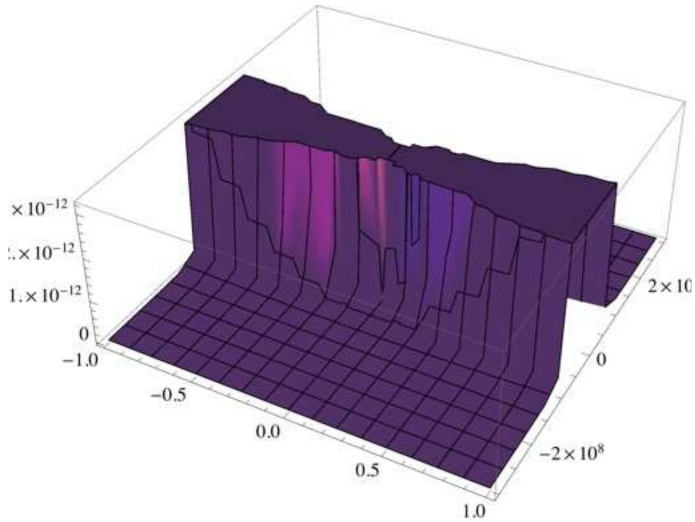
2.25882×10^{18}

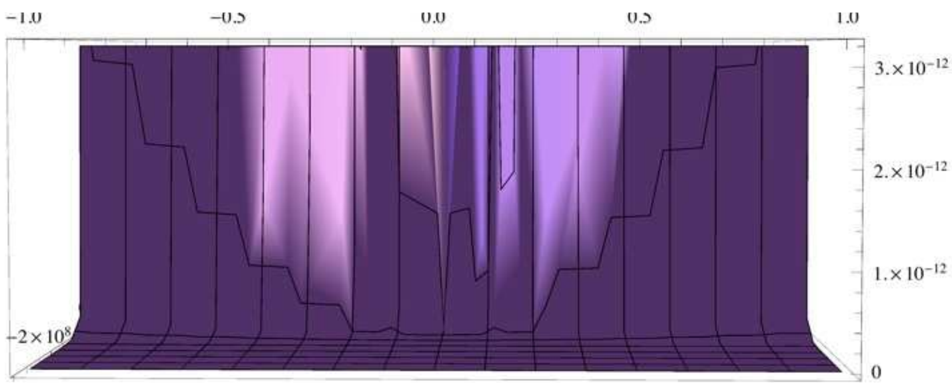
N[2.2588181335162944]

2.25882

35 481 432 270 250 993

```
Plot3D[((1.015062247661201`*^35 - 2.2588181335162944`*^18 c^2 + 12.566370614359172` c^4) r^2) /
  (c^6 + 8.07760871306249`*^33 r^2 - 1.7975103574736352`*^17 c^2 r^2 + c^4 r^2),
 {r, -1, 1}, {c, -3.0 * 10^8, 3.0 * 10^8}]
```





$$\text{Solve}\Big[\sqrt{-\frac{4\,c^2\,\pi\,r^2}{-4\,\pi\,r^2+r^2\,\theta}+\frac{c^2\,r^2\,\theta}{-4\,\pi\,r^2+r^2\,\theta}+\frac{\sqrt{4\,c^6\,\pi\,r^2\,\theta-c^6\,r^2\,\theta^2}}{-4\,\pi\,r^2+r^2\,\theta}} ==$$

$$\Big(\sqrt{}\,(-1.1294090667581471\text{`*^}18\,\theta+8.987551787368176\text{`*^}16\,\theta^2+3.5481432270250993\text{`*^}18\,\text{Sin}[\beta]^2)\Big)\Big/\left(\sqrt{-12.566370614359172\text{`}\,\theta+\theta^2+39.47841760435743\text{`}\,\text{Sin}[\beta]^2}\right),\ \beta\Big]$$

{{}}

$$\text{Solve}\Big[\sqrt{-\frac{4\,c^2\,\pi\,r^2}{-4\,\pi\,r^2+r^2\,\theta}+\frac{c^2\,r^2\,\theta}{-4\,\pi\,r^2+r^2\,\theta}+\frac{\sqrt{4\,c^6\,\pi\,r^2\,\theta-c^6\,r^2\,\theta^2}}{-4\,\pi\,r^2+r^2\,\theta}} ==$$

$$\Big(\sqrt{}\,(-1.1294090667581471\text{`*^}18\,\theta+8.987551787368176\text{`*^}16\,\theta^2+3.5481432270250993\text{`*^}18\,\text{Sin}[\beta]^2)\Big)\Big/\left(\sqrt{-12.566370614359172\text{`}\,\theta+\theta^2+39.47841760435743\text{`}\,\text{Sin}[\beta]^2}\right),\ c\Big]$$

$\{\{c \to$

$\text{Root}[1.65397036374417683783049055835 64 \times 10^{80}\, r^2\, \theta^2 - 5.26475118234771776821055787576 95 \times 10^{79}$

$r^2\, \theta^3 + 6.2843338111465477438022090943599 \times 10^{78}\, r^2\, \theta^4 -$

$3.3339426336116751332792945998413 \times 10^{77}\, r^2\, \theta^5 + 6.6326681265513728159832666914994 \times 10^{75}$

$r^2\, \theta^6 - 1.0392202287987887421152167160698 \times 10^{81}\, r^2\, \theta\, \text{Sin}[\beta]^2 +$

$2.4809555456162651150679275533446 \times 10^{80}\, r^2\, \theta^2\, \text{Sin}[\beta]^2 -$

$1.9742816933803941278907918143229 \times 10^{79}\, r^2\, \theta^3\, \text{Sin}[\beta]^2 +$

$5.2369448426221232322178822848048 \times 10^{77}\, r^2\, \theta^4\, \text{Sin}[\beta]^2 +$

$1.6324033181280892853402600163092 \times 10^{81}\, r^2\, \text{Sin}[\beta]^4 -$

$2.5980505719969718552880417901745 \times 10^{80}\, r^2\, \theta\, \text{Sin}[\beta]^4 +$

$1.0337314773401105236440565989727 \times 10^{79}\, r^2\, \theta^2\, \text{Sin}[\beta]^4 +$

$(-3.6805804358619639152542428946229 \times 10^{63}\, r^2\, \theta^2 + 1.1715651396295084343350277150183 \times 10^{63}$

$r^2\, \theta^3 - 1.3984528734463712417367400517893 \times 10^{62}\, r^2\, \theta^4 +$

$7.4190229163351575984264269033015 \times 10^{60}\, r^2\, \theta^5 - 1.4759677125584867290209028281013 \times$

$10^{59}\, r^2\, \theta^6 + 2.3125768916500528710263464701776 \times 10^{64}\, r^2\, \theta\, \text{Sin}[\beta]^2 -$

$5.5208706537929457853006379110093 \times 10^{63}\, r^2\, \theta^2\, \text{Sin}[\beta]^2 +$

$4.3933692736106565513513281036903 \times 10^{62}\, r^2\, \theta^3\, \text{Sin}[\beta]^2 -$

$1.1653773945386426788696894124088 \times 10^{61}\, r^2\, \theta^4\, \text{Sin}[\beta]^2 -$

$3.6325872868346624737226508141661 \times 10^{64}\, r^2\, \text{Sin}[\beta]^4 +$

$5.7814422291251321775658661754441 \times 10^{63}\, r^2\, \theta\, \text{Sin}[\beta]^4 -$

$2.3003627724137274470339018091393 \times 10^{62}\, r^2\, \theta^2\, \text{Sin}[\beta]^4)\, \#1^2 +$

$(2.0475990141358326793862229391694 \times 10^{46}\, r^2\, \theta^2 - 6.5177100913961894138291231822509 \times 10^{45}$

$r^2\, \theta^3 + 7.7799433401422439744648369436701 \times 10^{44}\, r^2\, \theta^4 -$

$4.1273881318616966226529531247028 \times 10^{43}\, r^2\, \theta^5 + 8.2111777905576565504833386109 90 \times$

$10^{41}\, r^2\, \theta^6 - 1.2865444040613669693619145948318 \times 10^{47}\, r^2\, \theta\, \text{Sin}[\beta]^2 +$

$3.0713985212037490970283649740038 \times 10^{46}\, r^2\, \theta^2\, \text{Sin}[\beta]^2 -$

$2.4441412842735709876238193943913 \times 10^{45}\, r^2\, \theta^3\, \text{Sin}[\beta]^2 +$

$6.4832861167852029217157336168513 \times 10^{43}\, r^2\, \theta^4\, \text{Sin}[\beta]^2 +$

$2.0208992241581244181716190885021 \times 10^{47}\, r^2\, \text{Sin}[\beta]^4 -$

$3.2163610101534174234047864870794 \times 10^{46}\, r^2\, \theta\, \text{Sin}[\beta]^4 +$

$1.2797493838348954246163893369809 \times 10^{45}\, r^2\, \theta^2\, \text{Sin}[\beta]^4)\, \#1^4 +$

$(-1.6294275228490474905680650795677 \times 10^{45}\, \theta^3 + 3.8899716700711220883495200210324 \times 10^{44}$

$\theta^4 - 3.0955410988962724606297290957727 \times 10^{43}\, \theta^5 +$

$8.2111777905576565504833386109 90 \times 10^{41}\, \theta^6 + 1.0237995070679164258423579964977 \times 10^{46}$

$\theta^2\, \text{Sin}[\beta]^2 - 1.6294275228490473484356373294105 \times 10^{45}\, \theta^3\, \text{Sin}[\beta]^2 +$

$6.4832861167852029217157336168513 \times 10^{43}\, \theta^4\, \text{Sin}[\beta]^2 -$

$1.6081805050767087858042877996039 \times 10^{46}\, \theta\, \text{Sin}[\beta]^4 +$

$1.2797493838348954246163893369809 \times 10^{45}\, \theta^2\, \text{Sin}[\beta]^4)\, \#1^6\, \&,\, 1]\},$

$\{c \to \text{Root}[1.6539703637441768378304905583564 \times 10^{80}\, r^2\, \theta^2 -$

$5.2647511823477177682105578757695 \times 10^{79}\, r^2\, \theta^3 +$

$6.2843338111465477438022090943599 \times 10^{78}\, r^2\, \theta^4 -$

$3.3339426336116751332792945998413 \times 10^{77}\, r^2\, \theta^5 +$
$6.6326681265513728159832666914994 \times 10^{75}\, r^2\, \theta^6 -$
$1.0392202287987887421152167160698 \times 10^{81}\, r^2\, \theta\, \text{Sin}[\beta]^2 +$
$2.4809555456162651150679275533446 \times 10^{80}\, r^2\, \theta^2\, \text{Sin}[\beta]^2 -$
$1.9742816933803941278907918143229 \times 10^{79}\, r^2\, \theta^3\, \text{Sin}[\beta]^2 +$
$5.2369448426221232322178822848048 \times 10^{77}\, r^2\, \theta^4\, \text{Sin}[\beta]^2 +$
$1.6324033181280892853402600163092 \times 10^{81}\, r^2\, \text{Sin}[\beta]^4 -$
$2.5980505719969718552880417901745 \times 10^{80}\, r^2\, \theta\, \text{Sin}[\beta]^4 +$
$1.0337314773401105236440565989727 \times 10^{79}\, r^2\, \theta^2\, \text{Sin}[\beta]^4 +$
$(-3.6805804358619639152542428946229 \times 10^{63}\, r^2\, \theta^2 + 1.1715651396295084343350277150183 \times 10^{63}$
$r^2\, \theta^3 - 1.3984528734463712417367400517893 \times 10^{62}\, r^2\, \theta^4 +$
$7.4190229163351575984264269033015 \times 10^{60}\, r^2\, \theta^5 - 1.4759677125584867290209028281013 \times$
$10^{59}\, r^2\, \theta^6 + 2.3125768916500528710263464701776 \times 10^{64}\, r^2\, \theta\, \text{Sin}[\beta]^2 -$
$5.5208706537929457853006379110093 \times 10^{63}\, r^2\, \theta^2\, \text{Sin}[\beta]^2 +$
$4.3933692736106565513513281036903 \times 10^{62}\, r^2\, \theta^3\, \text{Sin}[\beta]^2 -$
$1.1653773945386426788696894124088 \times 10^{61}\, r^2\, \theta^4\, \text{Sin}[\beta]^2 -$
$3.6325872868346624737226508141661 \times 10^{64}\, r^2\, \text{Sin}[\beta]^4 +$
$5.7814422291251321775658661754441 \times 10^{63}\, r^2\, \theta\, \text{Sin}[\beta]^4 -$
$2.3003627724137274470339018091393 \times 10^{62}\, r^2\, \theta^2\, \text{Sin}[\beta]^4)\, \#1^2 +$
$(2.0475990141358326793862229391694 \times 10^{46}\, r^2\, \theta^2 - 6.5177100913961894138291231822509 \times 10^{45}$
$r^2\, \theta^3 + 7.7799433401422439744648369436701 \times 10^{44}\, r^2\, \theta^4 -$
$4.1273881318616966226529531247028 \times 10^{43}\, r^2\, \theta^5 + 8.2111777905576565504833386109990 \times$
$10^{41}\, r^2\, \theta^6 - 1.2865444040613669693619145948318 \times 10^{47}\, r^2\, \theta\, \text{Sin}[\beta]^2 +$
$3.0713985212037490970283649740038 \times 10^{46}\, r^2\, \theta^2\, \text{Sin}[\beta]^2 -$
$2.4441412842735709876238193943913 \times 10^{45}\, r^2\, \theta^3\, \text{Sin}[\beta]^2 +$
$6.4832861167852029217157336168513 \times 10^{43}\, r^2\, \theta^4\, \text{Sin}[\beta]^2 +$
$2.0208992241581244181716190885021 \times 10^{47}\, r^2\, \text{Sin}[\beta]^4 -$
$3.2163610101534174234047864870794 \times 10^{46}\, r^2\, \theta\, \text{Sin}[\beta]^4 +$
$1.2797493838348954246163893369809 \times 10^{45}\, r^2\, \theta^2\, \text{Sin}[\beta]^4)\, \#1^4 +$
$(-1.6294275228490474905680650795677 \times 10^{45}\, \theta^3 + 3.8899716700711220883495200210324 \times 10^{44}$
$\theta^4 - 3.0955410988962724606297290957727 \times 10^{43}\, \theta^5 +$
$8.2111777905576565504833386109990 \times 10^{41}\, \theta^6 + 1.0237995070679164258423579964977 \times 10^{46}$
$\theta^2\, \text{Sin}[\beta]^2 - 1.6294275228490473484356373294105 \times 10^{45}\, \theta^3\, \text{Sin}[\beta]^2 +$
$6.4832861167852029217157336168513 \times 10^{43}\, \theta^4\, \text{Sin}[\beta]^2 -$
$1.6081805050767087858042877996039 \times 10^{46}\, \theta\, \text{Sin}[\beta]^4 +$
$1.2797493838348954246163893369809 \times 10^{45}\, \theta^2\, \text{Sin}[\beta]^4)\, \#1^6\ \&,\ 2]\},$
$\{c \to \text{Root}[1.6539703637441768378304905583564 \times 10^{80}\, r^2\, \theta^2 -$
$5.2647511823477177682105578757695 \times 10^{79}\, r^2\, \theta^3 +$
$6.2843338111465477438022090943599 \times 10^{78}\, r^2\, \theta^4 -$
$3.3339426336116751332792945998413 \times 10^{77}\, r^2\, \theta^5 +$
$6.6326681265513728159832666914994 \times 10^{75}\, r^2\, \theta^6 -$
$1.0392202287987887421152167160698 \times 10^{81}\, r^2\, \theta\, \text{Sin}[\beta]^2 +$

$2.4809555456162651150679275533446 \times 10^{80}\, r^2\, \theta^2\, \mathrm{Sin}[\beta]^2 -$
$1.9742816933803941278907918143229 \times 10^{79}\, r^2\, \theta^3\, \mathrm{Sin}[\beta]^2 +$
$5.2369448426222123232217882284804 8 \times 10^{77}\, r^2\, \theta^4\, \mathrm{Sin}[\beta]^2 +$
$1.6324033181280892853402600163092 \times 10^{81}\, r^2\, \mathrm{Sin}[\beta]^4 -$
$2.5980505719969718552880417901745 \times 10^{80}\, r^2\, \theta\, \mathrm{Sin}[\beta]^4 +$
$1.0337314773401105236440565989727 \times 10^{79}\, r^2\, \theta^2\, \mathrm{Sin}[\beta]^4 +$
$\big(-3.6805804358619639152542428946229 \times 10^{63}\, r^2\, \theta^2 + 1.1715651396295084343350277150183 \times 10^{63}$
$r^2\, \theta^3 - 1.3984528734463712417367400517893 \times 10^{62}\, r^2\, \theta^4 +$
$7.4190229163351575984264269033015 \times 10^{60}\, r^2\, \theta^5 - 1.4759677125584867290209028281013 \times$
$10^{59}\, r^2\, \theta^6 + 2.3125768916500528710263464701776 \times 10^{64}\, r^2\, \theta\, \mathrm{Sin}[\beta]^2 -$
$5.5208706537929457853006379110093 \times 10^{63}\, r^2\, \theta^2\, \mathrm{Sin}[\beta]^2 +$
$4.3933692736106565513513281036903 \times 10^{62}\, r^2\, \theta^3\, \mathrm{Sin}[\beta]^2 -$
$1.1653773945386426788696894124088 \times 10^{61}\, r^2\, \theta^4\, \mathrm{Sin}[\beta]^2 -$
$3.6325872868346624737226508141661 \times 10^{64}\, r^2\, \mathrm{Sin}[\beta]^4 +$
$5.7814422291251321775658661754441 \times 10^{63}\, r^2\, \theta\, \mathrm{Sin}[\beta]^4 -$
$2.3003627724137274470339018091393 \times 10^{62}\, r^2\, \theta^2\, \mathrm{Sin}[\beta]^4\big)\, \#1^2 +$
$\big(2.0475990141358326793862229391694 \times 10^{46}\, r^2\, \theta^2 - 6.5177100913961894138291231822509 \times 10^{45}$
$r^2\, \theta^3 + 7.7799433401422439744648369436701 \times 10^{44}\, r^2\, \theta^4 -$
$4.1273881318616966226529531247028 \times 10^{43}\, r^2\, \theta^5 + 8.2111777905576565504833386109 90 \times$
$10^{41}\, r^2\, \theta^6 - 1.2865444040613669693619145948318 \times 10^{47}\, r^2\, \theta\, \mathrm{Sin}[\beta]^2 +$
$3.0713985212037490970283649740038 \times 10^{46}\, r^2\, \theta^2\, \mathrm{Sin}[\beta]^2 -$
$2.4441412842735709876238193943913 \times 10^{45}\, r^2\, \theta^3\, \mathrm{Sin}[\beta]^2 +$
$6.4832861167852029217157336168513 \times 10^{43}\, r^2\, \theta^4\, \mathrm{Sin}[\beta]^2 +$
$2.0208992241581244181716190885021 \times 10^{47}\, r^2\, \mathrm{Sin}[\beta]^4 -$
$3.2163610101534174234047864870794 \times 10^{46}\, r^2\, \theta\, \mathrm{Sin}[\beta]^4 +$
$1.2797493838348954246163893369809 \times 10^{45}\, r^2\, \theta^2\, \mathrm{Sin}[\beta]^4\big)\, \#1^4 +$
$\big(-1.6294275228490474905680650795677 \times 10^{45}\, \theta^3 + 3.8899716700711220883495200210324 \times 10^{44}$
$\theta^4 - 3.0955410988962724606297290957727 \times 10^{43}\, \theta^5 +$
$8.2111777905576565504833386109 90 \times 10^{41}\, \theta^6 + 1.0237995070679164258423579964977 \times 10^{46}$
$\theta^2\, \mathrm{Sin}[\beta]^2 - 1.6294275228490473484356373294105 \times 10^{45}\, \theta^3\, \mathrm{Sin}[\beta]^2 +$
$6.4832861167852029217157336168513 \times 10^{43}\, \theta^4\, \mathrm{Sin}[\beta]^2 -$
$1.6081805050767087858042877996039 \times 10^{46}\, \theta\, \mathrm{Sin}[\beta]^4 +$
$1.2797493838348954246163893369809 \times 10^{45}\, \theta^2\, \mathrm{Sin}[\beta]^4\big)\, \#1^6\ \&,\ 3\big]\big\},$
$\big\{c \to \mathrm{Root}\big[1.6539703637441768378304905583564 \times 10^{80}\, r^2\, \theta^2 -$
$5.2647511823477177682105578757695 \times 10^{79}\, r^2\, \theta^3 +$
$6.2843338111465477438022090943599 \times 10^{78}\, r^2\, \theta^4 -$
$3.3339426336116751332792945998413 \times 10^{77}\, r^2\, \theta^5 +$
$6.6326681265513728159832666914994 \times 10^{75}\, r^2\, \theta^6 -$
$1.0392202287987887421152167160698 \times 10^{81}\, r^2\, \theta\, \mathrm{Sin}[\beta]^2 +$
$2.4809555456162651150679275533446 \times 10^{80}\, r^2\, \theta^2\, \mathrm{Sin}[\beta]^2 -$
$1.9742816933803941278907918143229 \times 10^{79}\, r^2\, \theta^3\, \mathrm{Sin}[\beta]^2 +$
$5.2369448426222123232217882284804 8 \times 10^{77}\, r^2\, \theta^4\, \mathrm{Sin}[\beta]^2 +$

$1.6324033181280892853402600163092\times10^{81}\ r^2\ \mathrm{Sin}[\beta]^4-$
$2.5980505719969718552880417901745\times10^{80}\ r^2\ \theta\ \mathrm{Sin}[\beta]^4+$
$1.0337314773401105236440565989727\times10^{79}\ r^2\ \theta^2\ \mathrm{Sin}[\beta]^4+$
$\big(-3.6805804358619639152542428946229\times10^{63}\ r^2\ \theta^2+1.1715651396295084343350277150183\times10^{63}$
$r^2\ \theta^3-1.3984528734463712417367400517893\times10^{62}\ r^2\ \theta^4+$
$7.4190229163351575984264269033015\times10^{60}\ r^2\ \theta^5-1.4759677125584867290209028281013\times$
$10^{59}\ r^2\ \theta^6+2.3125768916500528710263464701776\times10^{64}\ r^2\ \theta\ \mathrm{Sin}[\beta]^2-$
$5.5208706537929457853006379110093\times10^{63}\ r^2\ \theta^2\ \mathrm{Sin}[\beta]^2+$
$4.3933692736106565513513281036903\times10^{62}\ r^2\ \theta^3\ \mathrm{Sin}[\beta]^2-$
$1.1653773945386426788696894124088\times10^{61}\ r^2\ \theta^4\ \mathrm{Sin}[\beta]^2-$
$3.6325872868346624737226508141661\times10^{64}\ r^2\ \mathrm{Sin}[\beta]^4+$
$5.7814422291251321775658661754441\times10^{63}\ r^2\ \theta\ \mathrm{Sin}[\beta]^4-$
$2.3003627724137274470339018091393\times10^{62}\ r^2\ \theta^2\ \mathrm{Sin}[\beta]^4\big)\ \#1^2+$
$\big(2.0475990141358326793862229391694\times10^{46}\ r^2\ \theta^2-6.5177100913961894138291231822509\times10^{45}$
$r^2\ \theta^3+7.7799433401422439744648369436701\times10^{44}\ r^2\ \theta^4-$
$4.1273881318616966226529531247028\times10^{43}\ r^2\ \theta^5+8.2111777905576565504833386109 90\times$
$10^{41}\ r^2\ \theta^6-1.2865444040613669693619145948318\times10^{47}\ r^2\ \theta\ \mathrm{Sin}[\beta]^2+$
$3.0713985212037490970283649740038\times10^{46}\ r^2\ \theta^2\ \mathrm{Sin}[\beta]^2-$
$2.4441412842735709876238193943913\times10^{45}\ r^2\ \theta^3\ \mathrm{Sin}[\beta]^2+$
$6.4832861167852029217157336168513\times10^{43}\ r^2\ \theta^4\ \mathrm{Sin}[\beta]^2+$
$2.0208992241581244181716190885021\times10^{47}\ r^2\ \mathrm{Sin}[\beta]^4-$
$3.2163610101534174234047864870794\times10^{46}\ r^2\ \theta\ \mathrm{Sin}[\beta]^4+$
$1.2797493838348954246163893369809\times10^{45}\ r^2\ \theta^2\ \mathrm{Sin}[\beta]^4\big)\ \#1^4+$
$\big(-1.6294275228490474905680650795677\times10^{45}\ \theta^3+3.8899716700711220883495200210324\times10^{44}$
$\theta^4-3.0955410988962724606297290957727\times10^{43}\ \theta^5+$
$8.2111777905576565504833386109 90\times10^{41}\ \theta^6+1.0237995070679164258423579964977\times10^{46}$
$\theta^2\ \mathrm{Sin}[\beta]^2-1.6294275228490473484356373294105\times10^{45}\ \theta^3\ \mathrm{Sin}[\beta]^2+$
$6.4832861167852029217157336168513\times10^{43}\ \theta^4\ \mathrm{Sin}[\beta]^2-$
$1.6081805050767087858042877996039\times10^{46}\ \theta\ \mathrm{Sin}[\beta]^4+$
$1.2797493838348954246163893369809\times10^{45}\ \theta^2\ \mathrm{Sin}[\beta]^4\big)\ \#1^6\ \&,\ 4\big]\big\},$
$\big\{c\to\mathrm{Root}\big[1.6539703637441768378304905583564\times10^{80}\ r^2\ \theta^2-$
$5.2647511823477177682105578757695\times10^{79}\ r^2\ \theta^3+$
$6.2843338111465477438022090943599\times10^{78}\ r^2\ \theta^4-$
$3.3339426336116751332792945998413\times10^{77}\ r^2\ \theta^5+$
$6.6326681265513728159832666914994\times10^{75}\ r^2\ \theta^6-$
$1.0392202287987887421152167160698\times10^{81}\ r^2\ \theta\ \mathrm{Sin}[\beta]^2+$
$2.4809555456162651150679275533446\times10^{80}\ r^2\ \theta^2\ \mathrm{Sin}[\beta]^2-$
$1.9742816933803941278907918143229\times10^{79}\ r^2\ \theta^3\ \mathrm{Sin}[\beta]^2+$
$5.2369448426221232322178822848048\times10^{77}\ r^2\ \theta^4\ \mathrm{Sin}[\beta]^2+$
$1.6324033181280892853402600163092\times10^{81}\ r^2\ \mathrm{Sin}[\beta]^4-$
$2.5980505719969718552880417901745\times10^{80}\ r^2\ \theta\ \mathrm{Sin}[\beta]^4+$
$1.0337314773401105236440565989727\times10^{79}\ r^2\ \theta^2\ \mathrm{Sin}[\beta]^4+$

$$
\begin{aligned}
&\big(-3.6805804358619639152542428946229\times 10^{63}\, r^2\, \theta^2 + 1.1715651396295084343350277150183\times 10^{63}\\
&\quad r^2\, \theta^3 - 1.3984528734463712417367400517893\times 10^{62}\, r^2\, \theta^4 +\\
&\quad 7.4190229163351575984264269033015\times 10^{60}\, r^2\, \theta^5 - 1.4759677125584867290209028281013\times\\
&\quad 10^{59}\, r^2\, \theta^6 + 2.3125768916500528710263464701776\times 10^{64}\, r^2\, \theta\, \mathrm{Sin}[\beta]^2 -\\
&\quad 5.5208706537929457853006379110093\times 10^{63}\, r^2\, \theta^2\, \mathrm{Sin}[\beta]^2 +\\
&\quad 4.3933692736106565513513281036903\times 10^{62}\, r^2\, \theta^3\, \mathrm{Sin}[\beta]^2 -\\
&\quad 1.1653773945386426788696894124088\times 10^{61}\, r^2\, \theta^4\, \mathrm{Sin}[\beta]^2 -\\
&\quad 3.6325872868346624737226508141661\times 10^{64}\, r^2\, \mathrm{Sin}[\beta]^4 +\\
&\quad 5.7814422291251321775658661754441\times 10^{63}\, r^2\, \theta\, \mathrm{Sin}[\beta]^4 -\\
&\quad 2.3003627724137274470339018091393\times 10^{62}\, r^2\, \theta^2\, \mathrm{Sin}[\beta]^4\big)\, \#1^2 +\\
&\big(2.0475990141358326793862229391694\times 10^{46}\, r^2\, \theta^2 - 6.5177100913961894138291231822509\times 10^{45}\\
&\quad r^2\, \theta^3 + 7.7799433401422439744648369436701\times 10^{44}\, r^2\, \theta^4 -\\
&\quad 4.1273881318616966226529531247028\times 10^{43}\, r^2\, \theta^5 + 8.2111777905576565504833386109990\times\\
&\quad 10^{41}\, r^2\, \theta^6 - 1.2865444040613669693619145948318\times 10^{47}\, r^2\, \theta\, \mathrm{Sin}[\beta]^2 +\\
&\quad 3.0713985212037490970283649740038\times 10^{46}\, r^2\, \theta^2\, \mathrm{Sin}[\beta]^2 -\\
&\quad 2.4441412842735709876238193943913\times 10^{45}\, r^2\, \theta^3\, \mathrm{Sin}[\beta]^2 +\\
&\quad 6.4832861167852029217157336168513\times 10^{43}\, r^2\, \theta^4\, \mathrm{Sin}[\beta]^2 +\\
&\quad 2.0208992241581244181716190885021\times 10^{47}\, r^2\, \mathrm{Sin}[\beta]^4 -\\
&\quad 3.2163610101534174234047864870794\times 10^{46}\, r^2\, \theta\, \mathrm{Sin}[\beta]^4 +\\
&\quad 1.2797493838348954246163893369809\times 10^{45}\, r^2\, \theta^2\, \mathrm{Sin}[\beta]^4\big)\, \#1^4 +\\
&\big(-1.6294275228490474905680650795677\times 10^{45}\, \theta^3 + 3.8899716700711220883495200210324\times 10^{44}\\
&\quad \theta^4 - 3.0955410988962724606297290957727\times 10^{43}\, \theta^5 +\\
&\quad 8.2111777905576565504833386109990\times 10^{41}\, \theta^6 + 1.0237995070679164258423579964977\times 10^{46}\\
&\quad \theta^2\, \mathrm{Sin}[\beta]^2 - 1.6294275228490473484356373294105\times 10^{45}\, \theta^3\, \mathrm{Sin}[\beta]^2 +\\
&\quad 6.4832861167852029217157336168513\times 10^{43}\, \theta^4\, \mathrm{Sin}[\beta]^2 -\\
&\quad 1.6081805050767087858042877996039\times 10^{46}\, \theta\, \mathrm{Sin}[\beta]^4 +\\
&\quad 1.2797493838348954246163893369809\times 10^{45}\, \theta^2\, \mathrm{Sin}[\beta]^4\big)\, \#1^6\ \&,\ 5\big]\big\},
\end{aligned}
$$

$$
\begin{aligned}
&\big\{c \to \mathrm{Root}\big[1.6539703637441768378304905583564\times 10^{80}\, r^2\, \theta^2 -\\
&\quad 5.2647511823477177682105578757695\times 10^{79}\, r^2\, \theta^3 +\\
&\quad 6.2843338111465477438022090943599\times 10^{78}\, r^2\, \theta^4 -\\
&\quad 3.3339426336116751332792945998413\times 10^{77}\, r^2\, \theta^5 +\\
&\quad 6.6326681265513728159832666914994\times 10^{75}\, r^2\, \theta^6 -\\
&\quad 1.0392202287987887421152167160698\times 10^{81}\, r^2\, \theta\, \mathrm{Sin}[\beta]^2 +\\
&\quad 2.4809555456162651150679275533446\times 10^{80}\, r^2\, \theta^2\, \mathrm{Sin}[\beta]^2 -\\
&\quad 1.9742816933803941278907918143229\times 10^{79}\, r^2\, \theta^3\, \mathrm{Sin}[\beta]^2 +\\
&\quad 5.2369448426222123232217882284804 8\times 10^{77}\, r^2\, \theta^4\, \mathrm{Sin}[\beta]^2 +\\
&\quad 1.6324033181280892853402600163092\times 10^{81}\, r^2\, \mathrm{Sin}[\beta]^4 -\\
&\quad 2.5980505719969718552880417901745\times 10^{80}\, r^2\, \theta\, \mathrm{Sin}[\beta]^4 +\\
&\quad 1.0337314773401105236440565989727\times 10^{79}\, r^2\, \theta^2\, \mathrm{Sin}[\beta]^4 +\\
&\quad \big(-3.6805804358619639152542428946229\times 10^{63}\, r^2\, \theta^2 + 1.1715651396295084343350277150183\times 10^{63}\\
&\qquad r^2\, \theta^3 - 1.3984528734463712417367400517893\times 10^{62}\, r^2\, \theta^4 +\\
&\qquad 7.4190229163351575984264269033015\times 10^{60}\, r^2\, \theta^5 - 1.4759677125584867290209028281013\times
\end{aligned}
$$

$$10^{59}\, r^2\, \theta^6 + 2.3125768916500528710263464701776 \times 10^{64}\, r^2\, \theta\, \mathrm{Sin}[\beta]^2 -$$
$$5.5208706537929457853006379110093 \times 10^{63}\, r^2\, \theta^2\, \mathrm{Sin}[\beta]^2 +$$
$$4.3933692736106565513513281036903 \times 10^{62}\, r^2\, \theta^3\, \mathrm{Sin}[\beta]^2 -$$
$$1.1653773945386426788696894124088 \times 10^{61}\, r^2\, \theta^4\, \mathrm{Sin}[\beta]^2 -$$
$$3.6325872868346624737226508141661 \times 10^{64}\, r^2\, \mathrm{Sin}[\beta]^4 +$$
$$5.7814422291251321775658661754441 \times 10^{63}\, r^2\, \theta\, \mathrm{Sin}[\beta]^4 -$$
$$2.3003627724137274470339018091393 \times 10^{62}\, r^2\, \theta^2\, \mathrm{Sin}[\beta]^4\Big)\, \sharp 1^2 +$$
$$\Big(2.0475990141358326793862229391694 \times 10^{46}\, r^2\, \theta^2 - 6.5177100913961894138291231822509 \times 10^{45}$$
$$r^2\, \theta^3 + 7.7799433401422439744648369436701 \times 10^{44}\, r^2\, \theta^4 -$$
$$4.1273881318616966226529531247028 \times 10^{43}\, r^2\, \theta^5 + 8.2111777905576565504833386109900 \times$$
$$10^{41}\, r^2\, \theta^6 - 1.2865444040613669693619145948318 \times 10^{47}\, r^2\, \theta\, \mathrm{Sin}[\beta]^2 +$$
$$3.0713985212037490970283649740038 \times 10^{46}\, r^2\, \theta^2\, \mathrm{Sin}[\beta]^2 -$$
$$2.4441412842735709876238193943913 \times 10^{45}\, r^2\, \theta^3\, \mathrm{Sin}[\beta]^2 +$$
$$6.4832861167852029217157336168513 \times 10^{43}\, r^2\, \theta^4\, \mathrm{Sin}[\beta]^2 +$$
$$2.0208992241581244181716190885021 \times 10^{47}\, r^2\, \mathrm{Sin}[\beta]^4 -$$
$$3.2163610101534174234047864870794 \times 10^{46}\, r^2\, \theta\, \mathrm{Sin}[\beta]^4 +$$
$$1.2797493838348954246163893369809 \times 10^{45}\, r^2\, \theta^2\, \mathrm{Sin}[\beta]^4\Big)\, \sharp 1^4 +$$
$$\Big(-1.6294275228490474905680650795677 \times 10^{45}\, \theta^3 + 3.8899716700711220883495200210324 \times 10^{44}$$
$$\theta^4 - 3.0955410988962724606297290957727 \times 10^{43}\, \theta^5 +$$
$$8.2111777905576565504833386109900 \times 10^{41}\, \theta^6 + 1.0237995070679164258423579964977 \times 10^{46}$$
$$\theta^2\, \mathrm{Sin}[\beta]^2 - 1.6294275228490473484356373294105 \times 10^{45}\, \theta^3\, \mathrm{Sin}[\beta]^2 +$$
$$6.4832861167852029217157336168513 \times 10^{43}\, \theta^4\, \mathrm{Sin}[\beta]^2 -$$
$$1.6081805050767087858042877996039 \times 10^{46}\, \theta\, \mathrm{Sin}[\beta]^4 +$$
$$1.2797493838348954246163893369809 \times 10^{45}\, \theta^2\, \mathrm{Sin}[\beta]^4\Big)\, \sharp 1^6\ \&,\ 6\Big]\Big\}\Big\}$$

XX. On Formulations of Pythagorean Theorem

by Parker Emmerson

Can this system help us prove fermat's last theorem in form? Within even the smallest unit of time, there is an implied ten dimensionality. Because it is mathematically proven, it is no longer theoretical. The mathematics is directly correlated to the perceptual experience just as the geometer measures as he perceives. Please consult "A Geometric Pattern of Perception" (Emmerson, 2009-2012) for more information regarding proofs of the following statements.

Preface and Premise :

$$r\text{^}2 = \eta\text{^}2 + r_1\text{^}2$$

$$\theta\, r = 2\pi r - 2\pi r_1 = 2\pi r - 2\pi\sqrt{(r\text{^}2 - \eta\text{^}2)}$$

$$\theta\, r - 2\pi r = -2\pi\sqrt{(r\text{^}2 - \eta\text{^}2)}$$

$$\frac{\theta\, r - 2\pi r}{2\pi} = \sqrt{(r\text{^}2 - \eta\text{^}2)}$$

$$\left(\frac{\theta\, r - 2\pi r}{2\pi}\right)\text{^}2 = (r\text{^}2 - \eta\text{^}2)$$

$$\left(\frac{\theta\, r - 2\pi r}{2\pi}\right)\text{^}2 + \eta\text{^}2 = r\text{^}2 = \eta\text{^}2 + r_1\text{^}2$$

The Statement that Concludes in Paradox (Evidence of Euclidean Geometry's not Being Comprehensive of All Geometries)

Theorem 4 The initial radius of the circle, r, is a function of only the angle taken out of the initial circle, θ; i.e. r=f(θ). This cannot be proven with Euclidean geometry, thus, this is evidence that Euclidean geometry is consistent, because, any consistent system is not comprehensive, as shown by Gödel in his Incompleteness Theorems.

Proof.

Lemma 6 From Lemma 4, it can be shown that $1=\frac{2\pi\,\mathrm{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}}$

Proof.

$$r == \frac{2\pi\, r\, \mathrm{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}}$$; r cancels on both sides, therefore $1 == \frac{2\pi\,\mathrm{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}}$

Lemma 7 From Theorem 1 and lemma 1, it can be shown that:

$$0=\theta\, r-\left(2\pi r - 2\pi\sqrt{\left(r\,\hat{}\,2-\frac{\sqrt{4\pi r^2\theta-r^2\theta^2}}{2\pi}\,\hat{}\,2\right)}\right)$$

Proof.

θ r = 2 π r - 2 $\pi\sqrt{(r\,\hat{}\,2-\eta\,\hat{}\,2)}$

Subtract θ*r from both sides.

$$0=\theta\, r-\left(2\pi r - 2\pi\sqrt{\left(r\,\hat{}\,2-\frac{\sqrt{4\pi r^2\theta-r^2\theta^2}}{2\pi}\,\hat{}\,2\right)}\right)$$

Lemma 8 One can calculate the radius as a function of θ and β from constructing the equation 1-1=0

as $1-\frac{2\pi\,\mathrm{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}} == \theta\, r-\left(2\pi r - 2\pi\sqrt{\left(r\,\hat{}\,2-\frac{\sqrt{4\pi r^2\theta-r^2\theta^2}}{2\pi}\,\hat{}\,2\right)}\right)$.

Proof.

$$\text{Solve}\left[1 - \frac{2\pi\,\text{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}} == \theta\,r - \left(2\pi r - 2\pi\sqrt{\left(r\hat{}2 - \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}\hat{}2\right)}\right), r\right]$$

$$\left\{\left\{r \to \left(-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta] + 4\pi^2\,\text{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{4\pi-\theta} - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\right) \Big/ \left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\text{Sin}[\beta]^2 + 8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right\}\right\}$$

Lemma 9 From Lemma 5, θ is a function of β. From Lemma 4, β is a function of θ. Therefore, r is a function of theta or beta. This cannot be proven with Euclidean geometry, because one can draw a circle of any size and take any angle (arc length) from it.

Euclidean geometry is consistent, but not comprehensive of all geometries,
including the system related to difference in circumferences of two
circles equaling an arc length (applied to the Pythagorean theorem to form *a* cone).

The Formulation of New Pythagorean Expressions from Difference in Circumferences Equals an Arc Length of the Initial Circle (Which is Folded into a Cone) Applied to Pythagorean Theorem

$$r\hat{}2 = \eta\hat{}2 + r_1\hat{}2$$

$$r := \left(-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta] + 4\pi^2\,\text{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{4\pi-\theta} - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\right) \Big/ \left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\text{Sin}[\beta]^2 + 8\pi^2\theta\,\text{Sin}[\beta]^2\right)$$

This expression for r is referred to as the univocal radius solution.

$$\left(\left(-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta] + 4\pi^2\,\text{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{4\pi-\theta} - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\right) \Big/ \left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\text{Sin}[\beta]^2 + 8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)\hat{}2 = \eta\hat{}2 + r_1\hat{}2$$

$$r\hat{}2 = \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}\hat{}2 + r_1\hat{}2 = \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}\hat{}2 + \frac{\theta r - 2\pi r}{2\pi}\hat{}2$$

Canceling Solutions and Implicit Roots from Substituing the Univocal Radius Expression

$$\text{Solve}\left[r^{\wedge}2 == \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}{}^{\wedge}2 + \frac{\theta r - 2\pi r}{2\pi}{}^{\wedge}2,\ r\right]$$

{{}}

$$\text{Solve}\left[r^{\wedge}2 == \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}{}^{\wedge}2 + \frac{\theta r - 2\pi r}{2\pi}{}^{\wedge}2,\ \theta\right]$$

{{}}

$$\text{Solve}\Bigg[\Bigg(\Bigg(-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta] + 4\pi^2\,\text{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{4\pi-\theta} - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{\theta}\Bigg)\Bigg/\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\text{Sin}[\beta]^2 + 8\pi^2\theta\,\text{Sin}[\beta]^2\right)\Bigg)^{\wedge}2 ==$$

$$\frac{1}{2\pi}\Bigg(\sqrt{\Bigg(4\pi\Bigg(\Bigg(-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta] + 4\pi^2\,\text{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{4\pi-\theta} - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{\theta}\Bigg)\Bigg/\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\text{Sin}[\beta]^2 + 8\pi^2\theta\,\text{Sin}[\beta]^2\right)\Bigg)^2\theta - \Bigg(\Bigg(-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta] + 4\pi^2\,\text{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{4\pi-\theta} - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{\theta}\Bigg)\Bigg/\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\text{Sin}[\beta]^2 + 8\pi^2\theta\,\text{Sin}[\beta]^2\right)\Bigg)^2\theta^2\Bigg)}\Bigg)^{\wedge}2 +$$

$$\frac{1}{2\pi}\theta\Bigg(\Bigg(-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta] + 4\pi^2\,\text{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{4\pi-\theta} - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{\theta}\Bigg)\Bigg/\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\text{Sin}[\beta]^2 + 8\pi^2\theta\,\text{Sin}[\beta]^2\right)\Bigg) -$$

$$2\pi\Bigg(\Bigg(-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta] + 4\pi^2\,\text{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{4\pi-\theta} - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{\theta}\Bigg)\Bigg/\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\text{Sin}[\beta]^2 + 8\pi^2\theta\,\text{Sin}[\beta]^2\right)\Bigg)\ {}^{\wedge}2,\ \theta\Bigg]$$

$$\Big\{\big\{\theta \to \text{Root}\big[16\,\pi^4\,\text{Sin}[\beta]^2 + 64\,\pi^5\,\text{Sin}[\beta]^2 + 64\,\pi^6\,\text{Sin}[\beta]^2 + \left(-16\,\pi^3 - 64\,\pi^4 - 64\,\pi^5 - 64\,\pi^4\,\text{Sin}[\beta]^2 - 128\,\pi^5\,\text{Sin}[\beta]^2\right) \#1 + \left(4\,\pi^2 + 16\,\pi^3 + 16\,\pi^4 + 16\,\pi^3\,\text{Sin}[\beta]^2 + 96\,\pi^4\,\text{Sin}[\beta]^2\right) \#1^2 + \left(16\,\pi^2 + 32\,\pi^3 - 32\,\pi^3\,\text{Sin}[\beta]^2\right) \#1^3 + \left(-4\,\pi - 8\,\pi^2 + 4\,\pi^2\,\text{Sin}[\beta]^2\right) \#1^4 - 4\,\pi\,\#1^5 + \#1^6\,\&,\ 1\big]\big\},$$

$$\big\{\theta \to \text{Root}\big[16\,\pi^4\,\text{Sin}[\beta]^2 + 64\,\pi^5\,\text{Sin}[\beta]^2 + 64\,\pi^6\,\text{Sin}[\beta]^2 + \left(-16\,\pi^3 - 64\,\pi^4 - 64\,\pi^5 - 64\,\pi^4\,\text{Sin}[\beta]^2 - 128\,\pi^5\,\text{Sin}[\beta]^2\right) \#1 + \left(4\,\pi^2 + 16\,\pi^3 + 16\,\pi^4 + 16\,\pi^3\,\text{Sin}[\beta]^2 + 96\,\pi^4\,\text{Sin}[\beta]^2\right) \#1^2 + \left(16\,\pi^2 + 32\,\pi^3 - 32\,\pi^3\,\text{Sin}[\beta]^2\right) \#1^3 + \left(-4\,\pi - 8\,\pi^2 + 4\,\pi^2\,\text{Sin}[\beta]^2\right) \#1^4 - 4\,\pi\,\#1^5 + \#1^6\,\&,\ 2\big]\big\},$$

$$\big\{\theta \to \text{Root}\big[16\,\pi^4\,\text{Sin}[\beta]^2 + 64\,\pi^5\,\text{Sin}[\beta]^2 + 64\,\pi^6\,\text{Sin}[\beta]^2 + \left(-16\,\pi^3 - 64\,\pi^4 - 64\,\pi^5 - 64\,\pi^4\,\text{Sin}[\beta]^2 - 128\,\pi^5\,\text{Sin}[\beta]^2\right) \#1 + \left(4\,\pi^2 + 16\,\pi^3 + 16\,\pi^4 + 16\,\pi^3\,\text{Sin}[\beta]^2 + 96\,\pi^4\,\text{Sin}[\beta]^2\right) \#1^2 + \left(16\,\pi^2 + 32\,\pi^3 - 32\,\pi^3\,\text{Sin}[\beta]^2\right) \#1^3 + \left(-4\,\pi - 8\,\pi^2 + 4\,\pi^2\,\text{Sin}[\beta]^2\right) \#1^4 - 4\,\pi\,\#1^5 + \#1^6\,\&,\ 3\big]\big\},$$

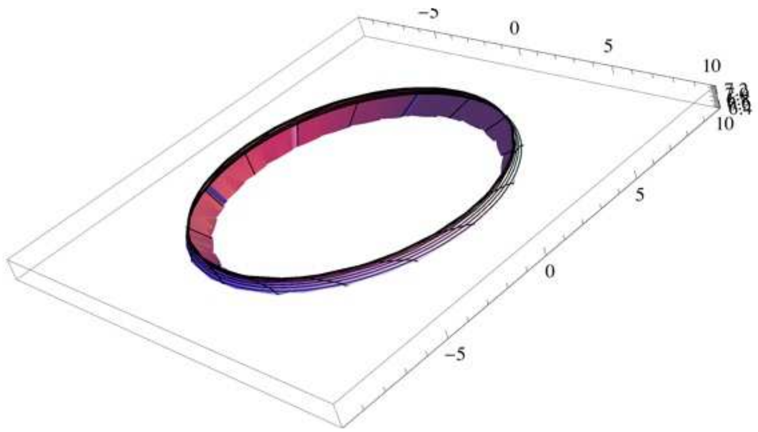
$$\big\{\theta \to \text{Root}\big[16\,\pi^4\,\text{Sin}[\beta]^2 + 64\,\pi^5\,\text{Sin}[\beta]^2 + 64\,\pi^6\,\text{Sin}[\beta]^2 + \left(-16\,\pi^3 - 64\,\pi^4 - 64\,\pi^5 - 64\,\pi^4\,\text{Sin}[\beta]^2 - 128\,\pi^5\,\text{Sin}[\beta]^2\right) \#1 + \left(4\,\pi^2 + 16\,\pi^3 + 16\,\pi^4 + 16\,\pi^3\,\text{Sin}[\beta]^2 + 96\,\pi^4\,\text{Sin}[\beta]^2\right) \#1^2 + \left(16\,\pi^2 + 32\,\pi^3 - 32\,\pi^3\,\text{Sin}[\beta]^2\right) \#1^3 + \left(-4\,\pi - 8\,\pi^2 + 4\,\pi^2\,\text{Sin}[\beta]^2\right) \#1^4 - 4\,\pi\,\#1^5 + \#1^6\,\&,\ 4\big]\big\},$$

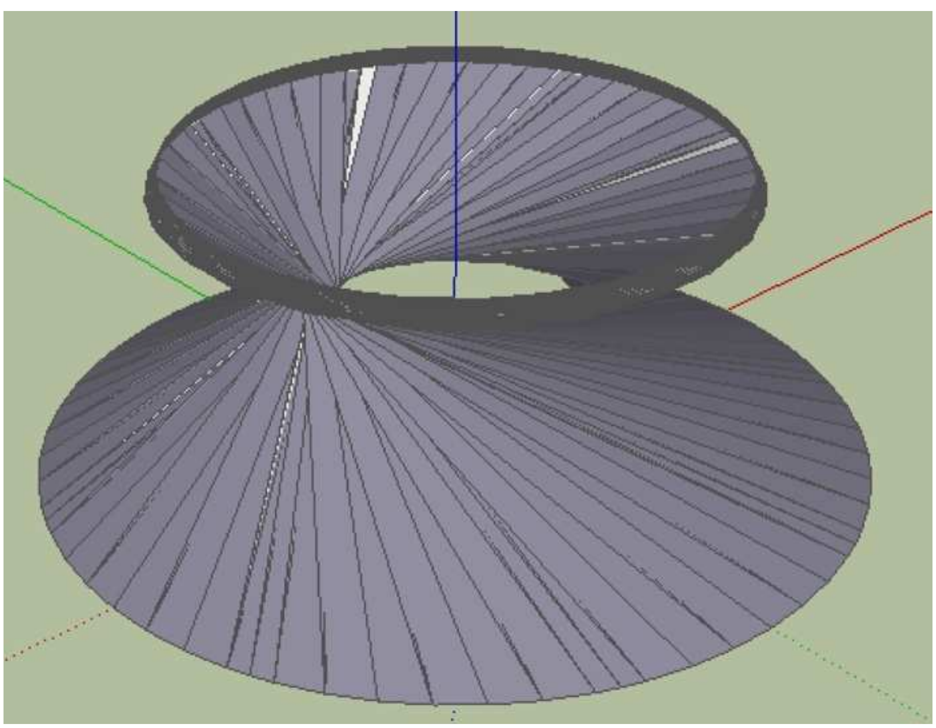
$$\big\{\theta \to \text{Root}\big[16\,\pi^4\,\text{Sin}[\beta]^2 + 64\,\pi^5\,\text{Sin}[\beta]^2 + 64\,\pi^6\,\text{Sin}[\beta]^2 + \left(-16\,\pi^3 - 64\,\pi^4 - 64\,\pi^5 - 64\,\pi^4\,\text{Sin}[\beta]^2 - 128\,\pi^5\,\text{Sin}[\beta]^2\right) \#1 + \left(4\,\pi^2 + 16\,\pi^3 + 16\,\pi^4 + 16\,\pi^3\,\text{Sin}[\beta]^2 + 96\,\pi^4\,\text{Sin}[\beta]^2\right) \#1^2 + \left(16\,\pi^2 + 32\,\pi^3 - 32\,\pi^3\,\text{Sin}[\beta]^2\right) \#1^3 + \left(-4\,\pi - 8\,\pi^2 + 4\,\pi^2\,\text{Sin}[\beta]^2\right) \#1^4 - 4\,\pi\,\#1^5 + \#1^6\,\&,\ 5\big]\big\},$$

$$\big\{\theta \to \text{Root}\big[16\,\pi^4\,\text{Sin}[\beta]^2 + 64\,\pi^5\,\text{Sin}[\beta]^2 + 64\,\pi^6\,\text{Sin}[\beta]^2 + \left(-16\,\pi^3 - 64\,\pi^4 - 64\,\pi^5 - 64\,\pi^4\,\text{Sin}[\beta]^2 - 128\,\pi^5\,\text{Sin}[\beta]^2\right) \#1 + \left(4\,\pi^2 + 16\,\pi^3 + 16\,\pi^4 + 16\,\pi^3\,\text{Sin}[\beta]^2 + 96\,\pi^4\,\text{Sin}[\beta]^2\right) \#1^2 + \left(16\,\pi^2 + 32\,\pi^3 - 32\,\pi^3\,\text{Sin}[\beta]^2\right) \#1^3 + \left(-4\,\pi - 8\,\pi^2 + 4\,\pi^2\,\text{Sin}[\beta]^2\right) \#1^4 - 4\,\pi\,\#1^5 + \#1^6\,\&,\ 6\big]\big\},$$

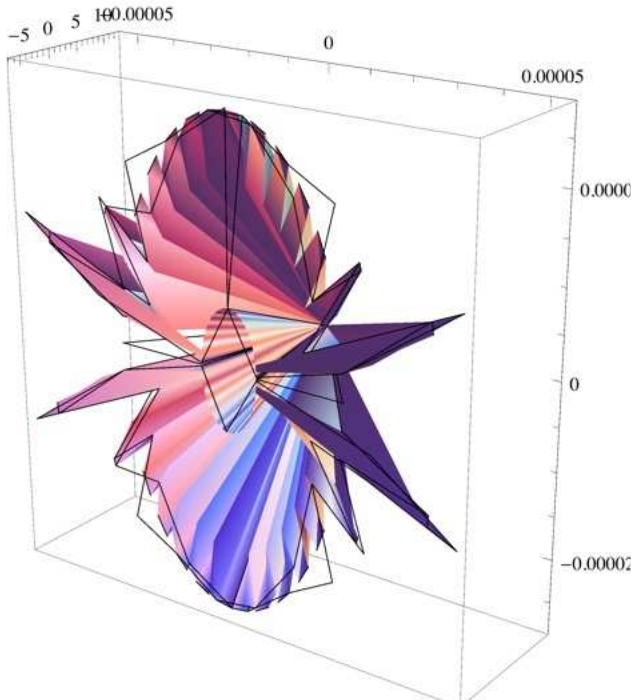
$$\left\{\theta \to 2\left(\pi - \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)\right\},\ \left\{\theta \to 2\left(\pi + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)\right\}\Big\}$$

```
RevolutionPlot3D[{Root[
    16 π^4 Sin[β]^2 + 64 π^5 Sin[β]^2 + 64 π^6 Sin[β]^2 + (-16 π^3 - 64 π^4 - 64 π^5 - 64 π^4 Sin[β]^2 - 128 π^5 Sin[β]^2)
       #1 + (4 π^2 + 16 π^3 + 16 π^4 + 16 π^3 Sin[β]^2 + 96 π^4 Sin[β]^2) #1^2 +
      (16 π^2 + 32 π^3 - 32 π^3 Sin[β]^2) #1^3 + (-4 π - 8 π^2 + 4 π^2 Sin[β]^2) #1^4 - 4 π #1^5 + #1^6 &, 1], Root[
    16 π^4 Sin[β]^2 + 64 π^5 Sin[β]^2 + 64 π^6 Sin[β]^2 + (-16 π^3 - 64 π^4 - 64 π^5 - 64 π^4 Sin[β]^2 - 128 π^5 Sin[β]^2)
       #1 + (4 π^2 + 16 π^3 + 16 π^4 + 16 π^3 Sin[β]^2 + 96 π^4 Sin[β]^2) #1^2 +
      (16 π^2 + 32 π^3 - 32 π^3 Sin[β]^2) #1^3 + (-4 π - 8 π^2 + 4 π^2 Sin[β]^2) #1^4 - 4 π #1^5 + #1^6 &, 2], Root[
    16 π^4 Sin[β]^2 + 64 π^5 Sin[β]^2 + 64 π^6 Sin[β]^2 + (-16 π^3 - 64 π^4 - 64 π^5 - 64 π^4 Sin[β]^2 - 128 π^5 Sin[β]^2)
       #1 + (4 π^2 + 16 π^3 + 16 π^4 + 16 π^3 Sin[β]^2 + 96 π^4 Sin[β]^2) #1^2 + (16 π^2 + 32 π^3 - 32 π^3 Sin[β]^2) #1^3 +
      (-4 π - 8 π^2 + 4 π^2 Sin[β]^2) #1^4 - 4 π #1^5 + #1^6 &, 3]}, {β, -π / 2, π / 2}]
```





```
RevolutionPlot3D[Root[
  16 π^4 Sin[β]^2 + 64 π^5 Sin[β]^2 + 64 π^6 Sin[β]^2 + (-16 π^3 - 64 π^4 - 64 π^5 - 64 π^4 Sin[β]^2 - 128 π^5 Sin[β]^2)
     #1 + (4 π^2 + 16 π^3 + 16 π^4 + 16 π^3 Sin[β]^2 + 96 π^4 Sin[β]^2) #1^2 + (16 π^2 + 32 π^3 - 32 π^3 Sin[β]^2) #1^3 +
    (-4 π - 8 π^2 + 4 π^2 Sin[β]^2) #1^4 - 4 π #1^5 + #1^6 &, 6], {β, -π / 2, π / 2}]
```



■ Substitutions that Don't Cancel

$$\text{Solve}\left[r^2 == \frac{\sqrt{4\pi\,(r)^2\,\theta - (r)^2\,\theta^2}}{2\pi}\,\hat{}\,2 + \frac{1}{2\pi}\,\theta\left(\left(-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta] + 4\pi^2\,\text{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta} - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\text{Sin}[\beta]^2 + 8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right) - 2\pi\left(\left(-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta] + 4\pi^2\,\text{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta} - \frac{2\pi^2\sqrt{(w4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\text{Sin}[\beta]^2 + 8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)\,\hat{}\,2,\ r\right]$$

$$\left\{\left\{r \to -\frac{1}{\sqrt{1-\frac{\theta}{\pi}+\frac{\theta^2}{4\pi^2}}}\left(\sqrt{\left(-\frac{32\pi^3\theta^2}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}+\right.}\right.\right.\right.$$

$$\frac{16\pi^2\theta^3}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{2\pi\theta^4}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{32\pi^3\theta\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{8\pi^2\theta^2\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{32\pi^4\theta\,\mathrm{Sin}[\beta]^2}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{8\pi^3\theta^2\,\mathrm{Sin}[\beta]^2}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{32\pi^4\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]^3}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{32\pi^4\theta\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]^3}{(4\pi-\theta)\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{8\pi^3\theta^2\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]^3}{(4\pi-\theta)\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{32\pi^4\sqrt{(\pi\,\mathrm{w4}-\theta)\,\theta}\,\mathrm{Sin}[\beta]^3}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{8\pi^3\theta\sqrt{(\pi\,\mathrm{w4}-\theta)\,\theta}\,\mathrm{Sin}[\beta]^3}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{32\,\pi^5\,\text{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{16\,\pi^4\,\theta\,\text{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{16\,\pi^4\,\sqrt{(4\,\pi-\theta)\,\theta}\,\sqrt{(\pi\,\text{w4}-\theta)\,\theta}\,\text{Sin}[\beta]^4}{\theta\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{32\,\pi^5\,\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^5}{(4\,\pi-\theta)\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{32\,\pi^5\,\sqrt{(\pi\,\text{w4}-\theta)\,\theta}\,\text{Sin}[\beta]^5}{\theta\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{8\,\pi^5\,\text{Sin}[\beta]^6}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{8\,\pi^6\,\text{w4}\,\text{Sin}[\beta]^6}{\theta\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{8\,\pi^5\,\theta\,\text{Sin}[\beta]^6}{(4\,\pi-\theta)\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{16\,\pi^5\,\sqrt{(4\,\pi-\theta)\,\theta}\,\sqrt{(\pi\,\text{w4}-\theta)\,\theta}\,\text{Sin}[\beta]^6}{(4\,\pi-\theta)\,\theta\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{2\,\theta^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}+$$

$$\frac{\theta^3}{2\,\pi\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)}+$$

$$\frac{\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}+$$

$$\frac{2\,\pi\,\theta\,\text{Sin}[\beta]^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}-$$

$$\frac{\pi \sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]^3}{16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2}-$$

$$\left.\left.\frac{\pi\theta\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]^3}{(4\pi-\theta)\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)}\right)\right)\Big\},$$

$$\Big\{r\to\frac{1}{\sqrt{1-\frac{\theta}{\pi}+\frac{\theta^2}{4\pi^2}}}\left(\sqrt{\left(-\frac{32\pi^3\theta^2}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}+\right.}\right.$$

$$\frac{16\pi^2\theta^3}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{2\pi\theta^4}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{32\pi^3\theta\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{8\pi^2\theta^2\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{32\pi^4\theta\,\mathrm{Sin}[\beta]^2}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{8\pi^3\theta^2\,\mathrm{Sin}[\beta]^2}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{32\pi^4\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]^3}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{32\pi^4\theta\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]^3}{(4\pi-\theta)\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{8\pi^3\theta^2\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]^3}{(4\pi-\theta)\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{32\,\pi^4\sqrt{(\pi\,\text{w4}-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{8\,\pi^3\,\theta\sqrt{(\pi\,\text{w4}-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{32\,\pi^5\,\text{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{16\,\pi^4\,\theta\,\text{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{16\,\pi^4\sqrt{(4\,\pi-\theta)\,\theta}\;\sqrt{(\pi\,\text{w4}-\theta)\,\theta}\;\text{Sin}[\beta]^4}{\theta\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{32\,\pi^5\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^5}{(4\,\pi-\theta)\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{32\,\pi^5\sqrt{(\pi\,\text{w4}-\theta)\,\theta}\;\text{Sin}[\beta]^5}{\theta\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{8\,\pi^5\,\text{Sin}[\beta]^6}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{8\,\pi^6\,\text{w4}\,\text{Sin}[\beta]^6}{\theta\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{8\,\pi^5\,\theta\,\text{Sin}[\beta]^6}{(4\,\pi-\theta)\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{16\,\pi^5\sqrt{(4\,\pi-\theta)\,\theta}\;\sqrt{(\pi\,\text{w4}-\theta)\,\theta}\;\text{Sin}[\beta]^6}{(4\,\pi-\theta)\,\theta\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{2\,\theta^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}+$$

$$\frac{\theta^3}{2\,\pi\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)}+$$

$$\frac{\theta \sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2}+$$
$$\frac{2\pi\theta\,\text{Sin}[\beta]^2}{16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2}-$$
$$\frac{\pi\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2}-$$
$$\left.\left.\frac{\pi\theta\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{(4\pi-\theta)\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)}\right)\right)\}\}$$

$$\textbf{SphericalPlot3D}\Big[\frac{1}{\sqrt{1-\frac{\theta}{\pi}+\frac{\theta^2}{4\pi^2}}}\left(\sqrt{\left(-\frac{32\pi^3\theta^2}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)^2}+\right.}\right.$$
$$\frac{16\pi^2\theta^3}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)^2}-$$
$$\frac{2\pi\theta^4}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)^2}+$$
$$\frac{32\pi^3\theta\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)^2}-$$
$$\frac{8\pi^2\theta^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)^2}+$$
$$\frac{32\pi^4\theta\,\text{Sin}[\beta]^2}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)^2}-$$
$$\frac{8\pi^3\theta^2\,\text{Sin}[\beta]^2}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)^2}-$$
$$\frac{64\pi^4\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{8\,\pi^3\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{32\,\pi^4\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{(4\,\pi-\theta)\,\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{8\,\pi^3\,\theta^2\,\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{(4\,\pi-\theta)\,\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{32\,\pi^5\,\text{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{32\,\pi^5\,\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^5}{(4\,\pi-\theta)\,\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{32\,\pi^5\,\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^5}{\theta\,\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{8\,\pi^5\,\text{Sin}[\beta]^6}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{32\,\pi^6\,\text{Sin}[\beta]^6}{\theta\,\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{8\,\pi^5\,\theta\,\text{Sin}[\beta]^6}{(4\,\pi-\theta)\,\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{2\,\theta^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}+$$

$$\frac{\theta^3}{2\,\pi\,\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)}+$$

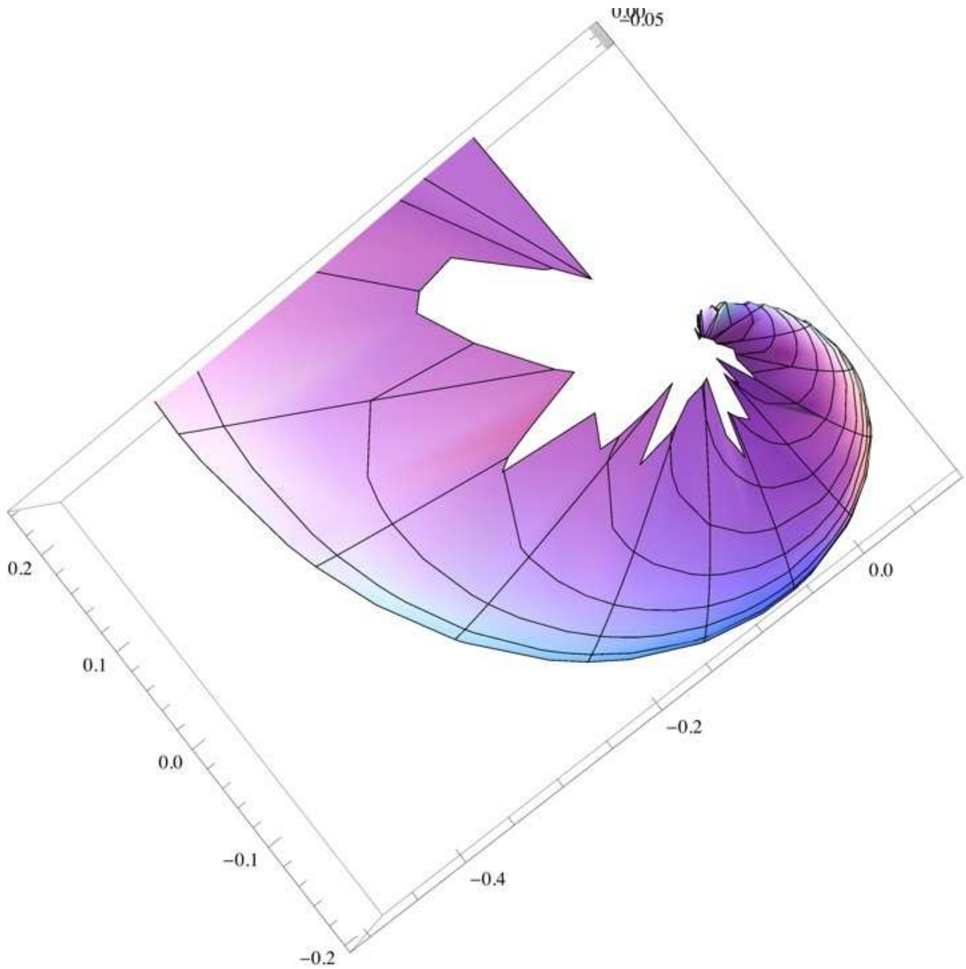
$$\frac{\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}+$$

$$\frac{2\,\pi\,\theta\,\text{Sin}[\beta]^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}-$$

$$\frac{\pi \sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]^3}{16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\mathrm{Sin}[\beta]^2 + 8\pi^2\theta\,\mathrm{Sin}[\beta]^2} -$$

$$\left.\left.\frac{\pi\theta \sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]^3}{(4\pi-\theta)\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\mathrm{Sin}[\beta]^2 + 8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)}\right)\right),$$

$\{\theta, -2\pi, 2\pi\}, \{\beta, -\pi/2, \pi/2\}]$



$$\text{Solve}\Big[r\text{^}2 == \frac{\sqrt{4\,\pi\,(r)^2\,\theta - (r)^2\,\theta^2}}{2\,\pi}\text{^}2 + \frac{1}{2\,\pi}\,\theta\left(\left(-4\,\pi\,\theta + \theta^2 + 2\,\pi\sqrt{(4\,\pi - \theta)\,\theta}\;\text{Sin}[\beta] + 4\,\pi^2\,\text{Sin}[\beta]^2 - \frac{2\,\pi^2\sqrt{(4\,\pi - \theta)\,\theta}\;\text{Sin}[\beta]^3}{4\,\pi - \theta} - \frac{2\,\pi^2\sqrt{(4\,\pi - \theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\text{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)\right) - 2\,\pi\left(\left(-4\,\pi\,\theta + \theta^2 + 2\,\pi\sqrt{(4\,\pi - \theta)\,\theta}\;\text{Sin}[\beta] + 4\,\pi^2\,\text{Sin}[\beta]^2 - \frac{2\,\pi^2\sqrt{(4\,\pi - \theta)\,\theta}\;\text{Sin}[\beta]^3}{4\,\pi - \theta} - \frac{2\,\pi^2\sqrt{(4\,\pi - \theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\text{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)\right)\text{^}2,\ \beta\Big]$$

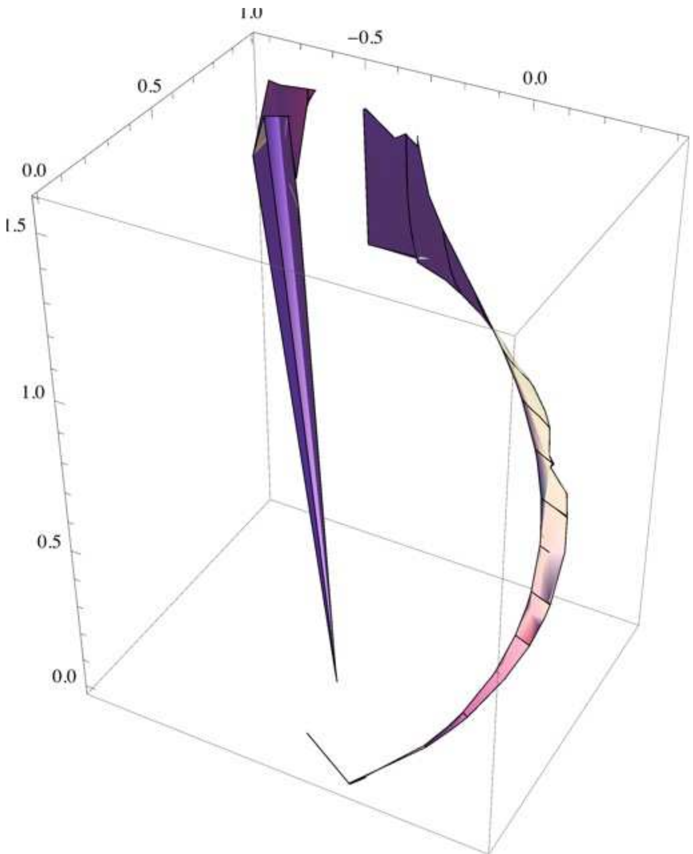
Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

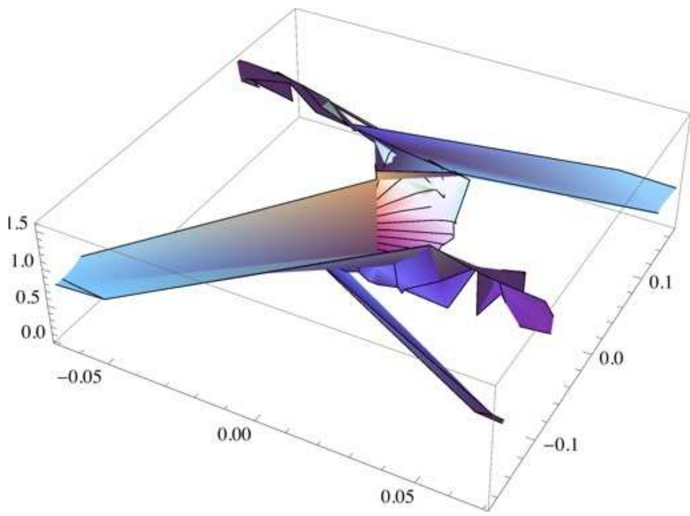
$$\Big\{\Big\{\beta \to \text{ArcSin}\Big[\frac{1}{16\,\pi^5}\left(-\pi^2\sqrt{(4\,\pi - \theta)\,\theta}\,\left(-8\,\pi^2 - 4\,\pi\,\theta + 2\,\theta^2\right) - 2\,\pi^2\sqrt{\left(-512\,\pi^6\,r^2\,\theta + 1152\,\pi^5\,r^2\,\theta^2 + 16\,\pi^3\,\theta^3 - 1024\,\pi^4\,r^2\,\theta^3 - 20\,\pi^2\,\theta^4 + 448\,\pi^3\,r^2\,\theta^4 + 8\,\pi\,\theta^5 - 96\,\pi^2\,r^2\,\theta^5 - \theta^6 + 8\,\pi\,r^2\,\theta^6\right)}\right)\Big]\Big\},$$

$$\Big\{\beta \to \text{ArcSin}\Big[\frac{1}{16\,\pi^5}\left(-\pi^2\sqrt{(4\,\pi - \theta)\,\theta}\,\left(-8\,\pi^2 - 4\,\pi\,\theta + 2\,\theta^2\right) + 2\,\pi^2\sqrt{\left(-512\,\pi^6\,r^2\,\theta + 1152\,\pi^5\,r^2\,\theta^2 + 16\,\pi^3\,\theta^3 - 1024\,\pi^4\,r^2\,\theta^3 - 20\,\pi^2\,\theta^4 + 448\,\pi^3\,r^2\,\theta^4 + 8\,\pi\,\theta^5 - 96\,\pi^2\,r^2\,\theta^5 - \theta^6 + 8\,\pi\,r^2\,\theta^6\right)}\right)\Big]\Big\}\Big\}$$

```
RevolutionPlot3D[
 {ArcSin[1/(16 π^5) (-π^2 Sqrt[(4 π - θ) θ] (-8 π^2 - 4 π θ + 2 θ^2) - 2 π^2 √(-512 π^6 r^2 θ + 1152 π^5 r^2 θ^2 + 16 π^3 θ^3 -
        1024 π^4 r^2 θ^3 - 20 π^2 θ^4 + 448 π^3 r^2 θ^4 + 8 π θ^5 - 96 π^2 r^2 θ^5 - θ^6 + 8 π r^2 θ^6))], ArcSin[1/(16 π^5)
   (-π^2 Sqrt[(4 π - θ) θ] (-8 π^2 - 4 π θ + 2 θ^2) + 2 π^2 √(-512 π^6 r^2 θ + 1152 π^5 r^2 θ^2 + 16 π^3 θ^3 - 1024 π^4 r^2 θ^3 -
        20 π^2 θ^4 + 448 π^3 r^2 θ^4 + 8 π θ^5 - 96 π^2 r^2 θ^5 - θ^6 + 8 π r^2 θ^6))]}, {r, -1, 1}, {θ, -2 π, 2 π}]
```



$$\text{RevolutionPlot3D}\Big[\text{ArcSin}\Big[\frac{1}{16\,\pi^5}\Big(-\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\left(-8\pi^2-4\pi\theta+2\theta^2\right)+2\pi^2\sqrt{\left(-512\pi^6 r^2\theta+1152\pi^5 r^2\theta^2+16\pi^3\theta^3-1024\pi^4 r^2\theta^3-20\pi^2\theta^4+448\pi^3 r^2\theta^4+8\pi\theta^5-96\pi^2 r^2\theta^5-\theta^6+8\pi r^2\theta^6\right)}\Big)\Big],\ \{r,-1,1\},\ \{\theta,-2\pi,2\pi\}\Big]$$



$$\text{Solve}\Big[r^{\wedge}2 == \frac{\sqrt{4\pi(r)^2\theta-(r)^2\theta^2}}{2\pi}{}^{\wedge}2+\frac{1}{2\pi}\theta\left(\left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)-2\pi\left(\left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right){}^{\wedge}2,\ \theta\Big]$$

$$\Big\{\Big\{\theta\to\text{Root}\Big[64\pi^{10}\,\text{Sin}[\beta]^4+\left(-128\pi^9\,\text{Sin}[\beta]^2+1024\pi^{10} r^2\,\text{Sin}[\beta]^2\right)\#1+$$
$$\left(64\pi^8+1024\pi^9 r^2+4096\pi^{10} r^4-96\pi^8\,\text{Sin}[\beta]^2-2304\pi^9 r^2\,\text{Sin}[\beta]^2\right)\#1^2+$$
$$\left(96\pi^7-1536\pi^8 r^2-18\,432\pi^9 r^4+32\pi^7\,\text{Sin}[\beta]^2+2048\pi^8 r^2\,\text{Sin}[\beta]^2\right)\#1^3+$$
$$\left(-60\pi^6-448\pi^7 r^2+37\,120\pi^8 r^4+64\pi^6\,\text{Sin}[\beta]^2-896\pi^7 r^2\,\text{Sin}[\beta]^2\right)\#1^4+$$

$$\left(-56\,\pi^5+2496\,\pi^6\,r^2-44\,032\,\pi^7\,r^4-32\,\pi^5\,\text{Sin}[\beta]^2+192\,\pi^6\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^5+$$
$$\left(48\,\pi^4-2304\,\pi^5\,r^2+34\,048\,\pi^6\,r^4+4\,\pi^4\,\text{Sin}[\beta]^2-16\,\pi^5\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^6+$$
$$\left(-12\,\pi^3+1056\,\pi^4\,r^2-17\,920\,\pi^5\,r^4\right)\,\#1^7+\left(\pi^2-268\,\pi^3\,r^2+6496\,\pi^4\,r^4\right)\,\#1^8+$$
$$\left(36\,\pi^2\,r^2-1600\,\pi^3\,r^4\right)\,\#1^9+\left(-2\,\pi\,r^2+256\,\pi^2\,r^4\right)\,\#1^{10}-24\,\pi\,r^4\,\#1^{11}+r^4\,\#1^{12}\,\&,\,1\Big]\Big\},$$

$$\Big\{\theta\to\text{Root}\Big[64\,\pi^{10}\,\text{Sin}[\beta]^4+\left(-128\,\pi^9\,\text{Sin}[\beta]^2+1024\,\pi^{10}\,r^2\,\text{Sin}[\beta]^2\right)\,\#1+$$
$$\left(64\,\pi^8+1024\,\pi^9\,r^2+4096\,\pi^{10}\,r^4-96\,\pi^8\,\text{Sin}[\beta]^2-2304\,\pi^9\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^2+$$
$$\left(96\,\pi^7-1536\,\pi^8\,r^2-18\,432\,\pi^9\,r^4+32\,\pi^7\,\text{Sin}[\beta]^2+2048\,\pi^8\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^3+$$
$$\left(-60\,\pi^6-448\,\pi^7\,r^2+37\,120\,\pi^8\,r^4+64\,\pi^6\,\text{Sin}[\beta]^2-896\,\pi^7\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^4+$$
$$\left(-56\,\pi^5+2496\,\pi^6\,r^2-44\,032\,\pi^7\,r^4-32\,\pi^5\,\text{Sin}[\beta]^2+192\,\pi^6\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^5+$$
$$\left(48\,\pi^4-2304\,\pi^5\,r^2+34\,048\,\pi^6\,r^4+4\,\pi^4\,\text{Sin}[\beta]^2-16\,\pi^5\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^6+$$
$$\left(-12\,\pi^3+1056\,\pi^4\,r^2-17\,920\,\pi^5\,r^4\right)\,\#1^7+\left(\pi^2-268\,\pi^3\,r^2+6496\,\pi^4\,r^4\right)\,\#1^8+$$
$$\left(36\,\pi^2\,r^2-1600\,\pi^3\,r^4\right)\,\#1^9+\left(-2\,\pi\,r^2+256\,\pi^2\,r^4\right)\,\#1^{10}-24\,\pi\,r^4\,\#1^{11}+r^4\,\#1^{12}\,\&,\,2\Big]\Big\},$$

$$\Big\{\theta\to\text{Root}\Big[64\,\pi^{10}\,\text{Sin}[\beta]^4+\left(-128\,\pi^9\,\text{Sin}[\beta]^2+1024\,\pi^{10}\,r^2\,\text{Sin}[\beta]^2\right)\,\#1+$$
$$\left(64\,\pi^8+1024\,\pi^9\,r^2+4096\,\pi^{10}\,r^4-96\,\pi^8\,\text{Sin}[\beta]^2-2304\,\pi^9\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^2+$$
$$\left(96\,\pi^7-1536\,\pi^8\,r^2-18\,432\,\pi^9\,r^4+32\,\pi^7\,\text{Sin}[\beta]^2+2048\,\pi^8\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^3+$$
$$\left(-60\,\pi^6-448\,\pi^7\,r^2+37\,120\,\pi^8\,r^4+64\,\pi^6\,\text{Sin}[\beta]^2-896\,\pi^7\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^4+$$
$$\left(-56\,\pi^5+2496\,\pi^6\,r^2-44\,032\,\pi^7\,r^4-32\,\pi^5\,\text{Sin}[\beta]^2+192\,\pi^6\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^5+$$
$$\left(48\,\pi^4-2304\,\pi^5\,r^2+34\,048\,\pi^6\,r^4+4\,\pi^4\,\text{Sin}[\beta]^2-16\,\pi^5\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^6+$$
$$\left(-12\,\pi^3+1056\,\pi^4\,r^2-17\,920\,\pi^5\,r^4\right)\,\#1^7+\left(\pi^2-268\,\pi^3\,r^2+6496\,\pi^4\,r^4\right)\,\#1^8+$$
$$\left(36\,\pi^2\,r^2-1600\,\pi^3\,r^4\right)\,\#1^9+\left(-2\,\pi\,r^2+256\,\pi^2\,r^4\right)\,\#1^{10}-24\,\pi\,r^4\,\#1^{11}+r^4\,\#1^{12}\,\&,\,3\Big]\Big\},$$

$$\Big\{\theta\to\text{Root}\Big[64\,\pi^{10}\,\text{Sin}[\beta]^4+\left(-128\,\pi^9\,\text{Sin}[\beta]^2+1024\,\pi^{10}\,r^2\,\text{Sin}[\beta]^2\right)\,\#1+$$
$$\left(64\,\pi^8+1024\,\pi^9\,r^2+4096\,\pi^{10}\,r^4-96\,\pi^8\,\text{Sin}[\beta]^2-2304\,\pi^9\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^2+$$
$$\left(96\,\pi^7-1536\,\pi^8\,r^2-18\,432\,\pi^9\,r^4+32\,\pi^7\,\text{Sin}[\beta]^2+2048\,\pi^8\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^3+$$
$$\left(-60\,\pi^6-448\,\pi^7\,r^2+37\,120\,\pi^8\,r^4+64\,\pi^6\,\text{Sin}[\beta]^2-896\,\pi^7\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^4+$$
$$\left(-56\,\pi^5+2496\,\pi^6\,r^2-44\,032\,\pi^7\,r^4-32\,\pi^5\,\text{Sin}[\beta]^2+192\,\pi^6\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^5+$$
$$\left(48\,\pi^4-2304\,\pi^5\,r^2+34\,048\,\pi^6\,r^4+4\,\pi^4\,\text{Sin}[\beta]^2-16\,\pi^5\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^6+$$
$$\left(-12\,\pi^3+1056\,\pi^4\,r^2-17\,920\,\pi^5\,r^4\right)\,\#1^7+\left(\pi^2-268\,\pi^3\,r^2+6496\,\pi^4\,r^4\right)\,\#1^8+$$
$$\left(36\,\pi^2\,r^2-1600\,\pi^3\,r^4\right)\,\#1^9+\left(-2\,\pi\,r^2+256\,\pi^2\,r^4\right)\,\#1^{10}-24\,\pi\,r^4\,\#1^{11}+r^4\,\#1^{12}\,\&,\,4\Big]\Big\},$$

$$\Big\{\theta\to\text{Root}\Big[64\,\pi^{10}\,\text{Sin}[\beta]^4+\left(-128\,\pi^9\,\text{Sin}[\beta]^2+1024\,\pi^{10}\,r^2\,\text{Sin}[\beta]^2\right)\,\#1+$$
$$\left(64\,\pi^8+1024\,\pi^9\,r^2+4096\,\pi^{10}\,r^4-96\,\pi^8\,\text{Sin}[\beta]^2-2304\,\pi^9\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^2+$$
$$\left(96\,\pi^7-1536\,\pi^8\,r^2-18\,432\,\pi^9\,r^4+32\,\pi^7\,\text{Sin}[\beta]^2+2048\,\pi^8\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^3+$$
$$\left(-60\,\pi^6-448\,\pi^7\,r^2+37\,120\,\pi^8\,r^4+64\,\pi^6\,\text{Sin}[\beta]^2-896\,\pi^7\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^4+$$
$$\left(-56\,\pi^5+2496\,\pi^6\,r^2-44\,032\,\pi^7\,r^4-32\,\pi^5\,\text{Sin}[\beta]^2+192\,\pi^6\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^5+$$
$$\left(48\,\pi^4-2304\,\pi^5\,r^2+34\,048\,\pi^6\,r^4+4\,\pi^4\,\text{Sin}[\beta]^2-16\,\pi^5\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^6+$$
$$\left(-12\,\pi^3+1056\,\pi^4\,r^2-17\,920\,\pi^5\,r^4\right)\,\#1^7+\left(\pi^2-268\,\pi^3\,r^2+6496\,\pi^4\,r^4\right)\,\#1^8+$$
$$\left(36\,\pi^2\,r^2-1600\,\pi^3\,r^4\right)\,\#1^9+\left(-2\,\pi\,r^2+256\,\pi^2\,r^4\right)\,\#1^{10}-24\,\pi\,r^4\,\#1^{11}+r^4\,\#1^{12}\,\&,\,5\Big]\Big\},$$

$$\Big\{\theta\to\text{Root}\Big[64\,\pi^{10}\,\text{Sin}[\beta]^4+\left(-128\,\pi^9\,\text{Sin}[\beta]^2+1024\,\pi^{10}\,r^2\,\text{Sin}[\beta]^2\right)\,\#1+$$

$$\left(64\,\pi^8 + 1024\,\pi^9\,r^2 + 4096\,\pi^{10}\,r^4 - 96\,\pi^8\,\text{Sin}[\beta]^2 - 2304\,\pi^9\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^2 +$$
$$\left(96\,\pi^7 - 1536\,\pi^8\,r^2 - 18\,432\,\pi^9\,r^4 + 32\,\pi^7\,\text{Sin}[\beta]^2 + 2048\,\pi^8\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^3 +$$
$$\left(-60\,\pi^6 - 448\,\pi^7\,r^2 + 37\,120\,\pi^8\,r^4 + 64\,\pi^6\,\text{Sin}[\beta]^2 - 896\,\pi^7\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^4 +$$
$$\left(-56\,\pi^5 + 2496\,\pi^6\,r^2 - 44\,032\,\pi^7\,r^4 - 32\,\pi^5\,\text{Sin}[\beta]^2 + 192\,\pi^6\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^5 +$$
$$\left(48\,\pi^4 - 2304\,\pi^5\,r^2 + 34\,048\,\pi^6\,r^4 + 4\,\pi^4\,\text{Sin}[\beta]^2 - 16\,\pi^5\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^6 +$$
$$\left(-12\,\pi^3 + 1056\,\pi^4\,r^2 - 17\,920\,\pi^5\,r^4\right)\,\#1^7 + \left(\pi^2 - 268\,\pi^3\,r^2 + 6496\,\pi^4\,r^4\right)\,\#1^8 +$$
$$\left(36\,\pi^2\,r^2 - 1600\,\pi^3\,r^4\right)\,\#1^9 + \left(-2\,\pi\,r^2 + 256\,\pi^2\,r^4\right)\,\#1^{10} - 24\,\pi\,r^4\,\#1^{11} + r^4\,\#1^{12}\,\&,\,6\right]\Big\},$$
$$\Big\{\theta \to \text{Root}\Big[64\,\pi^{10}\,\text{Sin}[\beta]^4 + \left(-128\,\pi^9\,\text{Sin}[\beta]^2 + 1024\,\pi^{10}\,r^2\,\text{Sin}[\beta]^2\right)\,\#1 +$$
$$\left(64\,\pi^8 + 1024\,\pi^9\,r^2 + 4096\,\pi^{10}\,r^4 - 96\,\pi^8\,\text{Sin}[\beta]^2 - 2304\,\pi^9\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^2 +$$
$$\left(96\,\pi^7 - 1536\,\pi^8\,r^2 - 18\,432\,\pi^9\,r^4 + 32\,\pi^7\,\text{Sin}[\beta]^2 + 2048\,\pi^8\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^3 +$$
$$\left(-60\,\pi^6 - 448\,\pi^7\,r^2 + 37\,120\,\pi^8\,r^4 + 64\,\pi^6\,\text{Sin}[\beta]^2 - 896\,\pi^7\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^4 +$$
$$\left(-56\,\pi^5 + 2496\,\pi^6\,r^2 - 44\,032\,\pi^7\,r^4 - 32\,\pi^5\,\text{Sin}[\beta]^2 + 192\,\pi^6\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^5 +$$
$$\left(48\,\pi^4 - 2304\,\pi^5\,r^2 + 34\,048\,\pi^6\,r^4 + 4\,\pi^4\,\text{Sin}[\beta]^2 - 16\,\pi^5\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^6 +$$
$$\left(-12\,\pi^3 + 1056\,\pi^4\,r^2 - 17\,920\,\pi^5\,r^4\right)\,\#1^7 + \left(\pi^2 - 268\,\pi^3\,r^2 + 6496\,\pi^4\,r^4\right)\,\#1^8 +$$
$$\left(36\,\pi^2\,r^2 - 1600\,\pi^3\,r^4\right)\,\#1^9 + \left(-2\,\pi\,r^2 + 256\,\pi^2\,r^4\right)\,\#1^{10} - 24\,\pi\,r^4\,\#1^{11} + r^4\,\#1^{12}\,\&,\,7\Big]\Big\},$$
$$\Big\{\theta \to \text{Root}\Big[64\,\pi^{10}\,\text{Sin}[\beta]^4 + \left(-128\,\pi^9\,\text{Sin}[\beta]^2 + 1024\,\pi^{10}\,r^2\,\text{Sin}[\beta]^2\right)\,\#1 +$$
$$\left(64\,\pi^8 + 1024\,\pi^9\,r^2 + 4096\,\pi^{10}\,r^4 - 96\,\pi^8\,\text{Sin}[\beta]^2 - 2304\,\pi^9\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^2 +$$
$$\left(96\,\pi^7 - 1536\,\pi^8\,r^2 - 18\,432\,\pi^9\,r^4 + 32\,\pi^7\,\text{Sin}[\beta]^2 + 2048\,\pi^8\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^3 +$$
$$\left(-60\,\pi^6 - 448\,\pi^7\,r^2 + 37\,120\,\pi^8\,r^4 + 64\,\pi^6\,\text{Sin}[\beta]^2 - 896\,\pi^7\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^4 +$$
$$\left(-56\,\pi^5 + 2496\,\pi^6\,r^2 - 44\,032\,\pi^7\,r^4 - 32\,\pi^5\,\text{Sin}[\beta]^2 + 192\,\pi^6\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^5 +$$
$$\left(48\,\pi^4 - 2304\,\pi^5\,r^2 + 34\,048\,\pi^6\,r^4 + 4\,\pi^4\,\text{Sin}[\beta]^2 - 16\,\pi^5\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^6 +$$
$$\left(-12\,\pi^3 + 1056\,\pi^4\,r^2 - 17\,920\,\pi^5\,r^4\right)\,\#1^7 + \left(\pi^2 - 268\,\pi^3\,r^2 + 6496\,\pi^4\,r^4\right)\,\#1^8 +$$
$$\left(36\,\pi^2\,r^2 - 1600\,\pi^3\,r^4\right)\,\#1^9 + \left(-2\,\pi\,r^2 + 256\,\pi^2\,r^4\right)\,\#1^{10} - 24\,\pi\,r^4\,\#1^{11} + r^4\,\#1^{12}\,\&,\,8\Big]\Big\},$$
$$\Big\{\theta \to \text{Root}\Big[64\,\pi^{10}\,\text{Sin}[\beta]^4 + \left(-128\,\pi^9\,\text{Sin}[\beta]^2 + 1024\,\pi^{10}\,r^2\,\text{Sin}[\beta]^2\right)\,\#1 +$$
$$\left(64\,\pi^8 + 1024\,\pi^9\,r^2 + 4096\,\pi^{10}\,r^4 - 96\,\pi^8\,\text{Sin}[\beta]^2 - 2304\,\pi^9\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^2 +$$
$$\left(96\,\pi^7 - 1536\,\pi^8\,r^2 - 18\,432\,\pi^9\,r^4 + 32\,\pi^7\,\text{Sin}[\beta]^2 + 2048\,\pi^8\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^3 +$$
$$\left(-60\,\pi^6 - 448\,\pi^7\,r^2 + 37\,120\,\pi^8\,r^4 + 64\,\pi^6\,\text{Sin}[\beta]^2 - 896\,\pi^7\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^4 +$$
$$\left(-56\,\pi^5 + 2496\,\pi^6\,r^2 - 44\,032\,\pi^7\,r^4 - 32\,\pi^5\,\text{Sin}[\beta]^2 + 192\,\pi^6\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^5 +$$
$$\left(48\,\pi^4 - 2304\,\pi^5\,r^2 + 34\,048\,\pi^6\,r^4 + 4\,\pi^4\,\text{Sin}[\beta]^2 - 16\,\pi^5\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^6 +$$
$$\left(-12\,\pi^3 + 1056\,\pi^4\,r^2 - 17\,920\,\pi^5\,r^4\right)\,\#1^7 + \left(\pi^2 - 268\,\pi^3\,r^2 + 6496\,\pi^4\,r^4\right)\,\#1^8 +$$
$$\left(36\,\pi^2\,r^2 - 1600\,\pi^3\,r^4\right)\,\#1^9 + \left(-2\,\pi\,r^2 + 256\,\pi^2\,r^4\right)\,\#1^{10} - 24\,\pi\,r^4\,\#1^{11} + r^4\,\#1^{12}\,\&,\,9\Big]\Big\},$$
$$\Big\{\theta \to \text{Root}\Big[64\,\pi^{10}\,\text{Sin}[\beta]^4 + \left(-128\,\pi^9\,\text{Sin}[\beta]^2 + 1024\,\pi^{10}\,r^2\,\text{Sin}[\beta]^2\right)\,\#1 +$$
$$\left(64\,\pi^8 + 1024\,\pi^9\,r^2 + 4096\,\pi^{10}\,r^4 - 96\,\pi^8\,\text{Sin}[\beta]^2 - 2304\,\pi^9\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^2 +$$
$$\left(96\,\pi^7 - 1536\,\pi^8\,r^2 - 18\,432\,\pi^9\,r^4 + 32\,\pi^7\,\text{Sin}[\beta]^2 + 2048\,\pi^8\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^3 +$$
$$\left(-60\,\pi^6 - 448\,\pi^7\,r^2 + 37\,120\,\pi^8\,r^4 + 64\,\pi^6\,\text{Sin}[\beta]^2 - 896\,\pi^7\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^4 +$$
$$\left(-56\,\pi^5 + 2496\,\pi^6\,r^2 - 44\,032\,\pi^7\,r^4 - 32\,\pi^5\,\text{Sin}[\beta]^2 + 192\,\pi^6\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^5 +$$
$$\left(48\,\pi^4 - 2304\,\pi^5\,r^2 + 34\,048\,\pi^6\,r^4 + 4\,\pi^4\,\text{Sin}[\beta]^2 - 16\,\pi^5\,r^2\,\text{Sin}[\beta]^2\right)\,\#1^6 +$$

$\left(-12\,\pi^3 + 1056\,\pi^4\,r^2 - 17\,920\,\pi^5\,r^4\right)\,\sharp 1^7 + \left(\pi^2 - 268\,\pi^3\,r^2 + 6496\,\pi^4\,r^4\right)\,\sharp 1^8 +$
$\left(36\,\pi^2\,r^2 - 1600\,\pi^3\,r^4\right)\,\sharp 1^9 + \left(-2\,\pi\,r^2 + 256\,\pi^2\,r^4\right)\,\sharp 1^{10} - 24\,\pi\,r^4\,\sharp 1^{11} + r^4\,\sharp 1^{12}\ \&,\ 10\Big]\Big\},$

$\Big\{\theta \to \text{Root}\Big[64\,\pi^{10}\,\text{Sin}[\beta]^4 + \left(-128\,\pi^9\,\text{Sin}[\beta]^2 + 1024\,\pi^{10}\,r^2\,\text{Sin}[\beta]^2\right)\,\sharp 1 +$
$\left(64\,\pi^8 + 1024\,\pi^9\,r^2 + 4096\,\pi^{10}\,r^4 - 96\,\pi^8\,\text{Sin}[\beta]^2 - 2304\,\pi^9\,r^2\,\text{Sin}[\beta]^2\right)\,\sharp 1^2 +$
$\left(96\,\pi^7 - 1536\,\pi^8\,r^2 - 18\,432\,\pi^9\,r^4 + 32\,\pi^7\,\text{Sin}[\beta]^2 + 2048\,\pi^8\,r^2\,\text{Sin}[\beta]^2\right)\,\sharp 1^3 +$
$\left(-60\,\pi^6 - 448\,\pi^7\,r^2 + 37\,120\,\pi^8\,r^4 + 64\,\pi^6\,\text{Sin}[\beta]^2 - 896\,\pi^7\,r^2\,\text{Sin}[\beta]^2\right)\,\sharp 1^4 +$
$\left(-56\,\pi^5 + 2496\,\pi^6\,r^2 - 44\,032\,\pi^7\,r^4 - 32\,\pi^5\,\text{Sin}[\beta]^2 + 192\,\pi^6\,r^2\,\text{Sin}[\beta]^2\right)\,\sharp 1^5 +$
$\left(48\,\pi^4 - 2304\,\pi^5\,r^2 + 34\,048\,\pi^6\,r^4 + 4\,\pi^4\,\text{Sin}[\beta]^2 - 16\,\pi^5\,r^2\,\text{Sin}[\beta]^2\right)\,\sharp 1^6 +$
$\left(-12\,\pi^3 + 1056\,\pi^4\,r^2 - 17\,920\,\pi^5\,r^4\right)\,\sharp 1^7 + \left(\pi^2 - 268\,\pi^3\,r^2 + 6496\,\pi^4\,r^4\right)\,\sharp 1^8 +$
$\left(36\,\pi^2\,r^2 - 1600\,\pi^3\,r^4\right)\,\sharp 1^9 + \left(-2\,\pi\,r^2 + 256\,\pi^2\,r^4\right)\,\sharp 1^{10} - 24\,\pi\,r^4\,\sharp 1^{11} + r^4\,\sharp 1^{12}\ \&,\ 11\Big]\Big\},$

$\Big\{\theta \to \text{Root}\Big[64\,\pi^{10}\,\text{Sin}[\beta]^4 + \left(-128\,\pi^9\,\text{Sin}[\beta]^2 + 1024\,\pi^{10}\,r^2\,\text{Sin}[\beta]^2\right)\,\sharp 1 +$
$\left(64\,\pi^8 + 1024\,\pi^9\,r^2 + 4096\,\pi^{10}\,r^4 - 96\,\pi^8\,\text{Sin}[\beta]^2 - 2304\,\pi^9\,r^2\,\text{Sin}[\beta]^2\right)\,\sharp 1^2 +$
$\left(96\,\pi^7 - 1536\,\pi^8\,r^2 - 18\,432\,\pi^9\,r^4 + 32\,\pi^7\,\text{Sin}[\beta]^2 + 2048\,\pi^8\,r^2\,\text{Sin}[\beta]^2\right)\,\sharp 1^3 +$
$\left(-60\,\pi^6 - 448\,\pi^7\,r^2 + 37\,120\,\pi^8\,r^4 + 64\,\pi^6\,\text{Sin}[\beta]^2 - 896\,\pi^7\,r^2\,\text{Sin}[\beta]^2\right)\,\sharp 1^4 +$
$\left(-56\,\pi^5 + 2496\,\pi^6\,r^2 - 44\,032\,\pi^7\,r^4 - 32\,\pi^5\,\text{Sin}[\beta]^2 + 192\,\pi^6\,r^2\,\text{Sin}[\beta]^2\right)\,\sharp 1^5 +$
$\left(48\,\pi^4 - 2304\,\pi^5\,r^2 + 34\,048\,\pi^6\,r^4 + 4\,\pi^4\,\text{Sin}[\beta]^2 - 16\,\pi^5\,r^2\,\text{Sin}[\beta]^2\right)\,\sharp 1^6 +$
$\left(-12\,\pi^3 + 1056\,\pi^4\,r^2 - 17\,920\,\pi^5\,r^4\right)\,\sharp 1^7 + \left(\pi^2 - 268\,\pi^3\,r^2 + 6496\,\pi^4\,r^4\right)\,\sharp 1^8 +$
$\left(36\,\pi^2\,r^2 - 1600\,\pi^3\,r^4\right)\,\sharp 1^9 + \left(-2\,\pi\,r^2 + 256\,\pi^2\,r^4\right)\,\sharp 1^{10} - 24\,\pi\,r^4\,\sharp 1^{11} + r^4\,\sharp 1^{12}\ \&,\ 12\Big]\Big\}\Big\}$

$$\textbf{Solve}\Big[r\,\hat{}\,2 == \frac{1}{2\pi}\left(\sqrt{\left(4\pi\,(r)^2\,\theta - \left(\left(\left(-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta] + 4\pi^2\,\text{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta} - \frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\text{Sin}[\beta]^2 + 8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)^2\theta^2\right)\right)}\right)\hat{}\,2 + \frac{\theta\,r - 2\pi\,r}{2\pi}\,\hat{}\,2,\ \theta\Big]$$

Solve::verif :
Potential solution $\{\theta \to 0\}$ (possibly discarded by verifier) should be checked by hand. May require use of limits. ≫

$$\Big\{\Big\{\theta \to -\frac{1 - 8\pi r}{4r} - \frac{1}{2}\sqrt{\left(\frac{(1-8\pi r)^2}{4r^2} - \frac{-1 + 24\pi r - 80\pi^2 r^2}{12 r^2} - \frac{1 - 24\pi r + 80\pi^2 r^2}{4 r^2} + \left(1 - 32\pi^2 r^2 + 256\pi^4 r^4 + 192\right.\right.}$$

$$\pi^2\,r^2\,\mathrm{Sin}[\beta]^2\Big)\Big/\Big(6\,2^{2/3}\,r^2\,\Big(2-96\,\pi^2\,r^2+1536\,\pi^4\,r^4-8192\,\pi^6\,r^6+576\,\pi^2\,r^2$$
$$\mathrm{Sin}[\beta]^2+18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2-\sqrt{}\Big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2-5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2+$$
$$84\,934\,656\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^2-452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2-110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4+$$
$$35\,389\,440\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^4+226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4-28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Big)+$$

$$\frac{1}{12\,2^{1/3}\,r^2}\,\Big(2-96\,\pi^2\,r^2+1536\,\pi^4\,r^4-8192\,\pi^6\,r^6+576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2+18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2-$$
$$\sqrt{}\Big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2-5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2+84\,934\,656\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^2-$$
$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2-110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4+35\,389\,440\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^4+$$
$$226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4-28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Bigg)-$$

$$\frac{1}{2}\sqrt{}\Bigg(\Bigg(\frac{(1-8\,\pi\,r)^2}{2\,r^2}+\frac{-1+24\,\pi\,r-80\,\pi^2\,r^2}{12\,r^2}-\frac{1-24\,\pi\,r+80\,\pi^2\,r^2}{4\,r^2}-$$
$$\Big(1-32\,\pi^2\,r^2+256\,\pi^4\,r^4+192\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2\Big)\Big/$$
$$\Big(6\,2^{2/3}\,r^2\,\Big(2-96\,\pi^2\,r^2+1536\,\pi^4\,r^4-8192\,\pi^6\,r^6+576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2+18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2-$$
$$\sqrt{}\Big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2-5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2+84\,934\,656\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^2-$$
$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2-110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4+35\,389\,440\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^4+$$
$$226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4-28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Big)-$$

$$\frac{1}{12\,2^{1/3}\,r^2}\,\Big(2-96\,\pi^2\,r^2+1536\,\pi^4\,r^4-8192\,\pi^6\,r^6+576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2+18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2-$$
$$\sqrt{}\Big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2-5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2+84\,934\,656\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^2-$$
$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2-110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4+35\,389\,440\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^4+$$
$$226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4-28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}-$$

$$\Bigg(-\frac{(1-8\,\pi\,r)^3}{r^3}+\frac{(1-8\,\pi\,r)\,\big(1-24\,\pi\,r+80\,\pi^2\,r^2\big)}{r^3}-\frac{8\,\big(-\pi+8\,\pi^2\,r-16\,\pi^3\,r^2\big)}{r^2}\Bigg)\Bigg/$$

$$\Bigg(4\,\sqrt{}\Bigg(\Bigg(\frac{(1-8\,\pi\,r)^2}{4\,r^2}-\frac{-1+24\,\pi\,r-80\,\pi^2\,r^2}{12\,r^2}-\frac{1-24\,\pi\,r+80\,\pi^2\,r^2}{4\,r^2}+$$
$$\Big(1-32\,\pi^2\,r^2+256\,\pi^4\,r^4+192\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2\Big)\Big/\Big(6\,2^{2/3}\,r^2$$
$$\Big(2-96\,\pi^2\,r^2+1536\,\pi^4\,r^4-8192\,\pi^6\,r^6+576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2+18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2-$$
$$\sqrt{}\Big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2-5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2+84\,934\,656\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^2-$$
$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2-110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4+35\,389\,440\,\pi^6\,r^6$$
$$\mathrm{Sin}[\beta]^4+226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4-28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Big)+$$

$$\frac{1}{12\,2^{1/3}\,r^2}\,\Big(2-96\,\pi^2\,r^2+1536\,\pi^4\,r^4-8192\,\pi^6\,r^6+576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2+18\,432$$
$$\pi^4\,r^4\,\mathrm{Sin}[\beta]^2-\sqrt{}\Big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2-5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2+84\,934\,656\,\pi^8$$

$$r^8\,\mathrm{Sin}[\beta]^2 - 452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440$$

$$\pi^6\,r^6\,\mathrm{Sin}[\beta]^4 + 226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6))^{1/3}))))\},$$

$$\{\theta \to -\frac{1-8\pi r}{4r} - \frac{1}{2}\sqrt{}\Bigg(\Bigg(\frac{(1-8\pi r)^2}{4r^2} - \frac{-1+24\pi r-80\pi^2 r^2}{12r^2} - \frac{1-24\pi r+80\pi^2 r^2}{4r^2} +$$

$$(1-32\pi^2 r^2+256\pi^4 r^4+192\pi^2 r^2\,\mathrm{Sin}[\beta]^2) \Big/$$

$$\Big(6\,2^{2/3}\,r^2\,\Big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\mathrm{Sin}[\beta]^2+18\,432\,\pi^4 r^4\,\mathrm{Sin}[\beta]^2 -$$

$$\sqrt{}\,(110\,592\,\pi^4 r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6 r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8 r^8\,\mathrm{Sin}[\beta]^2 -$$

$$452\,984\,832\,\pi^{10} r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4 r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6 r^6\,\mathrm{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8 r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6 r^6\,\mathrm{Sin}[\beta]^6)\Big)^{1/3}\Big) +$$

$$\frac{1}{12\,2^{1/3}\,r^2}\Big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\mathrm{Sin}[\beta]^2+18\,432\,\pi^4 r^4\,\mathrm{Sin}[\beta]^2 -$$

$$\sqrt{}\,(110\,592\,\pi^4 r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6 r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8 r^8\,\mathrm{Sin}[\beta]^2 -$$

$$452\,984\,832\,\pi^{10} r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4 r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6 r^6\,\mathrm{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8 r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6 r^6\,\mathrm{Sin}[\beta]^6)\Big)^{1/3}\Bigg) +$$

$$\frac{1}{2}\sqrt{}\Bigg(\Bigg(\frac{(1-8\pi r)^2}{2r^2} + \frac{-1+24\pi r-80\pi^2 r^2}{12r^2} - \frac{1-24\pi r+80\pi^2 r^2}{4r^2} -$$

$$(1-32\pi^2 r^2+256\pi^4 r^4+192\pi^2 r^2\,\mathrm{Sin}[\beta]^2) \Big/$$

$$\Big(6\,2^{2/3}\,r^2\,\Big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\mathrm{Sin}[\beta]^2+18\,432\,\pi^4 r^4\,\mathrm{Sin}[\beta]^2 -$$

$$\sqrt{}\,(110\,592\,\pi^4 r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6 r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8 r^8\,\mathrm{Sin}[\beta]^2 -$$

$$452\,984\,832\,\pi^{10} r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4 r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6 r^6\,\mathrm{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8 r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6 r^6\,\mathrm{Sin}[\beta]^6)\Big)^{1/3}\Big) -$$

$$\frac{1}{12\,2^{1/3}\,r^2}\Big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\mathrm{Sin}[\beta]^2+18\,432\,\pi^4 r^4\,\mathrm{Sin}[\beta]^2 -$$

$$\sqrt{}\,(110\,592\,\pi^4 r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6 r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8 r^8\,\mathrm{Sin}[\beta]^2 -$$

$$452\,984\,832\,\pi^{10} r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4 r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6 r^6\,\mathrm{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8 r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6 r^6\,\mathrm{Sin}[\beta]^6)\Big)^{1/3} -$$

$$\left(-\frac{(1-8\pi r)^3}{r^3} + \frac{(1-8\pi r)\,(1-24\pi r+80\pi^2 r^2)}{r^3} - \frac{8\,(-\pi+8\pi^2 r-16\pi^3 r^2)}{r^2}\right) \Big/$$

$$\Bigg(4\sqrt{}\Bigg(\Bigg(\frac{(1-8\pi r)^2}{4r^2} - \frac{-1+24\pi r-80\pi^2 r^2}{12r^2} - \frac{1-24\pi r+80\pi^2 r^2}{4r^2} +$$

$$(1-32\pi^2 r^2+256\pi^4 r^4+192\pi^2 r^2\,\mathrm{Sin}[\beta]^2) \Big/ \Big(6\,2^{2/3}\,r^2$$

$$\left(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2 \text{Sin}[\beta]^2+18432\pi^4 r^4 \text{Sin}[\beta]^2-\right.$$
$$\sqrt{\left(110592\pi^4 r^4 \text{Sin}[\beta]^2-5308416\pi^6 r^6 \text{Sin}[\beta]^2+84934656\pi^8 r^8 \text{Sin}[\beta]^2-\right.}$$
$$452984832\pi^{10} r^{10} \text{Sin}[\beta]^2-110592\pi^4 r^4 \text{Sin}[\beta]^4+35389440\pi^6 r^6$$
$$\left.\left.\text{Sin}[\beta]^4+226492416\pi^8 r^8 \text{Sin}[\beta]^4-28311552\pi^6 r^6 \text{Sin}[\beta]^6\right)\right)^{1/3}\Big)+$$

$$\frac{1}{12\, 2^{1/3} r^2}\left(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2 \text{Sin}[\beta]^2+18432\right.$$
$$\pi^4 r^4 \text{Sin}[\beta]^2-\sqrt{\left(110592\pi^4 r^4 \text{Sin}[\beta]^2-5308416\pi^6 r^6 \text{Sin}[\beta]^2+84934656\pi^8\right.}$$
$$r^8 \text{Sin}[\beta]^2-452984832\pi^{10} r^{10} \text{Sin}[\beta]^2-110592\pi^4 r^4 \text{Sin}[\beta]^4+35389440$$
$$\left.\left.\pi^6 r^6 \text{Sin}[\beta]^4+226492416\pi^8 r^8 \text{Sin}[\beta]^4-28311552\pi^6 r^6 \text{Sin}[\beta]^6\right)\right)^{1/3}\Big)\Big)\Big)\Big\},$$

$$\Big\{\theta\to-\frac{1-8\pi r}{4r}+\frac{1}{2}\sqrt{\left(\frac{(1-8\pi r)^2}{4r^2}-\frac{-1+24\pi r-80\pi^2 r^2}{12r^2}-\frac{1-24\pi r+80\pi^2 r^2}{4r^2}+\right.}$$
$$\left(1-32\pi^2 r^2+256\pi^4 r^4+192\pi^2 r^2 \text{Sin}[\beta]^2\right)\Big/$$
$$\left(6\, 2^{2/3} r^2\left(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2 \text{Sin}[\beta]^2+18432\pi^4 r^4 \text{Sin}[\beta]^2-\right.\right.$$
$$\sqrt{\left(110592\pi^4 r^4 \text{Sin}[\beta]^2-5308416\pi^6 r^6 \text{Sin}[\beta]^2+84934656\pi^8 r^8 \text{Sin}[\beta]^2-\right.}$$
$$452984832\pi^{10} r^{10} \text{Sin}[\beta]^2-110592\pi^4 r^4 \text{Sin}[\beta]^4+35389440\pi^6 r^6 \text{Sin}[\beta]^4+$$
$$\left.\left.226492416\pi^8 r^8 \text{Sin}[\beta]^4-28311552\pi^6 r^6 \text{Sin}[\beta]^6\right)\right)^{1/3}\Big)+$$

$$\frac{1}{12\, 2^{1/3} r^2}\left(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2 \text{Sin}[\beta]^2+18432\pi^4 r^4 \text{Sin}[\beta]^2-\right.$$
$$\sqrt{\left(110592\pi^4 r^4 \text{Sin}[\beta]^2-5308416\pi^6 r^6 \text{Sin}[\beta]^2+84934656\pi^8 r^8 \text{Sin}[\beta]^2-\right.}$$
$$452984832\pi^{10} r^{10} \text{Sin}[\beta]^2-110592\pi^4 r^4 \text{Sin}[\beta]^4+35389440\pi^6 r^6 \text{Sin}[\beta]^4+$$
$$\left.\left.226492416\pi^8 r^8 \text{Sin}[\beta]^4-28311552\pi^6 r^6 \text{Sin}[\beta]^6\right)\right)^{1/3}\Big)-$$

$$\frac{1}{2}\sqrt{\left(\frac{(1-8\pi r)^2}{2r^2}+\frac{-1+24\pi r-80\pi^2 r^2}{12r^2}-\frac{1-24\pi r+80\pi^2 r^2}{4r^2}-\right.}$$
$$\left(1-32\pi^2 r^2+256\pi^4 r^4+192\pi^2 r^2 \text{Sin}[\beta]^2\right)\Big/$$
$$\left(6\, 2^{2/3} r^2\left(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2 \text{Sin}[\beta]^2+18432\pi^4 r^4 \text{Sin}[\beta]^2-\right.\right.$$
$$\sqrt{\left(110592\pi^4 r^4 \text{Sin}[\beta]^2-5308416\pi^6 r^6 \text{Sin}[\beta]^2+84934656\pi^8 r^8 \text{Sin}[\beta]^2-\right.}$$
$$452984832\pi^{10} r^{10} \text{Sin}[\beta]^2-110592\pi^4 r^4 \text{Sin}[\beta]^4+35389440\pi^6 r^6 \text{Sin}[\beta]^4+$$
$$\left.\left.226492416\pi^8 r^8 \text{Sin}[\beta]^4-28311552\pi^6 r^6 \text{Sin}[\beta]^6\right)\right)^{1/3}\Big)-$$

$$\frac{1}{12\, 2^{1/3} r^2}\left(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2 \text{Sin}[\beta]^2+18432\pi^4 r^4 \text{Sin}[\beta]^2-\right.$$
$$\sqrt{\left(110592\pi^4 r^4 \text{Sin}[\beta]^2-5308416\pi^6 r^6 \text{Sin}[\beta]^2+84934656\pi^8 r^8 \text{Sin}[\beta]^2-\right.}$$
$$452984832\pi^{10} r^{10} \text{Sin}[\beta]^2-110592\pi^4 r^4 \text{Sin}[\beta]^4+35389440\pi^6 r^6 \text{Sin}[\beta]^4+$$

$$226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4-28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}+$$

$$\left(-\frac{(1-8\pi r)^3}{r^3}+\frac{(1-8\pi r)\left(1-24\pi r+80\pi^2 r^2\right)}{r^3}-\frac{8\left(-\pi+8\pi^2 r-16\pi^3 r^2\right)}{r^2}\right)\Big/$$

$$\Bigg(4\sqrt{\Bigg(\frac{(1-8\pi r)^2}{4\,r^2}-\frac{-1+24\pi r-80\pi^2 r^2}{12\,r^2}-\frac{1-24\pi r+80\pi^2 r^2}{4\,r^2}+}$$

$$\left(1-32\pi^2 r^2+256\pi^4 r^4+192\pi^2 r^2\,\mathrm{Sin}[\beta]^2\right)\Big/\Big(6\;2^{2/3}\,r^2$$

$$\Big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\mathrm{Sin}[\beta]^2+18\,432\,\pi^4 r^4\,\mathrm{Sin}[\beta]^2-$$

$$\sqrt{}\Big(110\,592\,\pi^4 r^4\,\mathrm{Sin}[\beta]^2-5\,308\,416\,\pi^6 r^6\,\mathrm{Sin}[\beta]^2+84\,934\,656\,\pi^8 r^8\,\mathrm{Sin}[\beta]^2-$$

$$452\,984\,832\,\pi^{10} r^{10}\,\mathrm{Sin}[\beta]^2-110\,592\,\pi^4 r^4\,\mathrm{Sin}[\beta]^4+35\,389\,440\,\pi^6 r^6$$

$$\mathrm{Sin}[\beta]^4+226\,492\,416\,\pi^8 r^8\,\mathrm{Sin}[\beta]^4-28\,311\,552\,\pi^6 r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Big)+$$

$$\frac{1}{12\;2^{1/3}\,r^2}\Big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\mathrm{Sin}[\beta]^2+18\,432$$

$$\pi^4 r^4\,\mathrm{Sin}[\beta]^2-\sqrt{}\Big(110\,592\,\pi^4 r^4\,\mathrm{Sin}[\beta]^2-5\,308\,416\,\pi^6 r^6\,\mathrm{Sin}[\beta]^2+84\,934\,656\,\pi^8$$

$$r^8\,\mathrm{Sin}[\beta]^2-452\,984\,832\,\pi^{10} r^{10}\,\mathrm{Sin}[\beta]^2-110\,592\,\pi^4 r^4\,\mathrm{Sin}[\beta]^4+35\,389\,440$$

$$\pi^6 r^6\,\mathrm{Sin}[\beta]^4+226\,492\,416\,\pi^8 r^8\,\mathrm{Sin}[\beta]^4-28\,311\,552\,\pi^6 r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Big)\Bigg)\Bigg)\Bigg\},$$

$$\Bigg\{\theta\to-\frac{1-8\pi r}{4\,r}+\frac{1}{2}\sqrt{\Bigg(\frac{(1-8\pi r)^2}{4\,r^2}-\frac{-1+24\pi r-80\pi^2 r^2}{12\,r^2}-\frac{1-24\pi r+80\pi^2 r^2}{4\,r^2}+}$$

$$\left(1-32\pi^2 r^2+256\pi^4 r^4+192\pi^2 r^2\,\mathrm{Sin}[\beta]^2\right)\Big/$$

$$\Big(6\;2^{2/3}\,r^2\,\Big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\mathrm{Sin}[\beta]^2+18\,432\,\pi^4 r^4\,\mathrm{Sin}[\beta]^2-$$

$$\sqrt{}\Big(110\,592\,\pi^4 r^4\,\mathrm{Sin}[\beta]^2-5\,308\,416\,\pi^6 r^6\,\mathrm{Sin}[\beta]^2+84\,934\,656\,\pi^8 r^8\,\mathrm{Sin}[\beta]^2-$$

$$452\,984\,832\,\pi^{10} r^{10}\,\mathrm{Sin}[\beta]^2-110\,592\,\pi^4 r^4\,\mathrm{Sin}[\beta]^4+35\,389\,440\,\pi^6 r^6\,\mathrm{Sin}[\beta]^4+$$

$$226\,492\,416\,\pi^8 r^8\,\mathrm{Sin}[\beta]^4-28\,311\,552\,\pi^6 r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Big)+$$

$$\frac{1}{12\;2^{1/3}\,r^2}\Big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\mathrm{Sin}[\beta]^2+18\,432\,\pi^4 r^4\,\mathrm{Sin}[\beta]^2-$$

$$\sqrt{}\Big(110\,592\,\pi^4 r^4\,\mathrm{Sin}[\beta]^2-5\,308\,416\,\pi^6 r^6\,\mathrm{Sin}[\beta]^2+84\,934\,656\,\pi^8 r^8\,\mathrm{Sin}[\beta]^2-$$

$$452\,984\,832\,\pi^{10} r^{10}\,\mathrm{Sin}[\beta]^2-110\,592\,\pi^4 r^4\,\mathrm{Sin}[\beta]^4+35\,389\,440\,\pi^6 r^6\,\mathrm{Sin}[\beta]^4+$$

$$226\,492\,416\,\pi^8 r^8\,\mathrm{Sin}[\beta]^4-28\,311\,552\,\pi^6 r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Bigg)+$$

$$\frac{1}{2}\sqrt{\Bigg(\frac{(1-8\pi r)^2}{2\,r^2}+\frac{-1+24\pi r-80\pi^2 r^2}{12\,r^2}-\frac{1-24\pi r+80\pi^2 r^2}{4\,r^2}-}$$

$$\left(1-32\pi^2 r^2+256\pi^4 r^4+192\pi^2 r^2\,\mathrm{Sin}[\beta]^2\right)\Big/$$

$$\Big(6\;2^{2/3}\,r^2\,\Big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\mathrm{Sin}[\beta]^2+18\,432\,\pi^4 r^4\,\mathrm{Sin}[\beta]^2-$$

$$\sqrt{(110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - 5308416\,\pi^6 r^6 \mathrm{Sin}[\beta]^2 + 84934656\,\pi^8 r^8 \mathrm{Sin}[\beta]^2 -}$$
$$452984832\,\pi^{10} r^{10} \mathrm{Sin}[\beta]^2 - 110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^4 + 35389440\,\pi^6 r^6 \mathrm{Sin}[\beta]^4 +$$
$$226492416\,\pi^8 r^8 \mathrm{Sin}[\beta]^4 - 28311552\,\pi^6 r^6 \mathrm{Sin}[\beta]^6))^{1/3}) -$$

$$\frac{1}{12\,2^{1/3}\,r^2}\,(2 - 96\,\pi^2 r^2 + 1536\,\pi^4 r^4 - 8192\,\pi^6 r^6 + 576\,\pi^2 r^2 \mathrm{Sin}[\beta]^2 + 18432\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 -$$
$$\sqrt{(110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - 5308416\,\pi^6 r^6 \mathrm{Sin}[\beta]^2 + 84934656\,\pi^8 r^8 \mathrm{Sin}[\beta]^2 -}$$
$$452984832\,\pi^{10} r^{10} \mathrm{Sin}[\beta]^2 - 110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^4 + 35389440\,\pi^6 r^6 \mathrm{Sin}[\beta]^4 +$$
$$226492416\,\pi^8 r^8 \mathrm{Sin}[\beta]^4 - 28311552\,\pi^6 r^6 \mathrm{Sin}[\beta]^6))^{1/3} +$$

$$\left(-\frac{(1 - 8\,\pi\,r)^3}{r^3} + \frac{(1 - 8\,\pi\,r)\,(1 - 24\,\pi\,r + 80\,\pi^2 r^2)}{r^3} - \frac{8\,(-\pi + 8\,\pi^2 r - 16\,\pi^3 r^2)}{r^2}\right) \Big/$$

$$\left(4\sqrt{\left(\frac{(1 - 8\,\pi\,r)^2}{4\,r^2} - \frac{-1 + 24\,\pi\,r - 80\,\pi^2 r^2}{12\,r^2} - \frac{1 - 24\,\pi\,r + 80\,\pi^2 r^2}{4\,r^2} +\right.}\right.$$
$$(1 - 32\,\pi^2 r^2 + 256\,\pi^4 r^4 + 192\,\pi^2 r^2 \mathrm{Sin}[\beta]^2) \Big/ (6\,2^{2/3}\,r^2$$
$$(2 - 96\,\pi^2 r^2 + 1536\,\pi^4 r^4 - 8192\,\pi^6 r^6 + 576\,\pi^2 r^2 \mathrm{Sin}[\beta]^2 + 18432\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 -$$
$$\sqrt{(110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - 5308416\,\pi^6 r^6 \mathrm{Sin}[\beta]^2 + 84934656\,\pi^8 r^8 \mathrm{Sin}[\beta]^2 -}$$
$$452984832\,\pi^{10} r^{10} \mathrm{Sin}[\beta]^2 - 110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^4 + 35389440\,\pi^6 r^6$$
$$\mathrm{Sin}[\beta]^4 + 226492416\,\pi^8 r^8 \mathrm{Sin}[\beta]^4 - 28311552\,\pi^6 r^6 \mathrm{Sin}[\beta]^6))^{1/3}) +$$

$$\frac{1}{12\,2^{1/3}\,r^2}\,(2 - 96\,\pi^2 r^2 + 1536\,\pi^4 r^4 - 8192\,\pi^6 r^6 + 576\,\pi^2 r^2 \mathrm{Sin}[\beta]^2 + 18432$$
$$\pi^4 r^4 \mathrm{Sin}[\beta]^2 - \sqrt{(110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - 5308416\,\pi^6 r^6 \mathrm{Sin}[\beta]^2 + 84934656\,\pi^8}$$
$$r^8 \mathrm{Sin}[\beta]^2 - 452984832\,\pi^{10} r^{10} \mathrm{Sin}[\beta]^2 - 110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^4 + 35389440$$
$$\pi^6 r^6 \mathrm{Sin}[\beta]^4 + 226492416\,\pi^8 r^8 \mathrm{Sin}[\beta]^4 - 28311552\,\pi^6 r^6 \mathrm{Sin}[\beta]^6))^{1/3}))))\},$$

$$\{\theta \to -\frac{-1 - 8\,\pi\,r}{4\,r} - \frac{1}{2}\sqrt{\left(\frac{(-1 - 8\,\pi\,r)^2}{4\,r^2} - \frac{-1 - 24\,\pi\,r - 80\,\pi^2 r^2}{12\,r^2} - \frac{1 + 24\,\pi\,r + 80\,\pi^2 r^2}{4\,r^2} +\right.}$$
$$(1 - 32\,\pi^2 r^2 + 256\,\pi^4 r^4 + 192\,\pi^2 r^2 \mathrm{Sin}[\beta]^2) \Big/$$
$$(6\,2^{2/3}\,r^2\,(2 - 96\,\pi^2 r^2 + 1536\,\pi^4 r^4 - 8192\,\pi^6 r^6 + 576\,\pi^2 r^2 \mathrm{Sin}[\beta]^2 + 18432\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 -$$
$$\sqrt{(110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - 5308416\,\pi^6 r^6 \mathrm{Sin}[\beta]^2 + 84934656\,\pi^8 r^8 \mathrm{Sin}[\beta]^2 -}$$
$$452984832\,\pi^{10} r^{10} \mathrm{Sin}[\beta]^2 - 110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^4 + 35389440\,\pi^6 r^6 \mathrm{Sin}[\beta]^4 +$$
$$226492416\,\pi^8 r^8 \mathrm{Sin}[\beta]^4 - 28311552\,\pi^6 r^6 \mathrm{Sin}[\beta]^6))^{1/3}) +$$

$$\frac{1}{12\,2^{1/3}\,r^2}\,(2 - 96\,\pi^2 r^2 + 1536\,\pi^4 r^4 - 8192\,\pi^6 r^6 + 576\,\pi^2 r^2 \mathrm{Sin}[\beta]^2 + 18432\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 -$$
$$\sqrt{(110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - 5308416\,\pi^6 r^6 \mathrm{Sin}[\beta]^2 + 84934656\,\pi^8 r^8 \mathrm{Sin}[\beta]^2 -}$$
$$452984832\,\pi^{10} r^{10} \mathrm{Sin}[\beta]^2 - 110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^4 + 35389440\,\pi^6 r^6 \mathrm{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8\,r^8\,\text{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\text{Sin}[\beta]^6\Big)\Big)^{1/3}\Bigg) -$$

$$\frac{1}{2}\sqrt{\Bigg(\frac{(-1-8\pi r)^2}{2\,r^2} + \frac{-1-24\pi r-80\pi^2 r^2}{12\,r^2} - \frac{1+24\pi r+80\pi^2 r^2}{4\,r^2} -}$$

$$\left(1 - 32\pi^2 r^2 + 256\pi^4 r^4 + 192\pi^2 r^2\,\text{Sin}[\beta]^2\right)\Big/$$

$$\Big(6\,2^{2/3}\,r^2\,\Big(2 - 96\pi^2 r^2 + 1536\pi^4 r^4 - 8192\pi^6 r^6 + 576\pi^2 r^2\,\text{Sin}[\beta]^2 + 18\,432\,\pi^4 r^4\,\text{Sin}[\beta]^2 -$$

$$\sqrt{}\Big(110\,592\,\pi^4 r^4\,\text{Sin}[\beta]^2 - 5\,308\,416\,\pi^6 r^6\,\text{Sin}[\beta]^2 + 84\,934\,656\,\pi^8 r^8\,\text{Sin}[\beta]^2 -$$

$$452\,984\,832\,\pi^{10} r^{10}\,\text{Sin}[\beta]^2 - 110\,592\,\pi^4 r^4\,\text{Sin}[\beta]^4 + 35\,389\,440\,\pi^6 r^6\,\text{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8 r^8\,\text{Sin}[\beta]^4 - 28\,311\,552\,\pi^6 r^6\,\text{Sin}[\beta]^6\Big)\Big)^{1/3}\Big) -$$

$$\frac{1}{12\,2^{1/3}\,r^2}\Big(2 - 96\pi^2 r^2 + 1536\pi^4 r^4 - 8192\pi^6 r^6 + 576\pi^2 r^2\,\text{Sin}[\beta]^2 + 18\,432\,\pi^4 r^4\,\text{Sin}[\beta]^2 -$$

$$\sqrt{}\Big(110\,592\,\pi^4 r^4\,\text{Sin}[\beta]^2 - 5\,308\,416\,\pi^6 r^6\,\text{Sin}[\beta]^2 + 84\,934\,656\,\pi^8 r^8\,\text{Sin}[\beta]^2 -$$

$$452\,984\,832\,\pi^{10} r^{10}\,\text{Sin}[\beta]^2 - 110\,592\,\pi^4 r^4\,\text{Sin}[\beta]^4 + 35\,389\,440\,\pi^6 r^6\,\text{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8 r^8\,\text{Sin}[\beta]^4 - 28\,311\,552\,\pi^6 r^6\,\text{Sin}[\beta]^6\Big)\Big)^{1/3} -$$

$$\left(-\frac{(-1-8\pi r)^3}{r^3} + \frac{(-1-8\pi r)\left(1+24\pi r+80\pi^2 r^2\right)}{r^3} - \frac{8\left(-\pi-8\pi^2 r-16\pi^3 r^2\right)}{r^2}\right)\Big/$$

$$\Bigg(4\sqrt{\Bigg(\frac{(-1-8\pi r)^2}{4\,r^2} - \frac{-1-24\pi r-80\pi^2 r^2}{12\,r^2} - \frac{1+24\pi r+80\pi^2 r^2}{4\,r^2} +}$$

$$\left(1 - 32\pi^2 r^2 + 256\pi^4 r^4 + 192\pi^2 r^2\,\text{Sin}[\beta]^2\right)\Big/\Big(6\,2^{2/3}\,r^2$$

$$\Big(2 - 96\pi^2 r^2 + 1536\pi^4 r^4 - 8192\pi^6 r^6 + 576\pi^2 r^2\,\text{Sin}[\beta]^2 + 18\,432\,\pi^4 r^4\,\text{Sin}[\beta]^2 -$$

$$\sqrt{}\Big(110\,592\,\pi^4 r^4\,\text{Sin}[\beta]^2 - 5\,308\,416\,\pi^6 r^6\,\text{Sin}[\beta]^2 + 84\,934\,656\,\pi^8 r^8\,\text{Sin}[\beta]^2 -$$

$$452\,984\,832\,\pi^{10} r^{10}\,\text{Sin}[\beta]^2 - 110\,592\,\pi^4 r^4\,\text{Sin}[\beta]^4 + 35\,389\,440\,\pi^6 r^6$$

$$\text{Sin}[\beta]^4 + 226\,492\,416\,\pi^8 r^8\,\text{Sin}[\beta]^4 - 28\,311\,552\,\pi^6 r^6\,\text{Sin}[\beta]^6\Big)\Big)^{1/3}\Big) +$$

$$\frac{1}{12\,2^{1/3}\,r^2}\Big(2 - 96\pi^2 r^2 + 1536\pi^4 r^4 - 8192\pi^6 r^6 + 576\pi^2 r^2\,\text{Sin}[\beta]^2 + 18\,432$$

$$\pi^4 r^4\,\text{Sin}[\beta]^2 - \sqrt{}\Big(110\,592\,\pi^4 r^4\,\text{Sin}[\beta]^2 - 5\,308\,416\,\pi^6 r^6\,\text{Sin}[\beta]^2 + 84\,934\,656\,\pi^8$$

$$r^8\,\text{Sin}[\beta]^2 - 452\,984\,832\,\pi^{10} r^{10}\,\text{Sin}[\beta]^2 - 110\,592\,\pi^4 r^4\,\text{Sin}[\beta]^4 + 35\,389\,440$$

$$\pi^6 r^6\,\text{Sin}[\beta]^4 + 226\,492\,416\,\pi^8 r^8\,\text{Sin}[\beta]^4 - 28\,311\,552\,\pi^6 r^6\,\text{Sin}[\beta]^6\Big)\Big)^{1/3}\Bigg)\Bigg)\Bigg)\Bigg\},$$

$$\Big\{\theta \to -\frac{-1-8\pi r}{4\,r} - \frac{1}{2}\sqrt{\Bigg(\frac{(-1-8\pi r)^2}{4\,r^2} - \frac{-1-24\pi r-80\pi^2 r^2}{12\,r^2} - \frac{1+24\pi r+80\pi^2 r^2}{4\,r^2} +}$$

$$\left(1 - 32\pi^2 r^2 + 256\pi^4 r^4 + 192\pi^2 r^2\,\text{Sin}[\beta]^2\right)\Big/$$

$$\Big(6\,2^{2/3}\,r^2\,\Big(2 - 96\pi^2 r^2 + 1536\pi^4 r^4 - 8192\pi^6 r^6 + 576\pi^2 r^2\,\text{Sin}[\beta]^2 + 18\,432\,\pi^4 r^4\,\text{Sin}[\beta]^2 -$$

$$\sqrt{}\left(110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\text{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\text{Sin}[\beta]^2 -\right.$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\text{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6\,\text{Sin}[\beta]^4 +$$

$$\left.\left.226\,492\,416\,\pi^8\,r^8\,\text{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\text{Sin}[\beta]^6\right)\right)^{1/3}\Big) +$$

$$\frac{1}{12\;2^{1/3}\,r^2}\,\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\text{Sin}[\beta]^2 + 18\,432\,\pi^4\,r^4\,\text{Sin}[\beta]^2 -$$

$$\sqrt{}\left(110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\text{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\text{Sin}[\beta]^2 -\right.$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\text{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6\,\text{Sin}[\beta]^4 +$$

$$\left.\left.226\,492\,416\,\pi^8\,r^8\,\text{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\text{Sin}[\beta]^6\right)\right)^{1/3}\Big) +$$

$$\frac{1}{2}\sqrt{}\left(\frac{(-1 - 8\,\pi\,r)^2}{2\,r^2} + \frac{-1 - 24\,\pi\,r - 80\,\pi^2\,r^2}{12\,r^2} - \frac{1 + 24\,\pi\,r + 80\,\pi^2\,r^2}{4\,r^2} -\right.$$

$$\left(1 - 32\,\pi^2\,r^2 + 256\,\pi^4\,r^4 + 192\,\pi^2\,r^2\,\text{Sin}[\beta]^2\right)\Big/$$

$$\Big(6\;2^{2/3}\,r^2\,\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\text{Sin}[\beta]^2 + 18\,432\,\pi^4\,r^4\,\text{Sin}[\beta]^2 -$$

$$\sqrt{}\left(110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\text{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\text{Sin}[\beta]^2 -\right.$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\text{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6\,\text{Sin}[\beta]^4 +$$

$$\left.\left.226\,492\,416\,\pi^8\,r^8\,\text{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\text{Sin}[\beta]^6\right)\right)^{1/3}\Big) -$$

$$\frac{1}{12\;2^{1/3}\,r^2}\,\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\text{Sin}[\beta]^2 + 18\,432\,\pi^4\,r^4\,\text{Sin}[\beta]^2 -$$

$$\sqrt{}\left(110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\text{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\text{Sin}[\beta]^2 -\right.$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\text{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6\,\text{Sin}[\beta]^4 +$$

$$\left.\left.226\,492\,416\,\pi^8\,r^8\,\text{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\text{Sin}[\beta]^6\right)\right)^{1/3} -$$

$$\left(-\frac{(-1 - 8\,\pi\,r)^3}{r^3} + \frac{(-1 - 8\,\pi\,r)\,\left(1 + 24\,\pi\,r + 80\,\pi^2\,r^2\right)}{r^3} - \frac{8\,\left(-\pi - 8\,\pi^2\,r - 16\,\pi^3\,r^2\right)}{r^2}\right)\Big/$$

$$\left(4\sqrt{}\left(\frac{(-1 - 8\,\pi\,r)^2}{4\,r^2} - \frac{-1 - 24\,\pi\,r - 80\,\pi^2\,r^2}{12\,r^2} - \frac{1 + 24\,\pi\,r + 80\,\pi^2\,r^2}{4\,r^2} +\right.\right.$$

$$\left(1 - 32\,\pi^2\,r^2 + 256\,\pi^4\,r^4 + 192\,\pi^2\,r^2\,\text{Sin}[\beta]^2\right)\Big/\Big(6\;2^{2/3}\,r^2$$

$$\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\text{Sin}[\beta]^2 + 18\,432\,\pi^4\,r^4\,\text{Sin}[\beta]^2 -$$

$$\sqrt{}\left(110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\text{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\text{Sin}[\beta]^2 -\right.$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\text{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6$$

$$\left.\left.\text{Sin}[\beta]^4 + 226\,492\,416\,\pi^8\,r^8\,\text{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\text{Sin}[\beta]^6\right)\right)^{1/3}\Big) +$$

$$\frac{1}{12\;2^{1/3}\,r^2}\,\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\text{Sin}[\beta]^2 + 18\,432$$

$$\pi^4\,r^4\,\text{Sin}[\beta]^2 - \sqrt{}\Big(110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\text{Sin}[\beta]^2 + 84\,934\,656\,\pi^8$$

$$r^8\,\text{Sin}[\beta]^2 - 452\,984\,832\,\pi^{10}\,r^{10}\,\text{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^4 + 35\,389\,440$$

$$\pi^6 r^6 \mathrm{Sin}[\beta]^4 + 226\,492\,416\,\pi^8 r^8 \mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6 r^6 \mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Big)\Big)\Big)\Big\},$$

$$\Big\{\theta \to -\frac{-1-8\pi r}{4r} + \frac{1}{2}\sqrt{\Big(\frac{(-1-8\pi r)^2}{4r^2} - \frac{-1-24\pi r-80\pi^2 r^2}{12r^2} - \frac{1+24\pi r+80\pi^2 r^2}{4r^2} +}$$

$$\big(1 - 32\pi^2 r^2 + 256\pi^4 r^4 + 192\pi^2 r^2 \mathrm{Sin}[\beta]^2\big) \Big/$$

$$\Big(6\, 2^{2/3} r^2 \Big(2 - 96\pi^2 r^2 + 1536\pi^4 r^4 - 8192\pi^6 r^6 + 576\pi^2 r^2 \mathrm{Sin}[\beta]^2 + 18\,432\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 -$$

$$\sqrt{\big(110\,592\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6 r^6 \mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8 r^8 \mathrm{Sin}[\beta]^2 -}$$

$$452\,984\,832\,\pi^{10} r^{10} \mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4 r^4 \mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6 r^6 \mathrm{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8 r^8 \mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6 r^6 \mathrm{Sin}[\beta]^6\big)\Big)^{1/3}\Big) +$$

$$\frac{1}{12\, 2^{1/3} r^2}\Big(2 - 96\pi^2 r^2 + 1536\pi^4 r^4 - 8192\pi^6 r^6 + 576\pi^2 r^2 \mathrm{Sin}[\beta]^2 + 18\,432\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 -$$

$$\sqrt{\big(110\,592\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6 r^6 \mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8 r^8 \mathrm{Sin}[\beta]^2 -}$$

$$452\,984\,832\,\pi^{10} r^{10} \mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4 r^4 \mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6 r^6 \mathrm{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8 r^8 \mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6 r^6 \mathrm{Sin}[\beta]^6\big)\Big)^{1/3}\Big) -$$

$$\frac{1}{2}\sqrt{\Big(\frac{(-1-8\pi r)^2}{2r^2} + \frac{-1-24\pi r-80\pi^2 r^2}{12r^2} - \frac{1+24\pi r+80\pi^2 r^2}{4r^2} -}$$

$$\big(1 - 32\pi^2 r^2 + 256\pi^4 r^4 + 192\pi^2 r^2 \mathrm{Sin}[\beta]^2\big) \Big/$$

$$\Big(6\, 2^{2/3} r^2 \Big(2 - 96\pi^2 r^2 + 1536\pi^4 r^4 - 8192\pi^6 r^6 + 576\pi^2 r^2 \mathrm{Sin}[\beta]^2 + 18\,432\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 -$$

$$\sqrt{\big(110\,592\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6 r^6 \mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8 r^8 \mathrm{Sin}[\beta]^2 -}$$

$$452\,984\,832\,\pi^{10} r^{10} \mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4 r^4 \mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6 r^6 \mathrm{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8 r^8 \mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6 r^6 \mathrm{Sin}[\beta]^6\big)\Big)^{1/3}\Big) -$$

$$\frac{1}{12\, 2^{1/3} r^2}\Big(2 - 96\pi^2 r^2 + 1536\pi^4 r^4 - 8192\pi^6 r^6 + 576\pi^2 r^2 \mathrm{Sin}[\beta]^2 + 18\,432\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 -$$

$$\sqrt{\big(110\,592\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6 r^6 \mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8 r^8 \mathrm{Sin}[\beta]^2 -}$$

$$452\,984\,832\,\pi^{10} r^{10} \mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4 r^4 \mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6 r^6 \mathrm{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8 r^8 \mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6 r^6 \mathrm{Sin}[\beta]^6\big)\Big)^{1/3} +$$

$$\Big(-\frac{(-1-8\pi r)^3}{r^3} + \frac{(-1-8\pi r)\big(1+24\pi r+80\pi^2 r^2\big)}{r^3} - \frac{8\big(-\pi-8\pi^2 r-16\pi^3 r^2\big)}{r^2}\Big) \Big/$$

$$\Big(4\sqrt{\Big(\frac{(-1-8\pi r)^2}{4r^2} - \frac{-1-24\pi r-80\pi^2 r^2}{12r^2} - \frac{1+24\pi r+80\pi^2 r^2}{4r^2} +}$$

$$\big(1 - 32\pi^2 r^2 + 256\pi^4 r^4 + 192\pi^2 r^2 \mathrm{Sin}[\beta]^2\big) \Big/ \Big(6\, 2^{2/3} r^2$$

$$\Big(2 - 96\pi^2 r^2 + 1536\pi^4 r^4 - 8192\pi^6 r^6 + 576\pi^2 r^2 \mathrm{Sin}[\beta]^2 + 18\,432\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 -$$

$$\sqrt{\left(110592\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - 5308416\,\pi^6\,r^6\,\text{Sin}[\beta]^2 + 84934656\,\pi^8\,r^8\,\text{Sin}[\beta]^2 - 452984832\,\pi^{10}\,r^{10}\,\text{Sin}[\beta]^2 - 110592\,\pi^4\,r^4\,\text{Sin}[\beta]^4 + 35389440\,\pi^6\,r^6\,\text{Sin}[\beta]^4 + 226492416\,\pi^8\,r^8\,\text{Sin}[\beta]^4 - 28311552\,\pi^6\,r^6\,\text{Sin}[\beta]^6\right)}\Big)^{1/3}\Big) +$$

$$\frac{1}{12\,2^{1/3}\,r^2}\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\text{Sin}[\beta]^2 + 18432\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - \sqrt{\left(110592\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - 5308416\,\pi^6\,r^6\,\text{Sin}[\beta]^2 + 84934656\,\pi^8\,r^8\,\text{Sin}[\beta]^2 - 452984832\,\pi^{10}\,r^{10}\,\text{Sin}[\beta]^2 - 110592\,\pi^4\,r^4\,\text{Sin}[\beta]^4 + 35389440\,\pi^6\,r^6\,\text{Sin}[\beta]^4 + 226492416\,\pi^8\,r^8\,\text{Sin}[\beta]^4 - 28311552\,\pi^6\,r^6\,\text{Sin}[\beta]^6\right)}\Big)^{1/3}\Big)\Big)\Big)\Big\},$$

$$\Big\{\theta \to -\frac{-1 - 8\,\pi\,r}{4\,r} + \frac{1}{2}\sqrt{\Big(\frac{(-1 - 8\,\pi\,r)^2}{4\,r^2} - \frac{-1 - 24\,\pi\,r - 80\,\pi^2\,r^2}{12\,r^2} - \frac{1 + 24\,\pi\,r + 80\,\pi^2\,r^2}{4\,r^2} +}$$

$$\left(1 - 32\,\pi^2\,r^2 + 256\,\pi^4\,r^4 + 192\,\pi^2\,r^2\,\text{Sin}[\beta]^2\right) \Big/$$

$$\Big(6\,2^{2/3}\,r^2\,\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\text{Sin}[\beta]^2 + 18432\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - \sqrt{\left(110592\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - 5308416\,\pi^6\,r^6\,\text{Sin}[\beta]^2 + 84934656\,\pi^8\,r^8\,\text{Sin}[\beta]^2 - 452984832\,\pi^{10}\,r^{10}\,\text{Sin}[\beta]^2 - 110592\,\pi^4\,r^4\,\text{Sin}[\beta]^4 + 35389440\,\pi^6\,r^6\,\text{Sin}[\beta]^4 + 226492416\,\pi^8\,r^8\,\text{Sin}[\beta]^4 - 28311552\,\pi^6\,r^6\,\text{Sin}[\beta]^6\right)}\Big)^{1/3}\Big) +$$

$$\frac{1}{12\,2^{1/3}\,r^2}\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\text{Sin}[\beta]^2 + 18432\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - \sqrt{\left(110592\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - 5308416\,\pi^6\,r^6\,\text{Sin}[\beta]^2 + 84934656\,\pi^8\,r^8\,\text{Sin}[\beta]^2 - 452984832\,\pi^{10}\,r^{10}\,\text{Sin}[\beta]^2 - 110592\,\pi^4\,r^4\,\text{Sin}[\beta]^4 + 35389440\,\pi^6\,r^6\,\text{Sin}[\beta]^4 + 226492416\,\pi^8\,r^8\,\text{Sin}[\beta]^4 - 28311552\,\pi^6\,r^6\,\text{Sin}[\beta]^6\right)}\Big)^{1/3}\Big) +$$

$$\frac{1}{2}\sqrt{\Big(\frac{(-1 - 8\,\pi\,r)^2}{2\,r^2} + \frac{-1 - 24\,\pi\,r - 80\,\pi^2\,r^2}{12\,r^2} - \frac{1 + 24\,\pi\,r + 80\,\pi^2\,r^2}{4\,r^2} -}$$

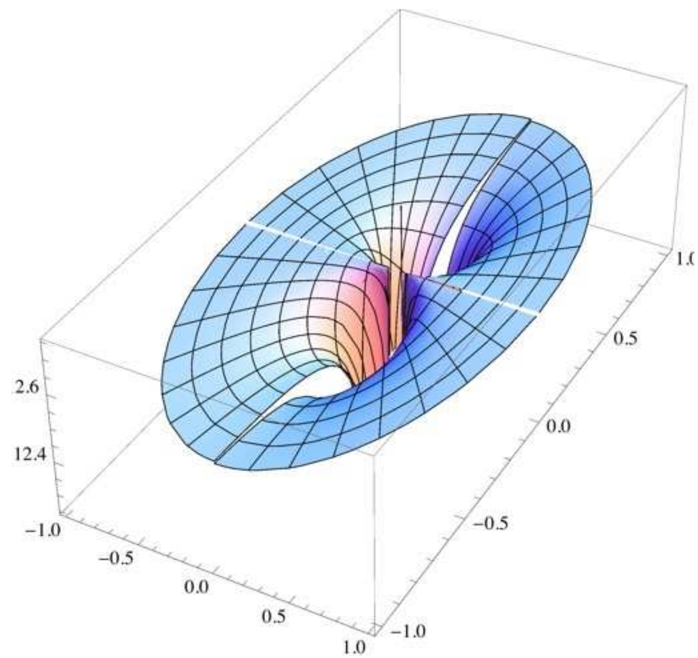
$$\left(1 - 32\,\pi^2\,r^2 + 256\,\pi^4\,r^4 + 192\,\pi^2\,r^2\,\text{Sin}[\beta]^2\right) \Big/$$

$$\Big(6\,2^{2/3}\,r^2\,\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\text{Sin}[\beta]^2 + 18432\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - \sqrt{\left(110592\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - 5308416\,\pi^6\,r^6\,\text{Sin}[\beta]^2 + 84934656\,\pi^8\,r^8\,\text{Sin}[\beta]^2 - 452984832\,\pi^{10}\,r^{10}\,\text{Sin}[\beta]^2 - 110592\,\pi^4\,r^4\,\text{Sin}[\beta]^4 + 35389440\,\pi^6\,r^6\,\text{Sin}[\beta]^4 + 226492416\,\pi^8\,r^8\,\text{Sin}[\beta]^4 - 28311552\,\pi^6\,r^6\,\text{Sin}[\beta]^6\right)}\Big)^{1/3}\Big) -$$

$$\frac{1}{12\,2^{1/3}\,r^2}\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\text{Sin}[\beta]^2 + 18432\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - \sqrt{\left(110592\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - 5308416\,\pi^6\,r^6\,\text{Sin}[\beta]^2 + 84934656\,\pi^8\,r^8\,\text{Sin}[\beta]^2 - 452984832\,\pi^{10}\,r^{10}\,\text{Sin}[\beta]^2 - 110592\,\pi^4\,r^4\,\text{Sin}[\beta]^4 + 35389440\,\pi^6\,r^6\,\text{Sin}[\beta]^4 + 226492416\,\pi^8\,r^8\,\text{Sin}[\beta]^4 - 28311552\,\pi^6\,r^6\,\text{Sin}[\beta]^6\right)}\Big)^{1/3} +$$

$$\left(-\frac{(-1-8\pi r)^3}{r^3}+\frac{(-1-8\pi r)\left(1+24\pi r+80\pi^2 r^2\right)}{r^3}-\frac{8\left(-\pi-8\pi^2 r-16\pi^3 r^2\right)}{r^2}\right)\Big/$$
$$\left(4\sqrt{\left(\frac{(-1-8\pi r)^2}{4 r^2}-\frac{-1-24\pi r-80\pi^2 r^2}{12 r^2}-\frac{1+24\pi r+80\pi^2 r^2}{4 r^2}+\right.}\right.$$
$$\left(1-32\pi^2 r^2+256\pi^4 r^4+192\pi^2 r^2 \mathrm{Sin}[\beta]^2\right)\Big/\left(6\, 2^{2/3} r^2\right.$$
$$\left(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2 \mathrm{Sin}[\beta]^2+18\,432\pi^4 r^4 \mathrm{Sin}[\beta]^2-\right.$$
$$\sqrt{\left(110\,592\pi^4 r^4 \mathrm{Sin}[\beta]^2-5\,308\,416\pi^6 r^6 \mathrm{Sin}[\beta]^2+84\,934\,656\pi^8 r^8 \mathrm{Sin}[\beta]^2-\right.}$$
$$452\,984\,832\pi^{10} r^{10} \mathrm{Sin}[\beta]^2-110\,592\pi^4 r^4 \mathrm{Sin}[\beta]^4+35\,389\,440\pi^6 r^6$$
$$\left.\left.\left.\mathrm{Sin}[\beta]^4+226\,492\,416\pi^8 r^8 \mathrm{Sin}[\beta]^4-28\,311\,552\pi^6 r^6 \mathrm{Sin}[\beta]^6\right)\right)^{1/3}\right)+$$
$$\frac{1}{12\, 2^{1/3} r^2}\left(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2 \mathrm{Sin}[\beta]^2+18\,432\right.$$
$$\pi^4 r^4 \mathrm{Sin}[\beta]^2-\sqrt{\left(110\,592\pi^4 r^4 \mathrm{Sin}[\beta]^2-5\,308\,416\pi^6 r^6 \mathrm{Sin}[\beta]^2+84\,934\,656\pi^8\right.}$$
$$r^8 \mathrm{Sin}[\beta]^2-452\,984\,832\pi^{10} r^{10} \mathrm{Sin}[\beta]^2-110\,592\pi^4 r^4 \mathrm{Sin}[\beta]^4+35\,389\,440$$
$$\left.\left.\left.\left.\pi^6 r^6 \mathrm{Sin}[\beta]^4+226\,492\,416\pi^8 r^8 \mathrm{Sin}[\beta]^4-28\,311\,552\pi^6 r^6 \mathrm{Sin}[\beta]^6\right)\right)^{1/3}\right)\right)\right)\}\}$$

$$
\begin{aligned}
&\textbf{RevolutionPlot3D}\Bigg[\\
&-\frac{-1-8\pi r}{4 r}+\frac{1}{2}\sqrt{\left(\frac{(-1-8\pi r)^2}{4 r^2}-\frac{-1-24\pi r-80\pi^2 r^2}{12 r^2}-\frac{1+24\pi r+80\pi^2 r^2}{4 r^2}+\left(1-32\pi^2 r^2+256\pi^4 r^4+\right.\right.}\\
&\quad 192\pi^2 r^2\,\text{Sin}[\beta]^2\big)\Big/\Big(6\,2^{2/3} r^2\,\big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\text{Sin}[\beta]^2+\\
&\quad 18432\pi^4 r^4\,\text{Sin}[\beta]^2-\sqrt{\big(110592\pi^4 r^4\,\text{Sin}[\beta]^2-5308416\pi^6 r^6\,\text{Sin}[\beta]^2+}\\
&\quad 84934656\pi^8 r^8\,\text{Sin}[\beta]^2-452984832\pi^{10} r^{10}\,\text{Sin}[\beta]^2-110592\pi^4 r^4\,\text{Sin}[\beta]^4+\\
&\quad 35389440\pi^6 r^6\,\text{Sin}[\beta]^4+226492416\pi^8 r^8\,\text{Sin}[\beta]^4-28311552\pi^6 r^6\,\text{Sin}[\beta]^6\big)\big)^{1/3}\Big)+\\
&\quad \frac{1}{12\,2^{1/3} r^2}\big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\text{Sin}[\beta]^2+18432\pi^4 r^4\,\text{Sin}[\beta]^2-\\
&\quad \sqrt{\big(110592\pi^4 r^4\,\text{Sin}[\beta]^2-5308416\pi^6 r^6\,\text{Sin}[\beta]^2+84934656\pi^8 r^8\,\text{Sin}[\beta]^2-}\\
&\quad 452984832\pi^{10} r^{10}\,\text{Sin}[\beta]^2-110592\pi^4 r^4\,\text{Sin}[\beta]^4+35389440\pi^6 r^6\,\text{Sin}[\beta]^4+\\
&\quad \left.226492416\pi^8 r^8\,\text{Sin}[\beta]^4-28311552\pi^6 r^6\,\text{Sin}[\beta]^6\big)\big)^{1/3}\right)+\\
&\frac{1}{2}\sqrt{\left(\frac{(-1-8\pi r)^2}{2 r^2}+\frac{-1-24\pi r-80\pi^2 r^2}{12 r^2}-\frac{1+24\pi r+80\pi^2 r^2}{4 r^2}-\right.}\\
&\quad \big(1-32\pi^2 r^2+256\pi^4 r^4+192\pi^2 r^2\,\text{Sin}[\beta]^2\big)\Big/\\
&\quad \Big(6\,2^{2/3} r^2\,\big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\text{Sin}[\beta]^2+18432\pi^4 r^4\,\text{Sin}[\beta]^2-\\
&\quad \sqrt{\big(110592\pi^4 r^4\,\text{Sin}[\beta]^2-5308416\pi^6 r^6\,\text{Sin}[\beta]^2+84934656\pi^8 r^8\,\text{Sin}[\beta]^2-}\\
&\quad 452984832\pi^{10} r^{10}\,\text{Sin}[\beta]^2-110592\pi^4 r^4\,\text{Sin}[\beta]^4+35389440\pi^6 r^6\,\text{Sin}[\beta]^4+\\
&\quad 226492416\pi^8 r^8\,\text{Sin}[\beta]^4-28311552\pi^6 r^6\,\text{Sin}[\beta]^6\big)\big)^{1/3}\Big)-\\
&\quad \frac{1}{12\,2^{1/3} r^2}\big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\text{Sin}[\beta]^2+18432\pi^4 r^4\,\text{Sin}[\beta]^2-\\
&\quad \sqrt{\big(110592\pi^4 r^4\,\text{Sin}[\beta]^2-5308416\pi^6 r^6\,\text{Sin}[\beta]^2+84934656\pi^8 r^8\,\text{Sin}[\beta]^2-}\\
&\quad 452984832\pi^{10} r^{10}\,\text{Sin}[\beta]^2-110592\pi^4 r^4\,\text{Sin}[\beta]^4+35389440\pi^6 r^6\,\text{Sin}[\beta]^4+\\
&\quad 226492416\pi^8 r^8\,\text{Sin}[\beta]^4-28311552\pi^6 r^6\,\text{Sin}[\beta]^6\big)\big)^{1/3}+\\
&\quad \left(-\frac{(-1-8\pi r)^3}{r^3}+\frac{(-1-8\pi r)\,\big(1+24\pi r+80\pi^2 r^2\big)}{r^3}-\frac{8\,\big(-\pi-8\pi^2 r-16\pi^3 r^2\big)}{r^2}\right)\Bigg/\\
&\quad \left(4\sqrt{\left(\frac{(-1-8\pi r)^2}{4 r^2}-\frac{-1-24\pi r-80\pi^2 r^2}{12 r^2}-\frac{1+24\pi r+80\pi^2 r^2}{4 r^2}+\right.}\right.\\
&\qquad \big(1-32\pi^2 r^2+256\pi^4 r^4+192\pi^2 r^2\,\text{Sin}[\beta]^2\big)\Big/\\
&\qquad \Big(6\,2^{2/3} r^2\,\big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\text{Sin}[\beta]^2+18432\pi^4 r^4\,\text{Sin}[\beta]^2-\\
&\qquad \sqrt{\big(110592\pi^4 r^4\,\text{Sin}[\beta]^2-5308416\pi^6 r^6\,\text{Sin}[\beta]^2+84934656\pi^8 r^8\,\text{Sin}[\beta]^2-}\\
&\qquad 452984832\pi^{10} r^{10}\,\text{Sin}[\beta]^2-110592\pi^4 r^4\,\text{Sin}[\beta]^4+35389440\pi^6 r^6\,\text{Sin}[\beta]^4+\\
&\qquad 226492416\pi^8 r^8\,\text{Sin}[\beta]^4-28311552\pi^6 r^6\,\text{Sin}[\beta]^6\big)\big)^{1/3}\Big)+\frac{1}{12\,2^{1/3} r^2}\\
&\qquad \big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\text{Sin}[\beta]^2+18432\pi^4 r^4\,\text{Sin}[\beta]^2-\sqrt{\big(110592}\\
&\qquad \pi^4 r^4\,\text{Sin}[\beta]^2-5308416\pi^6 r^6\,\text{Sin}[\beta]^2+84934656\pi^8 r^8\,\text{Sin}[\beta]^2-452984832\\
&\qquad \pi^{10} r^{10}\,\text{Sin}[\beta]^2-110592\pi^4 r^4\,\text{Sin}[\beta]^4+35389440\pi^6 r^6\,\text{Sin}[\beta]^4+226492416\pi^8\\
&\qquad \left.\left.r^8\,\text{Sin}[\beta]^4-28311552\pi^6 r^6\,\text{Sin}[\beta]^6\big)\big)^{1/3}\right)\right),\ \{r,-1,1\},\ \{\beta,-\pi/2,\pi/2\}\Bigg]
\end{aligned}
$$



Univocal Radius Solutions Continued

by Parker Emmerson © 2009 - 2010

- $r_1 = \sqrt{\left(r\text{^}2 - \frac{\sqrt{4\pi(r)^2\theta - (r)^2\theta^2}}{2\pi}\text{^}2\right)} == \frac{2\pi r - r\theta}{2\pi}$

$$\text{Solve}\left[\sqrt{\left(r\text{^}2 - \frac{\sqrt{4\pi(r)^2\theta - (r)^2\theta^2}}{2\pi}\text{^}2\right)} == \frac{2\pi r - r\theta}{2\pi}, \theta\right]$$

{{}}

$$\text{Solve}\left[\sqrt{\left(\left(\left(-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta] + 4\pi^2\,\text{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta} - \frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\text{Sin}[\beta]^2 + 8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)\text{^}2 - \frac{1}{2\pi}\left(\sqrt{\left(4\pi\left(\left(-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta] + 4\pi^2\,\text{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta} - \frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\text{Sin}[\beta]^2 + 8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)^2\theta - (r)^2\theta^2\right)}\right)\text{^}2\right)} == \frac{2\pi r - r\theta}{2\pi}, r\right]$$

$$\Big\{\Big\{r \to -\frac{1}{\sqrt{-16\pi + 16\theta}} \Bigg(\sqrt{\Bigg(-\frac{3\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]}{\pi\ (4\pi-\theta)} + \frac{\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]}{\pi\theta} +}$$

$$\frac{3\,\mathrm{Sin}[\beta]^2}{4\pi-\theta} - \frac{\mathrm{Sin}[\beta]^2}{\theta} + \frac{256\pi^4\theta}{\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\mathrm{Sin}[\beta]^2 + 8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2} -$$

$$\frac{64\pi^3\theta^2}{\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\mathrm{Sin}[\beta]^2 + 8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2} -$$

$$\frac{256\pi^3\theta\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]}{\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\mathrm{Sin}[\beta]^2 + 8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2} +$$

$$\frac{64\pi^2\theta^2\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]}{\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\mathrm{Sin}[\beta]^2 + 8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2} -$$

$$\frac{256\pi^5\,\mathrm{Sin}[\beta]^2}{\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\mathrm{Sin}[\beta]^2 + 8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2} +$$

$$\frac{256\pi^4\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]^3}{\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\mathrm{Sin}[\beta]^2 + 8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2} +$$

$$\frac{256\pi^3\theta\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]^3}{\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\mathrm{Sin}[\beta]^2 + 8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2} -$$

$$\frac{64\pi^2\theta^2\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]^3}{\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\mathrm{Sin}[\beta]^2 + 8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2} -$$

$$\frac{256\pi^4\theta\,\mathrm{Sin}[\beta]^4}{\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\mathrm{Sin}[\beta]^2 + 8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2} +$$

$$\frac{64\pi^3\theta^2\,\mathrm{Sin}[\beta]^4}{\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\mathrm{Sin}[\beta]^2 + 8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2} -$$

$$\frac{256\pi^4\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]^5}{\left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\mathrm{Sin}[\beta]^2 + 8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)^2} +$$

$$\frac{256\,\pi^5\,\text{Sin}[\beta]^6}{\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\text{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2} - \frac{16\,\pi^2}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\text{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\text{Sin}[\beta]^2} - \frac{24\,\pi\,\theta}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\text{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\text{Sin}[\beta]^2} + \frac{8\,\theta^2}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\text{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\text{Sin}[\beta]^2} + \frac{16\,\pi\,\sqrt{(4\,\pi - \theta)\,\theta}\,\text{Sin}[\beta]}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\text{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\text{Sin}[\beta]^2} + \frac{24\,\theta\,\sqrt{(4\,\pi - \theta)\,\theta}\,\text{Sin}[\beta]}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\text{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\text{Sin}[\beta]^2} - \frac{8\,\theta^2\,\sqrt{(4\,\pi - \theta)\,\theta}\,\text{Sin}[\beta]}{\pi\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\text{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)} + \frac{16\,\pi^2\,\text{Sin}[\beta]^2}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\text{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\text{Sin}[\beta]^2} - \frac{24\,\pi\,\theta\,\text{Sin}[\beta]^2}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\text{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\text{Sin}[\beta]^2} + \frac{8\,\theta^2\,\text{Sin}[\beta]^2}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\text{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\text{Sin}[\beta]^2} - \frac{32\,\pi\,\sqrt{(4\,\pi - \theta)\,\theta}\,\text{Sin}[\beta]^3}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\text{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\text{Sin}[\beta]^2} + \frac{32\,\pi^2\,\text{Sin}[\beta]^4}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\text{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}\Bigg)\Bigg)\Bigg\},$$

$$\Bigg\{r \to \frac{1}{\sqrt{-16\,\pi + 16\,\theta}}\left(\sqrt{\left(-\frac{3\,\sqrt{(4\,\pi - \theta)\,\theta}\,\text{Sin}[\beta]}{\pi\,(4\,\pi - \theta)} + \frac{\sqrt{(4\,\pi - \theta)\,\theta}\,\text{Sin}[\beta]}{\pi\,\theta} + \frac{3\,\text{Sin}[\beta]^2}{4\,\pi - \theta} - \frac{\text{Sin}[\beta]^2}{\theta} + \frac{256\,\pi^4\,\theta}{\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\text{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2} -\right.}\right.$$

$$\frac{64\,\pi^3\,\theta^2}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{256\,\pi^3\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\,\,\mathrm{Sin}[\beta]}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{64\,\pi^2\,\theta^2\,\sqrt{(4\,\pi-\theta)\,\theta}\,\,\mathrm{Sin}[\beta]}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{256\,\pi^5\,\mathrm{Sin}[\beta]^2}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{256\,\pi^4\,\sqrt{(4\,\pi-\theta)\,\theta}\,\,\mathrm{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{256\,\pi^3\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\,\,\mathrm{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{64\,\pi^2\,\theta^2\,\sqrt{(4\,\pi-\theta)\,\theta}\,\,\mathrm{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{256\,\pi^4\,\theta\,\mathrm{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{64\,\pi^3\,\theta^2\,\mathrm{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{256\,\pi^4\,\sqrt{(4\,\pi-\theta)\,\theta}\,\,\mathrm{Sin}[\beta]^5}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{256\,\pi^5\,\mathrm{Sin}[\beta]^6}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{16\,\pi^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}-$$

$$\frac{24\,\pi\,\theta}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}+$$

$$\frac{8\,\theta^2}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2} +$$

$$\frac{16\,\pi\,\sqrt{(4\,\pi - \theta)\,\theta}\,\,\mathrm{Sin}[\beta]}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2} +$$

$$\frac{24\,\theta\,\sqrt{(4\,\pi - \theta)\,\theta}\,\,\mathrm{Sin}[\beta]}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2} -$$

$$\frac{8\,\theta^2\,\sqrt{(4\,\pi - \theta)\,\theta}\,\,\mathrm{Sin}[\beta]}{\pi\,\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)} +$$

$$\frac{16\,\pi^2\,\mathrm{Sin}[\beta]^2}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2} -$$

$$\frac{24\,\pi\,\theta\,\mathrm{Sin}[\beta]^2}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2} +$$

$$\frac{8\,\theta^2\,\mathrm{Sin}[\beta]^2}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2} -$$

$$\frac{32\,\pi\,\sqrt{(4\,\pi - \theta)\,\theta}\,\,\mathrm{Sin}[\beta]^3}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2} +$$

$$\left.\left.\frac{32\,\pi^2\,\mathrm{Sin}[\beta]^4}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}\right)\right)\}\}$$

SphericalPlot3D[

$$\frac{1}{\sqrt{-16\,\pi + 16\,\theta}}\left(\sqrt{\left(-\frac{3\,\sqrt{(4\,\pi - \theta)\,\theta}\,\,\mathrm{Sin}[\beta]}{\pi\,(4\,\pi - \theta)} + \frac{\sqrt{(4\,\pi - \theta)\,\theta}\,\,\mathrm{Sin}[\beta]}{\pi\,\theta} + \frac{3\,\mathrm{Sin}[\beta]^2}{4\,\pi - \theta} - \frac{\mathrm{Sin}[\beta]^2}{\theta} + \right.}\right.$$

$$\frac{256\,\pi^4\,\theta}{\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2} -$$

$$\frac{64\,\pi^3\,\theta^2}{\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2} -$$

$$\frac{256\,\pi^3\,\theta\,\sqrt{(4\,\pi - \theta)\,\theta}\,\,\mathrm{Sin}[\beta]}{\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2} +$$

$$\frac{64\,\pi^2\,\theta^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{256\,\pi^5\,\text{Sin}[\beta]^2}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{256\,\pi^4\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{256\,\pi^3\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{64\,\pi^2\,\theta^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{256\,\pi^4\,\theta\,\text{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{64\,\pi^3\,\theta^2\,\text{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{256\,\pi^4\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^5}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{256\,\pi^5\,\text{Sin}[\beta]^6}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{16\,\pi^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}-$$

$$\frac{24\,\pi\,\theta}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}+$$

$$\frac{8\,\theta^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}+$$

$$\frac{16\,\pi\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}+$$

$$\frac{24\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}-$$

$$\frac{8\,\theta^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{\pi\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)}+$$

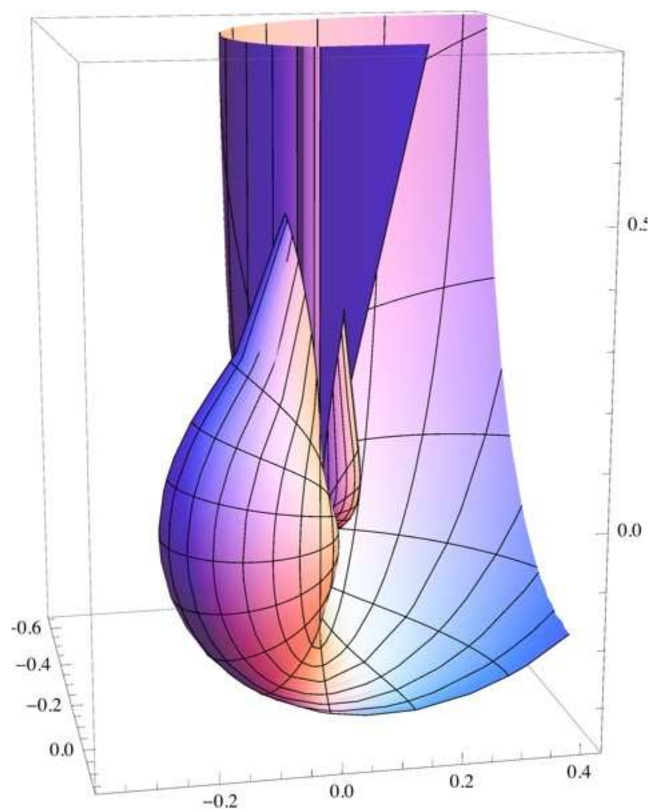
$$\frac{16\,\pi^2\,\text{Sin}[\beta]^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}-$$

$$\frac{24\,\pi\,\theta\,\text{Sin}[\beta]^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}+$$

$$\frac{8\,\theta^2\,\text{Sin}[\beta]^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}-$$

$$\frac{32\,\pi\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}+$$

$$\left.\left.\frac{32\,\pi^2\,\text{Sin}[\beta]^4}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}\right)\right),\ \{\theta,-2\,\pi,2\,\pi\},\ \{\beta,-\pi/2,\pi/2\}]$$



$$\mathbf{Solve}\Bigg[\sqrt{\Bigg(\Bigg(\Bigg(\Bigg(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\theta}\,\mathrm{Sin}[\beta]+4\pi^2\,\mathrm{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\mathrm{Sin}[\beta]^3}{4\pi-\theta}-}$$

$$\frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\mathrm{Sin}[\beta]^3}{\theta}\Bigg)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)\Bigg)\,\hat{}\,2-$$

$$\frac{1}{2\pi}\Bigg(\sqrt{\Bigg(4\pi\Bigg(\Bigg(\Bigg(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\theta}\,\mathrm{Sin}[\beta]+4\pi^2\,\mathrm{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\mathrm{Sin}[\beta]^3}{4\pi-\theta}-}$$

$$\frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\mathrm{Sin}[\beta]^3}{\theta}\Bigg)\Big/\Big(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\mathrm{Sin}[\beta]^2+$$

$$8\pi^2\theta\,\mathrm{Sin}[\beta]^2\Big)\Bigg)^2\theta-\Bigg(\Bigg(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\theta}\,\mathrm{Sin}[\beta]+$$

$$4\pi^2\,\mathrm{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\mathrm{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\mathrm{Sin}[\beta]^3}{\theta}\Bigg)\Big/$$

$$\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)\Bigg)^2\theta^2\Bigg)\Bigg)\,\hat{}\,2\Bigg)==\frac{2\pi r-r\theta}{2\pi},\ \beta\Bigg]$$

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

$$\Big\{\Big\{\beta\to\mathrm{ArcSin}\Big[$$

$$\Big(-\pi^2\sqrt{(4\pi-\theta)\theta}\,(-8\pi+16\pi\theta-4\theta^2)-\sqrt{\Big(\pi^4(4\pi-\theta)\theta\left(-8\pi+16\pi\theta-4\theta^2\right)^2+4\pi^3\theta\Big(8\pi-16\pi\theta+}$$

$$4\theta^2\Big)\Big(-8\pi^3+128\pi^5r^2+2\pi^2\theta+16\pi^3\theta-288\pi^4r^2\theta-8\pi^2\theta^2+256\pi^3r^2\theta^2+\pi\theta^3-$$

$$112\pi^2r^2\theta^3+24\pi r^2\theta^4-2r^2\theta^5\Big)\Big)\Big)\Big/\left(2\pi^3\left(8\pi-16\pi\theta+4\theta^2\right)\right)\Big]\Big\},$$

$$\Big\{\beta\to\mathrm{ArcSin}\Big[\Big(-\pi^2\sqrt{(4\pi-\theta)\theta}\,(-8\pi+16\pi\theta-4\theta^2)+\sqrt{\Big(\pi^4(4\pi-\theta)\theta\left(-8\pi+16\pi\theta-4\theta^2\right)^2+}$$

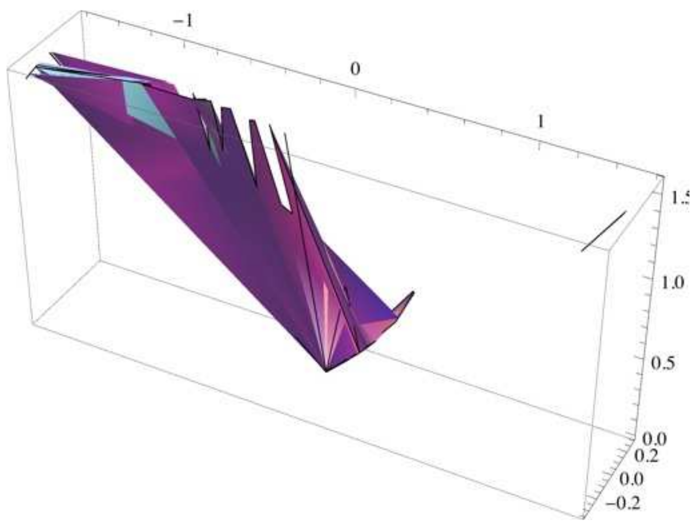
$$4\pi^3\theta\left(8\pi-16\pi\theta+4\theta^2\right)\Big(-8\pi^3+128\pi^5r^2+2\pi^2\theta+16\pi^3\theta-288\pi^4r^2\theta-8\pi^2\theta^2+$$

$$256\pi^3r^2\theta^2+\pi\theta^3-112\pi^2r^2\theta^3+24\pi r^2\theta^4-2r^2\theta^5\Big)\Big)\Big)\Big/\left(2\pi^3\left(8\pi-16\pi\theta+4\theta^2\right)\right)\Big]\Big\}\Big\}$$

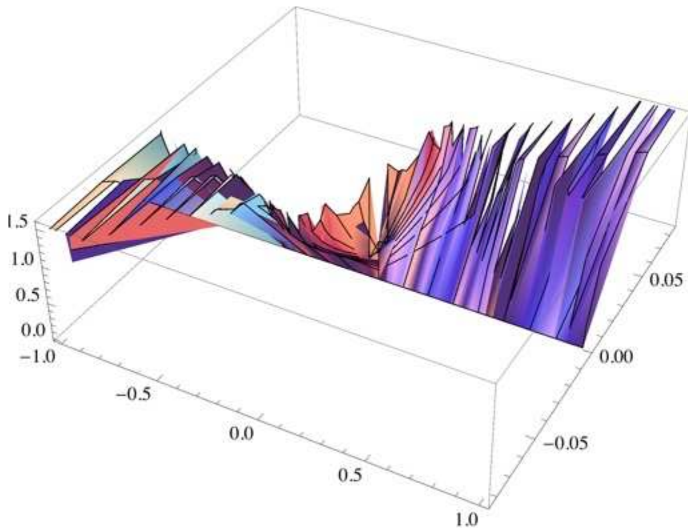
```
RevolutionPlot3D[
 {ArcSin[(-π^2 Sqrt[(4 π - θ) θ] (-8 π + 16 π θ - 4 θ^2) - Sqrt[(π^4 (4 π - θ) θ (-8 π + 16 π θ - 4 θ^2)^2 +
        4 π^3 θ (8 π - 16 π θ + 4 θ^2) (-8 π^3 + 128 π^5 r^2 + 2 π^2 θ + 16 π^3 θ - 288 π^4 r^2 θ -
          8 π^2 θ^2 + 256 π^3 r^2 θ^2 + π θ^3 - 112 π^2 r^2 θ^3 + 24 π r^2 θ^4 - 2 r^2 θ^5))]) /
    (2 π^3 (8 π - 16 π θ + 4 θ^2))], ArcSin[(-π^2 Sqrt[(4 π - θ) θ] (-8 π + 16 π θ - 4 θ^2) +
     Sqrt[(π^4 (4 π - θ) θ (-8 π + 16 π θ - 4 θ^2)^2 + 4 π^3 θ (8 π - 16 π θ + 4 θ^2) (-8 π^3 + 128 π^5 r^2 + 2 π^2 θ +
          16 π^3 θ - 288 π^4 r^2 θ - 8 π^2 θ^2 + 256 π^3 r^2 θ^2 + π θ^3 - 112 π^2 r^2 θ^3 + 24 π r^2 θ^4 - 2 r^2 θ^5))]) /
    (2 π^3 (8 π - 16 π θ + 4 θ^2))]}, {r, -1, 1}, {θ, -2 π, 2 π}]
```

ComputationalGeometry`Methods`SpatialSearch::spoutbounds :
 –– Message text not found –– ({10.4975, –65.509})

ComputationalGeometry`Methods`SpatialSearch::spoutbounds :
 –– Message text not found –– ({–10.4975, –65.509})



```
RevolutionPlot3D[
 ArcSin[(-π^2 Sqrt[(4 π - θ) θ] (-8 π + 16 π θ - 4 θ^2) + Sqrt[(π^4 (4 π - θ) θ (-8 π + 16 π θ - 4 θ^2)^2 +
        4 π^3 θ (8 π - 16 π θ + 4 θ^2) (-8 π^3 + 128 π^5 r^2 + 2 π^2 θ + 16 π^3 θ - 288 π^4 r^2 θ -
           8 π^2 θ^2 + 256 π^3 r^2 θ^2 + π θ^3 - 112 π^2 r^2 θ^3 + 24 π r^2 θ^4 - 2 r^2 θ^5))]) /
   (2 π^3 (8 π - 16 π θ + 4 θ^2))], {r, -1, 1}, {θ, -2 π, 2 π}]
```



$$\text{Solve}\Big[\sqrt{\left(\left(\left(\left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)\hat{}\,2-\frac{1}{2\pi}\left(\sqrt{\left(4\pi\left(\left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)^2\theta-\left(\left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)^2\theta^2\right)}\right)\hat{}\,2\right)}==$$

$$\frac{1}{2\pi}\,2\pi\left(\left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)-r\,\theta,\ r\Big]$$

$$
\Bigg\{\Bigg\{r \to \frac{1}{\theta}\Bigg(\Bigg(-4\,\pi\,\theta + \theta^2 + 2\,\pi\,\sqrt{(4\,\pi - \theta)\,\theta}\;\mathrm{Sin}[\beta] + 4\,\pi^2\,\mathrm{Sin}[\beta]^2 - \frac{2\,\pi^2\,\sqrt{(4\,\pi - \theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{4\,\pi - \theta} - \frac{2\,\pi^2\,\sqrt{(4\,\pi - \theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{\theta}\Bigg) \Big/ \left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right) -
$$

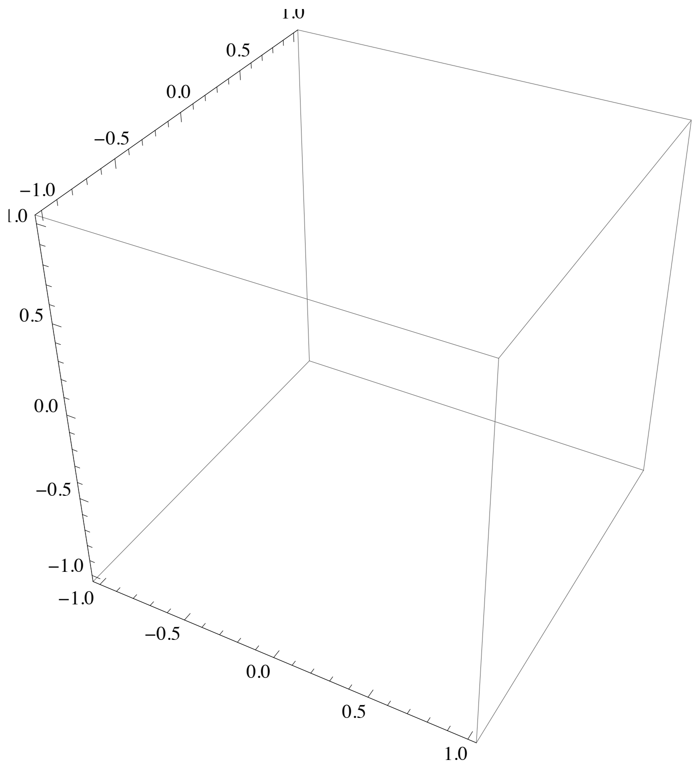
$$
\sqrt{\Bigg(\Bigg(-4\,\pi\,\theta + \theta^2 + 2\,\pi\,\sqrt{(4\,\pi - \theta)\,\theta}\;\mathrm{Sin}[\beta] + 4\,\pi^2\,\mathrm{Sin}[\beta]^2 - \frac{2\,\pi^2\,\sqrt{(4\,\pi - \theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{4\,\pi - \theta} - \frac{2\,\pi^2\,\sqrt{(4\,\pi - \theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{\theta}\Bigg)^2 \Big/ \left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2 - \cdots}
$$

$$
\frac{1}{2\,\pi}\Bigg(\Bigg(4\,\pi\,\theta\,\Bigg(-4\,\pi\,\theta + \theta^2 + 2\,\pi\,\sqrt{(4\,\pi - \theta)\,\theta}\;\mathrm{Sin}[\beta] + 4\,\pi^2\,\mathrm{Sin}[\beta]^2 - \frac{2\,\pi^2\,\sqrt{(4\,\pi - \theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{4\,\pi - \theta} - \frac{2\,\pi^2\,\sqrt{(4\,\pi - \theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{\theta}\Bigg)^2\Bigg) \Big/ \Big(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 +
$$

$$
8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\Big)^2 - \Bigg(\theta^2\,\Bigg(-4\,\pi\,\theta + \theta^2 + 2\,\pi\,\sqrt{(4\,\pi - \theta)\,\theta}\;\mathrm{Sin}[\beta] + 4\,\pi^2\,\mathrm{Sin}[\beta]^2 - \frac{2\,\pi^2\,\sqrt{(4\,\pi - \theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{4\,\pi - \theta} - \frac{2\,\pi^2\,\sqrt{(4\,\pi - \theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{\theta}\Bigg)^2\Bigg) \Big/
$$

$$
\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2\Bigg)\Bigg)\Bigg)\Bigg\}\Bigg\}
$$

$$\text{SphericalPlot3D}\Big[\frac{1}{\theta}\Bigg(\Bigg(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\Bigg)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)-\sqrt{\Bigg(\Bigg(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\Bigg)^2\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)^2-\frac{1}{2\pi}\Bigg(\Bigg(4\pi\theta\Bigg(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\Bigg)^2\Bigg)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)^2-\Bigg(\theta^2\Bigg(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\Bigg)^2\Bigg)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)^2\Bigg)\Bigg)}\Bigg),\ \{\theta,-2\pi,2\pi\},\ \{\beta,-\pi/2,\pi/2\}\Big]$$


1.0
0.5
0.0
−0.5
−1.0
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0

$$
\text{SphericalPlot3D}\Bigg[\frac{1}{\theta}\Bigg(\Bigg(\Bigg(-4\,\pi\,\theta+\theta^2+2\,\pi\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]+4\,\pi^2\,\text{Sin}[\beta]^2-\frac{2\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{4\,\pi-\theta}-
$$

$$
\frac{2\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\Bigg)\Bigg/\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)-
$$

$$
\sqrt{\Bigg(\Bigg(\Bigg(-4\,\pi\,\theta+\theta^2+2\,\pi\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]+4\,\pi^2\,\text{Sin}[\beta]^2-\frac{2\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{4\,\pi-\theta}-}
$$

$$
\frac{2\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\Bigg)^2\Bigg/\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2-
$$

$$
\frac{1}{2\,\pi}\Bigg(\Bigg(4\,\pi\,\theta\,\Bigg(-4\,\pi\,\theta+\theta^2+2\,\pi\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]+4\,\pi^2\,\text{Sin}[\beta]^2-
$$

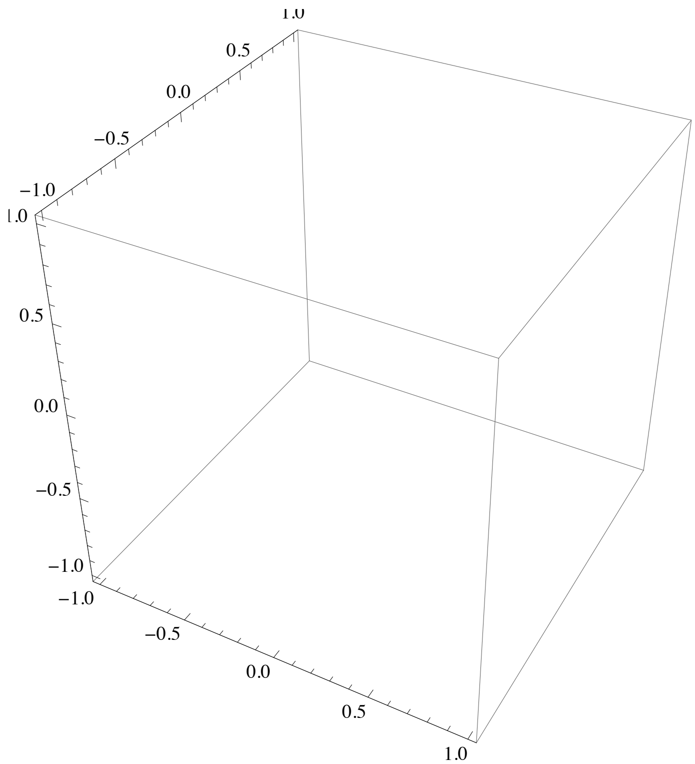
$$
\frac{2\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{4\,\pi-\theta}-\frac{2\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\Bigg)^2\Bigg)\Bigg/
$$

$$
\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2-
$$

$$
\Bigg(\theta^2\,\Bigg(-4\,\pi\,\theta+\theta^2+2\,\pi\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]+4\,\pi^2\,\text{Sin}[\beta]^2-\frac{2\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{4\,\pi-\theta}-
$$

$$
\frac{2\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\Bigg)^2\Bigg)\Bigg/\Big(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3
$$

$$
\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\Big)^2\Bigg)\Bigg)\Bigg),\ \{\beta,-\pi/2,\pi/2\},\ \{\theta,-2\,\pi,2\,\pi\}\Bigg]
$$


1.0
0.5
0.0
−0.5
−1.0
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0

$$
\begin{aligned}
\text{Solve}\Bigg[&\sqrt{\Bigg(\Bigg(\Bigg(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta}-}\\
&\quad\frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{\theta}\Bigg)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\Bigg)\,\hat{}\,2-\\
&\quad\frac{1}{2\pi}\Bigg(\sqrt{\Bigg(4\pi\Bigg(\Bigg(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta}-}\\
&\quad\quad\frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{\theta}\Bigg)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+\right.\\
&\quad\quad\left.8\pi^2\theta\,\text{Sin}[\beta]^2\right)\Bigg)^2\theta-\Bigg(\Bigg(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]+\\
&\quad\quad4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{\theta}\Bigg)\Big/\\
&\quad\quad\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\Bigg)^2\theta^2\Bigg)\Bigg)\,\hat{}\,2\Bigg)==\\
&\frac{1}{2\pi}2\pi\Bigg(\Bigg(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta}-\\
&\quad\frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{\theta}\Bigg)\Big/\\
&\quad\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\Bigg)-\\
&\Bigg(\Bigg(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta}-\\
&\quad\frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{\theta}\Bigg)\Big/\\
&\quad\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\Bigg)\theta,\ \beta\Bigg]
\end{aligned}
$$

```
{}
```

Solve[

$$\sqrt{\left(r^{\wedge}2-\frac{1}{2\pi}\left(\sqrt{\left(4\pi\left(\left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)^2\theta-\left(\left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)^2\theta^2\right)}\right)^{\wedge}2\right)}==$$

$$\frac{1}{2\pi}\,2\pi\left(\left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)-\left(\left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)\theta,\ \mathbf{r}$$

]

$$\Bigg\{\Bigg\{r \to -\frac{1}{4\sqrt{\pi}}\left(\sqrt{\left(-2+4\pi+\frac{4\pi}{(2\pi-\theta)^2}-\frac{8\pi^2}{(2\pi-\theta)^2}+\frac{16\pi^3}{(2\pi-\theta)^2}+\frac{8\pi}{2\pi-\theta}-\frac{16\pi^2}{2\pi-\theta}-\frac{4\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{(2\pi-\theta)^2}+\right.}\right.$$

$$\frac{8\pi\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{(2\pi-\theta)^2}-\frac{16\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{(2\pi-\theta)^2}-\frac{8\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{2\pi-\theta}+$$

$$\frac{16\pi\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{2\pi-\theta}+\frac{8\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{4\pi-\theta}-\frac{16\pi\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{4\pi-\theta}-$$

$$\frac{4\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{(4\pi-\theta)\,\theta}+\frac{4\pi\,\mathrm{Sin}[\beta]^2}{(2\pi-\theta)^2}-\frac{8\pi^2\,\mathrm{Sin}[\beta]^2}{(2\pi-\theta)^2}+\frac{16\pi^3\,\mathrm{Sin}[\beta]^2}{(2\pi-\theta)^2}+\frac{8\pi\,\mathrm{Sin}[\beta]^2}{2\pi-\theta}-$$

$$\left.\left.\frac{16\pi^2\,\mathrm{Sin}[\beta]^2}{2\pi-\theta}+\frac{\mathrm{Sin}[\beta]^2}{4\pi-\theta}-\frac{8\pi\,\mathrm{Sin}[\beta]^2}{4\pi-\theta}+\frac{16\pi^2\,\mathrm{Sin}[\beta]^2}{4\pi-\theta}+\frac{\mathrm{Sin}[\beta]^2}{\theta}\right)\right)\Bigg\},$$

$$\Bigg\{r \to \frac{1}{4\sqrt{\pi}}\left(\sqrt{\left(-2+4\pi+\frac{4\pi}{(2\pi-\theta)^2}-\frac{8\pi^2}{(2\pi-\theta)^2}+\frac{16\pi^3}{(2\pi-\theta)^2}+\frac{8\pi}{2\pi-\theta}-\frac{16\pi^2}{2\pi-\theta}-\frac{4\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{(2\pi-\theta)^2}+\right.}\right.$$

$$\frac{8\pi\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{(2\pi-\theta)^2}-\frac{16\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{(2\pi-\theta)^2}-\frac{8\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{2\pi-\theta}+$$

$$\frac{16\pi\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{2\pi-\theta}+\frac{8\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{4\pi-\theta}-\frac{16\pi\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{4\pi-\theta}-$$

$$\frac{4\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{(4\pi-\theta)\,\theta}+\frac{4\pi\,\mathrm{Sin}[\beta]^2}{(2\pi-\theta)^2}-\frac{8\pi^2\,\mathrm{Sin}[\beta]^2}{(2\pi-\theta)^2}+\frac{16\pi^3\,\mathrm{Sin}[\beta]^2}{(2\pi-\theta)^2}+\frac{8\pi\,\mathrm{Sin}[\beta]^2}{2\pi-\theta}-$$

$$\left.\left.\frac{16\pi^2\,\mathrm{Sin}[\beta]^2}{2\pi-\theta}+\frac{\mathrm{Sin}[\beta]^2}{4\pi-\theta}-\frac{8\pi\,\mathrm{Sin}[\beta]^2}{4\pi-\theta}+\frac{16\pi^2\,\mathrm{Sin}[\beta]^2}{4\pi-\theta}+\frac{\mathrm{Sin}[\beta]^2}{\theta}\right)\right)\Bigg\}\Bigg\}$$

$$\texttt{SphericalPlot3D}\Bigg[\frac{1}{4\sqrt{\pi}}\Bigg(\sqrt{\Bigg(-2+4\pi+\frac{4\pi}{(2\pi-\theta)^2}-\frac{8\pi^2}{(2\pi-\theta)^2}+\frac{16\pi^3}{(2\pi-\theta)^2}+\frac{8\pi}{2\pi-\theta}-\frac{16\pi^2}{2\pi-\theta}-\frac{4\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{(2\pi-\theta)^2}+}$$

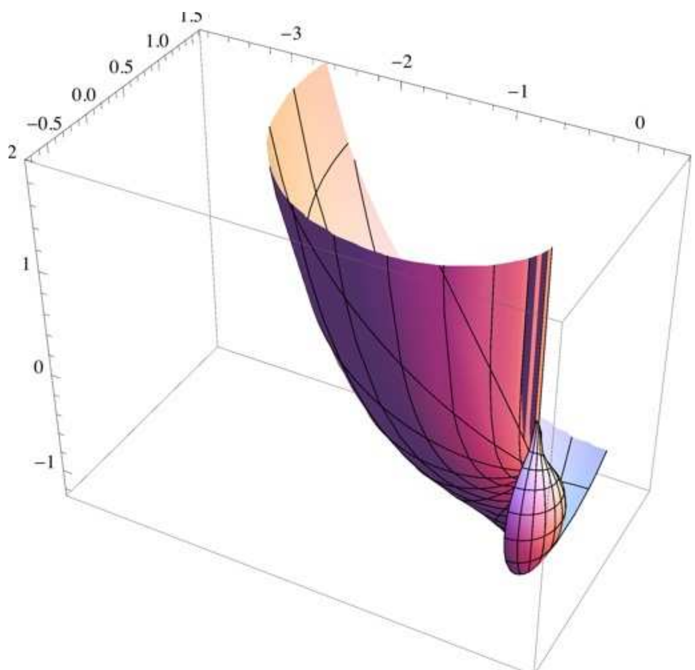
$$\frac{8\pi\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{(2\pi-\theta)^2}-\frac{16\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{(2\pi-\theta)^2}-$$

$$\frac{8\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{2\pi-\theta}+\frac{16\pi\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{2\pi-\theta}+\frac{8\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{4\pi-\theta}-$$

$$\frac{16\pi\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{4\pi-\theta}-\frac{4\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{(4\pi-\theta)\,\theta}+\frac{4\pi\,\text{Sin}[\beta]^2}{(2\pi-\theta)^2}-$$

$$\frac{8\pi^2\,\text{Sin}[\beta]^2}{(2\pi-\theta)^2}+\frac{16\pi^3\,\text{Sin}[\beta]^2}{(2\pi-\theta)^2}+\frac{8\pi\,\text{Sin}[\beta]^2}{2\pi-\theta}-\frac{16\pi^2\,\text{Sin}[\beta]^2}{2\pi-\theta}+\frac{\text{Sin}[\beta]^2}{4\pi-\theta}-$$

$$\frac{8\pi\,\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{16\pi^2\,\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{\text{Sin}[\beta]^2}{\theta}\Bigg)\Bigg),\ \{\theta,-2\pi,2\pi\},\ \{\beta,-\pi/2,\pi/2\}\Bigg]$$



SphericalPlot3D$\Bigg[$

$$\frac{1}{4\sqrt{\pi}}\left(\sqrt{\left(-2+4\pi+\frac{4\pi}{(2\pi-\theta)^2}-\frac{8\pi^2}{(2\pi-\theta)^2}+\frac{16\pi^3}{(2\pi-\theta)^2}+\frac{8\pi}{2\pi-\theta}-\frac{16\pi^2}{2\pi-\theta}-\frac{4\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{(2\pi-\theta)^2}+\right.}\right.$$

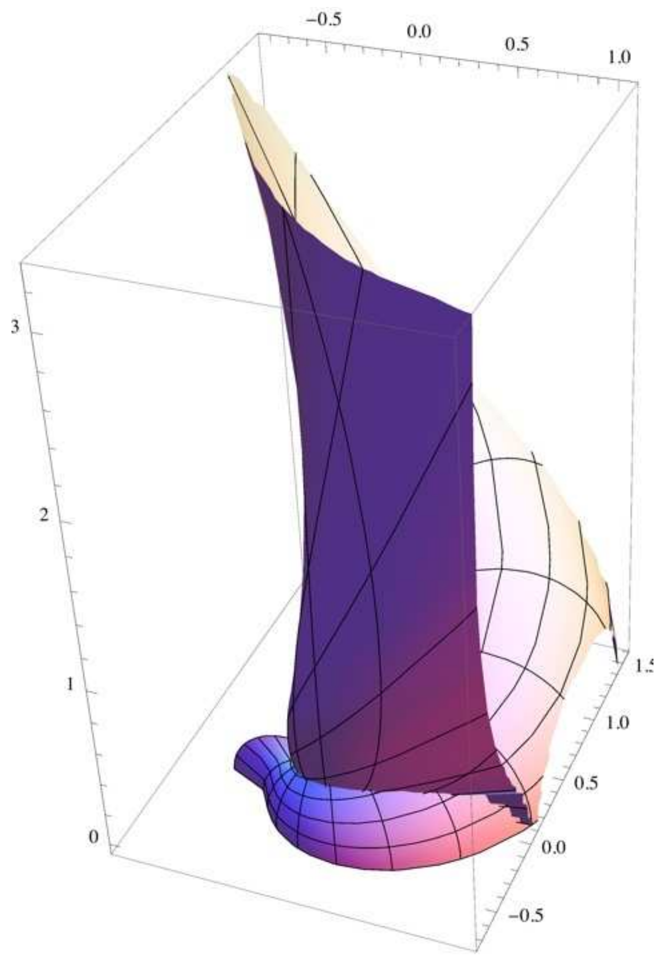
$$\frac{8\pi\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{(2\pi-\theta)^2}-\frac{16\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{(2\pi-\theta)^2}-$$

$$\frac{8\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{2\pi-\theta}+\frac{16\pi\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{2\pi-\theta}+\frac{8\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{4\pi-\theta}-$$

$$\frac{16\pi\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{4\pi-\theta}-\frac{4\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{(4\pi-\theta)\,\theta}+\frac{4\pi\,\mathrm{Sin}[\beta]^2}{(2\pi-\theta)^2}-$$

$$\frac{8\pi^2\,\mathrm{Sin}[\beta]^2}{(2\pi-\theta)^2}+\frac{16\pi^3\,\mathrm{Sin}[\beta]^2}{(2\pi-\theta)^2}+\frac{8\pi\,\mathrm{Sin}[\beta]^2}{2\pi-\theta}-\frac{16\pi^2\,\mathrm{Sin}[\beta]^2}{2\pi-\theta}+\frac{\mathrm{Sin}[\beta]^2}{4\pi-\theta}-$$

$$\left.\left.\frac{8\pi\,\mathrm{Sin}[\beta]^2}{4\pi-\theta}+\frac{16\pi^2\,\mathrm{Sin}[\beta]^2}{4\pi-\theta}+\frac{\mathrm{Sin}[\beta]^2}{\theta}\right)\right),\ \{\beta,-\pi/2,\pi/2\},\ \{\theta,-2\pi,2\pi\}\Bigg]$$



$$\textbf{Solve}\Bigg[\sqrt{\left(r\verb|^|2-\frac{1}{2\pi}\left(\sqrt{\left(4\pi\left(\left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)^2\theta-\left(\left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)^2\theta^2\right)}\right)\verb|^|2\right)}==$$

$$\frac{1}{2\pi}\left(2\pi\left(\left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)-\left(\left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)\theta\right),\ \beta\Bigg]$$

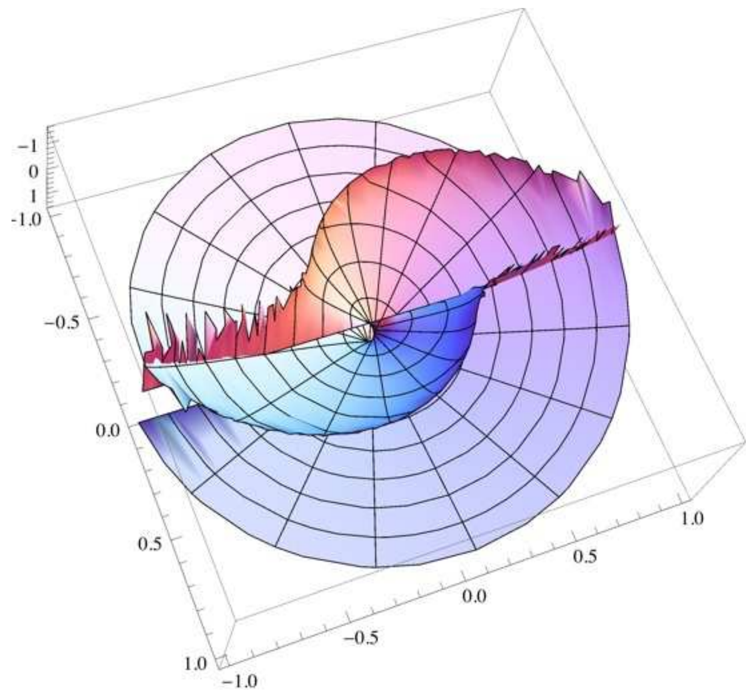
Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

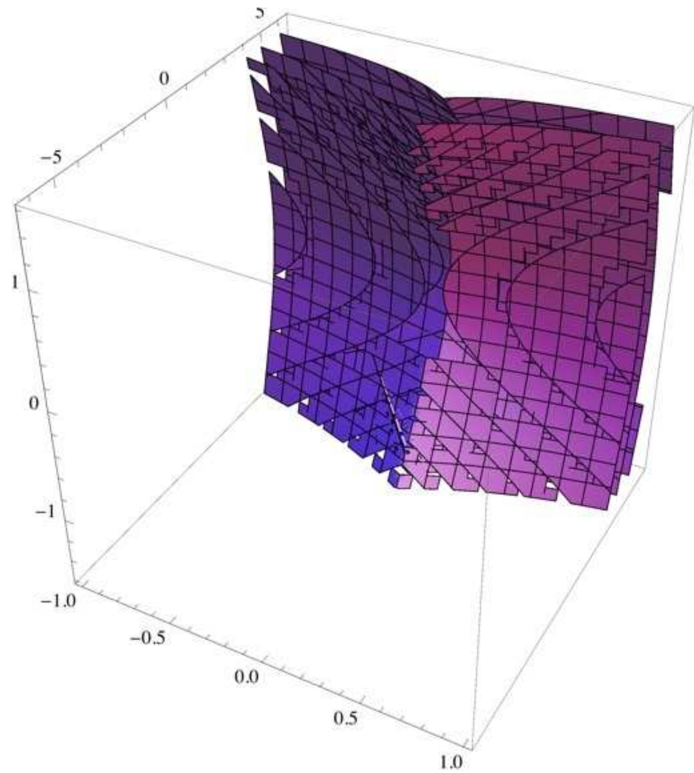
$$\Big\{\Big\{\beta\to\text{ArcSin}\Big[\Big(-8\pi^2\sqrt{(4\pi-\theta)\,\theta}+4\pi\theta^2\sqrt{(4\pi-\theta)\,\theta}-8\pi^2\theta^2\sqrt{(4\pi-\theta)\,\theta}-\sqrt{\left(4096\pi^7r^2\theta-5120\pi^6r^2\theta^2+2048\pi^5r^2\theta^3-2048\pi^6r^2\theta^3+4096\pi^7r^2\theta^3-256\pi^4r^2\theta^4+2560\pi^5r^2\theta^4-5120\pi^6r^2\theta^4-1024\pi^4r^2\theta^5+2048\pi^5r^2\theta^5+128\pi^3r^2\theta^6-256\pi^4r^2\theta^6\right)}\Big)\Big/\left(2\left(-8\pi^3+4\pi^2\theta^2-8\pi^3\theta^2\right)\right)\Big]\Big\},$$

$$\Big\{\beta\to\text{ArcSin}\Big[\Big(-8\pi^2\sqrt{(4\pi-\theta)\,\theta}+4\pi\theta^2\sqrt{(4\pi-\theta)\,\theta}-8\pi^2\theta^2\sqrt{(4\pi-\theta)\,\theta}+\sqrt{\left(4096\pi^7r^2\theta-5120\pi^6r^2\theta^2+2048\pi^5r^2\theta^3-2048\pi^6r^2\theta^3+4096\pi^7r^2\theta^3-256\pi^4r^2\theta^4+2560\pi^5r^2\theta^4-5120\pi^6r^2\theta^4-1024\pi^4r^2\theta^5+2048\pi^5r^2\theta^5+128\pi^3r^2\theta^6-256\pi^4r^2\theta^6\right)}\Big)\Big/\left(2\left(-8\pi^3+4\pi^2\theta^2-8\pi^3\theta^2\right)\right)\Big]\Big\}\Big\}$$

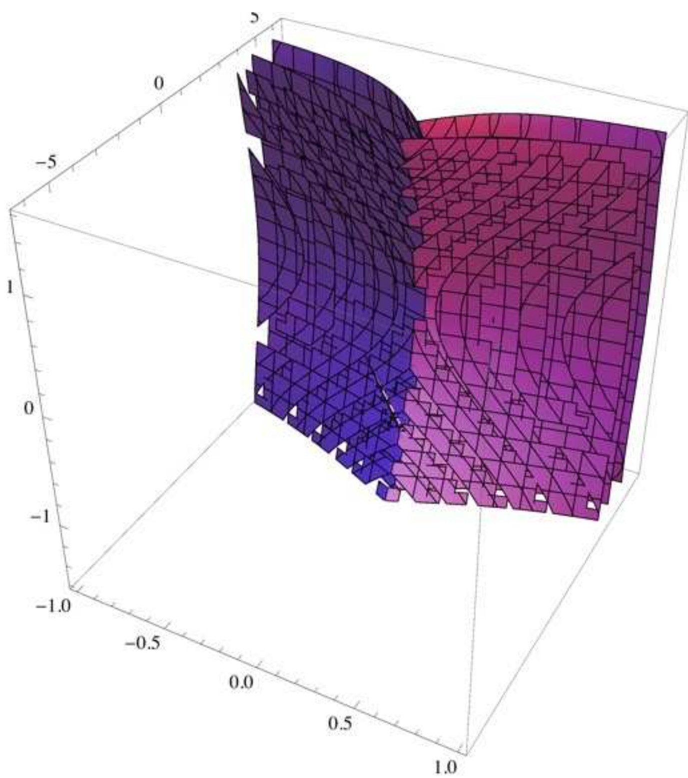
```
RevolutionPlot3D[ArcSin[(-8 π^2 Sqrt[(4 π - θ) θ] + 4 π θ^2 Sqrt[(4 π - θ) θ] - 8 π^2 θ^2 Sqrt[(4 π - θ) θ] +
    Sqrt[(4096 π^7 r^2 θ - 5120 π^6 r^2 θ^2 + 2048 π^5 r^2 θ^3 - 2048 π^6 r^2 θ^3 + 4096 π^7 r^2 θ^3 - 256 π^4 r^2 θ^4 +
        2560 π^5 r^2 θ^4 - 5120 π^6 r^2 θ^4 - 1024 π^4 r^2 θ^5 + 2048 π^5 r^2 θ^5 + 128 π^3 r^2 θ^6 - 256 π^4 r^2 θ^6)]) /
   (2 (-8 π^3 + 4 π^2 θ^2 - 8 π^3 θ^2))], {r, -1, 1}, {θ, -2 π, 2 π}]
```



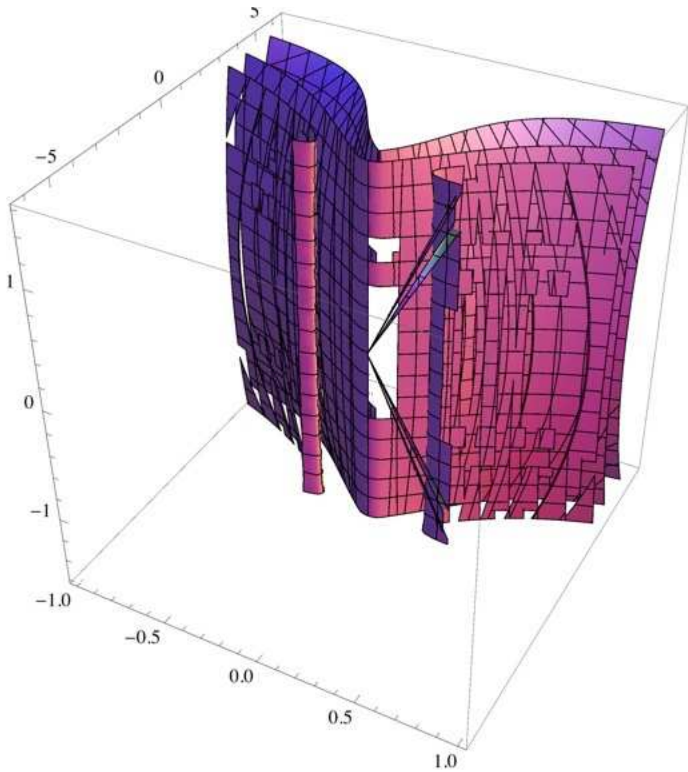
```
ContourPlot3D[
 ArcSin[(-8 π² √((4 π - θ) θ) + 4 π (2 (π + √(π² - π² Sin[β]²)))² √((4 π - θ) θ) - 8 π² θ² √((4 π - θ) θ) +
     √(4096 π⁷ r² θ - 5120 π⁶ r² θ² + 2048 π⁵ r² θ³ - 2048 π⁶ r² θ³ + 4096 π⁷ r² θ³ - 256 π⁴ r² θ⁴ +
        2560 π⁵ r² θ⁴ - 5120 π⁶ r² θ⁴ - 1024 π⁴ r² θ⁵ + 2048 π⁵ r² θ⁵ + 128 π³ r² θ⁶ - 256 π⁴ r² θ⁶)) /
   (2 (-8 π³ + 4 π² θ² - 8 π³ θ²))], {r, -1, 1}, {θ, -2 π, 2 π}, {β, -π / 2, π / 2}]
```



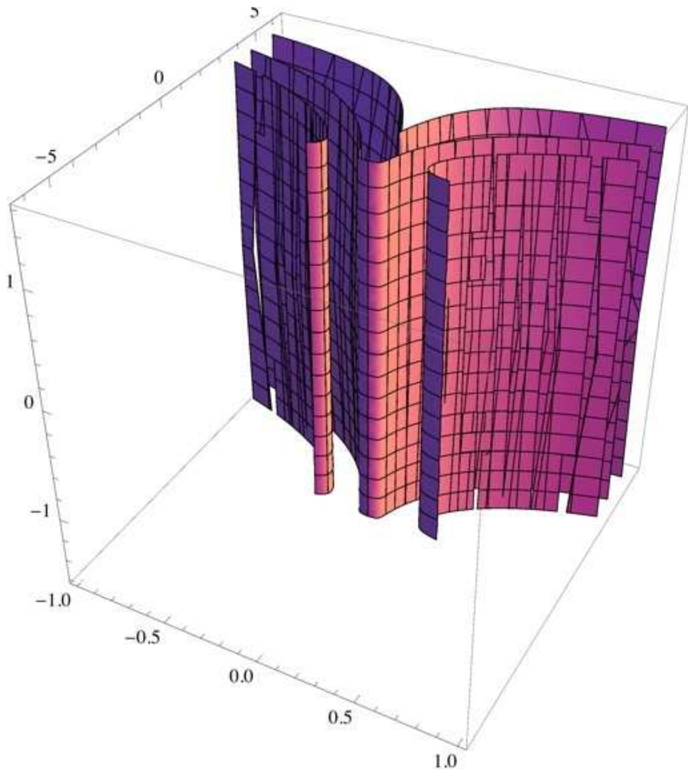
```
ContourPlot3D[ArcSin[(-8 π^2 √((4 π - θ) θ) +
    4 π (2 (π + √(π^2 - π^2 Sin[β]^2)))^2 √((4 π - θ) θ) - 8 π^2 θ^2 √((4 π - θ) (2 (π + √(π^2 - π^2 Sin[β]^2)))) +
    √(4096 π^7 r^2 θ - 5120 π^6 r^2 θ^2 + 2048 π^5 r^2 θ^3 - 2048 π^6 r^2 θ^3 + 4096 π^7 r^2 θ^3 - 256 π^4 r^2 θ^4 +
        2560 π^5 r^2 θ^4 - 5120 π^6 r^2 θ^4 - 1024 π^4 r^2 θ^5 + 2048 π^5 r^2 θ^5 + 128 π^3 r^2 θ^6 - 256 π^4 r^2 θ^6)) /
  (2 (-8 π^3 + 4 π^2 θ^2 - 8 π^3 θ^2))], {r, -1, 1}, {θ, -2 π, 2 π}, {β, -π / 2, π / 2}]
```



```
ContourPlot3D[
 ArcSin[(-8 π^2 √((4 π - θ) θ) + 4 π (θ)^2 √((4 π - θ) θ) - 8 π^2 θ^2 √((4 π - θ) (2 (π + √(π^2 - π^2 Sin[β]^2)))) +
     √(4096 π^7 r^2 θ - 5120 π^6 r^2 θ^2 + 2048 π^5 r^2 θ^3 - 2048 π^6 r^2 θ^3 + 4096 π^7 r^2 θ^3 - 256 π^4 r^2 θ^4 +
        2560 π^5 r^2 θ^4 - 5120 π^6 r^2 θ^4 - 1024 π^4 r^2 θ^5 + 2048 π^5 r^2 θ^5 + 128 π^3 r^2 θ^6 - 256 π^4 r^2 θ^6)) /
   (2 (-8 π^3 + 4 π^2 θ^2 - 8 π^3 θ^2))], {r, -1, 1}, {θ, -2 π, 2 π}, {β, -π / 2, π / 2}]
```



```
ContourPlot3D[
 ArcSin[(-8 π^2 √((4 π - θ) θ) + 4 π (θ)^2 √((4 π - (2 (π + √(π^2 - π^2 Sin[β]^2)))) θ) - 8 π^2 θ^2 √((4 π - θ) (θ)) +
     √(4096 π^7 r^2 θ - 5120 π^6 r^2 θ^2 + 2048 π^5 r^2 θ^3 - 2048 π^6 r^2 θ^3 + 4096 π^7 r^2 θ^3 - 256 π^4 r^2 θ^4 +
        2560 π^5 r^2 θ^4 - 5120 π^6 r^2 θ^4 - 1024 π^4 r^2 θ^5 + 2048 π^5 r^2 θ^5 + 128 π^3 r^2 θ^6 - 256 π^4 r^2 θ^6)) /
   (2 (-8 π^3 + 4 π^2 θ^2 - 8 π^3 θ^2))], {r, -1, 1}, {θ, -2 π, 2 π}, {β, -π / 2, π / 2}]
```



$$\text{Solve}\left[\sqrt{\left(r\hat{}2 - \frac{\sqrt{4\pi (r)^2 \theta - (r)^2 \theta^2}}{2\pi}\hat{}2\right)} ==\right.$$

$$\frac{1}{2\pi}\left(2\pi\left(\left(-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta] + 4\pi^2\,\text{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta} - \right.\right.\right.$$

$$\left.\frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{\theta}\right) \Big/ \left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\text{Sin}[\beta]^2 + 8\pi^2\theta\,\text{Sin}[\beta]^2\right)\Bigg) -$$

$$\left(\left(-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta] + 4\pi^2\,\text{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta} - \right.\right.$$

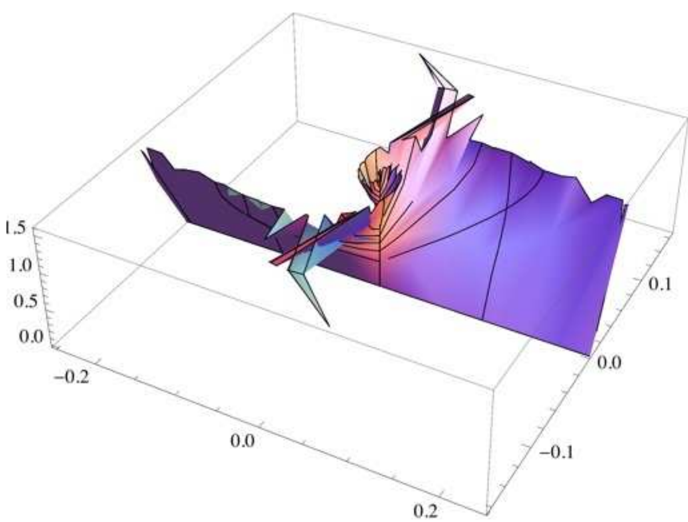
$$\left.\left.\left.\frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{\theta}\right) \Big/ \left(16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\text{Sin}[\beta]^2 + 8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)\theta\right), \beta\Big]$$

Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

$$\left\{\left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\theta} + 2\sqrt{r^2(2\pi-\theta)^2}\sqrt{(4\pi-\theta)\theta}}{2\pi}\right]\right\}\right\}$$

$$\text{RevolutionPlot3D}\left[\text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\theta} + 2\sqrt{r^2(2\pi-\theta)^2}\sqrt{(4\pi-\theta)\theta}}{2\pi}\right], \{r, -1, 1\}, \{\theta, -2\pi, 2\pi\}\right]$$



$$\text{SphericalPlot3D}\left[\text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\theta} + 2\sqrt{r^2(2\pi-\theta)^2}\sqrt{(4\pi-\theta)\theta}}{2\pi}\right], \{, -1, 1\}, \{\theta, -2\pi, 2\pi\}\right]$$

$$\mathbf{Solve}\left[\sqrt{\left(r^{\wedge}2-\frac{\sqrt{4\,\pi\,(r)^{2}\,\theta-(r)^{2}\,\theta^{2}}}{2\,\pi}{}^{\wedge}2\right)}==\right.$$

$$\frac{1}{2\,\pi}\left(2\,\pi\left(\left(-4\,\pi\,\theta+\theta^{2}+2\,\pi\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]+4\,\pi^{2}\,\mathrm{Sin}[\beta]^{2}-\frac{2\,\pi^{2}\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^{3}}{4\,\pi-\theta}-\right.\right.\right.$$

$$\left.\left.\frac{2\,\pi^{2}\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^{3}}{\theta}\right)\Big/\left(16\,\pi^{2}\,\theta-12\,\pi\,\theta^{2}+2\,\theta^{3}-16\,\pi^{3}\,\mathrm{Sin}[\beta]^{2}+8\,\pi^{2}\,\theta\,\mathrm{Sin}[\beta]^{2}\right)\right)-$$

$$\left(\left(-4\,\pi\,\theta+\theta^{2}+2\,\pi\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]+4\,\pi^{2}\,\mathrm{Sin}[\beta]^{2}-\frac{2\,\pi^{2}\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^{3}}{4\,\pi-\theta}-\right.\right.$$

$$\left.\left.\left.\frac{2\,\pi^{2}\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^{3}}{\theta}\right)\Big/\left(16\,\pi^{2}\,\theta-12\,\pi\,\theta^{2}+2\,\theta^{3}-16\,\pi^{3}\,\mathrm{Sin}[\beta]^{2}+8\,\pi^{2}\,\theta\,\mathrm{Sin}[\beta]^{2}\right)\theta\right),\theta\right]$$

$$\left\{\left\{\theta\to-\frac{1-8\,\pi\,r}{4\,r}-\right.\right.$$

$$\frac{1}{2}\sqrt{\left(\frac{(1-8\,\pi\,r)^{2}}{4\,r^{2}}-\frac{-1+24\,\pi\,r-80\,\pi^{2}\,r^{2}}{12\,r^{2}}-\frac{1-24\,\pi\,r+80\,\pi^{2}\,r^{2}}{4\,r^{2}}+\left(1-32\,\pi^{2}\,r^{2}+256\,\pi^{4}\,r^{4}+192\right.\right.}$$

$$\left.\pi^{2}\,r^{2}\,\mathrm{Sin}[\beta]^{2}\right)\Big/\left(6\,2^{2/3}\,r^{2}\,\left(2-96\,\pi^{2}\,r^{2}+1536\,\pi^{4}\,r^{4}-8192\,\pi^{6}\,r^{6}+576\,\pi^{2}\,r^{2}\right.\right.$$

$$\mathrm{Sin}[\beta]^{2}+18\,432\,\pi^{4}\,r^{4}\,\mathrm{Sin}[\beta]^{2}-\sqrt{\left(110\,592\,\pi^{4}\,r^{4}\,\mathrm{Sin}[\beta]^{2}-5\,308\,416\,\pi^{6}\,r^{6}\,\mathrm{Sin}[\beta]^{2}+\right.}$$

$$84\,934\,656\,\pi^{8}\,r^{8}\,\mathrm{Sin}[\beta]^{2}-452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^{2}-110\,592\,\pi^{4}\,r^{4}\,\mathrm{Sin}[\beta]^{4}+$$

$$\left.\left.\left.35\,389\,440\,\pi^{6}\,r^{6}\,\mathrm{Sin}[\beta]^{4}+226\,492\,416\,\pi^{8}\,r^{8}\,\mathrm{Sin}[\beta]^{4}-28\,311\,552\,\pi^{6}\,r^{6}\,\mathrm{Sin}[\beta]^{6}\right)\right)^{1/3}\right)+$$

$$\frac{1}{12\,2^{1/3}\,r^{2}}\left(2-96\,\pi^{2}\,r^{2}+1536\,\pi^{4}\,r^{4}-8192\,\pi^{6}\,r^{6}+576\,\pi^{2}\,r^{2}\,\mathrm{Sin}[\beta]^{2}+18\,432\,\pi^{4}\,r^{4}\,\mathrm{Sin}[\beta]^{2}-\right.$$

$$\sqrt{\left(110\,592\,\pi^{4}\,r^{4}\,\mathrm{Sin}[\beta]^{2}-5\,308\,416\,\pi^{6}\,r^{6}\,\mathrm{Sin}[\beta]^{2}+84\,934\,656\,\pi^{8}\,r^{8}\,\mathrm{Sin}[\beta]^{2}-\right.}$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^{2}-110\,592\,\pi^{4}\,r^{4}\,\mathrm{Sin}[\beta]^{4}+35\,389\,440\,\pi^{6}\,r^{6}\,\mathrm{Sin}[\beta]^{4}+$$

$$\left.\left.226\,492\,416\,\pi^{8}\,r^{8}\,\mathrm{Sin}[\beta]^{4}-28\,311\,552\,\pi^{6}\,r^{6}\,\mathrm{Sin}[\beta]^{6}\right)\right)^{1/3}\Bigg)-$$

$$\frac{1}{2}\sqrt{\left(\frac{(1-8\,\pi\,r)^{2}}{2\,r^{2}}+\frac{-1+24\,\pi\,r-80\,\pi^{2}\,r^{2}}{12\,r^{2}}-\frac{1-24\,\pi\,r+80\,\pi^{2}\,r^{2}}{4\,r^{2}}-\right.}$$

$$\left(1-32\,\pi^{2}\,r^{2}+256\,\pi^{4}\,r^{4}+192\,\pi^{2}\,r^{2}\,\mathrm{Sin}[\beta]^{2}\right)\Big/$$

$$\left(6\,2^{2/3}\,r^{2}\,\left(2-96\,\pi^{2}\,r^{2}+1536\,\pi^{4}\,r^{4}-8192\,\pi^{6}\,r^{6}+576\,\pi^{2}\,r^{2}\,\mathrm{Sin}[\beta]^{2}+18\,432\,\pi^{4}\,r^{4}\,\mathrm{Sin}[\beta]^{2}-\right.\right.$$

$$\sqrt{\left(110\,592\,\pi^{4}\,r^{4}\,\mathrm{Sin}[\beta]^{2}-5\,308\,416\,\pi^{6}\,r^{6}\,\mathrm{Sin}[\beta]^{2}+84\,934\,656\,\pi^{8}\,r^{8}\,\mathrm{Sin}[\beta]^{2}-\right.}$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^{2}-110\,592\,\pi^{4}\,r^{4}\,\mathrm{Sin}[\beta]^{4}+35\,389\,440\,\pi^{6}\,r^{6}\,\mathrm{Sin}[\beta]^{4}+$$

$$226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\big)\big)^{1/3}\Big) -$$

$$\frac{1}{12\,2^{1/3}\,r^2}\,\big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2 + 18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 -$$

$$\sqrt{}\big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^2 -$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\big)\big)^{1/3} -$$

$$\left(-\frac{(1-8\,\pi\,r)^3}{r^3} + \frac{(1-8\,\pi\,r)\,\left(1-24\,\pi\,r+80\,\pi^2\,r^2\right)}{r^3} - \frac{8\,\left(-\pi+8\,\pi^2\,r-16\,\pi^3\,r^2\right)}{r^2}\right)\Big/$$

$$\left(4\,\sqrt{\left(\frac{(1-8\,\pi\,r)^2}{4\,r^2} - \frac{-1+24\,\pi\,r-80\,\pi^2\,r^2}{12\,r^2} - \frac{1-24\,\pi\,r+80\,\pi^2\,r^2}{4\,r^2} + \right.}\right.$$

$$\left(1 - 32\,\pi^2\,r^2 + 256\,\pi^4\,r^4 + 192\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2\right)\Big/\Big(6\,2^{2/3}\,r^2$$

$$\big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2 + 18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 -$$

$$\sqrt{}\big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^2 -$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6$$

$$\mathrm{Sin}[\beta]^4 + 226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\big)\big)^{1/3}\Big) +$$

$$\frac{1}{12\,2^{1/3}\,r^2}\,\big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2 + 18\,432$$

$$\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - \sqrt{}\big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8$$

$$r^8\,\mathrm{Sin}[\beta]^2 - 452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440$$

$$\left.\left.\left.\pi^6\,r^6\,\mathrm{Sin}[\beta]^4 + 226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\big)\big)^{1/3}\right)\right)\right)\Big\},$$

$$\Big\{\theta \to -\frac{1-8\,\pi\,r}{4\,r} - \frac{1}{2}\,\sqrt{\left(\frac{(1-8\,\pi\,r)^2}{4\,r^2} - \frac{-1+24\,\pi\,r-80\,\pi^2\,r^2}{12\,r^2} - \frac{1-24\,\pi\,r+80\,\pi^2\,r^2}{4\,r^2} + \right.}$$

$$\left(1 - 32\,\pi^2\,r^2 + 256\,\pi^4\,r^4 + 192\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2\right)\Big/$$

$$\Big(6\,2^{2/3}\,r^2\,\big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2 + 18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 -$$

$$\sqrt{}\big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^2 -$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\big)\big)^{1/3}\Big) +$$

$$\frac{1}{12\,2^{1/3}\,r^2}\,\big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2 + 18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 -$$

$$\sqrt{}\big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^2 -$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\big)\big)^{1/3}\Big) +$$

$$\frac{1}{2}\sqrt{\Bigg(\frac{(1-8\pi r)^2}{2 r^2}+\frac{-1+24\pi r-80\pi^2 r^2}{12 r^2}-\frac{1-24\pi r+80\pi^2 r^2}{4 r^2}-}$$

$$\left(1-32\pi^2 r^2+256\pi^4 r^4+192\pi^2 r^2 \text{Sin}[\beta]^2\right)\Big/$$

$$\Big(6\, 2^{2/3} r^2 \big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2 \text{Sin}[\beta]^2+18\,432\pi^4 r^4 \text{Sin}[\beta]^2-$$

$$\sqrt{\big(110\,592\pi^4 r^4 \text{Sin}[\beta]^2-5\,308\,416\pi^6 r^6 \text{Sin}[\beta]^2+84\,934\,656\pi^8 r^8 \text{Sin}[\beta]^2-}$$

$$452\,984\,832\pi^{10} r^{10} \text{Sin}[\beta]^2-110\,592\pi^4 r^4 \text{Sin}[\beta]^4+35\,389\,440\pi^6 r^6 \text{Sin}[\beta]^4+$$

$$226\,492\,416\pi^8 r^8 \text{Sin}[\beta]^4-28\,311\,552\pi^6 r^6 \text{Sin}[\beta]^6\big)\big)^{1/3}\Big)-$$

$$\frac{1}{12\, 2^{1/3} r^2}\big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2 \text{Sin}[\beta]^2+18\,432\pi^4 r^4 \text{Sin}[\beta]^2-$$

$$\sqrt{\big(110\,592\pi^4 r^4 \text{Sin}[\beta]^2-5\,308\,416\pi^6 r^6 \text{Sin}[\beta]^2+84\,934\,656\pi^8 r^8 \text{Sin}[\beta]^2-}$$

$$452\,984\,832\pi^{10} r^{10} \text{Sin}[\beta]^2-110\,592\pi^4 r^4 \text{Sin}[\beta]^4+35\,389\,440\pi^6 r^6 \text{Sin}[\beta]^4+$$

$$226\,492\,416\pi^8 r^8 \text{Sin}[\beta]^4-28\,311\,552\pi^6 r^6 \text{Sin}[\beta]^6\big)\big)^{1/3}-$$

$$\left(-\frac{(1-8\pi r)^3}{r^3}+\frac{(1-8\pi r)\left(1-24\pi r+80\pi^2 r^2\right)}{r^3}-\frac{8\left(-\pi+8\pi^2 r-16\pi^3 r^2\right)}{r^2}\right)\Big/$$

$$\Bigg(4\sqrt{\Bigg(\frac{(1-8\pi r)^2}{4 r^2}-\frac{-1+24\pi r-80\pi^2 r^2}{12 r^2}-\frac{1-24\pi r+80\pi^2 r^2}{4 r^2}+}$$

$$\left(1-32\pi^2 r^2+256\pi^4 r^4+192\pi^2 r^2 \text{Sin}[\beta]^2\right)\Big/\Big(6\, 2^{2/3} r^2$$

$$\big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2 \text{Sin}[\beta]^2+18\,432\pi^4 r^4 \text{Sin}[\beta]^2-$$

$$\sqrt{\big(110\,592\pi^4 r^4 \text{Sin}[\beta]^2-5\,308\,416\pi^6 r^6 \text{Sin}[\beta]^2+84\,934\,656\pi^8 r^8 \text{Sin}[\beta]^2-}$$

$$452\,984\,832\pi^{10} r^{10} \text{Sin}[\beta]^2-110\,592\pi^4 r^4 \text{Sin}[\beta]^4+35\,389\,440\pi^6 r^6$$

$$\text{Sin}[\beta]^4+226\,492\,416\pi^8 r^8 \text{Sin}[\beta]^4-28\,311\,552\pi^6 r^6 \text{Sin}[\beta]^6\big)\big)^{1/3}\Big)+$$

$$\frac{1}{12\, 2^{1/3} r^2}\big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2 \text{Sin}[\beta]^2+18\,432$$

$$\pi^4 r^4 \text{Sin}[\beta]^2-\sqrt{\big(110\,592\pi^4 r^4 \text{Sin}[\beta]^2-5\,308\,416\pi^6 r^6 \text{Sin}[\beta]^2+84\,934\,656\pi^8}$$

$$r^8 \text{Sin}[\beta]^2-452\,984\,832\pi^{10} r^{10} \text{Sin}[\beta]^2-110\,592\pi^4 r^4 \text{Sin}[\beta]^4+35\,389\,440$$

$$\pi^6 r^6 \text{Sin}[\beta]^4+226\,492\,416\pi^8 r^8 \text{Sin}[\beta]^4-28\,311\,552\pi^6 r^6 \text{Sin}[\beta]^6\big)\big)^{1/3}\Bigg)\Bigg)\Bigg)\Bigg\},$$

$$\Bigg\{\Theta\to-\frac{1-8\pi r}{4 r}+\frac{1}{2}\sqrt{\Bigg(\frac{(1-8\pi r)^2}{4 r^2}-\frac{-1+24\pi r-80\pi^2 r^2}{12 r^2}-\frac{1-24\pi r+80\pi^2 r^2}{4 r^2}+}$$

$$\left(1-32\pi^2 r^2+256\pi^4 r^4+192\pi^2 r^2 \text{Sin}[\beta]^2\right)\Big/$$

$$\Big(6\, 2^{2/3} r^2 \big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2 \text{Sin}[\beta]^2+18\,432\pi^4 r^4 \text{Sin}[\beta]^2-$$

$$\sqrt{\big(110\,592\pi^4 r^4 \text{Sin}[\beta]^2-5\,308\,416\pi^6 r^6 \text{Sin}[\beta]^2+84\,934\,656\pi^8 r^8 \text{Sin}[\beta]^2-}$$

$$452\,984\,832\pi^{10} r^{10} \text{Sin}[\beta]^2-110\,592\pi^4 r^4 \text{Sin}[\beta]^4+35\,389\,440\pi^6 r^6 \text{Sin}[\beta]^4+$$

$$226\,492\,416\,\pi^8\,r^8\,\text{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\text{Sin}[\beta]^6\big)\big)^{1/3}\Big) +$$

$$\frac{1}{12\,2^{1/3}\,r^2}\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\text{Sin}[\beta]^2 + 18\,432\,\pi^4\,r^4\,\text{Sin}[\beta]^2 -$$

$$\sqrt{}\big(110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\text{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\text{Sin}[\beta]^2 -$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\text{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6\,\text{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8\,r^8\,\text{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\text{Sin}[\beta]^6\big)\big)^{1/3}\Big) -$$

$$\frac{1}{2}\sqrt{}\Bigg(\frac{(1 - 8\,\pi\,r)^2}{2\,r^2} + \frac{-1 + 24\,\pi\,r - 80\,\pi^2\,r^2}{12\,r^2} - \frac{1 - 24\,\pi\,r + 80\,\pi^2\,r^2}{4\,r^2} -$$

$$\big(1 - 32\,\pi^2\,r^2 + 256\,\pi^4\,r^4 + 192\,\pi^2\,r^2\,\text{Sin}[\beta]^2\big)\Big/$$

$$\Big(6\,2^{2/3}\,r^2\,\big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\text{Sin}[\beta]^2 + 18\,432\,\pi^4\,r^4\,\text{Sin}[\beta]^2 -$$

$$\sqrt{}\big(110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\text{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\text{Sin}[\beta]^2 -$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\text{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6\,\text{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8\,r^8\,\text{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\text{Sin}[\beta]^6\big)\big)^{1/3}\Big) -$$

$$\frac{1}{12\,2^{1/3}\,r^2}\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\text{Sin}[\beta]^2 + 18\,432\,\pi^4\,r^4\,\text{Sin}[\beta]^2 -$$

$$\sqrt{}\big(110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\text{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\text{Sin}[\beta]^2 -$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\text{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6\,\text{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8\,r^8\,\text{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\text{Sin}[\beta]^6\big)\big)^{1/3} +$$

$$\left(-\frac{(1 - 8\,\pi\,r)^3}{r^3} + \frac{(1 - 8\,\pi\,r)\,\left(1 - 24\,\pi\,r + 80\,\pi^2\,r^2\right)}{r^3} - \frac{8\,\left(-\pi + 8\,\pi^2\,r - 16\,\pi^3\,r^2\right)}{r^2}\right)\Big/$$

$$\Bigg(4\sqrt{}\Bigg(\frac{(1 - 8\,\pi\,r)^2}{4\,r^2} - \frac{-1 + 24\,\pi\,r - 80\,\pi^2\,r^2}{12\,r^2} - \frac{1 - 24\,\pi\,r + 80\,\pi^2\,r^2}{4\,r^2} +$$

$$\big(1 - 32\,\pi^2\,r^2 + 256\,\pi^4\,r^4 + 192\,\pi^2\,r^2\,\text{Sin}[\beta]^2\big)\Big/\Big(6\,2^{2/3}\,r^2$$

$$\big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\text{Sin}[\beta]^2 + 18\,432\,\pi^4\,r^4\,\text{Sin}[\beta]^2 -$$

$$\sqrt{}\big(110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\text{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\text{Sin}[\beta]^2 -$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\text{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6$$

$$\text{Sin}[\beta]^4 + 226\,492\,416\,\pi^8\,r^8\,\text{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\text{Sin}[\beta]^6\big)\big)^{1/3}\Big) +$$

$$\frac{1}{12\,2^{1/3}\,r^2}\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\text{Sin}[\beta]^2 + 18\,432$$

$$\pi^4\,r^4\,\text{Sin}[\beta]^2 - \sqrt{}\big(110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\text{Sin}[\beta]^2 + 84\,934\,656\,\pi^8$$

$$r^8\,\text{Sin}[\beta]^2 - 452\,984\,832\,\pi^{10}\,r^{10}\,\text{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\text{Sin}[\beta]^4 + 35\,389\,440$$

$$\pi^6\,r^6\,\text{Sin}[\beta]^4 + 226\,492\,416\,\pi^8\,r^8\,\text{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\text{Sin}[\beta]^6\big)\big)^{1/3}\Big)\Bigg)\Bigg)\Bigg)\Big\},$$

$$\{\theta \to -\frac{1-8\pi r}{4 r}+\frac{1}{2}\sqrt{\Big(}\frac{(1-8\pi r)^{2}}{4 r^{2}}-\frac{-1+24\pi r-80\pi^{2} r^{2}}{12 r^{2}}-\frac{1-24\pi r+80\pi^{2} r^{2}}{4 r^{2}}+$$

$$(1-32\pi^{2} r^{2}+256\pi^{4} r^{4}+192\pi^{2} r^{2}\,\mathrm{Sin}[\beta]^{2})\Big/$$

$$\Big(6\; 2^{2/3} r^{2}\,(2-96\pi^{2} r^{2}+1536\pi^{4} r^{4}-8192\pi^{6} r^{6}+576\pi^{2} r^{2}\,\mathrm{Sin}[\beta]^{2}+18\,432\pi^{4} r^{4}\,\mathrm{Sin}[\beta]^{2}-$$

$$\sqrt{}(110\,592\pi^{4} r^{4}\,\mathrm{Sin}[\beta]^{2}-5\,308\,416\pi^{6} r^{6}\,\mathrm{Sin}[\beta]^{2}+84\,934\,656\pi^{8} r^{8}\,\mathrm{Sin}[\beta]^{2}-$$

$$452\,984\,832\pi^{10} r^{10}\,\mathrm{Sin}[\beta]^{2}-110\,592\pi^{4} r^{4}\,\mathrm{Sin}[\beta]^{4}+35\,389\,440\pi^{6} r^{6}\,\mathrm{Sin}[\beta]^{4}+$$

$$226\,492\,416\pi^{8} r^{8}\,\mathrm{Sin}[\beta]^{4}-28\,311\,552\pi^{6} r^{6}\,\mathrm{Sin}[\beta]^{6}))^{1/3}\Big)+$$

$$\frac{1}{12\; 2^{1/3} r^{2}}(2-96\pi^{2} r^{2}+1536\pi^{4} r^{4}-8192\pi^{6} r^{6}+576\pi^{2} r^{2}\,\mathrm{Sin}[\beta]^{2}+18\,432\pi^{4} r^{4}\,\mathrm{Sin}[\beta]^{2}-$$

$$\sqrt{}(110\,592\pi^{4} r^{4}\,\mathrm{Sin}[\beta]^{2}-5\,308\,416\pi^{6} r^{6}\,\mathrm{Sin}[\beta]^{2}+84\,934\,656\pi^{8} r^{8}\,\mathrm{Sin}[\beta]^{2}-$$

$$452\,984\,832\pi^{10} r^{10}\,\mathrm{Sin}[\beta]^{2}-110\,592\pi^{4} r^{4}\,\mathrm{Sin}[\beta]^{4}+35\,389\,440\pi^{6} r^{6}\,\mathrm{Sin}[\beta]^{4}+$$

$$226\,492\,416\pi^{8} r^{8}\,\mathrm{Sin}[\beta]^{4}-28\,311\,552\pi^{6} r^{6}\,\mathrm{Sin}[\beta]^{6}))^{1/3}\Big)+$$

$$\frac{1}{2}\sqrt{\Big(}\frac{(1-8\pi r)^{2}}{2 r^{2}}+\frac{-1+24\pi r-80\pi^{2} r^{2}}{12 r^{2}}-\frac{1-24\pi r+80\pi^{2} r^{2}}{4 r^{2}}-$$

$$(1-32\pi^{2} r^{2}+256\pi^{4} r^{4}+192\pi^{2} r^{2}\,\mathrm{Sin}[\beta]^{2})\Big/$$

$$\Big(6\; 2^{2/3} r^{2}\,(2-96\pi^{2} r^{2}+1536\pi^{4} r^{4}-8192\pi^{6} r^{6}+576\pi^{2} r^{2}\,\mathrm{Sin}[\beta]^{2}+18\,432\pi^{4} r^{4}\,\mathrm{Sin}[\beta]^{2}-$$

$$\sqrt{}(110\,592\pi^{4} r^{4}\,\mathrm{Sin}[\beta]^{2}-5\,308\,416\pi^{6} r^{6}\,\mathrm{Sin}[\beta]^{2}+84\,934\,656\pi^{8} r^{8}\,\mathrm{Sin}[\beta]^{2}-$$

$$452\,984\,832\pi^{10} r^{10}\,\mathrm{Sin}[\beta]^{2}-110\,592\pi^{4} r^{4}\,\mathrm{Sin}[\beta]^{4}+35\,389\,440\pi^{6} r^{6}\,\mathrm{Sin}[\beta]^{4}+$$

$$226\,492\,416\pi^{8} r^{8}\,\mathrm{Sin}[\beta]^{4}-28\,311\,552\pi^{6} r^{6}\,\mathrm{Sin}[\beta]^{6}))^{1/3}\Big)-$$

$$\frac{1}{12\; 2^{1/3} r^{2}}(2-96\pi^{2} r^{2}+1536\pi^{4} r^{4}-8192\pi^{6} r^{6}+576\pi^{2} r^{2}\,\mathrm{Sin}[\beta]^{2}+18\,432\pi^{4} r^{4}\,\mathrm{Sin}[\beta]^{2}-$$

$$\sqrt{}(110\,592\pi^{4} r^{4}\,\mathrm{Sin}[\beta]^{2}-5\,308\,416\pi^{6} r^{6}\,\mathrm{Sin}[\beta]^{2}+84\,934\,656\pi^{8} r^{8}\,\mathrm{Sin}[\beta]^{2}-$$

$$452\,984\,832\pi^{10} r^{10}\,\mathrm{Sin}[\beta]^{2}-110\,592\pi^{4} r^{4}\,\mathrm{Sin}[\beta]^{4}+35\,389\,440\pi^{6} r^{6}\,\mathrm{Sin}[\beta]^{4}+$$

$$226\,492\,416\pi^{8} r^{8}\,\mathrm{Sin}[\beta]^{4}-28\,311\,552\pi^{6} r^{6}\,\mathrm{Sin}[\beta]^{6}))^{1/3}+$$

$$\Big(-\frac{(1-8\pi r)^{3}}{r^{3}}+\frac{(1-8\pi r)\,(1-24\pi r+80\pi^{2} r^{2})}{r^{3}}-\frac{8\,(-\pi+8\pi^{2} r-16\pi^{3} r^{2})}{r^{2}}\Big)\Big/$$

$$\Big(4\sqrt{\Big(}\frac{(1-8\pi r)^{2}}{4 r^{2}}-\frac{-1+24\pi r-80\pi^{2} r^{2}}{12 r^{2}}-\frac{1-24\pi r+80\pi^{2} r^{2}}{4 r^{2}}+$$

$$(1-32\pi^{2} r^{2}+256\pi^{4} r^{4}+192\pi^{2} r^{2}\,\mathrm{Sin}[\beta]^{2})\Big/\Big(6\; 2^{2/3} r^{2}$$

$$(2-96\pi^{2} r^{2}+1536\pi^{4} r^{4}-8192\pi^{6} r^{6}+576\pi^{2} r^{2}\,\mathrm{Sin}[\beta]^{2}+18\,432\pi^{4} r^{4}\,\mathrm{Sin}[\beta]^{2}-$$

$$\sqrt{}(110\,592\pi^{4} r^{4}\,\mathrm{Sin}[\beta]^{2}-5\,308\,416\pi^{6} r^{6}\,\mathrm{Sin}[\beta]^{2}+84\,934\,656\pi^{8} r^{8}\,\mathrm{Sin}[\beta]^{2}-$$

$$452\,984\,832\pi^{10} r^{10}\,\mathrm{Sin}[\beta]^{2}-110\,592\pi^{4} r^{4}\,\mathrm{Sin}[\beta]^{4}+35\,389\,440\pi^{6} r^{6}$$

$$\mathrm{Sin}[\beta]^4 + 226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Big) +$$

$$\frac{1}{12\,2^{1/3}\,r^2}\,\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2 + 18\,432$$

$$\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - \sqrt{\Big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8}$$

$$r^8\,\mathrm{Sin}[\beta]^2 - 452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440$$

$$\pi^6\,r^6\,\mathrm{Sin}[\beta]^4 + 226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Big)\Big)\Big)\Big)\Big\},$$

$$\Big\{\theta \to -\frac{-1 - 8\,\pi\,r}{4\,r} - \frac{1}{2}\sqrt{\Big(\frac{(-1 - 8\,\pi\,r)^2}{4\,r^2} - \frac{-1 - 24\,\pi\,r - 80\,\pi^2\,r^2}{12\,r^2} - \frac{1 + 24\,\pi\,r + 80\,\pi^2\,r^2}{4\,r^2} +}$$

$$\Big(1 - 32\,\pi^2\,r^2 + 256\,\pi^4\,r^4 + 192\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2\Big)\Big/$$

$$\Big(6\,2^{2/3}\,r^2\,\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2 + 18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 -$$

$$\sqrt{\Big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^2 -}$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Big) +$$

$$\frac{1}{12\,2^{1/3}\,r^2}\,\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2 + 18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 -$$

$$\sqrt{\Big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^2 -}$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Big) -$$

$$\frac{1}{2}\sqrt{\Big(\frac{(-1 - 8\,\pi\,r)^2}{2\,r^2} + \frac{-1 - 24\,\pi\,r - 80\,\pi^2\,r^2}{12\,r^2} - \frac{1 + 24\,\pi\,r + 80\,\pi^2\,r^2}{4\,r^2} -}$$

$$\Big(1 - 32\,\pi^2\,r^2 + 256\,\pi^4\,r^4 + 192\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2\Big)\Big/$$

$$\Big(6\,2^{2/3}\,r^2\,\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2 + 18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 -$$

$$\sqrt{\Big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^2 -}$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Big) -$$

$$\frac{1}{12\,2^{1/3}\,r^2}\,\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2 + 18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 -$$

$$\sqrt{\Big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^2 -}$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3} -$$

$$\left(-\frac{(-1-8\pi r)^3}{r^3}+\frac{(-1-8\pi r)\left(1+24\pi r+80\pi^2 r^2\right)}{r^3}-\frac{8\left(-\pi-8\pi^2 r-16\pi^3 r^2\right)}{r^2}\right)\Big/$$

$$\left(4\sqrt{\left(\frac{(-1-8\pi r)^2}{4 r^2}-\frac{-1-24\pi r-80\pi^2 r^2}{12 r^2}-\frac{1+24\pi r+80\pi^2 r^2}{4 r^2}+\right.}\right.$$
$$\left(1-32\pi^2 r^2+256\pi^4 r^4+192\pi^2 r^2 \text{Sin}[\beta]^2\right)\Big/\left(6\ 2^{2/3} r^2\right.$$
$$\left(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2 \text{Sin}[\beta]^2+18432\pi^4 r^4 \text{Sin}[\beta]^2-\right.$$
$$\sqrt{\left(110592\pi^4 r^4 \text{Sin}[\beta]^2-5308416\pi^6 r^6 \text{Sin}[\beta]^2+84934656\pi^8 r^8 \text{Sin}[\beta]^2-\right.}$$
$$452984832\pi^{10} r^{10} \text{Sin}[\beta]^2-110592\pi^4 r^4 \text{Sin}[\beta]^4+35389440\pi^6 r^6$$
$$\left.\left.\left.\text{Sin}[\beta]^4+226492416\pi^8 r^8 \text{Sin}[\beta]^4-28311552\pi^6 r^6 \text{Sin}[\beta]^6\right)\right)^{1/3}\right)+$$
$$\frac{1}{12\ 2^{1/3} r^2}\left(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2 \text{Sin}[\beta]^2+18432\right.$$
$$\pi^4 r^4 \text{Sin}[\beta]^2-\sqrt{\left(110592\pi^4 r^4 \text{Sin}[\beta]^2-5308416\pi^6 r^6 \text{Sin}[\beta]^2+84934656\pi^8\right.}$$
$$r^8 \text{Sin}[\beta]^2-452984832\pi^{10} r^{10} \text{Sin}[\beta]^2-110592\pi^4 r^4 \text{Sin}[\beta]^4+35389440$$
$$\left.\left.\left.\left.\pi^6 r^6 \text{Sin}[\beta]^4+226492416\pi^8 r^8 \text{Sin}[\beta]^4-28311552\pi^6 r^6 \text{Sin}[\beta]^6\right)\right)^{1/3}\right)\right)\Big\}\text{,}$$

$$\Big\{\theta\to-\frac{-1-8\pi r}{4 r}-\frac{1}{2}\sqrt{\left(\frac{(-1-8\pi r)^2}{4 r^2}-\frac{-1-24\pi r-80\pi^2 r^2}{12 r^2}-\frac{1+24\pi r+80\pi^2 r^2}{4 r^2}+\right.}$$
$$\left(1-32\pi^2 r^2+256\pi^4 r^4+192\pi^2 r^2 \text{Sin}[\beta]^2\right)\Big/$$
$$\left(6\ 2^{2/3} r^2\left(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2 \text{Sin}[\beta]^2+18432\pi^4 r^4 \text{Sin}[\beta]^2-\right.\right.$$
$$\sqrt{\left(110592\pi^4 r^4 \text{Sin}[\beta]^2-5308416\pi^6 r^6 \text{Sin}[\beta]^2+84934656\pi^8 r^8 \text{Sin}[\beta]^2-\right.}$$
$$452984832\pi^{10} r^{10} \text{Sin}[\beta]^2-110592\pi^4 r^4 \text{Sin}[\beta]^4+35389440\pi^6 r^6 \text{Sin}[\beta]^4+$$
$$\left.\left.\left.226492416\pi^8 r^8 \text{Sin}[\beta]^4-28311552\pi^6 r^6 \text{Sin}[\beta]^6\right)\right)^{1/3}\right)+$$
$$\frac{1}{12\ 2^{1/3} r^2}\left(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2 \text{Sin}[\beta]^2+18432\pi^4 r^4 \text{Sin}[\beta]^2-\right.$$
$$\sqrt{\left(110592\pi^4 r^4 \text{Sin}[\beta]^2-5308416\pi^6 r^6 \text{Sin}[\beta]^2+84934656\pi^8 r^8 \text{Sin}[\beta]^2-\right.}$$
$$452984832\pi^{10} r^{10} \text{Sin}[\beta]^2-110592\pi^4 r^4 \text{Sin}[\beta]^4+35389440\pi^6 r^6 \text{Sin}[\beta]^4+$$
$$\left.\left.\left.226492416\pi^8 r^8 \text{Sin}[\beta]^4-28311552\pi^6 r^6 \text{Sin}[\beta]^6\right)\right)^{1/3}\right)+$$

$$\frac{1}{2}\sqrt{\left(\frac{(-1-8\pi r)^2}{2 r^2}+\frac{-1-24\pi r-80\pi^2 r^2}{12 r^2}-\frac{1+24\pi r+80\pi^2 r^2}{4 r^2}-\right.}$$
$$\left(1-32\pi^2 r^2+256\pi^4 r^4+192\pi^2 r^2 \text{Sin}[\beta]^2\right)\Big/$$
$$\left(6\ 2^{2/3} r^2\left(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2 \text{Sin}[\beta]^2+18432\pi^4 r^4 \text{Sin}[\beta]^2-\right.\right.$$
$$\sqrt{\left(110592\pi^4 r^4 \text{Sin}[\beta]^2-5308416\pi^6 r^6 \text{Sin}[\beta]^2+84934656\pi^8 r^8 \text{Sin}[\beta]^2-\right.}$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^4 +$$
$$226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Big) -$$

$$\frac{1}{12\;2^{1/3}\,r^2}\,\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2 + 18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 -$$
$$\sqrt{}\Big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^2 -$$
$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^4 +$$
$$226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3} -$$

$$\left(-\frac{(-1-8\,\pi\,r)^3}{r^3} + \frac{(-1-8\,\pi\,r)\,\left(1+24\,\pi\,r+80\,\pi^2\,r^2\right)}{r^3} - \frac{8\,\left(-\pi-8\,\pi^2\,r-16\,\pi^3\,r^2\right)}{r^2}\right)\Big/$$

$$\Bigg(4\,\sqrt{\Bigg(\frac{(-1-8\,\pi\,r)^2}{4\,r^2} - \frac{-1-24\,\pi\,r-80\,\pi^2\,r^2}{12\,r^2} - \frac{1+24\,\pi\,r+80\,\pi^2\,r^2}{4\,r^2} +}$$
$$\left(1 - 32\,\pi^2\,r^2 + 256\,\pi^4\,r^4 + 192\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2\right)\Big/\Big(6\;2^{2/3}\,r^2$$
$$\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2 + 18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 -$$
$$\sqrt{}\Big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^2 -$$
$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6$$
$$\mathrm{Sin}[\beta]^4 + 226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Big) +$$

$$\frac{1}{12\;2^{1/3}\,r^2}\,\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2 + 18\,432$$
$$\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - \sqrt{}\Big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8$$
$$r^8\,\mathrm{Sin}[\beta]^2 - 452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440$$
$$\pi^6\,r^6\,\mathrm{Sin}[\beta]^4 + 226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Bigg)\Bigg)\Bigg)\Bigg\},$$

$$\Bigg\{\theta \to -\frac{-1-8\,\pi\,r}{4\,r} + \frac{1}{2}\,\sqrt{\Bigg(\frac{(-1-8\,\pi\,r)^2}{4\,r^2} - \frac{-1-24\,\pi\,r-80\,\pi^2\,r^2}{12\,r^2} - \frac{1+24\,\pi\,r+80\,\pi^2\,r^2}{4\,r^2} +}$$
$$\left(1 - 32\,\pi^2\,r^2 + 256\,\pi^4\,r^4 + 192\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2\right)\Big/$$
$$\Big(6\;2^{2/3}\,r^2\,\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2 + 18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 -$$
$$\sqrt{}\Big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^2 -$$
$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^4 +$$
$$226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Big) +$$

$$\frac{1}{12\;2^{1/3}\,r^2}\,\Big(2 - 96\,\pi^2\,r^2 + 1536\,\pi^4\,r^4 - 8192\,\pi^6\,r^6 + 576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2 + 18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 -$$
$$\sqrt{}\Big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^2 -$$
$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Bigg) -$$

$$\frac{1}{2}\sqrt{\Bigg(\frac{(-1-8\,\pi\,r)^2}{2\,r^2} + \frac{-1-24\,\pi\,r-80\,\pi^2\,r^2}{12\,r^2} - \frac{1+24\,\pi\,r+80\,\pi^2\,r^2}{4\,r^2} -}$$

$$\left(1-32\,\pi^2\,r^2+256\,\pi^4\,r^4+192\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2\right)\Big/$$

$$\Bigg(6\,2^{2/3}\,r^2\,\Big(2-96\,\pi^2\,r^2+1536\,\pi^4\,r^4-8192\,\pi^6\,r^6+576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2+18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 -$$

$$\sqrt{\Big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^2 -}$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Bigg) -$$

$$\frac{1}{12\,2^{1/3}\,r^2}\Big(2-96\,\pi^2\,r^2+1536\,\pi^4\,r^4-8192\,\pi^6\,r^6+576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2+18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 -$$

$$\sqrt{\Big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^2 -}$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^4 +$$

$$226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3} +$$

$$\left(-\frac{(-1-8\,\pi\,r)^3}{r^3} + \frac{(-1-8\,\pi\,r)\left(1+24\,\pi\,r+80\,\pi^2\,r^2\right)}{r^3} - \frac{8\left(-\pi-8\,\pi^2\,r-16\,\pi^3\,r^2\right)}{r^2}\right)\Big/$$

$$\Bigg(4\sqrt{\Bigg(\frac{(-1-8\,\pi\,r)^2}{4\,r^2} - \frac{-1-24\,\pi\,r-80\,\pi^2\,r^2}{12\,r^2} - \frac{1+24\,\pi\,r+80\,\pi^2\,r^2}{4\,r^2} +}$$

$$\left(1-32\,\pi^2\,r^2+256\,\pi^4\,r^4+192\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2\right)\Big/\Big(6\,2^{2/3}\,r^2$$

$$\Big(2-96\,\pi^2\,r^2+1536\,\pi^4\,r^4-8192\,\pi^6\,r^6+576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2+18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 -$$

$$\sqrt{\Big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^2 -}$$

$$452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440\,\pi^6\,r^6$$

$$\mathrm{Sin}[\beta]^4 + 226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Big) +$$

$$\frac{1}{12\,2^{1/3}\,r^2}\Big(2-96\,\pi^2\,r^2+1536\,\pi^4\,r^4-8192\,\pi^6\,r^6+576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2+18\,432$$

$$\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - \sqrt{\Big(110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 - 5\,308\,416\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^2 + 84\,934\,656\,\pi^8}$$

$$r^8\,\mathrm{Sin}[\beta]^2 - 452\,984\,832\,\pi^{10}\,r^{10}\,\mathrm{Sin}[\beta]^2 - 110\,592\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^4 + 35\,389\,440$$

$$\pi^6\,r^6\,\mathrm{Sin}[\beta]^4 + 226\,492\,416\,\pi^8\,r^8\,\mathrm{Sin}[\beta]^4 - 28\,311\,552\,\pi^6\,r^6\,\mathrm{Sin}[\beta]^6\Big)\Big)^{1/3}\Bigg)\Bigg)\Bigg)\Bigg\},$$

$$\Bigg\{\theta \to -\frac{-1-8\,\pi\,r}{4\,r} + \frac{1}{2}\sqrt{\Bigg(\frac{(-1-8\,\pi\,r)^2}{4\,r^2} - \frac{-1-24\,\pi\,r-80\,\pi^2\,r^2}{12\,r^2} - \frac{1+24\,\pi\,r+80\,\pi^2\,r^2}{4\,r^2} +}$$

$$\left(1-32\,\pi^2\,r^2+256\,\pi^4\,r^4+192\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2\right)\Big/$$

$$\Bigg(6\,2^{2/3}\,r^2\,\Big(2-96\,\pi^2\,r^2+1536\,\pi^4\,r^4-8192\,\pi^6\,r^6+576\,\pi^2\,r^2\,\mathrm{Sin}[\beta]^2+18\,432\,\pi^4\,r^4\,\mathrm{Sin}[\beta]^2 -$$

$$\sqrt{}\left(110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - 5308416\,\pi^6 r^6 \mathrm{Sin}[\beta]^2 + 84934656\,\pi^8 r^8 \mathrm{Sin}[\beta]^2 - 452984832\,\pi^{10} r^{10} \mathrm{Sin}[\beta]^2 - 110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^4 + 35389440\,\pi^6 r^6 \mathrm{Sin}[\beta]^4 + 226492416\,\pi^8 r^8 \mathrm{Sin}[\beta]^4 - 28311552\,\pi^6 r^6 \mathrm{Sin}[\beta]^6)\right)^{1/3}\Big) +$$

$$\frac{1}{12\,2^{1/3}\,r^2}\Big(2 - 96\,\pi^2 r^2 + 1536\,\pi^4 r^4 - 8192\,\pi^6 r^6 + 576\,\pi^2 r^2 \mathrm{Sin}[\beta]^2 + 18432\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - \sqrt{}\big(110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - 5308416\,\pi^6 r^6 \mathrm{Sin}[\beta]^2 + 84934656\,\pi^8 r^8 \mathrm{Sin}[\beta]^2 - 452984832\,\pi^{10} r^{10} \mathrm{Sin}[\beta]^2 - 110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^4 + 35389440\,\pi^6 r^6 \mathrm{Sin}[\beta]^4 + 226492416\,\pi^8 r^8 \mathrm{Sin}[\beta]^4 - 28311552\,\pi^6 r^6 \mathrm{Sin}[\beta]^6)\big)^{1/3}\Big) +$$

$$\frac{1}{2}\sqrt{\Bigg(\frac{(-1-8\,\pi\,r)^2}{2\,r^2} + \frac{-1-24\,\pi\,r-80\,\pi^2 r^2}{12\,r^2} - \frac{1+24\,\pi\,r+80\,\pi^2 r^2}{4\,r^2} -}$$

$$\left(1 - 32\,\pi^2 r^2 + 256\,\pi^4 r^4 + 192\,\pi^2 r^2 \mathrm{Sin}[\beta]^2\right) \Big/$$

$$\Big(6\,2^{2/3}\,r^2\,\big(2 - 96\,\pi^2 r^2 + 1536\,\pi^4 r^4 - 8192\,\pi^6 r^6 + 576\,\pi^2 r^2 \mathrm{Sin}[\beta]^2 + 18432\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - \sqrt{}\big(110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - 5308416\,\pi^6 r^6 \mathrm{Sin}[\beta]^2 + 84934656\,\pi^8 r^8 \mathrm{Sin}[\beta]^2 - 452984832\,\pi^{10} r^{10} \mathrm{Sin}[\beta]^2 - 110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^4 + 35389440\,\pi^6 r^6 \mathrm{Sin}[\beta]^4 + 226492416\,\pi^8 r^8 \mathrm{Sin}[\beta]^4 - 28311552\,\pi^6 r^6 \mathrm{Sin}[\beta]^6)\big)^{1/3}\Big) -$$

$$\frac{1}{12\,2^{1/3}\,r^2}\Big(2 - 96\,\pi^2 r^2 + 1536\,\pi^4 r^4 - 8192\,\pi^6 r^6 + 576\,\pi^2 r^2 \mathrm{Sin}[\beta]^2 + 18432\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - \sqrt{}\big(110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - 5308416\,\pi^6 r^6 \mathrm{Sin}[\beta]^2 + 84934656\,\pi^8 r^8 \mathrm{Sin}[\beta]^2 - 452984832\,\pi^{10} r^{10} \mathrm{Sin}[\beta]^2 - 110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^4 + 35389440\,\pi^6 r^6 \mathrm{Sin}[\beta]^4 + 226492416\,\pi^8 r^8 \mathrm{Sin}[\beta]^4 - 28311552\,\pi^6 r^6 \mathrm{Sin}[\beta]^6)\big)^{1/3} +$$

$$\left(-\frac{(-1-8\,\pi\,r)^3}{r^3} + \frac{(-1-8\,\pi\,r)\,(1+24\,\pi\,r+80\,\pi^2 r^2)}{r^3} - \frac{8\,(-\pi-8\,\pi^2 r-16\,\pi^3 r^2)}{r^2}\right) \Big/$$

$$\Bigg(4\sqrt{\Bigg(\frac{(-1-8\,\pi\,r)^2}{4\,r^2} - \frac{-1-24\,\pi\,r-80\,\pi^2 r^2}{12\,r^2} - \frac{1+24\,\pi\,r+80\,\pi^2 r^2}{4\,r^2} +}$$

$$\left(1 - 32\,\pi^2 r^2 + 256\,\pi^4 r^4 + 192\,\pi^2 r^2 \mathrm{Sin}[\beta]^2\right) \Big/ \Big(6\,2^{2/3}\,r^2$$

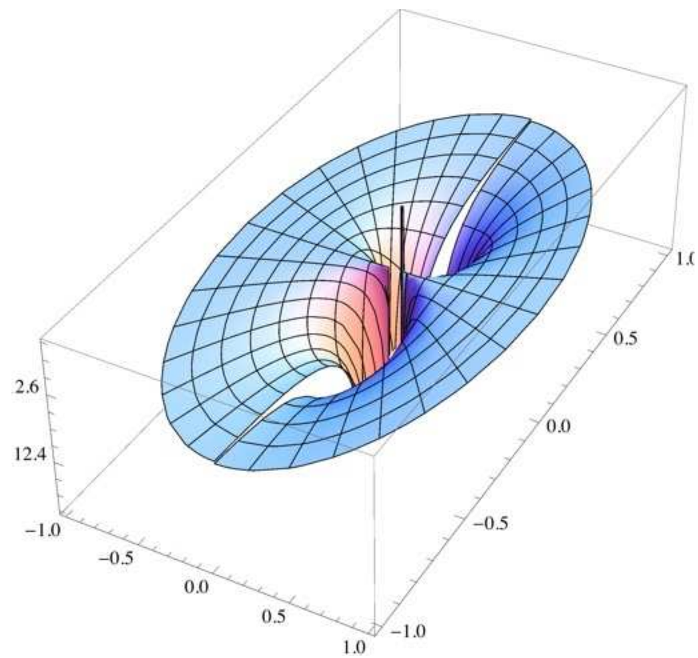
$$\big(2 - 96\,\pi^2 r^2 + 1536\,\pi^4 r^4 - 8192\,\pi^6 r^6 + 576\,\pi^2 r^2 \mathrm{Sin}[\beta]^2 + 18432\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - \sqrt{}\big(110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - 5308416\,\pi^6 r^6 \mathrm{Sin}[\beta]^2 + 84934656\,\pi^8 r^8 \mathrm{Sin}[\beta]^2 - 452984832\,\pi^{10} r^{10} \mathrm{Sin}[\beta]^2 - 110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^4 + 35389440\,\pi^6 r^6 \mathrm{Sin}[\beta]^4 + 226492416\,\pi^8 r^8 \mathrm{Sin}[\beta]^4 - 28311552\,\pi^6 r^6 \mathrm{Sin}[\beta]^6)\big)^{1/3}\big) +$$

$$\frac{1}{12\,2^{1/3}\,r^2}\Big(2 - 96\,\pi^2 r^2 + 1536\,\pi^4 r^4 - 8192\,\pi^6 r^6 + 576\,\pi^2 r^2 \mathrm{Sin}[\beta]^2 + 18432\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - \sqrt{}\big(110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^2 - 5308416\,\pi^6 r^6 \mathrm{Sin}[\beta]^2 + 84934656\,\pi^8 r^8 \mathrm{Sin}[\beta]^2 - 452984832\,\pi^{10} r^{10} \mathrm{Sin}[\beta]^2 - 110592\,\pi^4 r^4 \mathrm{Sin}[\beta]^4 + 35389440\,\pi^6 r^6 \mathrm{Sin}[\beta]^4 + 226492416\,\pi^8 r^8 \mathrm{Sin}[\beta]^4 - 28311552\,\pi^6 r^6 \mathrm{Sin}[\beta]^6)\big)^{1/3}\Big)\Bigg)\Bigg)\Bigg)\}\}$$

RevolutionPlot3D

$$\text{RevolutionPlot3D}\Bigg[-\frac{-1-8\pi r}{4r}+\frac{1}{2}\sqrt{\Bigg(\frac{(-1-8\pi r)^2}{4r^2}-\frac{-1-24\pi r-80\pi^2 r^2}{12r^2}-\frac{1+24\pi r+80\pi^2 r^2}{4r^2}+\big(1-32\pi^2 r^2+256\pi^4 r^4+192\pi^2 r^2\,\text{Sin}[\beta]^2\big)\Big/\Big(6\,2^{2/3}r^2\big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\text{Sin}[\beta]^2+18432\pi^4 r^4\,\text{Sin}[\beta]^2-\sqrt{\big(110592\pi^4 r^4\,\text{Sin}[\beta]^2-5308416\pi^6 r^6\,\text{Sin}[\beta]^2+84934656\pi^8 r^8\,\text{Sin}[\beta]^2-452984832\pi^{10} r^{10}\,\text{Sin}[\beta]^2-110592\pi^4 r^4\,\text{Sin}[\beta]^4+35389440\pi^6 r^6\,\text{Sin}[\beta]^4+226492416\pi^8 r^8\,\text{Sin}[\beta]^4-28311552\pi^6 r^6\,\text{Sin}[\beta]^6\big)}\big)^{1/3}\Big)+\frac{1}{12\,2^{1/3}r^2}\big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\text{Sin}[\beta]^2+18432\pi^4 r^4\,\text{Sin}[\beta]^2-\sqrt{\big(110592\pi^4 r^4\,\text{Sin}[\beta]^2-5308416\pi^6 r^6\,\text{Sin}[\beta]^2+84934656\pi^8 r^8\,\text{Sin}[\beta]^2-452984832\pi^{10} r^{10}\,\text{Sin}[\beta]^2-110592\pi^4 r^4\,\text{Sin}[\beta]^4+35389440\pi^6 r^6\,\text{Sin}[\beta]^4+226492416\pi^8 r^8\,\text{Sin}[\beta]^4-28311552\pi^6 r^6\,\text{Sin}[\beta]^6\big)}\big)^{1/3}\Bigg)}+$$

$$\frac{1}{2}\sqrt{\Bigg(\frac{(-1-8\pi r)^2}{2r^2}+\frac{-1-24\pi r-80\pi^2 r^2}{12r^2}-\frac{1+24\pi r+80\pi^2 r^2}{4r^2}-\big(1-32\pi^2 r^2+256\pi^4 r^4+192\pi^2 r^2\,\text{Sin}[\beta]^2\big)\Big/\Big(6\,2^{2/3}r^2\big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\text{Sin}[\beta]^2+18432\pi^4 r^4\,\text{Sin}[\beta]^2-\sqrt{\big(110592\pi^4 r^4\,\text{Sin}[\beta]^2-5308416\pi^6 r^6\,\text{Sin}[\beta]^2+84934656\pi^8 r^8\,\text{Sin}[\beta]^2-452984832\pi^{10} r^{10}\,\text{Sin}[\beta]^2-110592\pi^4 r^4\,\text{Sin}[\beta]^4+35389440\pi^6 r^6\,\text{Sin}[\beta]^4+226492416\pi^8 r^8\,\text{Sin}[\beta]^4-28311552\pi^6 r^6\,\text{Sin}[\beta]^6\big)}\big)^{1/3}\Big)-\frac{1}{12\,2^{1/3}r^2}\big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\text{Sin}[\beta]^2+18432\pi^4 r^4\,\text{Sin}[\beta]^2-\sqrt{\big(110592\pi^4 r^4\,\text{Sin}[\beta]^2-5308416\pi^6 r^6\,\text{Sin}[\beta]^2+84934656\pi^8 r^8\,\text{Sin}[\beta]^2-452984832\pi^{10} r^{10}\,\text{Sin}[\beta]^2-110592\pi^4 r^4\,\text{Sin}[\beta]^4+35389440\pi^6 r^6\,\text{Sin}[\beta]^4+226492416\pi^8 r^8\,\text{Sin}[\beta]^4-28311552\pi^6 r^6\,\text{Sin}[\beta]^6\big)}\big)^{1/3}+\Bigg(-\frac{(-1-8\pi r)^3}{r^3}+\frac{(-1-8\pi r)\,(1+24\pi r+80\pi^2 r^2)}{r^3}-\frac{8\,(-\pi-8\pi^2 r-16\pi^3 r^2)}{r^2}\Bigg)\Bigg/\Bigg(4\sqrt{\Bigg(\frac{(-1-8\pi r)^2}{4r^2}-\frac{-1-24\pi r-80\pi^2 r^2}{12r^2}-\frac{1+24\pi r+80\pi^2 r^2}{4r^2}+\big(1-32\pi^2 r^2+256\pi^4 r^4+192\pi^2 r^2\,\text{Sin}[\beta]^2\big)\Big/\Big(6\,2^{2/3}r^2\big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\text{Sin}[\beta]^2+18432\pi^4 r^4\,\text{Sin}[\beta]^2-\sqrt{\big(110592\pi^4 r^4\,\text{Sin}[\beta]^2-5308416\pi^6 r^6\,\text{Sin}[\beta]^2+84934656\pi^8 r^8\,\text{Sin}[\beta]^2-452984832\pi^{10} r^{10}\,\text{Sin}[\beta]^2-110592\pi^4 r^4\,\text{Sin}[\beta]^4+35389440\pi^6 r^6\,\text{Sin}[\beta]^4+226492416\pi^8 r^8\,\text{Sin}[\beta]^4-28311552\pi^6 r^6\,\text{Sin}[\beta]^6\big)}\big)^{1/3}\Big)+\frac{1}{12\,2^{1/3}r^2}\big(2-96\pi^2 r^2+1536\pi^4 r^4-8192\pi^6 r^6+576\pi^2 r^2\,\text{Sin}[\beta]^2+18432\pi^4 r^4\,\text{Sin}[\beta]^2-\sqrt{\big(110592\pi^4 r^4\,\text{Sin}[\beta]^2-5308416\pi^6 r^6\,\text{Sin}[\beta]^2+84934656\pi^8 r^8\,\text{Sin}[\beta]^2-452984832\pi^{10} r^{10}\,\text{Sin}[\beta]^2-110592\pi^4 r^4\,\text{Sin}[\beta]^4+35389440\pi^6 r^6\,\text{Sin}[\beta]^4+226492416\pi^8 r^8\,\text{Sin}[\beta]^4-28311552\pi^6 r^6\,\text{Sin}[\beta]^6\big)}\big)^{1/3}\Bigg)}\Bigg)\Bigg)},\ \{r,-1,1\},\ \{\beta,-\pi/2,\pi/2\}\Bigg]$$



$$\textbf{Solve}\left[\sqrt{\left(r\text{^}2-\frac{\sqrt{4\,\pi\,(r)^2\,\theta-(r)^2\,\theta^2}}{2\,\pi}\text{^}2\right)}==\right.$$

$$\frac{1}{2\,\pi}\left(2\,\pi\left(\left(-4\,\pi\,\theta+\theta^2+2\,\pi\,\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]+4\,\pi^2\,\text{Sin}[\beta]^2-\frac{2\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{4\,\pi-\theta}-\frac{2\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)\right)-\right.$$

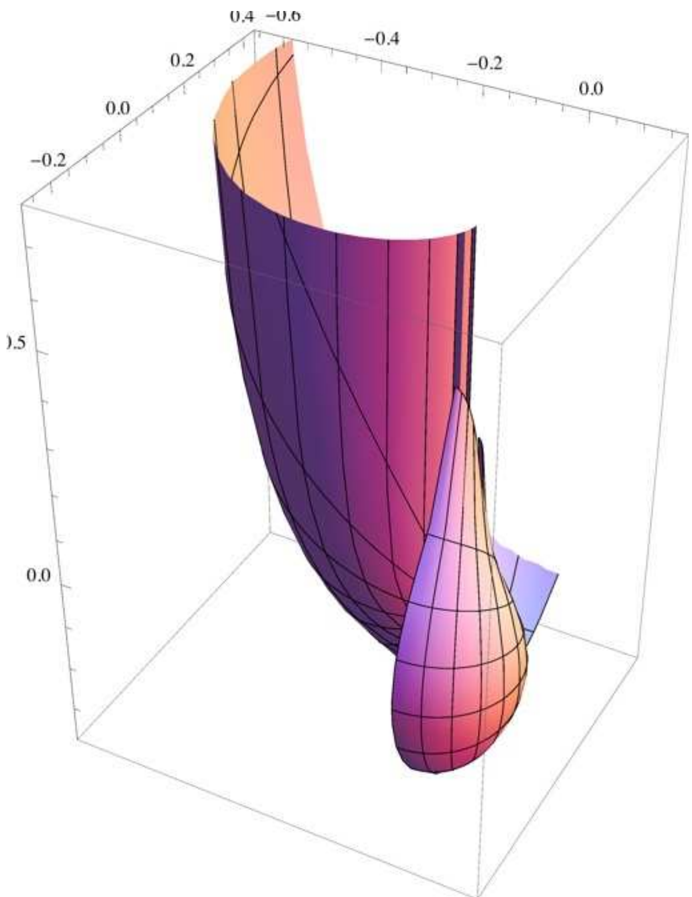
$$\left.\left(\left(-4\,\pi\,\theta+\theta^2+2\,\pi\,\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]+4\,\pi^2\,\text{Sin}[\beta]^2-\frac{2\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{4\,\pi-\theta}-\frac{2\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)\right)\theta\right),\ r\Big]$$

$$\left\{\left\{r\to-\frac{\sqrt{1-\frac{\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{4\pi-\theta}-\frac{\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{\theta}}}{\sqrt{16\,\pi^2-16\,\pi\,\theta+4\,\theta^2}}\right\},\right.$$

$$\left.\left\{r\to\frac{\sqrt{1-\frac{\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{4\pi-\theta}-\frac{\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{\theta}}}{\sqrt{16\,\pi^2-16\,\pi\,\theta+4\,\theta^2}}\right\}\right\}$$

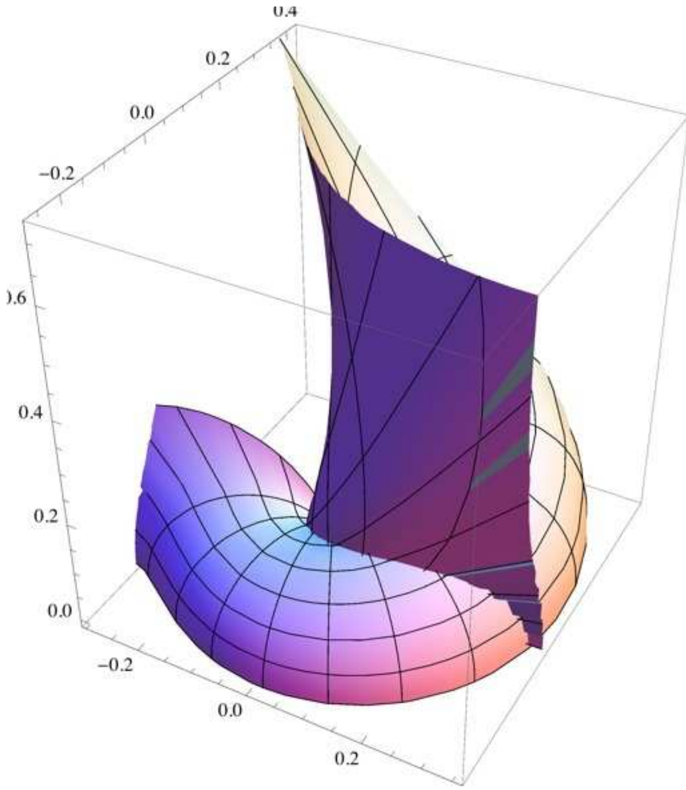
$$\text{SphericalPlot3D}\left[\frac{\sqrt{1-\frac{\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]}{4\pi-\theta}-\frac{\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]}{\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{\theta}}}{\sqrt{16\,\pi^2-16\,\pi\,\theta+4\,\theta^2}},\right.$$

$$\left.\{\theta,\,-2\,\pi,\,2\,\pi\},\,\{\beta,\,-\pi/2,\,\pi/2\}\right]$$



$$\texttt{SphericalPlot3D}\left[\frac{\sqrt{1-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \texttt{Sin}[\beta]}{4\pi-\theta}-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \texttt{Sin}[\beta]}{\theta}+\frac{\pi\,\texttt{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\texttt{Sin}[\beta]^2}{\theta}}}{\sqrt{16\,\pi^2-16\,\pi\,\theta+4\,\theta^2}},\right.$$

$$\left.\{\beta,\ -\pi/2,\ \pi/2\},\ \{\theta,\ -2\,\pi,\ 2\,\pi\}\right]$$



$$\texttt{Solve}\left[\sqrt{\left(\texttt{r}\hat{}2-\frac{\sqrt{4\,\pi\,(\texttt{r})^2\,\theta-(\texttt{r})^2\,\theta^2}}{2\,\pi}\hat{}2\right)}\ ==\right.$$

$$\frac{1}{2\,\pi}\left(2\,\pi\,\texttt{r}-\left(\left(-4\,\pi\,\theta+\theta^2+2\,\pi\sqrt{(4\,\pi-\theta)\,\theta}\ \texttt{Sin}[\beta]+4\,\pi^2\,\texttt{Sin}[\beta]^2-\frac{2\,\pi^2\sqrt{(4\,\pi-\theta)\,\theta}\ \texttt{Sin}[\beta]^3}{4\,\pi-\theta}-\right.\right.\right.$$

$$\left.\left.\left.\left.\frac{2\,\pi^2\sqrt{(4\,\pi-\theta)\,\theta}\ \texttt{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\texttt{Sin}[\beta]^2+8\,\pi^2\,\theta\,\texttt{Sin}[\beta]^2\right)\right)\theta\right),\ \beta\right]$$

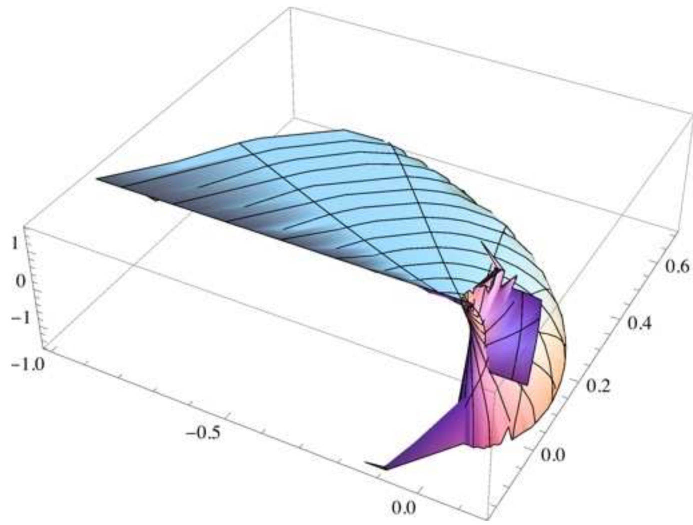
Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

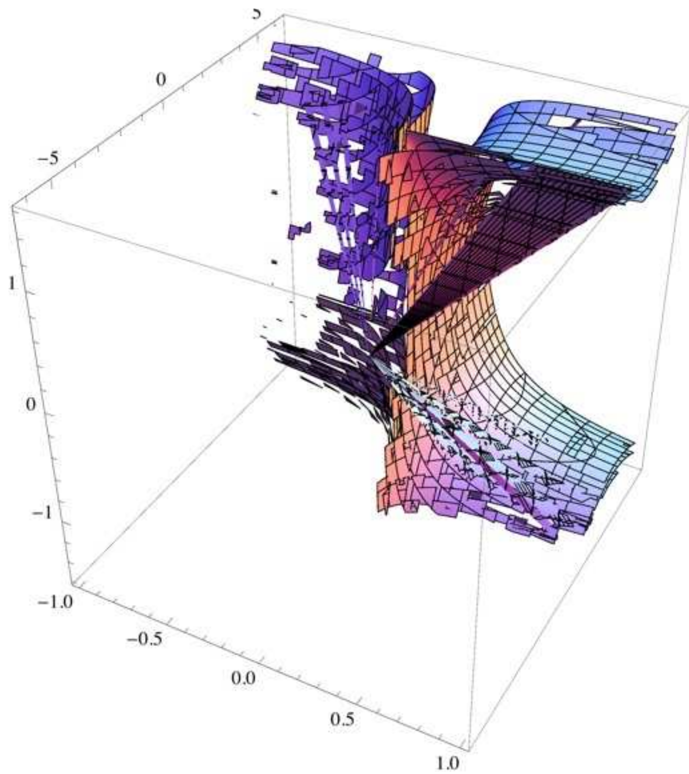
$$\left\{\left\{\beta\to\texttt{ArcSin}\left[\frac{1}{2\,\pi\,\theta}\left(8\,\pi^2\,\texttt{r}\sqrt{(4\,\pi-\theta)\,\theta}-4\,\pi\sqrt{\texttt{r}^2\,(2\,\pi-\theta)^2}\sqrt{(4\,\pi-\theta)\,\theta}+\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\theta\sqrt{(4\,\pi-\theta)\,\theta}-4\,\pi\,\texttt{r}\,\theta\sqrt{(4\,\pi-\theta)\,\theta}+2\sqrt{\texttt{r}^2\,(2\,\pi-\theta)^2}\,\theta\sqrt{(4\,\pi-\theta)\,\theta}\right)\right]\right\}\right\}$$

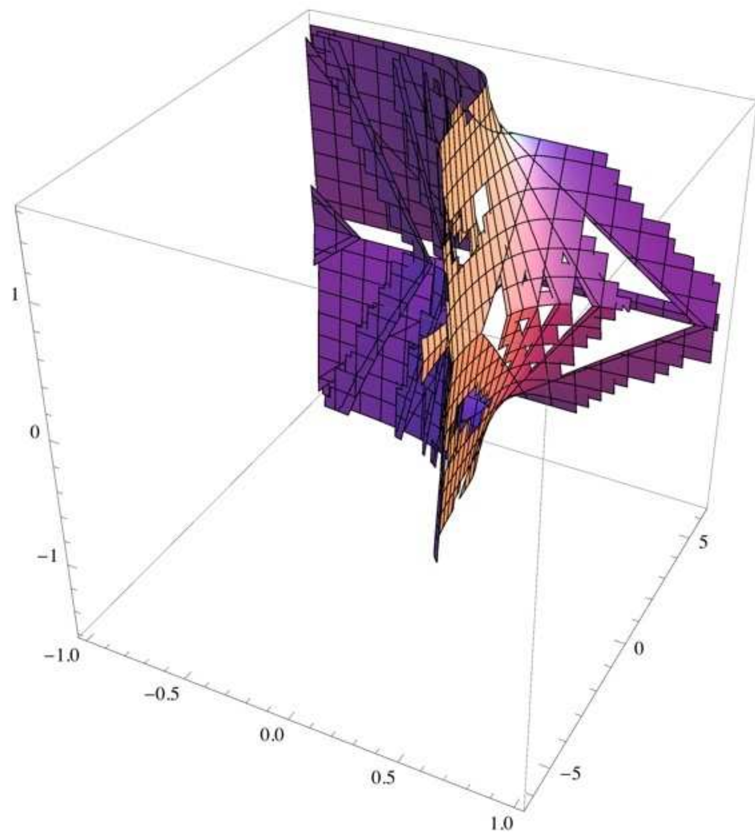
```
RevolutionPlot3D[ArcSin[
  1/(2 π θ) (8 π^2 r Sqrt[(4 π - θ) θ] - 4 π Sqrt[r^2 (2 π - θ)^2] Sqrt[(4 π - θ) θ] + θ Sqrt[(4 π - θ) θ] - 4 π r θ Sqrt[(4 π - θ) θ] +
    2 Sqrt[r^2 (2 π - θ)^2] θ Sqrt[(4 π - θ) θ])], {r, -1, 1}, {θ, -2 π, 2 π}]
```

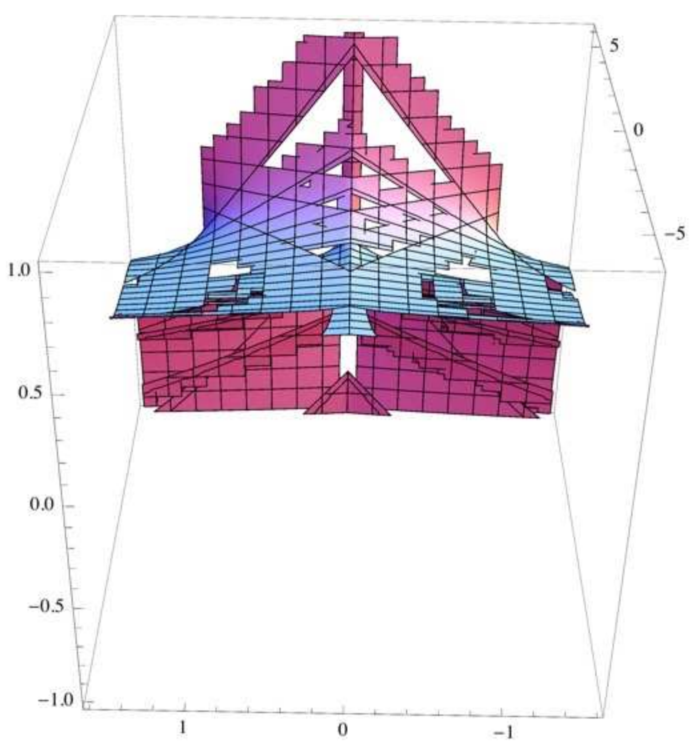


$$\text{ContourPlot3D}\Big[\text{ArcSin}\Big[\frac{1}{2\pi\theta}\Big(8\pi^2 r\sqrt{\Big(4\pi-\Big(2\Big(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\Big)\Big)\Big)\theta}-4\pi\sqrt{r^2(2\pi-\theta)^2}\sqrt{(4\pi-\theta)\theta}+\theta\sqrt{(4\pi-\theta)\theta}-4\pi r\theta\sqrt{(4\pi-\theta)\theta}+2\sqrt{r^2(2\pi-\theta)^2}\,\theta\sqrt{(4\pi-\theta)\theta}\Big)\Big],\ \{r,-1,1\},\{\theta,-2\pi,2\pi\},\{\beta,-\pi/2,\pi/2\}\Big]$$

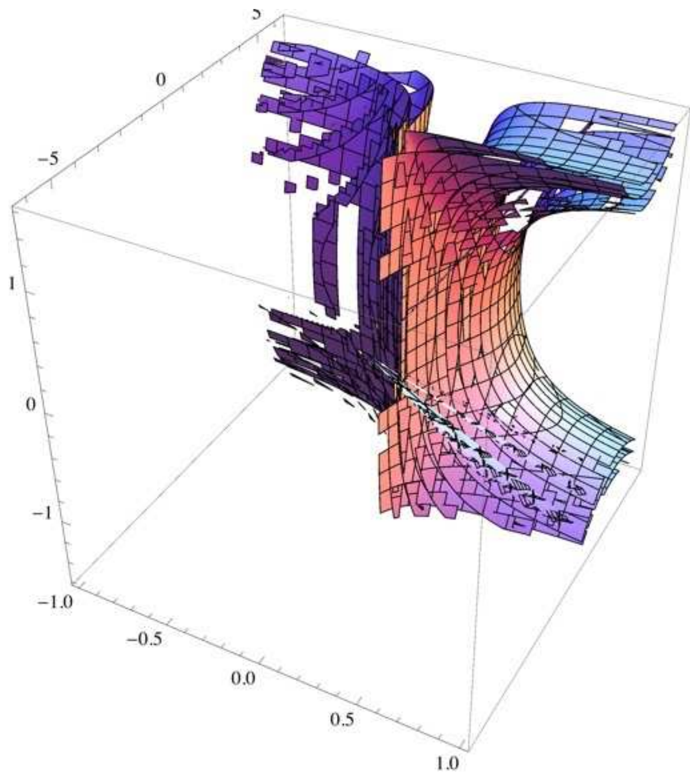


$$\text{ContourPlot3D}\Big[\text{ArcSin}\Big[\frac{1}{2\pi\theta}\Big(8\pi^2 r\sqrt{(4\pi-\theta)\theta}-4\pi\sqrt{r^2(2\pi-\theta)^2}\sqrt{(4\pi-\theta)\theta}+\theta\sqrt{\Big(4\pi-\Big(2\Big(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\Big)\Big)\Big)\theta}-4\pi r\theta\sqrt{(4\pi-\theta)(\theta)}+2\sqrt{r^2(2\pi-\theta)^2}\,\theta\sqrt{(4\pi-\theta)\theta}\Big)\Big],\ \{r,-1,1\},\{\theta,-2\pi,2\pi\},\{\beta,-\pi/2,\pi/2\}\Big]$$

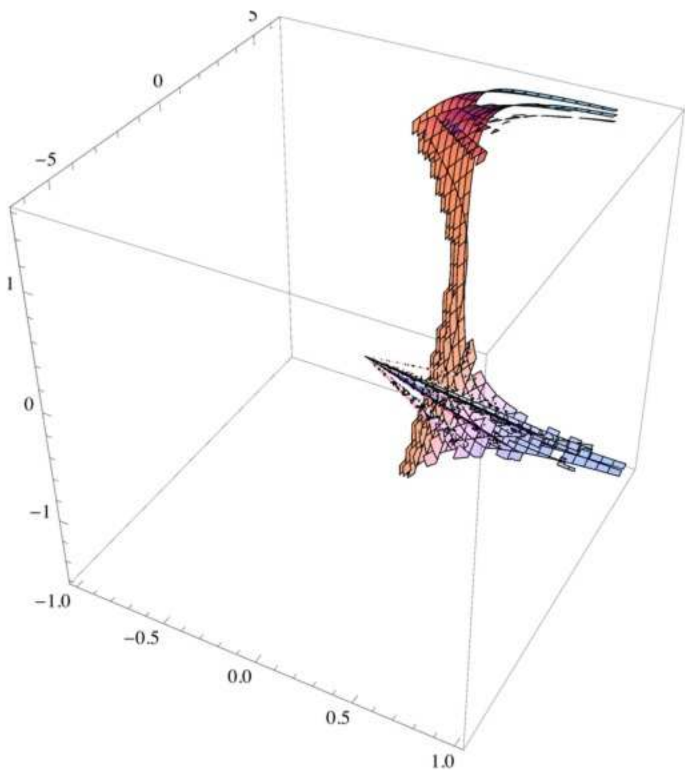

1
0
−1
−1.0
−0.5
0.0
0.5
1.0
5
0
−5


5
0
−5
1.0
0.5
0.0
−0.5
−1.0
1
0
−1

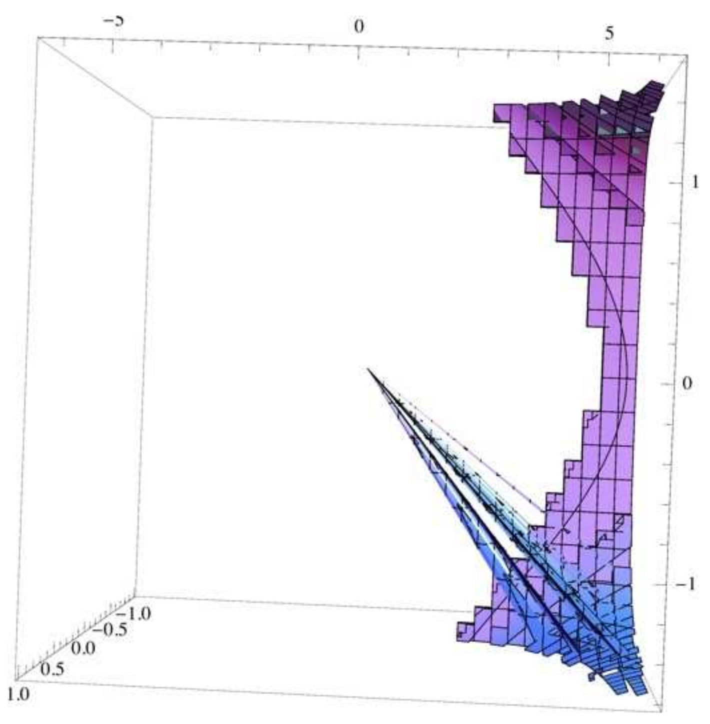
$$\textbf{ContourPlot3D}\left[\textbf{ArcSin}\left[\frac{1}{2\pi\theta}\left(8\pi^2 r\sqrt{(4\pi-\theta)\theta}-4\pi\sqrt{r^2(2\pi-\theta)^2}\sqrt{(4\pi-\theta)\theta}+\theta\sqrt{(4\pi-\theta)\theta}-4\pi r\theta\sqrt{(4\pi-\theta)\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\textbf{Sin}[\beta]^2}\right)\right)}+2\sqrt{r^2(2\pi-\theta)^2}\,\theta\sqrt{(4\pi-\theta)\theta}\right)\right],\{r,-1,1\},\{\theta,-2\pi,2\pi\},\{\beta,-\pi/2,\pi/2\}\right]$$



$$\textbf{ContourPlot3D}\left[\textbf{ArcSin}\left[\frac{1}{2\pi\theta}\left(8\pi^2 r\sqrt{(4\pi-\theta)\theta}-4\pi\sqrt{r^2(2\pi-\theta)^2}\sqrt{(4\pi-\theta)\theta}+\theta\sqrt{(4\pi-\theta)\theta}-4\pi r\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\textbf{Sin}[\beta]^2}\right)\right)\sqrt{(4\pi-\theta)\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\textbf{Sin}[\beta]^2}\right)\right)}+2\sqrt{r^2(2\pi-\theta)^2}\,\theta\sqrt{(4\pi-\theta)\theta}\right)\right],\{r,-1,1\},\{\theta,-2\pi,2\pi\},\{\beta,-\pi/2,\pi/2\}\right]$$

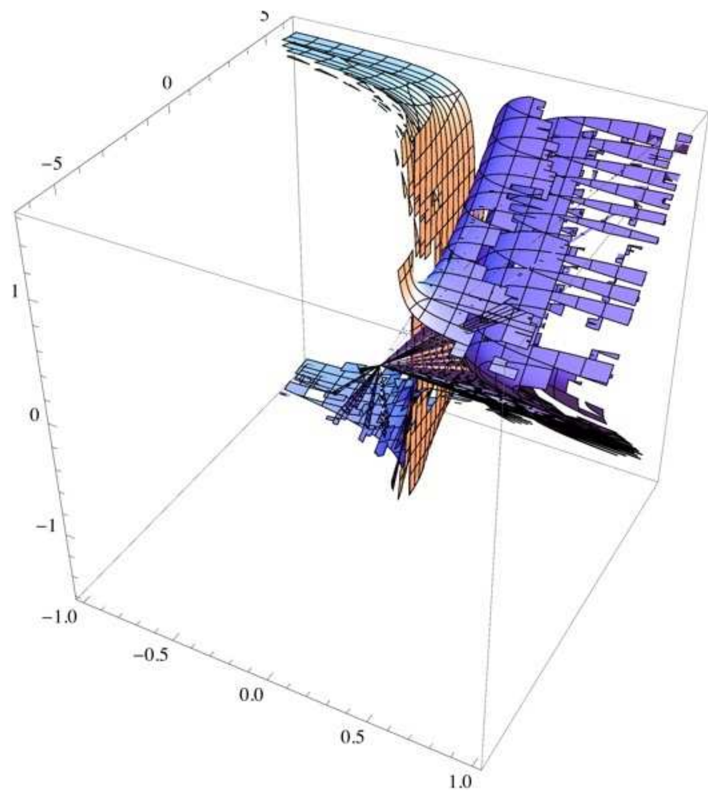


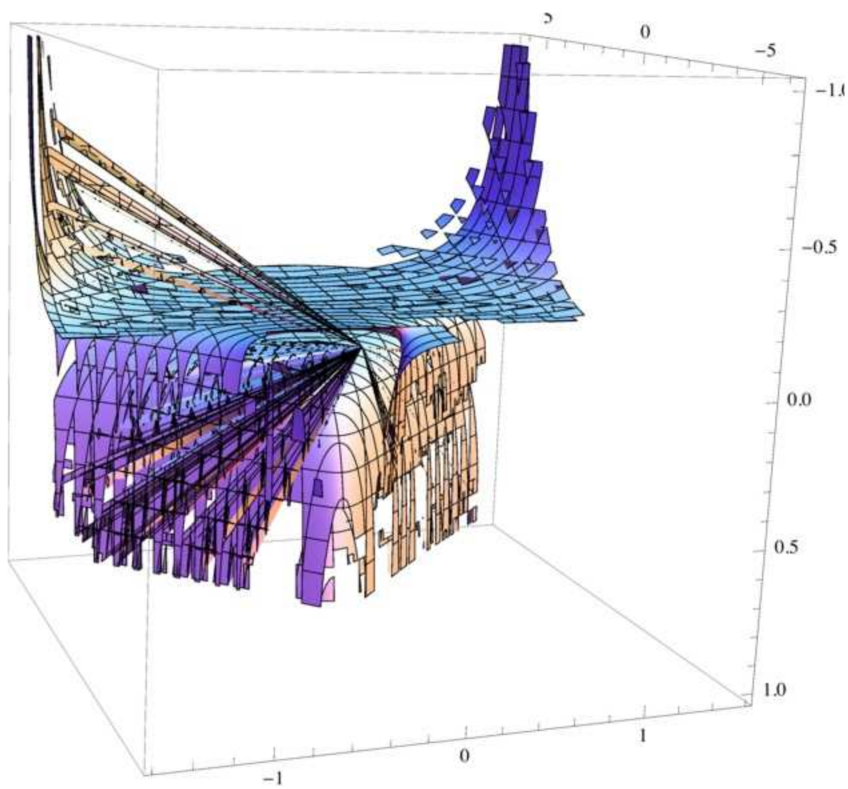
0
−5
1
0
−1
−1.0
−0.5
0.0
0.5
1.0



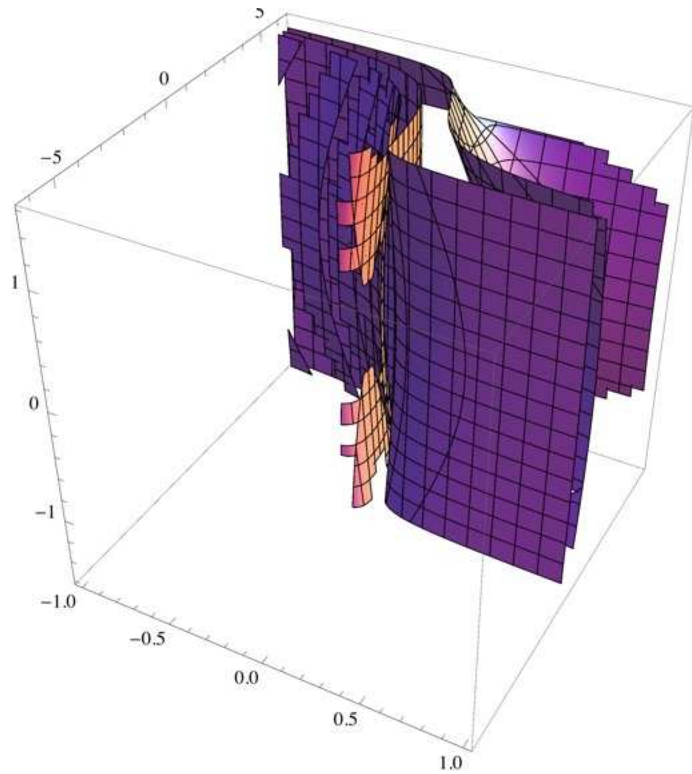
0
5
1
0
−1
−1.0
−0.5
0.0
0.5
1.0

$$\text{ContourPlot3D}\Big[\text{ArcSin}\Big[\frac{1}{2\pi\theta}\Big(8\pi^2 r\sqrt{(4\pi-\theta)\theta} - 4\pi\sqrt{r^2\left(2\pi-2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right)^2}\sqrt{(4\pi-\theta)\theta} + \theta\sqrt{(4\pi-\theta)\theta} - 4\pi r(\theta)\sqrt{(4\pi-\theta)(\theta)} + 2\sqrt{r^2(2\pi-\theta)^2}\,\theta\sqrt{(4\pi-\theta)\theta}\Big)\Big],$$
$$\{r,-1,1\},\{\theta,-2\pi,2\pi\},\{\beta,-\pi/2,\pi/2\}\Big]$$

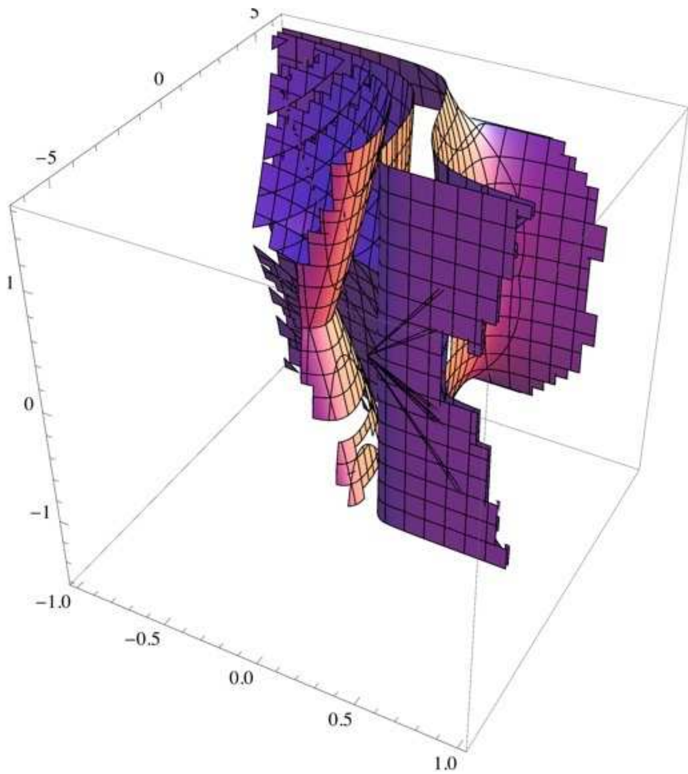

0
−5
1
0
−1
−1.0
−0.5
0.0
0.5
1.0


0
−5
−0.5
0.0
0.5
1.0
−1
0
1

```
ContourPlot3D[
 ArcSin[1/(2 π (2 (π + Sqrt[π^2 - π^2 Sin[β]^2]))) (8 π^2 r Sqrt[(4 π - (θ)) θ] - 4 π Sqrt[r^2 (2 π - θ)^2] Sqrt[(4 π - θ) θ] +
     θ Sqrt[(4 π - θ) θ] - 4 π r θ Sqrt[(4 π - θ) θ] + 2 Sqrt[r^2 (2 π - θ)^2] θ Sqrt[(4 π - θ) θ])],
 {r, -1, 1}, {θ, -2 π, 2 π}, {β, -π / 2, π / 2}]
```

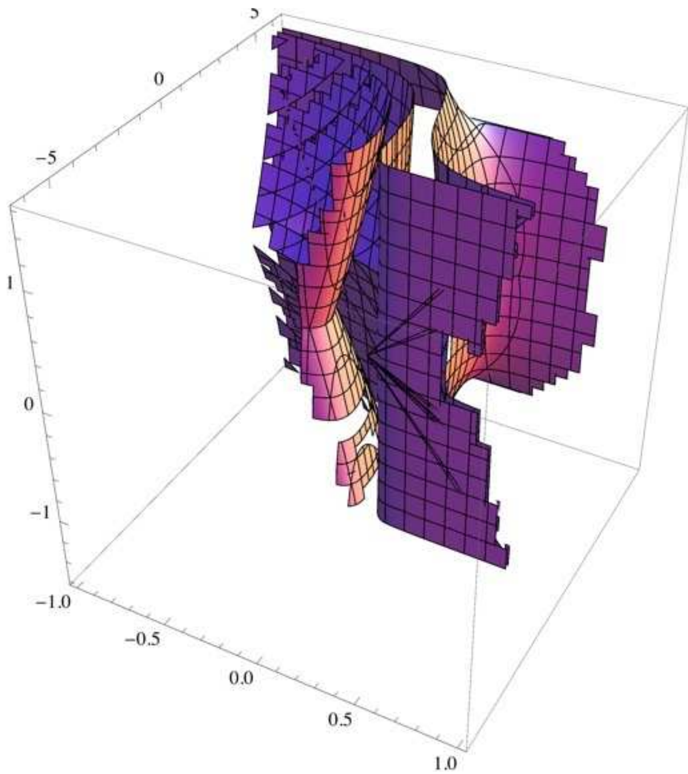


```
ContourPlot3D[ArcSin[1 / (2 π (2 (π + Sqrt[π^2 - π^2 Sin[β]^2])))
   (8 π^2 r Sqrt[(4 π - (θ)) θ] - 4 π Sqrt[r^2 (2 π - θ)^2] Sqrt[(4 π - (2 (π + Sqrt[π^2 - π^2 Sin[β]^2]))) θ] +
     θ Sqrt[(4 π - θ) θ] - 4 π r θ Sqrt[(4 π - θ) θ] + 2 Sqrt[r^2 (2 π - θ)^2] θ Sqrt[(4 π - θ) θ])],
  {r, -1, 1}, {θ, -2 π, 2 π}, {β, -π / 2, π / 2}]
```



$$\text{ContourPlot3D}\Big[\text{ArcSin}\Big[\frac{1}{2\,\pi\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right)}\left(8\,\pi^2\,r\,\sqrt{(4\,\pi-(\theta))\,\theta}-4\,\pi\,\sqrt{r^2\,(2\,\pi-\theta)^2}\,\sqrt{\left(4\,\pi-\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right)\right)\theta}+\theta\,\sqrt{(4\,\pi-\theta)\,\theta}-4\,\pi\,r\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}+2\,\sqrt{r^2\,(2\,\pi-\theta)^2}\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\right)\Big],$$

$$\{r,-1,1\},\{\theta,-2\,\pi,2\,\pi\},\{\beta,-\pi/2,\pi/2\}\Big]$$

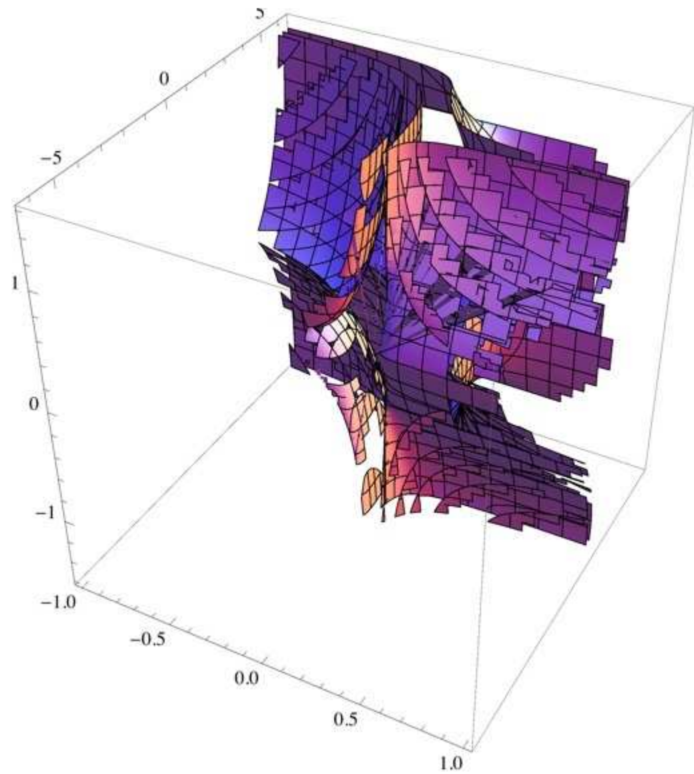


$$\text{ContourPlot3D}\Big[\text{ArcSin}\Big[\frac{1}{2\,\pi\,\left(2\,\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right)}\Big(8\,\pi^2\,r\,\sqrt{(4\,\pi-(\theta))\,\theta}\,-$$

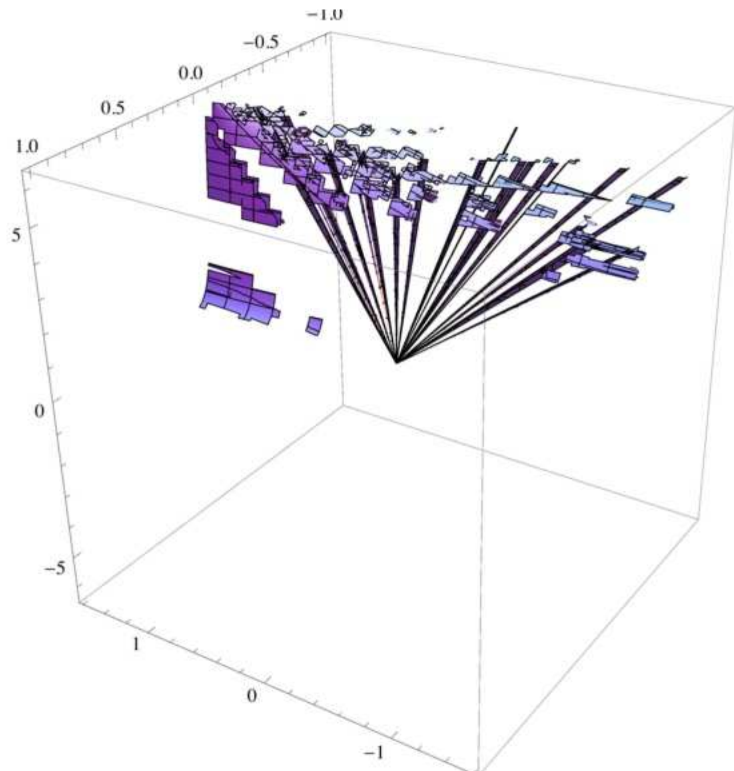
$$4\,\pi\,\sqrt{r^2\,(2\,\pi-\theta)^2}\,\sqrt{\left(4\,\pi-\left(2\,\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right)\right)\left(2\,\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right)}\,+$$

$$\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\,-4\,\pi\,r\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\,+2\,\sqrt{r^2\,(2\,\pi-\theta)^2}\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\Big)\Big],$$

$$\{r,\,-1,\,1\},\,\{\theta,\,-2\,\pi,\,2\,\pi\},\,\{\beta,\,-\pi/2,\,\pi/2\}\Big]$$



$$\textbf{ContourPlot3D}\Big[\textbf{ArcSin}\Big[\frac{1}{2\pi(\theta)}\Big(8\pi^2 r\sqrt{(4\pi-(\theta))\theta}-4\pi\sqrt{r^2(2\pi-\theta)^2}\sqrt{(4\pi-(\theta))(\theta)}+\theta\sqrt{(4\pi-\theta)\theta}-4\pi\Big(\Big(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]+4\pi^2\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{\theta}\Big)\Big/\Big(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\Big)\Big)(\theta)\sqrt{(4\pi-\theta)\theta}+2\sqrt{r^2(2\pi-\theta)^2}\,\theta\sqrt{(4\pi-\theta)\theta}\Big)\Big],\{r,-1,1\},\{\theta,-2\pi,2\pi\},\{\beta,-\pi/2,\pi/2\}\Big]$$



$$\textbf{Solve}\Big[\sqrt{\Big(r\hat{}2-\frac{\sqrt{4\pi(r)^2\theta-(r)^2\theta^2}}{2\pi}\hat{}2\Big)}==\frac{2\pi r-r\theta}{2\pi},\theta\Big]$$

{{}}

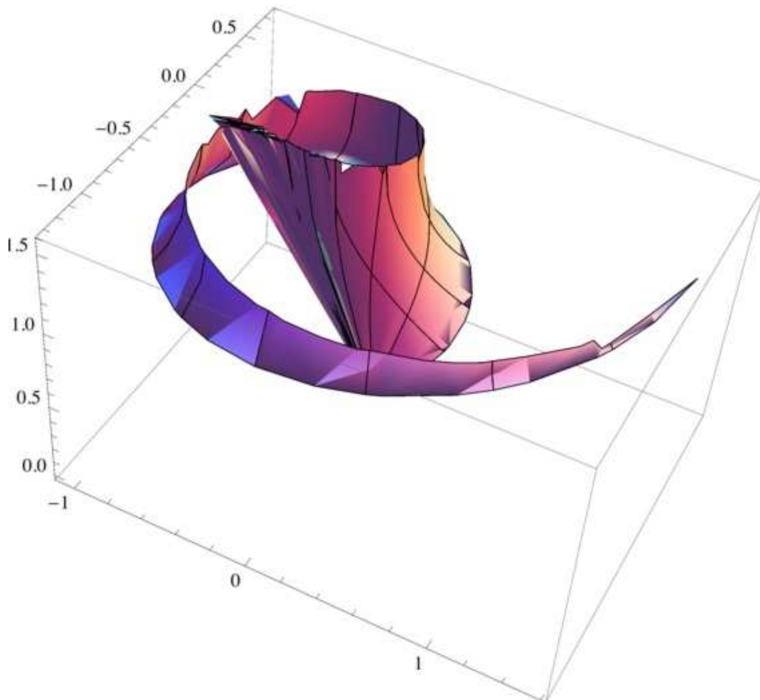
$$\text{Solve}\Bigg[\sqrt{\left(r\hat{}2-\frac{1}{2\pi}\left(\sqrt{\left(4\pi\left(\left(\left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)^2\theta-\left(\left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)^2\theta^2\right)}\right)\hat{}2\right)}==\frac{2\pi r-r\theta}{2\pi},\ \beta\Bigg]$$

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

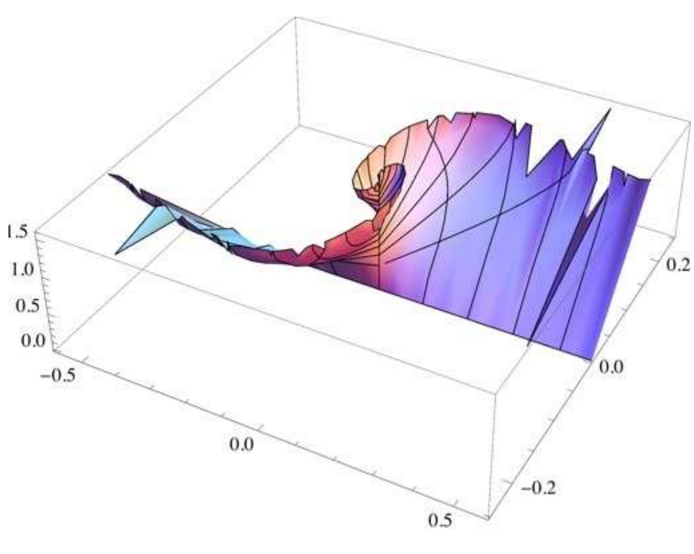
$$\left\{\left\{\beta\to\text{ArcSin}\left[\frac{\pi^2\sqrt{(4\pi-\theta)\,\theta}-\sqrt{2}\,\pi^{3/2}\sqrt{16\pi^3 r^2\theta-20\pi^2 r^2\theta^2+8\pi r^2\theta^3-r^2\theta^4}}{2\pi^3}\right]\right\},\right.$$
$$\left.\left\{\beta\to\text{ArcSin}\left[\frac{\pi^2\sqrt{(4\pi-\theta)\,\theta}+\sqrt{2}\,\pi^{3/2}\sqrt{16\pi^3 r^2\theta-20\pi^2 r^2\theta^2+8\pi r^2\theta^3-r^2\theta^4}}{2\pi^3}\right]\right\}\right\}$$

```
RevolutionPlot3D[
 {ArcSin[(π^2 Sqrt[(4 π - θ) θ] - Sqrt[2] π^(3/2) Sqrt[16 π^3 r^2 θ - 20 π^2 r^2 θ^2 + 8 π r^2 θ^3 - r^2 θ^4])/(2 π^3)], ArcSin[
   (π^2 Sqrt[(4 π - θ) θ] + Sqrt[2] π^(3/2) Sqrt[16 π^3 r^2 θ - 20 π^2 r^2 θ^2 + 8 π r^2 θ^3 - r^2 θ^4])/(2 π^3)]}, {r, -1, 1}, {θ, -2 π, 2 π}]
```

$$\text{RevolutionPlot3D}\left[\left\{\text{ArcSin}\left[\frac{\pi^2\sqrt{(4\pi-\theta)\,\theta}-\sqrt{2}\,\pi^{3/2}\sqrt{16\,\pi^3 r^2\theta-20\,\pi^2 r^2\theta^2+8\,\pi\, r^2\theta^3-r^2\theta^4}}{2\,\pi^3}\right],\ \text{ArcSin}\left[\frac{\pi^2\sqrt{(4\pi-\theta)\,\theta}+\sqrt{2}\,\pi^{3/2}\sqrt{16\,\pi^3 r^2\theta-20\,\pi^2 r^2\theta^2+8\,\pi\, r^2\theta^3-r^2\theta^4}}{2\,\pi^3}\right]\right\},\ \{r,-1,1\},\ \{\theta,-2\pi,2\pi\}\right]$$



$$\text{RevolutionPlot3D}\left[\text{ArcSin}\left[\frac{\pi^2\sqrt{(4\pi-\theta)\,\theta}+\sqrt{2}\,\pi^{3/2}\sqrt{16\,\pi^3 r^2\theta-20\,\pi^2 r^2\theta^2+8\,\pi\, r^2\theta^3-r^2\theta^4}}{2\,\pi^3}\right],\ \{r,-1,1\},\ \{\theta,-2\pi,2\pi\}\right]$$



```
Solve[
 Sqrt[(r^2 - 1/(2 π) (Sqrt[(4 π (((-4 π θ + θ^2 + 2 π Sqrt[(4 π - θ) θ] Sin[β] + 4 π^2 Sin[β]^2 - (2 π^2 Sqrt[(4 π - θ) θ] Sin[β]^3)/(4 π - θ) -
        (2 π^2 Sqrt[(4 π - θ) θ] Sin[β]^3)/θ) / (16 π^2 θ - 12 π θ^2 + 2 θ^3 - 16 π^3 Sin[β]^2 +
        8 π^2 θ Sin[β]^2))^2 θ - (((-4 π θ + θ^2 + 2 π Sqrt[(4 π - θ) θ] Sin[β] +
        4 π^2 Sin[β]^2 - (2 π^2 Sqrt[(4 π - θ) θ] Sin[β]^3)/(4 π - θ) - (2 π^2 Sqrt[(4 π - θ) θ] Sin[β]^3)/θ) /
      (16 π^2 θ - 12 π θ^2 + 2 θ^3 - 16 π^3 Sin[β]^2 + 8 π^2 θ Sin[β]^2))^2 θ^2))])^2)] == (2 π r - r θ)/(2 π), r]
```

$$\Bigg\{\Bigg\{r \to -\frac{1}{2\sqrt{2}}\left(\sqrt{\left(-\frac{\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{\pi\,(4\pi-\theta)} - \frac{\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{\pi\,\theta} + \right.}\right.$$

$$\frac{\mathrm{Sin}[\beta]^2}{4\pi-\theta} + \frac{\mathrm{Sin}[\beta]^2}{\theta} + \frac{256\,\pi^4\,\theta}{\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2} -$$

$$\frac{64\,\pi^3\,\theta^2}{\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2} -$$

$$\frac{256\,\pi^3\,\theta\,\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2} +$$

$$\frac{64\,\pi^2\,\theta^2\,\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2} -$$

$$\frac{256\,\pi^5\,\mathrm{Sin}[\beta]^2}{\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2} +$$

$$\frac{256\,\pi^4\,\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2} +$$

$$\frac{256\,\pi^3\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{64\,\pi^2\,\theta^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{256\,\pi^4\,\theta\,\mathrm{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{64\,\pi^3\,\theta^2\,\mathrm{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{256\,\pi^4\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^5}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{256\,\pi^5\,\mathrm{Sin}[\beta]^6}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{16\,\pi^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}+$$

$$\frac{8\,\pi\,\theta}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}+$$

$$\frac{16\,\pi\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}-$$

$$\frac{8\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}-$$

$$\frac{16\,\pi^2\,\mathrm{Sin}[\beta]^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}+$$

$$\left.\left.\frac{8\,\pi\,\theta\,\mathrm{Sin}[\beta]^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}\right)\right)\Big\},$$

$$\Big\{r\to\frac{1}{2\,\sqrt{2}}\left(\sqrt{\left(-\frac{\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{\pi\,(4\,\pi-\theta)}-\frac{\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{\pi\,\theta}+\frac{\mathrm{Sin}[\beta]^2}{4\,\pi-\theta}+\frac{\mathrm{Sin}[\beta]^2}{\theta}+\right.}\right.$$

$$\frac{256\,\pi^4\,\theta}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{64\,\pi^3\,\theta^2}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{256\,\pi^3\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{64\,\pi^2\,\theta^2\,\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{256\,\pi^5\,\text{Sin}[\beta]^2}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{256\,\pi^4\,\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{256\,\pi^3\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{64\,\pi^2\,\theta^2\,\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{256\,\pi^4\,\theta\,\text{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{64\,\pi^3\,\theta^2\,\text{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{256\,\pi^4\,\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^5}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{256\,\pi^5\,\text{Sin}[\beta]^6}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{16\,\pi^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}+$$

$$\frac{8\,\pi\,\theta}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}+\frac{16\,\pi\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}-\frac{8\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}-\frac{16\,\pi^2\,\mathrm{Sin}[\beta]^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}+\frac{8\,\pi\,\theta\,\mathrm{Sin}[\beta]^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}\Bigg)\Bigg)\}\}$$

$$\mathbf{SphericalPlot3D}\Big[\frac{1}{2\,\sqrt{2}}\left(\sqrt{\left(-\frac{\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{\pi\,(4\,\pi-\theta)}-\frac{\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{\pi\,\theta}+\right.}\right.$$

$$\frac{\mathrm{Sin}[\beta]^2}{4\,\pi-\theta}+\frac{\mathrm{Sin}[\beta]^2}{\theta}+\frac{256\,\pi^4\,\theta}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{64\,\pi^3\,\theta^2}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-\frac{256\,\pi^3\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{64\,\pi^2\,\theta^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-\frac{256\,\pi^5\,\mathrm{Sin}[\beta]^2}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{256\,\pi^4\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}+\frac{256\,\pi^3\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{64\,\pi^2\,\theta^2\,\sqrt{(4\,\pi-\theta)\,\theta}\,\,\text{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{256\,\pi^4\,\theta\,\text{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{64\,\pi^3\,\theta^2\,\text{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{256\,\pi^4\,\sqrt{(4\,\pi-\theta)\,\theta}\,\,\text{Sin}[\beta]^5}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{256\,\pi^5\,\text{Sin}[\beta]^6}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{16\,\pi^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}+$$

$$\frac{8\,\pi\,\theta}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}+$$

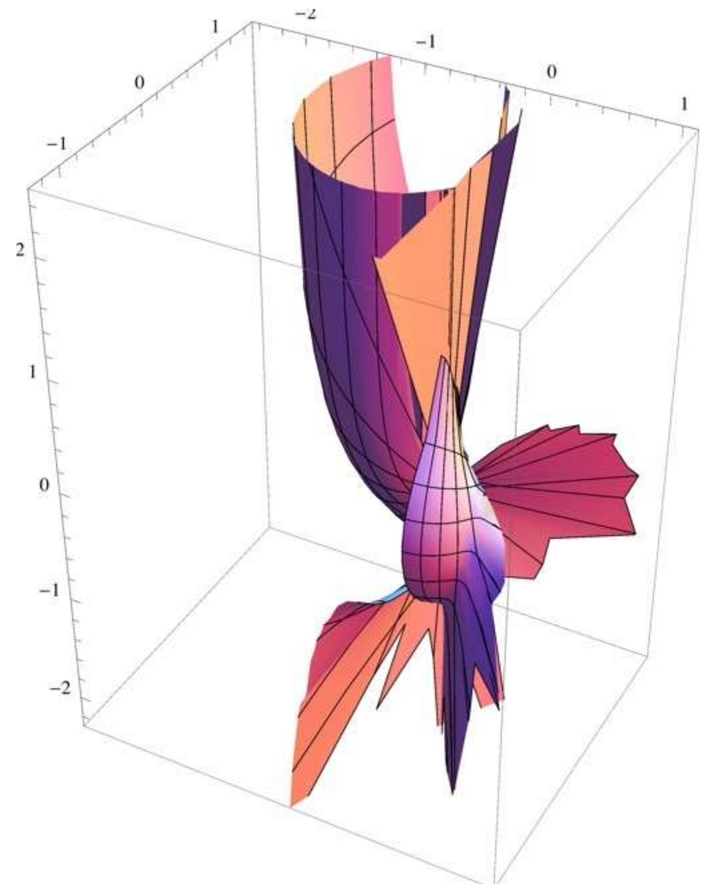
$$\frac{16\,\pi\,\sqrt{(4\,\pi-\theta)\,\theta}\,\,\text{Sin}[\beta]}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}-$$

$$\frac{8\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\,\,\text{Sin}[\beta]}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}-$$

$$\frac{16\,\pi^2\,\text{Sin}[\beta]^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2}+$$

$$\left(8\,\pi\,\theta\,\text{Sin}[\beta]^2\right)\Big/\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\left(2\,\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\,\right)\right)^3-\right.$$

$$\left.\left.\left.16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)\right)\right),\ \{\theta,\ -2\,\pi,\ 2\,\pi\},\ \{\beta,\ -\pi\,/\,2,\ \pi\,/\,2\}\Big]$$



$$\text{SphericalPlot3D}\Big[\frac{1}{2\sqrt{2}}\left(\sqrt{\left(-\frac{\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{\pi\,(4\pi-\theta)}-\frac{\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{\pi\,\theta}+\right.}\right.$$

$$\frac{\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{\text{Sin}[\beta]^2}{\theta}+\frac{256\,\pi^4\,\theta}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{64\,\pi^3\,\theta^2}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{256\,\pi^3\,\theta\,\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{64\,\pi^2\,\theta^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{256\,\pi^5\,\mathrm{Sin}[\beta]^2}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{256\,\pi^4\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{256\,\pi^3\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{64\,\pi^2\,\theta^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{256\,\pi^4\,\theta\,\mathrm{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{64\,\pi^3\,\theta^2\,\mathrm{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{256\,\pi^4\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^5}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{256\,\pi^5\,\mathrm{Sin}[\beta]^6}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{16\,\pi^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}+$$

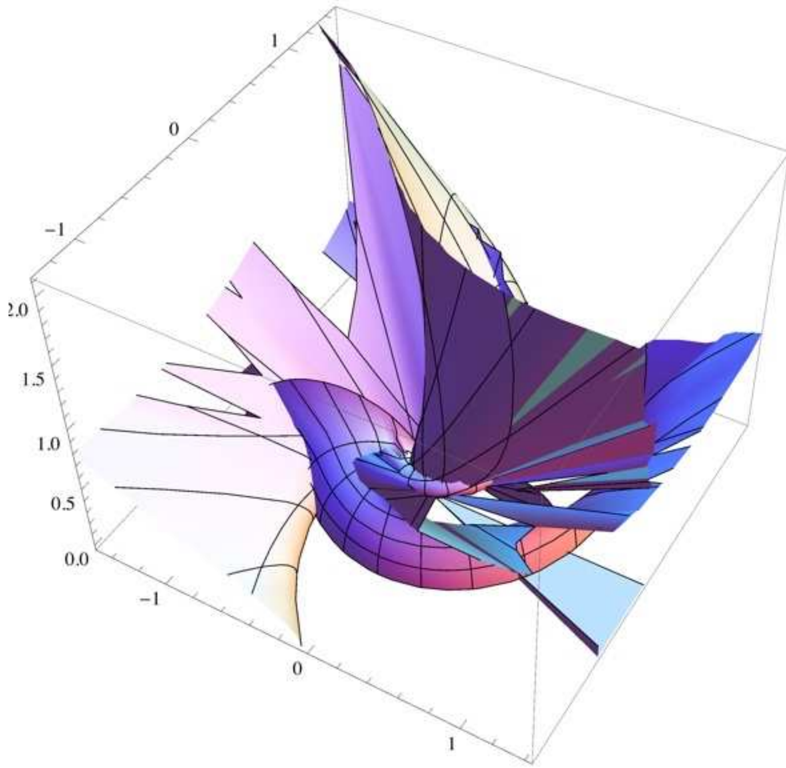
$$\frac{8\,\pi\,\theta}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}+$$

$$\frac{16\,\pi\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}-$$

$$\frac{8\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}-$$

$$\frac{16\,\pi^2\,\mathrm{Sin}[\beta]^2}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2} +$$

$$\left(8\,\pi\,\theta\,\mathrm{Sin}[\beta]^2\right) \Big/ \left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\left(2\left(\pi + \sqrt{\pi^2 - \pi^2\,\mathrm{Sin}[\beta]^2}\right)\right)^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)\Bigg)\Bigg), \{\beta, -\pi/2, \pi/2\}, \{\theta, -2\,\pi, 2\,\pi\}\Big]$$

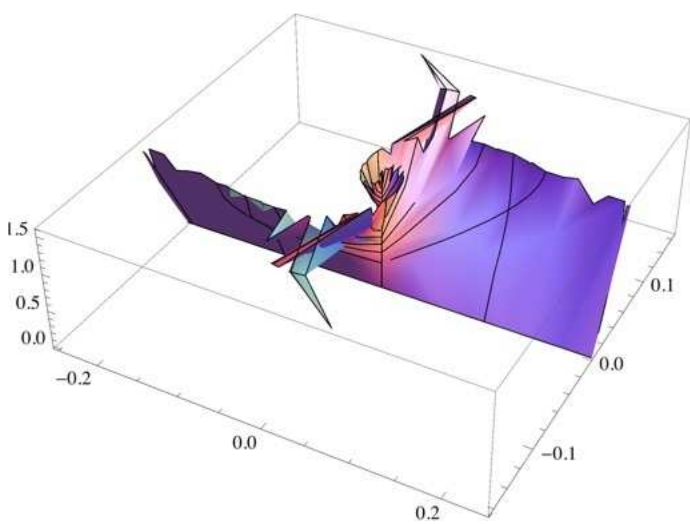


```
Solve[Sqrt[((((-4 π θ + θ^2 + 2 π Sqrt[(4 π - θ) θ] Sin[β] + 4 π^2 Sin[β]^2 - (2 π^2 Sqrt[(4 π - θ) θ] Sin[β]^3)/(4 π - θ) -
      (2 π^2 Sqrt[(4 π - θ) θ] Sin[β]^3)/θ) / (16 π^2 θ - 12 π θ^2 + 2 θ^3 - 16 π^3 Sin[β]^2 + 8 π^2 θ Sin[β]^2)) ^2 -
   1/(2 π) (Sqrt[(4 π (r)^2 θ - (((-4 π θ + θ^2 + 2 π Sqrt[(4 π - θ) θ] Sin[β] + 4 π^2 Sin[β]^2 -
            (2 π^2 Sqrt[(4 π - θ) θ] Sin[β]^3)/(4 π - θ) - (2 π^2 Sqrt[(4 π - θ) θ] Sin[β]^3)/θ) /
          (16 π^2 θ - 12 π θ^2 + 2 θ^3 - 16 π^3 Sin[β]^2 + 8 π^2 θ Sin[β]^2))^2 θ^2))]) ^2)] == (2 π r - r θ)/(2 π), β]
```

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

$$\left\{\left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\,\theta} - 2\sqrt{16\pi^3 r^2\theta - 20\pi^2 r^2\theta^2 + 8\pi r^2\theta^3 - r^2\theta^4}}{2\pi}\right]\right\},\right.$$
$$\left.\left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\,\theta} + 2\sqrt{16\pi^3 r^2\theta - 20\pi^2 r^2\theta^2 + 8\pi r^2\theta^3 - r^2\theta^4}}{2\pi}\right]\right\}\right\}$$

```
RevolutionPlot3D[
 ArcSin[(Sqrt[(4 π - θ) θ] + 2 Sqrt[16 π^3 r^2 θ - 20 π^2 r^2 θ^2 + 8 π r^2 θ^3 - r^2 θ^4])/(2 π)], {r, -1, 1}, {θ, -2 π, 2 π}]
```



$$\text{Solve}\left[\sqrt{\left(\left(\left(\left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)\text{^}2-\frac{1}{2\pi}\left(\sqrt{\left(4\pi\,(r)^2\,\theta-\left(\left(\left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)^2\theta^2\right)\right)\text{^}2\right)}==\frac{2\pi r-r\theta}{2\pi},\ r\right]$$

$$\Bigg\{\Bigg\{r\to-\frac{1}{\sqrt{16\pi^2+4\theta^2}}\Bigg(\sqrt{\Bigg(1-\frac{5\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{4\pi-\theta}-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{\theta}+\frac{5\pi\,\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{\theta}+\frac{512\pi^5\theta}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)^2}-}$$

$$\frac{128\pi^4\theta^2}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{512\pi^4\theta\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{128\pi^3\theta^2\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)^2}-$$

$$\frac{512\pi^6\,\text{Sin}[\beta]^2}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{512\pi^5\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)^2}+$$

$$\frac{512\,\pi^4\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{128\,\pi^3\,\theta^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{512\,\pi^5\,\theta\,\mathrm{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{128\,\pi^4\,\theta^2\,\mathrm{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{512\,\pi^5\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^5}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{512\,\pi^6\,\mathrm{Sin}[\beta]^6}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{32\,\pi^3}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}-$$

$$\frac{16\,\pi^2\,\theta}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}+$$

$$\frac{8\,\pi\,\theta^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}+$$

$$\frac{32\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}+$$

$$\frac{16\,\pi\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}-$$

$$\frac{8\,\theta^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}-$$

$$\frac{16\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}+$$

$$\frac{8\,\pi\,\theta^2\,\mathrm{Sin}[\beta]^2}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2} -$$

$$\frac{32\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2} +$$

$$\left.\left.\left.\frac{32\,\pi^3\,\mathrm{Sin}[\beta]^4}{16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}\right)\right)\right\},$$

$$\left\{r \to \frac{1}{\sqrt{16\,\pi^2 + 4\,\theta^2}}\left(\sqrt{\left(1 - \frac{5\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{4\,\pi-\theta} - \frac{\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{\theta} + \frac{5\,\pi\,\mathrm{Sin}[\beta]^2}{4\,\pi-\theta} +\right.}\right.\right.$$

$$\frac{\pi\,\mathrm{Sin}[\beta]^2}{\theta} + \frac{512\,\pi^5\,\theta}{\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2} -$$

$$\frac{128\,\pi^4\,\theta^2}{\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2} -$$

$$\frac{512\,\pi^4\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2} +$$

$$\frac{128\,\pi^3\,\theta^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2} -$$

$$\frac{512\,\pi^6\,\mathrm{Sin}[\beta]^2}{\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2} +$$

$$\frac{512\,\pi^5\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2} +$$

$$\frac{512\,\pi^4\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2} -$$

$$\frac{128\,\pi^3\,\theta^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]^3}{\left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\mathrm{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2} -$$

$$\frac{512\,\pi^5\,\theta\,\mathrm{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{128\,\pi^4\,\theta^2\,\mathrm{Sin}[\beta]^4}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{512\,\pi^5\,\sqrt{(4\,\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]^5}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}+$$

$$\frac{512\,\pi^6\,\mathrm{Sin}[\beta]^6}{\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)^2}-$$

$$\frac{32\,\pi^3}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}-$$

$$\frac{16\,\pi^2\,\theta}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}+$$

$$\frac{8\,\pi\,\theta^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}+$$

$$\frac{32\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}+$$

$$\frac{16\,\pi\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}-$$

$$\frac{8\,\theta^2\,\sqrt{(4\,\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}-$$

$$\frac{16\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}+$$

$$\frac{8\,\pi\,\theta^2\,\mathrm{Sin}[\beta]^2}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}-$$

$$\frac{32\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]^3}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}+$$

$$\left.\left.\frac{32\,\pi^3\,\mathrm{Sin}[\beta]^4}{16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\mathrm{Sin}[\beta]^2+8\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2}\right)\right)\}\}$$

■ New Solution 1

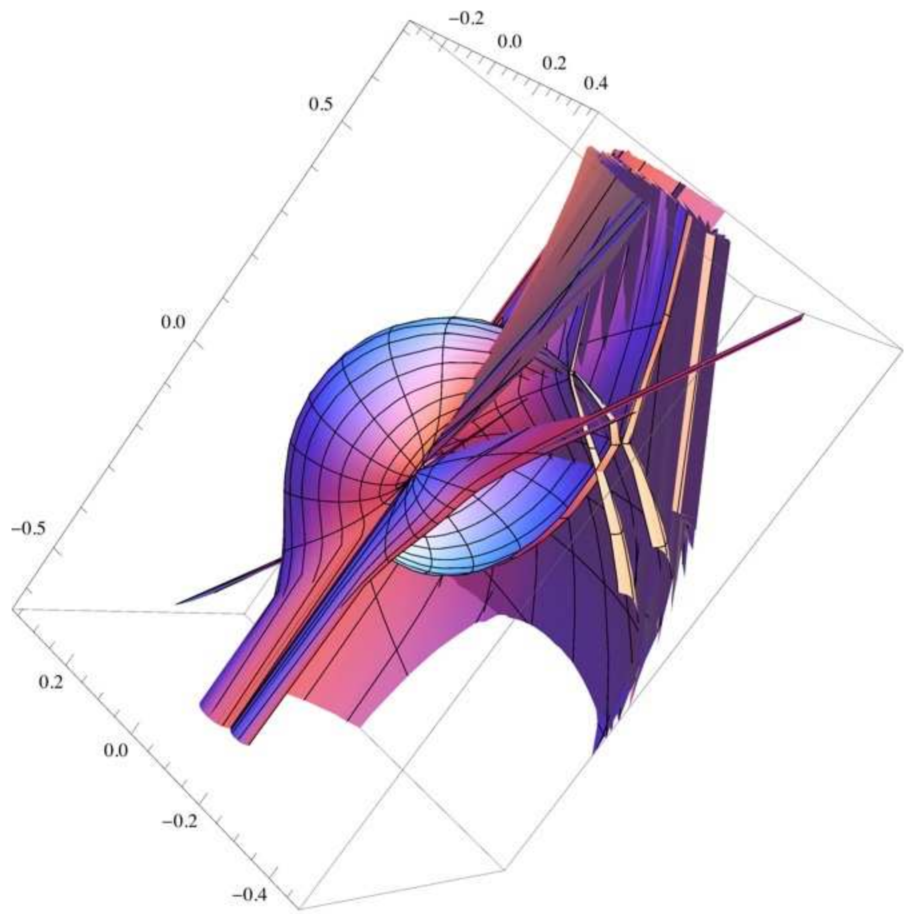
$$\mathbf{Solve}\Big[\sqrt{\left(r^{\wedge}2-\frac{1}{2\pi}\left(\sqrt{\left(4\pi\,(r)^2\,\theta-\left(\left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]+4\pi^2\,\mathrm{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)\right)^2\theta^2\right)}\right)^{\wedge}2\right)}==$$

$$\frac{1}{2\pi}\left(2\pi\left(\left(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]+4\pi^2\,\mathrm{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\mathrm{Sin}[\beta]^2+8\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)\right)-r\,\theta\right),\ r\Big]$$

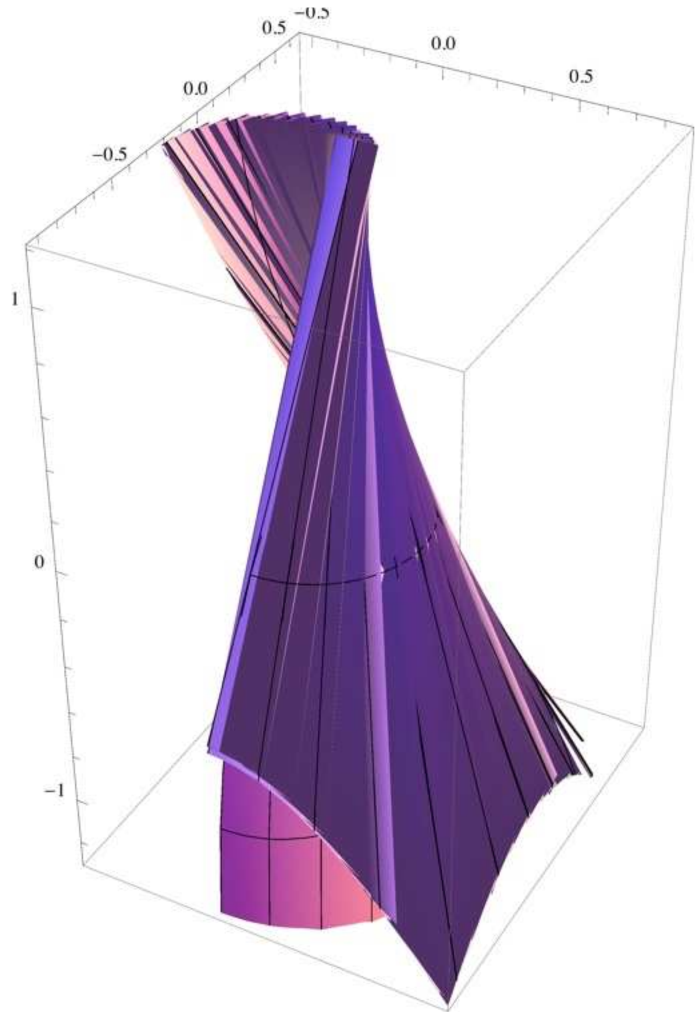
$$\Big\{\Big\{r\to\Big(32\pi^2\theta^2-8\pi\theta^3-16\pi^2\theta\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]-\sqrt{\Big(\left(-32\pi^2\theta^2+8\pi\theta^3+16\pi^2\theta\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]\right)^2-4\left(128\pi^4\theta-224\pi^3\theta^2+80\pi^2\theta^3+8\pi\theta^4-4\theta^5\right)\left(-8\pi^2\theta-2\pi\theta^2+\theta^3+8\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]+4\pi\theta\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]-8\pi^3\,\mathrm{Sin}[\beta]^2-4\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)\Big)}\Big)\Big/\left(2\left(128\pi^4\theta-224\pi^3\theta^2+80\pi^2\theta^3+8\pi\theta^4-4\theta^5\right)\right)\Big\},$$

$$\Big\{r\to\Big(32\pi^2\theta^2-8\pi\theta^3-16\pi^2\theta\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]+\sqrt{\Big(\left(-32\pi^2\theta^2+8\pi\theta^3+16\pi^2\theta\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]\right)^2-4\left(128\pi^4\theta-224\pi^3\theta^2+80\pi^2\theta^3+8\pi\theta^4-4\theta^5\right)\left(-8\pi^2\theta-2\pi\theta^2+\theta^3+8\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]+4\pi\theta\sqrt{(4\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]-8\pi^3\,\mathrm{Sin}[\beta]^2-4\pi^2\theta\,\mathrm{Sin}[\beta]^2\right)\Big)}\Big)\Big/\left(2\left(128\pi^4\theta-224\pi^3\theta^2+80\pi^2\theta^3+8\pi\theta^4-4\theta^5\right)\right)\Big\}\Big\}$$

```
SphericalPlot3D[
 {(32 π^2 θ^2 - 8 π θ^3 - 16 π^2 θ √((4 π - θ) θ) Sin[β] - √(((-32 π^2 θ^2 + 8 π θ^3 + 16 π^2 θ √((4 π - θ) θ) Sin[β])^2 -
        4 (128 π^4 θ - 224 π^3 θ^2 + 80 π^2 θ^3 + 8 π θ^4 - 4 θ^5) (-8 π^2 θ - 2 π θ^2 + θ^3 +
           8 π^2 √((4 π - θ) θ) Sin[β] + 4 π θ √((4 π - θ) θ) Sin[β] - 8 π^3 Sin[β]^2 - 4 π^2 θ Sin[β]^2)))) /
   (2 (128 π^4 θ - 224 π^3 θ^2 + 80 π^2 θ^3 + 8 π θ^4 - 4 θ^5)), (32 π^2 θ^2 - 8 π θ^3 - 16 π^2 θ √((4 π - θ) θ) Sin[β] +
     √(((-32 π^2 θ^2 + 8 π θ^3 + 16 π^2 θ √((4 π - θ) θ) Sin[β])^2 -
        4 (128 π^4 θ - 224 π^3 θ^2 + 80 π^2 θ^3 + 8 π θ^4 - 4 θ^5) (-8 π^2 θ - 2 π θ^2 + θ^3 +
           8 π^2 √((4 π - θ) θ) Sin[β] + 4 π θ √((4 π - θ) θ) Sin[β] - 8 π^3 Sin[β]^2 - 4 π^2 θ Sin[β]^2)))) /
   (2 (128 π^4 θ - 224 π^3 θ^2 + 80 π^2 θ^3 + 8 π θ^4 - 4 θ^5))}, {θ, -2 π, 2 π}, {β, -π /
   2, π / 2}]
```

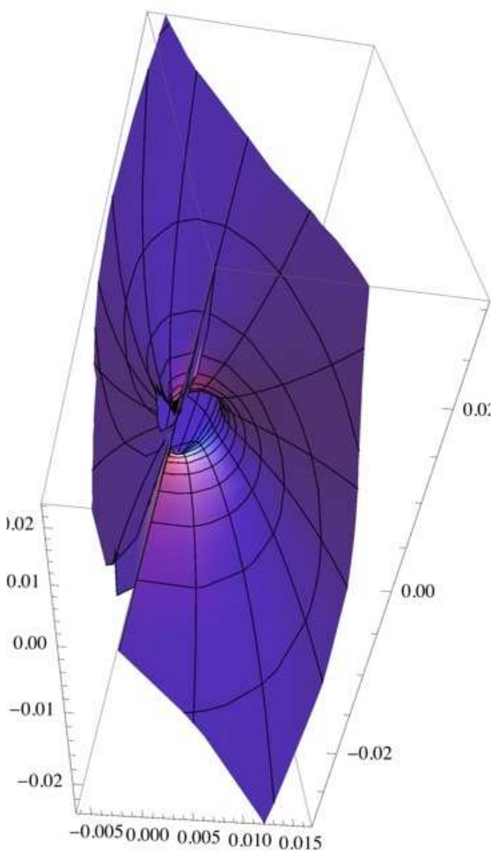


```
SphericalPlot3D[
 (32 π² θ² - 8 π θ³ - 16 π² θ √((4 π - θ) θ) Sin[β] + √((-32 π² θ² + 8 π θ³ + 16 π² θ √((4 π - θ) θ) Sin[β])² -
      4 (128 π⁴ θ - 224 π³ θ² + 80 π² θ³ + 8 π θ⁴ - 4 θ⁵) (-8 π² θ - 2 π θ² + θ³ +
         8 π² √((4 π - θ) θ) Sin[β] + 4 π θ √((4 π - θ) θ) Sin[β] - 8 π³ Sin[β]² - 4 π² θ Sin[β]²))) /
  (2 (128 π⁴ θ - 224 π³ θ² + 80 π² θ³ + 8 π θ⁴ - 4 θ⁵)), {θ, -2 π, 2 π}, {β, -π / 2, π / 2}]
```

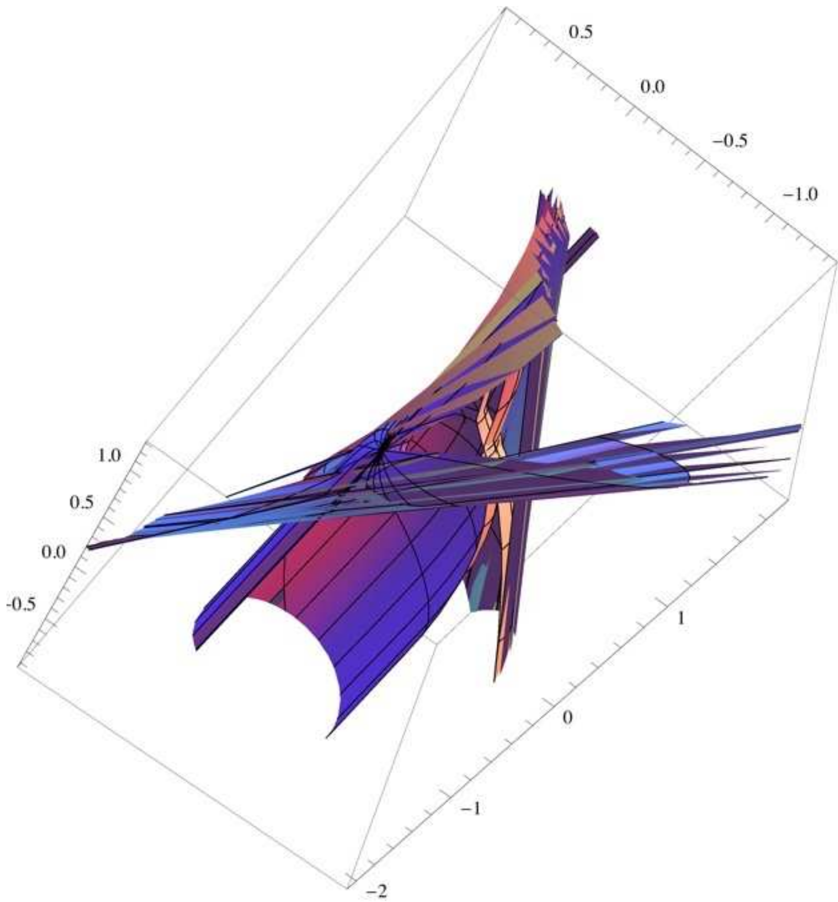


■ Solution 1 Substitutions

$$\textbf{SphericalPlot3D}\Big[\Big(32\,\pi^2\,\theta^2 - 8\,\pi\,\theta^3 - 16\,\pi^2\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\,\textbf{Sin}[\beta] + \sqrt{\Big(\Big(-32\,\pi^2\,\theta^2 + 8\,\pi\,\theta^3 + 16\,\pi^2\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\,\textbf{Sin}[\beta]\Big)^2 - 4\,\big(128\,\pi^4\,\theta - 224\,\pi^3\,\theta^2 + 80\,\pi^2\,\theta^3 + 8\,\pi\,\theta^4 - 4\,\theta^5\big)\,\Big(-8\,\pi^2\,\theta - 2\,\pi\,\theta^2 + \Big(2\,\Big(\pi + \sqrt{\pi^2 - \pi^2\,\textbf{Sin}[\beta]^2}\Big)\Big)^3 + 8\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\,\textbf{Sin}[\beta] + 4\,\pi\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\,\textbf{Sin}[\beta] - 8\,\pi^3\,\textbf{Sin}[\beta]^2 - 4\,\pi^2\,\theta\,\textbf{Sin}[\beta]^2\Big)\Big)}\Big) \Big/ \Big(2\,\Big(128\,\pi^4\,\theta - 224\,\pi^3\,\theta^2 + 80\,\pi^2\,\theta^3 + 8\,\pi\,\theta^4 - 4\,\Big(2\,\Big(\pi + \sqrt{\pi^2 - \pi^2\,\textbf{Sin}[\beta]^2}\Big)\Big)^5\Big)\Big),\ \{\theta,\ -2\,\pi,\ 2\,\pi\},\ \{\beta,\ -\pi/2,\ \pi/2\}\Big]$$

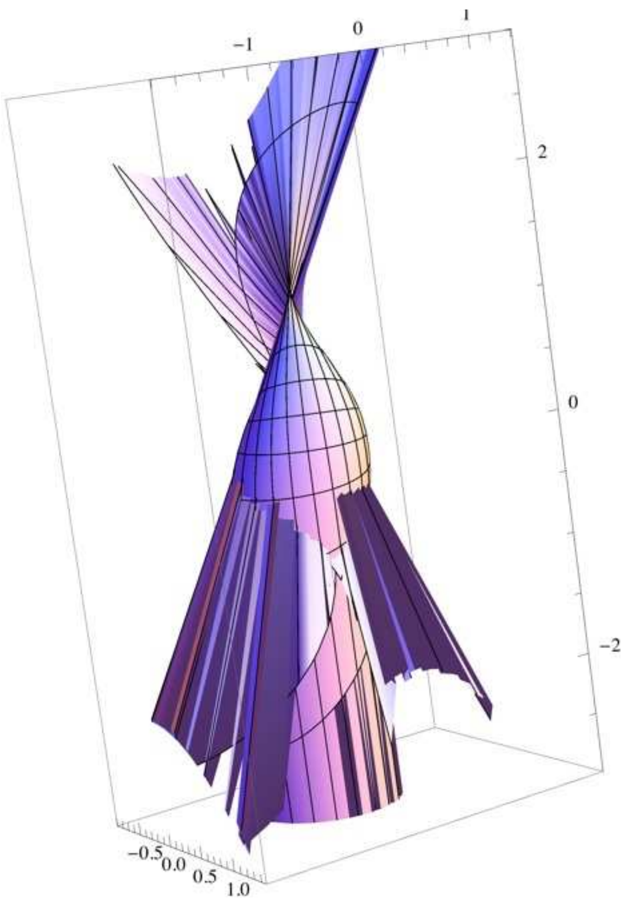


```
SphericalPlot3D[(32 π^2 θ^2 - 8 π θ^3 - 16 π^2 (2 (π + Sqrt[π^2 - π^2 Sin[β]^2])) Sqrt[(4 π - θ) θ] Sin[β] +
    Sqrt[((-32 π^2 θ^2 + 8 π θ^3 + 16 π^2 θ Sqrt[(4 π - θ) θ] Sin[β])^2 -
      4 (128 π^4 θ - 224 π^3 θ^2 + 80 π^2 θ^3 + 8 π θ^4 - 4 θ^5) (-8 π^2 θ - 2 π θ^2 + θ^3 +
        8 π^2 Sqrt[(4 π - θ) θ] Sin[β] + 4 π θ Sqrt[(4 π - θ) θ] Sin[β] - 8 π^3 Sin[β]^2 - 4 π^2 θ Sin[β]^2))]) /
  (2 (128 π^4 θ - 224 π^3 θ^2 + 80 π^2 θ^3 + 8 π θ^4 - 4 θ^5)), {θ, -2 π, 2 π}, {β,
  -π / 2,
  π / 2}]
```

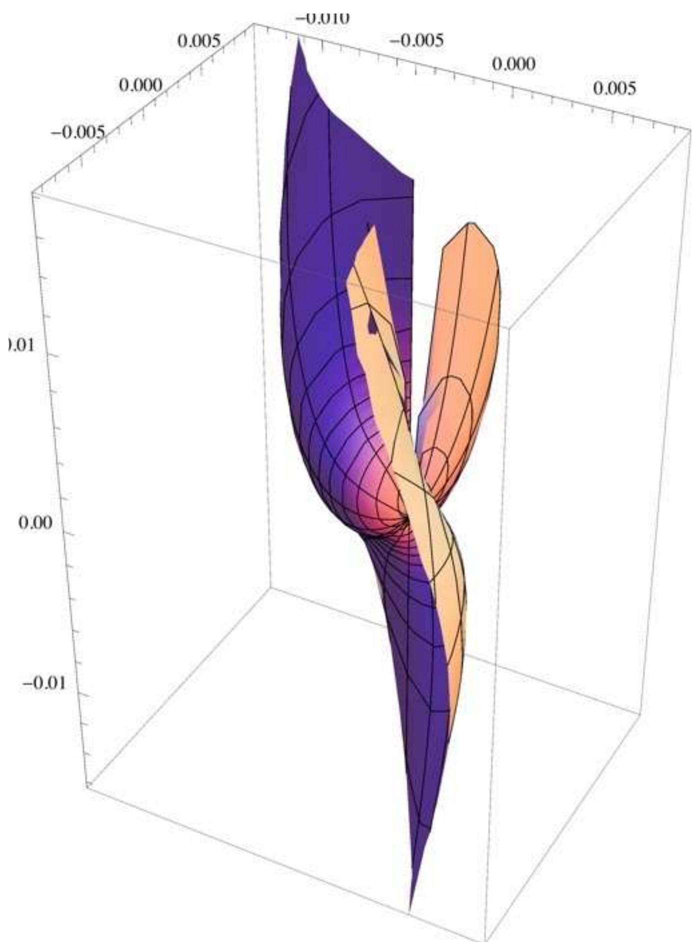


SphericalPlot3D[

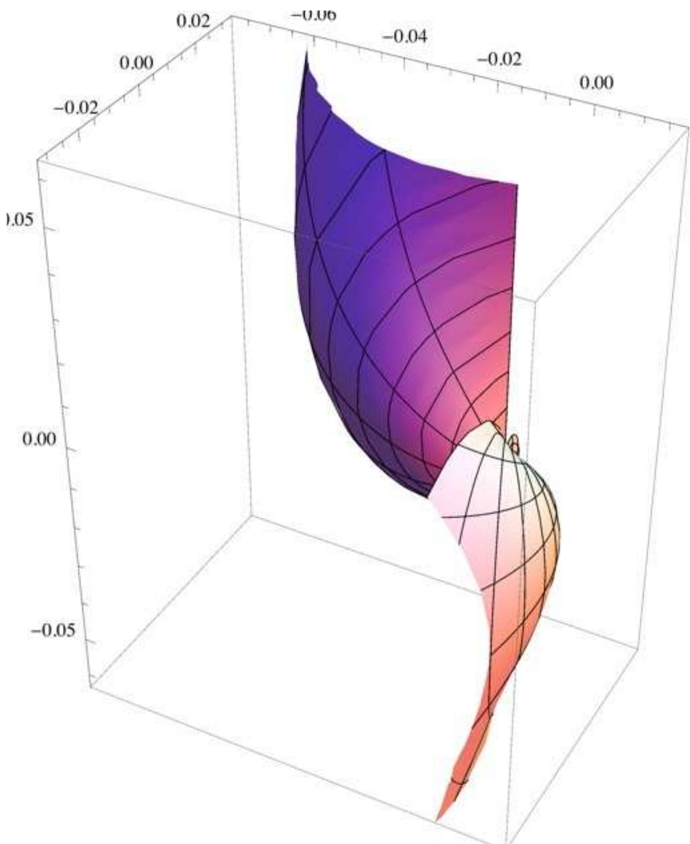
$$\left(32\,\pi^2\,\theta^2 - 8\,\pi\,\theta^3 - 16\,\pi^2\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta] + \sqrt{\left(\left(-32\,\pi^2\,\theta^2 + 8\,\pi\,\theta^3 + 16\,\pi^2\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta]\right)^2 - 4\,\left(128\,\pi^4\,\theta - 224\,\pi^3\,\theta^2 + 80\,\pi^2\,\theta^3 + 8\,\pi\,\theta^4 - 4\,\theta^5\right)\,\left(-8\,\pi^2\,\theta - 2\,\pi\,\theta^2 + \left(2\,\left(\pi + \sqrt{\pi^2 - \pi^2\,\mathrm{Sin}[\beta]^2}\right)\right)^3 + 8\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta] + 4\,\pi\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\,\mathrm{Sin}[\beta] - 8\,\pi^3\,\mathrm{Sin}[\beta]^2 - 4\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)\right)}\right) \Big/ \left(2\,\left(128\,\pi^4\,\theta - 224\,\pi^3\,\theta^2 + 80\,\pi^2\,\theta^3 + 8\,\pi\,\theta^4 - 4\,\theta^5\right)\right),\ \{\theta, -2\,\pi, 2\,\pi\},\ \{\beta, -\pi/2, \pi/2\}]$$



$$\text{SphericalPlot3D}\Big[\Big(32\pi^2\theta^2 - 8\pi\theta^3 - 16\pi^2\theta\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta] + \sqrt{\Big(\Big(-32\pi^2\theta^2 + 8\pi\theta^3 + 16\pi^2\theta\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]\Big)^2 - 4\left(128\pi^4\theta - 224\pi^3\theta^2 + 80\pi^2\theta^3 + 8\pi\theta^4 - 4\theta^5\right)\Big(-8\pi^2\theta - 2\pi\theta^2 + \theta^3 + 8\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta] + 4\pi\theta\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta] - 8\pi^3\text{Sin}[\beta]^2 - 4\pi^2\theta\,\text{Sin}[\beta]^2\Big)\Big)}\Big)\Big/ \Big(2\Big(128\pi^4\theta - 224\pi^3\theta^2 + 80\pi^2\theta^3 + 8\pi\Big(2\Big(\pi + \sqrt{\pi^2 - \pi^2\text{Sin}[\beta]^2}\Big)\Big)^4 - 4\theta^5\Big)\Big),$$
$$\{\theta, -2\pi, 2\pi\}, \{\beta, -\pi/2, \pi/2\}\Big]$$



```
SphericalPlot3D[
 (32 π^2 θ^2 - 8 π θ^3 - 16 π^2 θ Sqrt[(4 π - θ) θ] Sin[β] + Sqrt[((-32 π^2 θ^2 + 8 π θ^3 + 16 π^2 θ Sqrt[(4 π - θ) θ] Sin[β])^2 -
      4 (128 π^4 θ - 224 π^3 θ^2 + 80 π^2 θ^3 + 8 π θ^4 - 4 θ^5) (-8 π^2 θ - 2 π θ^2 + θ^3 +
         8 π^2 Sqrt[(4 π - θ) θ] Sin[β] + 4 π θ Sqrt[(4 π - θ) θ] Sin[β] - 8 π^3 Sin[β]^2 - 4 π^2 θ Sin[β]^2))]) /
  (2 (128 π^4 (2 (π + Sqrt[π^2 - π^2 Sin[β]^2])) - 224 π^3 θ^2 + 80 π^2 θ^3 + 8 π θ^4 - 4 θ^5)),
 {θ, -2 π, 2 π}, {β, -π / 2, π / 2}]
```



```
SphericalPlot3D[
 (32 π^2 θ^2 - 8 π θ^3 - 16 π^2 θ Sqrt[(4 π - θ) θ] Sin[β] + Sqrt[((-32 π^2 θ^2 + 8 π θ^3 + 16 π^2 θ Sqrt[(4 π - θ) θ] Sin[β])^2 - 4
      (128 π^4 θ - 224 π^3 (2 (π + Sqrt[π^2 - π^2 Sin[β]^2]))^2 + 80 π^2 θ^3 + 8 π (2 (π + Sqrt[π^2 - π^2 Sin[β]^2]))^4 - 4 θ^5)
      (-8 π^2 θ - 2 π θ^2 + θ^3 + 8 π^2 Sqrt[(4 π - θ) θ] Sin[β] +
        4 π θ Sqrt[(4 π - θ) θ] Sin[β] - 8 π^3 Sin[β]^2 - 4 π^2 θ Sin[β]^2))]) /
  (2 (128 π^4 (2 (π + Sqrt[π^2 - π^2 Sin[β]^2])) - 224 π^3 θ^2 + 80 π^2 (2 (π + Sqrt[π^2 - π^2 Sin[β]^2]))^3 +
      8 π θ^4 - 4 (2 (π + Sqrt[π^2 - π^2 Sin[β]^2]))^5)), {θ, -2 π, 2 π}, {β, -π / 2, π / 2}]
```

$$\mathbf{SphericalPlot3D}\Big[\Big(32\,\pi^2\,\theta^2 - 8\,\pi\,\theta^3 - 16\,\pi^2\,\theta\,\sqrt{(4\,\pi - \theta)\,\theta}\;\mathrm{Sin}[\beta] + \sqrt{\Big(\Big(-32\,\pi^2\,\theta^2 + 8\,\pi\,\theta^3 + 16\,\pi^2\,\theta\,\sqrt{(4\,\pi - \theta)\,\theta}\;\mathrm{Sin}[\beta]\Big)^2 - 4\,\Big(128\,\pi^4\,\theta - 224\,\pi^3\,\theta^2 + 80\,\pi^2\,\theta^3 + 8\,\pi\,\Big(2\,\Big(\pi + \sqrt{\pi^2 - \pi^2\,\mathrm{Sin}[\beta]^2}\Big)\Big)^4 - 4\,\theta^5\Big)\,\Big(-8\,\pi^2\,\theta - 2\,\pi\,\theta^2 + \theta^3 + 8\,\pi^2\,\sqrt{(4\,\pi - \theta)\,\theta}\;\mathrm{Sin}[\beta] + 4\,\pi\,\theta\,\sqrt{(4\,\pi - \theta)\,\theta}\;\mathrm{Sin}[\beta] - 8\,\pi^3\,\mathrm{Sin}[\beta]^2 - 4\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\Big)\Big)}\Big) \Big/ \Big(2\,\Big(128\,\pi^4\,\theta - 224\,\pi^3\,\theta^2 + 80\,\pi^2\,\theta^3 + 8\,\pi\,\theta^4 - 4\,\theta^5\Big)\Big),\ \{\theta,\ -2\,\pi,\ 2\,\pi\},\ \{\beta,\ -\pi/2,\ \pi/2\}\Big]$$

- **New Solution 2**

$$\texttt{Solve}\left[\sqrt{\left(\texttt{r}^\wedge 2-\frac{\sqrt{4\,\pi\,(\texttt{r})^2\,\theta-(\texttt{r})^2\,\theta^2}}{2\,\pi}{}^\wedge 2\right)}==\right.$$

$$\frac{1}{2\,\pi}\left(2\,\pi\left(\left(-4\,\pi\,\theta+\theta^2+2\,\pi\sqrt{(4\,\pi-\theta)\,\theta}\;\texttt{Sin}[\beta]+4\,\pi^2\,\texttt{Sin}[\beta]^2-\frac{2\,\pi^2\sqrt{(4\,\pi-\theta)\,\theta}\;\texttt{Sin}[\beta]^3}{4\,\pi-\theta}-\right.\right.\right.$$

$$\left.\frac{2\,\pi^2\sqrt{(4\,\pi-\theta)\,\theta}\;\texttt{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\texttt{Sin}[\beta]^2+8\,\pi^2\,\theta\,\texttt{Sin}[\beta]^2\right)\Bigg)-$$

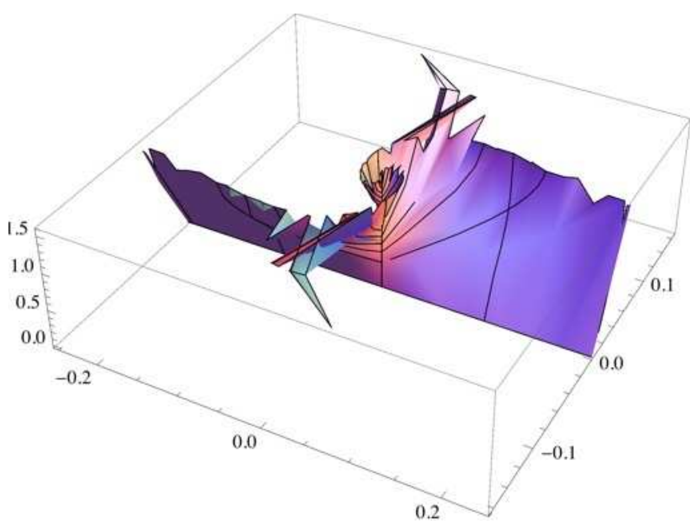
$$\left(\left(-4\,\pi\,\theta+\theta^2+2\,\pi\sqrt{(4\,\pi-\theta)\,\theta}\;\texttt{Sin}[\beta]+4\,\pi^2\,\texttt{Sin}[\beta]^2-\frac{2\,\pi^2\sqrt{(4\,\pi-\theta)\,\theta}\;\texttt{Sin}[\beta]^3}{4\,\pi-\theta}-\right.\right.$$

$$\left.\left.\left.\frac{2\,\pi^2\sqrt{(4\,\pi-\theta)\,\theta}\;\texttt{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\texttt{Sin}[\beta]^2+8\,\pi^2\,\theta\,\texttt{Sin}[\beta]^2\right)\right)\theta\right),\,\beta\Bigg]$$

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

$$\left\{\left\{\beta\to\texttt{ArcSin}\left[\frac{\sqrt{(4\,\pi-\theta)\,\theta}+2\sqrt{\texttt{r}^2\,(2\,\pi-\theta)^2}\;\sqrt{(4\,\pi-\theta)\,\theta}}{2\,\pi}\right]\right\}\right\}$$

$$\texttt{RevolutionPlot3D}\left[\texttt{ArcSin}\left[\frac{\sqrt{(4\,\pi-\theta)\,\theta}+2\sqrt{\texttt{r}^2\,(2\,\pi-\theta)^2}\;\sqrt{(4\,\pi-\theta)\,\theta}}{2\,\pi}\right],\,\{\texttt{r},\,-1,\,1\},\,\{\theta,\,-2\,\pi,\,2\,\pi\}\right]$$



$$\text{Solve}\left[\sqrt{\left(r^{\wedge}2-\left(\frac{\sqrt{4\pi\,(r)^2\,\theta-(r)^2\,\theta^2}}{2\pi}\right)^{\wedge}2\right)}==\right.$$
$$\left(\left(32\pi^2\theta^2-8\pi\theta^3-16\pi^2\theta\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]+\sqrt{\left(\left(-32\pi^2\theta^2+8\pi\theta^3+16\pi^2\theta\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]\right)^2-4\left(128\pi^4\theta-224\pi^3\theta^2+80\pi^2\theta^3+8\pi\theta^4-4\theta^5\right)\left(-8\pi^2\theta-2\pi\theta^2+\theta^3+8\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]+4\pi\theta\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]-8\pi^3\text{Sin}[\beta]^2-4\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)}\right)\Big/$$
$$\left(2\left(128\pi^4\theta-224\pi^3\theta^2+80\pi^2\theta^3+8\pi\theta^4-4\theta^5\right)\right)\right)-\frac{1}{2\pi}\left(\left(32\pi^2\theta^2-8\pi\theta^3-16\pi^2\theta\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]+\sqrt{\left(\left(-32\pi^2\theta^2+8\pi\theta^3+16\pi^2\theta\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]\right)^2-4\left(128\pi^4\theta-224\pi^3\theta^2+80\pi^2\theta^3+8\pi\theta^4-4\theta^5\right)\left(-8\pi^2\theta-2\pi\theta^2+\theta^3+8\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]+4\pi\theta\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]-8\pi^3\text{Sin}[\beta]^2-4\pi^2\theta\,\text{Sin}[\beta]^2\right)\right)}\right)\Big/$$
$$\left.\left(2\left(128\pi^4\theta-224\pi^3\theta^2+80\pi^2\theta^3+8\pi\theta^4-4\theta^5\right)\right)\right)\theta,\ r\right]$$

$$\left\{\left\{r\to-\left(\sqrt{\left(392+\frac{12544\,\pi^4}{\left(4\pi^2-4\pi\theta-\theta^2\right)^2}-\frac{12544\,\pi^3\theta}{\left(4\pi^2-4\pi\theta-\theta^2\right)^2}-\frac{3136\,\pi^2}{4\pi^2-4\pi\theta-\theta^2}+\right.}\right.\right.\right.$$
$$\frac{1568\,\pi\theta}{4\pi^2-4\pi\theta-\theta^2}-\frac{232\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{4\pi-\theta}-\frac{392\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{\theta}-$$
$$\frac{1792\,\pi^3\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{\left(4\pi^2-4\pi\theta-\theta^2\right)^2}-\frac{1792\,\pi^2\theta\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{\left(4\pi^2-4\pi\theta-\theta^2\right)^2}-$$
$$\frac{1280\,\pi\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{4\pi^2-4\pi\theta-\theta^2}-\frac{160\,\theta\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{4\pi^2-4\pi\theta-\theta^2}+\frac{232\,\pi\,\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{392\,\pi\,\text{Sin}[\beta]^2}{\theta}+$$
$$\frac{1792\,\pi^4\,\text{Sin}[\beta]^2}{\left(4\pi^2-4\pi\theta-\theta^2\right)^2}+\frac{1792\,\pi^3\theta\,\text{Sin}[\beta]^2}{\left(4\pi^2-4\pi\theta-\theta^2\right)^2}+\frac{1280\,\pi^2\,\text{Sin}[\beta]^2}{4\pi^2-4\pi\theta-\theta^2}+\frac{160\,\pi\theta\,\text{Sin}[\beta]^2}{4\pi^2-4\pi\theta-\theta^2}+$$
$$\left(392\,\pi\sqrt{\left(\left(-32\pi^2\theta^2+8\pi\theta^3+16\pi^2\theta\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]\right)^2-4\left(128\pi^4\theta-224\pi^3\theta^2+80\pi^2\theta^3+\right.\right.}\right.$$
$$8\pi\theta^4-4\theta^5\Big)\left(-8\pi^2\theta-2\pi\theta^2+\theta^3+8\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]+4\pi\theta\sqrt{(4\pi-\theta)\,\theta}\right.$$

$$\text{Sin}[\beta] - 8\,\pi^3\,\text{Sin}[\beta]^2 - 4\,\pi^2\,\theta\,\text{Sin}[\beta]^2\Big)\Big)\Big)\Big/\ \Big((4\,\pi-\theta)\ \big(4\,\pi^2 - 4\,\pi\,\theta - \theta^2\big)^2\Big) -$$

$$\Big(196\,\pi\,\sqrt{(4\,\pi-\theta)\,\theta}\ \text{Sin}[\beta]\,\sqrt{\Big(\Big(-32\,\pi^2\,\theta^2 + 8\,\pi\,\theta^3 + 16\,\pi^2\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\ \text{Sin}[\beta]\Big)^2 -}$$

$$4\,\big(128\,\pi^4\,\theta - 224\,\pi^3\,\theta^2 + 80\,\pi^2\,\theta^3 + 8\,\pi\,\theta^4 - 4\,\theta^5\big)$$

$$\Big(-8\,\pi^2\,\theta - 2\,\pi\,\theta^2 + \theta^3 + 8\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\ \text{Sin}[\beta] + 4\,\pi\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\ \text{Sin}[\beta] -$$

$$8\,\pi^3\,\text{Sin}[\beta]^2 - 4\,\pi^2\,\theta\,\text{Sin}[\beta]^2\Big)\Big)\Big)\Big/\ \Big((4\,\pi-\theta)^2\ \big(4\,\pi^2 - 4\,\pi\,\theta - \theta^2\big)^2\Big) -$$

$$\Big(49\,\sqrt{(4\,\pi-\theta)\,\theta}\ \text{Sin}[\beta]\,\sqrt{\Big(\Big(-32\,\pi^2\,\theta^2 + 8\,\pi\,\theta^3 + 16\,\pi^2\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\ \text{Sin}[\beta]\Big)^2 - 4}$$

$$\big(128\,\pi^4\,\theta - 224\,\pi^3\,\theta^2 + 80\,\pi^2\,\theta^3 + 8\,\pi\,\theta^4 - 4\,\theta^5\big)\ \Big(-8\,\pi^2\,\theta - 2\,\pi\,\theta^2 + \theta^3 + 8\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}$$

$$\text{Sin}[\beta] + 4\,\pi\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\ \text{Sin}[\beta] - 8\,\pi^3\,\text{Sin}[\beta]^2 - 4\,\pi^2\,\theta\,\text{Sin}[\beta]^2\Big)\Big)\Big)\Big/$$

$$\Big((4\,\pi-\theta)\ \big(4\,\pi^2 - 4\,\pi\,\theta - \theta^2\big)^2\Big) - \frac{1}{\theta\,\big(4\,\pi^2 - 4\,\pi\,\theta - \theta^2\big)^2}\,49\,\sqrt{(4\,\pi-\theta)\,\theta}\ \text{Sin}[\beta]$$

$$\sqrt{\Big(\Big(-32\,\pi^2\,\theta^2 + 8\,\pi\,\theta^3 + 16\,\pi^2\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\ \text{Sin}[\beta]\Big)^2 - 4\,\big(128\,\pi^4\,\theta - 224\,\pi^3\,\theta^2 + 80\,\pi^2\,\theta^3 +}$$

$$8\,\pi\,\theta^4 - 4\,\theta^5\big)\ \Big(-8\,\pi^2\,\theta - 2\,\pi\,\theta^2 + \theta^3 + 8\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\ \text{Sin}[\beta] + 4\,\pi\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}$$

$$\text{Sin}[\beta] - 8\,\pi^3\,\text{Sin}[\beta]^2 - 4\,\pi^2\,\theta\,\text{Sin}[\beta]^2\Big)\Big)\Big)\Big)\Big/\ \Big(\sqrt{6272\,\pi^2 - 6272\,\pi\,\theta + 1568\,\theta^2}\Big)\Big\},$$

$$\Big\{r \to \Big(\sqrt{\Big(392 + \frac{12\,544\,\pi^4}{\big(4\,\pi^2 - 4\,\pi\,\theta - \theta^2\big)^2} - \frac{12\,544\,\pi^3\,\theta}{\big(4\,\pi^2 - 4\,\pi\,\theta - \theta^2\big)^2} - \frac{3136\,\pi^2}{4\,\pi^2 - 4\,\pi\,\theta - \theta^2} + \frac{1568\,\pi\,\theta}{4\,\pi^2 - 4\,\pi\,\theta - \theta^2} -}$$

$$\frac{232\,\sqrt{(4\,\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{4\,\pi-\theta} - \frac{392\,\sqrt{(4\,\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{\theta} -$$

$$\frac{1792\,\pi^3\,\sqrt{(4\,\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{\big(4\,\pi^2 - 4\,\pi\,\theta - \theta^2\big)^2} -$$

$$\frac{1792\,\pi^2\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]}{\left(4\,\pi^2-4\,\pi\,\theta-\theta^2\right)^2}-$$

$$\frac{1280\,\pi\,\sqrt{(4\,\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]}{4\,\pi^2-4\,\pi\,\theta-\theta^2}-\frac{160\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]}{4\,\pi^2-4\,\pi\,\theta-\theta^2}+$$

$$\frac{232\,\pi\,\mathrm{Sin}[\beta]^2}{4\,\pi-\theta}+\frac{392\,\pi\,\mathrm{Sin}[\beta]^2}{\theta}+\frac{1792\,\pi^4\,\mathrm{Sin}[\beta]^2}{\left(4\,\pi^2-4\,\pi\,\theta-\theta^2\right)^2}+$$

$$\frac{1792\,\pi^3\,\theta\,\mathrm{Sin}[\beta]^2}{\left(4\,\pi^2-4\,\pi\,\theta-\theta^2\right)^2}+\frac{1280\,\pi^2\,\mathrm{Sin}[\beta]^2}{4\,\pi^2-4\,\pi\,\theta-\theta^2}+\frac{160\,\pi\,\theta\,\mathrm{Sin}[\beta]^2}{4\,\pi^2-4\,\pi\,\theta-\theta^2}+$$

$$\left(392\,\pi\sqrt{\left(\left(-32\,\pi^2\,\theta^2+8\,\pi\,\theta^3+16\,\pi^2\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]\right)^2-4\,\left(128\,\pi^4\,\theta-224\,\pi^3\,\theta^2+80\,\pi^2\,\theta^3+\right.\right.}\right.$$

$$\left.8\,\pi\,\theta^4-4\,\theta^5\right)\,\left(-8\,\pi^2\,\theta-2\,\pi\,\theta^2+\theta^3+8\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]+4\,\pi\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\right.$$

$$\left.\left.\left.\mathrm{Sin}[\beta]-8\,\pi^3\,\mathrm{Sin}[\beta]^2-4\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)\right)\right)\Big/\left((4\,\pi-\theta)\,\left(4\,\pi^2-4\,\pi\,\theta-\theta^2\right)^2\right)-$$

$$\left(196\,\pi\,\sqrt{(4\,\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]\,\sqrt{\left(\left(-32\,\pi^2\,\theta^2+8\,\pi\,\theta^3+16\,\pi^2\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]\right)^2-\right.}\right.$$

$$4\,\left(128\,\pi^4\,\theta-224\,\pi^3\,\theta^2+80\,\pi^2\,\theta^3+8\,\pi\,\theta^4-4\,\theta^5\right)$$

$$\left(-8\,\pi^2\,\theta-2\,\pi\,\theta^2+\theta^3+8\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]+4\,\pi\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]-\right.$$

$$\left.\left.\left.8\,\pi^3\,\mathrm{Sin}[\beta]^2-4\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)\right)\right)\Big/\left((4\,\pi-\theta)^2\,\left(4\,\pi^2-4\,\pi\,\theta-\theta^2\right)^2\right)-$$

$$\left(49\,\sqrt{(4\,\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]\,\sqrt{\left(\left(-32\,\pi^2\,\theta^2+8\,\pi\,\theta^3+16\,\pi^2\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]\right)^2-\right.}\right.$$

$$4\,\left(128\,\pi^4\,\theta-224\,\pi^3\,\theta^2+80\,\pi^2\,\theta^3+8\,\pi\,\theta^4-4\,\theta^5\right)\,\left(-8\,\pi^2\,\theta-2\,\pi\,\theta^2+\theta^3+8\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\right.$$

$$\left.\left.\left.\mathrm{Sin}[\beta]+4\,\pi\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]-8\,\pi^3\,\mathrm{Sin}[\beta]^2-4\,\pi^2\,\theta\,\mathrm{Sin}[\beta]^2\right)\right)\right)\Big/$$

$$\left((4\,\pi-\theta)\,\left(4\,\pi^2-4\,\pi\,\theta-\theta^2\right)^2\right)-\frac{1}{\theta\,\left(4\,\pi^2-4\,\pi\,\theta-\theta^2\right)^2}\,49\,\sqrt{(4\,\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]$$

$$\sqrt{\left(\left(-32\,\pi^2\,\theta^2+8\,\pi\,\theta^3+16\,\pi^2\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]\right)^2-4\,\left(128\,\pi^4\,\theta-224\,\pi^3\,\theta^2+80\,\pi^2\,\theta^3+8\,\pi\,\theta^4-4\,\theta^5\right)\,\left(-8\,\pi^2\,\theta-2\,\pi\,\theta^2+\theta^3+8\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]+4\,\pi\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]-8\,\pi^3\,\text{Sin}[\beta]^2-4\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)\right)}\Big/\left(\sqrt{6272\,\pi^2-6272\,\pi\,\theta+1568\,\theta^2}\right)\}\}$$

$$\textbf{Solve}\Big[\sqrt{\left(r\hat{}2-\frac{\sqrt{4\,\pi\,(r)^2\,\theta-(r)^2\,\theta^2}}{2\,\pi}\hat{}2\right)}==$$

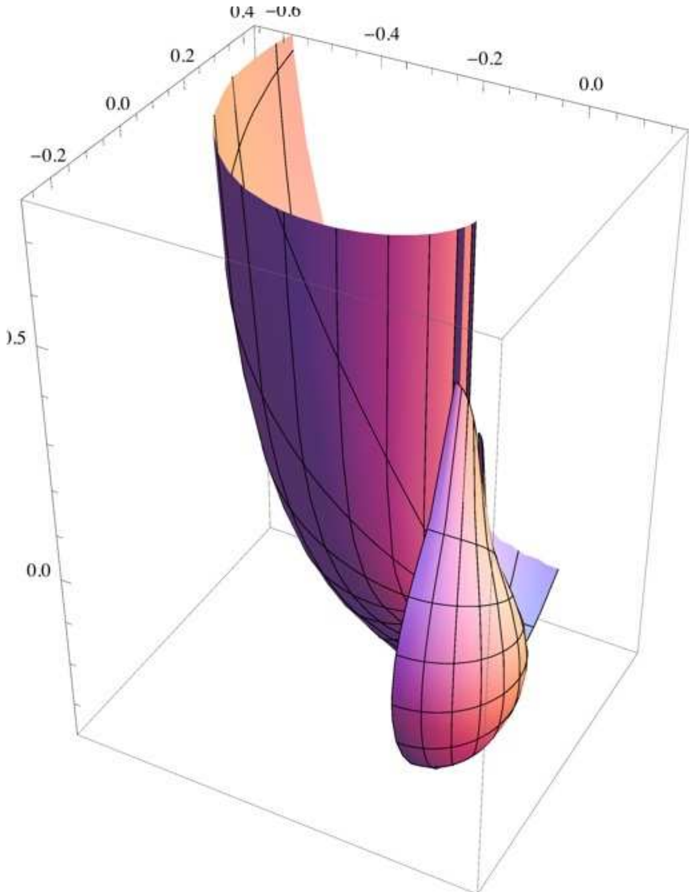
$$\frac{1}{2\,\pi}\left(2\,\pi\left(\left(-4\,\pi\,\theta+\theta^2+2\,\pi\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]+4\,\pi^2\,\text{Sin}[\beta]^2-\frac{2\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{4\,\pi-\theta}-\frac{2\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)\right)-\left(\left(-4\,\pi\,\theta+\theta^2+2\,\pi\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]+4\,\pi^2\,\text{Sin}[\beta]^2-\frac{2\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{4\,\pi-\theta}-\frac{2\,\pi^2\,\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)\right)\theta\right),\ r\Big]$$

$$\left\{\left\{r\to-\frac{\sqrt{1-\frac{\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]}{4\,\pi-\theta}-\frac{\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]}{\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{4\,\pi-\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{\theta}}}{\sqrt{16\,\pi^2-16\,\pi\,\theta+4\,\theta^2}}\right\},\right.$$

$$\left.\left\{r\to\frac{\sqrt{1-\frac{\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]}{4\,\pi-\theta}-\frac{\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]}{\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{4\,\pi-\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{\theta}}}{\sqrt{16\,\pi^2-16\,\pi\,\theta+4\,\theta^2}}\right\}\right\}$$

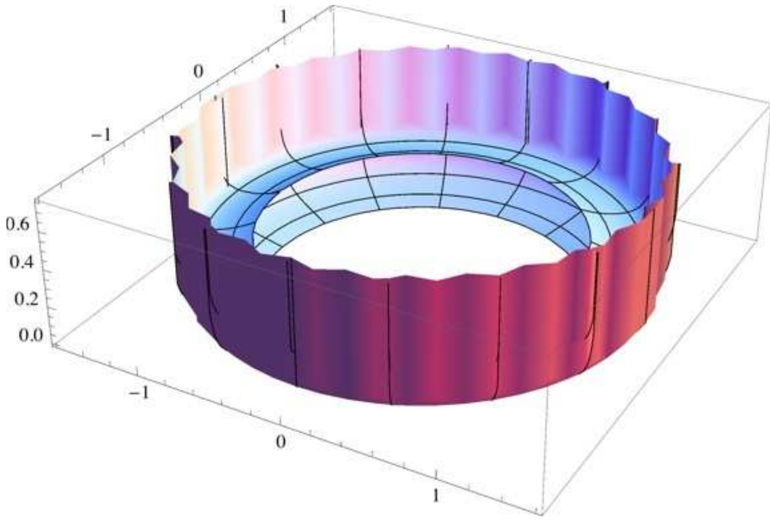
$$\text{SphericalPlot3D}\left[\frac{\sqrt{1-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{4\pi-\theta}-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{\theta}}}{\sqrt{16\,\pi^2-16\,\pi\,\theta+4\,\theta^2}},\right.$$

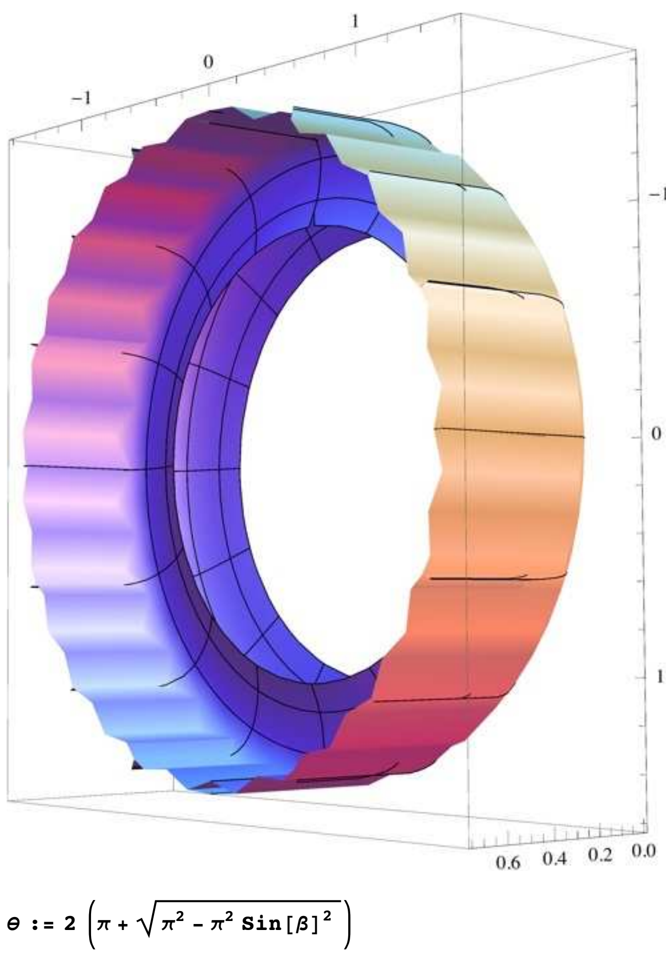
$$\left.\{\theta,\ -2\,\pi,\ 2\,\pi\},\ \{\beta,\ -\pi/2,\ \pi/2\}\right]$$

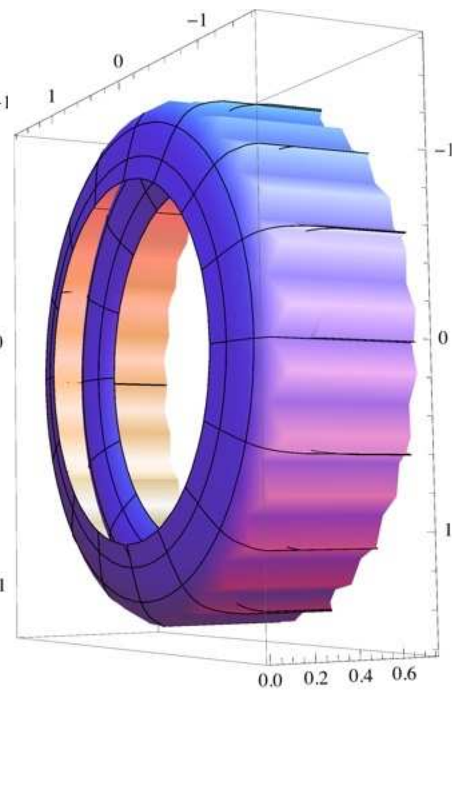


$$\theta := -i\,\text{Log}\left[\text{Cos}\left[2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right]+i\,\text{Sin}\left[2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right]\right]$$

$$\text{RevolutionPlot3D}\left[\frac{\sqrt{1-\frac{\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{4\pi-\theta}-\frac{\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{\theta}}}{\sqrt{16\pi^2-16\pi\theta+4\theta^2}},\ \{\beta,\ -\pi/2,\ \pi/2\}\right]$$

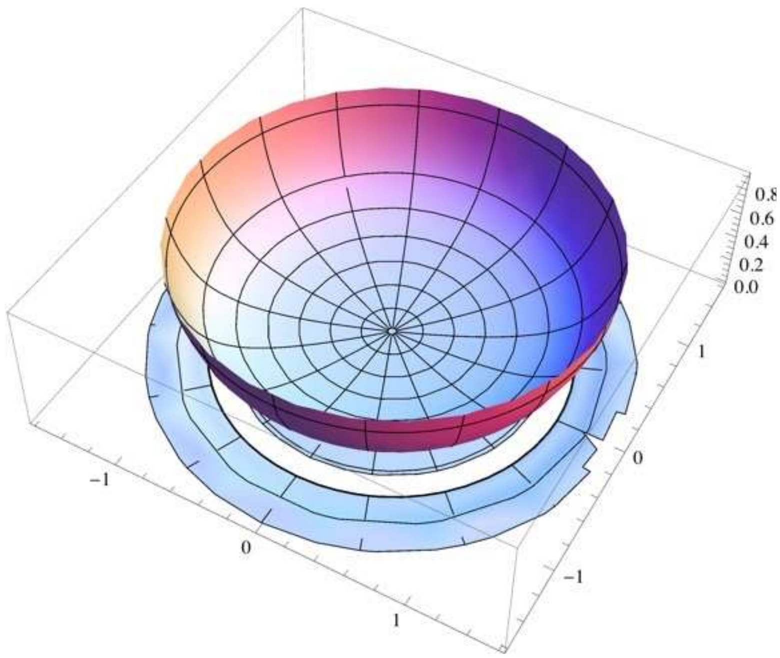






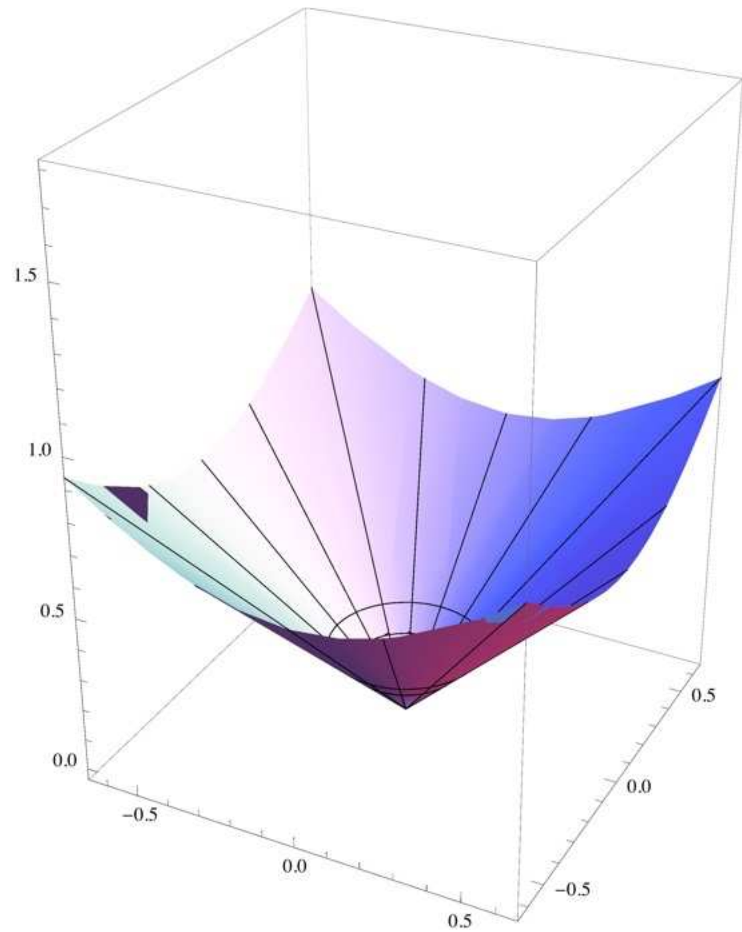
$$\theta := 2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)$$

$$\text{RevolutionPlot3D}\left[\frac{\sqrt{1-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{4\pi-\theta}-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{\theta}}}{\sqrt{16\,\pi^2-16\,\pi\,\theta+4\,\theta^2}},\ \{\beta,\ -\pi/2,\ \pi/2\}\right]$$



$$\text{RevolutionPlot3D}\left[\left\{-\frac{\sqrt{1-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{4\pi-\theta}-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{\theta}}}{\sqrt{16\pi^2-16\pi\theta+4\theta^2}},\right.\right.$$

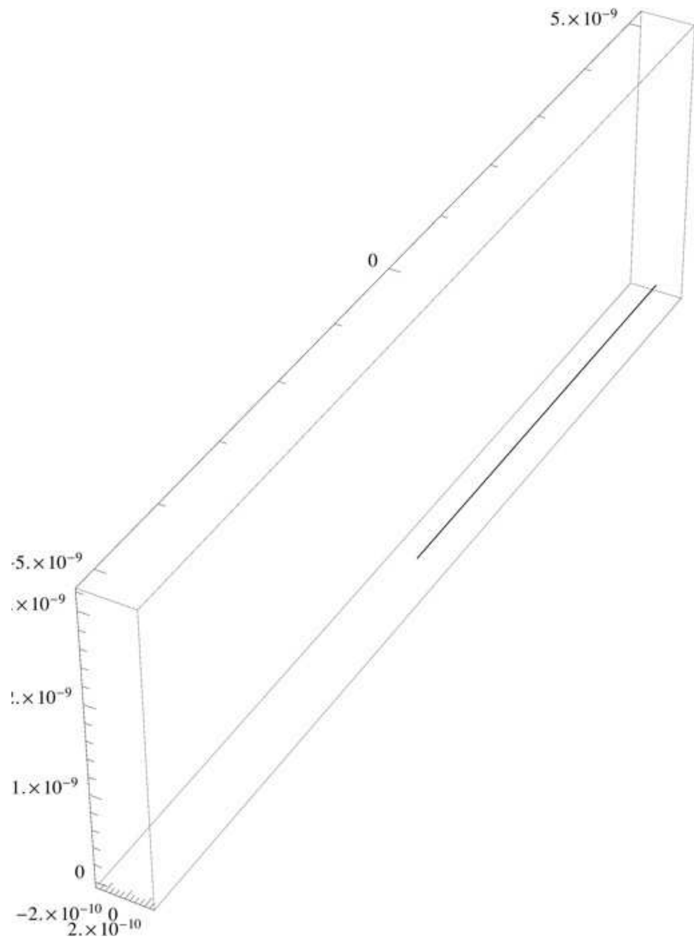
$$\left.\left.\frac{\sqrt{1-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{4\pi-\theta}-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{\theta}}}{\sqrt{16\pi^2-16\pi\theta+4\theta^2}}\right\},\ \{\beta,\ -2\pi,\ 2\pi\}\right]$$



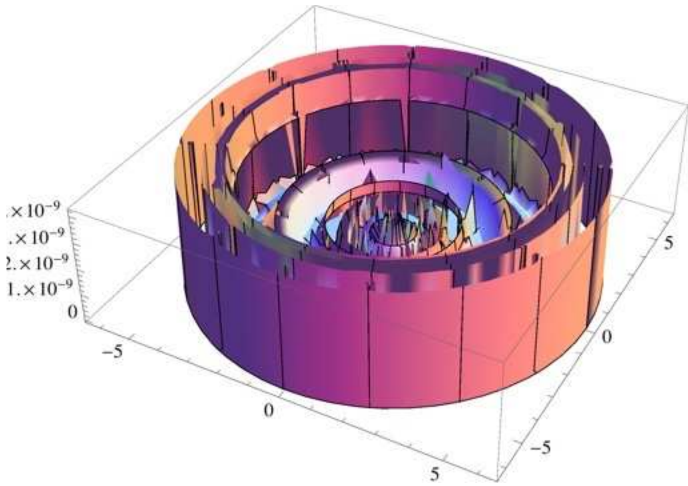
$$\beta := \text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\,\theta}}{2\pi}\right]$$

$$\text{RevolutionPlot3D}\left[\left\{-\frac{\sqrt{1-\frac{\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]}{4\pi-\theta}-\frac{\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]}{\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{\theta}}}{\sqrt{16\pi^2-16\pi\theta+4\theta^2}},\right.\right.$$

$$\left.\left.\frac{\sqrt{1-\frac{\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]}{4\pi-\theta}-\frac{\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]}{\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{\theta}}}{\sqrt{16\pi^2-16\pi\theta+4\theta^2}}\right\},\{\theta,-2\pi,2\pi\}\right]$$

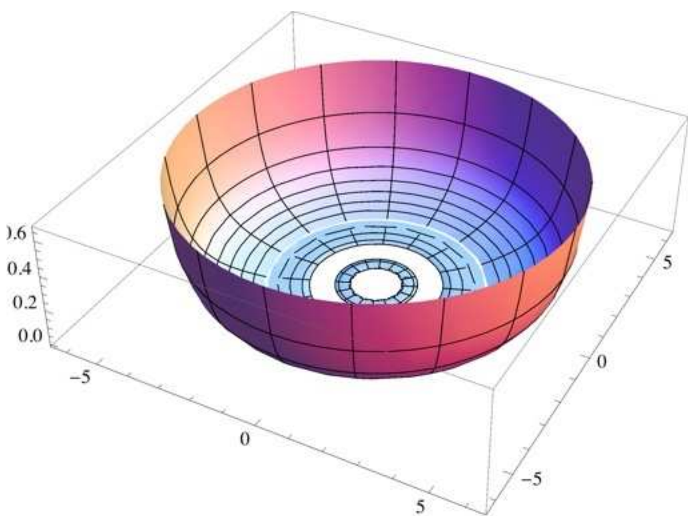


$$\textbf{RevolutionPlot3D}\left[\frac{\sqrt{1-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{4\pi-\theta}-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{\theta}}}{\sqrt{16\,\pi^2-16\,\pi\,\theta+4\,\theta^2}},\ \{\theta,\ -2\,\pi,\ 2\,\pi\}\right]$$



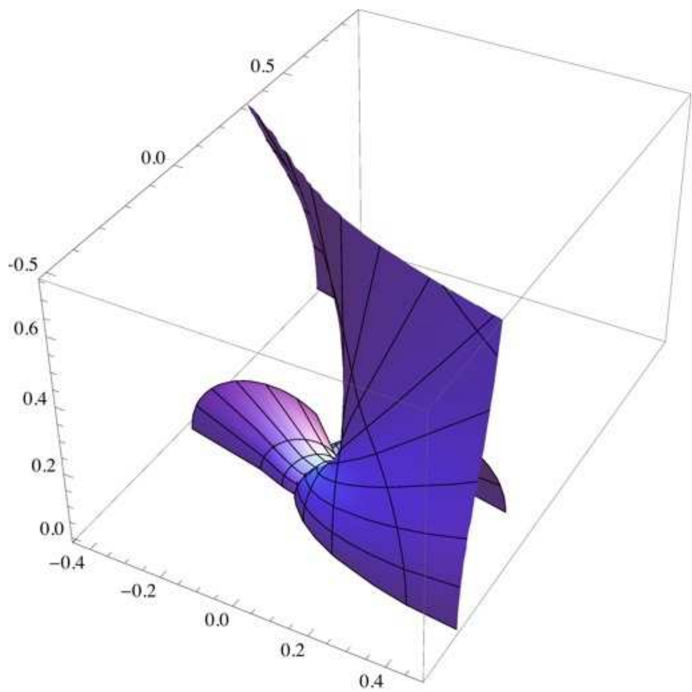
$$\beta := \textbf{ArcSin}\left[\frac{\sqrt{\pi^2+\textbf{Log}\left[-\sqrt{\textbf{Cos}[\theta]+\mathbb{i}\ \textbf{Sin}[\theta]}\right]^2}}{\pi}\right]$$

$$\textbf{RevolutionPlot3D}\left[\frac{\sqrt{1-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{4\pi-\theta}-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{\theta}}}{\sqrt{16\,\pi^2-16\,\pi\,\theta+4\,\theta^2}},\ \{\theta,\ -2\,\pi,\ 2\,\pi\}\right]$$

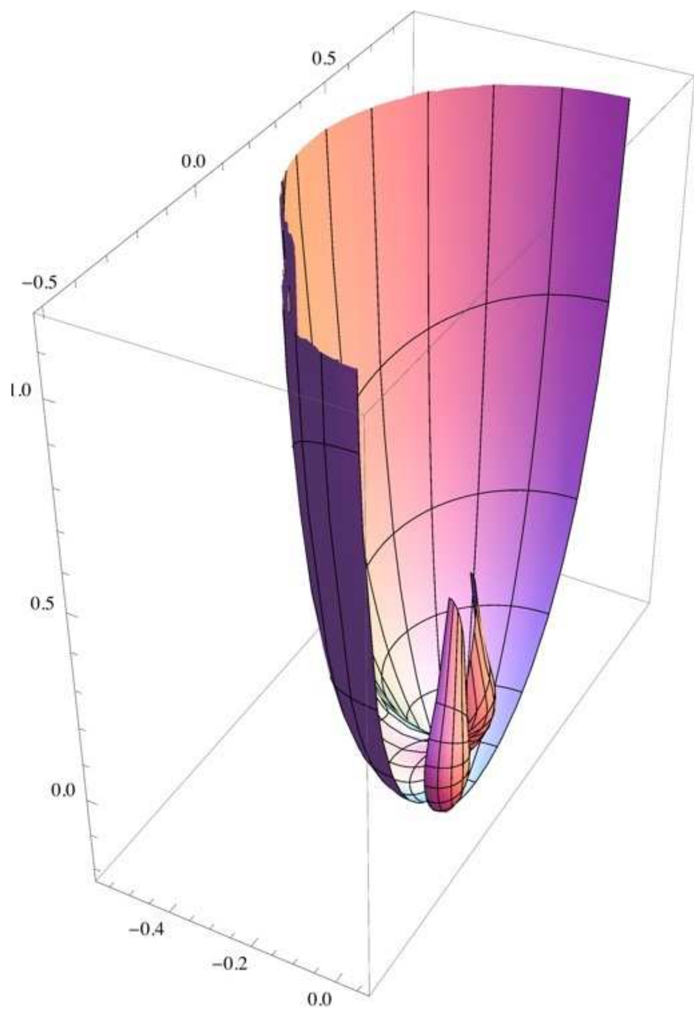


Solution 2 Substitutions

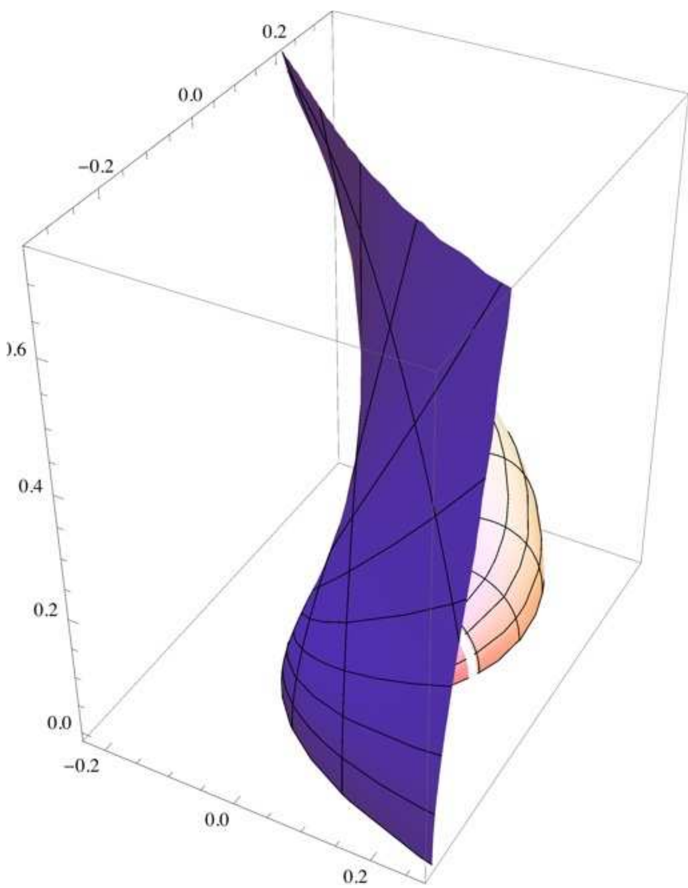
$$\text{SphericalPlot3D}\left[\left(\sqrt{\left(1-\frac{\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{4\pi-(\theta)}-\frac{\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{\pi^2+\text{Log}\left[-\sqrt{\text{Cos}[\theta]+i\,\text{Sin}[\theta]}\right]^2}}{\pi}\right]\right]}{\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{\theta}\right)}\right)\Big/\left(\sqrt{16\pi^2-16\pi\theta+4\theta^2}\right),\ \{\beta,-\pi/2,\pi/2\},\ \{\theta,-2\pi,2\pi\}\right]$$



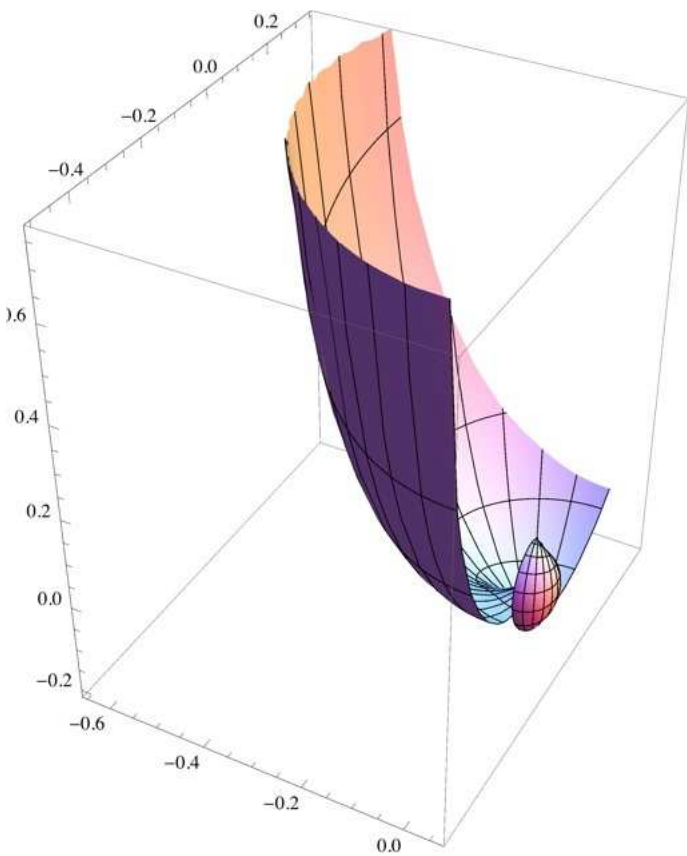
$$\textbf{SphericalPlot3D}\left[\left(\sqrt{\left(1-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]}{4\pi-(\theta)}-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}\left[\mathrm{ArcSin}\left[\frac{\sqrt{\pi^2+\mathrm{Log}\left[-\sqrt{\mathrm{Cos}[\theta]+i\,\mathrm{Sin}[\theta]}\right]^2}}{\pi}\right]\right]}{\theta}+\frac{\pi\,\mathrm{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\mathrm{Sin}[\beta]^2}{\theta}\right)}\right)\Big/\left(\sqrt{16\pi^2-16\pi\theta+4\theta^2}\right),\ \{\theta,-2\pi,2\pi\},\ \{\beta,-\pi/2,\pi/2\}\right]$$



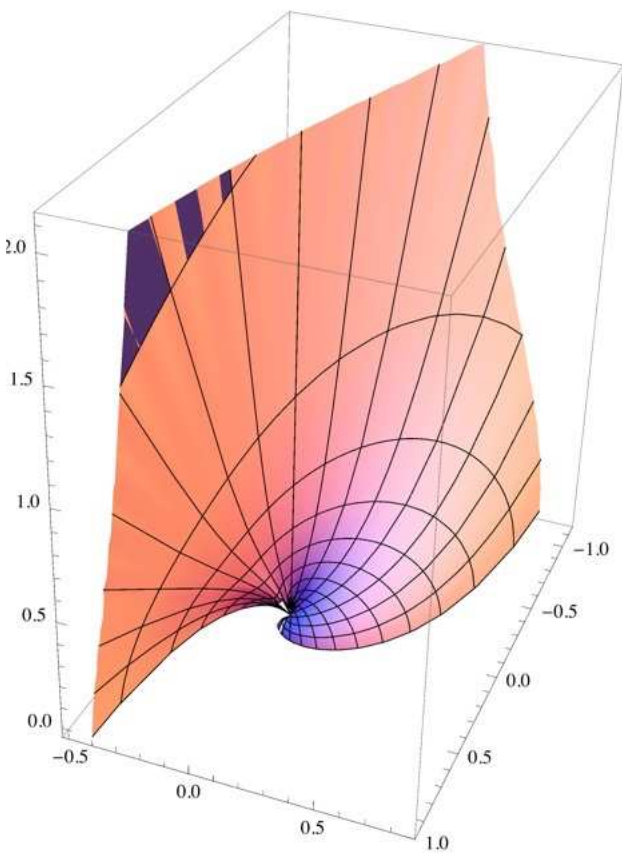
$$\textbf{SphericalPlot3D}\Big[\left(\sqrt{\left(1-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{4\pi-(\theta)}-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}\Big[\text{ArcSin}\Big[\frac{\sqrt{\pi^2+\text{Log}\left[-\sqrt{\text{Cos}[\theta]+i\,\text{Sin}[\theta]}\,\right]^2}}{\pi}\Big]\Big]}{\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\text{Sin}\Big[\text{ArcSin}\Big[\frac{\sqrt{(4\pi-\theta)\,\theta}}{2\pi}\Big]\Big]^2}{\theta}\right)}\right)\Big/\left(\sqrt{16\pi^2-16\pi\theta+4\theta^2}\right),\ \{\beta,-\pi/2,\pi/2\},\ \{\theta,-2\pi,2\pi\}\Big]$$



$$\text{SphericalPlot3D}\Bigg[\left(\sqrt{\left(1-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{4\pi-(\theta)}-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{\pi^2+\text{Log}\left[-\sqrt{\text{Cos}[\theta]+i\,\text{Sin}[\theta]}\right]^2}}{\pi}\right]\right]}{\theta}+\frac{\pi\,\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\,\theta}}{2\pi}\right]\right]^2}{\theta}\right)}\right)\Big/\left(\sqrt{16\,\pi^2-16\,\pi\,\theta+4\,\theta^2}\right),\ \{\theta,-2\pi,2\pi\},\ \{\beta,-\pi/2,\pi/2\}\Bigg]$$



$$\text{SphericalPlot3D}\left[\left(\sqrt{\left(1 - \frac{\sqrt{(4\pi-\theta)\,\theta}\;\text{Sin}[\beta]}{4\pi-(\theta)} - \frac{1}{\theta}\sqrt{(4\pi-\theta)\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right)}\;\text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{\pi^2+\text{Log}\left[-\sqrt{\text{Cos}[\theta]+i\,\text{Sin}[\theta]}\right]^2}}{\pi}\right]\right] + \frac{\pi\,\text{Sin}[\beta]^2}{4\pi-\theta} + \frac{\pi\,\text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\,\theta}}{2\pi}\right]\right]^2}{\theta}\right)}\right)\Big/\left(\sqrt{16\,\pi^2-16\,\pi\,\theta+4\,\theta^2}\right),\ \{\beta,\,-\pi/2,\,\pi/2\},\ \{\theta,\,-2\,\pi,\,2\,\pi\}\right]$$

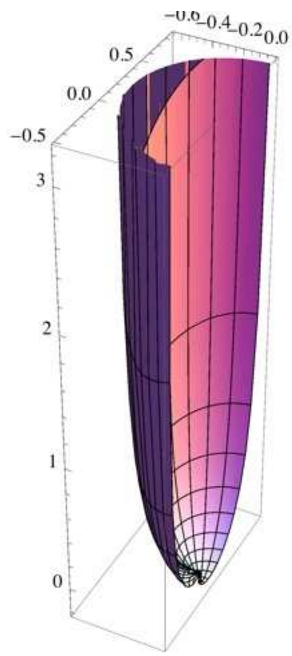


$$\text{SphericalPlot3D}\Bigg[\Bigg(\sqrt{\Bigg(1-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \text{Sin}[\beta]}{4\pi-(\theta)}-}$$

$$\frac{1}{\theta}\sqrt{(4\pi-\theta)\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right)}\ \text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{\pi^2+\text{Log}\left[-\sqrt{\text{Cos}[\theta]+\mathrm{i}\,\text{Sin}[\theta]}\right]^2}}{\pi}\right]\right]+$$

$$\frac{\pi\,\text{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\,\theta}}{2\pi}\right]\right]^2}{\theta}\Bigg)}\Bigg)\Big/$$

$$\left(\sqrt{16\pi^2-16\pi\theta+4\theta^2}\right),\ \{\theta,\ -2\pi,\ 2\pi\},\ \{\beta,\ -\pi/2,\ \pi/2\}\Bigg]$$



$$\texttt{SphericalPlot3D}\left[\left(\sqrt{\left(1-\frac{\sqrt{(4\pi-\theta)\,\theta}\ \texttt{Sin}[\beta]}{4\pi-(\theta)}-\frac{1}{\theta}\sqrt{(4\pi-\theta)\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\texttt{Sin}[\beta]^2}\right)\right)}\ \texttt{Sin}\left[\texttt{ArcSin}\left[\frac{\sqrt{\pi^2+\texttt{Log}\left[-\sqrt{\texttt{Cos}[\theta]+i\,\texttt{Sin}[\theta]}\right]^2}}{\pi}\right]\right]+\frac{\pi\,\texttt{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\texttt{Sin}\left[\texttt{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\,\theta}}{2\pi}\right]\right]^2}{\theta}\right)}\right)\Big/\left(\sqrt{\left(16\pi^2-16\pi\theta+4\left(-i\,\texttt{Log}\left[\texttt{Cos}\left[2\left(\pi+\sqrt{\pi^2-\pi^2\,\texttt{Sin}[\beta]^2}\right)\right]+i\,\texttt{Sin}\left[2\left(\pi+\sqrt{\pi^2-\pi^2\,\texttt{Sin}[\beta]^2}\right)\right]\right]\right)^2\right)}\right),\ \{\beta,-\pi/2,\pi/2\},\ \{\theta,-2\pi,2\pi\}\right]$$

ComputationalGeometry`Methods`SpatialSearch::spoutbounds :
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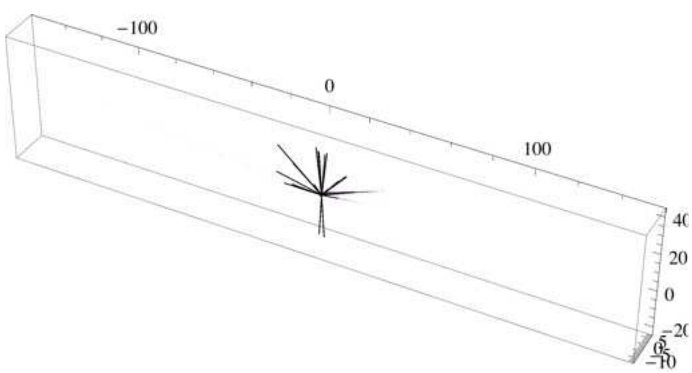
ComputationalGeometry`Methods`SpatialSearch::spoutbounds :
-- Message text not found -- ({-3.91763, 24.6465})

ComputationalGeometry`Methods`SpatialSearch::spoutbounds :
-- Message text not found -- ({-5.32253, -13.6605})

General::stop :
Further output of ComputationalGeometry`Methods`SpatialSearch::spoutbounds will be suppressed during this calculation. ≫



$$\mathbf{SphericalPlot3D}\Bigg[\left(\sqrt{\left(1-\frac{\sqrt{(4\pi-\theta)\,\theta}\;\mathrm{Sin}[\beta]}{4\pi-(\theta)}-\frac{1}{\theta}\sqrt{(4\pi-\theta)\left(2\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)\right)}\;\mathrm{Sin}\left[\mathrm{ArcSin}\left[\frac{\sqrt{\pi^2+\mathrm{Log}\left[-\sqrt{\mathrm{Cos}[\theta]+i\,\mathrm{Sin}[\theta]}\right]^2}}{\pi}\right]\right]+\frac{\pi\,\mathrm{Sin}[\beta]^2}{4\pi-\theta}+\frac{\pi\,\mathrm{Sin}\left[\mathrm{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\,\theta}}{2\pi}\right]\right]^2}{\theta}\right)}\right)\Big/$$

$$\left(\sqrt{\left(16\pi^2-16\pi\theta+4\left(-i\,\mathrm{Log}\left[\mathrm{Cos}\left[2\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)\right]+i\,\mathrm{Sin}\left[2\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)\right]\right]\right)^2\right)}\right),$$

$$\{\theta,-2\pi,2\pi\},\{\beta,-\pi/2,\pi/2\}\Bigg]$$

ComputationalGeometry`Methods`SpatialSearch::spoutbounds :
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ComputationalGeometry`Methods`SpatialSearch::spoutbounds :
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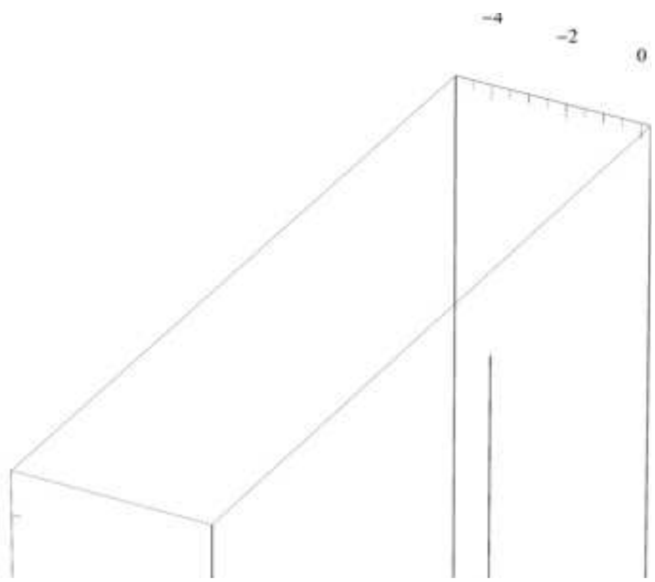

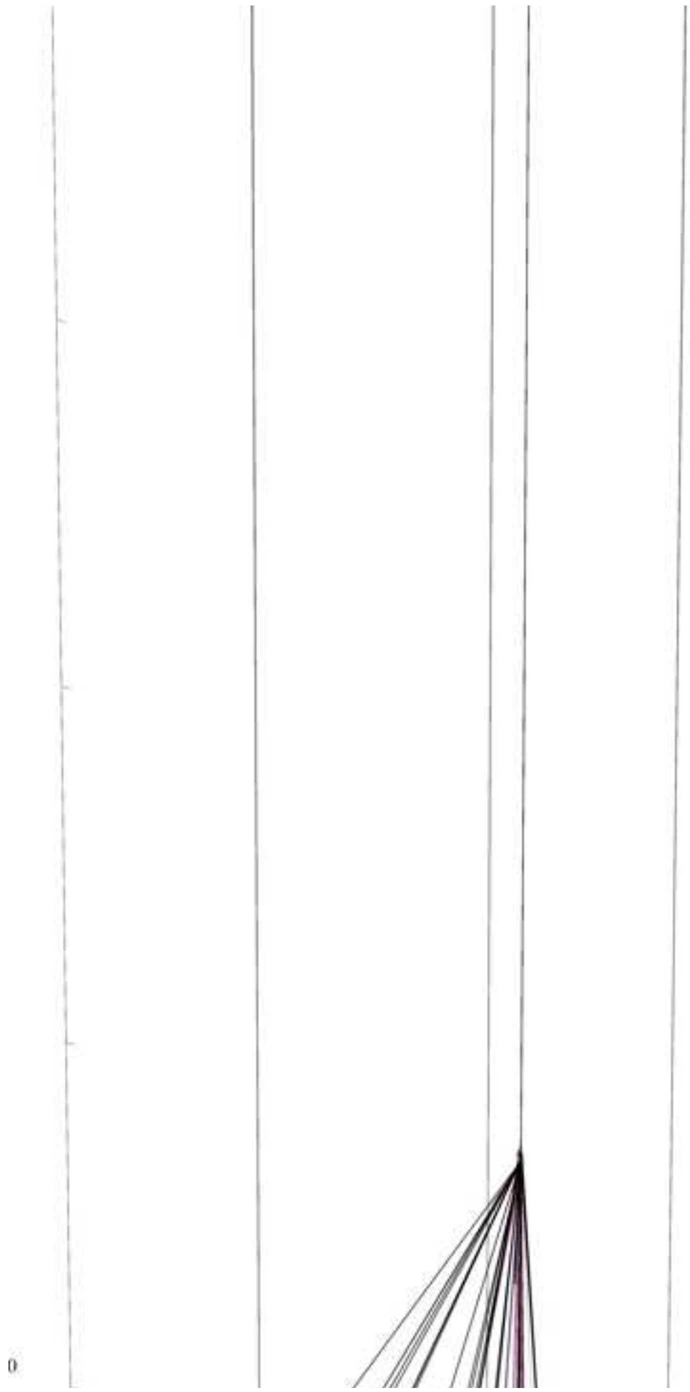
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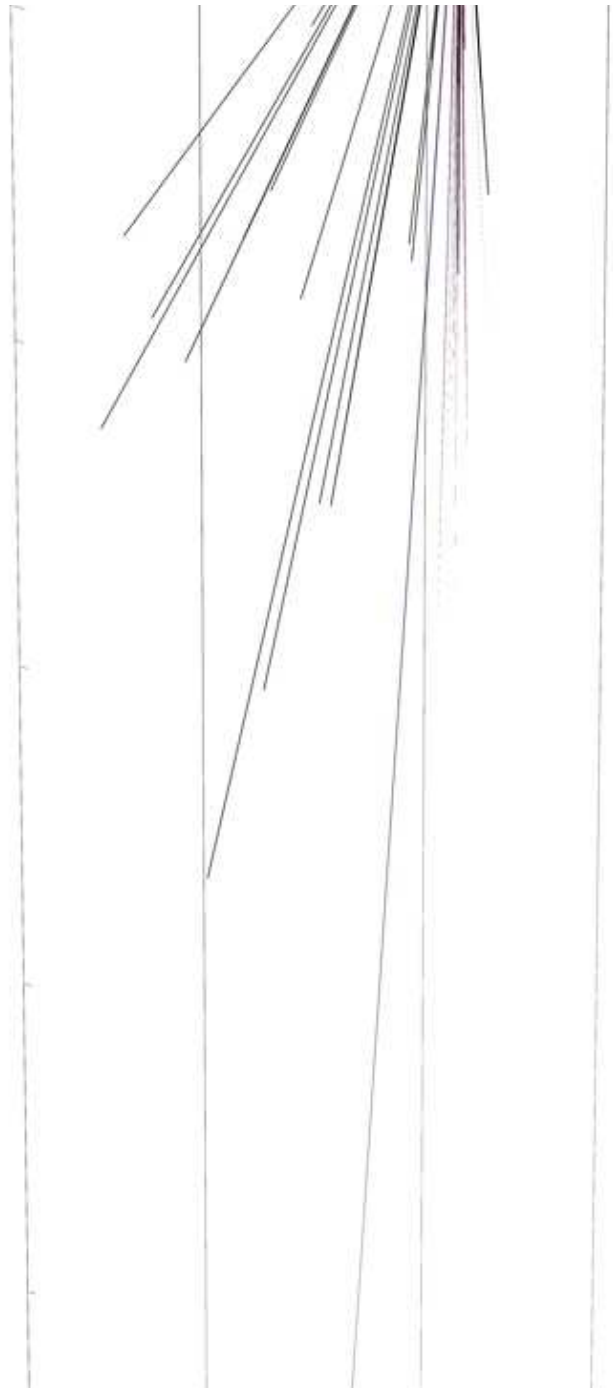


- **New Solution 3**

$$\text{Solve}\Big[\sqrt{\Big(\Big(\Big(\Big(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta}\Big)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\Big)\wedge 2-\frac{\sqrt{4\pi\,(r)^2\,\theta-(r)^2\,\theta^2}}{2\pi}\wedge 2\Big)}==$$

$$\frac{1}{2\pi}\Big(2\pi-\Big(\Big(-4\pi\theta+\theta^2+2\pi\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]+4\pi^2\,\text{Sin}[\beta]^2-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta}-\frac{2\pi^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta}\Big)\Big/\left(16\pi^2\theta-12\pi\theta^2+2\theta^3-16\pi^3\,\text{Sin}[\beta]^2+8\pi^2\theta\,\text{Sin}[\beta]^2\right)\Big)\theta\Big),\ r\Big]$$

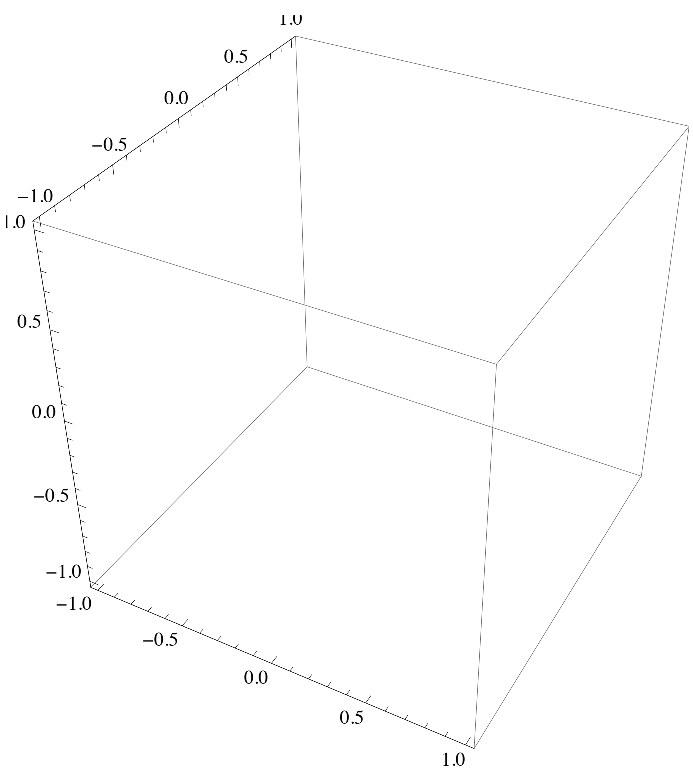
$$\Big\{\Big\{r\to-\frac{1}{4}\sqrt{\Big(-\frac{16}{2\pi-\theta}+\frac{16}{4\pi-\theta}-\frac{16\pi}{4\pi-\theta}+\frac{8}{(2\pi-\theta)\,(4\pi-\theta)}-\frac{16\pi}{\theta}+\frac{4}{(4\pi-\theta)\,\theta}-\frac{8\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{(4\pi-\theta)^2}-\frac{16\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{(2\pi-\theta)\,(4\pi-\theta)^2}+\frac{16\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{(2\pi-\theta)\,(4\pi-\theta)}-\frac{4\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{(4\pi-\theta)\,\theta^2}-\frac{20\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{(4\pi-\theta)^2\,\theta}+\frac{8\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{(4\pi-\theta)\,\theta}-\frac{3\,\text{Sin}[\beta]^2}{(4\pi-\theta)^2}+\frac{8\,\text{Sin}[\beta]^2}{(2\pi-\theta)\,(4\pi-\theta)}+\frac{\text{Sin}[\beta]^2}{\theta^2}+\frac{6\,\text{Sin}[\beta]^2}{(4\pi-\theta)\,\theta}\Big)}\Big\},$$

$$\Big\{r\to\frac{1}{4}\sqrt{\Big(-\frac{16}{2\pi-\theta}+\frac{16}{4\pi-\theta}-\frac{16\pi}{4\pi-\theta}+\frac{8}{(2\pi-\theta)\,(4\pi-\theta)}-\frac{16\pi}{\theta}+\frac{4}{(4\pi-\theta)\,\theta}-\frac{8\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{(4\pi-\theta)^2}-\frac{16\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{(2\pi-\theta)\,(4\pi-\theta)^2}+\frac{16\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{(2\pi-\theta)\,(4\pi-\theta)}-\frac{4\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{(4\pi-\theta)\,\theta^2}-\frac{20\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{(4\pi-\theta)^2\,\theta}+\frac{8\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{(4\pi-\theta)\,\theta}-\frac{3\,\text{Sin}[\beta]^2}{(4\pi-\theta)^2}+\frac{8\,\text{Sin}[\beta]^2}{(2\pi-\theta)\,(4\pi-\theta)}+\frac{\text{Sin}[\beta]^2}{\theta^2}+\frac{6\,\text{Sin}[\beta]^2}{(4\pi-\theta)\,\theta}\Big)}\Big\}\Big\}$$

SphericalPlot3D[

$$\left\{-\frac{1}{4}\sqrt{\left(-\frac{16}{2\pi-\theta}+\frac{16}{4\pi-\theta}-\frac{16\pi}{4\pi-\theta}+\frac{8}{(2\pi-\theta)\,(4\pi-\theta)}-\frac{16\pi}{\theta}+\frac{4}{(4\pi-\theta)\,\theta}-\frac{8\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]}{(4\pi-\theta)^2}-\frac{16\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]}{(2\pi-\theta)\,(4\pi-\theta)^2}+\frac{16\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]}{(2\pi-\theta)\,(4\pi-\theta)}-\frac{4\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]}{(4\pi-\theta)\,\theta^2}-\frac{20\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]}{(4\pi-\theta)^2\,\theta}+\frac{8\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]}{(4\pi-\theta)\,\theta}-\frac{3\,\mathrm{Sin}[\beta]^2}{(4\pi-\theta)^2}+\frac{8\,\mathrm{Sin}[\beta]^2}{(2\pi-\theta)\,(4\pi-\theta)}+\frac{\mathrm{Sin}[\beta]^2}{\theta^2}+\frac{6\,\mathrm{Sin}[\beta]^2}{(4\pi-\theta)\,\theta}\right)},\right.$$

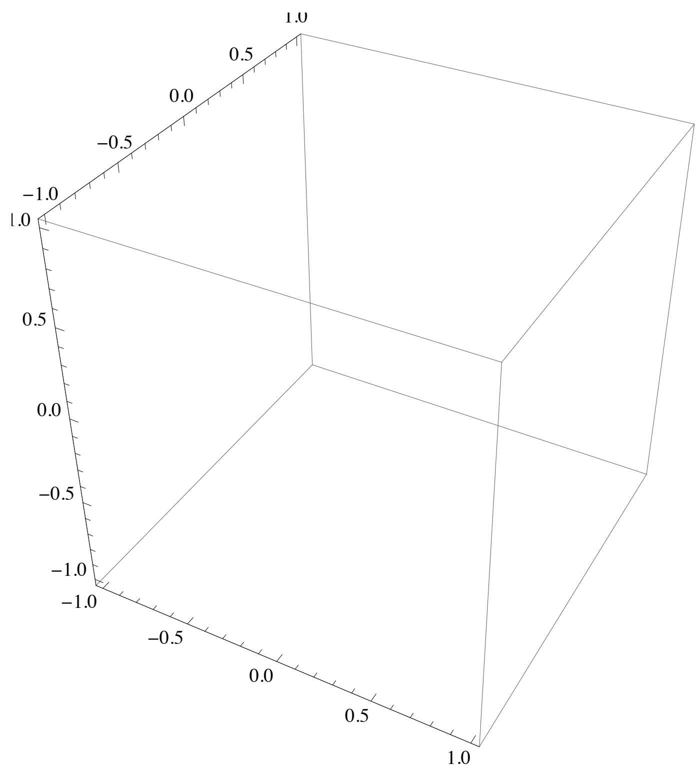
$$\left.\frac{1}{4}\sqrt{\left(-\frac{16}{2\pi-\theta}+\frac{16}{4\pi-\theta}-\frac{16\pi}{4\pi-\theta}+\frac{8}{(2\pi-\theta)\,(4\pi-\theta)}-\frac{16\pi}{\theta}+\frac{4}{(4\pi-\theta)\,\theta}-\frac{8\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]}{(4\pi-\theta)^2}-\frac{16\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]}{(2\pi-\theta)\,(4\pi-\theta)^2}+\frac{16\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]}{(2\pi-\theta)\,(4\pi-\theta)}-\frac{4\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]}{(4\pi-\theta)\,\theta^2}-\frac{20\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]}{(4\pi-\theta)^2\,\theta}+\frac{8\sqrt{(4\pi-\theta)\,\theta}\ \mathrm{Sin}[\beta]}{(4\pi-\theta)\,\theta}-\frac{3\,\mathrm{Sin}[\beta]^2}{(4\pi-\theta)^2}+\frac{8\,\mathrm{Sin}[\beta]^2}{(2\pi-\theta)\,(4\pi-\theta)}+\frac{\mathrm{Sin}[\beta]^2}{\theta^2}+\frac{6\,\mathrm{Sin}[\beta]^2}{(4\pi-\theta)\,\theta}\right)}\right\},\ \{\theta,-2\pi,2\pi\},\ \{\beta,-\pi/2,\pi/2\}\Big]$$



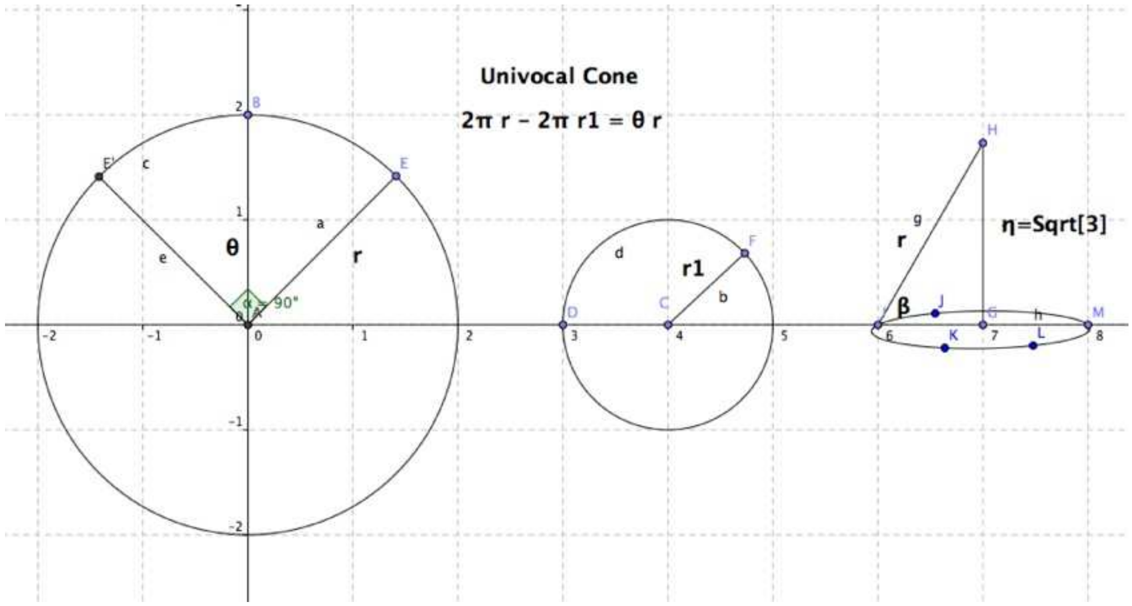
$$\textbf{SphericalPlot3D}\Big[$$

$$\Big\{-\frac{1}{4}\sqrt{\left(-\frac{16}{2\pi-\theta}+\frac{16}{4\pi-\theta}-\frac{16\pi}{4\pi-\theta}+\frac{8}{(2\pi-\theta)\ (4\pi-\theta)}-\frac{16\pi}{\theta}+\frac{4}{(4\pi-\theta)\ \theta}-\frac{8\sqrt{(4\pi-\theta)\ \theta}\ \text{Sin}[\beta]}{(4\pi-\theta)^2}-\frac{16\sqrt{(4\pi-\theta)\ \theta}\ \text{Sin}[\beta]}{(2\pi-\theta)\ (4\pi-\theta)^2}+\frac{16\sqrt{(4\pi-\theta)\ \theta}\ \text{Sin}[\beta]}{(2\pi-\theta)\ (4\pi-\theta)}-\frac{4\sqrt{(4\pi-\theta)\ \theta}\ \text{Sin}[\beta]}{(4\pi-\theta)\ \theta^2}-\frac{20\sqrt{(4\pi-\theta)\ \theta}\ \text{Sin}[\beta]}{(4\pi-\theta)^2\ \theta}+\frac{8\sqrt{(4\pi-\theta)\ \theta}\ \text{Sin}[\beta]}{(4\pi-\theta)\ \theta}-\frac{3\,\text{Sin}[\beta]^2}{(4\pi-\theta)^2}+\frac{8\,\text{Sin}[\beta]^2}{(2\pi-\theta)\ (4\pi-\theta)}+\frac{\text{Sin}[\beta]^2}{\theta^2}+\frac{6\,\text{Sin}[\beta]^2}{(4\pi-\theta)\ \theta}\right)},$$

$$\frac{1}{4}\sqrt{\left(-\frac{16}{2\pi-\theta}+\frac{16}{4\pi-\theta}-\frac{16\pi}{4\pi-\theta}+\frac{8}{(2\pi-\theta)\ (4\pi-\theta)}-\frac{16\pi}{\theta}+\frac{4}{(4\pi-\theta)\ \theta}-\frac{8\sqrt{(4\pi-\theta)\ \theta}\ \text{Sin}[\beta]}{(4\pi-\theta)^2}-\frac{16\sqrt{(4\pi-\theta)\ \theta}\ \text{Sin}[\beta]}{(2\pi-\theta)\ (4\pi-\theta)^2}+\frac{16\sqrt{(4\pi-\theta)\ \theta}\ \text{Sin}[\beta]}{(2\pi-\theta)\ (4\pi-\theta)}-\frac{4\sqrt{(4\pi-\theta)\ \theta}\ \text{Sin}[\beta]}{(4\pi-\theta)\ \theta^2}-\frac{20\sqrt{(4\pi-\theta)\ \theta}\ \text{Sin}[\beta]}{(4\pi-\theta)^2\ \theta}+\frac{8\sqrt{(4\pi-\theta)\ \theta}\ \text{Sin}[\beta]}{(4\pi-\theta)\ \theta}-\frac{3\,\text{Sin}[\beta]^2}{(4\pi-\theta)^2}+\frac{8\,\text{Sin}[\beta]^2}{(2\pi-\theta)\ (4\pi-\theta)}+\frac{\text{Sin}[\beta]^2}{\theta^2}+\frac{6\,\text{Sin}[\beta]^2}{(4\pi-\theta)\ \theta}\right)}\Big\},\ \{\beta,-\pi/2,\pi/2\},\ \{\theta,-2\pi,2\pi\}\Big]$$



XXI. Parker' s Commentary on Fermat' s Last Theorem



Please see The Geometric Pattern of Perception to reference the preliminary investigations.

Lemma 5 *The height of the cone can be calculated in terms of only r and* θ, thus θ is a function of β alone.

Proof. Since we have shown that θ r = 2 π r - 2 π r_1 and $r_1 \to \sqrt{r^2 - \eta^2}$, we can substitute the expression for r_1, calculated from the Pythagorean theorem in terms of the height of the cone and the initial radius of the circle, into the expression for θ r in terms of the change in circumference of the initial circle to the circle that is the base of the cone into which the circle was transformed.

θ r = 2 π r - 2 π $\sqrt{(r\hat{}2 - \eta\hat{}2)}$, thus, $\eta = \frac{\sqrt{4 \pi r^2 \theta - r^2 \theta^2}}{2 \pi}$ = (r Sin[β]). From $\frac{2 \pi \eta}{\sqrt{4 \pi \theta - \theta^2}}$ = r, we note that: r = $\frac{2 \pi r \operatorname{Sin}[\beta]}{\sqrt{4 \pi \theta - \theta^2}}$. So we solve the equation,

$$\text{Solve}\left[r == \frac{2 \pi r \operatorname{Sin}[\beta]}{\sqrt{4 \pi \theta - \theta^2}}, \theta\right]$$

$$\left\{\left\{\theta \to 2 \left(\pi - \sqrt{\pi^2 - \pi^2 \operatorname{Sin}[\beta]^2}\right)\right\}, \left\{\theta \to 2 \left(\pi + \sqrt{\pi^2 - \pi^2 \operatorname{Sin}[\beta]^2}\right)\right\}\right\}$$

$$1 = \frac{2 \pi \operatorname{Sin}[\beta]}{\sqrt{4 \pi \theta - \theta^2}}$$

The Meaning of n > 2 is $n > \frac{r\,\theta}{\pi\,(r - r_1)}$; $n > \frac{2\,\pi\,\mathrm{Sin}[\beta]}{\sqrt{4\,\pi\,\theta - \theta^2}} + \frac{2\,\pi\,\mathrm{Sin}[\beta]}{\sqrt{4\,\pi\,\theta - \theta^2}} >$

$$\frac{\frac{r\,\theta}{\pi\,(r-r_1)}\,\pi\,\mathrm{Sin}[\beta]}{\sqrt{\left(\frac{r\,\theta}{\pi\,(r-r_1)} + \frac{r\,\theta}{\pi\,(r-r_1)}\right)\pi\,\theta - \theta^2}} + \frac{\frac{r\,\theta}{\pi\,(r-r_1)}\,\pi\,\mathrm{Sin}[\beta]}{\sqrt{\left(\frac{r\,\theta}{\pi\,(r-r_1)} + \frac{r\,\theta}{\pi\,(r-r_1)}\right)\pi\,\theta - \theta^2}}$$

x^n + y^n == z^n, where n > 2

$\theta\,r = 2\,\pi\,r - 2\,\pi\,r_1$

What is 2?

Solve[$\theta\,r == \varrho\,\pi\,r - \varrho\,\pi\,r_1$, ϱ]

$$\left\{\left\{\varrho \to \frac{r\,\theta}{\pi\,(r - r_1)}\right\}\right\}$$

$$2 == \frac{r\,\theta}{\pi\,(r - r_1)}$$

x^n + y^n == z^n, where $n > \frac{r\,\theta}{\pi\,(r - r_1)}$

x^n + y^n == z^n, where $n > \left\{\frac{r\,\theta}{\pi\left(r - \frac{2\,\pi\,r - r\,\theta}{2\,\pi}\right)} = \frac{2\,\pi\,\mathrm{Sin}[\beta]}{\sqrt{4\,\pi\,\theta - \theta^2}} + \frac{2\,\pi\,\mathrm{Sin}[\beta]}{\sqrt{4\,\pi\,\theta - \theta^2}}\right\}$

$$\mathrm{Solve}\left[\frac{r\,\theta}{\pi\left(r - \frac{2\,\pi\,r - r\,\theta}{2\,\pi}\right)} == \frac{2\,\pi\,\mathrm{Sin}[\beta]}{\sqrt{4\,\pi\,\theta - \theta^2}} + \frac{2\,\pi\,\mathrm{Sin}[\beta]}{\sqrt{4\,\pi\,\theta - \theta^2}},\ r\right]$$

{}

$$\mathrm{Solve}\left[\frac{r\,\theta}{\pi\left(r - \sqrt{\left(r\wedge 2 - \frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi}\wedge 2\right)}\right)} == \frac{2\,\pi\,\mathrm{Sin}[\beta]}{\sqrt{4\,\pi\,\theta - \theta^2}} + \frac{2\,\pi\,\mathrm{Sin}[\beta]}{\sqrt{4\,\pi\,\theta - \theta^2}},\ r\right]$$

{}

$$\text{Solve}\left[\frac{r\,\theta}{\pi\left(r-\sqrt{(r\verb|^|2-\eta\verb|^|2)}\right)}==\frac{2\,\pi\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\theta-\theta^2}}+\frac{2\,\pi\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\theta-\theta^2}},\,r\right]$$

$$\left\{\left\{r\to-\left(2\sqrt{2}\sqrt{\left(2\,\pi^4\,\eta^2\,\text{Sin}[\beta]^2+\frac{\pi^4\,\eta^2\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{4\,\pi-\theta}+\frac{4\,\pi^5\,\eta^2\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta^2}+\frac{\pi^4\,\eta^2\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta}\right)}\right)\Big/\left(\sqrt{-4\,\pi\,\theta^3+\theta^4+64\,\pi^4\,\text{Sin}[\beta]^2}\right)\right\},\right.$$

$$\left.\left\{r\to\left(2\sqrt{2}\sqrt{\left(2\,\pi^4\,\eta^2\,\text{Sin}[\beta]^2+\frac{\pi^4\,\eta^2\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{4\,\pi-\theta}+\frac{4\,\pi^5\,\eta^2\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta^2}+\frac{\pi^4\,\eta^2\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta}\right)}\right)\Big/\left(\sqrt{-4\,\pi\,\theta^3+\theta^4+64\,\pi^4\,\text{Sin}[\beta]^2}\right)\right\}\right\}$$

$$\text{Solve}\left[\left(-4\,\pi\,\theta+\theta^2+2\,\pi\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]+4\,\pi^2\,\text{Sin}[\beta]^2-\frac{2\,\pi^2\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{4\,\pi-\theta}-\frac{2\,\pi^2\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)==\left(2\sqrt{2}\sqrt{\left(2\,\pi^4\,\eta^2\,\text{Sin}[\beta]^2+\frac{\pi^4\,\eta^2\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{4\,\pi-\theta}+\frac{4\,\pi^5\,\eta^2\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta^2}+\frac{\pi^4\,\eta^2\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta}\right)}\right)\Big/\left(\sqrt{-4\,\pi\,\theta^3+\theta^4+64\,\pi^4\,\text{Sin}[\beta]^2}\right),\,\eta\right]$$

$$\left\{\left\{\eta\to\right.\right.$$

$$-\left(\text{Csc}[\beta]\sqrt{\left(192\,\pi^6+\frac{256\,\pi^8}{(2\,\pi-\theta)^2}-\frac{512\,\pi^7}{2\,\pi-\theta}+64\,\pi^5\,\theta+16\,\pi^4\,\theta^2-4\,\pi^2\,\theta^4-4\,\pi\,\theta^5+\theta^6-320\,\pi^5\sqrt{(4\,\pi-\theta)\,\theta}\right.}\right.$$

$$\text{Sin}[\beta]-\frac{512\,\pi^7\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{(2\,\pi-\theta)^2}+\frac{896\,\pi^6\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{2\,\pi-\theta}-$$

$$96\,\pi^4\,\theta\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]-16\,\pi^3\,\theta^2\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]+8\,\pi^2\,\theta^3\sqrt{(4\,\pi-\theta)\,\theta}\,\text{Sin}[\beta]+$$

$$4\pi\theta^4\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta] + 448\pi^6\,\mathrm{Sin}[\beta]^2 + \frac{256\pi^8\,\mathrm{Sin}[\beta]^2}{(2\pi-\theta)^2} - \frac{1024\pi^7\,\mathrm{Sin}[\beta]^2}{2\pi-\theta} +$$

$$192\pi^5\theta\,\mathrm{Sin}[\beta]^2 + 80\pi^4\theta^2\,\mathrm{Sin}[\beta]^2 - 32\pi^3\theta^3\,\mathrm{Sin}[\beta]^2 - 4\pi^2\theta^4\,\mathrm{Sin}[\beta]^2 +$$

$$128\pi^5\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]^3 + \frac{1792\pi^7\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]^3}{(2\pi-\theta)^2} - \frac{1152\pi^6\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]^3}{2\pi-\theta} -$$

$$32\pi^4\theta\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]^3 - 256\pi^6\,\mathrm{Sin}[\beta]^4 - \frac{5120\pi^8\,\mathrm{Sin}[\beta]^4}{(2\pi-\theta)^2} +$$

$$\frac{3072\pi^7\,\mathrm{Sin}[\beta]^4}{2\pi-\theta} + \frac{1024\pi^7\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]^5}{(2\pi-\theta)^2} - \frac{512\pi^6\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]^5}{2\pi-\theta} +$$

$$\left.\left.\frac{512\pi^6\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]^5}{4\pi-\theta}\right)\right) \Big/ \left(\pi^2\sqrt{-256\pi\theta^3 + 64\theta^4 + 4096\pi^4\,\mathrm{Sin}[\beta]^2}\right)\Big\},$$

$$\Big\{\eta \to \left(\mathrm{Csc}[\beta]\sqrt{\left(192\pi^6 + \frac{256\pi^8}{(2\pi-\theta)^2} - \frac{512\pi^7}{2\pi-\theta} + 64\pi^5\theta + 16\pi^4\theta^2 - 4\pi^2\theta^4 - 4\pi\theta^5 + \theta^6 - \right.}\right.$$

$$320\pi^5\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta] - \frac{512\pi^7\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]}{(2\pi-\theta)^2} + \frac{896\pi^6\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]}{2\pi-\theta} -$$

$$96\pi^4\theta\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta] - 16\pi^3\theta^2\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta] + 8\pi^2\theta^3\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta] +$$

$$4\pi\theta^4\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta] + 448\pi^6\,\mathrm{Sin}[\beta]^2 + \frac{256\pi^8\,\mathrm{Sin}[\beta]^2}{(2\pi-\theta)^2} -$$

$$\frac{1024\pi^7\,\mathrm{Sin}[\beta]^2}{2\pi-\theta} + 192\pi^5\theta\,\mathrm{Sin}[\beta]^2 + 80\pi^4\theta^2\,\mathrm{Sin}[\beta]^2 - 32\pi^3\theta^3\,\mathrm{Sin}[\beta]^2 -$$

$$4\pi^2\theta^4\,\mathrm{Sin}[\beta]^2 + 128\pi^5\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]^3 + \frac{1792\pi^7\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]^3}{(2\pi-\theta)^2} -$$

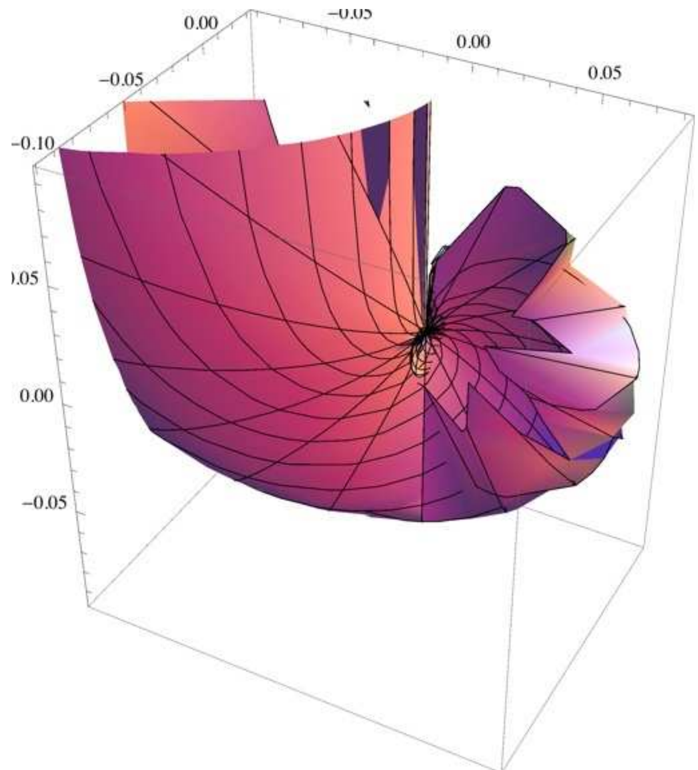
$$\frac{1152\pi^6\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]^3}{2\pi-\theta} - 32\pi^4\theta\sqrt{(4\pi-\theta)\theta}\ \mathrm{Sin}[\beta]^3 - 256\pi^6\,\mathrm{Sin}[\beta]^4 -$$

$$\frac{5120\,\pi^8\,\text{Sin}[\beta]^4}{(2\pi-\theta)^2}+\frac{3072\,\pi^7\,\text{Sin}[\beta]^4}{2\pi-\theta}+\frac{1024\,\pi^7\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^5}{(2\pi-\theta)^2}-\frac{512\,\pi^6\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^5}{2\pi-\theta}+\frac{512\,\pi^6\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^5}{4\pi-\theta}\Bigg)\Bigg)\Bigg/\left(\pi^2\sqrt{-256\,\pi\,\theta^3+64\,\theta^4+4096\,\pi^4\,\text{Sin}[\beta]^2}\right)\}\}$$

SphericalPlot3D[

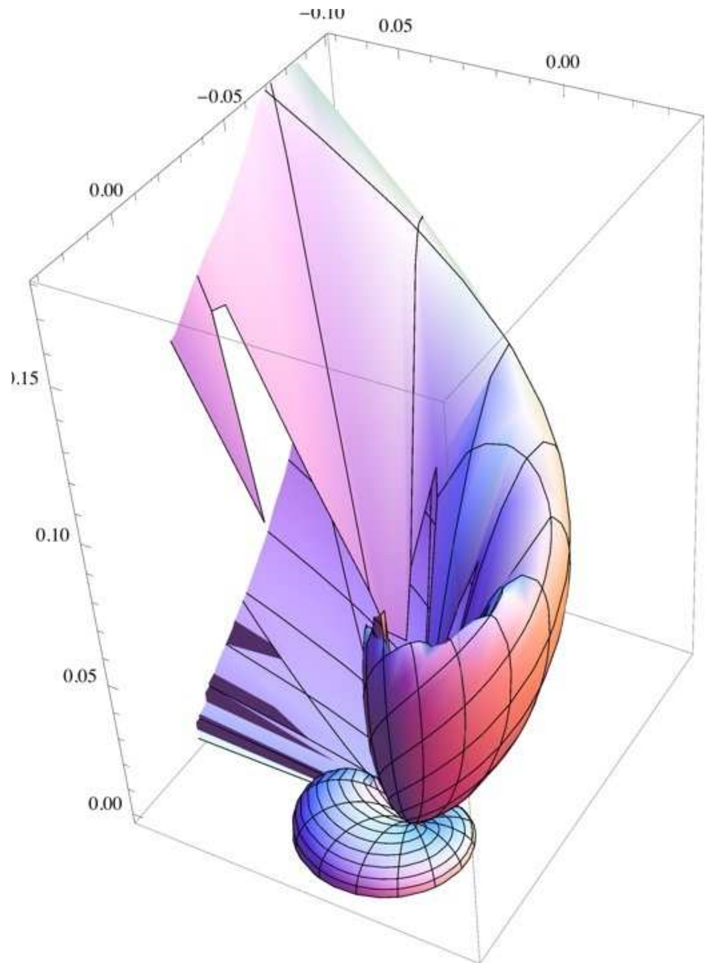
$$\left(\text{Csc}[\beta]\sqrt{\left(192\,\pi^6+\frac{256\,\pi^8}{(2\pi-\theta)^2}-\frac{512\,\pi^7}{2\pi-\theta}+64\,\pi^5\,\theta+16\,\pi^4\,\theta^2-4\,\pi^2\,\theta^4-4\,\pi\,\theta^5+\theta^6-320\,\pi^5\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]-\frac{512\,\pi^7\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{(2\pi-\theta)^2}+\frac{896\,\pi^6\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]}{2\pi-\theta}-96\,\pi^4\,\theta\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]-16\,\pi^3\,\theta^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]+8\,\pi^2\,\theta^3\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]+4\,\pi\,\theta^4\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]+448\,\pi^6\,\text{Sin}[\beta]^2+\frac{256\,\pi^8\,\text{Sin}[\beta]^2}{(2\pi-\theta)^2}-\frac{1024\,\pi^7\,\text{Sin}[\beta]^2}{2\pi-\theta}+192\,\pi^5\,\theta\,\text{Sin}[\beta]^2+80\,\pi^4\,\theta^2\,\text{Sin}[\beta]^2-32\,\pi^3\,\theta^3\,\text{Sin}[\beta]^2-4\,\pi^2\,\theta^4\,\text{Sin}[\beta]^2+128\,\pi^5\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3+\frac{1792\,\pi^7\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{(2\pi-\theta)^2}-\frac{1152\,\pi^6\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{2\pi-\theta}-32\,\pi^4\,\theta\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3-256\,\pi^6\,\text{Sin}[\beta]^4-\frac{5120\,\pi^8\,\text{Sin}[\beta]^4}{(2\pi-\theta)^2}+\frac{3072\,\pi^7\,\text{Sin}[\beta]^4}{2\pi-\theta}+\frac{1024\,\pi^7\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^5}{(2\pi-\theta)^2}-\frac{512\,\pi^6\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^5}{2\pi-\theta}+\frac{512\,\pi^6\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^5}{4\pi-\theta}\right)}\right)\Bigg/\left(\pi^2\sqrt{-256\,\pi\,\theta^3+64\,\theta^4+4096\,\pi^4\,\text{Sin}[\beta]^2}\right)$$

, {θ,
-2 π,
2 π}, {β,
-π / 2,
π / 2}]

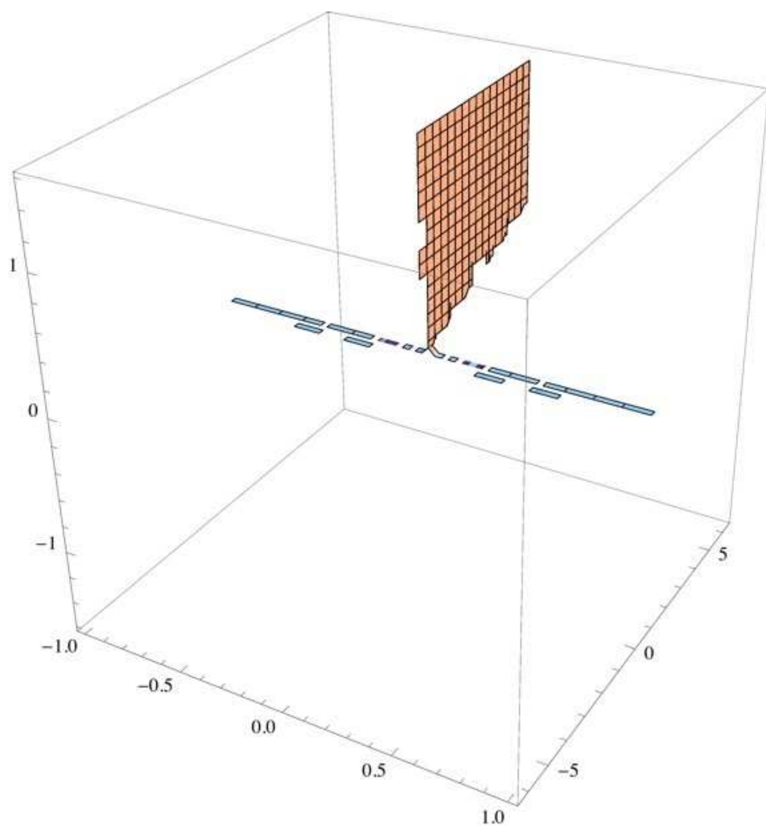


SphericalPlot3D[

$$
\begin{aligned}
&\Bigg(\text{Csc}[\beta]\sqrt{\Bigg(192\,\pi^6+\frac{256\,\pi^8}{(2\,\pi-\theta)^2}-\frac{512\,\pi^7}{2\,\pi-\theta}+64\,\pi^5\,\theta+16\,\pi^4\,\theta^2-4\,\pi^2\,\theta^4-4\,\pi\,\theta^5+\theta^6-320\,\pi^5\sqrt{(4\,\pi-\theta)\,\theta}}\\
&\quad \text{Sin}[\beta]-\frac{512\,\pi^7\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]}{(2\,\pi-\theta)^2}+\frac{896\,\pi^6\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]}{2\,\pi-\theta}-\\
&\quad 96\,\pi^4\,\theta\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]-16\,\pi^3\,\theta^2\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]+8\,\pi^2\,\theta^3\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]+\\
&\quad 4\,\pi\,\theta^4\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]+448\,\pi^6\,\text{Sin}[\beta]^2+\frac{256\,\pi^8\,\text{Sin}[\beta]^2}{(2\,\pi-\theta)^2}-\frac{1024\,\pi^7\,\text{Sin}[\beta]^2}{2\,\pi-\theta}+\\
&\quad 192\,\pi^5\,\theta\,\text{Sin}[\beta]^2+80\,\pi^4\,\theta^2\,\text{Sin}[\beta]^2-32\,\pi^3\,\theta^3\,\text{Sin}[\beta]^2-4\,\pi^2\,\theta^4\,\text{Sin}[\beta]^2+\\
&\quad 128\,\pi^5\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3+\frac{1792\,\pi^7\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{(2\,\pi-\theta)^2}-\frac{1152\,\pi^6\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3}{2\,\pi-\theta}-\\
&\quad 32\,\pi^4\,\theta\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^3-256\,\pi^6\,\text{Sin}[\beta]^4-\frac{5120\,\pi^8\,\text{Sin}[\beta]^4}{(2\,\pi-\theta)^2}+\frac{3072\,\pi^7\,\text{Sin}[\beta]^4}{2\,\pi-\theta}+\\
&\quad \frac{1024\,\pi^7\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^5}{(2\,\pi-\theta)^2}-\frac{512\,\pi^6\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^5}{2\,\pi-\theta}+\frac{512\,\pi^6\sqrt{(4\,\pi-\theta)\,\theta}\;\text{Sin}[\beta]^5}{4\,\pi-\theta}\Bigg)\Bigg)\Big/\\
&\left(\pi^2\sqrt{-256\,\pi\,\theta^3+64\,\theta^4+4096\,\pi^4\,\text{Sin}[\beta]^2}\right),\ \{\beta,\,-\pi/2,\\
&\pi/2\},\ \{\theta,\\
&-2\,\pi,\\
&2\,\pi\}\Big]
\end{aligned}
$$


−0.10
0.05
0.00
−0.05
0.00
).15
0.10
0.05
0.00

$$\text{ContourPlot3D}\Big[\Big\{-\left(2\sqrt{2}\,\sqrt{\left(2\pi^4\eta^2\,\text{Sin}[\beta]^2+\frac{\pi^4\eta^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta}+\frac{4\pi^5\eta^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta^2}+\frac{\pi^4\eta^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta}\right)}\right)\Big/\left(\sqrt{-4\pi\theta^3+\theta^4+64\pi^4\,\text{Sin}[\beta]^2}\right),\ \left(2\sqrt{2}\,\sqrt{\left(2\pi^4\eta^2\,\text{Sin}[\beta]^2+\frac{\pi^4\eta^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta}+\frac{4\pi^5\eta^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta^2}+\frac{\pi^4\eta^2\sqrt{(4\pi-\theta)\,\theta}\,\text{Sin}[\beta]^3}{\theta}\right)}\right)\Big/\left(\sqrt{-4\pi\theta^3+\theta^4+64\pi^4\,\text{Sin}[\beta]^2}\right)\Big\},\ \{\eta,-1,1\},\ \{\theta,-2\pi,2\pi\},\ \{\beta,-\pi/2,\pi/2\}\Big]$$



$$\text{Solve}\left[\frac{r\,\theta}{\pi\left(r-\sqrt{\left(r\,\hat{}\,2-\frac{\sqrt{4\pi r^2\theta-r^2\theta^2}}{2\pi}\,\hat{}\,2\right)}\right)}==\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}}+\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}},\ r\right]$$

{}

$$\text{Solve}\left[\frac{r\,\theta}{\pi\left(r-\sqrt{\left(r\verb|^|2-\frac{\sqrt{4\pi r^2\theta-r^2\theta^2}}{2\pi}\verb|^|2\right)}\right)}==\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}}+\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}},\theta\right]$$

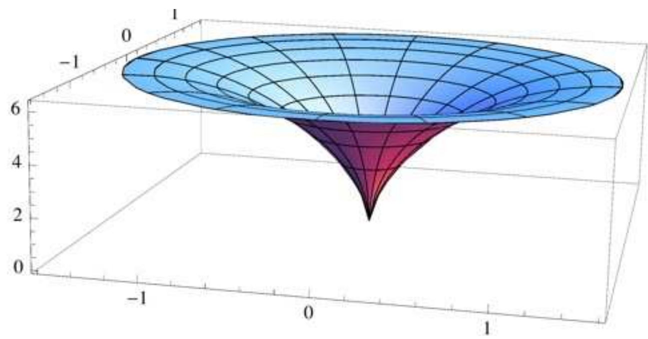
$$\left\{\left\{\theta\to 2\left(\pi-\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right\},\left\{\theta\to 2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)\right\},\right.$$

$$\left\{\theta\to 2\left(\frac{\pi^2\,\text{Sin}[\beta]^2}{3^{1/3}\left(-9\pi^3\,\text{Sin}[\beta]^2+\sqrt{3}\sqrt{27\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}}-\frac{\left(-9\pi^3\,\text{Sin}[\beta]^2+\sqrt{3}\sqrt{27\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}}{3^{2/3}}\right)\right\},$$

$$\left\{\theta\to-\frac{\left(1+i\sqrt{3}\right)\pi^2\,\text{Sin}[\beta]^2}{3^{1/3}\left(-9\pi^3\,\text{Sin}[\beta]^2+\sqrt{3}\sqrt{27\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}}+\frac{\left(1-i\sqrt{3}\right)\left(-9\pi^3\,\text{Sin}[\beta]^2+\sqrt{3}\sqrt{27\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}}{3^{2/3}}\right\},$$

$$\left.\left\{\theta\to-\frac{\left(1-i\sqrt{3}\right)\pi^2\,\text{Sin}[\beta]^2}{3^{1/3}\left(-9\pi^3\,\text{Sin}[\beta]^2+\sqrt{3}\sqrt{27\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}}+\frac{\left(1+i\sqrt{3}\right)\left(-9\pi^3\,\text{Sin}[\beta]^2+\sqrt{3}\sqrt{27\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}}{3^{2/3}}\right\}\right\}$$

$$\texttt{RevolutionPlot3D}\left[2\left(\frac{\pi^2\,\texttt{Sin}[\beta]^2}{3^{1/3}\left(-9\,\pi^3\,\texttt{Sin}[\beta]^2+\sqrt{3}\,\sqrt{27\,\pi^6\,\texttt{Sin}[\beta]^4+\pi^6\,\texttt{Sin}[\beta]^6}\right)^{1/3}}-\frac{\left(-9\,\pi^3\,\texttt{Sin}[\beta]^2+\sqrt{3}\,\sqrt{27\,\pi^6\,\texttt{Sin}[\beta]^4+\pi^6\,\texttt{Sin}[\beta]^6}\right)^{1/3}}{3^{2/3}}\right),\ \{\beta,\ -\pi/2,\ \pi/2\}\right]$$

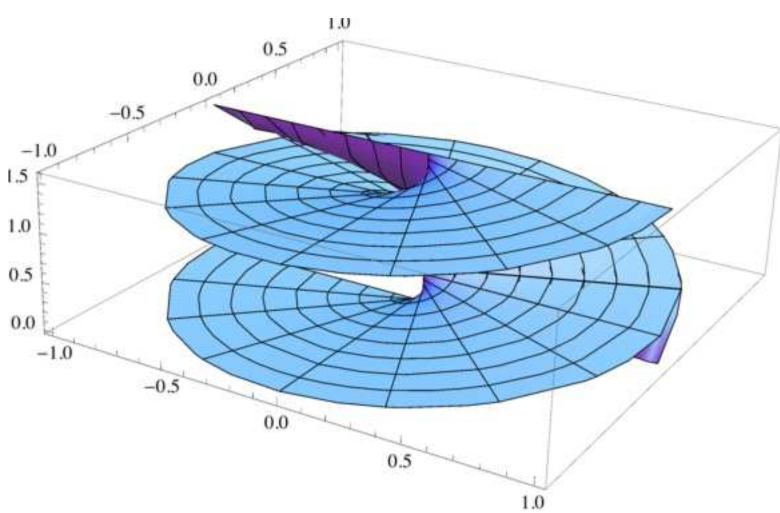


$$\texttt{Solve}\left[\frac{r\,\theta}{\pi\left(r-\sqrt{\left(r\,\hat{}\,2-\frac{\sqrt{4\,\pi\,r^2\,\theta-r^2\,\theta^2}}{2\,\pi}\,\hat{}\,2\right)}\right)}==\frac{2\,\pi\,\texttt{Sin}[\beta]}{\sqrt{4\,\pi\,\theta-\theta^2}}+\frac{2\,\pi\,\texttt{Sin}[\beta]}{\sqrt{4\,\pi\,\theta-\theta^2}},\ \beta\right]$$

Solve::ifun :
Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

$$\left\{\left\{\beta\to\texttt{ArcSin}\left[\frac{r\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}}{2\,\pi\left(2\,\pi\,r-\sqrt{r^2\,(2\,\pi-\theta)^2}\right)}\right]\right\}\right\}$$

$$\texttt{RevolutionPlot3D}\left[\texttt{ArcSin}\left[\frac{r\,\theta\,\sqrt{(4\,\pi-\theta)\,\theta}}{2\,\pi\left(2\,\pi\,r-\sqrt{r^2\,(2\,\pi-\theta)^2}\right)}\right],\ \{r,\ -1,\ 1\},\ \{\theta,\ -2\,\pi,\ 2\,\pi\}\right]$$



For n = 3,

$$\text{Solve}\left[x^\wedge\left(\frac{r\,\theta}{\pi\left(r-\frac{2\,\pi\,r-r\,\theta}{2\,\pi}\right)}+\frac{2\,\pi\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\theta-\theta^2}}\right)+y^\wedge\left(\frac{r\,\theta}{\pi\left(r-\frac{2\,\pi\,r-r\,\theta}{2\,\pi}\right)}+\frac{2\,\pi\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\theta-\theta^2}}\right)==\right.$$

$$\left.z^\wedge\left(\frac{r\,\theta}{\pi\left(r-\frac{2\,\pi\,r-r\,\theta}{2\,\pi}\right)}+\frac{2\,\pi\,\text{Sin}[\beta]}{\sqrt{4\,\pi\,\theta-\theta^2}}\right),\ x\right]$$

Solve::ifun :

Inverse functions are being used by Solve, so some solutions may not be found; use Reduce for complete solution information. ≫

$$\left\{\left\{x\to\left(-y^{\frac{r\,\theta}{\pi\left(r-\frac{2\,\pi\,r-r\,\theta}{2\,\pi}\right)}+\frac{2\,\pi\,\text{Sin}[\beta]}{\sqrt{(4\,\pi-\theta)\,\theta}}}+z^{\frac{r\,\theta}{\pi\left(r-\frac{2\,\pi\,r-r\,\theta}{2\,\pi}\right)}+\frac{2\,\pi\,\text{Sin}[\beta]}{\sqrt{(4\,\pi-\theta)\,\theta}}}\right)^{\frac{1}{2+\frac{2\,\pi\,\text{Sin}[\beta]}{\sqrt{(4\,\pi-\theta)\,\theta}}}}\right\}\right\}$$

$$r:=\left(-4\,\pi\,\theta+\theta^2+2\,\pi\sqrt{(4\,\pi-\theta)\,\theta}\ \text{Sin}[\beta]+4\,\pi^2\,\text{Sin}[\beta]^2-\frac{2\,\pi^2\sqrt{(4\,\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{4\,\pi-\theta}-\right.$$

$$\left.\frac{2\,\pi^2\sqrt{(4\,\pi-\theta)\,\theta}\ \text{Sin}[\beta]^3}{\theta}\right)\Big/\left(16\,\pi^2\,\theta-12\,\pi\,\theta^2+2\,\theta^3-16\,\pi^3\,\text{Sin}[\beta]^2+8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)$$

$$\theta:=2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)$$

$$\textbf{ContourPlot3D}\left[\left(-y^{\frac{r\,\theta}{\pi\left(r-\frac{2\pi r - r\theta}{2\pi}\right)}+\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{(4\pi-\theta)\,\theta}}}+z^{\frac{r\,\theta}{\pi\left(r-\frac{2\pi r - r\theta}{2\pi}\right)}+\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{(4\pi-\theta)\,\theta}}}\right)^{\frac{1}{2+\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{(4\pi-\theta)\,\theta}}}},\right.$$

$$\left.\{y,-5,5\},\{z,-5,5\},\{\beta,-\pi/2,\pi/2\}\right]$$

Power::infy : Infinite expression $\frac{1}{0.}$ encountered. ≫

∞::indet : Indeterminate expression 0. ComplexInfinity encountered. ≫

Power::infy : Infinite expression $\frac{1}{0.}$ encountered. ≫

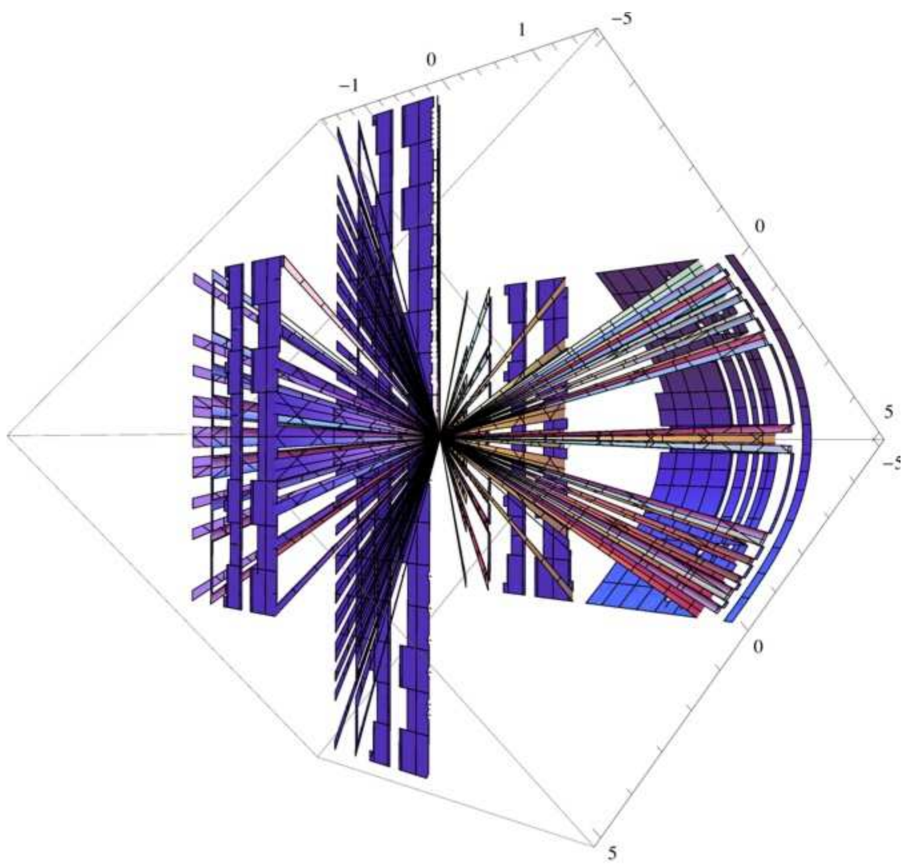
∞::indet : Indeterminate expression 0. ComplexInfinity encountered. ≫

Power::infy : Infinite expression $\frac{1}{0.}$ encountered. ≫

General::stop : Further output of Power::infy will be suppressed during this calculation. ≫

∞::indet : Indeterminate expression 0. ComplexInfinity encountered. ≫

General::stop : Further output of ∞::indet will be suppressed during this calculation. ≫



$$\beta := \textbf{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\,\theta}}{2\pi}\right]$$

$$\texttt{ContourPlot3D}\left[\left(-y^{\frac{r\theta}{\pi\left(r-\frac{2\pi r-r\theta}{2\pi}\right)}+\frac{2\pi \operatorname{Sin}[\beta]}{\sqrt{(4\pi-\theta)\theta}}}+z^{\frac{r\theta}{\pi\left(r-\frac{2\pi r-r\theta}{2\pi}\right)}+\frac{2\pi \operatorname{Sin}[\beta]}{\sqrt{(4\pi-\theta)\theta}}}\right)^{\frac{1}{2+\frac{2\pi \operatorname{Sin}[\beta]}{\sqrt{(4\pi-\theta)\theta}}}},\right.$$

$$\left.\{y,-5,5\},\{z,-5,5\},\{\theta,-2\pi,2\pi\}\right]$$

Power::infy : Infinite expression $\frac{1}{0.+0.i}$ encountered. ≫

∞::indet : Indeterminate expression $(0.+0.i)$ ComplexInfinity encountered. ≫

Power::infy : Infinite expression $\frac{1}{0.+0.i}$ encountered. ≫

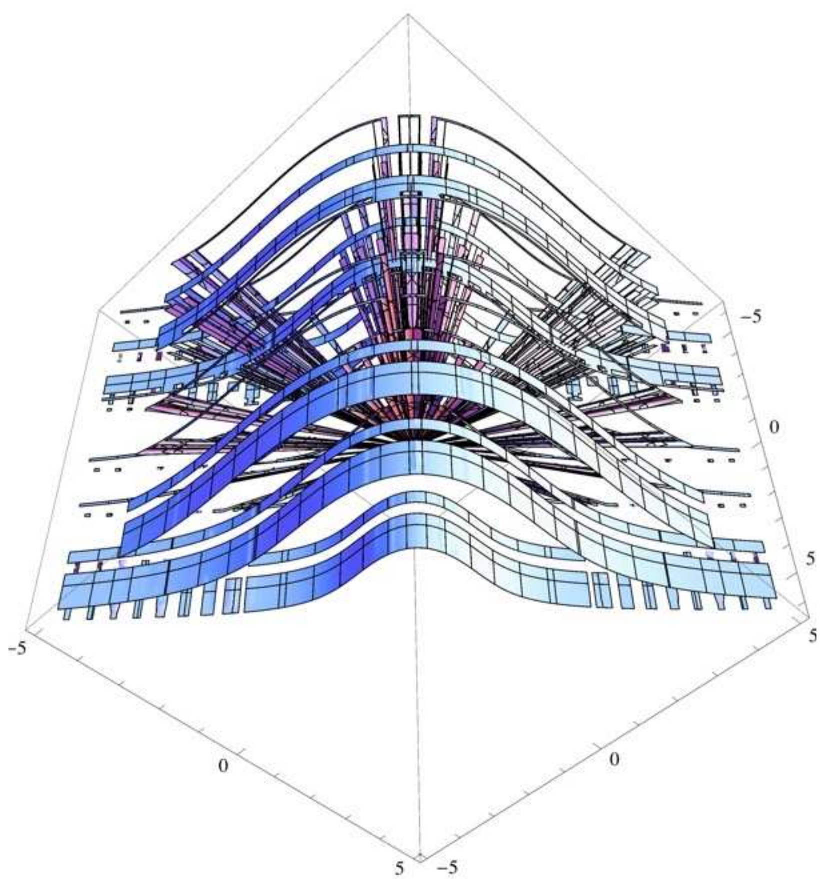
∞::indet : Indeterminate expression $(0.+0.i)$ ComplexInfinity encountered. ≫

Power::infy : Infinite expression $\frac{1}{0.+0.i}$ encountered. ≫

General::stop : Further output of Power::infy will be suppressed during this calculation. ≫

∞::indet : Indeterminate expression $(0.+0.i)$ ComplexInfinity encountered. ≫

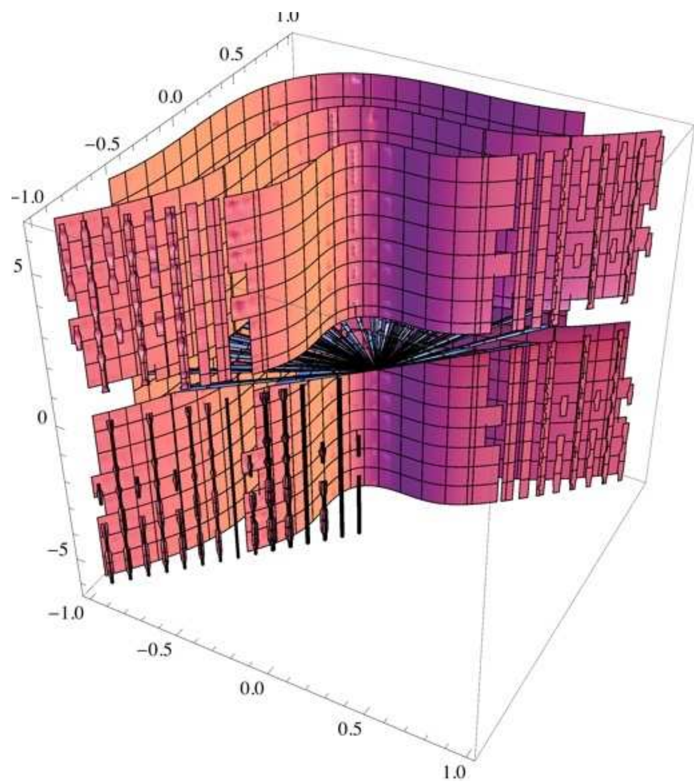
General::stop : Further output of ∞::indet will be suppressed during this calculation. ≫



$$\text{Solve}\left[x^{\wedge}\left(\frac{r\,\theta}{\pi\left(r-\frac{2\pi r-r\theta}{2\pi}\right)}+\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^{2}}}\right)+y^{\wedge}\left(\frac{r\,\theta}{\pi\left(r-\frac{2\pi r-r\theta}{2\pi}\right)}+\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^{2}}}\right)==z^{\wedge}\left(\frac{r\,\theta}{\pi\left(r-\frac{2\pi r-r\theta}{2\pi}\right)}+\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^{2}}}\right),\ y\right]$$

$$\left\{\left\{y\to\left(-x^{\frac{r\theta}{\pi\left(r-\frac{2\pi r-r\theta}{2\pi}\right)}+\frac{2\pi\text{Sin}[\beta]}{\sqrt{(4\pi-\theta)\theta}}}+z^{\frac{r\theta}{\pi\left(r-\frac{2\pi r-r\theta}{2\pi}\right)}+\frac{2\pi\text{Sin}[\beta]}{\sqrt{(4\pi-\theta)\theta}}}\right)^{\frac{1}{2+\frac{2\pi\text{Sin}[\beta]}{\sqrt{(4\pi-\theta)\theta}}}}\right\}\right\}$$

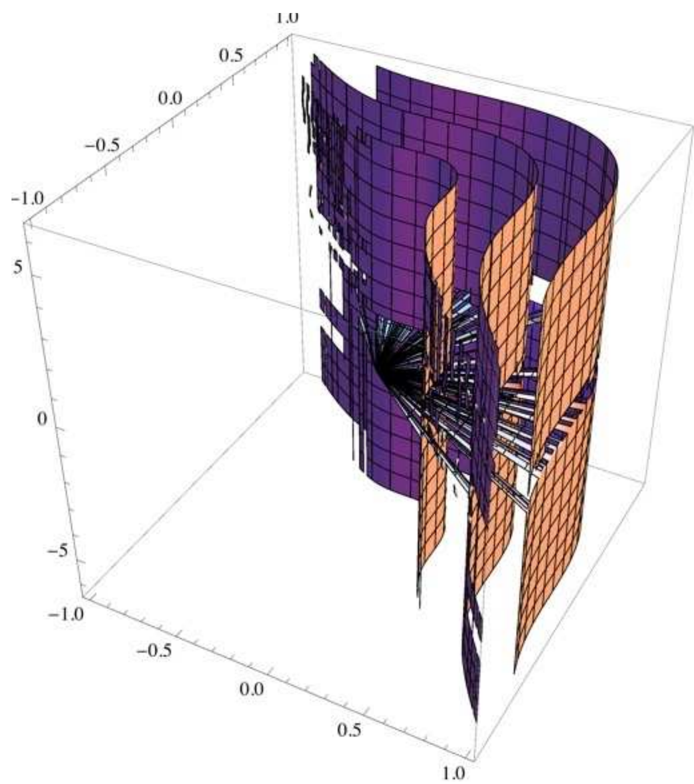
$$\text{ContourPlot3D}\left[\left(-x^{\frac{r\theta}{\pi\left(r-\frac{2\pi r-r\theta}{2\pi}\right)}+\frac{2\pi\text{Sin}[\beta]}{\sqrt{(4\pi-\theta)\theta}}}+z^{\frac{r\theta}{\pi\left(r-\frac{2\pi r-r\theta}{2\pi}\right)}+\frac{2\pi\text{Sin}[\beta]}{\sqrt{(4\pi-\theta)\theta}}}\right)^{\frac{1}{2+\frac{2\pi\text{Sin}[\beta]}{\sqrt{(4\pi-\theta)\theta}}}},\ \{x,-1,1\},\{z,-1,1\},\{\theta,-2\pi,2\pi\}\right]$$



$$\text{Solve}\left[x^{\wedge}\left(\frac{r\,\theta}{\pi\left(r-\frac{2\pi r-r\theta}{2\pi}\right)}+\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}}\right)+y^{\wedge}\left(\frac{r\,\theta}{\pi\left(r-\frac{2\pi r-r\theta}{2\pi}\right)}+\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}}\right)==z^{\wedge}\left(\frac{r\,\theta}{\pi\left(r-\frac{2\pi r-r\theta}{2\pi}\right)}+\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{4\pi\theta-\theta^2}}\right),z\right]$$

$$\left\{\left\{z\to\left(x^{\frac{r\,\theta}{\pi\left(r-\frac{2\pi r-r\theta}{2\pi}\right)}+\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{(4\pi-\theta)\,\theta}}}+y^{\frac{r\,\theta}{\pi\left(r-\frac{2\pi r-r\theta}{2\pi}\right)}+\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{(4\pi-\theta)\,\theta}}}\right)^{\frac{1}{2+\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{(4\pi-\theta)\,\theta}}}}\right\}\right\}$$

$$\text{ContourPlot3D}\left[\left(x^{\frac{r\,\theta}{\pi\left(r-\frac{2\pi r-r\theta}{2\pi}\right)}+\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{(4\pi-\theta)\,\theta}}}+y^{\frac{r\,\theta}{\pi\left(r-\frac{2\pi r-r\theta}{2\pi}\right)}+\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{(4\pi-\theta)\,\theta}}}\right)^{\frac{1}{2+\frac{2\pi\,\text{Sin}[\beta]}{\sqrt{(4\pi-\theta)\,\theta}}}},\{x,-1,1\},\{y,-1,1\},\{\theta,-2\pi,2\pi\}\right]$$



Form and Formality (Gibson - Stillwell - Emmerson Formulation of Fermat's Last Theorem)

It is important to note that it is an area of the retina that is affected by radiant energy, not just a single point. Gibson's reason for believing that the geometry of transformations is important to visual perception is that, "transformations are usually represented on a plane, however, whereas the retinal image is a projection on a curved surface," (The Visual World, 153) **(Gibson, James, J.. The Perception of the Visual World. Cambridge, Mass.: The Riverside Press, 1950. Print. (All further references to this source will be cited parenthetically in the text)**.), and that, "the actual retinal image on a curved surface is related to the hypothetical image on a picture-plane only by such a non-rigid transformation" (The Visual World, 153). A transformation in geometry is the mapping of one point onto another. Isometries, "are defined as the transformations that preserve distance" (The Four Pillars of Geometry, 145) **(Stillwell, John. The Four Pillars of Geometry (Undergraduate Texts in Mathematics). 1 ed. New York: Springer, 2005. Print.)**. In essence, the distance of the initial radius is preserved through the transformation of a circle into a cone so long as the height is orthogonal to the base of the cone and the initial radius is always the slant of the cone. Next, we see the diagram to which Gibson was referring when considering the notion of a transformation onto a picture plane.

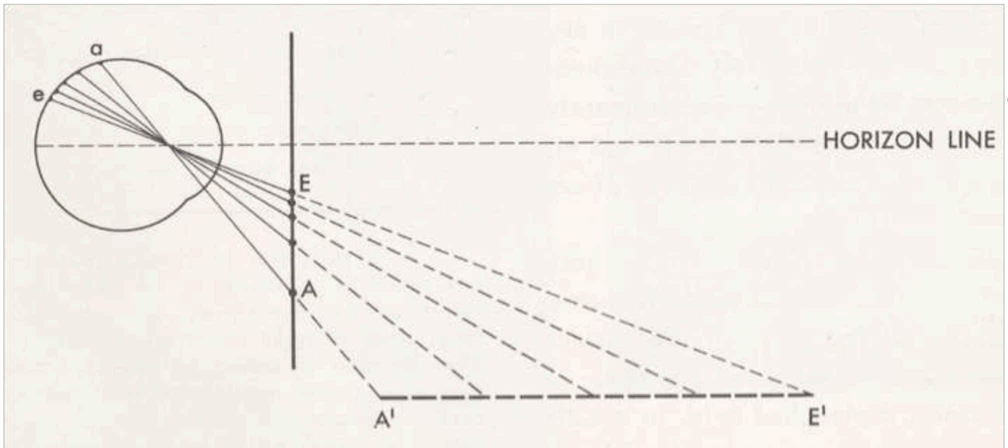


FIGURE 28. A Picture-Gradient, its Retinal Gradient, and the Longitudinal Surface Corresponding to Both

(The Visual World, 79).

In being preserved, the initial radius is considered an invariant. Stillwell comments about the picture plane that, "the line from (-1, 1) to (n, 0) crosses the y-axis at y = n/(n+1)" (The Four Pillars of Geometry, 91). This supposes that the eye is approximated like a point and that it is at the position of (-1, 1) in the Cartesian coordinate system. In the "coordinate system" described by The Geometric Pattern of Perception Theorems (Emmerson, 2009), the y-axis in general is described by the height of a cone. In relation to this diagram, in terms of the y intercept, the height of the cone would be changing with respect to both the initial radius (slant of the cone) and the angle taken out of the initial circle (the angle made between the line from the eye to the x-axis changes is a function of solely the angular amount taken out of the initial circle). Further mathematical analysis of optical infinity with relation to the horizon line and geometric system is needed, but perceived difference in circumferences as an arc length will be a useful formula. Gibson says that, "only because light is structured by the substantial environment can it contain information about it" (Ecological Approach, 86). The basic equation for an arc length as a difference in circumferences describes an even surface layout. Thus, for even surfaces, the equation that delivers that surface may be used as a linguistic device (in combination with rotation, or specifying the "adumbration" of the viewed surface) for describing the structuring of the light in the environment relevant to the perception of even surface layout. The expression for "phenomenal velocity" tells us "how" motion in general is essentially structured, thus this includes the motion of light. However, this still needs specific interpretation.

From Stillwell's statement,
y = n / (n + 1). From The Geometric Pattern of Perception (Emmerson, 2009),

$$\eta = \frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi}$$

$$\text{Solve}\left[\left\{\frac{2\,\pi\,r - r\,\theta}{2\,\pi}\text{^}n + \frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi}\text{^}n == r\text{^}n,\ n > 2\right\},\ \{\theta\}\right]$$

$$\text{Solve}\left[\left\{(2\,\pi)^{-n}\,(2\,\pi\,r - r\,\theta)^n + (2\,\pi)^{-n}\left(4\,\pi\,r^2\,\theta - r^2\,\theta^2\right)^{n/2} == r^n,\ n > 2\right\},\ \{\theta\}\right]$$

$$\frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi} = n\,/\,(n+1) \qquad (59)$$

$$\text{Solve}\left[\frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi} == n\,/\,(n+1),\ n\right]$$

$$\left\{\left\{n \to \frac{\sqrt{r^2\,(4\,\pi - \theta)\,\theta}}{2\,\pi - \sqrt{r^2\,(4\,\pi - \theta)\,\theta}}\right\}\right\} \qquad (60)$$

$$\text{Solve}\left[2 == \frac{\sqrt{r^2\,(4\,\pi - \theta)\,\theta}}{2\,\pi - \sqrt{r^2\,(4\,\pi - \theta)\,\theta}},\ r\right]$$

$$\left\{\left\{r \to -\frac{4\,\pi}{3\,\sqrt{4\,\pi\,\theta - \theta^2}}\right\},\ \left\{r \to \frac{4\,\pi}{3\,\sqrt{4\,\pi\,\theta - \theta^2}}\right\}\right\}$$

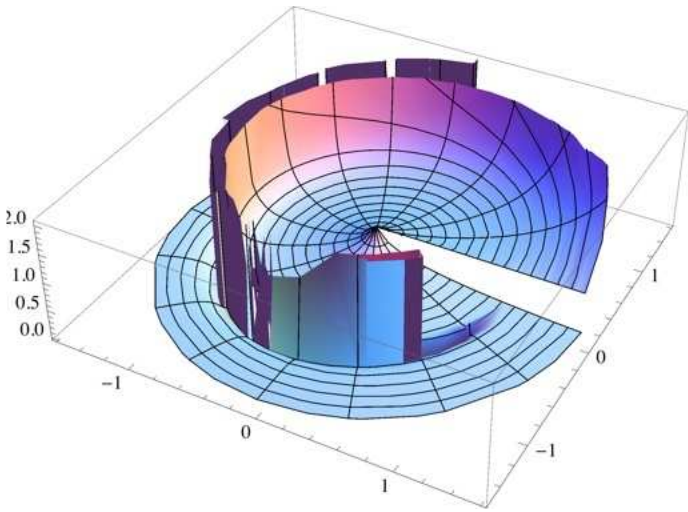
Therefore, $r > \dfrac{4\,\pi}{3\,\sqrt{4\,\pi\,\theta - \theta^2}}$, **and** $\theta > \dfrac{2\,\pi\left(3\,r^2 + \sqrt{-4\,r^2 + 9\,r^4}\right)}{3\,r^2}$

x^n + y^n =

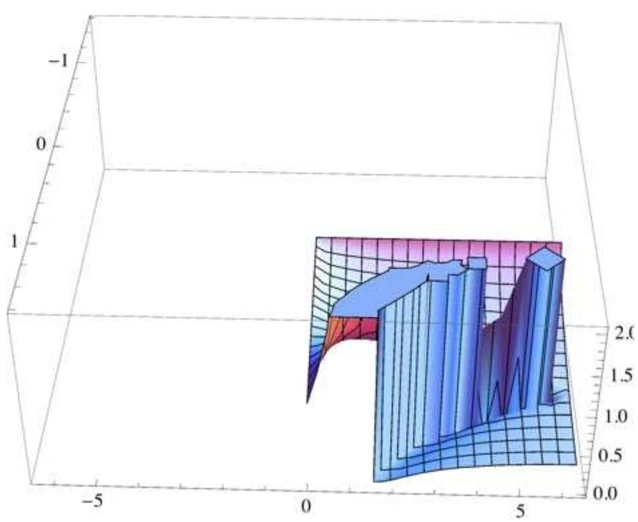
$$z\text{^}n = \frac{2\,\pi\,r - r\,\theta}{2\,\pi}\text{^}n + \frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi}\text{^}n == r\text{^}n = r\text{^}\left(\frac{\sqrt{r^2\,(4\,\pi - \theta)\,\theta}}{2\,\pi - \sqrt{r^2\,(4\,\pi - \theta)\,\theta}}\right) \qquad (61)$$

$$r := \left(-4\,\pi\,\theta + \theta^2 + 2\,\pi\,\sqrt{(4\,\pi - \theta)\,\theta}\ \text{Sin}[\beta] + 4\,\pi^2\,\text{Sin}[\beta]^2 - \frac{2\,\pi^2\,\sqrt{(4\,\pi - \theta)\,\theta}\ \text{Sin}[\beta]^3}{4\,\pi - \theta} - \frac{2\,\pi^2\,\sqrt{(4\,\pi - \theta)\,\theta}\ \text{Sin}[\beta]^3}{\theta}\right) \Big/ \left(16\,\pi^2\,\theta - 12\,\pi\,\theta^2 + 2\,\theta^3 - 16\,\pi^3\,\text{Sin}[\beta]^2 + 8\,\pi^2\,\theta\,\text{Sin}[\beta]^2\right)$$

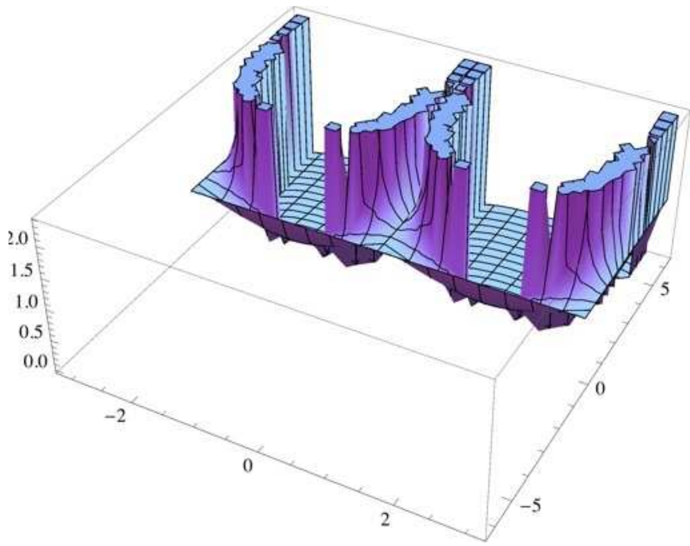
$$\text{RevolutionPlot3D}\left[r\wedge\frac{\sqrt{r^2\,(4\,\pi-\theta)\,\theta}}{2\,\pi-\sqrt{r^2\,(4\,\pi-\theta)\,\theta}},\ \{r,\ -\pi/2,\ \pi/2\},\ \{\theta,\ -2\,\pi,\ 2\,\pi\}\right]$$



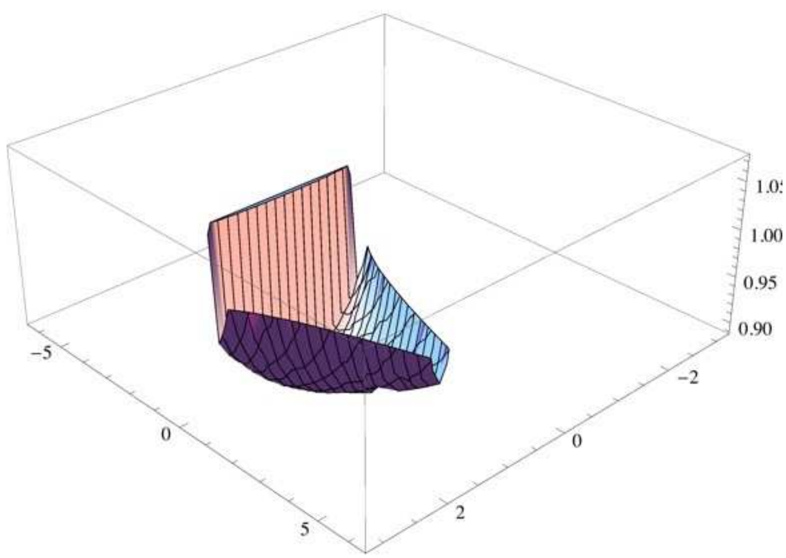
$$\text{Plot3D}\left[r\wedge\frac{\sqrt{r^2\,(4\,\pi-\theta)\,\theta}}{2\,\pi-\sqrt{r^2\,(4\,\pi-\theta)\,\theta}},\ \{r,\ -\pi/2,\ \pi/2\},\ \{\theta,\ -2\,\pi,\ 2\,\pi\}\right]$$



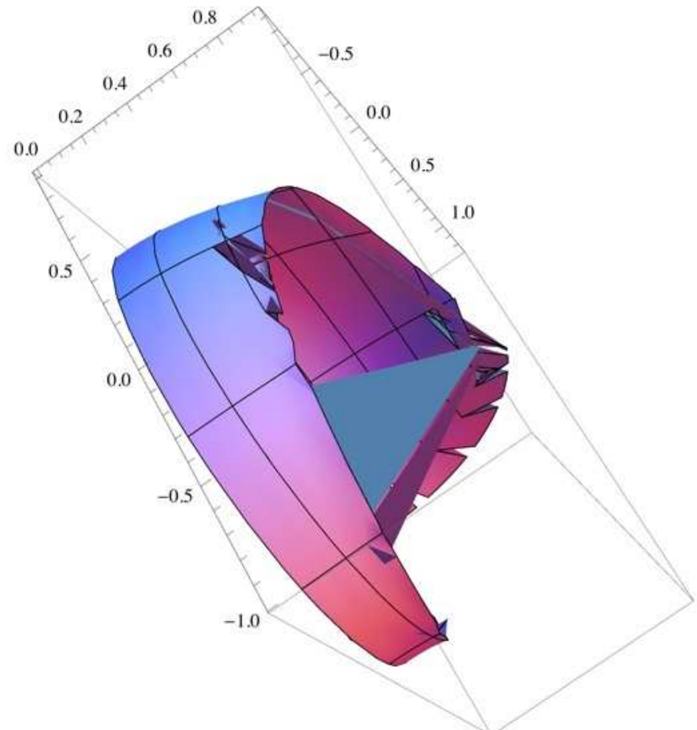
$$\text{Plot3D}\left[r^{\wedge}\frac{\sqrt{r^2\ (4\,\pi-\theta)\ \theta}}{2\,\pi-\sqrt{r^2\ (4\,\pi-\theta)\ \theta}},\ \{\beta,\ -\pi,\ \pi\},\ \{r,\ -2\,\pi,\ 2\,\pi\}\right]$$

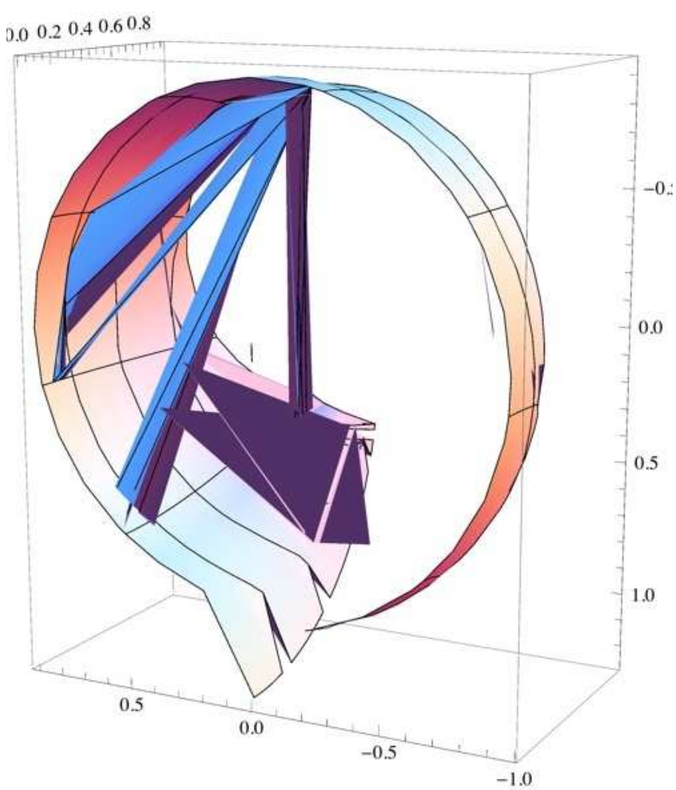


$$\text{Plot3D}\left[r^{\wedge}\frac{\sqrt{r^2\ (4\,\pi-\theta)\ \theta}}{2\,\pi-\sqrt{r^2\ (4\,\pi-\theta)\ \theta}},\ \{\beta,\ -\pi,\ \pi\},\ \{\theta,\ -2\,\pi,\ 2\,\pi\}\right]$$

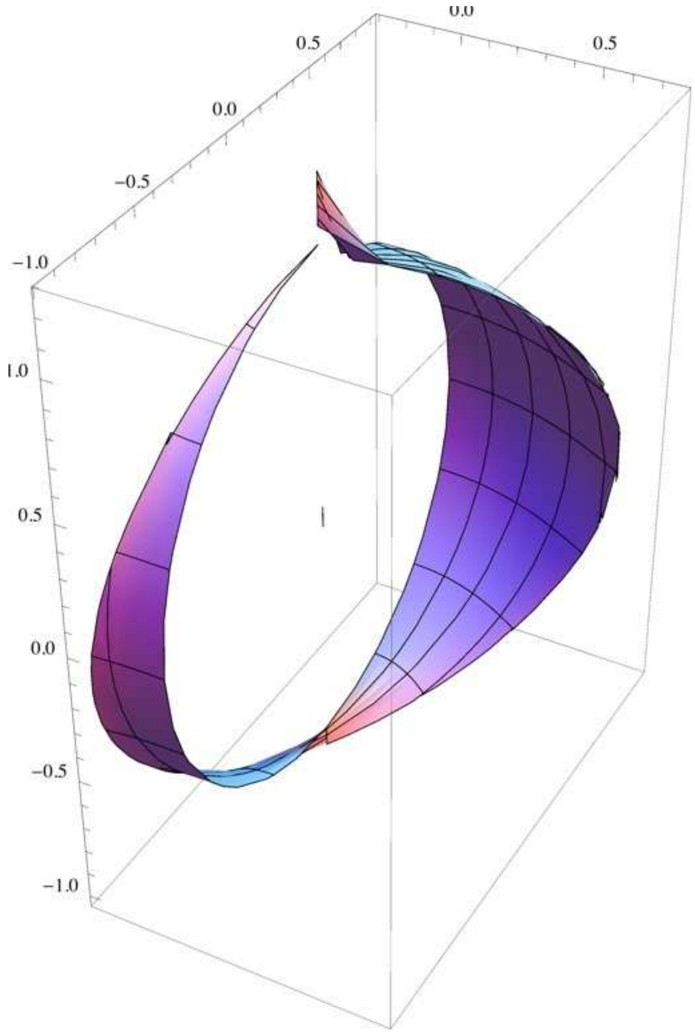


$$\text{SphericalPlot3D}\left[r\,\hat{}\,\frac{\sqrt{r^2\,(4\,\pi-\theta)\,\theta}}{2\,\pi-\sqrt{r^2\,(4\,\pi-\theta)\,\theta}},\ \{\beta,\ -\pi/2,\ \pi/2\},\ \{\theta,\ -2\,\pi,\ 2\,\pi\}\right]$$

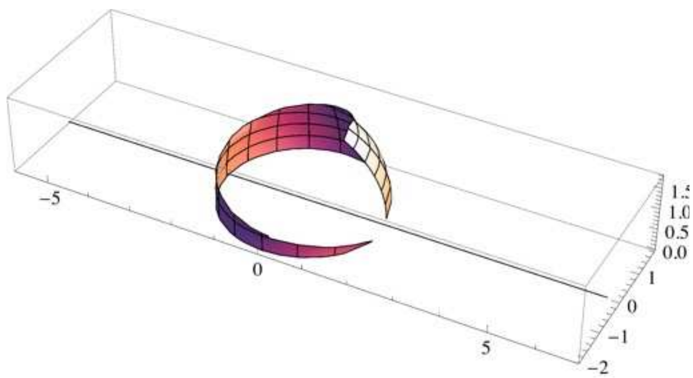




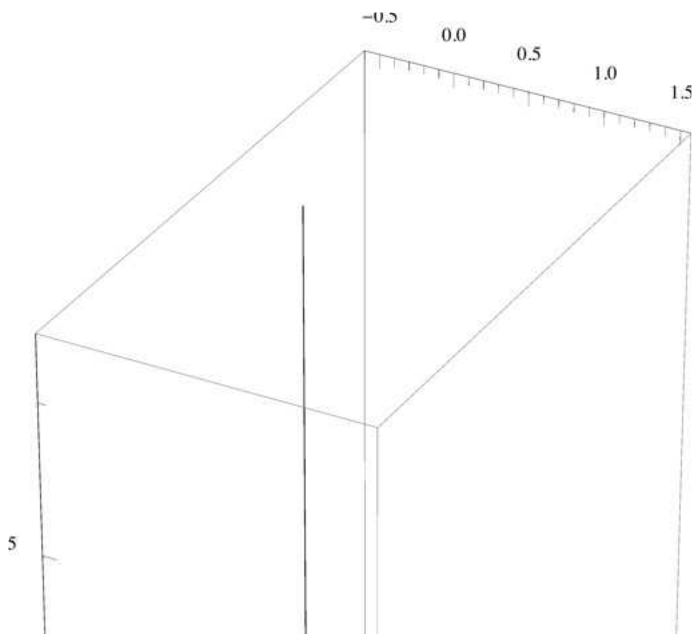
$$\texttt{SphericalPlot3D}\left[r\,\hat{}\,\frac{\sqrt{r^2\,(4\,\pi-\theta)\,\theta}}{2\,\pi-\sqrt{r^2\,(4\,\pi-\theta)\,\theta}},\ \{\theta,\ -2\,\pi,\ 2\,\pi\},\ \{\beta,\ -\pi/2,\ \pi/2\}\right]$$

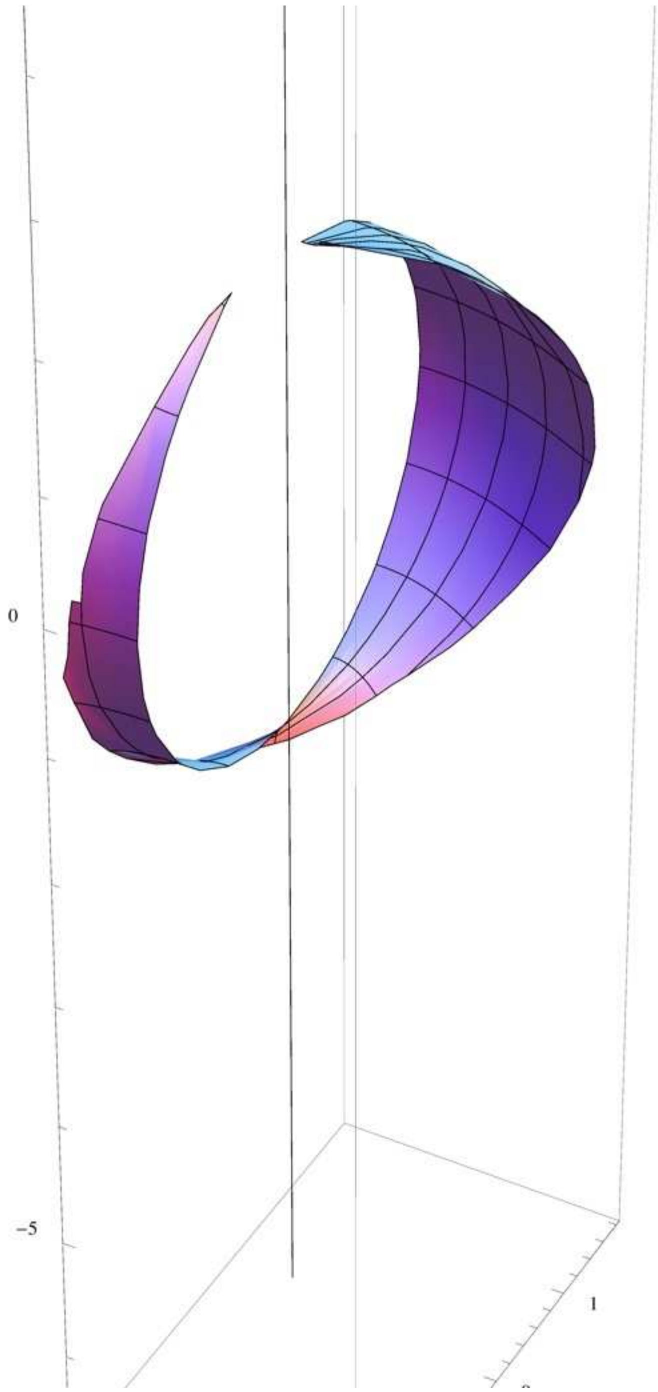


$$\text{SphericalPlot3D}\left[\frac{2\,\pi\,r - r\,\theta}{2\,\pi}{}^{\wedge}\frac{\sqrt{r^2\,(4\,\pi - \theta)\,\theta}}{2\,\pi - \sqrt{r^2\,(4\,\pi - \theta)\,\theta}} + \frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi}{}^{\wedge}\frac{\sqrt{r^2\,(4\,\pi - \theta)\,\theta}}{2\,\pi - \sqrt{r^2\,(4\,\pi - \theta)\,\theta}},\right.$$
$$\left.\{\beta,\,-\pi\,/\,2,\,\pi\,/\,2\},\,\{\theta,\,-2\,\pi,\,2\,\pi\}\right]$$



$$\text{SphericalPlot3D}\left[\frac{2\,\pi\,r - r\,\theta}{2\,\pi}{}^{\wedge}\frac{\sqrt{r^2\,(4\,\pi - \theta)\,\theta}}{2\,\pi - \sqrt{r^2\,(4\,\pi - \theta)\,\theta}} + \frac{\sqrt{4\,\pi\,r^2\,\theta - r^2\,\theta^2}}{2\,\pi}{}^{\wedge}\frac{\sqrt{r^2\,(4\,\pi - \theta)\,\theta}}{2\,\pi - \sqrt{r^2\,(4\,\pi - \theta)\,\theta}},\right.$$
$$\left.\{\theta,\,-2\,\pi,\,2\,\pi\},\,\{\beta,\,-\pi\,/\,2,\,\pi\,/\,2\}\right]$$

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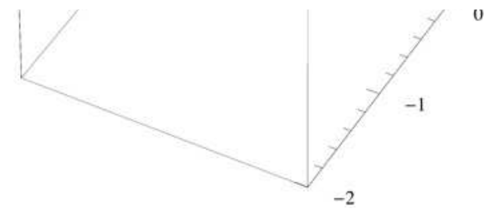
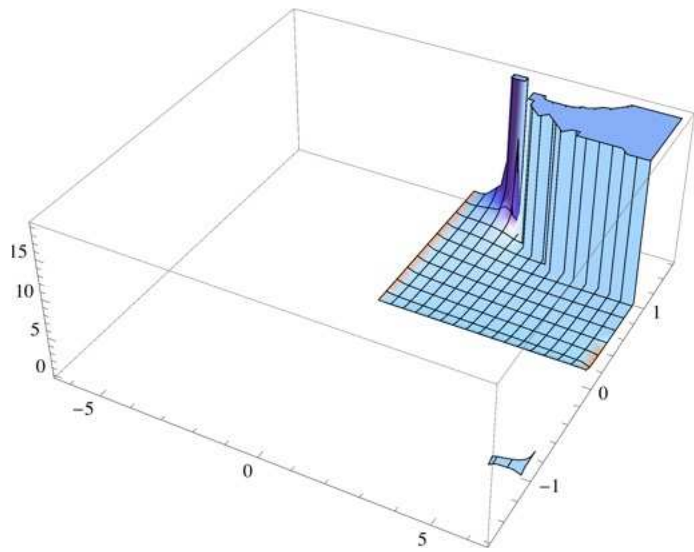
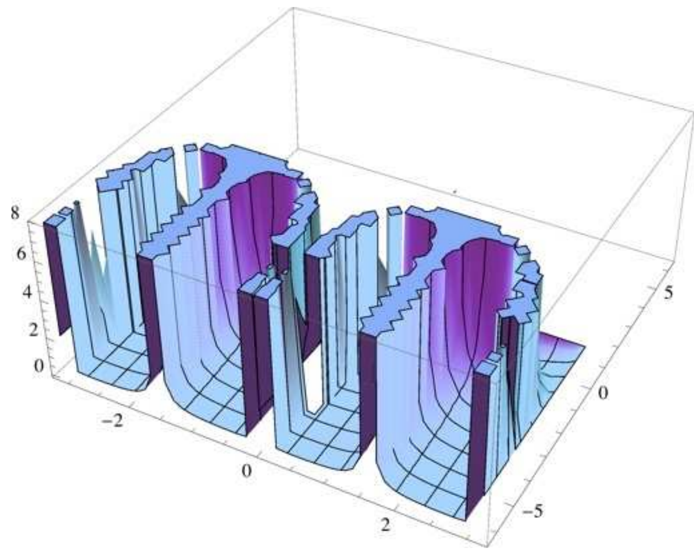


$$\text{Plot3D}\left[\frac{2\pi r - r\theta}{2\pi}\text{^}\frac{\sqrt{r^2 (4\pi-\theta)\theta}}{2\pi - \sqrt{r^2 (4\pi-\theta)\theta}} + \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}\text{^}\frac{\sqrt{r^2 (4\pi-\theta)\theta}}{2\pi - \sqrt{r^2 (4\pi-\theta)\theta}},\right.$$
$$\left.\{\theta, -2\pi, 2\pi\}, \{r, -\pi/2, \pi/2\}\right]$$

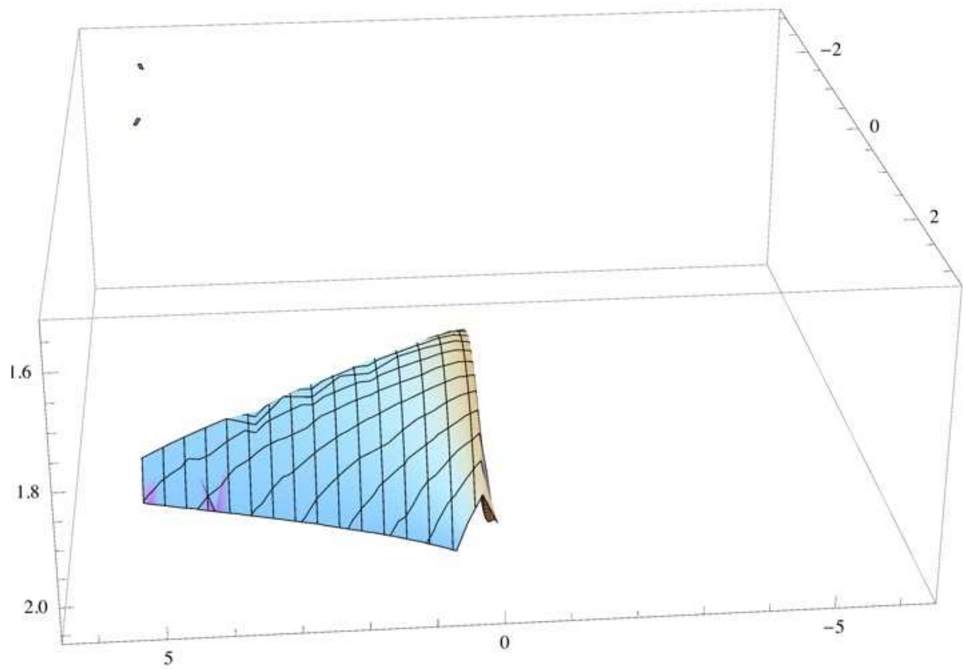


$$\theta := 2\left(\pi + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)$$

$$\text{Plot3D}\left[\frac{2\pi r - r\theta}{2\pi}\,\hat{}\,\frac{\sqrt{r^2\,(4\pi-\theta)\,\theta}}{2\pi-\sqrt{r^2\,(4\pi-\theta)\,\theta}} + \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}\,\hat{}\,\frac{\sqrt{r^2\,(4\pi-\theta)\,\theta}}{2\pi-\sqrt{r^2\,(4\pi-\theta)\,\theta}},\right.$$
$$\left.\{\beta, -\pi, \pi\}, \{r, -2\pi, 2\pi\}\right]$$



$$\text{Plot3D}\left[\frac{2\pi r - r\theta}{2\pi}\,\hat{}\,\frac{\sqrt{r^2\ (4\pi-\theta)\ \theta}}{2\pi-\sqrt{r^2\ (4\pi-\theta)\ \theta}} + \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}\,\hat{}\,\frac{\sqrt{r^2\ (4\pi-\theta)\ \theta}}{2\pi-\sqrt{r^2\ (4\pi-\theta)\ \theta}},\ \{\beta, -\pi, \pi\},\ \{\theta, -2\pi, 2\pi\}\right]$$



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XXII. Revelations of an Infinite Angle

In previous works, I have focused primarily on how the difference in circumferences of two circles equals an arc length of the initial circle. I have expounded in depth upon all of the implicaitons of this system that I have invented so far. I had gone along to show all sorts of different configurations of that system, including expressions of those equations that yielded mysterious results like the solution to the Lorentz coefficient when it should cancel out with itself. In addition, I had shown how there was a paradox in the fundamental geometry of space - time described by that system. I had also shown how to apply that system to complex analysis and also how to find a solution to the radius of the initial circle in terms an angle to the 10 th dimension (note: x^2 has a different meaning than "x squared" i.e. the latter secifically implies a geometric relationship). All of this and more have been the focus of my past 2 - 3 years. I now embark on a modification of the initial system and how this new, modified system connects and interacts with the previous insight about the difference in circumferences of two circles' equaling an arc length of the initial circle.

I ***have now realized*** **that the difference in circumferences of two circles can equal an arc length of either circle.** If that difference equals an arc length of the initial circle, we say, $\theta r == 2\pi r - 2\pi x == 2\pi r - 2\pi\sqrt{(r^2 - \eta^2)}$. On the other hand, if we say that the difference in the circumferences of the two circles equals an arc length of the "changed" circle, then we say : $\gamma x == 2\pi r - 2\pi x == 2\pi r - 2\pi\sqrt{(r^2 - \eta^2)}$. In this paper, I will show how :

1. There is also a paradox in the new expression of the difference in circumferences of two circles.

2. I can solve for the Lorentz coefficient using the system of the difference in circumferences of two circles equaling an arc length of the changed circle in a similar way as that shown in "A Geometric Pattern of Perception (Emmerson, 2009-2012)"

4. The difference in circumferences of two circles added together equals the arc lengths added together.

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1. There is also a paradox in the new expression of the difference in circumferences of two circles, $2\pi r - 2\pi x = \gamma x$.

$$2\pi r - 2\pi x = \gamma x$$

$$2\pi r = \gamma x + 2\pi x$$

$$2\pi r = 2\pi x + \gamma x$$

$$2\pi r - 2\pi x - \gamma x = 0$$

$$2\pi r - 2\pi x - \gamma x == 2\pi r - 2\pi x - \theta r$$

A brief exposition of solutions :

Solve[2 π r - γ x - 2 π x == 0, x]

$$\left\{\left\{x \to \frac{2\pi r}{2\pi + \gamma}\right\}\right\}$$

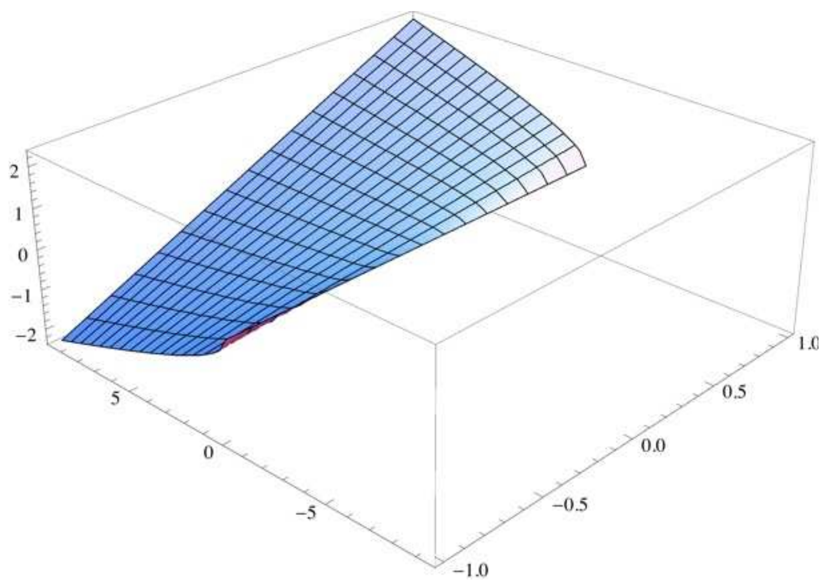
r == √x^2 + h^2

$$r == \sqrt{h^2 + x^2}$$

Solve$\left[2\pi\sqrt{h^2 + x^2} - 2\pi x == \gamma x, h\right]$

$$\left\{\left\{h \to -\frac{x\sqrt{\gamma}\sqrt{4\pi + \gamma}}{2\pi}\right\}, \left\{h \to \frac{x\sqrt{\gamma}\sqrt{4\pi + \gamma}}{2\pi}\right\}\right\}$$

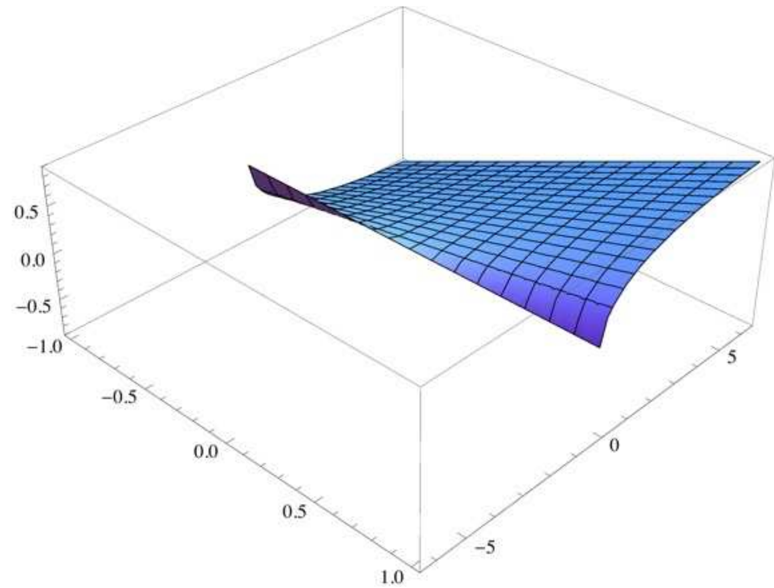
Plot3D$\left[\frac{x\sqrt{\gamma}\sqrt{4\pi + \gamma}}{2\pi}, \{x, -1, 1\}, \{\gamma, -3\pi, 3\pi\}\right]$



Solve$\left[2\pi r - 2\pi\sqrt{r^2 - h^2} == \gamma\sqrt{r^2 - h^2}, h\right]$

$$\left\{\left\{h \to -\frac{r\sqrt{\gamma}\sqrt{4\pi + \gamma}}{\sqrt{4\pi^2 + 4\pi\gamma + \gamma^2}}\right\}, \left\{h \to \frac{r\sqrt{\gamma}\sqrt{4\pi + \gamma}}{\sqrt{4\pi^2 + 4\pi\gamma + \gamma^2}}\right\}\right\}$$

$$\texttt{Plot3D}\left[\frac{r\sqrt{\gamma}\sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}, \{r,-1,1\}, \{\gamma,-2\pi,2\pi\}\right]$$



$$h == r\,\texttt{Sin}[\beta] = \frac{r\sqrt{\gamma}\sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}$$

$$\frac{\sqrt{\gamma}\sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}} == \texttt{Sin}[\beta]$$

$$1 == \frac{\sqrt{\gamma}\sqrt{4\pi+\gamma}}{\texttt{Sin}[\beta]\sqrt{4\pi^2+4\pi\gamma+\gamma^2}} = \frac{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}\,\texttt{Sin}[\beta]}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}$$

- **I want the reader of this paper to be aware that there are different solutions implied by the two equally valid algebraic expressions for the digit one from the geometry of the system.**

- **The solutions are as follows:**

$$\texttt{Solve}\left[1 == \frac{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}\,\texttt{Sin}[\beta]}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}, \beta\right]$$

$$\left\{\left\{\beta \to \texttt{ArcSin}\left[\frac{\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]\right\}\right\}$$

$$\textbf{Solve}\left[1 == \frac{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}\ \text{Sin}[\beta]}{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}, \gamma\right]$$

$$\left\{\left\{\gamma \to \frac{2\left(\pi - \pi\,\text{Sin}[\beta]^2 - \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)}{-1+\text{Sin}[\beta]^2}\right\}, \left\{\gamma \to \frac{2\left(\pi - \pi\,\text{Sin}[\beta]^2 + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)}{-1+\text{Sin}[\beta]^2}\right\}\right\}$$

$$\text{Solve}\left[1 == \frac{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}\ \text{Sin}[\beta]}, \beta\right]$$

$$\left\{\left\{\beta \to \text{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}\right]\right\}\right\}$$

$$\textbf{Solve}\left[1 == \frac{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}\ \text{Sin}[\beta]}, \gamma\right]$$

$$\left\{\left\{\gamma \to \frac{2\left(\pi - \pi\,\text{Csc}[\beta]^2 - \sqrt{-\pi^2\,\text{Csc}[\beta]^2 + \pi^2\,\text{Csc}[\beta]^4}\right)}{-1+\text{Csc}[\beta]^2}\right\},\right.$$
$$\left.\left\{\gamma \to \frac{2\left(\pi - \pi\,\text{Csc}[\beta]^2 + \sqrt{-\pi^2\,\text{Csc}[\beta]^2 + \pi^2\,\text{Csc}[\beta]^4}\right)}{-1+\text{Csc}[\beta]^2}\right\}\right\}$$

- **Later, when we describe the equation 1 - 1 = 0 in its algebraic expressions, this will be important, because we are going to want to see the visualizations for each of the following configurations :**

A.

$$1 - \frac{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}\ \text{Sin}[\beta]} == 2\pi r - 2\pi\sqrt{r^2 - \frac{r\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\text{^}2} - \gamma\sqrt{r^2 - \frac{r\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\text{^}2}$$

Substituting : $\left\{\left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]\right\}\right\}$ and

$\left\{\gamma \to \frac{2\left(\pi - \pi\,\text{Sin}[\beta]^2 + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)}{-1+\text{Sin}[\beta]^2}\right\}$ into the resulting radius solution.

B.

$$1 - \frac{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}\ \text{Sin}[\beta]}{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}} == 2\pi r - 2\pi\sqrt{r^2 - \frac{r\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\text{^}2} - \gamma\sqrt{r^2 - \frac{r\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\text{^}2}$$

Substituting : $\left\{\left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]\right\}\right\}$ and

$\left\{\gamma \to \frac{2\left(\pi - \pi\,\text{Sin}[\beta]^2 + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)}{-1+\text{Sin}[\beta]^2}\right\}$ into the resulting radius solution.

C.

$$1 - \frac{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}\ \text{Sin}[\beta]} == 2\pi r - 2\pi\sqrt{r^2 - \frac{r\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\ \hat{}\ 2} - \gamma\sqrt{r^2 - \frac{r\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\ \hat{}\ 2}$$

Substituting : $\left\{\left\{\beta \to \text{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}\right]\right\}\right\}$ and

$\left\{\gamma \to \frac{2\left(\pi - \pi\,\text{Csc}[\beta]^2 + \sqrt{-\pi^2\,\text{Csc}[\beta]^2 + \pi^2\,\text{Csc}[\beta]^4}\right)}{-1+\text{Csc}[\beta]^2}\right\}$ into the resulting radius solution

D.

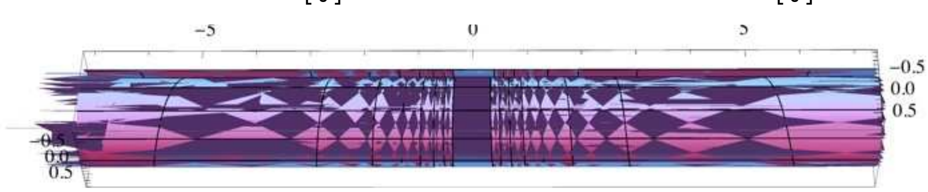
$$1 - \frac{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}\ \text{Sin}[\beta]}{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}} == 2\pi r - 2\pi\sqrt{r^2 - \frac{r\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\ \hat{}\ 2} - \gamma\sqrt{r^2 - \frac{r\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\ \hat{}\ 2}$$

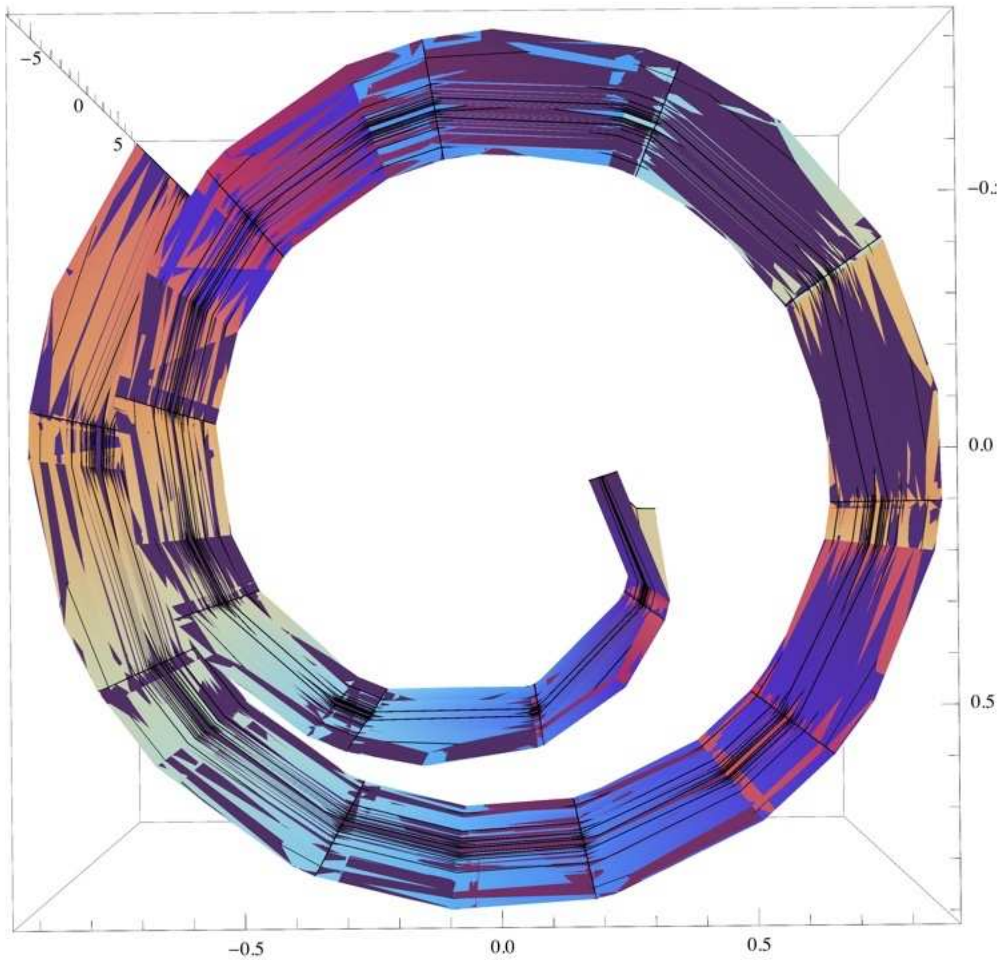
Substituting : $\left\{\left\{\beta \to \text{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}\right]\right\}\right\}$ and

$\left\{\gamma \to \frac{2\left(\pi - \pi\,\text{Csc}[\beta]^2 + \sqrt{-\pi^2\,\text{Csc}[\beta]^2 + \pi^2\,\text{Csc}[\beta]^4}\right)}{-1+\text{Csc}[\beta]^2}\right\}$ into the resulting radius solution

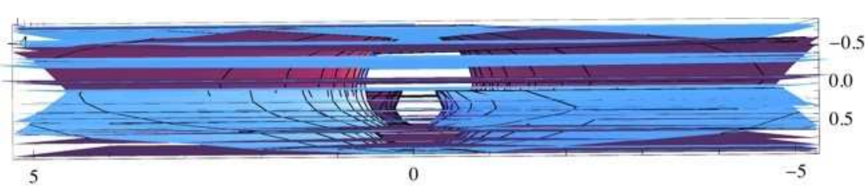
- **First, we will graph a few of the algebraic expressions for the number one.**

$$\textbf{SphericalPlot3D}\left[\frac{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\text{Sin}[\beta]\ \sqrt{4\pi^2+4\pi\gamma+\gamma^2}},\ \{\beta,\ -(3/8)\,\pi,\ (3/8)\,\pi\},\ \left\{\gamma,\ -\frac{2\left(\pi-\pi\,\text{Cos}\left[\frac{\pi}{8}\right]^2-\sqrt{\pi^2-\pi^2\,\text{Cos}\left[\frac{\pi}{8}\right]^2}\right)}{-1+\text{Cos}\left[\frac{\pi}{8}\right]^2},\ \frac{2\left(\pi-\pi\,\text{Cos}\left[\frac{\pi}{8}\right]^2-\sqrt{\pi^2-\pi^2\,\text{Cos}\left[\frac{\pi}{8}\right]^2}\right)}{-1+\text{Cos}\left[\frac{\pi}{8}\right]^2}\right\}\right]$$

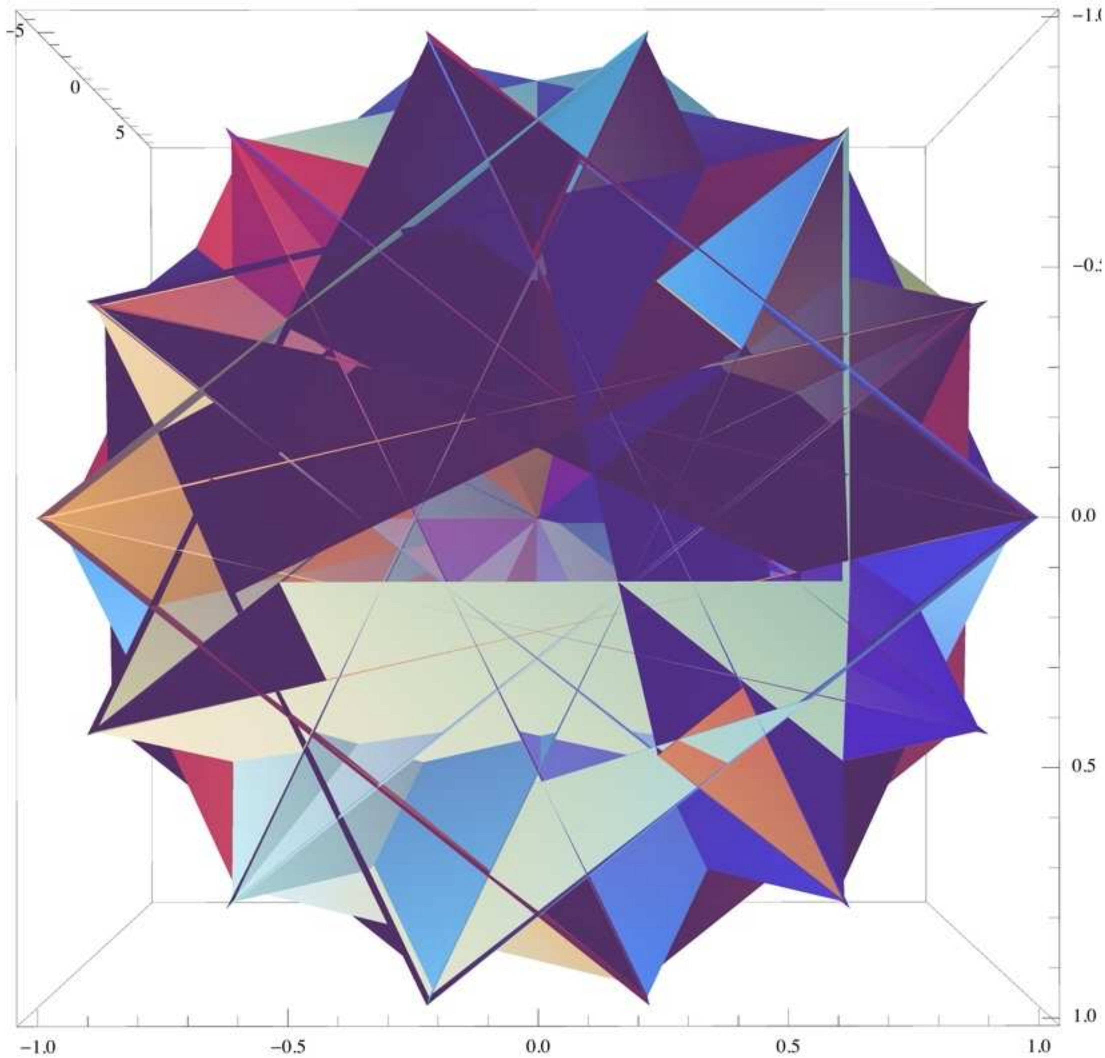


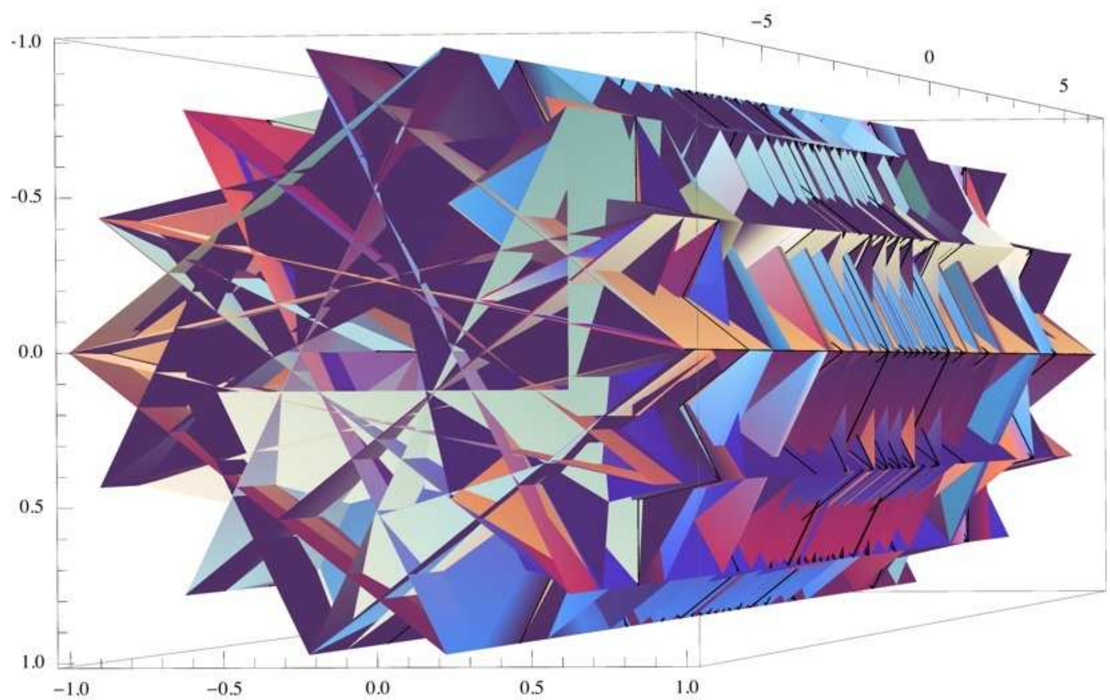


$$\text{SphericalPlot3D}\left[\frac{\sqrt{\gamma}\,\sqrt{4\,\pi+\gamma}}{\text{Sin}[\beta]\,\sqrt{4\,\pi^2+4\,\pi\,\gamma+\gamma^2}},\left\{\gamma,-\frac{2\left(\pi-\pi\,\text{Cos}\left[\frac{\pi}{8}\right]^2-\sqrt{\pi^2-\pi^2\,\text{Cos}\left[\frac{\pi}{8}\right]^2}\right)}{-1+\text{Cos}\left[\frac{\pi}{8}\right]^2},\frac{2\left(\pi-\pi\,\text{Cos}\left[\frac{\pi}{8}\right]^2-\sqrt{\pi^2-\pi^2\,\text{Cos}\left[\frac{\pi}{8}\right]^2}\right)}{-1+\text{Cos}\left[\frac{\pi}{8}\right]^2}\right\},\{\beta,-(3/8)\,\pi,(3/8)\,\pi\}\right]$$

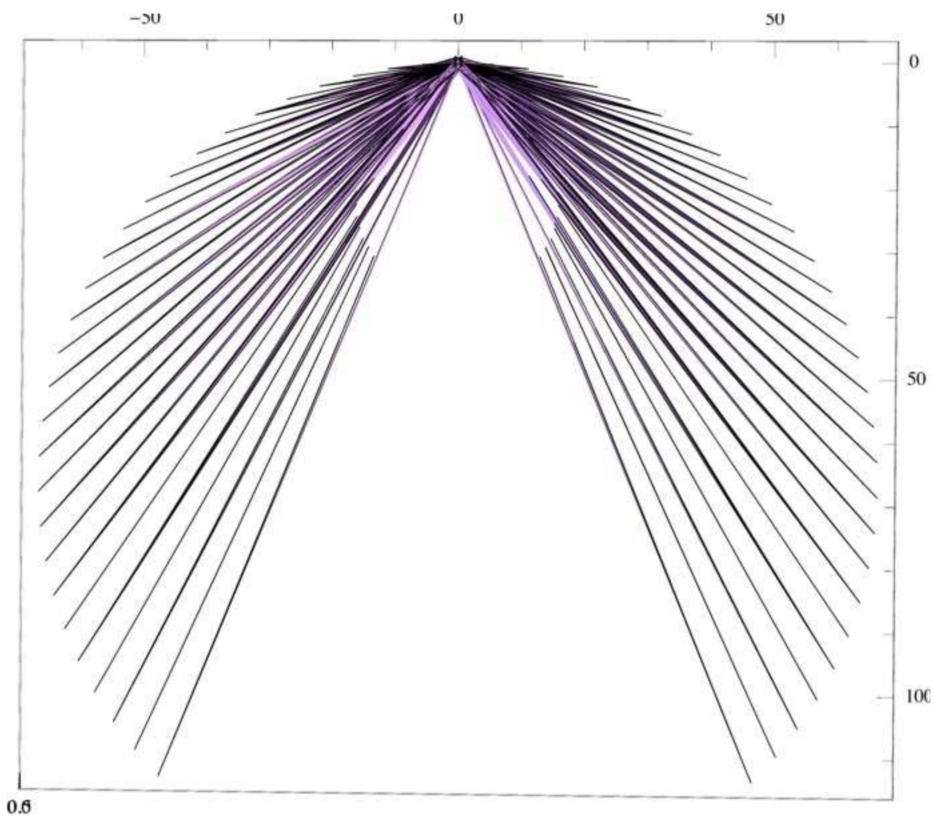


$$\text{SphericalPlot3D}\left[\frac{\sqrt{\gamma}\,\sqrt{4\,\pi+\gamma}}{\text{Sin}[\beta]\,\sqrt{4\,\pi^2+4\,\pi\,\gamma+\gamma^2}},\right.$$
$$\left.\left\{\beta,\,-\text{ArcSin}\left[\frac{16\,\sqrt{65}}{129}\right],\,\text{ArcSin}\left[\frac{16\,\sqrt{65}}{129}\right]\right\},\,\{\gamma,\,-256\,\pi,\,256\,\pi\}\right]$$

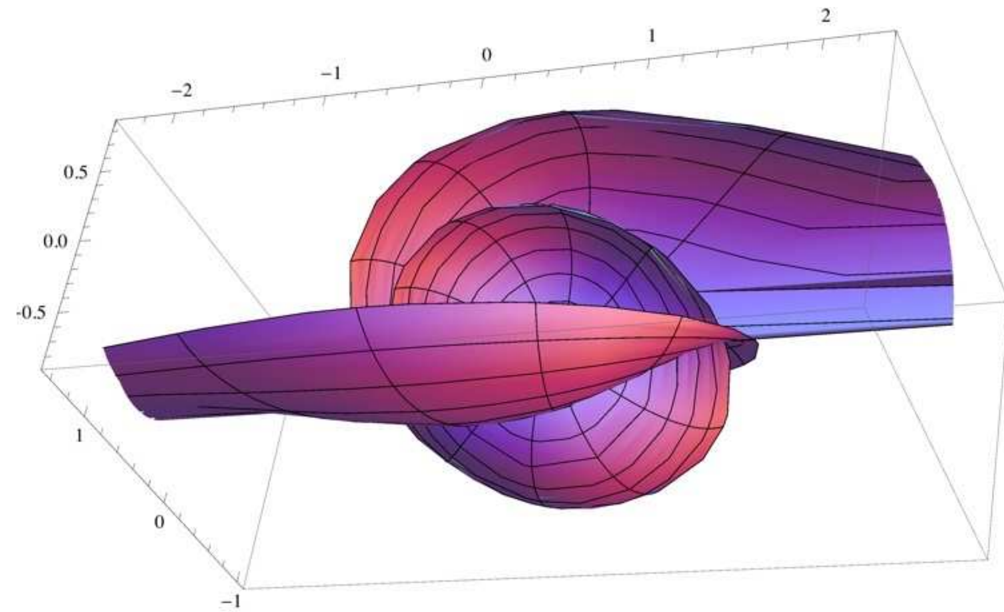



-1.0
-0.5
0.0
0.5
1.0
−1.0
−0.5
0.0
0.5
1.0
−5
0
5

$$\text{SphericalPlot3D}\left[\frac{\sqrt{4\,\pi^2+4\,\pi\,\gamma+\gamma^2}\ \text{Sin}[\beta]}{\sqrt{\gamma}\ \sqrt{4\,\pi+\gamma}},\ \{\beta,\ -(3/8)\,\pi,\ (3/8)\,\pi\},\ \left\{\gamma,\ -\frac{2\left(\pi-\pi\,\text{Cos}\left[\frac{\pi}{8}\right]^2-\sqrt{\pi^2-\pi^2\,\text{Cos}\left[\frac{\pi}{8}\right]^2}\right)}{-1+\text{Cos}\left[\frac{\pi}{8}\right]^2},\ \frac{2\left(\pi-\pi\,\text{Cos}\left[\frac{\pi}{8}\right]^2-\sqrt{\pi^2-\pi^2\,\text{Cos}\left[\frac{\pi}{8}\right]^2}\right)}{-1+\text{Cos}\left[\frac{\pi}{8}\right]^2}\right\}\right]$$



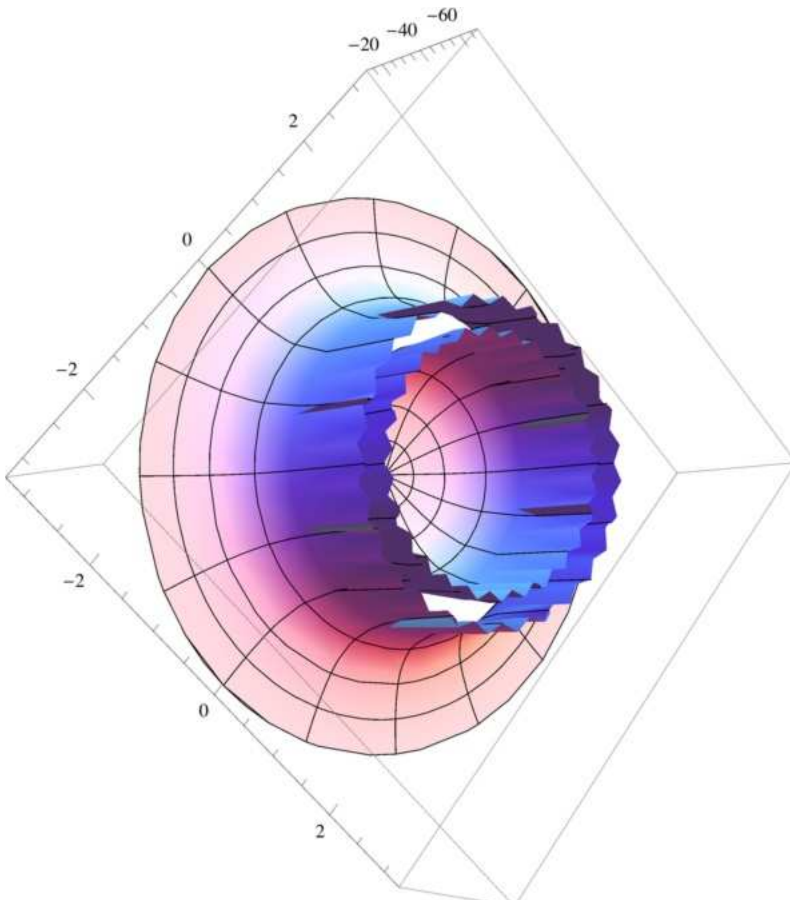
$$\text{SphericalPlot3D}\left[\frac{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}\ \text{Sin}[\beta]}{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}},\ \left\{\gamma,\ -\frac{2\left(\pi-\pi\,\text{Cos}\left[\frac{\pi}{8}\right]^2-\sqrt{\pi^2-\pi^2\,\text{Cos}\left[\frac{\pi}{8}\right]^2}\right)}{-1+\text{Cos}\left[\frac{\pi}{8}\right]^2},\ \frac{2\left(\pi-\pi\,\text{Cos}\left[\frac{\pi}{8}\right]^2-\sqrt{\pi^2-\pi^2\,\text{Cos}\left[\frac{\pi}{8}\right]^2}\right)}{-1+\text{Cos}\left[\frac{\pi}{8}\right]^2}\right\},\ \{\beta,\ -(3/8)\,\pi,\ (3/8)\,\pi\}\right]$$



$$\text{Solve}\left[1==\frac{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}\ \text{Sin}[\beta]}{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}},\ \gamma\right]$$

$$\left\{\left\{\gamma\to\frac{2\left(\pi-\pi\,\text{Sin}[\beta]^2-\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)}{-1+\text{Sin}[\beta]^2}\right\},\ \left\{\gamma\to\frac{2\left(\pi-\pi\,\text{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)}{-1+\text{Sin}[\beta]^2}\right\}\right\}$$

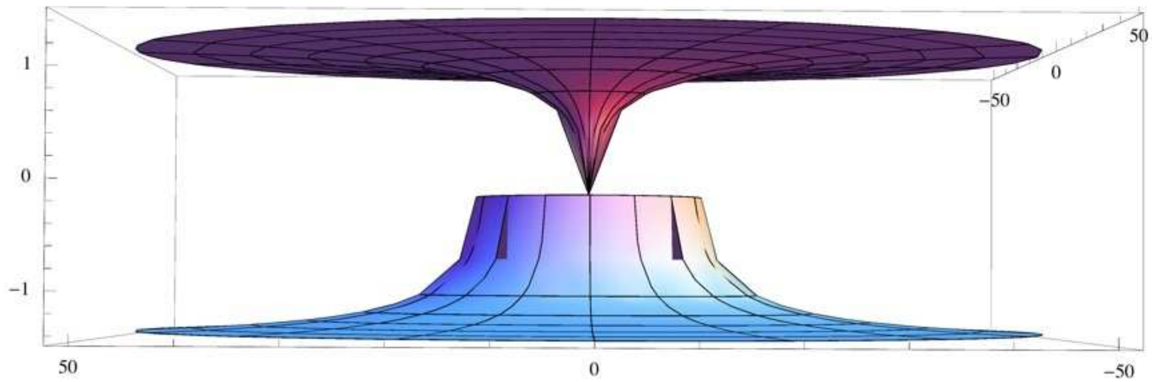
$$\text{RevolutionPlot3D}\left[\frac{2\left(\pi - \pi\,\text{Sin}[\beta]^2 + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)}{-1 + \text{Sin}[\beta]^2}, \{\beta, -\pi, \pi\}\right]$$

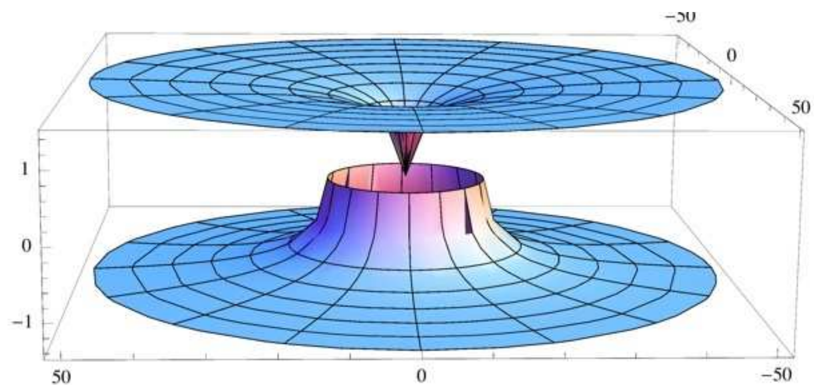


$$\text{Solve}\left[1 == \frac{\sqrt{4\,\pi^2 + 4\,\pi\,\gamma + \gamma^2}\;\text{Sin}[\beta]}{\sqrt{\gamma}\,\sqrt{4\,\pi + \gamma}}, \beta\right]$$

$$\left\{\left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{\gamma}\,\sqrt{(2\,\pi + \gamma)^2}\,\sqrt{4\,\pi + \gamma}}{4\,\pi^2 + 4\,\pi\,\gamma + \gamma^2}\right]\right\}\right\}$$

$$\text{RevolutionPlot3D}\left[\text{ArcSin}\left[\frac{\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right],\ \{\gamma,\ -16\pi,\ 16\pi\}\right]$$





I will now demonstrate how we can solve for the variable γ in terms of θ and then substitute the subsequent solution into our solutions for β in terms of γ.

$$\beta = \text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\ \theta}}{2\pi}\right]$$

$$\text{Solve}\left[1 == \frac{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}\ \text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\ \theta}}{2\pi}\right]\right]}{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}},\ \theta\right]$$

$$\left\{\left\{\theta \to \frac{2\pi\gamma}{2\pi+\gamma}\right\},\ \left\{\theta \to \frac{2\left(4\pi^2+\pi\gamma\right)}{2\pi+\gamma}\right\}\right\}$$

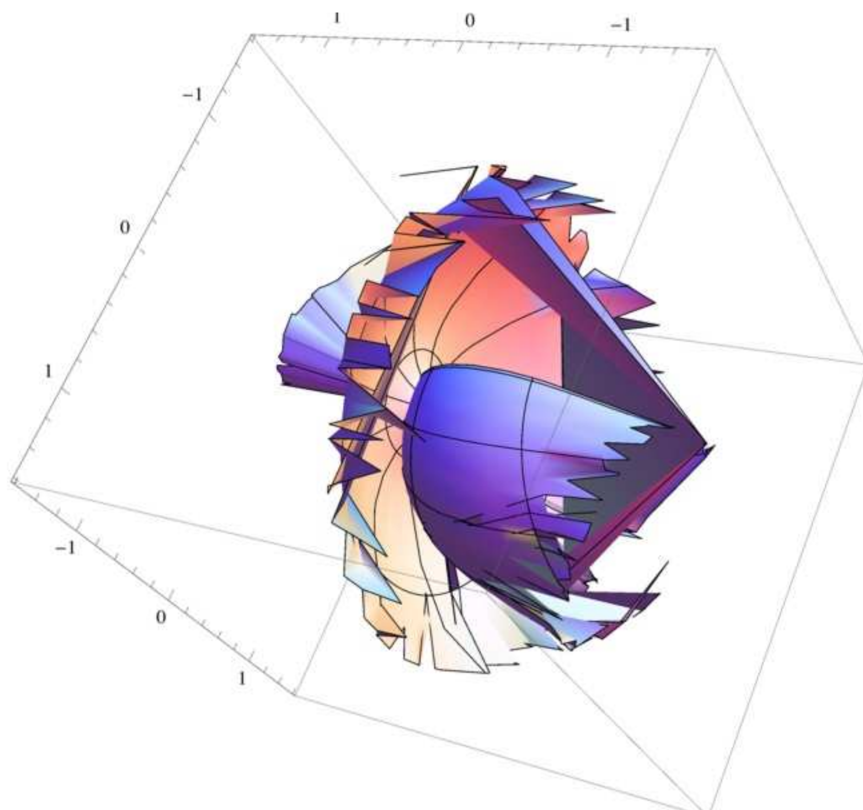
$$\textbf{Solve}\left[1 == \frac{\sqrt{4\pi^2 + 4\pi\gamma + \gamma^2}\ \textbf{Sin}\left[\textbf{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\,\theta}}{2\pi}\right]\right]}{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}, \gamma\right]$$

$$\left\{\left\{\gamma \to -\frac{2\pi\,(4\pi-\theta)}{2\pi-\theta}\right\}, \left\{\gamma \to \frac{2\pi\theta}{2\pi-\theta}\right\}\right\}$$

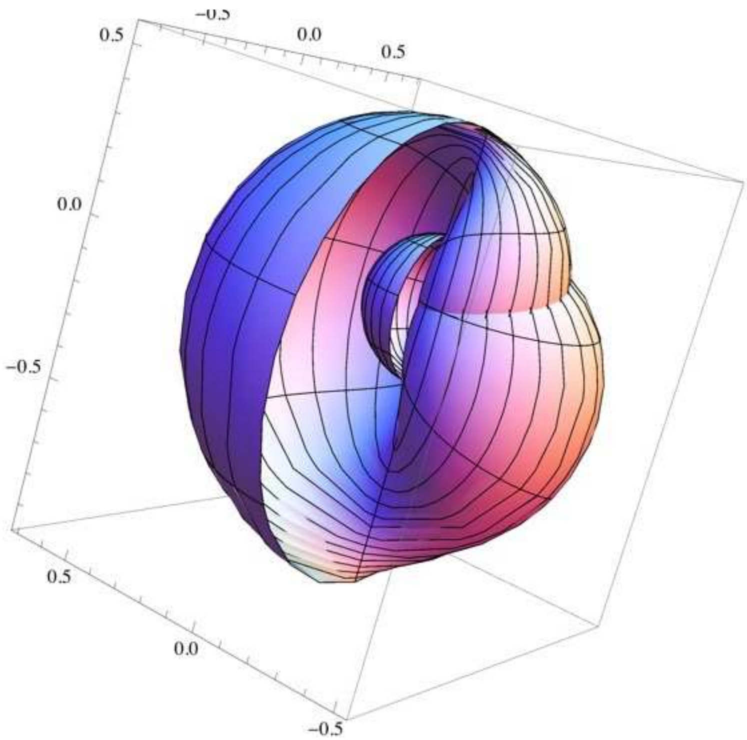
So, from the solutions, $\left\{\left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]\right\}\right\}$ and $\left\{\gamma \to \frac{2\pi\theta}{2\pi-\theta}\right\}$, we can generate the following graphs by substituting

$\gamma = \frac{2\pi\theta}{2\pi-\theta}$ into $\left\{\left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]\right\}\right\}$.

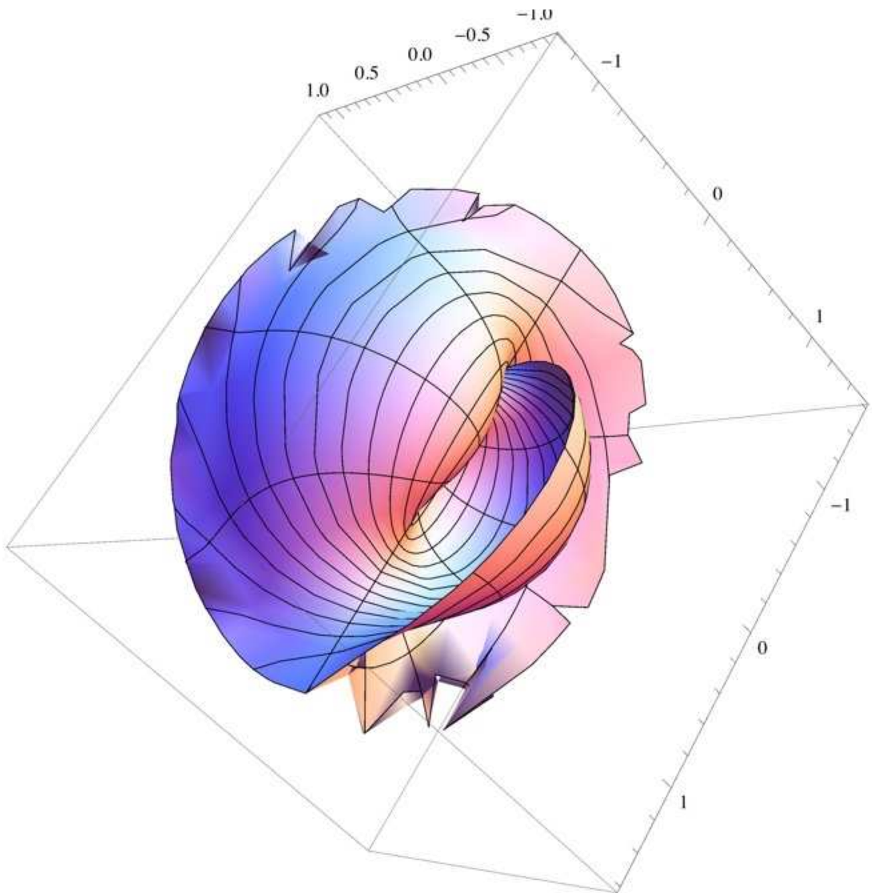
$$\textbf{SphericalPlot3D}\left[\textbf{ArcSin}\left[\frac{\sqrt{\frac{2\pi\theta}{2\pi-\theta}}\ \sqrt{\left(2\pi+\frac{2\pi\theta}{2\pi-\theta}\right)^2}\ \sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\left(\frac{2\pi\theta}{2\pi-\theta}\right)^2}\right], \{\theta, -2\pi, 2\pi\}, \{\gamma, -\pi, \pi\}\right]$$



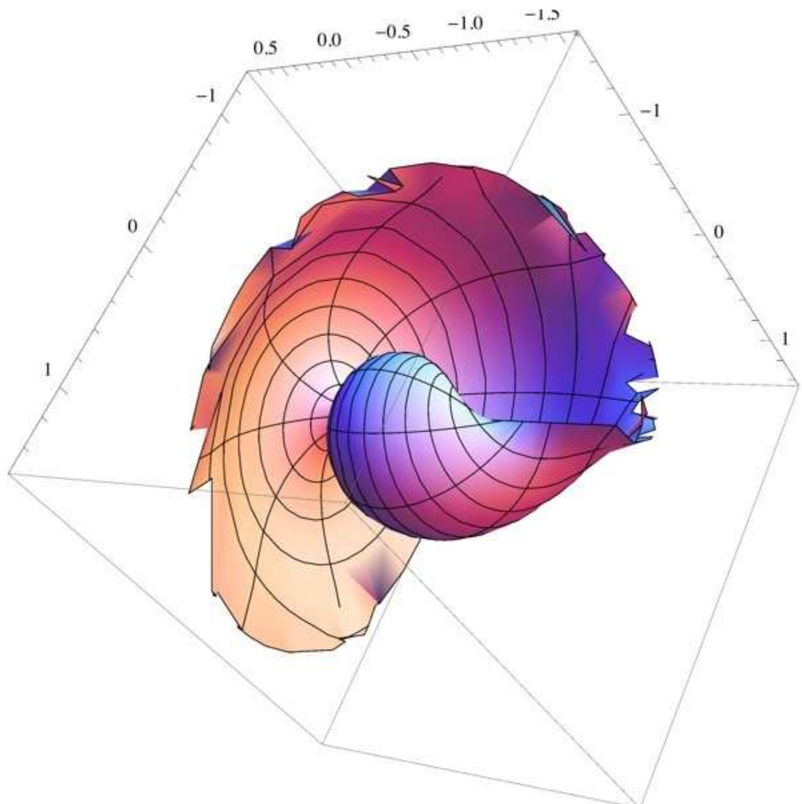
$$\texttt{SphericalPlot3D}\left[\texttt{ArcSin}\left[\frac{\sqrt{\gamma}\,\sqrt{\left(2\,\pi+\frac{2\,\pi\,\theta}{2\,\pi-\theta}\right)^{2}}\,\sqrt{4\,\pi+\frac{2\,\pi\,\theta}{2\,\pi-\theta}}}{4\,\pi^{2}+4\,\pi\,\gamma+\left(\frac{2\,\pi\,\theta}{2\,\pi-\theta}\right)^{2}}\right],\ \{\theta,\ -2\,\pi,\ 2\,\pi\},\ \{\gamma,\ -\pi,\ \pi\}\right]$$



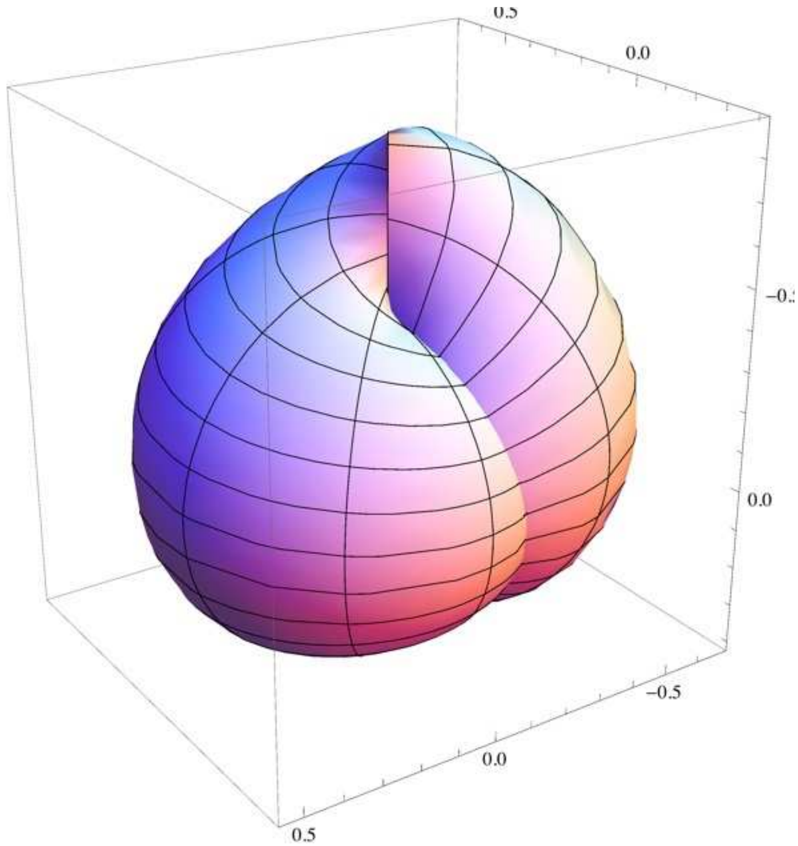
$$\text{SphericalPlot3D}\left[\text{ArcSin}\left[\frac{\sqrt{\gamma}\sqrt{\left(2\pi+\frac{2\pi\theta}{2\pi-\theta}\right)^2}\sqrt{4\pi+\frac{2\pi\theta}{2\pi-\theta}}}{4\pi^2+4\pi\frac{2\pi\theta}{2\pi-\theta}+\left(\frac{2\pi\theta}{2\pi-\theta}\right)^2}\right],\{\theta,-2\pi,2\pi\},\{\gamma,-\pi,\pi\}\right]$$



$$\text{SphericalPlot3D}\left[\text{ArcSin}\left[\frac{\sqrt{\gamma}\ \sqrt{\left(2\,\pi+\frac{2\,\pi\,\theta}{2\,\pi-\theta}\right)^{2}}\ \sqrt{4\,\pi+\frac{2\,\pi\,\theta}{2\,\pi-\theta}}}{4\,\pi^{2}+4\,\pi\,\frac{2\,\pi\,\theta}{2\,\pi-\theta}+\left(\frac{2\,\pi\,\theta}{2\,\pi-\theta}\right)^{2}}\right],\ \{\gamma,\ -\pi,\ \pi\},\ \{\theta,\ -2\,\pi,\ 2\,\pi\}\right]$$



$$\text{SphericalPlot3D}\left[\text{ArcSin}\left[\frac{\sqrt{\gamma}\,\sqrt{\left(2\pi+\frac{2\pi\theta}{2\pi-\theta}\right)^2}\,\sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\left(\frac{2\pi\theta}{2\pi-\theta}\right)^2}\right],\{\gamma,-\pi,\pi\},\{\theta,-2\pi,2\pi\}\right]$$



Now that we have performed a brief exposition of the functions related to the solutions to β and γ as well as the algebraic expression of the numeric one, I ' ll now reveal the paradoxical statement within this system of a circle ' s folding up into a cone as described by the difference in circumferenes of two circles ' equaling an arc length of the changed circle.

- **I. For $1 = \frac{\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}\,\text{Sin}[\beta]}$, the construction of the formula is:**

$$1-\frac{\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}\,\text{Sin}[\beta]} == 2\pi r-2\pi\sqrt{r^2-\frac{r\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\,\hat{}\,2}-\gamma\sqrt{r^2-\frac{r\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\,\hat{}\,2}$$

$$\text{Solve}\left[1-\frac{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}\ \text{Sin}[\beta]} ==\right.$$

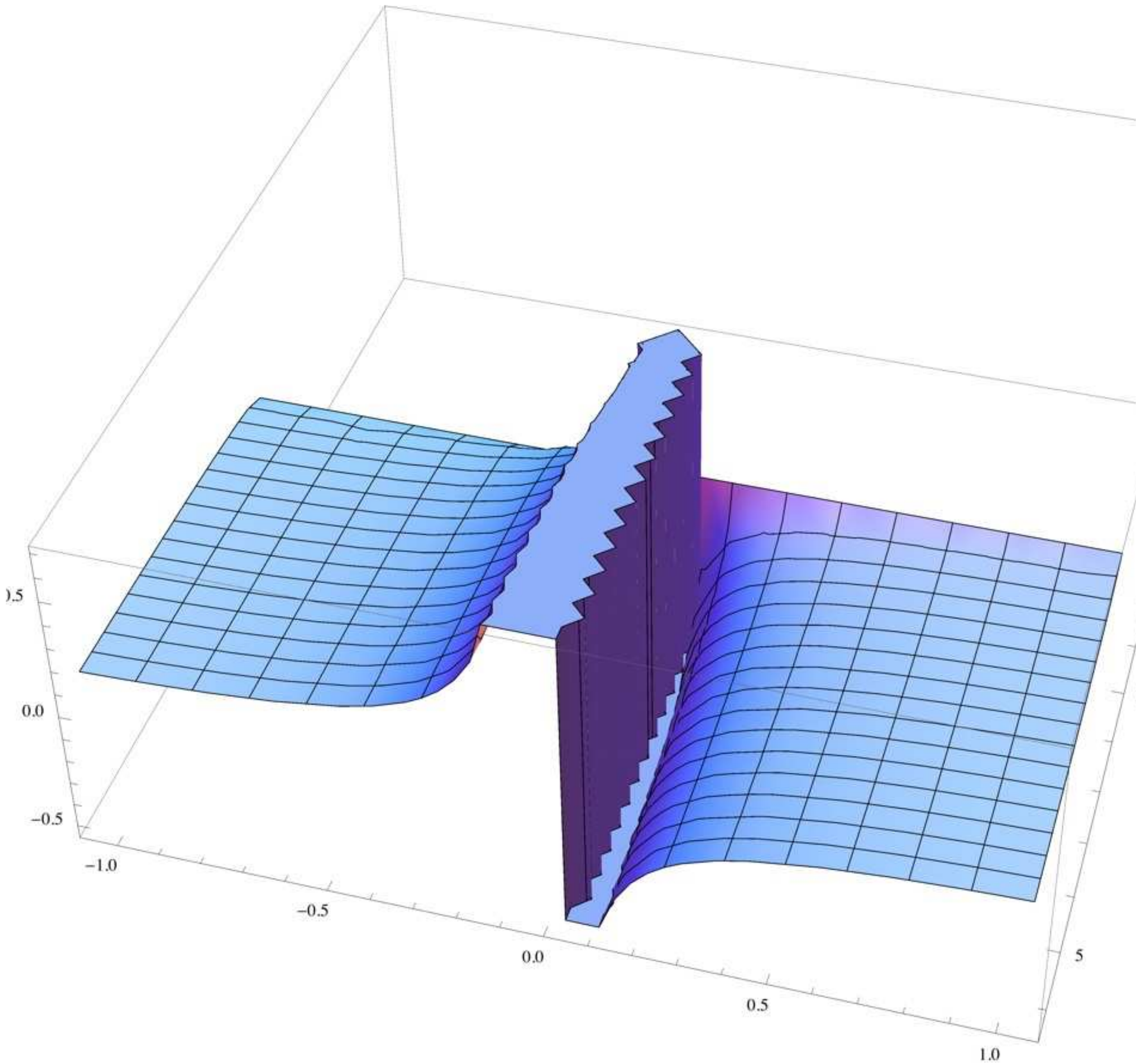
$$\left. 2\pi r-2\pi\sqrt{r^2-\frac{r\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\text{^}2}-\gamma\sqrt{r^2-\frac{r\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\text{^}2}\text{ , r}\right]$$

$$\left\{\left\{r\to\left(-4\pi^2-4\pi\gamma-\gamma^2+\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}\ \text{Csc}[\beta]+4\pi\gamma\,\text{Csc}[\beta]^2+\gamma^2\,\text{Csc}[\beta]^2-\right.\right.\right.$$

$$\left.\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}\ \text{Csc}[\beta]^3+\frac{4\pi^2\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}\ \text{Csc}[\beta]^3}{(2\pi+\gamma)^2}\right)\Big/$$

$$\left.\left.\left(\pi\left(-16\pi^2-16\pi\gamma-4\gamma^2+16\pi\gamma\,\text{Csc}[\beta]^2+4\gamma^2\,\text{Csc}[\beta]^2\right)\right)\right\}\right\}$$

```
Plot3D[(-4 π^2 - 4 π γ - γ^2 + Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Csc[β] + 4 π γ Csc[β]^2 + γ^2 Csc[β]^2 -
    Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Csc[β]^3 + (4 π^2 Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Csc[β]^3)/(2 π + γ)^2) /
  (π (-16 π^2 - 16 π γ - 4 γ^2 + 16 π γ Csc[β]^2 + 4 γ^2 Csc[β]^2)), {γ, -2 π, 2 π}, {β, -π / 3, π / 3}]
```

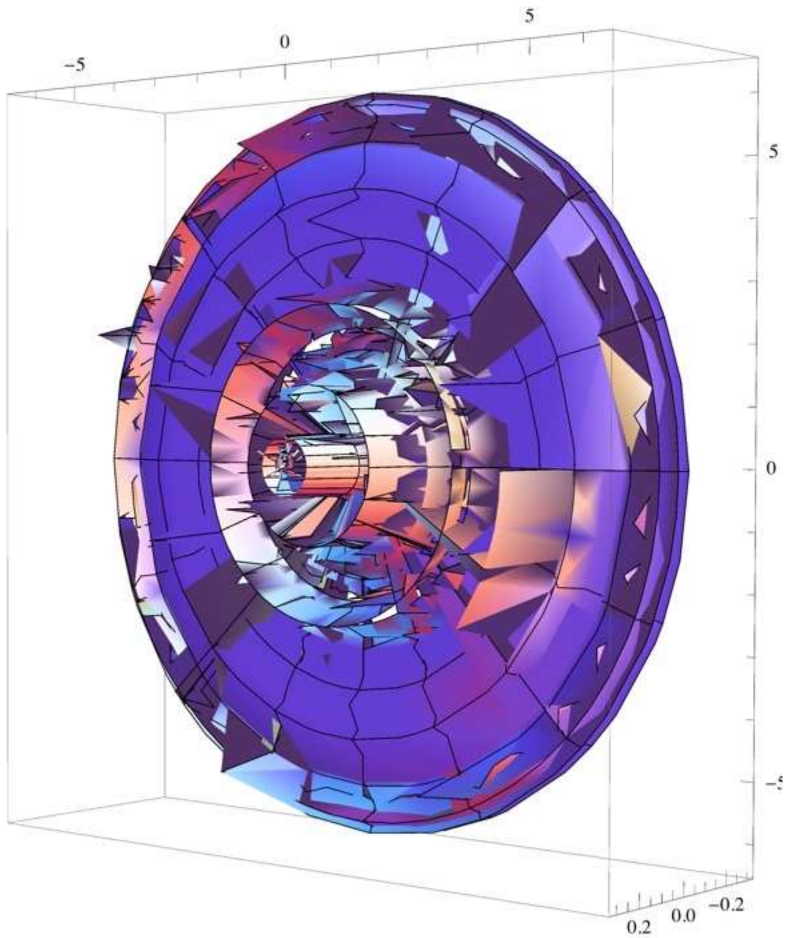


Letter A includes `Substituting :` $\left\{\left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]\right\}\right\}$ **and** $\left\{\gamma \to \frac{2\left(\pi-\pi\,\text{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)}{-1+\text{Sin}[\beta]^2}\right\}$ **into the resulting radius solution.**

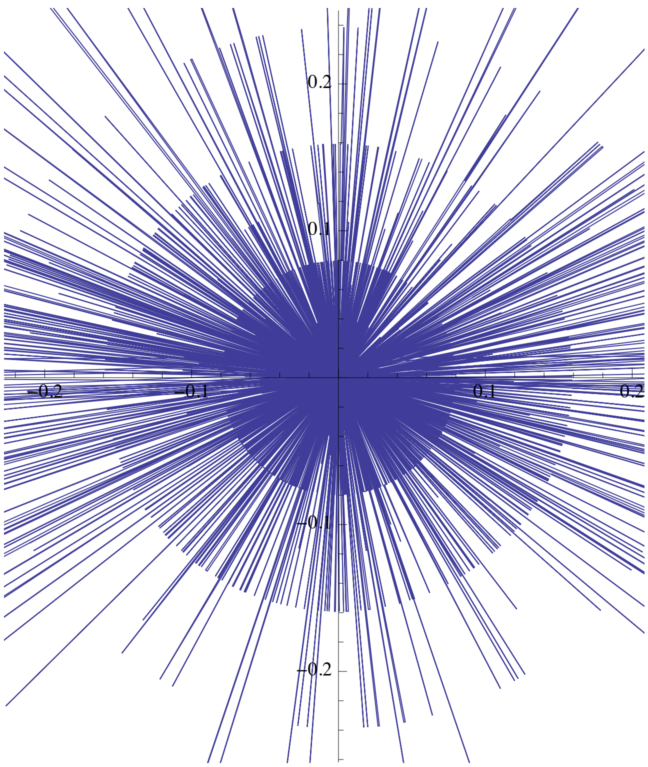
- γ **only**

$$\beta := \text{ArcSin}\left[\frac{\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]$$

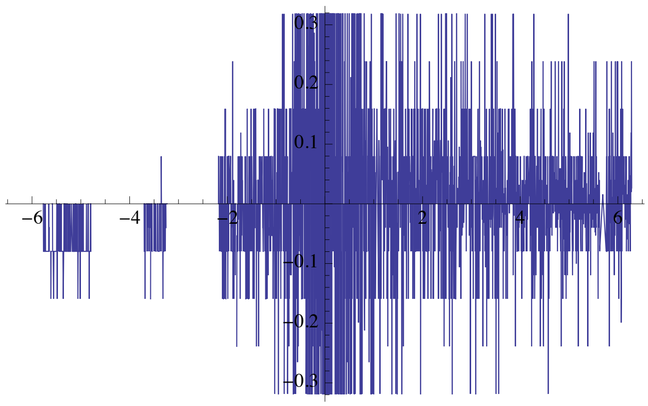
$$\text{RevolutionPlot3D}\left[\left(-4\pi^2-4\pi\gamma-\gamma^2+\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]+4\pi\gamma\,\text{Csc}[\beta]^2+\gamma^2\,\text{Csc}[\beta]^2-\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3+\frac{4\pi^2\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3}{(2\pi+\gamma)^2}\right)\Big/\left(\pi\left(-16\pi^2-16\pi\gamma-4\gamma^2+16\pi\gamma\,\text{Csc}[\beta]^2+4\gamma^2\,\text{Csc}[\beta]^2\right)\right),\ \{\gamma,-2\pi,2\pi\}\right]$$



$$\text{PolarPlot}\Bigg[\left(-4\pi^2-4\pi\gamma-\gamma^2+\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]+4\pi\gamma\,\text{Csc}[\beta]^2+\gamma^2\,\text{Csc}[\beta]^2-\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3+\frac{4\pi^2\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3}{(2\pi+\gamma)^2}\right)\Big/\left(\pi\left(-16\pi^2-16\pi\gamma-4\gamma^2+16\pi\gamma\,\text{Csc}[\beta]^2+4\gamma^2\,\text{Csc}[\beta]^2\right)\right),\ \{\gamma,-2\pi,2\pi\}\Bigg]$$



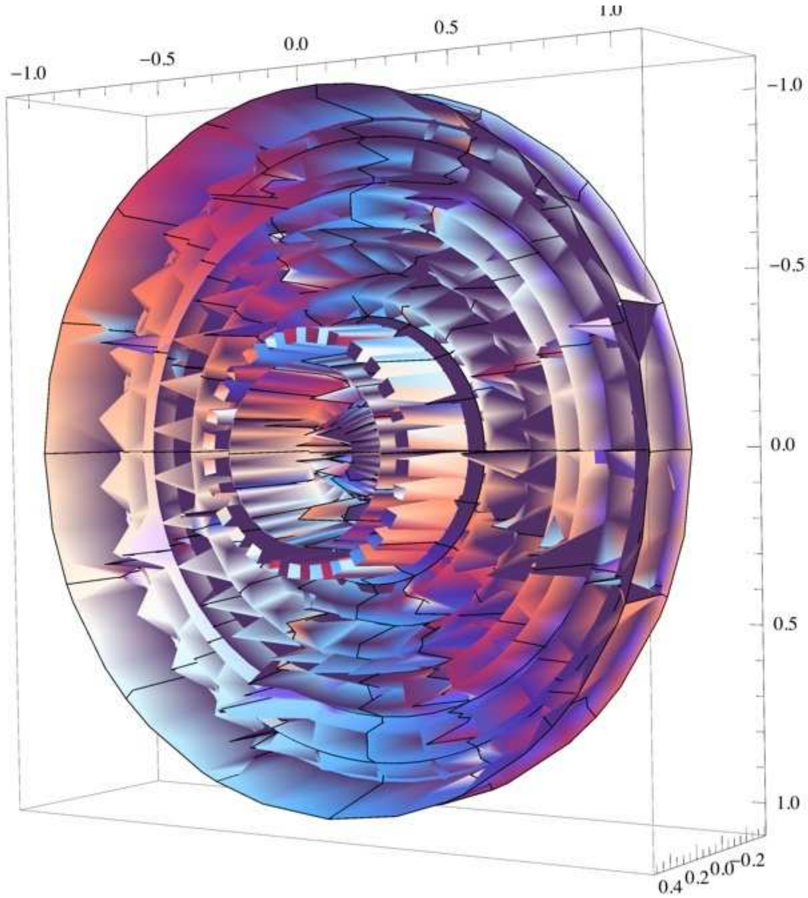
$$\text{Plot}\Big[\Big(-4\pi^2-4\pi\gamma-\gamma^2+\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]+4\pi\gamma\,\text{Csc}[\beta]^2+\gamma^2\,\text{Csc}[\beta]^2-\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3+\frac{4\pi^2\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3}{(2\pi+\gamma)^2}\Big)\Big/\left(\pi\left(-16\pi^2-16\pi\gamma-4\gamma^2+16\pi\gamma\,\text{Csc}[\beta]^2+4\gamma^2\,\text{Csc}[\beta]^2\right)\right),\ \{\gamma,-2\pi,2\pi\}\Big]$$



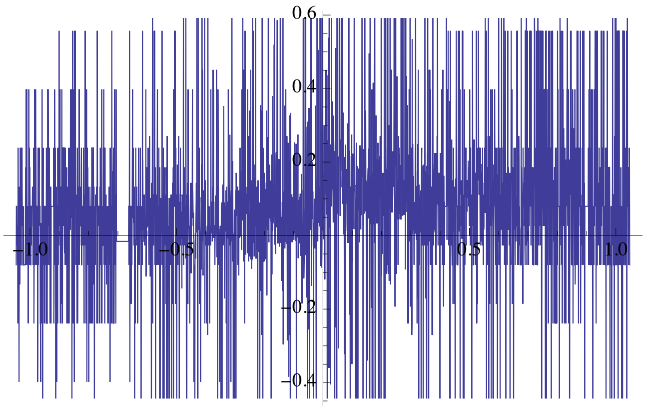
- **β only**

$$\gamma := \frac{2\left(\pi-\pi\,\text{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)}{-1+\text{Sin}[\beta]^2}$$

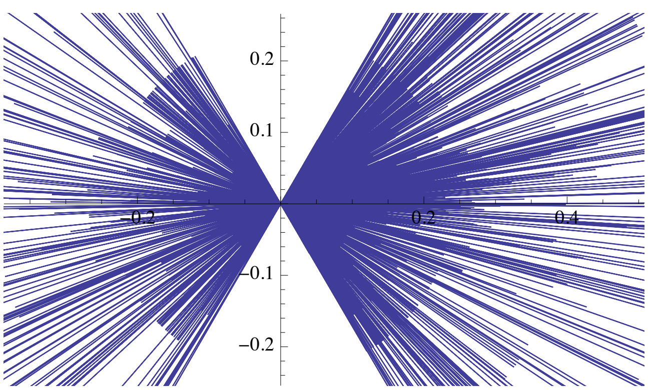
$$\text{RevolutionPlot3D}\Big[\Big(-4\pi^2-4\pi\gamma-\gamma^2+\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}\ \text{Csc}[\beta]+4\pi\gamma\,\text{Csc}[\beta]^2+\gamma^2\,\text{Csc}[\beta]^2-\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}\ \text{Csc}[\beta]^3+\frac{4\pi^2\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}\ \text{Csc}[\beta]^3}{(2\pi+\gamma)^2}\Big)\Big/\left(\pi\left(-16\pi^2-16\pi\gamma-4\gamma^2+16\pi\gamma\,\text{Csc}[\beta]^2+4\gamma^2\,\text{Csc}[\beta]^2\right)\right),\ \{\beta,-\pi/3,\pi/3\}\Big]$$



```
Plot[(-4 π^2 - 4 π γ - γ^2 + √γ √((2 π + γ)^2) √(4 π + γ) Csc[β] + 4 π γ Csc[β]^2 + γ^2 Csc[β]^2 -
    √γ √((2 π + γ)^2) √(4 π + γ) Csc[β]^3 + (4 π^2 √γ √((2 π + γ)^2) √(4 π + γ) Csc[β]^3)/(2 π + γ)^2) /
  (π (-16 π^2 - 16 π γ - 4 γ^2 + 16 π γ Csc[β]^2 + 4 γ^2 Csc[β]^2)), {β, -π / 3, π / 3}]
```

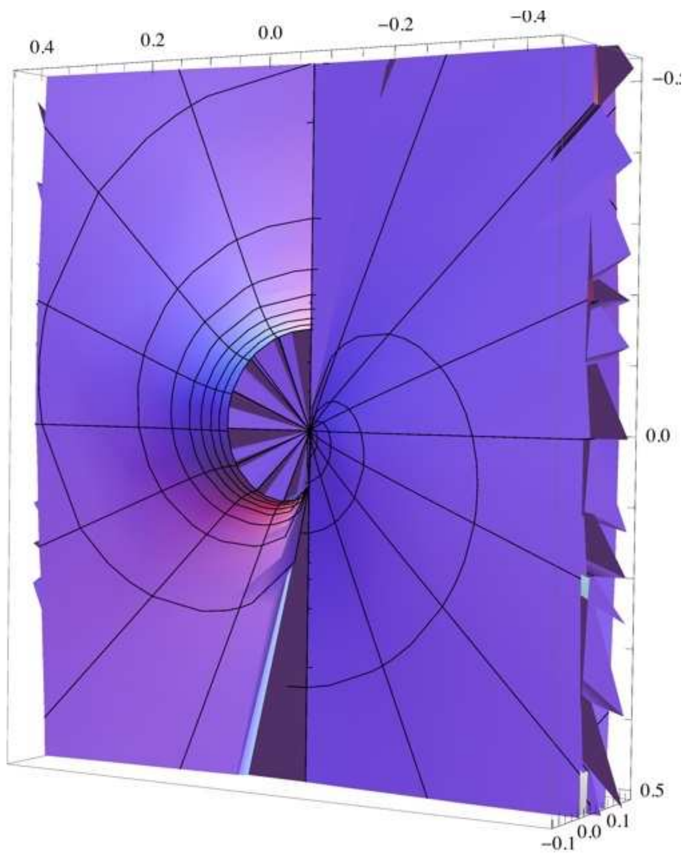



```
PolarPlot[(-4 π^2 - 4 π γ - γ^2 + √γ √((2 π + γ)^2) √(4 π + γ) Csc[β] + 4 π γ Csc[β]^2 + γ^2 Csc[β]^2 -
    √γ √((2 π + γ)^2) √(4 π + γ) Csc[β]^3 + (4 π^2 √γ √((2 π + γ)^2) √(4 π + γ) Csc[β]^3)/(2 π + γ)^2) /
  (π (-16 π^2 - 16 π γ - 4 γ^2 + 16 π γ Csc[β]^2 + 4 γ^2 Csc[β]^2)), {β, -π / 3, π / 3}]
```



- γ,β

```
SphericalPlot3D[(-4 π² - 4 π γ - γ² + √γ √((2 π + γ)²) √(4 π + γ) Csc[β] + 4 π γ Csc[β]² + γ² Csc[β]² -
    √γ √((2 π + γ)²) √(4 π + γ) Csc[β]³ + (4 π² √γ √((2 π + γ)²) √(4 π + γ) Csc[β]³)/(2 π + γ)²) /
  (π (-16 π² - 16 π γ - 4 γ² + 16 π γ Csc[β]² + 4 γ² Csc[β]²)), {γ, -2 π, 2 π}, {β, -π / 3, π / 3}]
```



Substituting $\left\{\gamma \to \frac{2\left(\pi - \pi \text{Sin}[\beta]^2 + \sqrt{\pi^2 - \pi^2 \text{Sin}[\beta]^2}\right)}{-1 + \text{Sin}[\beta]^2}\right\}$

SphericalPlot3D$\Big[$

$$\left(-4\pi^2-4\pi\gamma-\left(\frac{2\left(\pi-\pi\,\mathrm{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)}{-1+\mathrm{Sin}[\beta]^2}\right)^2+\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\mathrm{Csc}[\beta]+\right.$$

$$4\pi\left(\frac{2\left(\pi-\pi\,\mathrm{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)}{-1+\mathrm{Sin}[\beta]^2}\right)\mathrm{Csc}[\beta]^2+\gamma^2\,\mathrm{Csc}[\beta]^2-$$

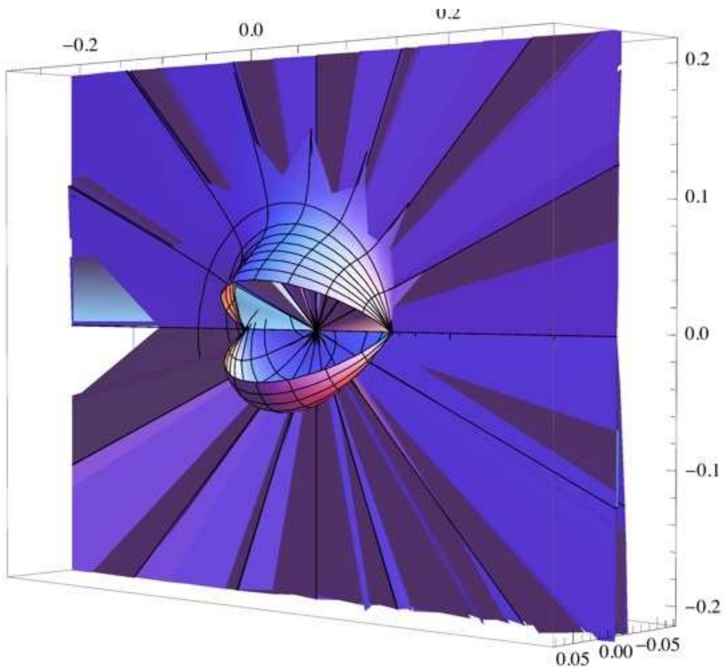
$$\sqrt{\gamma}\,\sqrt{\left(2\pi+\left(\frac{2\left(\pi-\pi\,\mathrm{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)}{-1+\mathrm{Sin}[\beta]^2}\right)\right)^2}\,\sqrt{4\pi+\gamma}\,\mathrm{Csc}[\beta]^3+$$

$$\left.\frac{4\pi^2\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\mathrm{Csc}[\beta]^3}{(2\pi+\gamma)^2}\right)\Bigg/$$

$$\left(\pi\left(-16\pi^2-16\pi\gamma-4\left(\frac{2\left(\pi-\pi\,\mathrm{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)}{-1+\mathrm{Sin}[\beta]^2}\right)^2+\right.\right.$$

$$16\pi\left(\frac{2\left(\pi-\pi\,\mathrm{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)}{-1+\mathrm{Sin}[\beta]^2}\right)\mathrm{Csc}[\beta]^2+$$

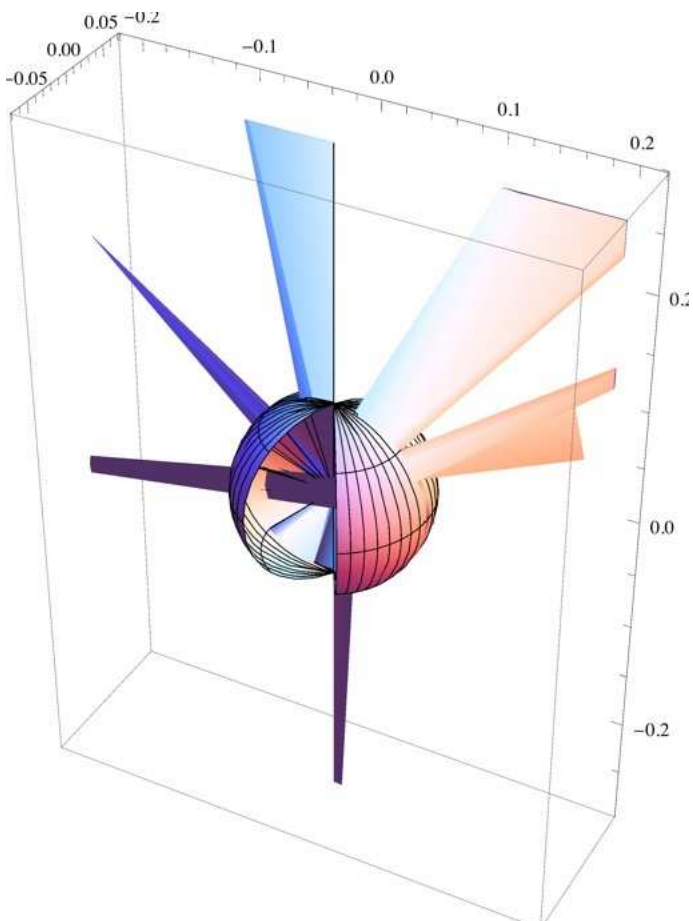
$$\left.\left.4\gamma^2\,\mathrm{Csc}[\beta]^2\right)\right),\ \{\gamma,\,-2\pi,\,2\pi\},\ \{\beta,\,-\pi/3,\,\pi/3\}\Big]$$


−0.2
0.0
0.2
0.2
0.1
0.0
−0.1
−0.2
0.05
0.00
−0.05

$$\text{SphericalPlot3D}\Big[\Bigg(-4\pi^2-4\pi\gamma-\gamma^2+\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]+4\pi\gamma\,\text{Csc}[\beta]^2+\gamma^2\,\text{Csc}[\beta]^2-$$

$$\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3+\frac{4\pi^2\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+(\gamma)}\,\text{Csc}[\beta]^3}{\left(2\pi+\left(\frac{2\left(\pi-\pi\,\text{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)}{-1+\text{Sin}[\beta]^2}\right)\right)^2}\Bigg)\Big/$$

$$\left(\pi\left(-16\pi^2-16\pi\gamma-4\gamma^2+16\pi\gamma\,\text{Csc}[\beta]^2+4\gamma^2\,\text{Csc}[\beta]^2\right)\right),\ \{\gamma,-2\pi,2\pi\},\ \{\beta,-\pi/3,\pi/3\}\Big]$$



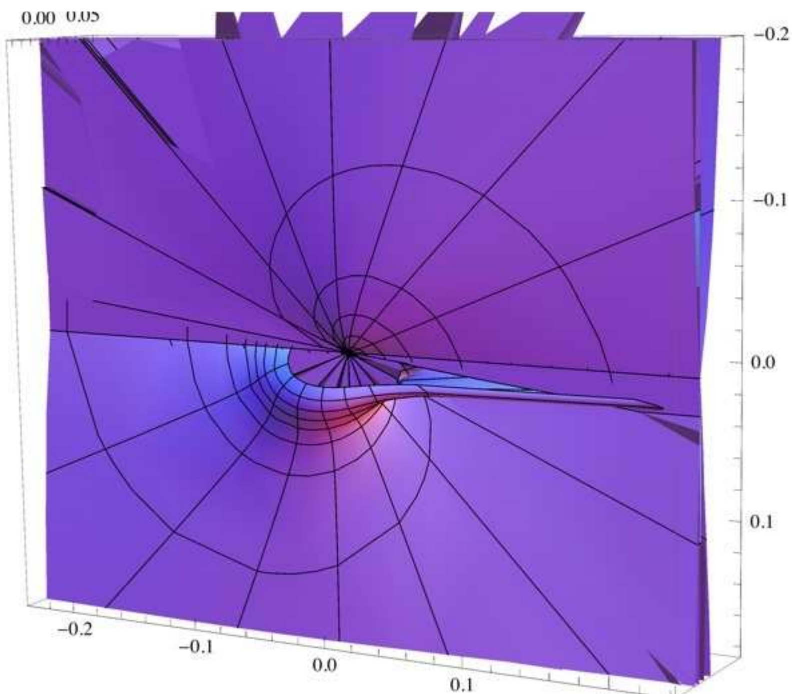
$$\text{Substituting}\left\{\beta\to\text{ArcSin}\left[\frac{\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]\right\}$$

SphericalPlot3D[

$$\left(-4\pi^2-4\pi\gamma-\gamma^2+\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\mathrm{Csc}\left[\mathrm{ArcSin}\left[\frac{\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]\right]+4\pi\gamma\,\mathrm{Csc}[\beta]^2+\gamma^2\,\mathrm{Csc}[\beta]^2-\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\mathrm{Csc}[\beta]^3+\frac{4\pi^2\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\mathrm{Csc}\left[\mathrm{ArcSin}\left[\frac{\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]\right]^3}{(2\pi+\gamma)^2}\right)\Bigg/$$

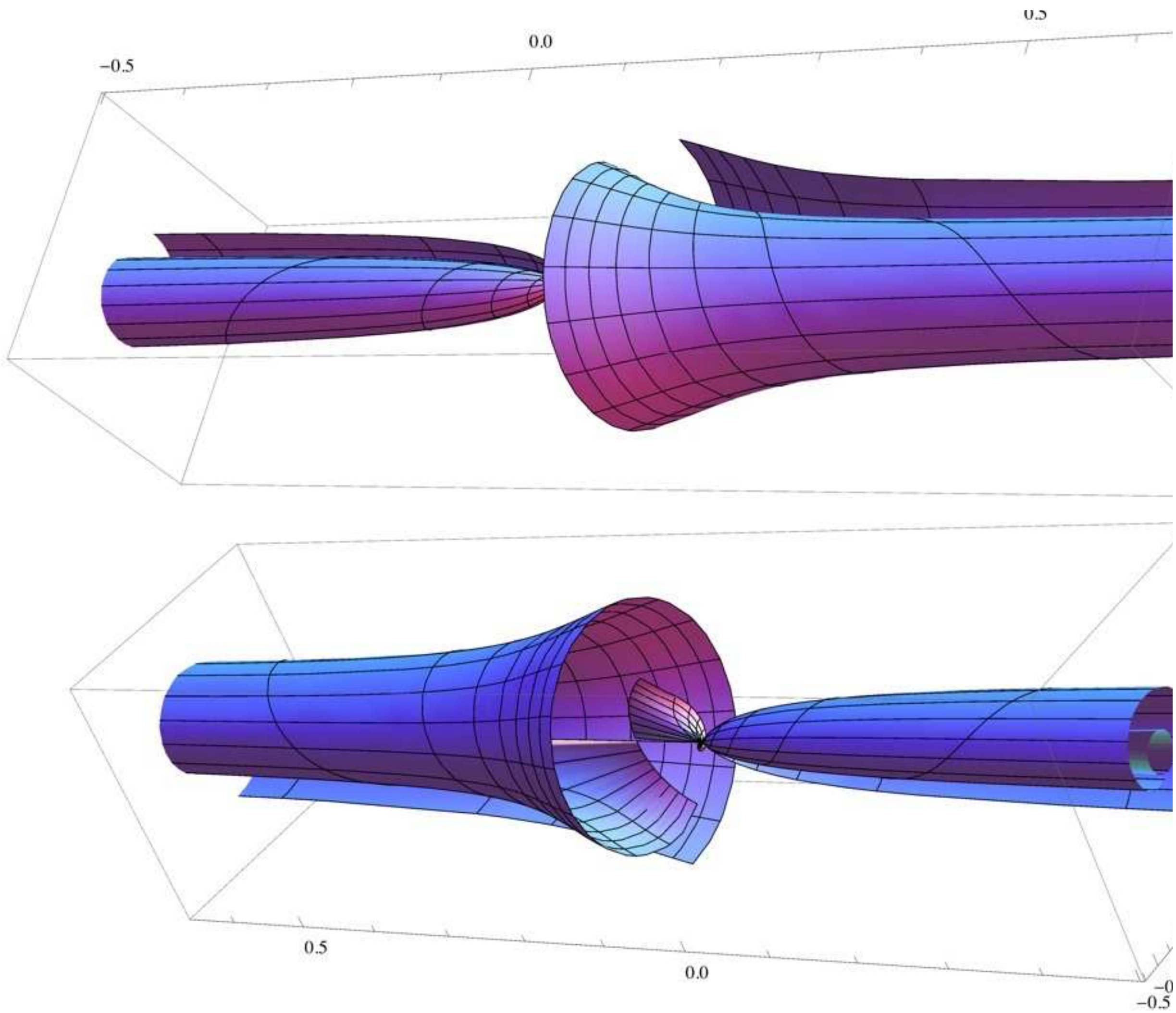
$$\left(\pi\left(-16\pi^2-16\pi\gamma-4\gamma^2+16\pi\gamma\,\mathrm{Csc}[\beta]^2+4\left(\frac{2\left(\pi-\pi\,\mathrm{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)}{-1+\mathrm{Sin}[\beta]^2}\right)^2\mathrm{Csc}[\beta]^2\right)\right),$$

{γ, -2 π, 2 π}, {β, -π / 3, π / 3}]



- β, γ

$$\text{SphericalPlot3D}\left[\left(-4\pi^2 - 4\pi\gamma - \gamma^2 + \sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta] + 4\pi\gamma\,\text{Csc}[\beta]^2 + \gamma^2\,\text{Csc}[\beta]^2 - \sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3 + \frac{4\pi^2\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3}{(2\pi+\gamma)^2}\right) \Big/ \left(\pi\left(-16\pi^2 - 16\pi\gamma - 4\gamma^2 + 16\pi\gamma\,\text{Csc}[\beta]^2 + 4\gamma^2\,\text{Csc}[\beta]^2\right)\right), \{\beta, -\pi/3, \pi/3\}, \{\gamma, -2\pi, 2\pi\}\right]$$



Substituting $\left\{\gamma \to \frac{2\left(\pi - \pi\,\text{Sin}[\beta]^2 + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)}{-1 + \text{Sin}[\beta]^2}\right\}$

SphericalPlot3D[

$$\left(-4\pi^2-4\pi\gamma-\left(\frac{2\left(\pi-\pi\,\mathrm{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)}{-1+\mathrm{Sin}[\beta]^2}\right)^2+\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\mathrm{Csc}[\beta]+\right.$$

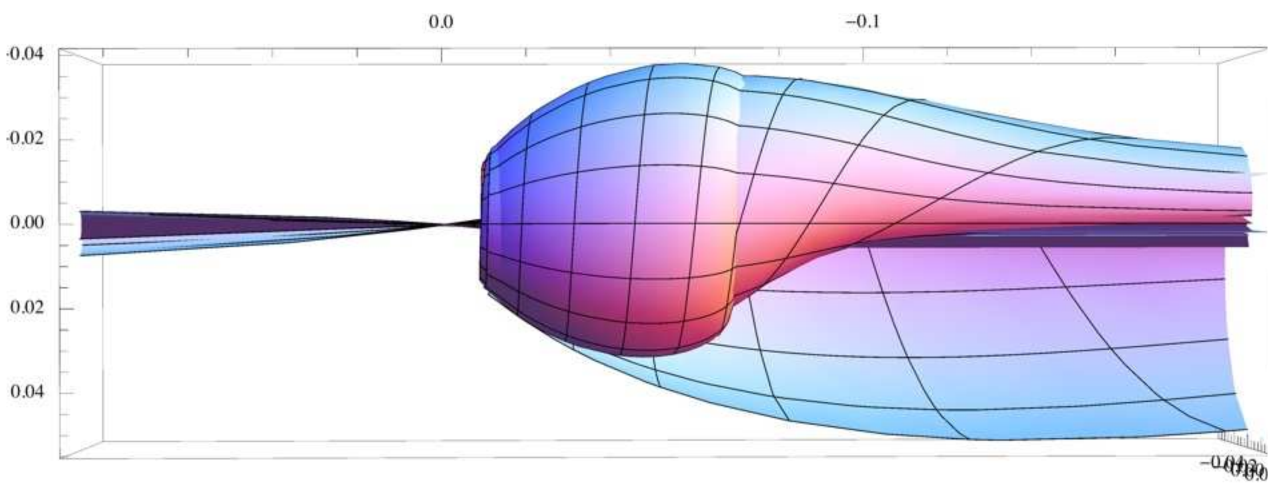
$$4\pi\gamma\,\mathrm{Csc}[\beta]^2+\left(\frac{2\left(\pi-\pi\,\mathrm{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)}{-1+\mathrm{Sin}[\beta]^2}\right)^2\mathrm{Csc}[\beta]^2-$$

$$\sqrt{\gamma}\,\sqrt{\left(2\pi+\frac{2\left(\pi-\pi\,\mathrm{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)}{-1+\mathrm{Sin}[\beta]^2}\right)^2}\,\sqrt{4\pi+\gamma}\,\mathrm{Csc}[\beta]^3+$$

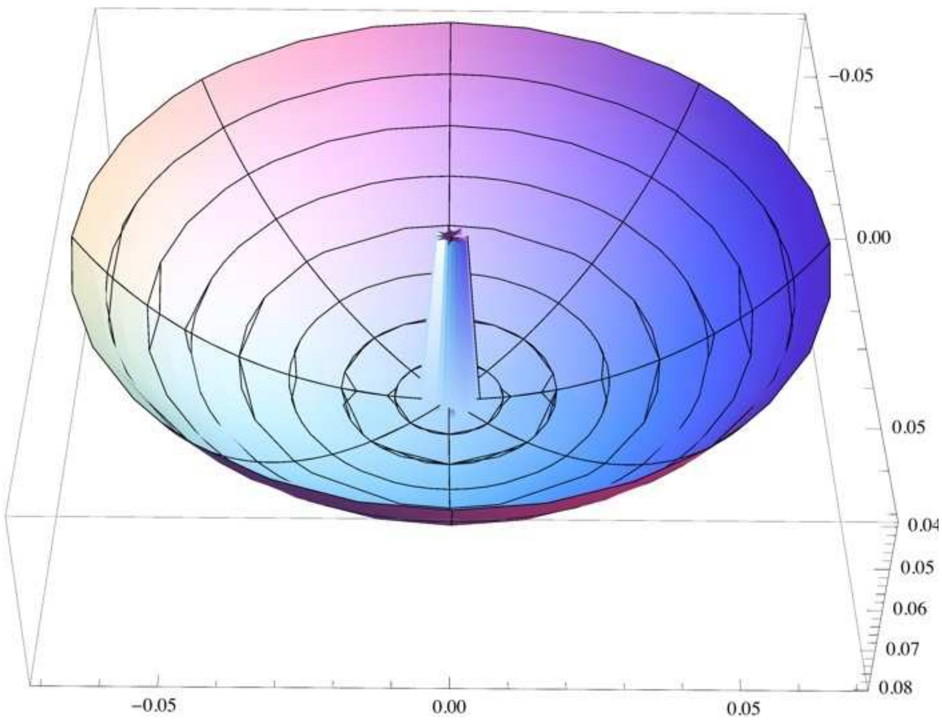
$$\left.\frac{4\pi^2\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\mathrm{Csc}[\beta]^3}{(2\pi+\gamma)^2}\right)\Big/$$

$$\left(\pi\left(-16\pi^2-16\pi\gamma-4\gamma^2+16\pi\,\frac{2\left(\pi-\pi\,\mathrm{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)}{-1+\mathrm{Sin}[\beta]^2}\,\mathrm{Csc}[\beta]^2+4\gamma^2\,\mathrm{Csc}[\beta]^2\right)\right),$$

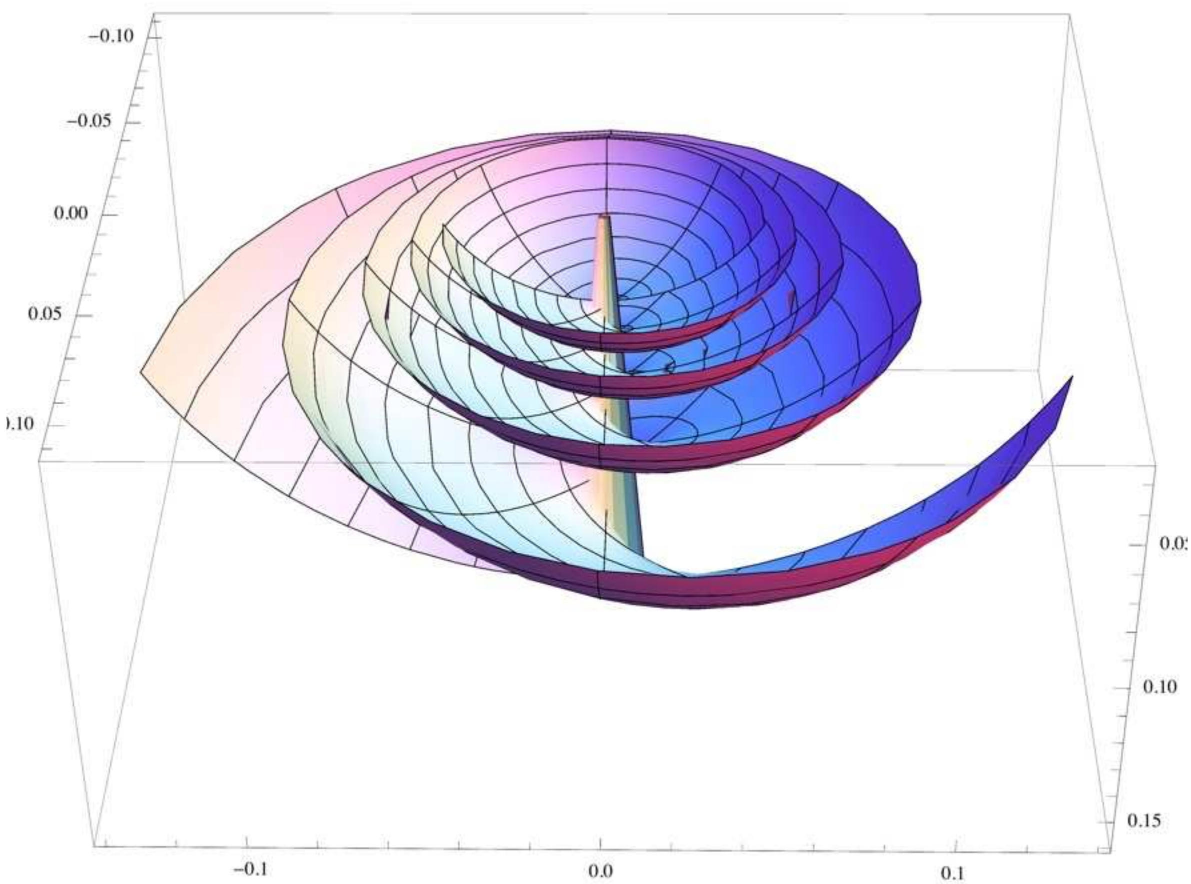
{β, -π / 3, π / 3}, {γ, -2 π, 2 π}]



```
SphericalPlot3D[
 (-4 π^2 - 4 π γ - γ^2 + Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Csc[ArcSin[(Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ])/(4 π^2 + 4 π γ + γ^2)]] + 4 π γ Csc[β]^2 +
    γ^2 Csc[β]^2 - Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Csc[ArcSin[(Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ])/(4 π^2 + 4 π γ + γ^2)]]^3 +
    (4 π^2 Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Csc[ArcSin[(Sqrt[γ] Sqrt[(2 π+γ)^2] Sqrt[4 π+γ])/(4 π^2+4 π γ+γ^2)]]^3)/(2 π + γ)^2) /
  (π (-16 π^2 - 16 π γ - 4 γ^2 + 16 π γ Csc[β]^2 + 4 γ^2 Csc[β]^2)), {β, -π / 3, π / 3}, {γ, -2 π, 2 π}]
```



```
SphericalPlot3D[
 (-4 π^2 - 4 π γ - γ^2 + Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Csc[ArcSin[(Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ])/(4 π^2 + 4 π γ + γ^2)]] +
   4 π γ Csc[β]^2 + γ^2 Csc[ArcSin[(Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ])/(4 π^2 + 4 π γ + γ^2)]]^2 -
   Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Csc[ArcSin[(Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ])/(4 π^2 + 4 π γ + γ^2)]]^3 +
   (4 π^2 Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Csc[ArcSin[(Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ])/(4 π^2 + 4 π γ + γ^2)]]^3)/(2 π + γ)^2) /
  (π (-16 π^2 - 16 π γ - 4 γ^2 + 16 π γ Csc[β]^2 + 4 γ^2 Csc[β]^2)), {β, -π / 3, π / 3}, {γ, -2 π, 2 π}]
```



$$\text{SphericalPlot3D}\Bigg[\Bigg(-4\pi^2-4\pi\gamma-\left(\frac{2\left(\pi-\pi\,\text{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)}{-1+\text{Sin}[\beta]^2}\right)^2+$$

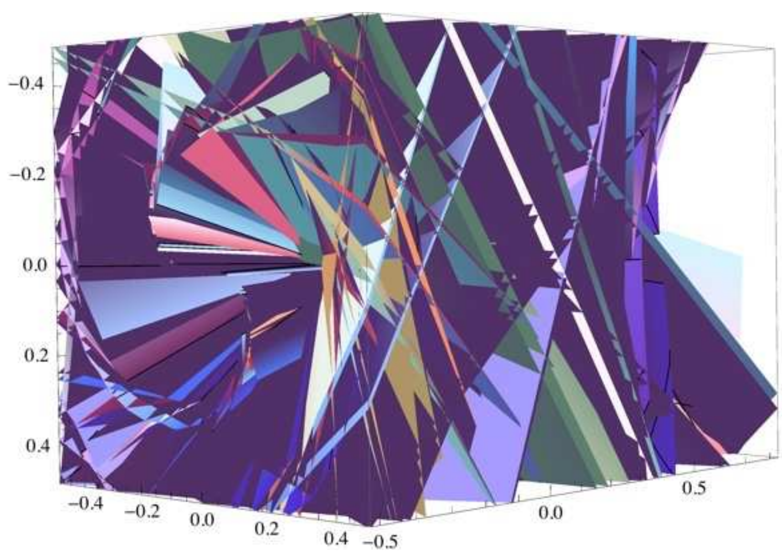
$$\sqrt{\gamma}\;\sqrt{(2\pi+\gamma)^2}\;\sqrt{4\pi+\gamma}\;\text{Csc}\left[\text{ArcSin}\left[\frac{\sqrt{\gamma}\;\sqrt{(2\pi+\gamma)^2}\;\sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]\right]+$$

$$4\pi\gamma\,\text{Csc}[\beta]^2+\gamma^2\,\text{Csc}\left[\text{ArcSin}\left[\frac{\sqrt{\gamma}\;\sqrt{(2\pi+\gamma)^2}\;\sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]\right]^2-$$

$$\sqrt{\gamma}\;\sqrt{(2\pi+\gamma)^2}\;\sqrt{4\pi+\gamma}\;\text{Csc}\left[\text{ArcSin}\left[\frac{\sqrt{\gamma}\;\sqrt{(2\pi+\gamma)^2}\;\sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]\right]^3+$$

$$\frac{4\pi^2\sqrt{\gamma}\;\sqrt{(2\pi+\gamma)^2}\;\sqrt{4\pi+\gamma}\;\text{Csc}\left[\text{ArcSin}\left[\frac{\sqrt{\gamma}\;\sqrt{(2\pi+\gamma)^2}\;\sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]\right]^3}{(2\pi+\gamma)^2}\Bigg)\Big/$$

$$\left(\pi\left(-16\pi^2-16\pi\gamma-4\gamma^2+16\pi\gamma\,\text{Csc}[\beta]^2+4\gamma^2\,\text{Csc}[\beta]^2\right)\right),\ \{\beta,-\pi/3,\pi/3\},\ \{\gamma,-2\pi,2\pi\}\Bigg]$$



- **Letter C includes**

Substituting : $\left\{\left\{\beta \to \text{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}\right]\right\}\right\}$ **and**

$\left\{\gamma \to \frac{2\left(\pi - \pi\,\text{Csc}[\beta]^2 + \sqrt{-\pi^2\,\text{Csc}[\beta]^2 + \pi^2\,\text{Csc}[\beta]^4}\right)}{-1+\text{Csc}[\beta]^2}\right\}$ **into the resulting radius solution**

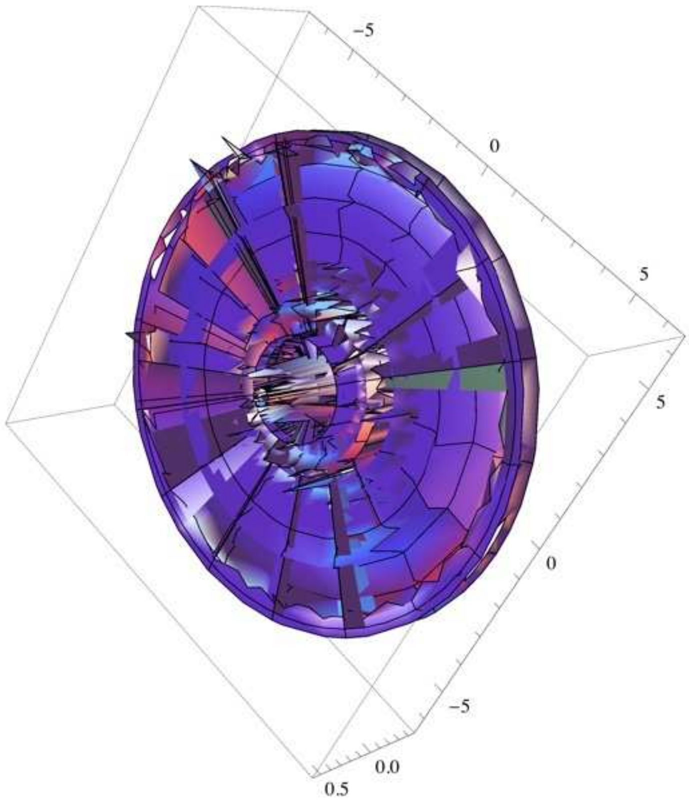
$$\text{Solve}\left[1 - \frac{\sqrt{\gamma}\sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}\,\text{Sin}[\beta]} == 2\pi r - 2\pi\sqrt{r^2 - \frac{r\sqrt{\gamma}\sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\text{^}2} - \gamma\sqrt{r^2 - \frac{r\sqrt{\gamma}\sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\text{^}2}\text{ , r}\right]$$

$$\Bigg\{\Bigg\{r \to \Bigg(-4\pi^2 - 4\pi\gamma - \gamma^2 + \sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta] + 4\pi\gamma\,\text{Csc}[\beta]^2 + \gamma^2\,\text{Csc}[\beta]^2 - \sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3 + \frac{4\pi^2\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3}{(2\pi+\gamma)^2}\Bigg) \Big/ \left(\pi\left(-16\pi^2 - 16\pi\gamma - 4\gamma^2 + 16\pi\gamma\,\text{Csc}[\beta]^2 + 4\gamma^2\,\text{Csc}[\beta]^2\right)\right)\Bigg\}\Bigg\}$$

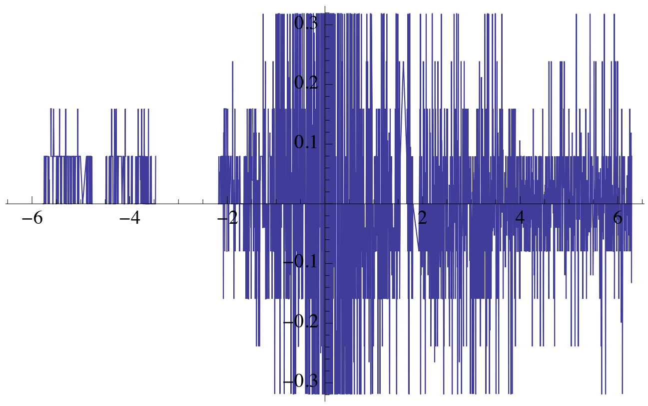
- **γ only substituting β := ArcCsc**$\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}\right]$

$$\beta := \text{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}\right]$$

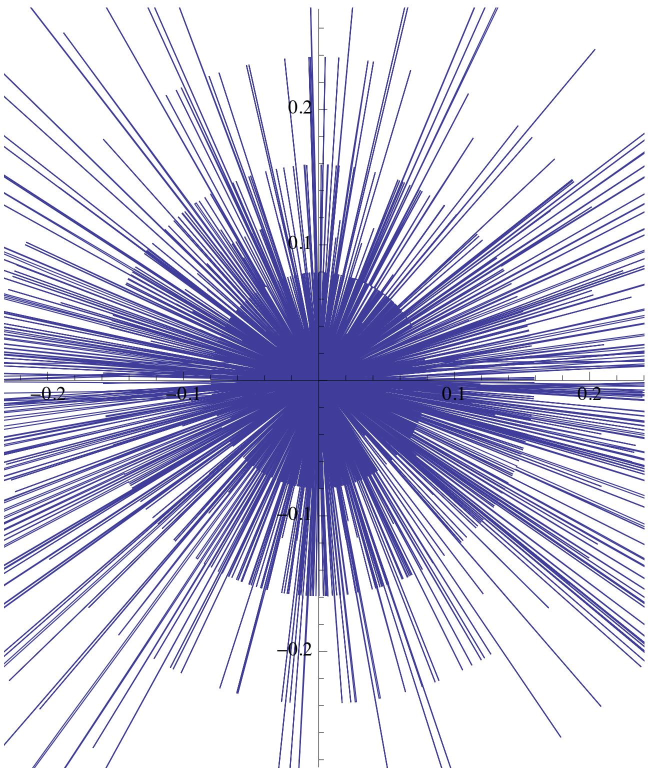
```
RevolutionPlot3D[(-4 π^2 - 4 π γ - γ^2 + Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Csc[β] + 4 π γ Csc[β]^2 +
    γ^2 Csc[β]^2 - Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Csc[β]^3 + (4 π^2 Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Csc[β]^3)/(2 π + γ)^2)/
  (π (-16 π^2 - 16 π γ - 4 γ^2 + 16 π γ Csc[β]^2 + 4 γ^2 Csc[β]^2)), {γ, -2 π, 2 π}]
```



$$\text{Plot}\left[\left(-4\,\pi^2-4\,\pi\,\gamma-\gamma^2+\sqrt{\gamma}\,\sqrt{(2\,\pi+\gamma)^2}\,\sqrt{4\,\pi+\gamma}\,\text{Csc}[\beta]+4\,\pi\,\gamma\,\text{Csc}[\beta]^2+\gamma^2\,\text{Csc}[\beta]^2-\sqrt{\gamma}\,\sqrt{(2\,\pi+\gamma)^2}\,\sqrt{4\,\pi+\gamma}\,\text{Csc}[\beta]^3+\frac{4\,\pi^2\,\sqrt{\gamma}\,\sqrt{(2\,\pi+\gamma)^2}\,\sqrt{4\,\pi+\gamma}\,\text{Csc}[\beta]^3}{(2\,\pi+\gamma)^2}\right)\Big/\left(\pi\left(-16\,\pi^2-16\,\pi\,\gamma-4\,\gamma^2+16\,\pi\,\gamma\,\text{Csc}[\beta]^2+4\,\gamma^2\,\text{Csc}[\beta]^2\right)\right),\ \{\gamma,\ -2\,\pi,\ 2\,\pi\}\right]$$



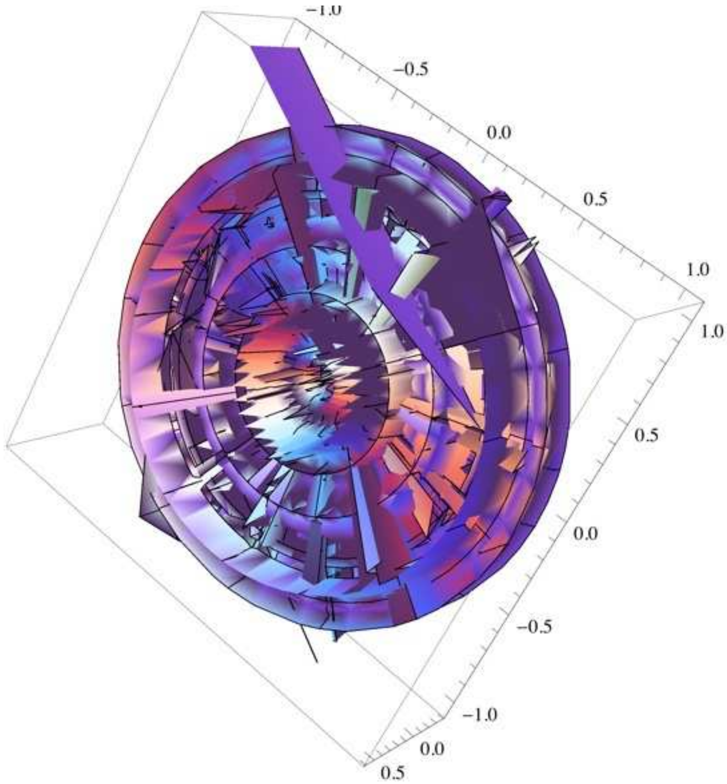
$$\text{PolarPlot}\Big[\Big(-4\pi^2 - 4\pi\gamma - \gamma^2 + \sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}\ \text{Csc}[\beta] + 4\pi\gamma\,\text{Csc}[\beta]^2 + \gamma^2\,\text{Csc}[\beta]^2 - \sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}\ \text{Csc}[\beta]^3 + \frac{4\pi^2\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}\ \text{Csc}[\beta]^3}{(2\pi+\gamma)^2}\Big) \Big/ \left(\pi\left(-16\pi^2 - 16\pi\gamma - 4\gamma^2 + 16\pi\gamma\,\text{Csc}[\beta]^2 + 4\gamma^2\,\text{Csc}[\beta]^2\right)\right),\ \{\gamma, -2\pi, 2\pi\}\Big]$$



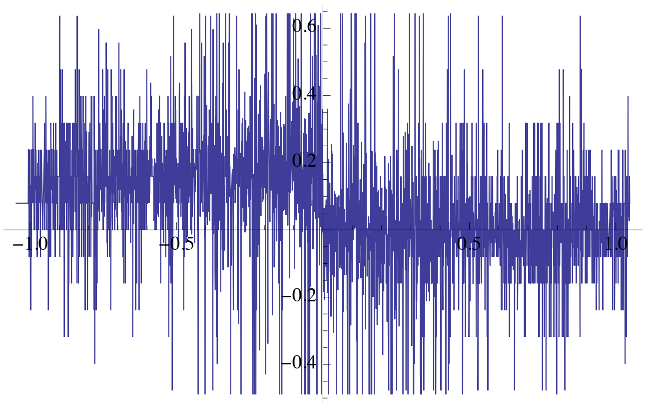
- β **only substituting** $\gamma := \dfrac{2\left(\pi - \pi\,\text{Csc}[\beta]^2 + \sqrt{-\pi^2\,\text{Csc}[\beta]^2 + \pi^2\,\text{Csc}[\beta]^4}\right)}{-1 + \text{Csc}[\beta]^2}$

$$\gamma := \frac{2\left(\pi - \pi\,\text{Csc}[\beta]^2 + \sqrt{-\pi^2\,\text{Csc}[\beta]^2 + \pi^2\,\text{Csc}[\beta]^4}\right)}{-1 + \text{Csc}[\beta]^2}$$

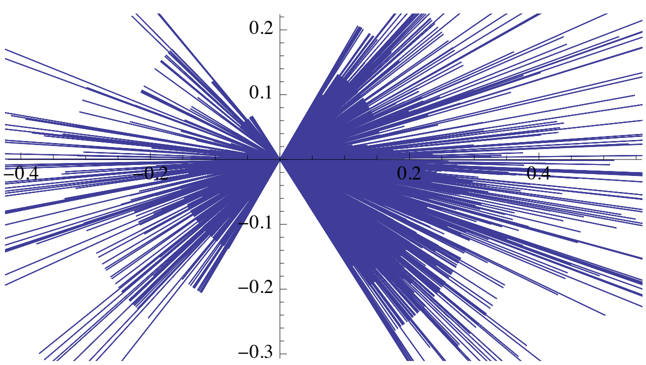
$$\text{RevolutionPlot3D}\Big[\Big(-4\pi^2 - 4\pi\gamma - \gamma^2 + \sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta] + 4\pi\gamma\,\text{Csc}[\beta]^2 + \gamma^2\,\text{Csc}[\beta]^2 - \sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3 + \frac{4\pi^2\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3}{(2\pi+\gamma)^2}\Big) \Big/ \Big(\pi\left(-16\pi^2 - 16\pi\gamma - 4\gamma^2 + 16\pi\gamma\,\text{Csc}[\beta]^2 + 4\gamma^2\,\text{Csc}[\beta]^2\right)\Big),\ \{\beta, -\pi/3, \pi/3\}\Big]$$



$$\text{Plot}\Big[\Big(-4\pi^2-4\pi\gamma-\gamma^2+\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]+4\pi\gamma\,\text{Csc}[\beta]^2+\gamma^2\,\text{Csc}[\beta]^2-\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3+\frac{4\pi^2\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3}{(2\pi+\gamma)^2}\Big)\Big/\left(\pi\left(-16\pi^2-16\pi\gamma-4\gamma^2+16\pi\gamma\,\text{Csc}[\beta]^2+4\gamma^2\,\text{Csc}[\beta]^2\right)\right),\ \{\beta,-\pi/3,\pi/3\}\Big]$$



$$\text{PolarPlot}\Big[\Big(-4\pi^2-4\pi\gamma-\gamma^2+\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]+4\pi\gamma\,\text{Csc}[\beta]^2+\gamma^2\,\text{Csc}[\beta]^2-\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3+\frac{4\pi^2\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3}{(2\pi+\gamma)^2}\Big)\Big/\left(\pi\left(-16\pi^2-16\pi\gamma-4\gamma^2+16\pi\gamma\,\text{Csc}[\beta]^2+4\gamma^2\,\text{Csc}[\beta]^2\right)\right),\ \{\beta,-\pi/3,\pi/3\}\Big]$$



- γ,β

Substituting $\beta = \text{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}\right]$

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SphericalPlot3D[
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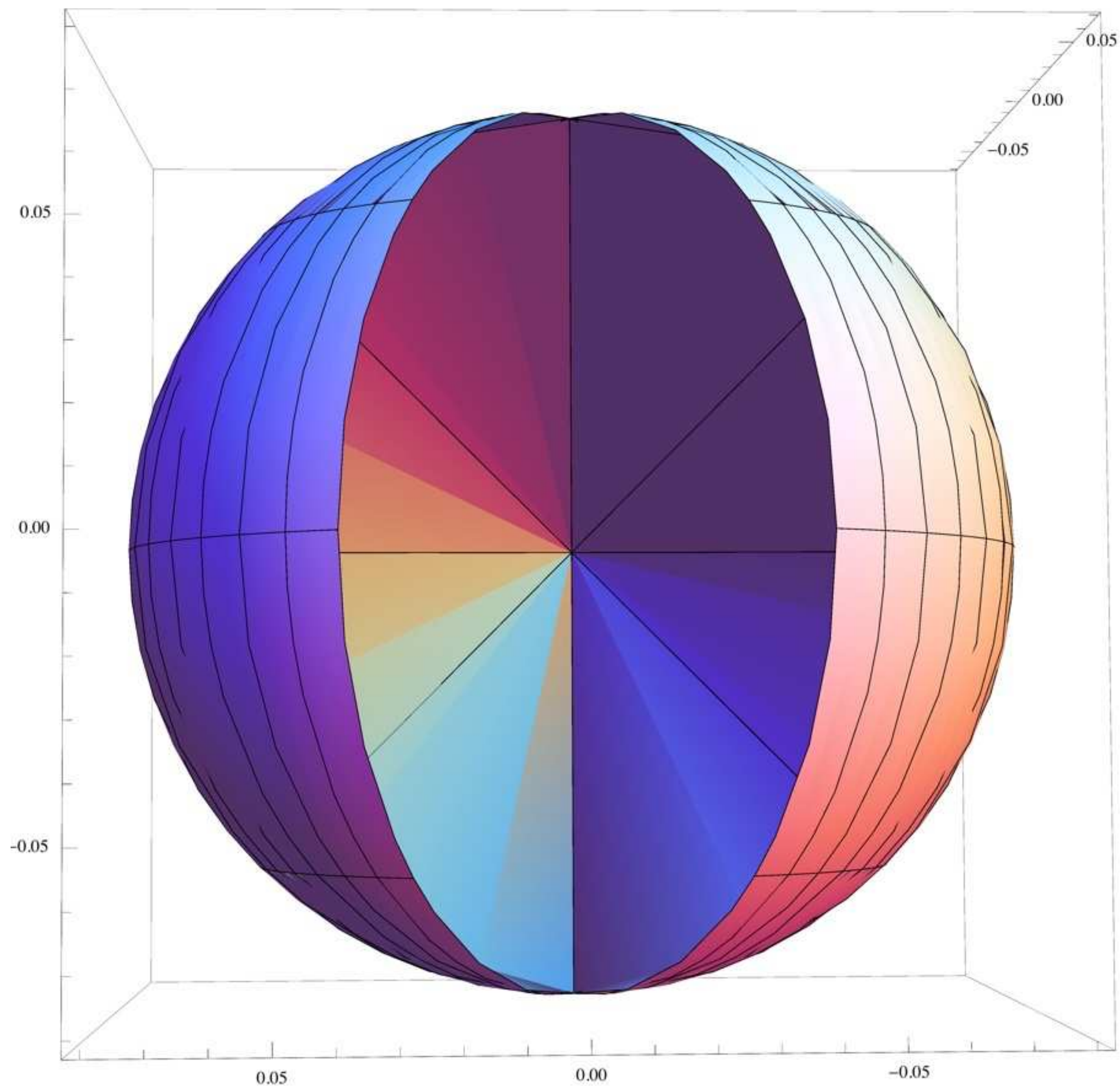
$$\left(-4\pi^2-4\pi\gamma-\gamma^2+\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\mathrm{Csc}\left[\mathrm{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}\right]\right]+4\pi\gamma\,\mathrm{Csc}[\beta]^2+\gamma^2\,\mathrm{Csc}[\beta]^2-\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\mathrm{Csc}\left[\mathrm{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}\right]\right]^3+\frac{4\pi^2\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\mathrm{Csc}\left[\mathrm{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}\right]\right]^3}{(2\pi+\gamma)^2}\right)\Big/$$

$$\left(\pi\left(-16\pi^2-16\pi\gamma-4\gamma^2+16\pi\gamma\,\mathrm{Csc}[\beta]^2+4\gamma^2\,\mathrm{Csc}[\beta]^2\right)\right),\ \{\gamma,-2\pi,2\pi\},\ \{\beta,-\pi/3,\pi/3\}]$$

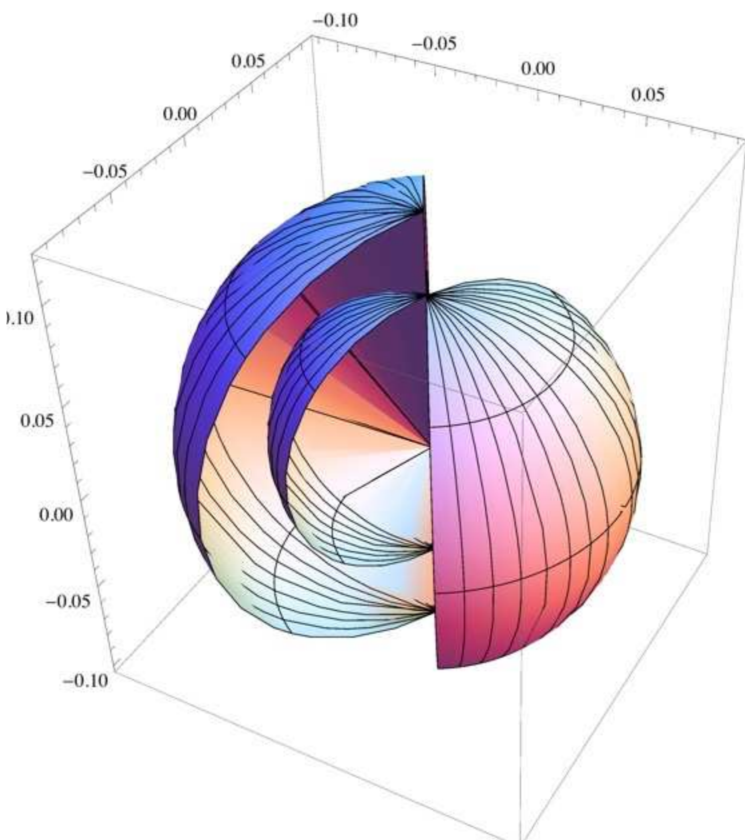

0.05
0.00
-0.05
0.05
0.00
-0.05
0.05
0.00
-0.05

SphericalPlot3D[

$$\left(-4\pi^2 - 4\pi\gamma - \gamma^2 + \sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\;\mathrm{Csc}\left[\mathrm{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}\right]\right] + 4\pi\gamma\,\mathrm{Csc}[\beta]^2 + \gamma^2\,\mathrm{Csc}[\beta]^2 - \sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\;\mathrm{Csc}\left[\mathrm{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}\right]\right]^3 + \frac{4\pi^2\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\;\mathrm{Csc}\left[\mathrm{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}\right]\right]^3}{(2\pi+\gamma)^2}\right) \Big/$$

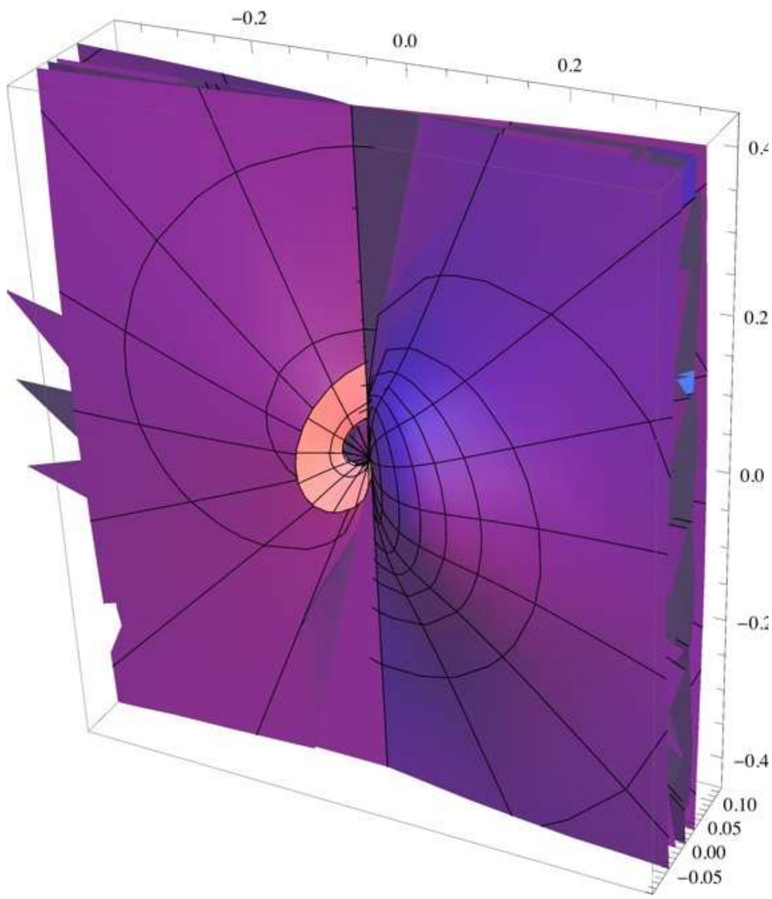
$$\left(\pi\left(-16\pi^2 - 16\pi\gamma - 4\gamma^2 + 16\pi\gamma\,\mathrm{Csc}[\beta]^2 + 4\gamma^2\,\mathrm{Csc}\left[\mathrm{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}\right]\right]^2\right)\right),$$

{γ, -2 π, 2 π}, {β, -π / 3, π / 3}]

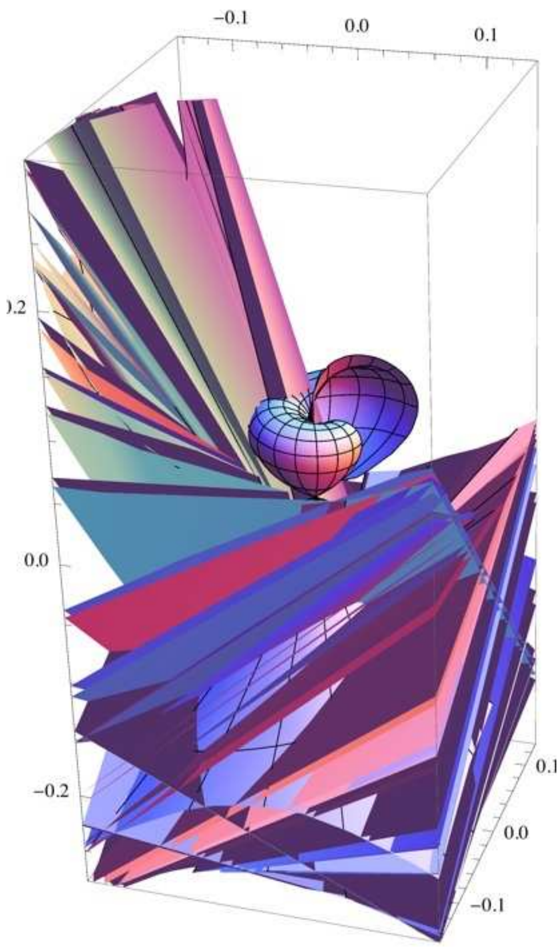


Substituting $\gamma = \dfrac{2\left(\pi - \pi\,\mathrm{Csc}[\beta]^2 + \sqrt{-\pi^2\,\mathrm{Csc}[\beta]^2 + \pi^2\,\mathrm{Csc}[\beta]^4}\right)}{-1 + \mathrm{Csc}[\beta]^2}$

$$\text{SphericalPlot3D}\Big[\Big(-4\pi^2-4\pi\gamma-\gamma^2+\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]+4\pi\gamma\,\text{Csc}[\beta]^2+\left(\frac{2\left(\pi-\pi\,\text{Csc}[\beta]^2+\sqrt{-\pi^2\,\text{Csc}[\beta]^2+\pi^2\,\text{Csc}[\beta]^4}\right)}{-1+\text{Csc}[\beta]^2}\right)^2\text{Csc}[\beta]^2-\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3+\frac{4\pi^2\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3}{(2\pi+\gamma)^2}\Big)\Big/\left(\pi\left(-16\pi^2-16\pi\gamma-4\gamma^2+16\pi\gamma\,\text{Csc}[\beta]^2+4\gamma^2\,\text{Csc}[\beta]^2\right)\right),\ \{\gamma,-2\pi,2\pi\},\ \{\beta,-\pi/3,\pi/3\}\Big]$$



$$\text{SphericalPlot3D}\Big[\Big(-4\pi^2-4\pi\gamma-\gamma^2+\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]+4\pi\gamma\,\text{Csc}[\beta]^2+\left(\frac{2\left(\pi-\pi\,\text{Csc}[\beta]^2+\sqrt{-\pi^2\,\text{Csc}[\beta]^2+\pi^2\,\text{Csc}[\beta]^4}\right)}{-1+\text{Csc}[\beta]^2}\right)^2\text{Csc}[\beta]^2-\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}\left[\text{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}\right]\right]^3+\frac{4\pi^2\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}\left[\text{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}\right]\right]^3}{(2\pi+\gamma)^2}\Big)\Big/\left(\pi\left(-16\pi^2-16\pi\gamma-4\gamma^2+16\pi\gamma\,\text{Csc}[\beta]^2+4\gamma^2\,\text{Csc}[\beta]^2\right)\right),\{\beta,-\pi/3,\pi/3\},\{\gamma,-2\pi,2\pi\}\Big]$$



$$\text{SphericalPlot3D}\Bigg[\Bigg(-4\,\pi^2 - 4\,\pi\,\frac{2\left(\pi - \pi\,\text{Csc}[\beta]^2 + \sqrt{-\pi^2\,\text{Csc}[\beta]^2 + \pi^2\,\text{Csc}[\beta]^4}\right)}{-1 + \text{Csc}[\beta]^2} -$$

$$\left(\frac{2\left(\pi - \pi\,\text{Csc}[\beta]^2 + \sqrt{-\pi^2\,\text{Csc}[\beta]^2 + \pi^2\,\text{Csc}[\beta]^4}\right)}{-1 + \text{Csc}[\beta]^2}\right)^2 +$$

$$\sqrt{\frac{2\left(\pi - \pi\,\text{Csc}[\beta]^2 + \sqrt{-\pi^2\,\text{Csc}[\beta]^2 + \pi^2\,\text{Csc}[\beta]^4}\right)}{-1 + \text{Csc}[\beta]^2}}$$

$$\sqrt{\left(2\,\pi + \frac{2\left(\pi - \pi\,\text{Csc}[\beta]^2 + \sqrt{-\pi^2\,\text{Csc}[\beta]^2 + \pi^2\,\text{Csc}[\beta]^4}\right)}{-1 + \text{Csc}[\beta]^2}\right)^2}$$

$$\sqrt{4\,\pi + \frac{2\left(\pi - \pi\,\text{Csc}[\beta]^2 + \sqrt{-\pi^2\,\text{Csc}[\beta]^2 + \pi^2\,\text{Csc}[\beta]^4}\right)}{-1 + \text{Csc}[\beta]^2}}\;\text{Csc}\left[\text{ArcCsc}\left[\frac{\sqrt{(2\,\pi + \gamma)^2}}{\sqrt{\gamma}\,\sqrt{4\,\pi + \gamma}}\right]\right] +$$

$$4\,\pi\,\frac{2\left(\pi - \pi\,\text{Csc}[\beta]^2 + \sqrt{-\pi^2\,\text{Csc}[\beta]^2 + \pi^2\,\text{Csc}[\beta]^4}\right)}{-1 + \text{Csc}[\beta]^2}\,\text{Csc}\left[\text{ArcCsc}\left[\frac{\sqrt{(2\,\pi + \gamma)^2}}{\sqrt{\gamma}\,\sqrt{4\,\pi + \gamma}}\right]\right]^2 +$$

$$\left(\frac{2\left(\pi - \pi\,\text{Csc}[\beta]^2 + \sqrt{-\pi^2\,\text{Csc}[\beta]^2 + \pi^2\,\text{Csc}[\beta]^4}\right)}{-1 + \text{Csc}[\beta]^2}\right)^2\,\text{Csc}\left[\text{ArcCsc}\left[\frac{\sqrt{(2\,\pi + \gamma)^2}}{\sqrt{\gamma}\,\sqrt{4\,\pi + \gamma}}\right]\right]^2 -$$

$$\sqrt{\frac{2\left(\pi - \pi\,\text{Csc}[\beta]^2 + \sqrt{-\pi^2\,\text{Csc}[\beta]^2 + \pi^2\,\text{Csc}[\beta]^4}\right)}{-1 + \text{Csc}[\beta]^2}}$$

$$\sqrt{\left(2\pi+\frac{2\left(\pi-\pi\,\text{Csc}[\beta]^2+\sqrt{-\pi^2\,\text{Csc}[\beta]^2+\pi^2\,\text{Csc}[\beta]^4}\right)}{-1+\text{Csc}[\beta]^2}\right)^2}$$

$$\sqrt{4\pi+\frac{2\left(\pi-\pi\,\text{Csc}[\beta]^2+\sqrt{-\pi^2\,\text{Csc}[\beta]^2+\pi^2\,\text{Csc}[\beta]^4}\right)}{-1+\text{Csc}[\beta]^2}}\ \text{Csc}\left[\text{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}\right]\right]^3+$$

$$\frac{1}{\left(2\pi+\frac{2\left(\pi-\pi\,\text{Csc}[\beta]^2+\sqrt{-\pi^2\,\text{Csc}[\beta]^2+\pi^2\,\text{Csc}[\beta]^4}\right)}{-1+\text{Csc}[\beta]^2}\right)^2}\,4\pi^2\sqrt{\frac{2\left(\pi-\pi\,\text{Csc}[\beta]^2+\sqrt{-\pi^2\,\text{Csc}[\beta]^2+\pi^2\,\text{Csc}[\beta]^4}\right)}{-1+\text{Csc}[\beta]^2}}$$

$$\sqrt{\left(2\pi+\frac{2\left(\pi-\pi\,\text{Csc}[\beta]^2+\sqrt{-\pi^2\,\text{Csc}[\beta]^2+\pi^2\,\text{Csc}[\beta]^4}\right)}{-1+\text{Csc}[\beta]^2}\right)^2}$$

$$\left.\sqrt{4\pi+\frac{2\left(\pi-\pi\,\text{Csc}[\beta]^2+\sqrt{-\pi^2\,\text{Csc}[\beta]^2+\pi^2\,\text{Csc}[\beta]^4}\right)}{-1+\text{Csc}[\beta]^2}}\ \text{Csc}\left[\text{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}\right]\right]^3\right)\Big/$$

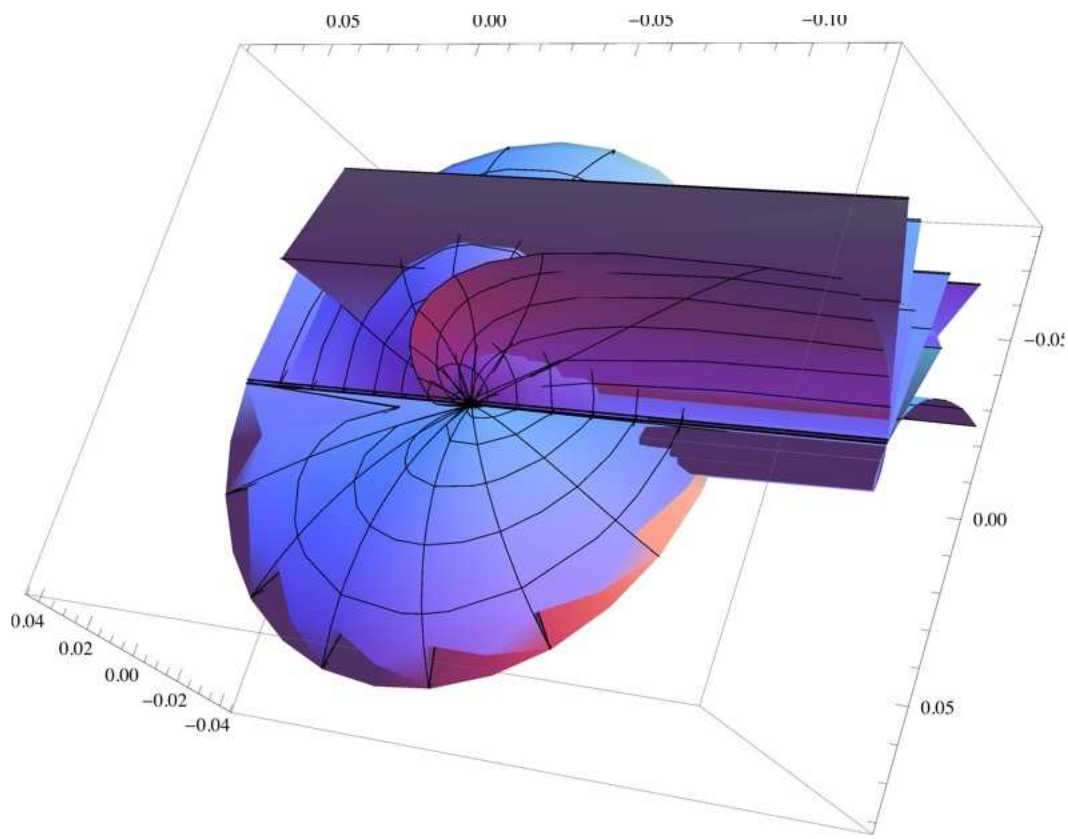
$$\left(\pi\left(-16\pi^2-16\pi\,\frac{2\left(\pi-\pi\,\text{Csc}[\beta]^2+\sqrt{-\pi^2\,\text{Csc}[\beta]^2+\pi^2\,\text{Csc}[\beta]^4}\right)}{-1+\text{Csc}[\beta]^2}-\right.\right.$$

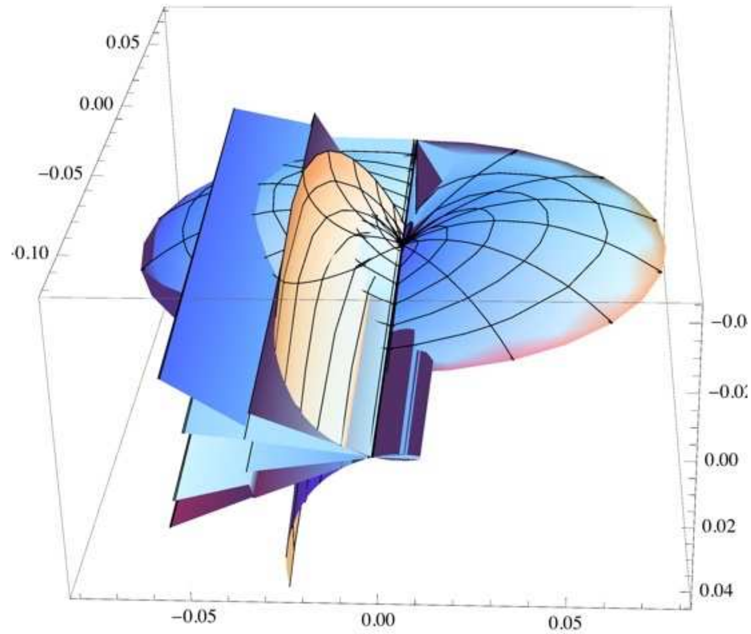
$$4\left(\frac{2\left(\pi-\pi\,\text{Csc}[\beta]^2+\sqrt{-\pi^2\,\text{Csc}[\beta]^2+\pi^2\,\text{Csc}[\beta]^4}\right)}{-1+\text{Csc}[\beta]^2}\right)^2+$$

$$16\,\pi\,\frac{2\left(\pi-\pi\,\text{Csc}[\beta]^2+\sqrt{-\pi^2\,\text{Csc}[\beta]^2+\pi^2\,\text{Csc}[\beta]^4}\right)}{-1+\text{Csc}[\beta]^2}\,\text{Csc}\left[\text{ArcCsc}\left[\frac{\sqrt{(2\,\pi+\gamma)^2}}{\sqrt{\gamma}\,\sqrt{4\,\pi+\gamma}}\right]\right]^2+$$

$$4\left(\frac{2\left(\pi-\pi\,\text{Csc}[\beta]^2+\sqrt{-\pi^2\,\text{Csc}[\beta]^2+\pi^2\,\text{Csc}[\beta]^4}\right)}{-1+\text{Csc}[\beta]^2}\right)^2$$

$$\left.\left.\text{Csc}\left[\text{ArcCsc}\left[\frac{\sqrt{(2\,\pi+\gamma)^2}}{\sqrt{\gamma}\,\sqrt{4\,\pi+\gamma}}\right]\right]^2\right)\right),\ \{\gamma,\ -2\,\pi,\ 2\,\pi\},\ \{\beta,\ -\pi/3,\ \pi/3\}\Big]$$





- β, γ

```
SphericalPlot3D[(-4 π^2 - 4 π γ - γ^2 + Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Csc[β] + 4 π γ Csc[β]^2 + γ^2 Csc[β]^2 -
   Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Csc[β]^3 + (4 π^2 Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Csc[β]^3)/(2 π + γ)^2) /
  (π (-16 π^2 - 16 π γ - 4 γ^2 + 16 π γ Csc[β]^2 + 4 γ^2 Csc[β]^2)), {β, -π / 3, π / 3}, {γ, -2 π, 2 π}]
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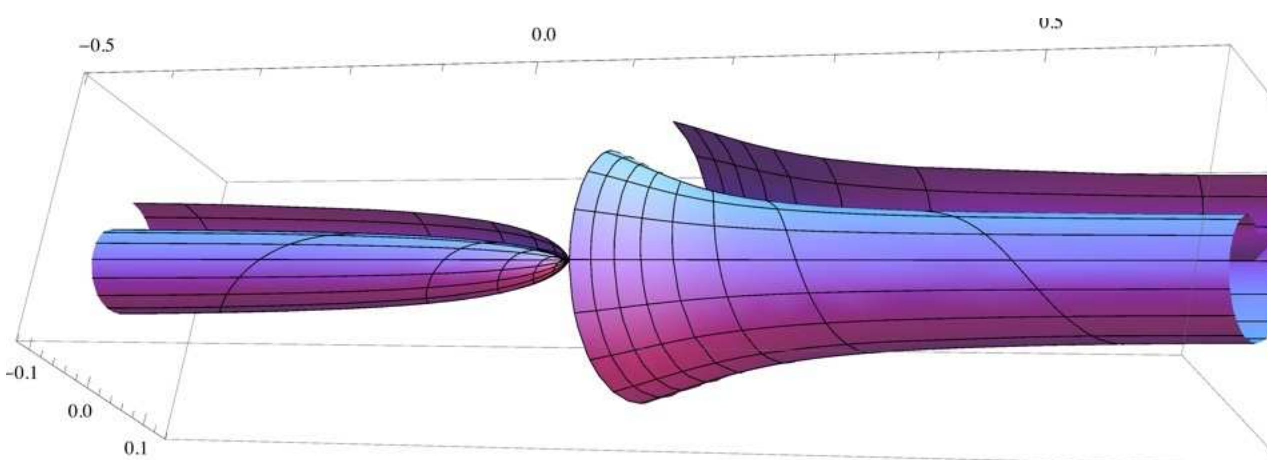

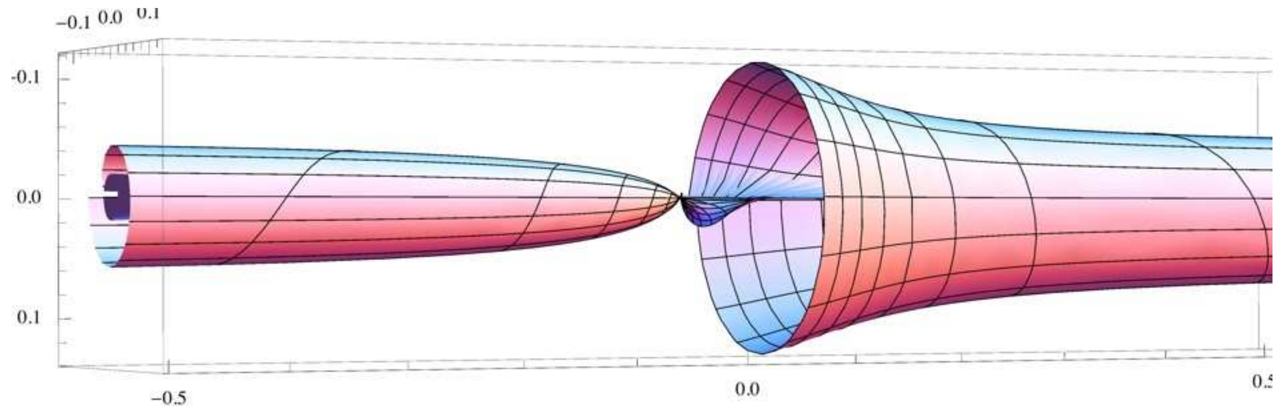

Substituting β = ArcCsc$\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}\right]$

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SphericalPlot3D[
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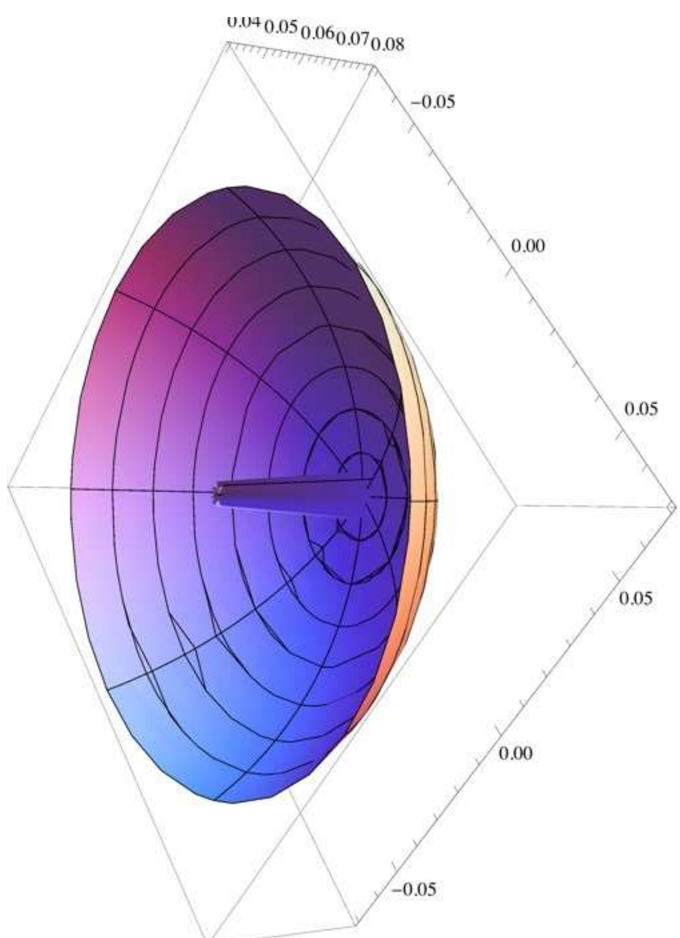
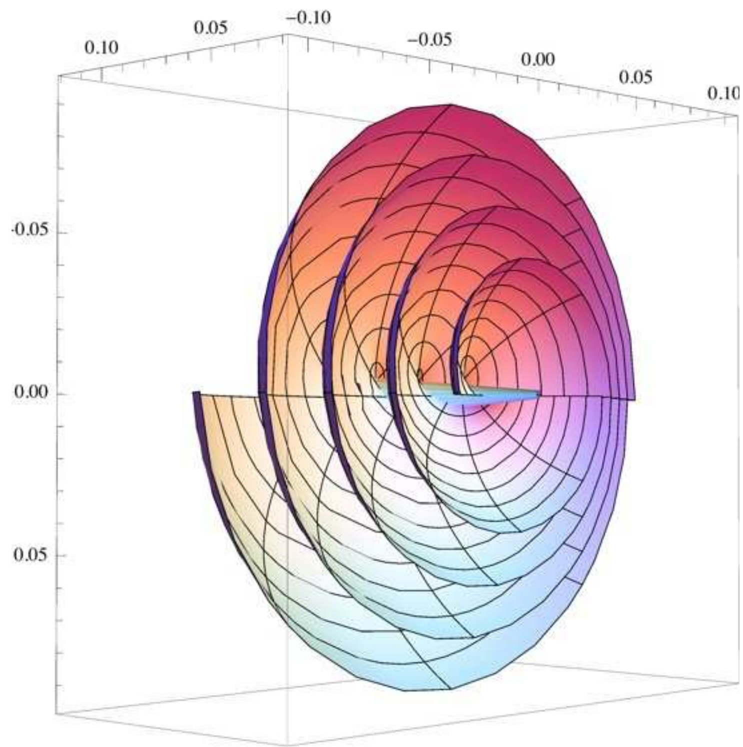
$$\left(-4\pi^2 - 4\pi\gamma - \gamma^2 + \sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\mathrm{Csc}\left[\mathrm{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}\right]\right] + 4\pi\gamma\,\mathrm{Csc}[\beta]^2 + \gamma^2\,\mathrm{Csc}[\beta]^2 - \sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\mathrm{Csc}\left[\mathrm{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}\right]\right]^3 + \frac{4\pi^2\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\mathrm{Csc}\left[\mathrm{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}\right]\right]^3}{(2\pi+\gamma)^2}\right) \Big/$$

$$\left(\pi\left(-16\pi^2 - 16\pi\gamma - 4\gamma^2 + 16\pi\gamma\,\mathrm{Csc}[\beta]^2 + 4\gamma^2\,\mathrm{Csc}[\beta]^2\right)\right), \{\beta, -\pi/3, \pi/3\}, \{\gamma, -2\pi, 2\pi\}]$$



$$\text{SphericalPlot3D}\Big[\left(-4\pi^2-4\pi\gamma-\gamma^2+\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}\left[\text{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}\right]\right]+4\pi\gamma\,\text{Csc}[\beta]^2+\gamma^2\,\text{Csc}[\beta]^2-\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}\left[\text{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}\right]\right]^3+\frac{4\pi^2\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}\left[\text{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}\right]\right]^3}{(2\pi+\gamma)^2}\right)\Big/\left(\pi\left(-16\pi^2-16\pi\gamma-4\gamma^2+16\pi\gamma\,\text{Csc}[\beta]^2+4\gamma^2\,\text{Csc}\left[\text{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}\right]\right]^2\right)\right),$$

$$\{\beta,-\pi/3,\pi/3\},\{\gamma,-2\pi,2\pi\}\Big]$$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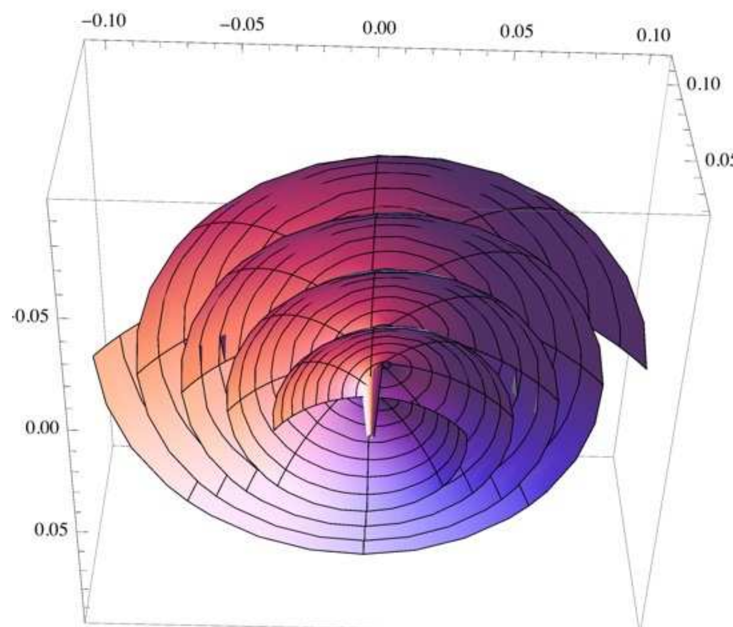
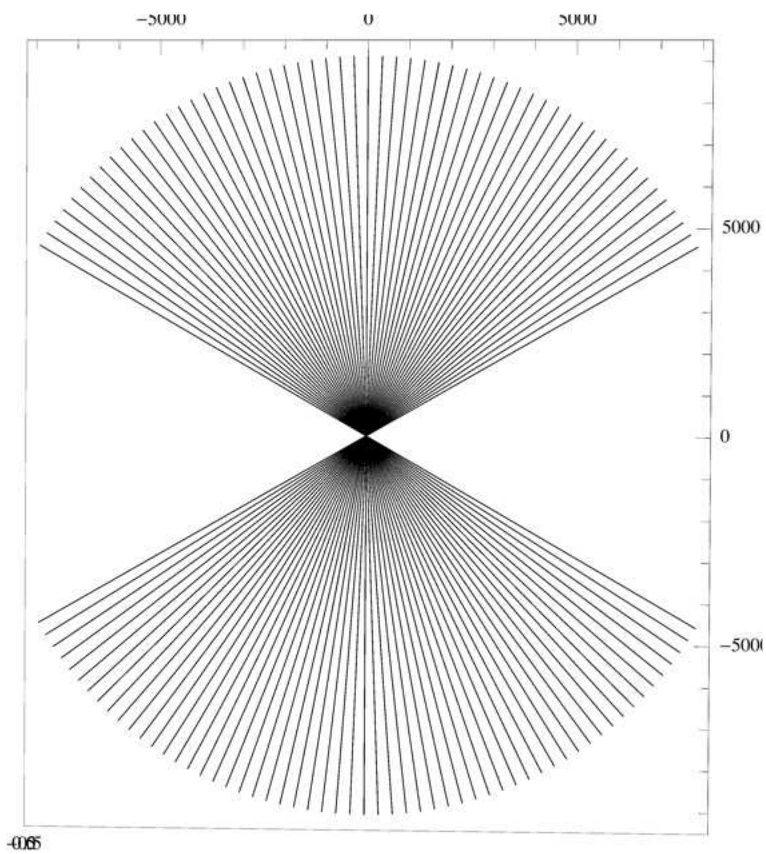
0.05
−0.10
−0.05
0.00
0.05
0.10
-0.05
0.00
0.05


−0.10
−0.05
0.00
0.05
0.10
0.10
-0.05
0.00
0.05

$$\textbf{SphericalPlot3D}\Bigg[\left(-4\pi^2-4\pi\gamma-\gamma^2+\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Csc}\left[\text{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}\right]\right]+4\pi\gamma\,\text{Csc}[\beta]^2+\gamma^2\,\text{Csc}[\beta]^2-\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Csc}\left[\text{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}\right]\right]^3+\frac{4\pi^2\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Csc}\left[\text{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}\right]\right]^3}{(2\pi+\gamma)^2}\right)\Big/\left(\pi\left(-16\pi^2-16\pi\gamma-4\gamma^2+16\pi\gamma\,\text{Csc}\left[\text{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}\right]\right]^2+4\gamma^2\,\text{Csc}[\beta]^2\right)\right),\ \{\beta,-\pi/3,\pi/3\},\ \{\gamma,-2\pi,2\pi\}\Bigg]$$

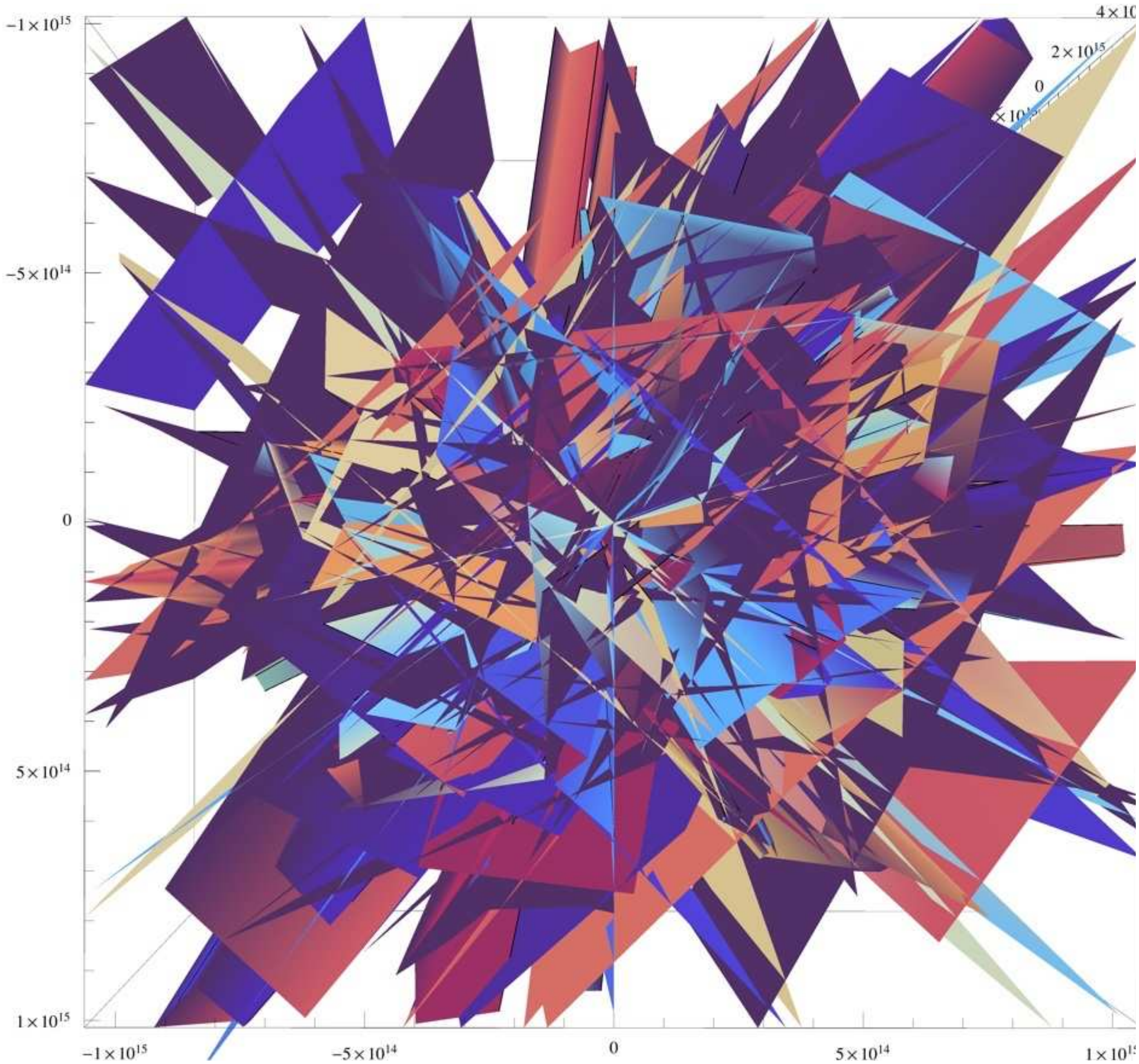


```
SphericalPlot3D[
```

$$\left(-4\pi^2-4\pi\gamma-\gamma^2+\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\mathrm{Csc}\left[\mathrm{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}\right]\right]+4\pi\gamma\,\mathrm{Csc}[\beta]^2+\gamma^2\,\mathrm{Csc}[\beta]^2-\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\mathrm{Csc}\left[\mathrm{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}\right]\right]^3+\frac{4\pi^2\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\mathrm{Csc}\left[\mathrm{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}\right]\right]^3}{(2\pi+\gamma)^2}\right)\Big/$$

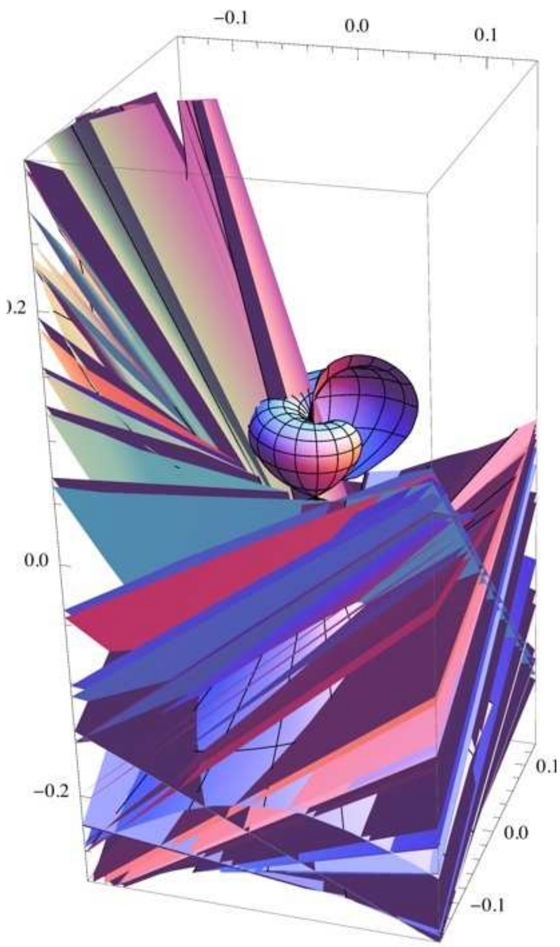
$$\left(\pi\left(-16\pi^2-16\pi\gamma-4\gamma^2+16\pi\gamma\,\mathrm{Csc}\left[\mathrm{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}\right]\right]^2+4\gamma^2\,\mathrm{Csc}\left[\mathrm{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}\right]\right]^2\right)\right),$$

```
{β, -π / 3, π / 3}, {γ, -2 π, 2 π}]
```

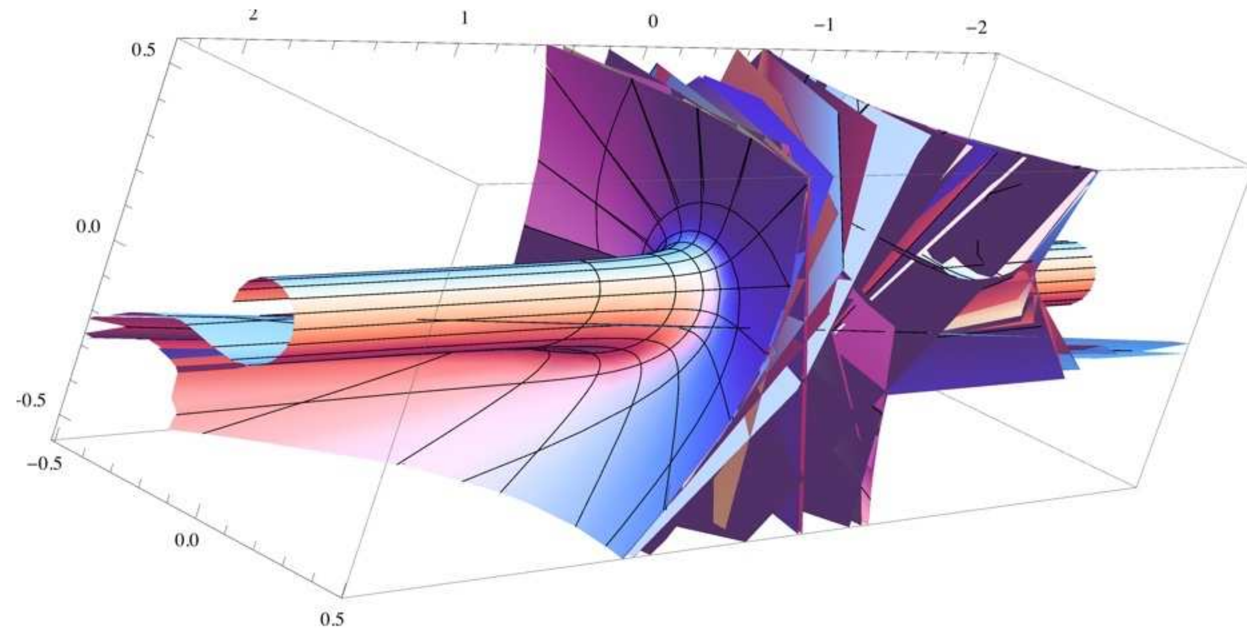


Substituting $\gamma = \frac{2\left(\pi - \pi\,\text{Csc}[\beta]^2 + \sqrt{-\pi^2\,\text{Csc}[\beta]^2 + \pi^2\,\text{Csc}[\beta]^4}\right)}{-1 + \text{Csc}[\beta]^2}$

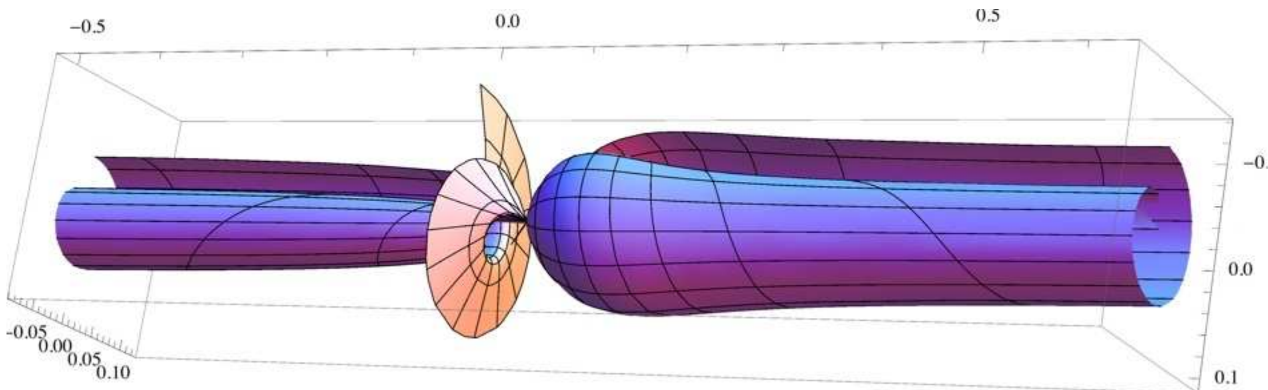
```
SphericalPlot3D[(-4 π^2 - 4 π γ - γ^2 + Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Csc[β] + 4 π γ Csc[β]^2 +
   ((2 (π - π Csc[β]^2 + Sqrt[-π^2 Csc[β]^2 + π^2 Csc[β]^4]))/(-1 + Csc[β]^2))^2 Csc[β]^2 - Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ]
   Csc[ArcCsc[Sqrt[(2 π + γ)^2]/(Sqrt[γ] Sqrt[4 π + γ])]]^3 + (4 π^2 Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Csc[ArcCsc[Sqrt[(2 π + γ)^2]/(Sqrt[γ] Sqrt[4 π + γ])]]^3)/(2 π + γ)^2) /
  (π (-16 π^2 - 16 π γ - 4 γ^2 + 16 π γ Csc[β]^2 + 4 γ^2 Csc[β]^2)), {β, -π / 3, π / 3}, {γ, -2 π, 2 π}]
```

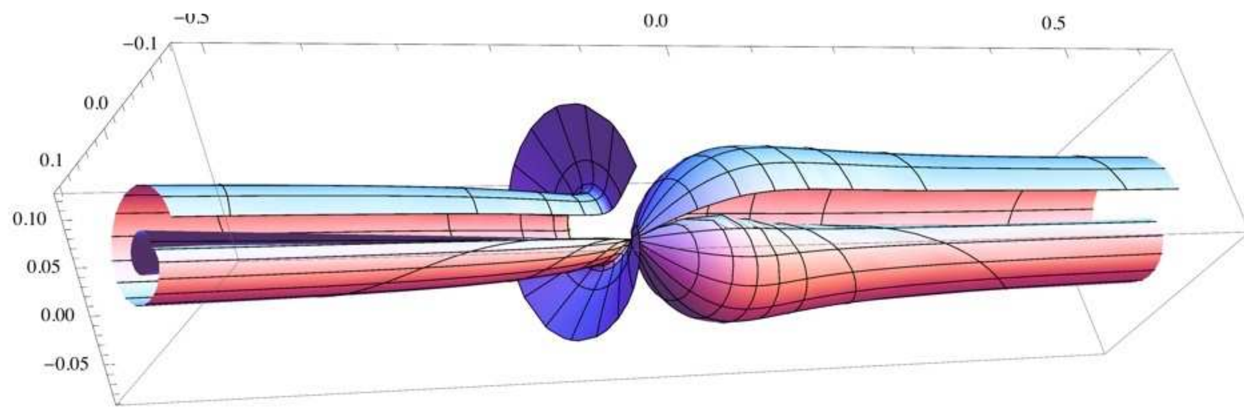


$$\text{SphericalPlot3D}\Bigg[\Bigg(-4\pi^2-4\pi\gamma-\gamma^2+\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]+4\pi\gamma\,\text{Csc}[\beta]^2+\left(\frac{2\left(\pi-\pi\,\text{Csc}[\beta]^2+\sqrt{-\pi^2\,\text{Csc}[\beta]^2+\pi^2\,\text{Csc}[\beta]^4}\right)}{-1+\text{Csc}[\beta]^2}\right)^2\text{Csc}[\beta]^2-\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3+\frac{4\pi^2\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Csc}\left[\text{ArcCsc}\left[\frac{\sqrt{(2\pi+\gamma)^2}}{\sqrt{\gamma}\sqrt{4\pi+\gamma}}\right]\right]^3}{(2\pi+\gamma)^2}\Bigg)\Big/\left(\pi\left(-16\pi^2-16\pi\gamma-4\gamma^2+16\pi\gamma\,\text{Csc}[\beta]^2+4\gamma^2\,\text{Csc}[\beta]^2\right)\right),\ \{\beta,-\pi/3,\pi/3\},\ \{\gamma,-2\pi,2\pi\}\Bigg]$$



```
SphericalPlot3D[(-4 π^2 - 4 π γ - γ^2 + Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Csc[β] +
   4 π γ Csc[β]^2 + ((2 (π - π Csc[β]^2 + Sqrt[-π^2 Csc[β]^2 + π^2 Csc[β]^4]))/(-1 + Csc[β]^2))^2 Csc[β]^2 -
   Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Csc[β]^3 + (4 π^2 Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Csc[β]^3)/(2 π + γ)^2)/
  (π (-16 π^2 - 16 π γ - 4 γ^2 + 16 π γ Csc[β]^2 + 4 γ^2 Csc[β]^2)), {β, -π / 3, π / 3}, {γ, -2 π, 2 π}]
```





- **II. For 1 = $\frac{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}\,\mathrm{Sin}[\beta]}{\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}$, the construction of the formula is:**

$$\text{Solve}\Big[\frac{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}\,\text{Sin}[\beta]}{\sqrt{\gamma}\,\sqrt{4\pi+\gamma}} - 1 ==$$

$$2\pi r - 2\pi\sqrt{r^2 - \frac{r\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\text{^}2} - \gamma\sqrt{r^2 - \frac{r\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\text{^}2}\text{ , } r\Big]$$

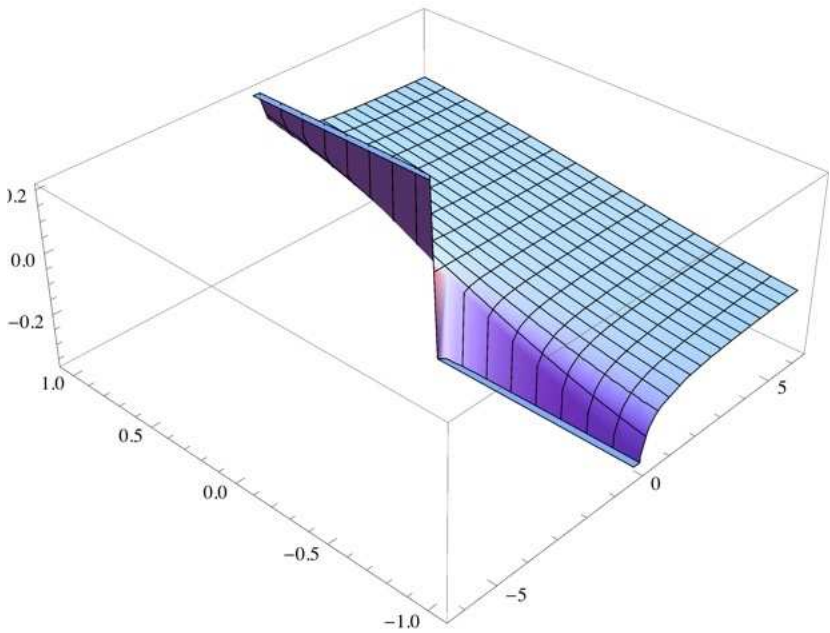
$$\Bigg\{\Bigg\{r \to \Bigg(4\pi\gamma+\gamma^2-\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Sin}[\beta] -$$

$$4\pi^2\,\text{Sin}[\beta]^2 - 4\pi\gamma\,\text{Sin}[\beta]^2 - \gamma^2\,\text{Sin}[\beta]^2 - \frac{\pi\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\text{Sin}[\beta]^3}{\sqrt{4\pi+\gamma}} +$$

$$\frac{\pi\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3}{\sqrt{\gamma}} + \sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3\Bigg)\Big/$$

$$\left(\pi\left(-16\pi\gamma - 4\gamma^2 + 16\pi^2\,\text{Sin}[\beta]^2 + 16\pi\gamma\,\text{Sin}[\beta]^2 + 4\gamma^2\,\text{Sin}[\beta]^2\right)\right)\Bigg\}\Bigg\}$$

$$\texttt{Plot3D}\Big[\Big(4\pi\gamma+\gamma^2-\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\texttt{Sin}[\beta]-4\pi^2\texttt{Sin}[\beta]^2-4\pi\gamma\,\texttt{Sin}[\beta]^2-\gamma^2\texttt{Sin}[\beta]^2-\frac{\pi\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\,\texttt{Sin}[\beta]^3}{\sqrt{4\pi+\gamma}}+\frac{\pi\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\texttt{Sin}[\beta]^3}{\sqrt{\gamma}}+\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\texttt{Sin}[\beta]^3\Big)\Big/\Big(\pi\left(-16\pi\gamma-4\gamma^2+16\pi^2\texttt{Sin}[\beta]^2+16\pi\gamma\,\texttt{Sin}[\beta]^2+4\gamma^2\texttt{Sin}[\beta]^2\right)\Big),\ \{\gamma,-2\pi,2\pi\},\ \{\beta,-\pi/3,\pi/3\}\Big]$$



- **Letter B includes substituting:**

B.

$$1-\frac{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}\ \texttt{Sin}[\beta]}{\sqrt{\gamma}\sqrt{4\pi+\gamma}} == 2\pi r-2\pi\sqrt{r^2-\frac{r\sqrt{\gamma}\sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\,\hat{}\,2}-\gamma\sqrt{r^2-\frac{r\sqrt{\gamma}\sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\,\hat{}\,2}$$

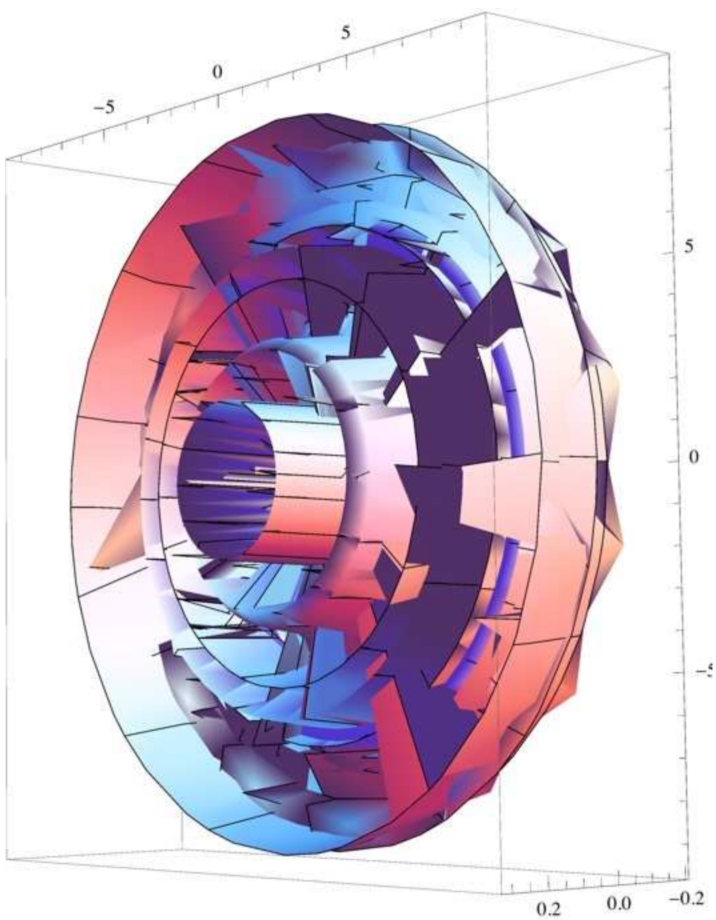
Substituting : $\left\{\left\{\beta\to\texttt{ArcSin}\left[\frac{\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]\right\}\right\}$ and

$\left\{\gamma\to\frac{2\left(\pi-\pi\,\texttt{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\texttt{Sin}[\beta]^2}\right)}{-1+\texttt{Sin}[\beta]^2}\right\}$ into the resulting radius solution.

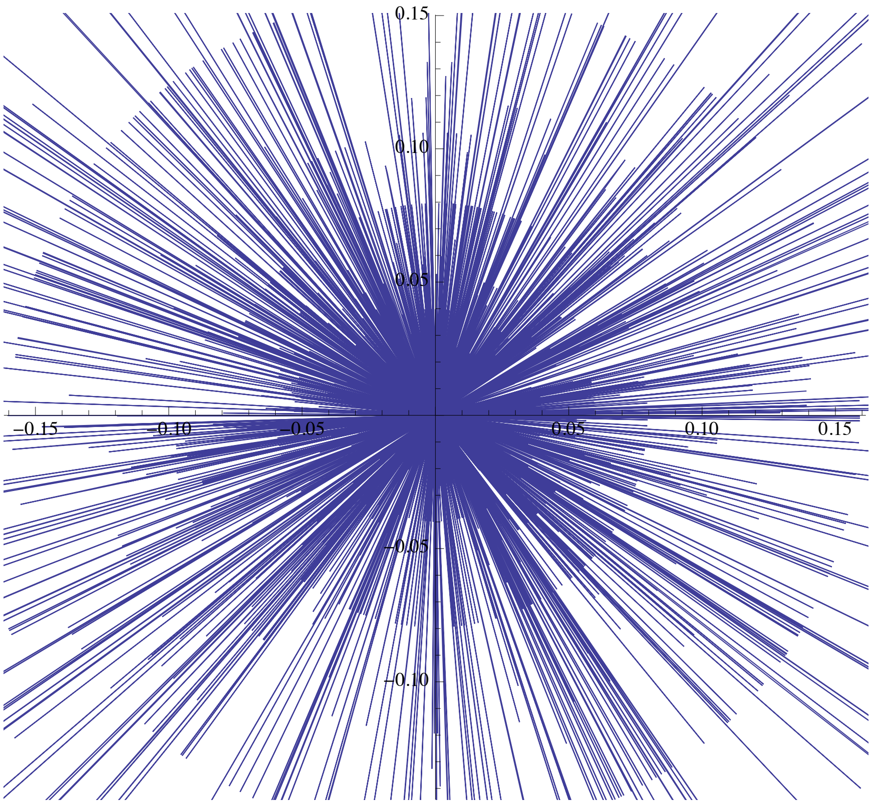
- **γ only**

$$\beta := \text{ArcSin}\left[\frac{\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]$$

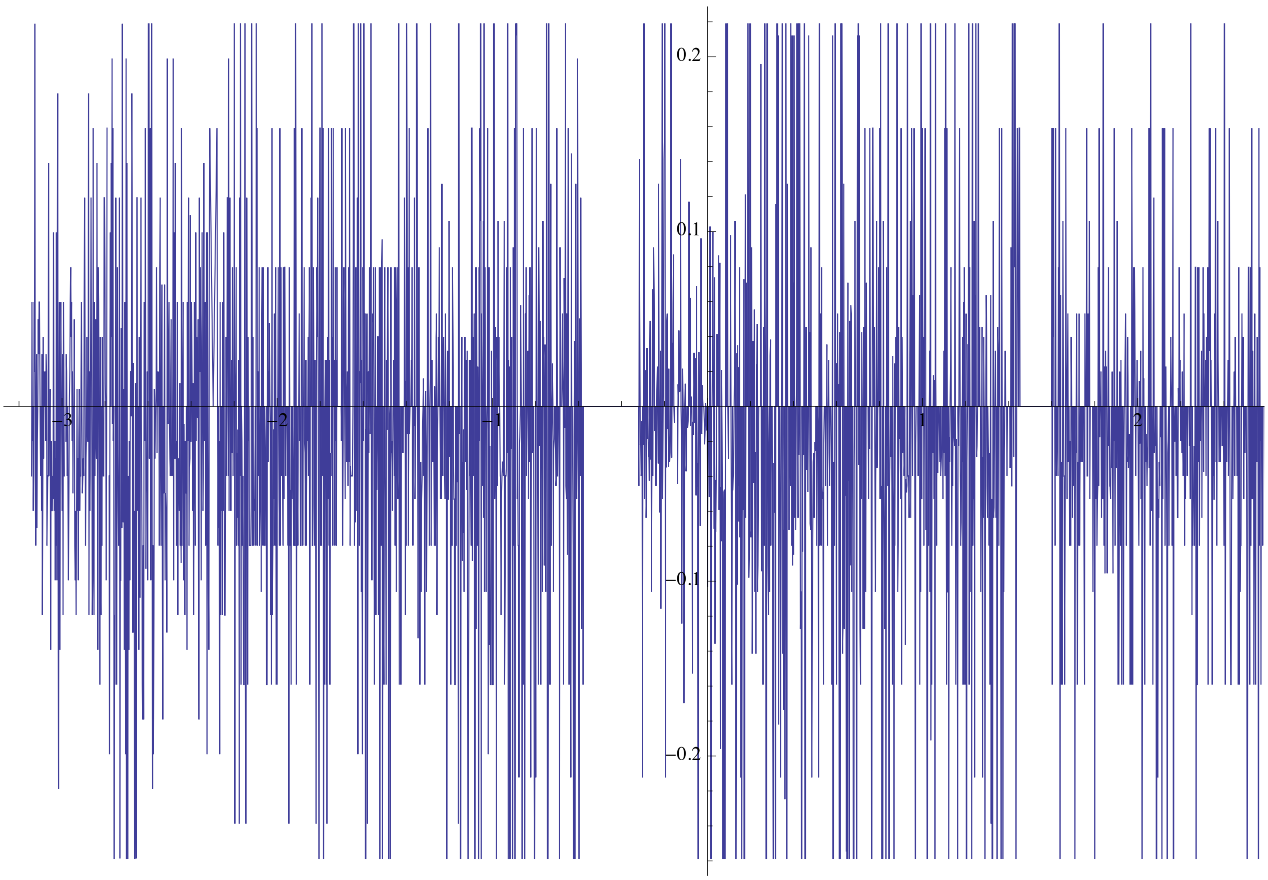
$$\text{RevolutionPlot3D}\Big[\Big(4\pi\gamma+\gamma^2-\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]-4\pi^2\,\text{Sin}[\beta]^2-4\pi\gamma\,\text{Sin}[\beta]^2-\gamma^2\,\text{Sin}[\beta]^2-\frac{\pi\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\text{Sin}[\beta]^3}{\sqrt{4\pi+\gamma}}+\frac{\pi\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3}{\sqrt{\gamma}}+\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3\Big)\Big/\left(\pi\left(-16\pi\gamma-4\gamma^2+16\pi^2\,\text{Sin}[\beta]^2+16\pi\gamma\,\text{Sin}[\beta]^2+4\gamma^2\,\text{Sin}[\beta]^2\right)\right),\ \{\gamma,-3\pi,3\pi\}\Big]$$



$$\text{PolarPlot}\Big[\Big(4\pi\gamma+\gamma^2-\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]-4\pi^2\,\text{Sin}[\beta]^2-4\pi\gamma\,\text{Sin}[\beta]^2-\gamma^2\,\text{Sin}[\beta]^2-\frac{\pi\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\,\text{Sin}[\beta]^3}{\sqrt{4\pi+\gamma}}+\frac{\pi\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3}{\sqrt{\gamma}}+\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3\Big)\Big/\left(\pi\left(-16\pi\gamma-4\gamma^2+16\pi^2\,\text{Sin}[\beta]^2+16\pi\gamma\,\text{Sin}[\beta]^2+4\gamma^2\,\text{Sin}[\beta]^2\right)\right),\ \{\gamma,-\pi,\pi\}\Big]$$



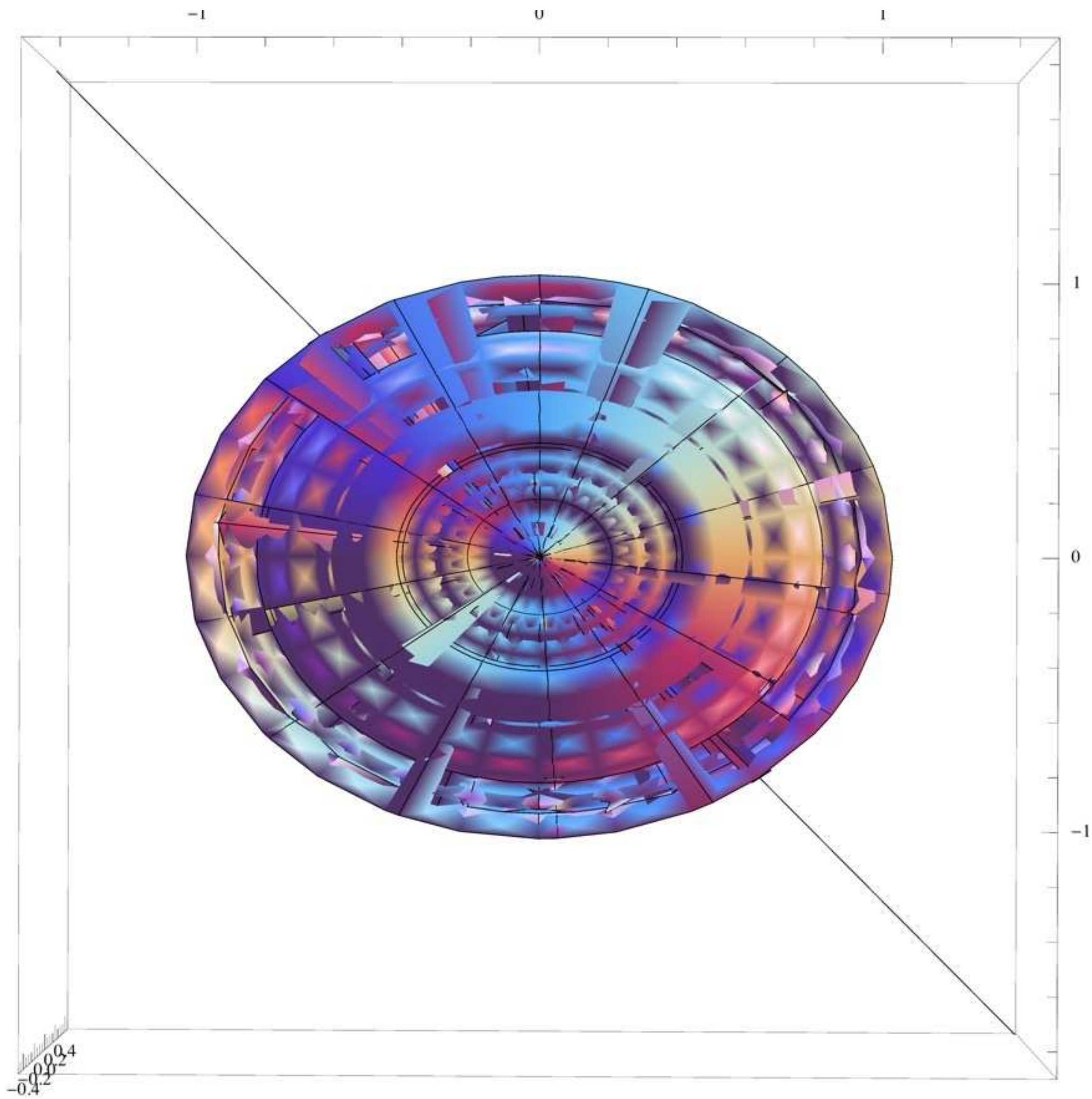
$$\text{Plot}\Big[\Big(4\pi\gamma+\gamma^2-\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]-4\pi^2\,\text{Sin}[\beta]^2-4\pi\gamma\,\text{Sin}[\beta]^2-\gamma^2\,\text{Sin}[\beta]^2-\frac{\pi\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\,\text{Sin}[\beta]^3}{\sqrt{4\pi+\gamma}}+\frac{\pi\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3}{\sqrt{\gamma}}+\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3\Big)\Big/\left(\pi\left(-16\pi\gamma-4\gamma^2+16\pi^2\,\text{Sin}[\beta]^2+16\pi\gamma\,\text{Sin}[\beta]^2+4\gamma^2\,\text{Sin}[\beta]^2\right)\right),\ \{\gamma,-\pi,\pi\}\Big]$$



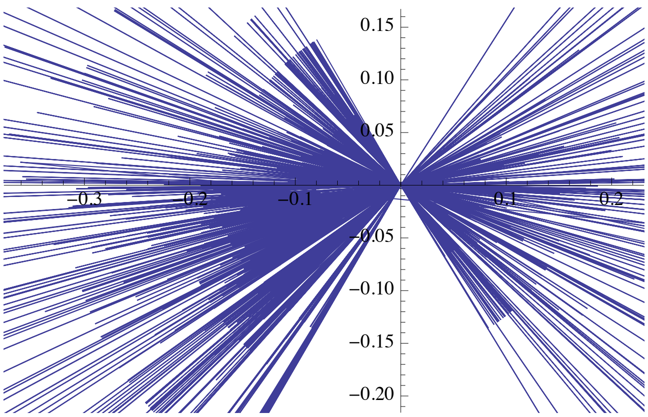
- ***β* only**

$$\gamma := \frac{2\left(\pi-\pi\,\text{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)}{-1+\text{Sin}[\beta]^2}$$

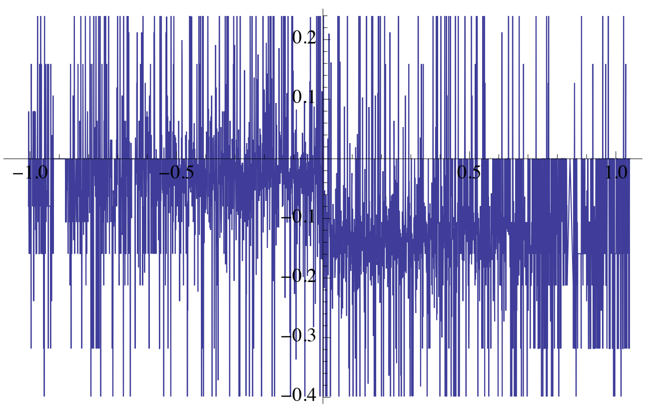
```
RevolutionPlot3D[(4 π γ + γ^2 - Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Sin[β] -
    4 π^2 Sin[β]^2 - 4 π γ Sin[β]^2 - γ^2 Sin[β]^2 - (π Sqrt[γ] Sqrt[(2 π + γ)^2] Sin[β]^3)/Sqrt[4 π + γ] +
    (π Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Sin[β]^3)/Sqrt[γ] + Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Sin[β]^3) /
  (π (-16 π γ - 4 γ^2 + 16 π^2 Sin[β]^2 + 16 π γ Sin[β]^2 + 4 γ^2 Sin[β]^2)), {β, -π / 3, π / 3}]
```



```
PolarPlot[(4 π γ + γ^2 - Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Sin[β] -
    4 π^2 Sin[β]^2 - 4 π γ Sin[β]^2 - γ^2 Sin[β]^2 - (π Sqrt[γ] Sqrt[(2 π + γ)^2] Sin[β]^3)/Sqrt[4 π + γ] +
    (π Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Sin[β]^3)/Sqrt[γ] + Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Sin[β]^3) /
  (π (-16 π γ - 4 γ^2 + 16 π^2 Sin[β]^2 + 16 π γ Sin[β]^2 + 4 γ^2 Sin[β]^2)), {β, -π / 3, π / 3}]
```

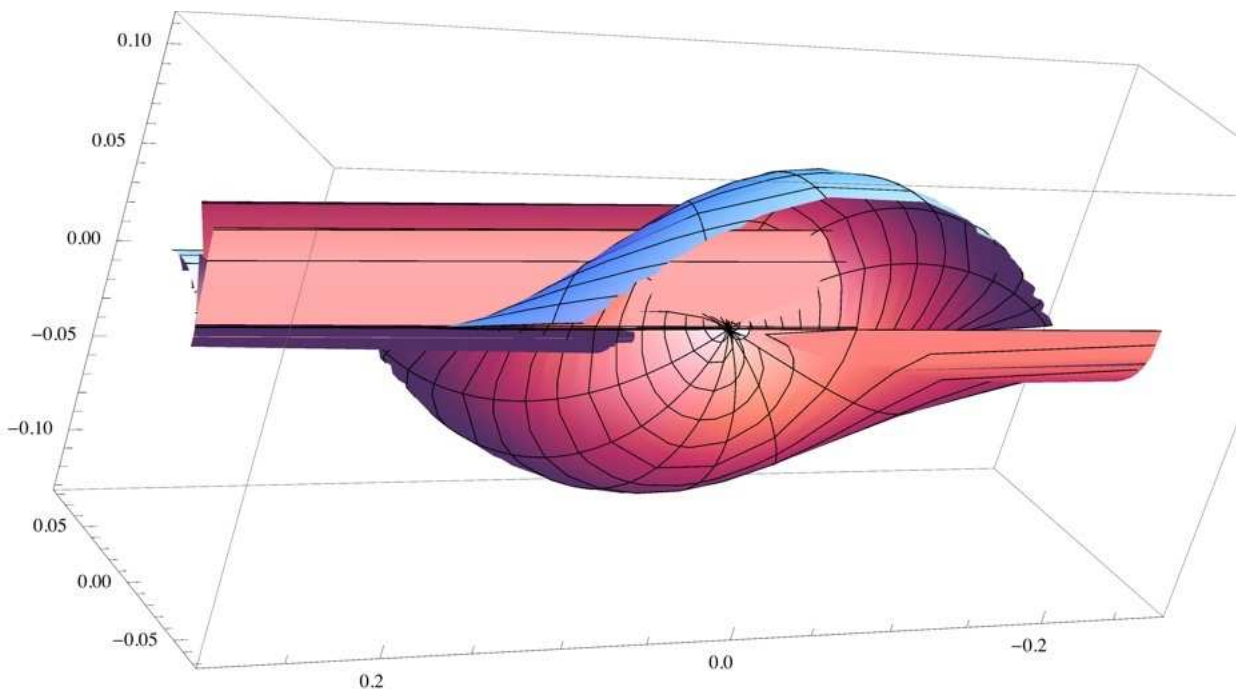



```
Plot[(4 π γ + γ^2 - Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Sin[β] -
    4 π^2 Sin[β]^2 - 4 π γ Sin[β]^2 - γ^2 Sin[β]^2 - (π Sqrt[γ] Sqrt[(2 π + γ)^2] Sin[β]^3)/Sqrt[4 π + γ] +
    (π Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Sin[β]^3)/Sqrt[γ] + Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Sin[β]^3) /
  (π (-16 π γ - 4 γ^2 + 16 π^2 Sin[β]^2 + 16 π γ Sin[β]^2 + 4 γ^2 Sin[β]^2)), {β, -π / 3, π / 3}]
```

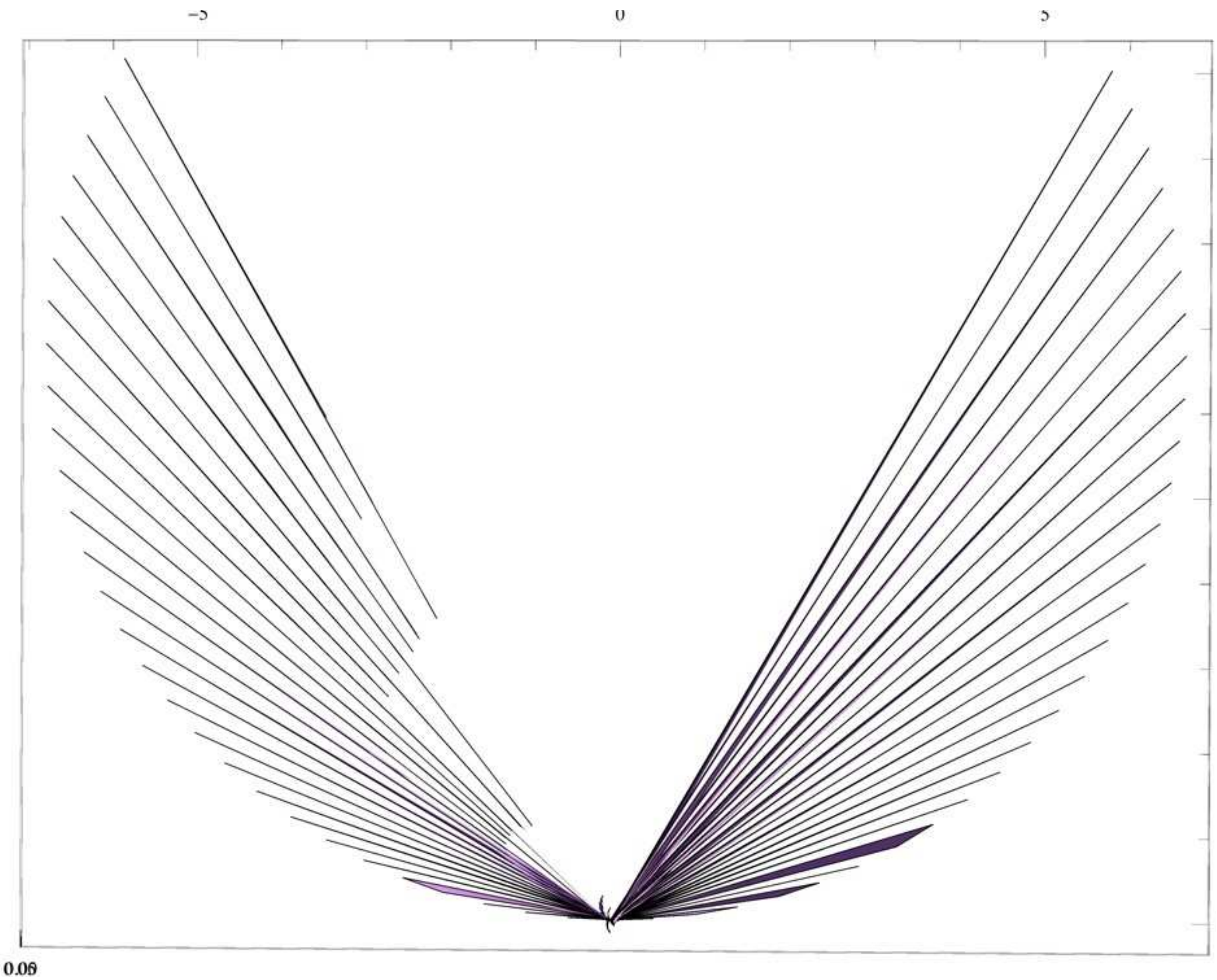


γ, β

$$\text{SphericalPlot3D}\Big[\Big(4\,\pi\,\gamma + \gamma^2 - \sqrt{\gamma}\,\sqrt{(2\,\pi + \gamma)^2}\,\sqrt{4\,\pi + \gamma}\,\text{Sin}[\beta] - 4\,\pi^2\,\text{Sin}[\beta]^2 - 4\,\pi\,\gamma\,\text{Sin}[\beta]^2 - \gamma^2\,\text{Sin}[\beta]^2 - \frac{\pi\,\sqrt{\gamma}\,\sqrt{(2\,\pi + \gamma)^2}\,\text{Sin}[\beta]^3}{\sqrt{4\,\pi + \gamma}} + \frac{\pi\,\sqrt{(2\,\pi + \gamma)^2}\,\sqrt{4\,\pi + \gamma}\,\text{Sin}[\beta]^3}{\sqrt{\gamma}} + \sqrt{\gamma}\,\sqrt{(2\,\pi + \gamma)^2}\,\sqrt{4\,\pi + \gamma}\,\text{Sin}[\beta]^3\Big) \Big/ \Big(\pi\,\big(-16\,\pi\,\gamma - 4\,\gamma^2 + 16\,\pi^2\,\text{Sin}[\beta]^2 + 16\,\pi\,\gamma\,\text{Sin}[\beta]^2 + 4\,\gamma^2\,\text{Sin}[\beta]^2\big)\Big),\ \{\gamma, -2\,\pi, 2\,\pi\},\ \{\beta, -\pi/3, \pi/3\}\Big]$$



$$\textbf{SphericalPlot3D}\Big[\Big(4\pi\gamma+\gamma^2-\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]-4\pi^2\,\text{Sin}[\beta]^2-4\pi\gamma\,\text{Sin}[\beta]^2-\gamma^2\,\text{Sin}[\beta]^2-\frac{\pi\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\text{Sin}[\beta]^3}{\sqrt{4\pi+\gamma}}+\frac{\pi\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3}{\sqrt{\gamma}}+\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3\Big)\Big/\Big(\pi\left(-16\pi\gamma-4\gamma^2+16\pi^2\,\text{Sin}[\beta]^2+16\pi\gamma\,\text{Sin}[\beta]^2+4\gamma^2\,\text{Sin}[\beta]^2\right)\Big),\ \{\beta,-\pi/3,\pi/3\},\ \{\gamma,-2\pi,2\pi\}\Big]$$



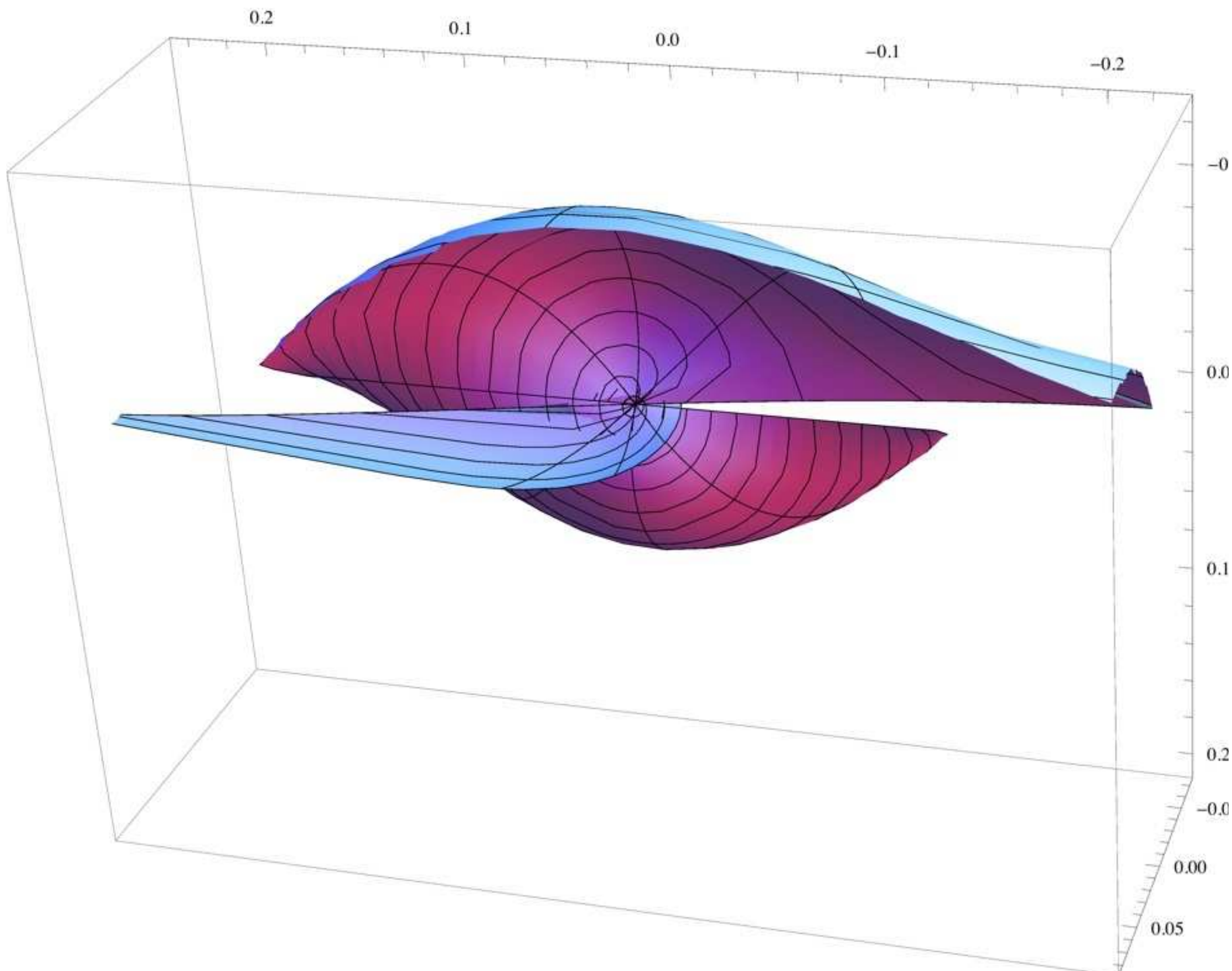
We can use the following substitutions to make the graphs more aesthetically pleasing :

Substituting :

$$\left\{\left\{\beta\to\text{ArcSin}\left[\frac{\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]\right\}\right\}\ \textbf{and}\ \left\{\gamma\to\frac{2\left(\pi-\pi\,\text{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)}{-1+\text{Sin}[\beta]^2}\right\}$$

$$\beta \to \text{ArcSin}\left[\frac{\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]$$

$$\text{SphericalPlot3D}\left[\left(4\pi\gamma+\gamma^2-\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}\ \text{Sin}[\beta]-4\pi^2\,\text{Sin}[\beta]^2-\right.\right.$$
$$4\pi\gamma\,\text{Sin}[\beta]^2-\gamma^2\,\text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]\right]^2-\frac{\pi\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \text{Sin}[\beta]^3}{\sqrt{4\pi+\gamma}}+$$
$$\left.\frac{\pi\sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}\ \text{Sin}[\beta]^3}{\sqrt{\gamma}}+\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}\ \text{Sin}[\beta]^3\right)\Big/$$
$$\left.\left(\pi\left(-16\pi\gamma-4\gamma^2+16\pi^2\,\text{Sin}[\beta]^2+16\pi\gamma\,\text{Sin}[\beta]^2+4\gamma^2\,\text{Sin}[\beta]^2\right)\right),\ \{\gamma,-2\pi,2\pi\},\ \{\beta,-\pi/3,\pi/3\}\right]$$



$$\gamma=\frac{2\left(\pi-\pi\,\text{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)}{-1+\text{Sin}[\beta]^2}$$

SphericalPlot3D$\Bigg[$

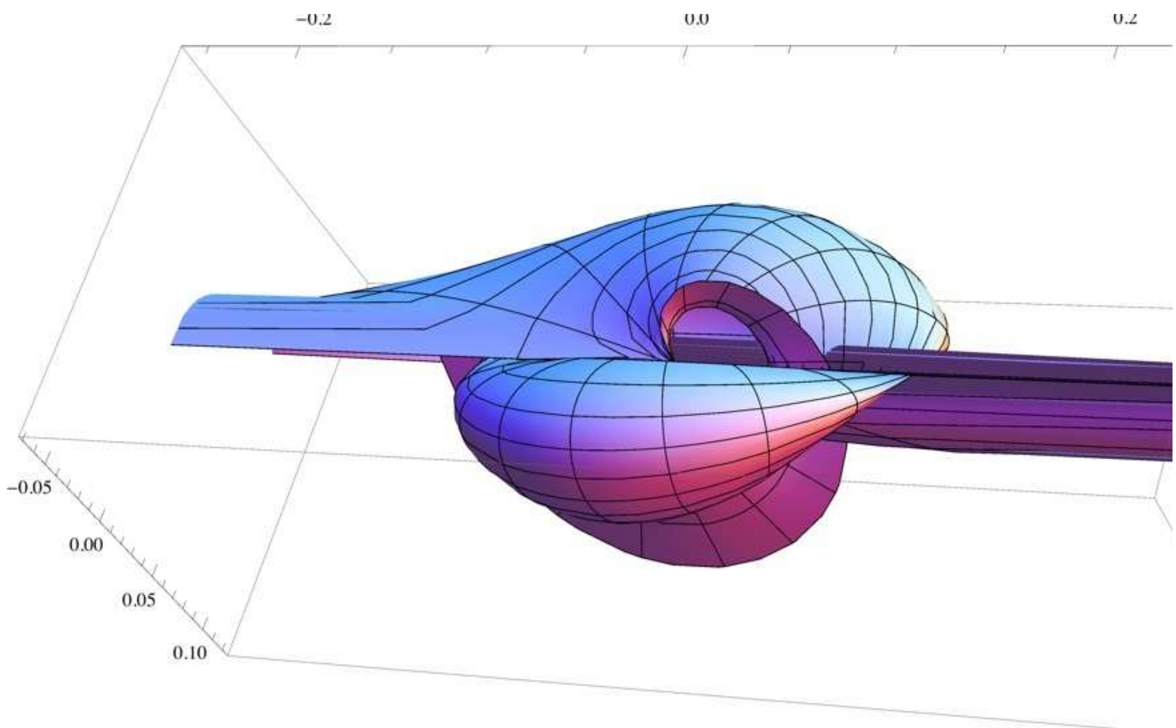
$$\left(4\,\pi\,\gamma+\gamma^2-\sqrt{\gamma}\,\sqrt{\left(2\,\pi+\frac{2\left(\pi-\pi\,\text{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)}{-1+\text{Sin}[\beta]^2}\right)^2}\,\sqrt{4\,\pi+\gamma}\,\text{Sin}[\beta]-4\,\pi^2\,\text{Sin}[\beta]^2-\right.$$

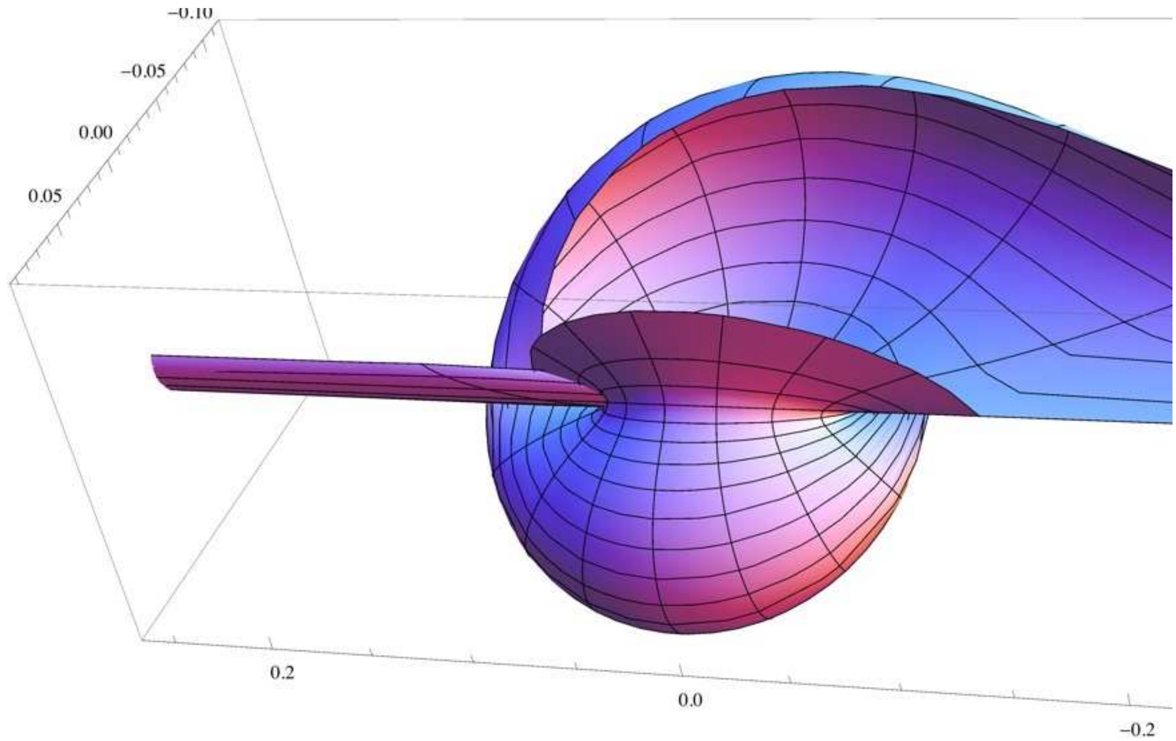
$$4\,\pi\,\gamma\,\text{Sin}[\beta]^2-\gamma^2\,\text{Sin}[\beta]^2-\frac{\pi\,\sqrt{\gamma}\,\sqrt{(2\,\pi+\gamma)^2}\,\text{Sin}[\beta]^3}{\sqrt{4\,\pi+\gamma}}+$$

$$\left.\frac{\pi\,\sqrt{(2\,\pi+\gamma)^2}\,\sqrt{4\,\pi+\gamma}\,\text{Sin}[\beta]^3}{\sqrt{\gamma}}+\sqrt{\gamma}\,\sqrt{(2\,\pi+\gamma)^2}\,\sqrt{4\,\pi+\gamma}\,\text{Sin}[\beta]^3\right)\Big/$$

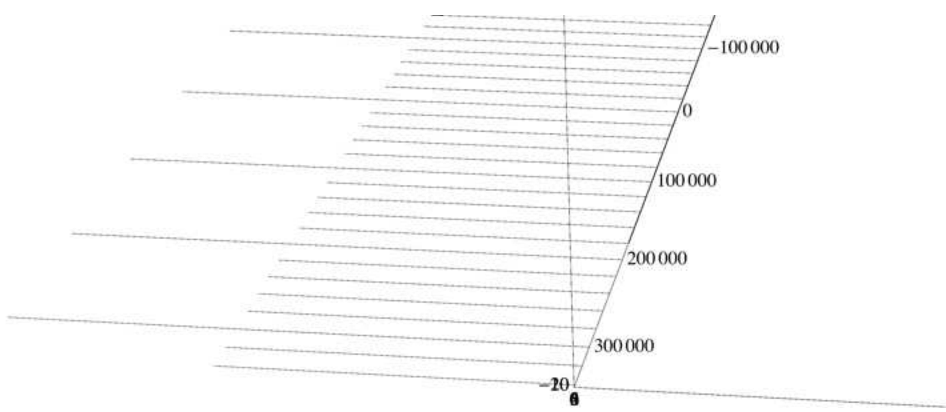
$$\left(\pi\left(-16\,\pi\,\gamma-4\,(\gamma)^2+16\,\pi^2\,\text{Sin}[\beta]^2+16\,\pi\,\gamma\,\text{Sin}[\beta]^2+4\,(\gamma)^2\,\text{Sin}[\beta]^2\right)\right),$$

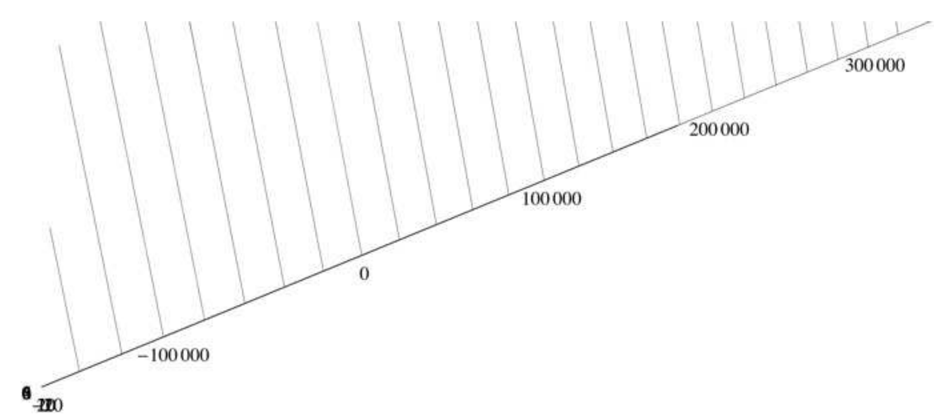
$\{\gamma,\ -2\,\pi,\ 2\,\pi\},\ \{\beta,\ -\pi\,/\,3,\ \pi\,/\,3\}\Big]$




-0.10
-0.05
0.00
0.05
0.2
0.0
-0.2

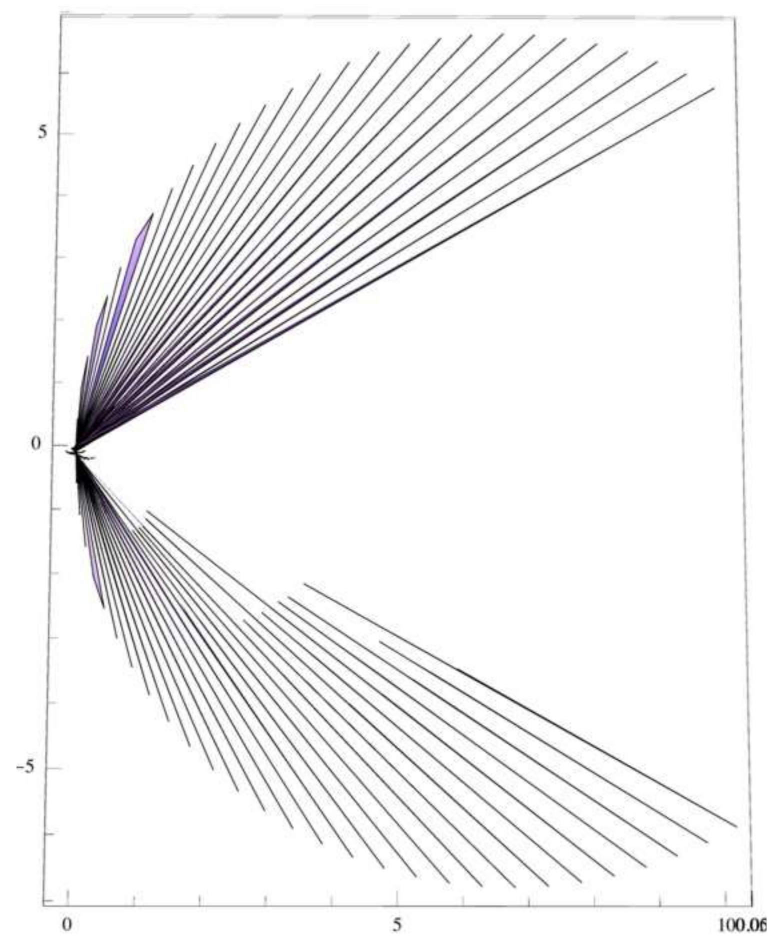
$$\text{SphericalPlot3D}\Big[\left(4\pi\gamma+\gamma^2-\sqrt{\gamma}\sqrt{\left(2\pi+\frac{2\left(\pi-\pi\,\text{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)}{-1+\text{Sin}[\beta]^2}\right)^2}\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]-4\pi^2\,\text{Sin}[\beta]^2-4\pi\gamma\,\text{Sin}[\beta]^2-\gamma^2\,\text{Sin}[\beta]^2-\frac{\pi\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\,\text{Sin}[\beta]^3}{\sqrt{4\pi+\gamma}}+\frac{\pi\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3}{\sqrt{\gamma}}+\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3\right)\Big/\left(\pi\left(-16\pi\gamma-4(\gamma)^2+16\pi^2\,\text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]\right]^2+16\pi\gamma\,\text{Sin}[\beta]^2+4(\gamma)^2\,\text{Sin}[\beta]^2\right)\right),\{\gamma,-2\pi,2\pi\},\{\beta,-\pi/3,\pi/3\}\Big]$$





β, γ

```
SphericalPlot3D[(4 π γ + γ^2 - Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Sin[β] -
   4 π^2 Sin[β]^2 - 4 π γ Sin[β]^2 - γ^2 Sin[β]^2 - (π Sqrt[γ] Sqrt[(2 π + γ)^2] Sin[β]^3)/Sqrt[4 π + γ] +
   (π Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Sin[β]^3)/Sqrt[γ] + Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Sin[β]^3) /
  (π (-16 π γ - 4 γ^2 + 16 π^2 Sin[β]^2 + 16 π γ Sin[β]^2 + 4 γ^2 Sin[β]^2)), {β, -π / 3, π / 3}, {γ, -2 π, 2 π}]
```

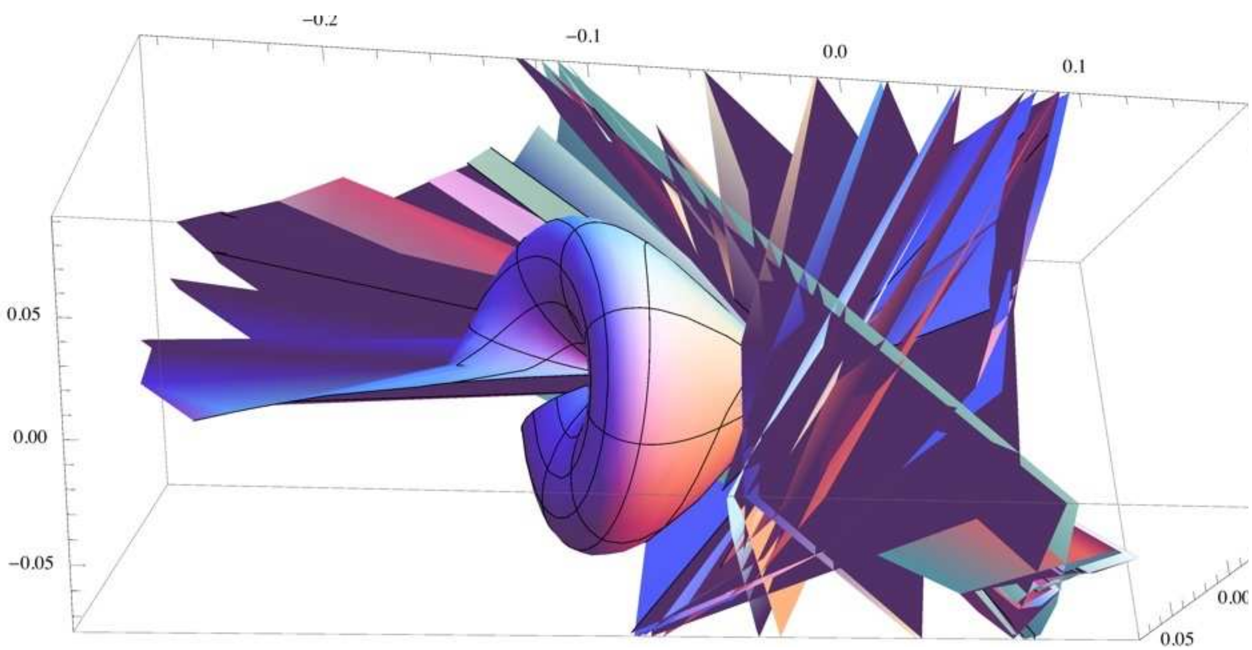


$$\text{SphericalPlot3D}\Bigg[\Bigg(4\pi\gamma+\gamma^2-\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]-4\pi^2\,\text{Sin}[\beta]^2-$$

$$4\pi\gamma\,\text{Sin}[\beta]^2-\gamma^2\,\text{Sin}[\beta]^2-\frac{\pi\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]\right]^3}{\sqrt{4\pi+\gamma}}+$$

$$\frac{\pi\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3}{\sqrt{\gamma}}+\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3\Bigg)\Big/$$

$$\left(\pi\left(-16\pi\gamma-4\gamma^2+16\pi^2\,\text{Sin}[\beta]^2+16\pi\gamma\,\text{Sin}[\beta]^2+4\gamma^2\,\text{Sin}[\beta]^2\right)\right),\ \{\beta,-\pi/3,\pi/3\},\ \{\gamma,-2\pi,2\pi\}\Bigg]$$



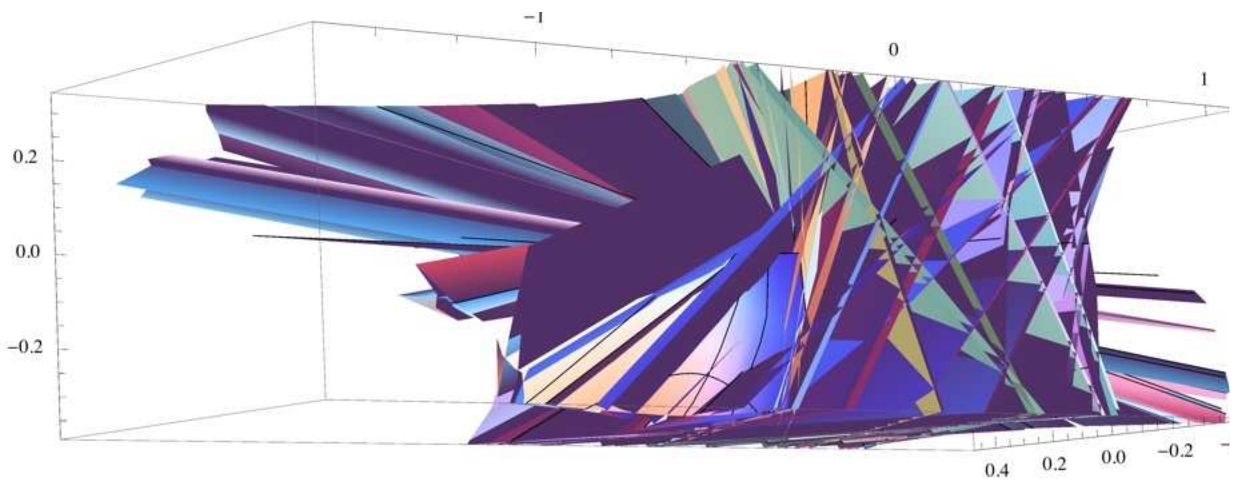
$$\text{SphericalPlot3D}\Bigg[\Bigg(4\pi\gamma+\gamma^2-\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]-$$

$$4\pi^2\,\text{Sin}[\beta]^2-4\pi\gamma\,\text{Sin}[\beta]^2-\left(\frac{2\left(\pi-\pi\,\text{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)}{-1+\text{Sin}[\beta]^2}\right)^2\text{Sin}[\beta]^2-$$

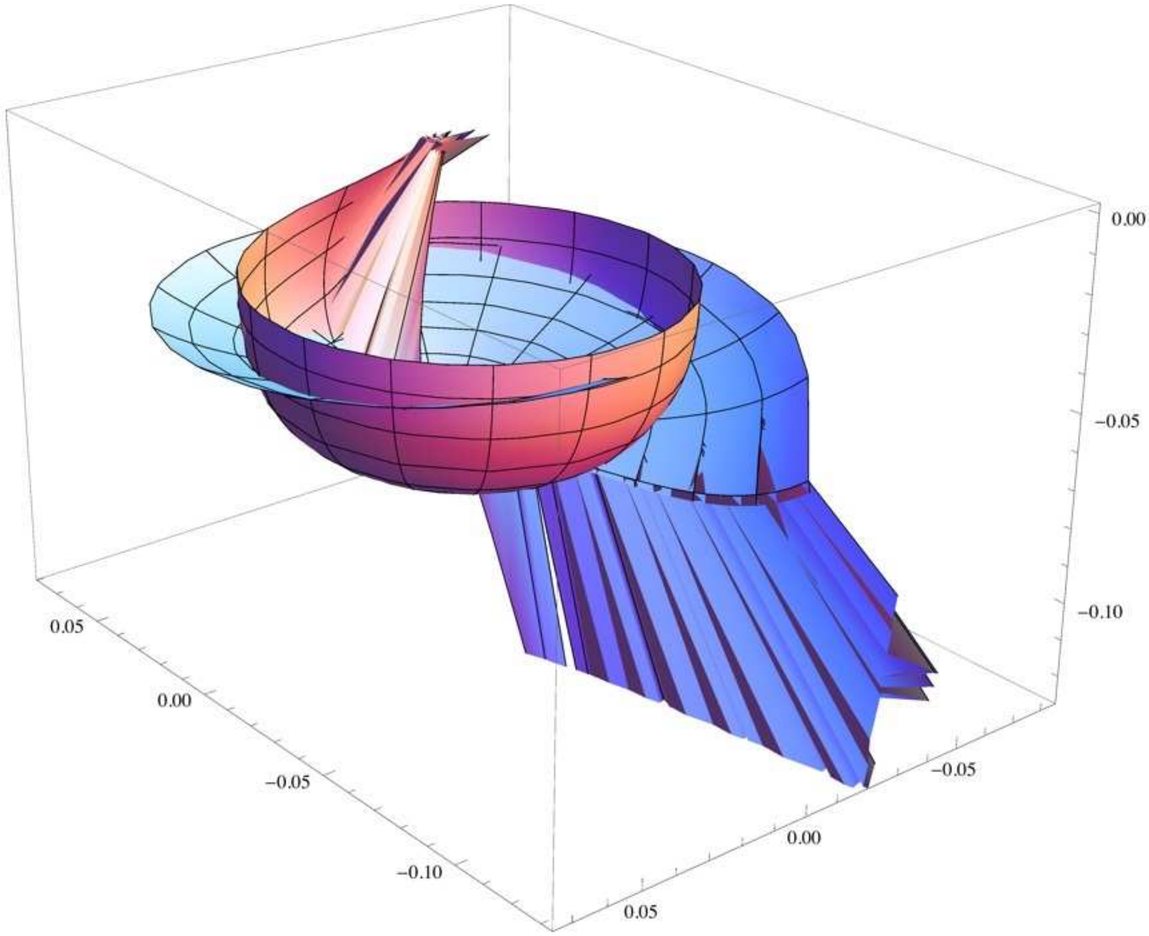
$$\frac{\pi\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\,\text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]\right]^3}{\sqrt{4\pi+\gamma}}+\frac{\pi\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3}{\sqrt{\gamma}}+$$

$$\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}}{4\pi^2+4\pi\gamma+\gamma^2}\right]\right]^3\Bigg)\Bigg/$$

$$\left(\pi\left(-16\pi\gamma-4\gamma^2+16\pi^2\,\text{Sin}[\beta]^2+16\pi\gamma\,\text{Sin}[\beta]^2+4\gamma^2\,\text{Sin}[\beta]^2\right)\right),\ \{\beta,-\pi/3,\pi/3\},\ \{\gamma,-2\pi,2\pi\}\Bigg]$$



$$\text{SphericalPlot3D}\Bigg[\Bigg(4\pi\gamma + \left(\frac{2\left(\pi - \pi\,\text{Sin}[\beta]^2 + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)}{-1 + \text{Sin}[\beta]^2}\right)^2 - \sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Sin}[\beta] - 4\pi^2\,\text{Sin}[\beta]^2 - 4\pi\gamma\,\text{Sin}[\beta]^2 - \gamma^2\,\text{Sin}[\beta]^2 - \frac{\pi\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\text{Sin}[\beta]^3}{\sqrt{4\pi+\gamma}} + \frac{\pi\sqrt{\left(2\pi + \frac{2\left(\pi-\pi\,\text{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right)}{-1+\text{Sin}[\beta]^2}\right)^2}\,\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3}{\sqrt{\gamma}} + \sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3\Bigg) \Big/ \left(\pi\left(-16\pi\gamma - 4\left(\frac{2\left(\pi - \pi\,\text{Sin}[\beta]^2 + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)}{-1 + \text{Sin}[\beta]^2}\right)^2 + 16\pi^2\,\text{Sin}[\beta]^2 + 16\pi\gamma\,\text{Sin}[\beta]^2 + 4\gamma^2\,\text{Sin}[\beta]^2\right)\right), \{\beta, -\pi/3, \pi/3\}, \{\gamma, -2\pi, 2\pi\}\Bigg]$$



- **For** $1 = \frac{\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}{\text{Sin}[\beta]\,\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}$

$$\text{Solve}\Big[\frac{\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}\,\text{Sin}[\beta]} - 1 ==$$

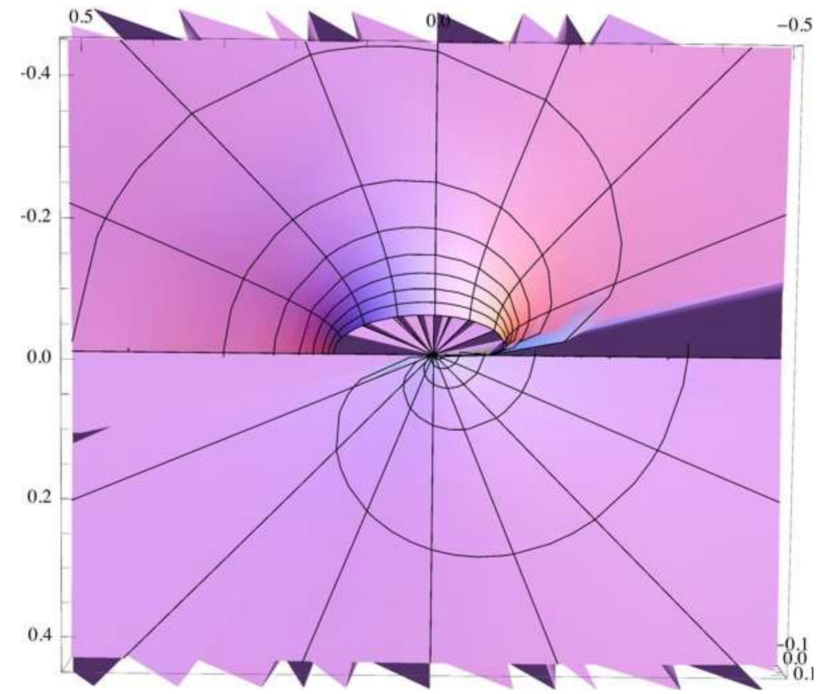
$$2\pi r - 2\pi\sqrt{r^2 - \frac{r\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\text{^}2} - \gamma\sqrt{r^2 - \frac{r\sqrt{\gamma}\,\sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\text{^}2}\text{ , r}\Big]$$

$$\Big\{\Big\{r \to \Big(4\pi^2 + 4\pi\gamma + \gamma^2 - \sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Csc}[\beta] - 4\pi\gamma\,\text{Csc}[\beta]^2 - \gamma^2\,\text{Csc}[\beta]^2 +$$

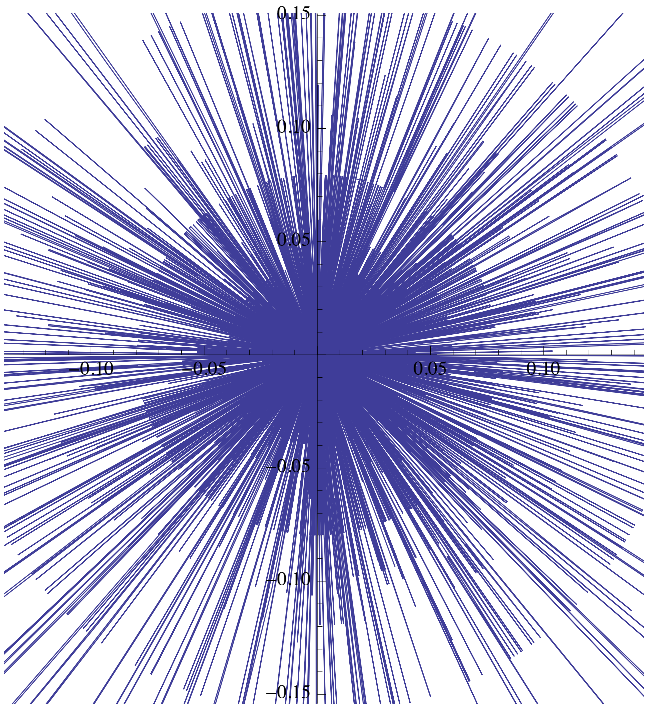
$$\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3 - \frac{4\pi^2\sqrt{\gamma}\,\sqrt{(2\pi+\gamma)^2}\,\sqrt{4\pi+\gamma}\,\text{Csc}[\beta]^3}{(2\pi+\gamma)^2}\Big)\Big/$$

$$\left(\pi\left(-16\pi^2 - 16\pi\gamma - 4\gamma^2 + 16\pi\gamma\,\text{Csc}[\beta]^2 + 4\gamma^2\,\text{Csc}[\beta]^2\right)\right)\Big\}\Big\}$$

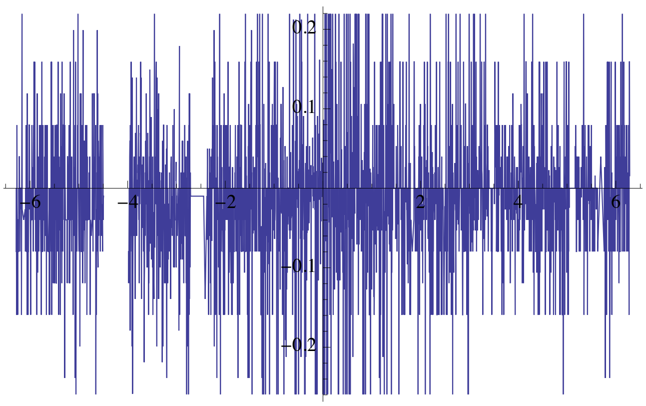
```
SphericalPlot3D[(4 π² + 4 π γ + γ² - √γ √((2 π + γ)²) √(4 π + γ) Csc[β] - 4 π γ Csc[β]² - γ² Csc[β]² +
    √γ √((2 π + γ)²) √(4 π + γ) Csc[β]³ - (4 π² √γ √((2 π + γ)²) √(4 π + γ) Csc[β]³)/(2 π + γ)²) /
  (π (-16 π² - 16 π γ - 4 γ² + 16 π γ Csc[β]² + 4 γ² Csc[β]²)), {γ, -2 π, 2 π}, {β, -π / 3, π / 3}]
```



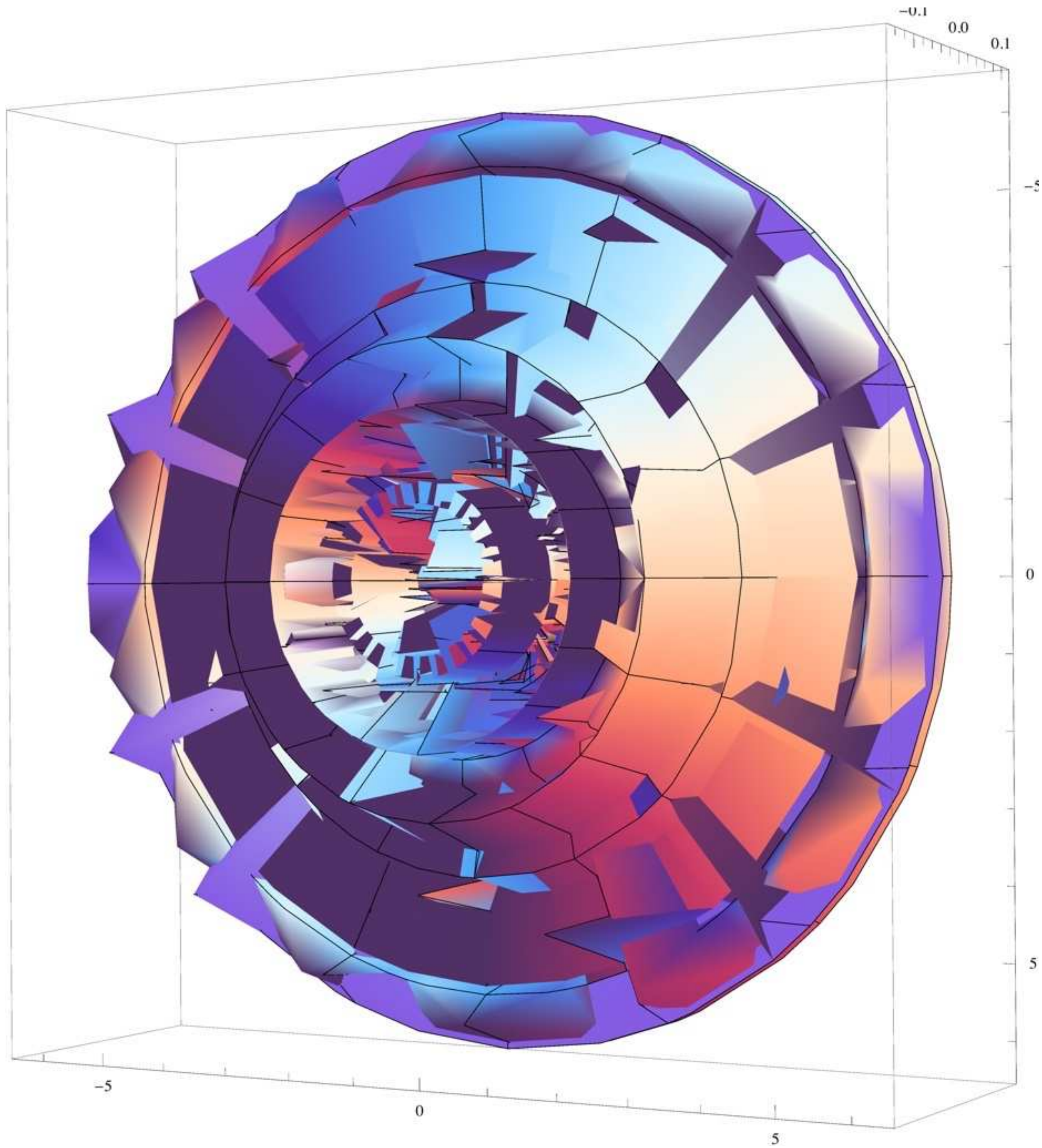
$$\text{PolarPlot}\Big[\Big(4\,\pi\,\gamma + \gamma^2 - \sqrt{\gamma}\,\sqrt{(2\,\pi+\gamma)^2}\,\sqrt{4\,\pi+\gamma}\,\text{Sin}[\beta] - 4\,\pi^2\,\text{Sin}[\beta]^2 - 4\,\pi\,\gamma\,\text{Sin}[\beta]^2 - \gamma^2\,\text{Sin}[\beta]^2 - \frac{\pi\,\sqrt{\gamma}\,\sqrt{(2\,\pi+\gamma)^2}\,\text{Sin}[\beta]^3}{\sqrt{4\,\pi+\gamma}} + \frac{\pi\,\sqrt{(2\,\pi+\gamma)^2}\,\sqrt{4\,\pi+\gamma}\,\text{Sin}[\beta]^3}{\sqrt{\gamma}} + \sqrt{\gamma}\,\sqrt{(2\,\pi+\gamma)^2}\,\sqrt{4\,\pi+\gamma}\,\text{Sin}[\beta]^3\Big) \Big/ \left(\pi\,\left(-16\,\pi\,\gamma - 4\,\gamma^2 + 16\,\pi^2\,\text{Sin}[\beta]^2 + 16\,\pi\,\gamma\,\text{Sin}[\beta]^2 + 4\,\gamma^2\,\text{Sin}[\beta]^2\right)\right),\ \{\gamma,\ -2\,\pi,\ 2\,\pi\}\Big]$$



$$\text{Plot}\Bigg[\Bigg(4\pi\gamma+\gamma^2-\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]-4\pi^2\,\text{Sin}[\beta]^2-4\pi\gamma\,\text{Sin}[\beta]^2-\gamma^2\,\text{Sin}[\beta]^2-\frac{\pi\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\,\text{Sin}[\beta]^3}{\sqrt{4\pi+\gamma}}+\frac{\pi\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3}{\sqrt{\gamma}}+\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3\Bigg)\Big/\left(\pi\left(-16\pi\gamma-4\gamma^2+16\pi^2\,\text{Sin}[\beta]^2+16\pi\gamma\,\text{Sin}[\beta]^2+4\gamma^2\,\text{Sin}[\beta]^2\right)\right),\ \{\gamma,\ -2\pi,\ 2\pi\}\Bigg]$$



$$\text{RevolutionPlot3D}\Bigg[\Bigg(4\pi\gamma+\gamma^2-\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]-4\pi^2\,\text{Sin}[\beta]^2-4\pi\gamma\,\text{Sin}[\beta]^2-\gamma^2\,\text{Sin}[\beta]^2-\frac{\pi\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\,\text{Sin}[\beta]^3}{\sqrt{4\pi+\gamma}}+\frac{\pi\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3}{\sqrt{\gamma}}+\sqrt{\gamma}\sqrt{(2\pi+\gamma)^2}\sqrt{4\pi+\gamma}\,\text{Sin}[\beta]^3\Bigg)\Big/\left(\pi\left(-16\pi\gamma-4\gamma^2+16\pi^2\,\text{Sin}[\beta]^2+16\pi\gamma\,\text{Sin}[\beta]^2+4\gamma^2\,\text{Sin}[\beta]^2\right)\right),\ \{\gamma,\ -2\pi,\ 2\pi\}\Bigg]$$



Remembering that :

$$\text{Solve}\left[1 == \frac{\sqrt{4\,\pi^2 + 4\,\pi\,\gamma + \gamma^2}\ \text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{(4\,\pi - \theta)\,\theta}}{2\,\pi}\right]\right]}{\sqrt{\gamma}\ \sqrt{4\,\pi + \gamma}},\ \theta\right]$$

$$\left\{\left\{\theta \to \frac{2\,\pi\,\gamma}{2\,\pi + \gamma}\right\},\ \left\{\theta \to \frac{2\,\left(4\,\pi^2 + \pi\,\gamma\right)}{2\,\pi + \gamma}\right\}\right\}$$

$$\text{Solve}\left[1 == \frac{\sqrt{4\pi^2 + 4\pi\gamma + \gamma^2}\ \text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\theta}}{2\pi}\right]\right]}{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}, \gamma\right]$$

$$\left\{\left\{\gamma \to -\frac{2\pi(4\pi-\theta)}{2\pi-\theta}\right\}, \left\{\gamma \to \frac{2\pi\theta}{2\pi-\theta}\right\}\right\}$$

We reformulate the 1 - 1 = 0 "structural set up equation" from

$$\text{Solve}\left[\frac{\sqrt{4\pi^2 + 4\pi\gamma + \gamma^2}\ \text{Sin}[\beta]}{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}} - 1 == 2\pi r - 2\pi\sqrt{r^2 - \frac{r\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\text{^}2} - \gamma\sqrt{r^2 - \frac{r\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\text{^}2}, r\right]$$

to

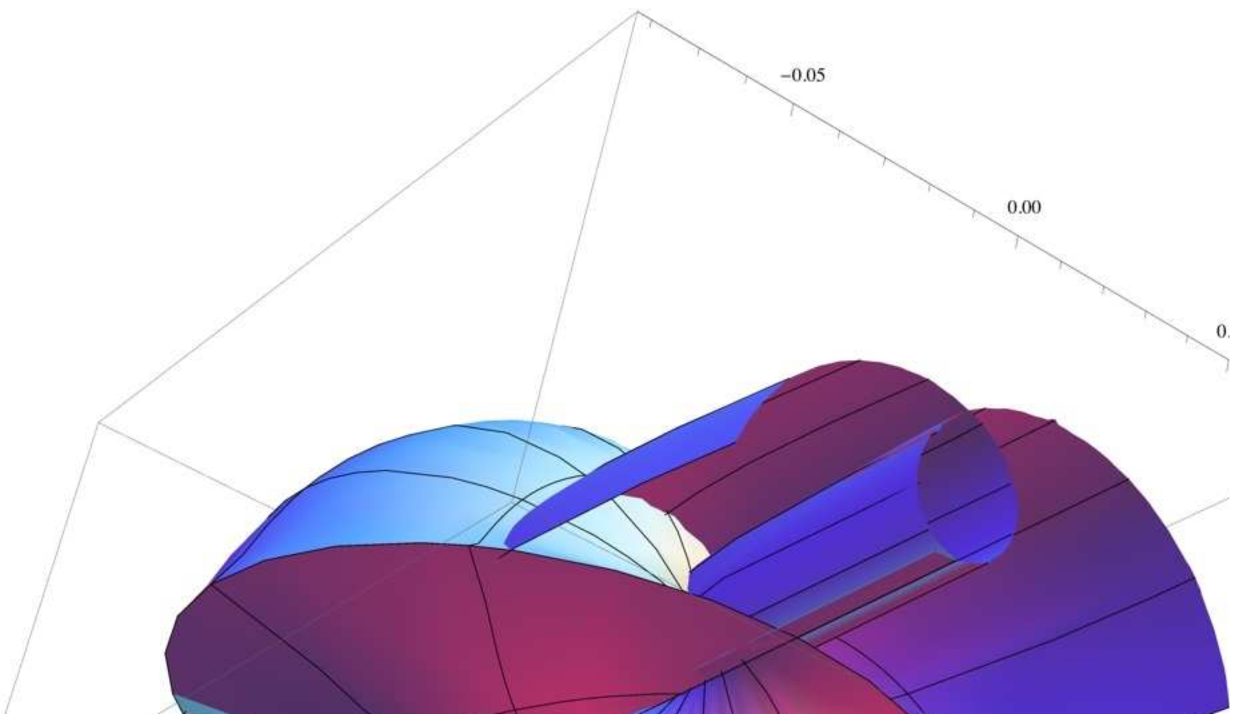
$$1 == \frac{\sqrt{4\pi^2 + 4\pi\gamma + \gamma^2}\ \text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\theta}}{2\pi}\right]\right]}{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}$$

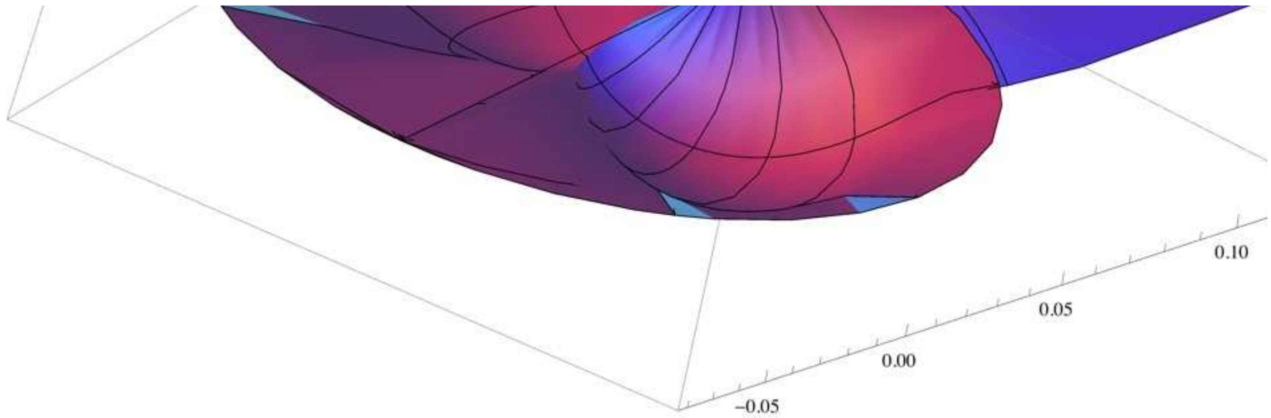
This yields :

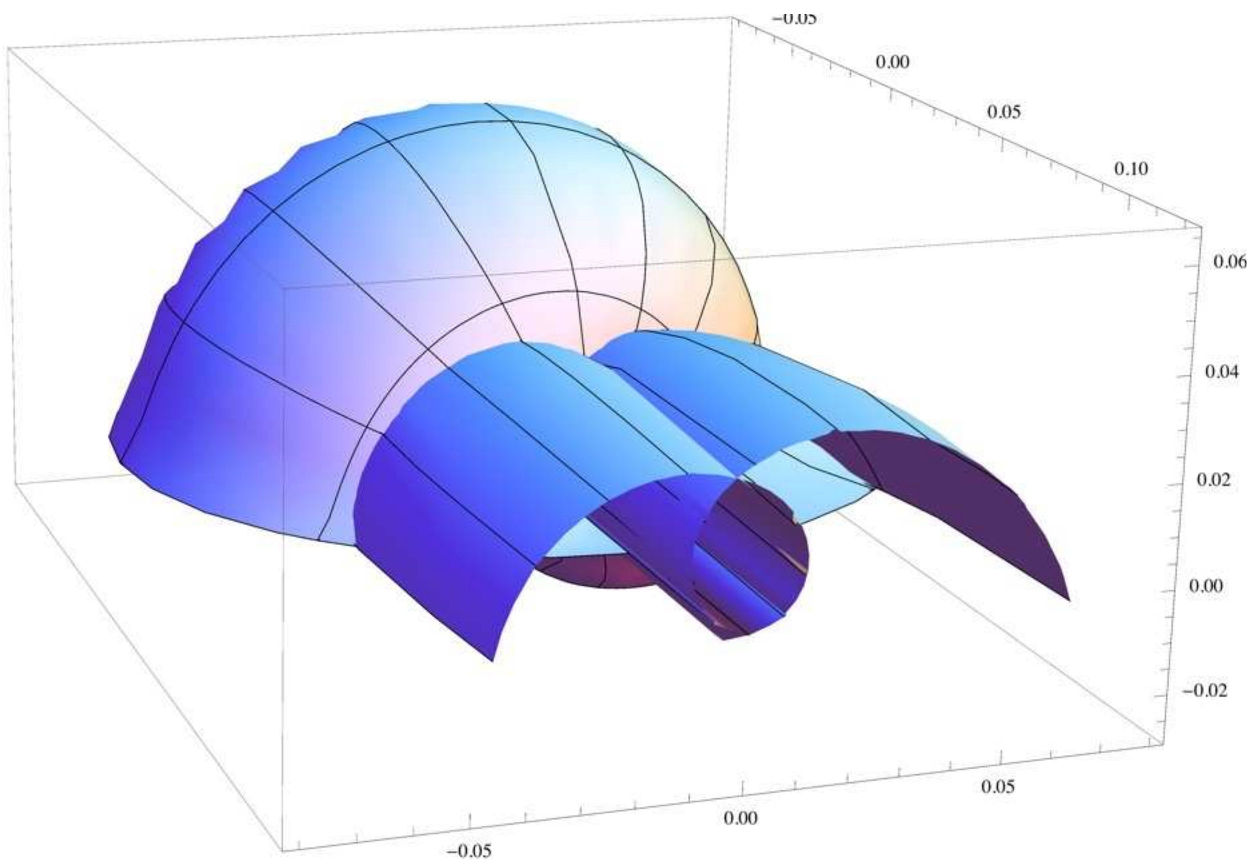
$$\text{Solve}\left[\frac{\sqrt{4\pi^2 + 4\pi\gamma + \gamma^2}\ \text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\theta}}{2\pi}\right]\right]}{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}} - 1 == 2\pi r - 2\pi\sqrt{r^2 - \frac{r\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\text{^}2} - \gamma\sqrt{r^2 - \frac{r\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\sqrt{4\pi^2+4\pi\gamma+\gamma^2}}\text{^}2}, r\right]$$

$$\Bigg\{\Bigg\{r \to \Bigg(-32\pi^4\gamma - 8\pi^3\gamma^2 + 32\pi^4\theta + 32\pi^3\gamma\theta + 8\pi^2\gamma^2\theta - 8\pi^3\theta^2 - 8\pi^2\gamma\theta^2 - 2\pi\gamma^2\theta^2 + 4\pi^2\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}\ \sqrt{(4\pi-\theta)\theta} + \frac{4\pi^2\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \theta\sqrt{(4\pi-\theta)\theta}}{\sqrt{4\pi+\gamma}} - \frac{4\pi^2\sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}\ \theta\sqrt{(4\pi-\theta)\theta}}{\sqrt{\gamma}} - 4\pi\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}\ \theta\sqrt{(4\pi-\theta)\theta} - \frac{\pi\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \theta^2\sqrt{(4\pi-\theta)\theta}}{\sqrt{4\pi+\gamma}} + \frac{\pi\sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}\ \theta^2\sqrt{(4\pi-\theta)\theta}}{\sqrt{\gamma}} + \sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}\ \theta^2\sqrt{(4\pi-\theta)\theta}\Bigg) \Big/ \left(\pi^2\left(128\pi^3\gamma + 32\pi^2\gamma^2 - 128\pi^3\theta - 128\pi^2\gamma\theta - 32\pi\gamma^2\theta + 32\pi^2\theta^2 + 32\pi\gamma\theta^2 + 8\gamma^2\theta^2\right)\right)\Bigg\}\Bigg\}$$

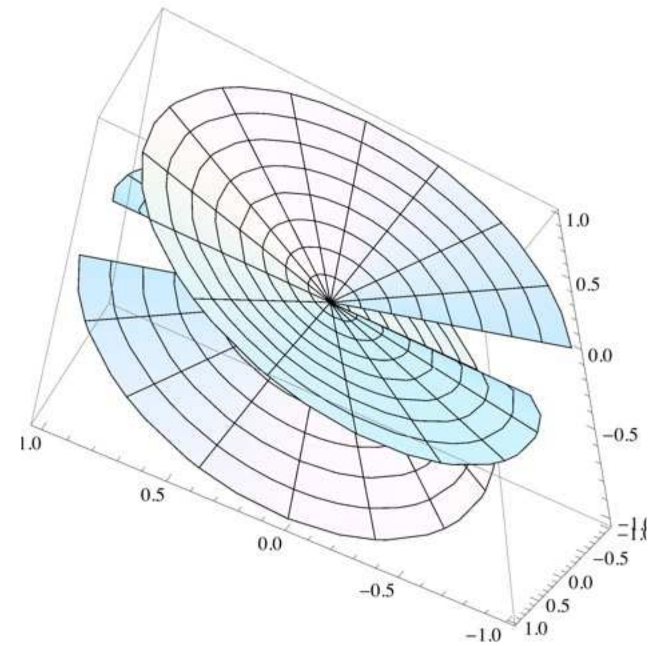
```
SphericalPlot3D[((-32 π^4 γ - 8 π^3 γ^2 + 32 π^4 θ + 32 π^3 γ θ + 8 π^2 γ^2 θ -
    8 π^3 θ^2 - 8 π^2 γ θ^2 - 2 π γ^2 θ^2 + 4 π^2 Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] Sqrt[(4 π - θ) θ] +
    (4 π^2 Sqrt[γ] Sqrt[(2 π + γ)^2] θ Sqrt[(4 π - θ) θ])/Sqrt[4 π + γ] - (4 π^2 Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] θ Sqrt[(4 π - θ) θ])/Sqrt[γ] -
    4 π Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] θ Sqrt[(4 π - θ) θ] - (π Sqrt[γ] Sqrt[(2 π + γ)^2] θ^2 Sqrt[(4 π - θ) θ])/Sqrt[4 π + γ] +
    (π Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] θ^2 Sqrt[(4 π - θ) θ])/Sqrt[γ] + Sqrt[γ] Sqrt[(2 π + γ)^2] Sqrt[4 π + γ] θ^2 Sqrt[(4 π - θ) θ]) /
  (π^2 (128 π^3 γ + 32 π^2 γ^2 - 128 π^3 θ - 128 π^2 γ θ - 32 π γ^2 θ + 32 π^2 θ^2 + 32 π γ θ^2 + 8 γ^2 θ^2)),
 {γ, -2 π, 2 π}, {θ, -π, π}]
```







XXIII. Long solution for theta with higher dimensionality (14D).



Theorem 2 When we designate that a single unit of time passes per revolution of the angle through the total number of radians in a circle, instantaneous velocity through the distance of the height of the cone can be found by taking the first derivative of the expression for that distance in terms of r and θ with respect to t = $\frac{\theta}{2\pi}$. There is also a velocity through the height of the cone, which is equal to wavelength times frequency = $\lambda f = \frac{\eta}{\left(\frac{\theta}{(2\pi)}\right)}$. Under the condition that one unit of time passes with one revolution of the circle, these two velocities are equal to each other at the position where a 30-60-90 triangle is formed between the apex, center of the base of the cone, and point on the circumference of the circle of the base of the cone.

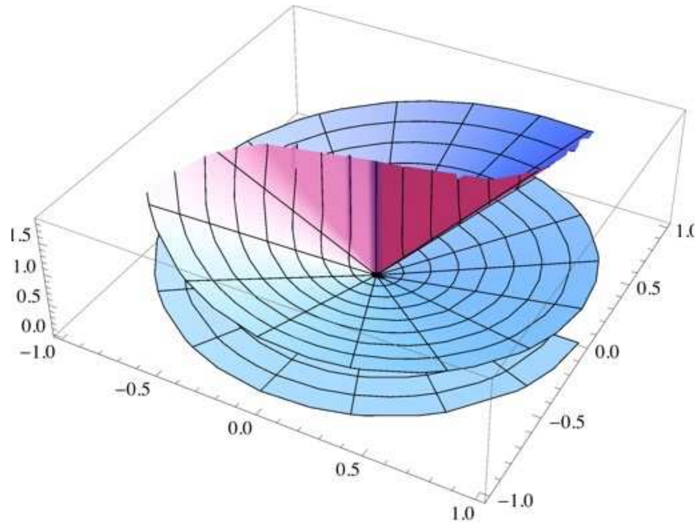
Proof.

To prove this, we can substitute r Sin[β] for the height of the cone in the expression of velocity = $\frac{2\pi\eta}{\theta}$ and find a real and two complex solutions for theta in terms of β , thus from Lemma 4, we can solve for β exactly.

$$D\left[2\pi\frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}, \theta\right] = \frac{4\pi r^2 - 2 r^2\theta}{2\sqrt{4\pi r^2\theta - r^2\theta^2}}$$

$$\text{Instantaneous Velocity} = \frac{4\pi r^2 - 2 r^2\theta}{2\sqrt{4\pi r^2\theta - r^2\theta^2}} \tag{62}$$

$$\text{RevolutionPlot3D}\left[\frac{4\pi r^2 - 2r^2\theta}{2\sqrt{4\pi r^2\theta - r^2\theta^2}}, \{r, -1, 1\}, \{\theta, -2\pi, 2\pi\}\right]$$

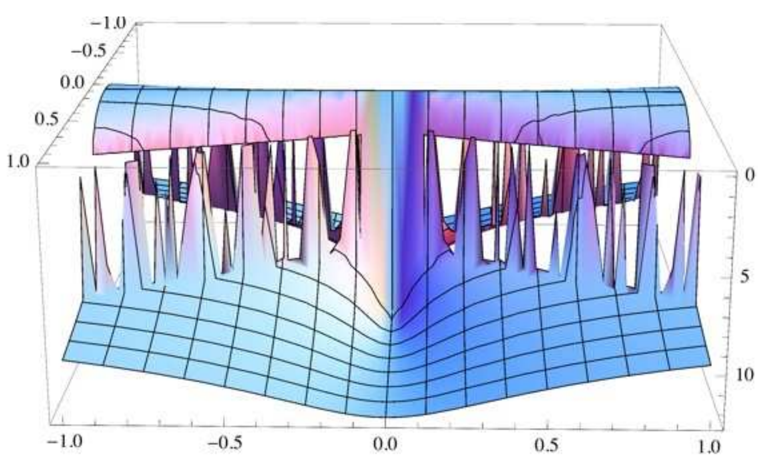


$$\text{Solve}\left[\frac{4\pi r^2 - 2r^2\theta}{2\sqrt{4\pi r^2\theta - r^2\theta^2}} == \frac{2\pi\eta}{\theta}, \theta\right]$$

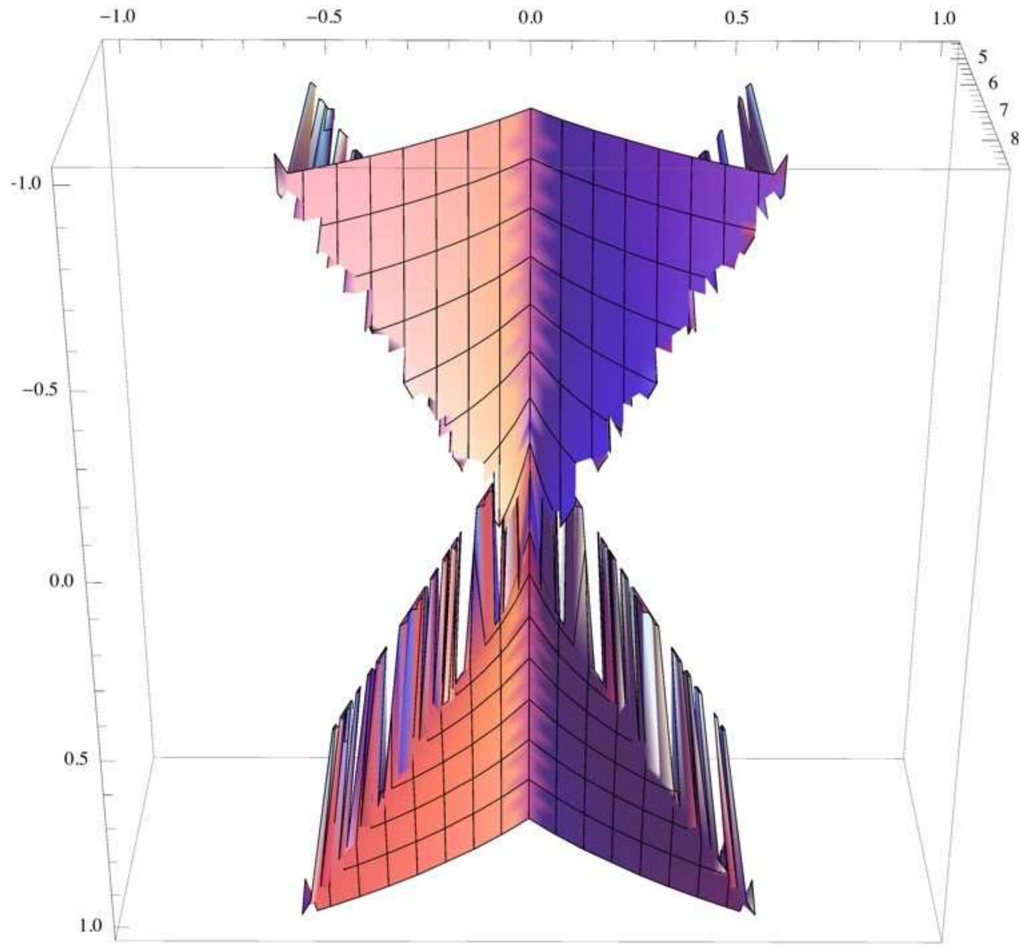
$$\left\{\left\{\theta \to \frac{4\pi}{3} + \frac{-4\pi^2 r^4 + 12\pi^2 r^2 \eta^2}{6\pi r^2\left(r^6 - 18r^4\eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11r^8\eta^4 + r^6\eta^6}\right)^{1/3}} - \frac{2\pi\left(r^6 - 18r^4\eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11r^8\eta^4 + r^6\eta^6}\right)^{1/3}}{3r^2}\right\},\right.$$

$$\left\{\theta \to \frac{4\pi}{3} - \frac{\left(1 + i\sqrt{3}\right)\left(-4\pi^2 r^4 + 12\pi^2 r^2\eta^2\right)}{12\pi r^2\left(r^6 - 18r^4\eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11r^8\eta^4 + r^6\eta^6}\right)^{1/3}} + \frac{\left(1 - i\sqrt{3}\right)\pi\left(r^6 - 18r^4\eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11r^8\eta^4 + r^6\eta^6}\right)^{1/3}}{3r^2}\right\},$$

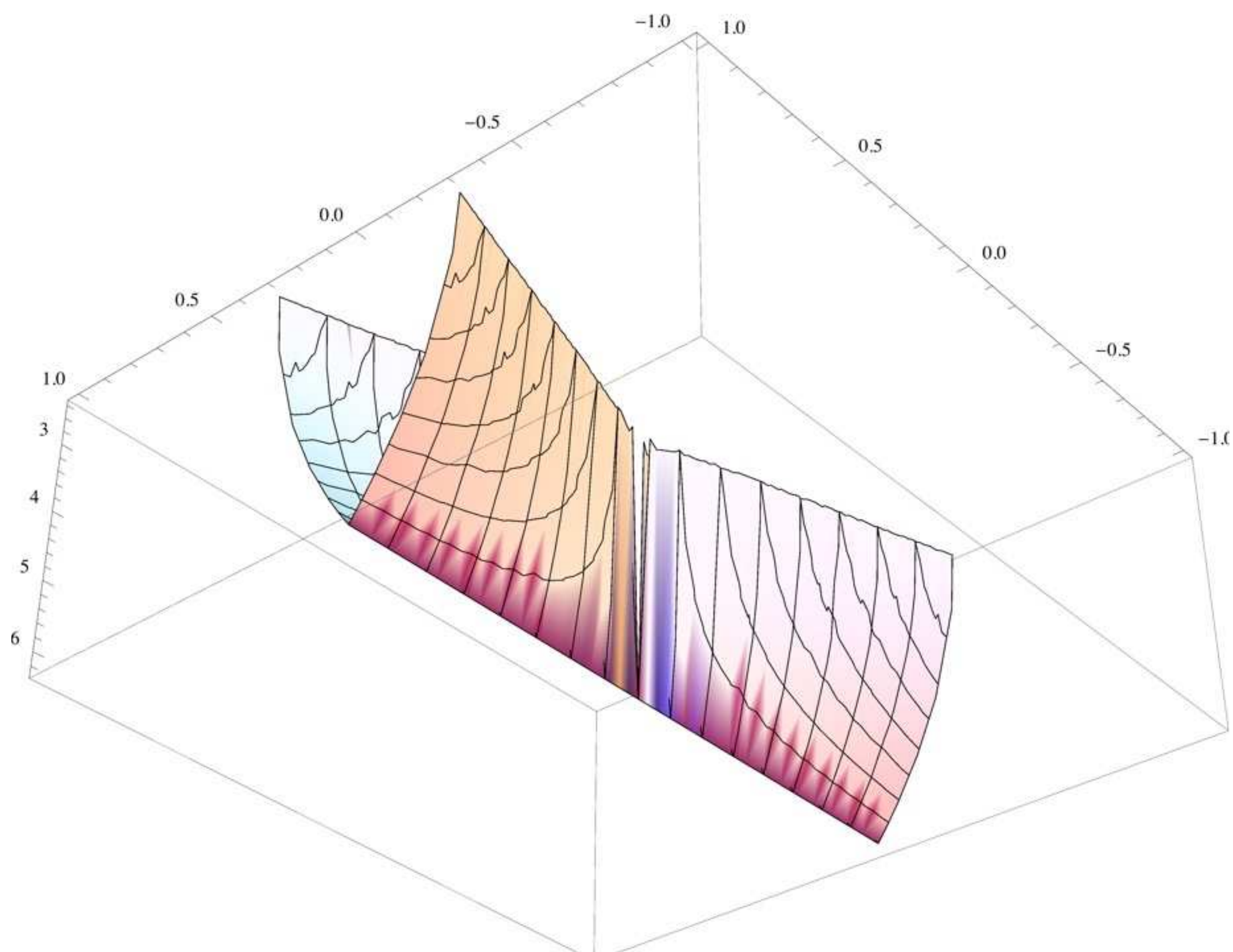
$$\left.\left\{\theta \to \frac{4\pi}{3} - \frac{\left(1 - i\sqrt{3}\right)\left(-4\pi^2 r^4 + 12\pi^2 r^2\eta^2\right)}{12\pi r^2\left(r^6 - 18r^4\eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11r^8\eta^4 + r^6\eta^6}\right)^{1/3}} + \frac{\left(1 + i\sqrt{3}\right)\pi\left(r^6 - 18r^4\eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11r^8\eta^4 + r^6\eta^6}\right)^{1/3}}{3r^2}\right\}\right\}$$



$$\text{Plot3D}\Big[\frac{4\pi}{3} - \frac{\left(1 + i\sqrt{3}\right)\left(-4\pi^2 r^4 + 12\pi^2 r^2 \eta^2\right)}{12\pi r^2 \left(r^6 - 18 r^4 \eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11 r^8 \eta^4 + r^6 \eta^6}\right)^{1/3}} + \frac{\left(1 - i\sqrt{3}\right)\pi\left(r^6 - 18 r^4 \eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11 r^8 \eta^4 + r^6 \eta^6}\right)^{1/3}}{3 r^2}, \{r, -1, 1\}, \{\eta, -1, 1\}\Big]$$



$$\text{Plot3D}\left[\frac{4\pi}{3} - \frac{\left(1 - i\sqrt{3}\right)\left(-4\pi^2 r^4 + 12\pi^2 r^2 \eta^2\right)}{12\pi r^2 \left(r^6 - 18 r^4 \eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11 r^8 \eta^4 + r^6 \eta^6}\right)^{1/3}} + \frac{\left(1 + i\sqrt{3}\right)\pi\left(r^6 - 18 r^4 \eta^2 + 3\sqrt{3}\sqrt{-r^{10}\eta^2 + 11 r^8 \eta^4 + r^6 \eta^6}\right)^{1/3}}{3 r^2}, \{r, -1, 1\}, \{\eta, -1, 1\}\right]$$



$$\text{Solve}\left[\frac{k\left(4\pi r^2 - 2 r^2 \theta\right)}{4\pi\sqrt{4\pi r^2 \theta - r^2 \theta^2}} == \frac{k\, r\, \text{Sin}[\beta]}{\theta}, \theta\right]$$

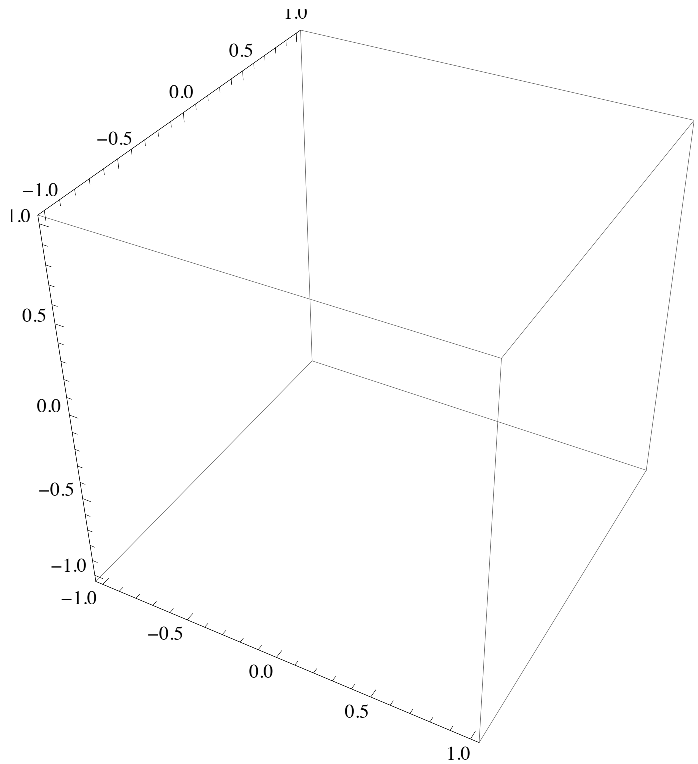
$$\left\{\left\{\theta \to \frac{4\pi}{3} - \frac{-4\pi^2 + 12\pi^2 \text{Sin}[\beta]^2}{6\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}} + \frac{2}{3}\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}\right\},\right.$$

$$\left\{\theta \to \frac{4\pi}{3} + \frac{\left(1 + i\sqrt{3}\right)\left(-4\pi^2 + 12\pi^2 \text{Sin}[\beta]^2\right)}{12\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}} - \frac{1}{3}\left(1 - i\sqrt{3}\right)\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}\right\},$$

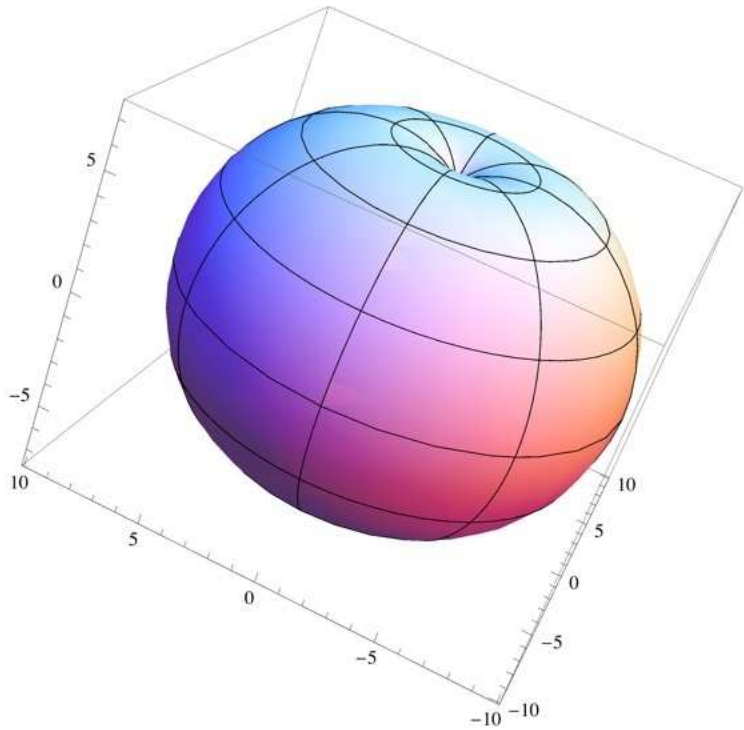
$$\left.\left\{\theta \to \frac{4\pi}{3} + \frac{\left(1 - i\sqrt{3}\right)\left(-4\pi^2 + 12\pi^2 \text{Sin}[\beta]^2\right)}{12\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}} - \frac{1}{3}\left(1 + i\sqrt{3}\right)\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}\right\}\right\}$$

$$\textbf{SphericalPlot3D}\left[\left(\left(\frac{4\pi}{3} + \frac{\left(1 + i\sqrt{3}\right)\left(-4\pi^2 + 12\pi^2 \text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta + 4\pi\theta^2 - \theta^3}}{2\pi\sqrt{-4\pi + \theta}}\right]\right]^2\right)}{12\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}} - \frac{1}{3}\left(1 - i\sqrt{3}\right)\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}\right)\right),\right.$$

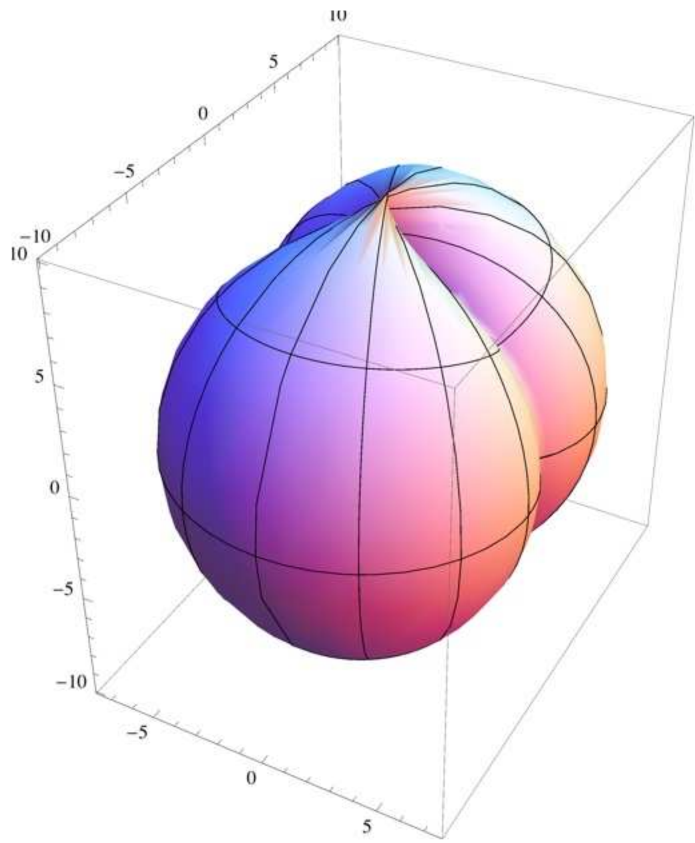
$$\left.\{\theta, -2\pi, 2\pi\}, \{\beta, -1\pi, 1\pi\}\right]$$


1.0
0.5
0.0
−0.5
−1.0
1.0
0.5
0.0
−0.5
−1.0
−1.0
−0.5
0.0
0.5
1.0

$$\texttt{SphericalPlot3D}\left[\frac{4\pi}{3}-\frac{-4\pi^2+12\pi^2\,\texttt{Sin}[\beta]^2}{6\left(-\pi^3+18\pi^3\,\texttt{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{-\pi^6\,\texttt{Sin}[\beta]^2+11\pi^6\,\texttt{Sin}[\beta]^4+\pi^6\,\texttt{Sin}[\beta]^6}\right)^{1/3}}+\frac{2}{3}\left(-\pi^3+18\pi^3\,\texttt{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{-\pi^6\,\texttt{Sin}[\beta]^2+11\pi^6\,\texttt{Sin}[\beta]^4+\pi^6\,\texttt{Sin}[\beta]^6}\right)^{1/3},\ \{\beta,-\pi,\pi\},\ \{\theta,-2\pi,2\pi\}\right]$$



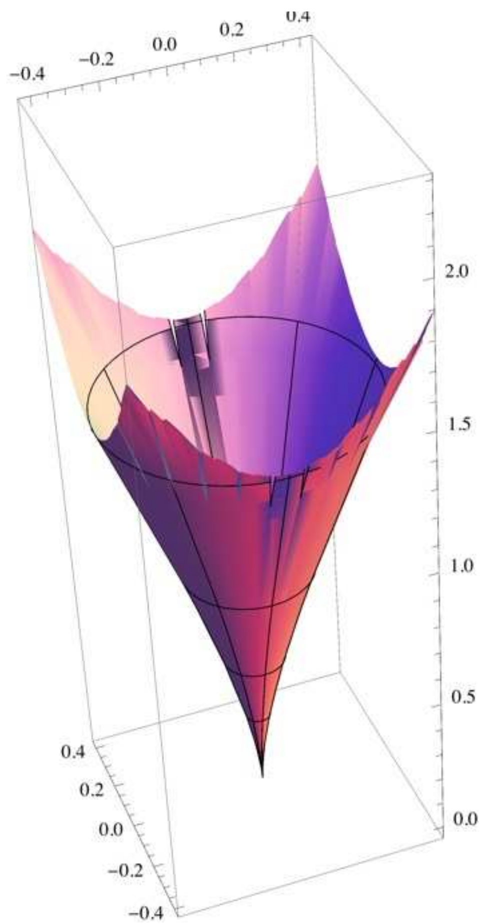
$$\text{SphericalPlot3D}\Big[\frac{4\pi}{3} - \frac{-4\pi^2 + 12\pi^2 \text{Sin}[\beta]^2}{6\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}} + \frac{2}{3}\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}, \{\theta, -2\pi, 2\pi\}, \{\beta, -\pi, \pi\}\Big]$$



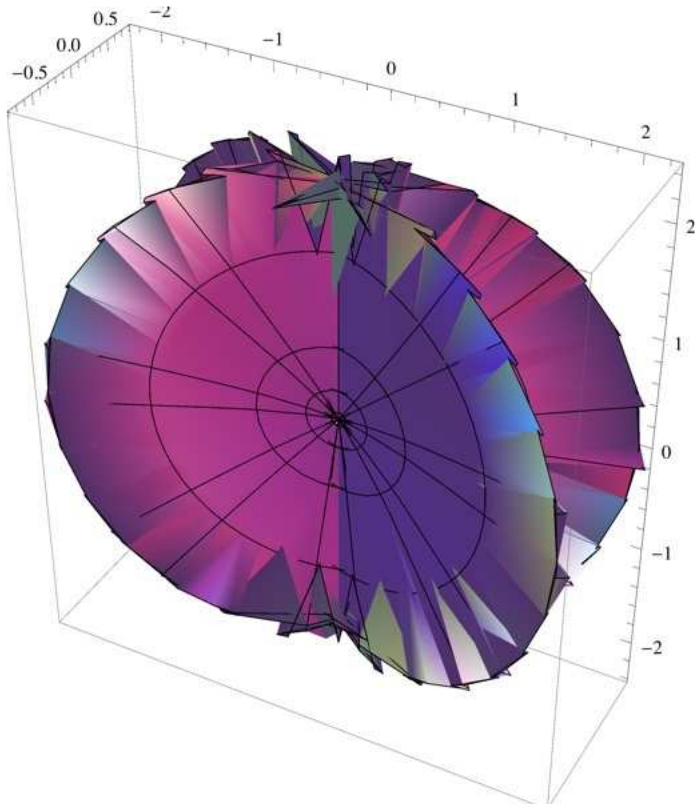
$$\text{SphericalPlot3D}\Big[\frac{4\pi}{3} + \frac{\left(1 + i\sqrt{3}\right)\left(-4\pi^2 + 12\pi^2 \text{Sin}[\beta]^2\right)}{12\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}} -$$

$$\frac{1}{3}\left(1 - i\sqrt{3}\right)\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3},$$

$$\{\beta, -\pi, \pi\}, \{\theta, -2\pi, 2\pi\}\Big]$$



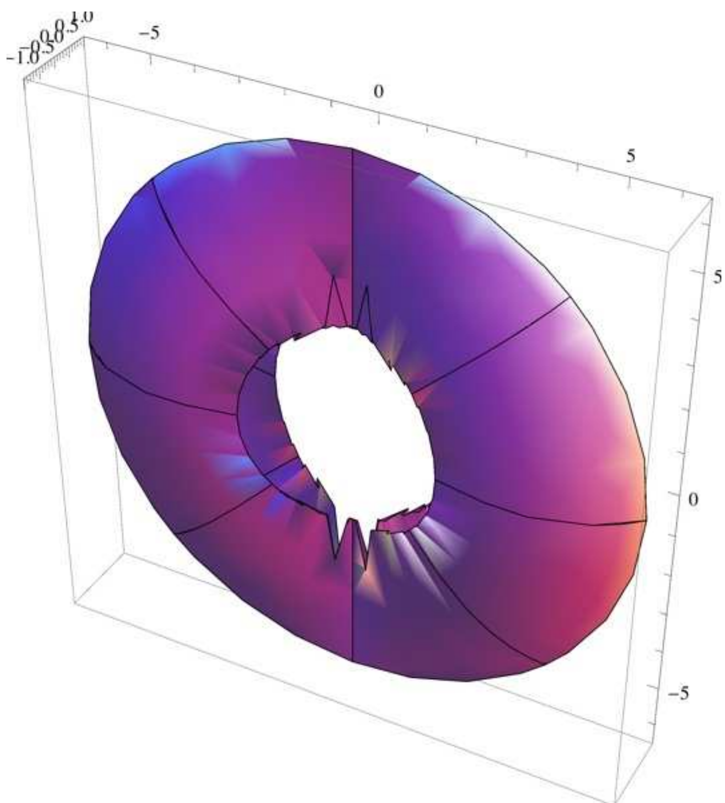
$$\mathbf{SphericalPlot3D}\Big[\frac{4\pi}{3}+\frac{\left(1+i\sqrt{3}\right)\left(-4\pi^2+12\pi^2\,\mathrm{Sin}[\beta]^2\right)}{12\left(-\pi^3+18\pi^3\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{-\pi^6\,\mathrm{Sin}[\beta]^2+11\,\pi^6\,\mathrm{Sin}[\beta]^4+\pi^6\,\mathrm{Sin}[\beta]^6}\right)^{1/3}}-\frac{1}{3}\left(1-i\sqrt{3}\right)\left(-\pi^3+18\pi^3\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{-\pi^6\,\mathrm{Sin}[\beta]^2+11\,\pi^6\,\mathrm{Sin}[\beta]^4+\pi^6\,\mathrm{Sin}[\beta]^6}\right)^{1/3},$$
$$\{\theta,-2\pi,2\pi\},\{\beta,-\pi,\pi\}\Big]$$



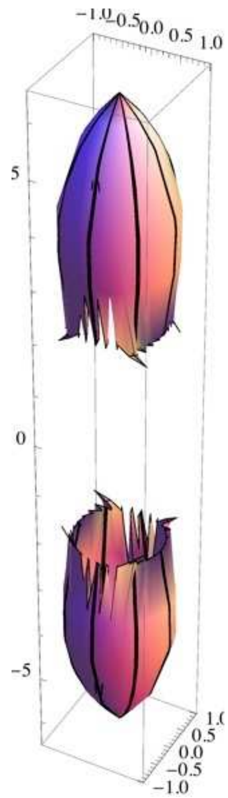
$$\text{SphericalPlot3D}\Big[\frac{4\pi}{3}+\frac{\left(1-i\sqrt{3}\right)\left(-4\pi^2+12\pi^2\,\text{Sin}[\beta]^2\right)}{12\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}}-$$

$$\frac{1}{3}\left(1+i\sqrt{3}\right)\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3},$$

$$\{\theta,-2\pi,2\pi\},\{\beta,-\pi,\pi\}\Big]$$



$$\texttt{SphericalPlot3D}\Big[\frac{4\pi}{3}+\frac{\left(1-i\sqrt{3}\right)\left(-4\pi^2+12\pi^2\,\text{Sin}[\beta]^2\right)}{12\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}}-\frac{1}{3}\left(1+i\sqrt{3}\right)\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3},\{\beta,-\pi,\pi\},\{\theta,-2\pi,2\pi\}\Big]$$



The real solution for θ, solved from equating the instantaneous velocity to the average velocity, can be equated with the real solution for the expression for θ from Lemma 4 to yield an exact solution for β that tells us that when these solutions for theta are equal, a 30-60-90 triangle is formed between the azimuth of the cone, the point on the base of the cone and the center of the base of the cone.

$$\text{Solve}\Big[\frac{4\pi}{3}-\frac{-4\pi^2+12\pi^2\,\text{Sin}[\beta]^2}{6\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}}+\frac{2}{3}\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}==2\left(\pi+\sqrt{\pi^2-\pi^2\,\text{Sin}[\beta]^2}\right),\beta\Big]$$

$$\left\{\left\{\beta\to-\frac{\pi}{3}\right\},\left\{\beta\to\frac{\pi}{3}\right\}\right\}$$

We know that the height of the cone is perpendicular to the center of the base of the cone, so this proves a 30-60-90 triangle, because the sum of the angles of the triangle must be 180.

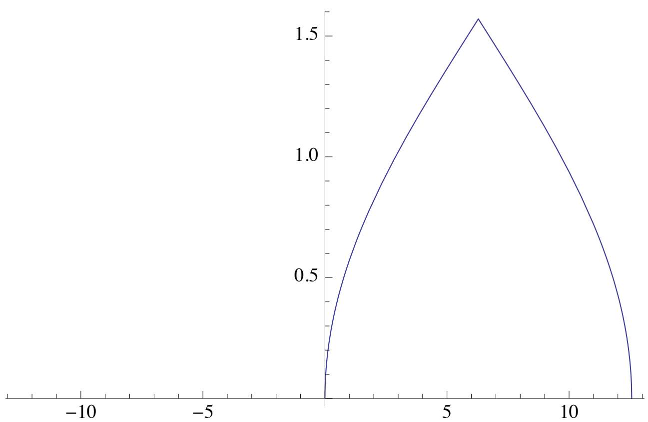
Lemma 6 We can show that $\beta = \frac{\pi}{3}$, thus we can show that there are two solutions to θ at which this occurs.

$$\text{ArcSin}\left[\frac{\sqrt{(4\pi - \theta)\,\theta}}{2\pi}\right] = \beta$$

$$\text{Solve}\left[\text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\,\theta}}{2\pi}\right] == \frac{\pi}{3},\ \theta\right]$$

$$\{\{\theta \to \pi\},\ \{\theta \to 3\pi\}\}$$

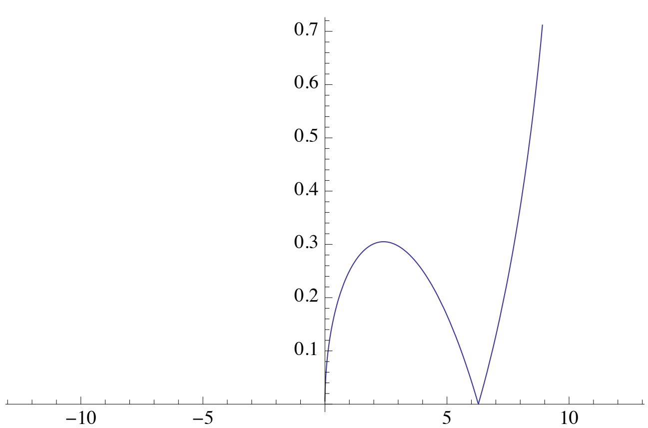
$$\textbf{Plot}\left[\text{ArcSin}\left[\frac{\sqrt{(4\pi-\theta)\,\theta}}{2\pi}\right],\ \{\theta,\ -4\pi,\ 4\pi\}\right]$$



Lemma 7 We can show can show that

$$\left\{\left\{\beta \to -\text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta + 4\pi\theta^2 - \theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right]\right\},\ \left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta + 4\pi\theta^2 - \theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right]\right\}\right\}.$$

$$\textbf{Plot}\left[\text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta + 4\pi\theta^2 - \theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right],\ \{\theta,\ -4\pi,\ 4\pi\}\right]$$



$$\text{Solve}\Big[\Theta == \frac{4\pi}{3} - \frac{-4\pi^2 + 12\pi^2 \text{Sin}[\beta]^2}{6\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}} + \frac{2}{3}\left(-\pi^3 + 18\pi^3 \text{Sin}[\beta]^2 + 3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2 + 11\pi^6 \text{Sin}[\beta]^4 + \pi^6 \text{Sin}[\beta]^6}\right)^{1/3}, \beta\Big]$$

$$\left\{\left\{\beta \to -\text{ArcSin}\left[\frac{\sqrt{-4\pi^2\Theta + 4\pi\Theta^2 - \Theta^3}}{2\pi\sqrt{-4\pi + \Theta}}\right]\right\}, \left\{\beta \to \text{ArcSin}\left[\frac{\sqrt{-4\pi^2\Theta + 4\pi\Theta^2 - \Theta^3}}{2\pi\sqrt{-4\pi + \Theta}}\right]\right\}\right\}$$

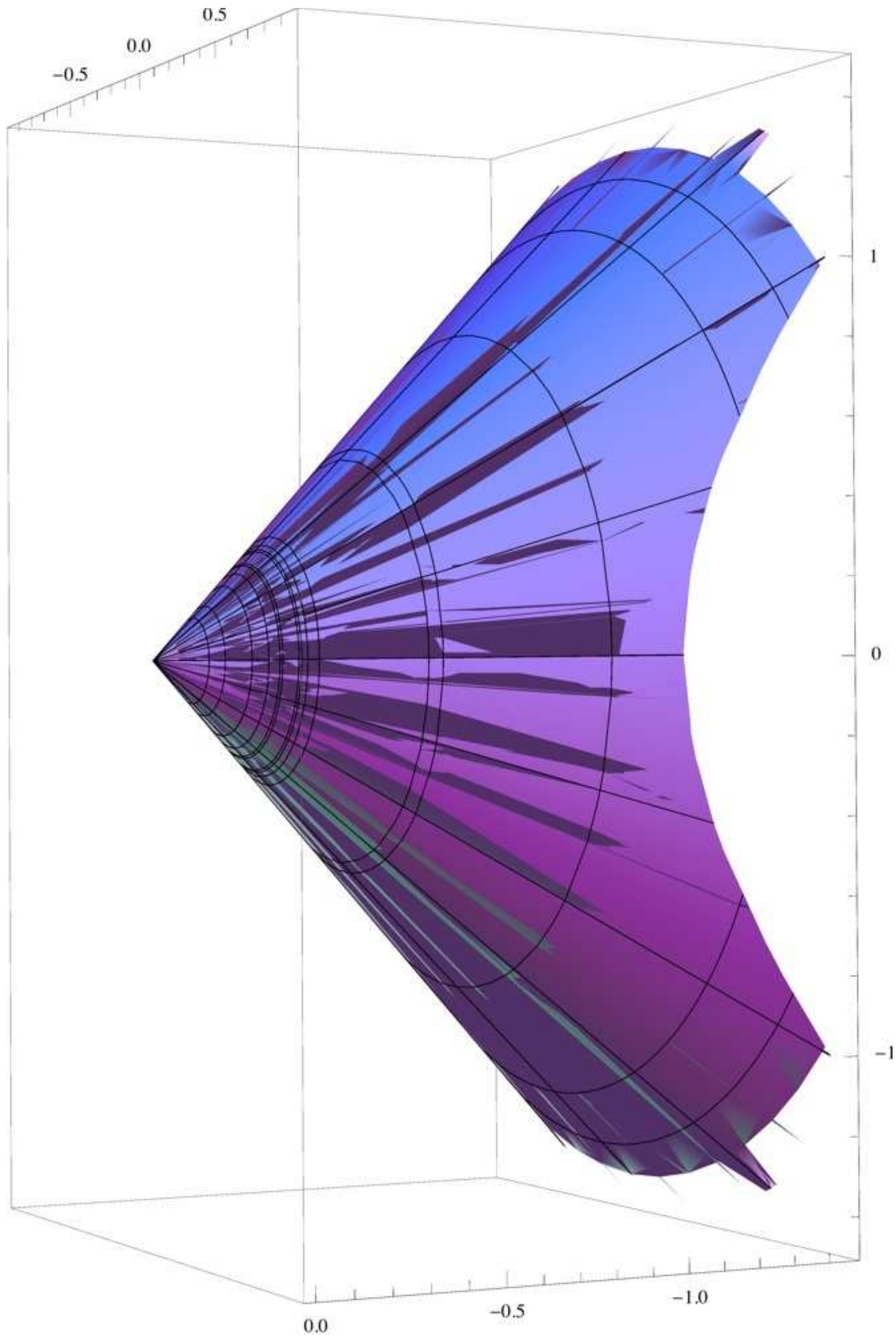
$$\eta = \text{ArcSin}\left[\frac{\sqrt{(4\pi - \Theta)\Theta}}{2\pi}\right] = \text{r Sin}[\beta]$$

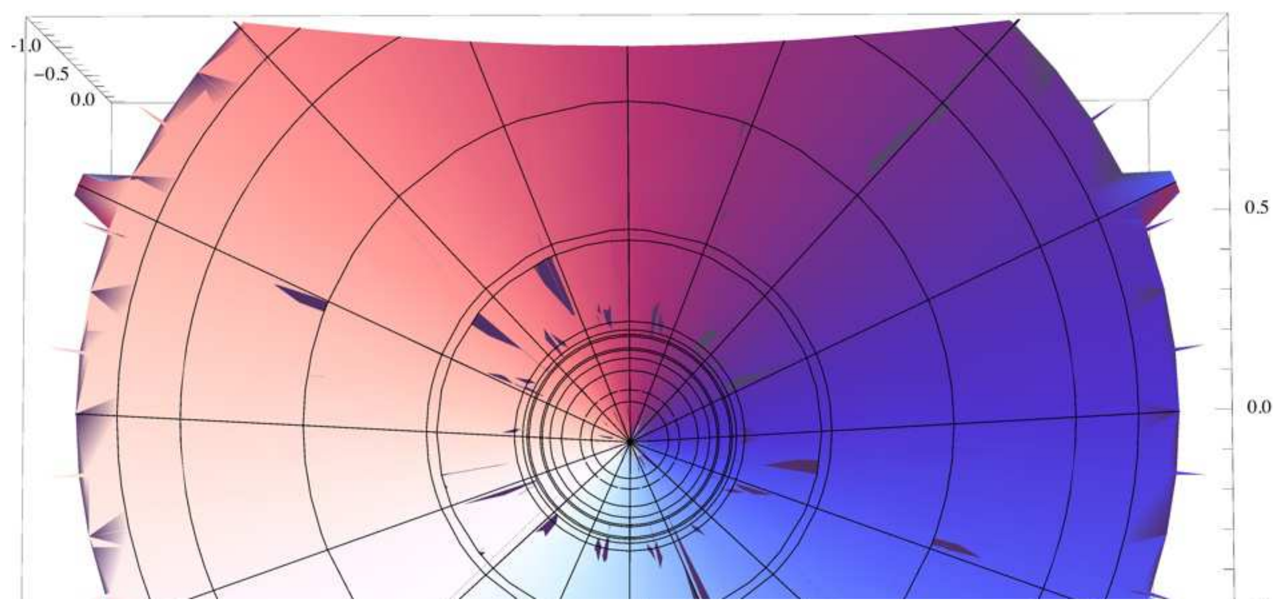
$$\text{Solve}\left[\frac{\sqrt{4\pi r^2\left(2\left(\pi + \sqrt{\pi^2 - \pi^2 \text{Sin}[\beta]^2}\right)\right) - r^2\left(2\left(\pi + \sqrt{\pi^2 - \pi^2 \text{Sin}[\beta]^2}\right)\right)^2}}{2\pi} == \text{r Sin}\left[\frac{\sqrt{-4\pi^2\Theta + 4\pi\Theta^2 - \Theta^3}}{2\pi\sqrt{-4\pi + \Theta}}\right], \beta\right]$$

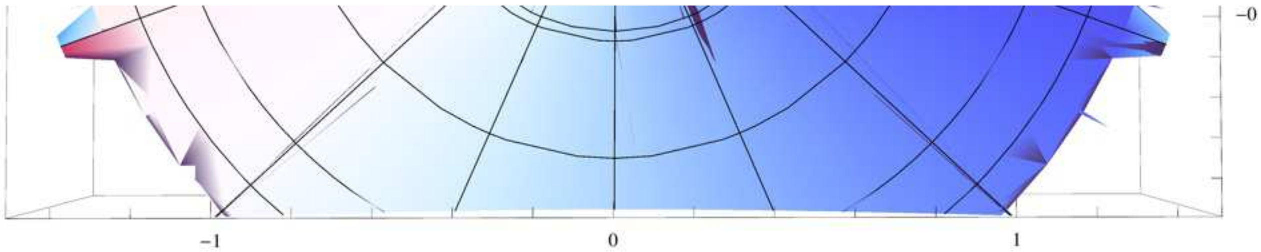
$$\left\{\left\{\beta \to -\text{ArcSin}\left[\sqrt{\text{Sin}\left[\frac{\sqrt{-4\pi^2\Theta + 4\pi\Theta^2 - \Theta^3}}{2\pi\sqrt{-4\pi + \Theta}}\right]^2}\right]\right\}, \left\{\beta \to \text{ArcSin}\left[\sqrt{\text{Sin}\left[\frac{\sqrt{-4\pi^2\Theta + 4\pi\Theta^2 - \Theta^3}}{2\pi\sqrt{-4\pi + \Theta}}\right]^2}\right]\right\}\right\}$$

$$\textbf{RevolutionPlot3D}\left[\left\{\textbf{ArcSin}\left[\sqrt{\textbf{Sin}\left[\frac{\sqrt{-4\,\pi^2\,\theta+4\,\pi\,\theta^2-\theta^3}}{2\,\pi\,\sqrt{-4\,\pi+\theta}}\right]^2}\right],\right.\right.$$

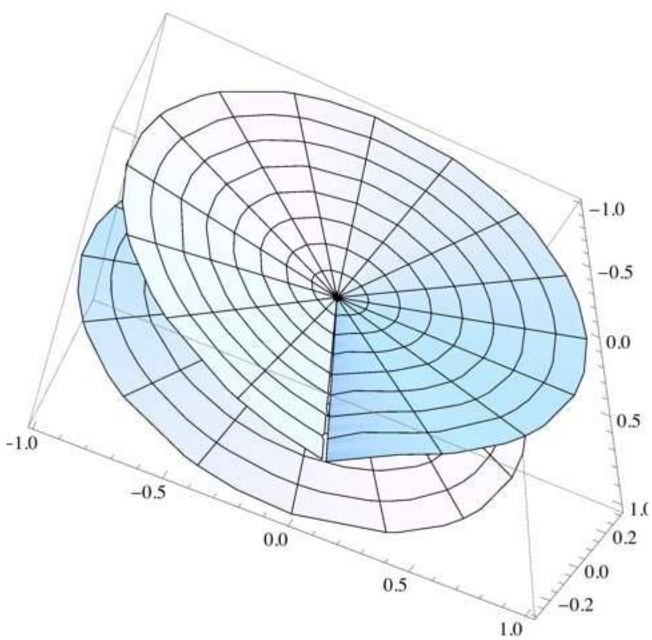
$$\left.\left.-\textbf{ArcSin}\left[\sqrt{\textbf{Sin}\left[\frac{\sqrt{-4\,\pi^2\,\theta+4\,\pi\,\theta^2-\theta^3}}{2\,\pi\,\sqrt{-4\,\pi+\theta}}\right]^2}\right]\right\},\ \{\theta,\ -4\,\pi,\ 4\,\pi\}\right]$$



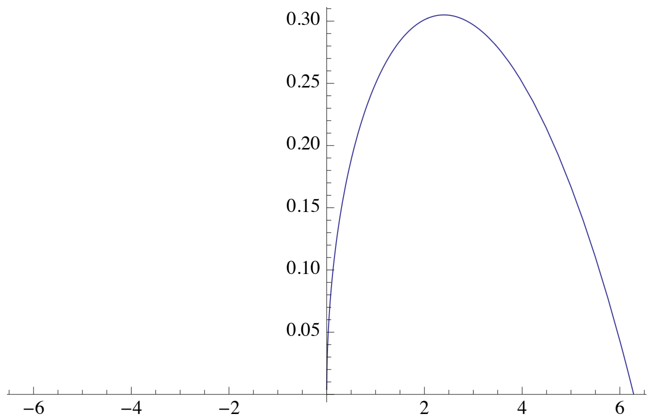

-1.0
−0.5
0.0
0.5
0.0



RevolutionPlot3D$\left[r\,\text{ArcSin}\left[\frac{\sqrt{-4\,\pi^2\,\theta+4\,\pi\,\theta^2-\theta^3}}{2\,\pi\,\sqrt{-4\,\pi+\theta}}\right],\ \{r,-1,1\},\ \{\theta,-2\,\pi,2\,\pi\}\right]$



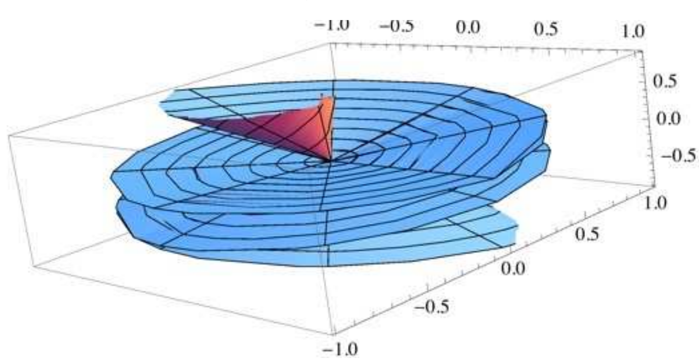
$$\text{Plot}\left[\text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right],\{\theta,-2\pi,2\pi\}\right]$$



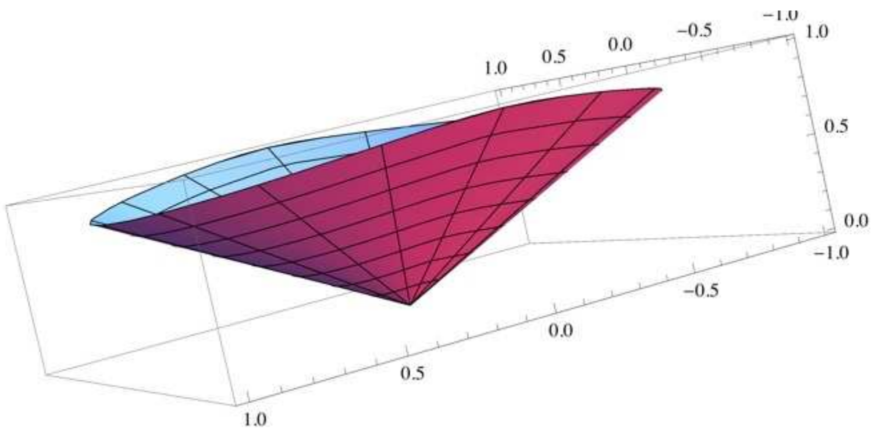
Lemma 7 From showing that $\eta = r\,\text{Sin}[\beta]$, thus we can show that $\eta = \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi} = r\,\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}} =$

$$\frac{1}{2\pi}\left(\sqrt{\left(4\pi r^2\left(\frac{4\pi}{3}-\frac{-4\pi^2+12\pi^2\,\text{Sin}[\beta]^2}{6\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}}+\frac{2}{3}\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}\right)-r^2\left(\frac{4\pi}{3}-\frac{-4\pi^2+12\pi^2\,\text{Sin}[\beta]^2}{6\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}}+\frac{2}{3}\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}\right)^2\right)}\right)$$

$$\text{RevolutionPlot3D}\left[r\,\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}},\{r,-1,1\},\{\theta,-4\pi,4\pi\}\right]$$



$$\text{RevolutionPlot3D}\Big[\frac{1}{2\pi}\Big(\sqrt{\Big(4\pi r^2\Big(\frac{4\pi}{3}-\left(-4\pi^2+12\pi^2 \text{Sin}[\beta]^2\right)\Big/\Big(6\left(-\pi^3+18\pi^3 \text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2+11\pi^6 \text{Sin}[\beta]^4+\pi^6 \text{Sin}[\beta]^6}\right)^{1/3}\Big)+\frac{2}{3}\left(-\pi^3+18\pi^3 \text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2+11\pi^6 \text{Sin}[\beta]^4+\pi^6 \text{Sin}[\beta]^6}\right)^{1/3}\Big)-r^2\Big(\frac{4\pi}{3}-\left(-4\pi^2+12\pi^2 \text{Sin}[\beta]^2\right)\Big/\Big(6\left(-\pi^3+18\pi^3 \text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2+11\pi^6 \text{Sin}[\beta]^4+\pi^6 \text{Sin}[\beta]^6}\right)^{1/3}\Big)+\frac{2}{3}\left(-\pi^3+18\pi^3 \text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2+11\pi^6 \text{Sin}[\beta]^4+\pi^6 \text{Sin}[\beta]^6}\right)^{1/3}\Big)^2\Big)}\Big), \{r, -1, 1\}, \{\beta, -\pi, \pi\}\Big]$$



we can also show that there are three other solutions to theta by solving the equation.

Proof.

$$\eta = \frac{\sqrt{4\pi r^2 \Theta - r^2 \Theta^2}}{2\pi} = r\,\text{Sin}[\beta] = r\,\text{Sin}\Big[\text{ArcSin}\Big[\frac{\sqrt{-4\pi^2\Theta+4\pi\Theta^2-\Theta^3}}{2\pi\sqrt{-4\pi+\Theta}}\Big]\Big]$$

$$\Theta == \frac{4\pi}{3} - \frac{-4\pi^2+12\pi^2 \text{Sin}[\beta]^2}{6\left(-\pi^3+18\pi^3 \text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2+11\pi^6 \text{Sin}[\beta]^4+\pi^6 \text{Sin}[\beta]^6}\right)^{1/3}} + \frac{2}{3}\left(-\pi^3+18\pi^3 \text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6 \text{Sin}[\beta]^2+11\pi^6 \text{Sin}[\beta]^4+\pi^6 \text{Sin}[\beta]^6}\right)^{1/3} == 2\left(\pi \pm \sqrt{\pi^2-\pi^2 \text{Sin}[\beta]^2}\right)$$

We can substitute either theta in the expression for the height of the cone.

$$\text{Solve}\Bigg[\frac{1}{2\pi}\Bigg(\sqrt{\Bigg(4\pi r^2\left(\frac{4\pi}{3}-\left(-4\pi^2+12\pi^2\,\text{Sin}[\beta]^2\right)\Big/\left(6\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}\right)+\frac{2}{3}\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}\right)-r^2\left(\frac{4\pi}{3}-\left(-4\pi^2+12\pi^2\,\text{Sin}[\beta]^2\right)\Big/\left(6\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}\right)+\frac{2}{3}\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}\right)^2\Bigg)}\Bigg) == r\,\text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right]\right],\ \theta\Bigg]$$

$$\text{Solve}\Big[\frac{1}{2\pi}\Bigg(\sqrt{\Bigg(4\pi r^2\left(\frac{4\pi}{3}-\left(-4\pi^2+12\pi^2\,\text{Sin}[\beta]^2\right)\Big/\left(6\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}\right)+\frac{2}{3}\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}\right)-r^2\left(\frac{4\pi}{3}-\left(-4\pi^2+12\pi^2\,\text{Sin}[\beta]^2\right)\Big/\left(6\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}\right)+\frac{2}{3}\left(-\pi^3+18\pi^3\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{-\pi^6\,\text{Sin}[\beta]^2+11\pi^6\,\text{Sin}[\beta]^4+\pi^6\,\text{Sin}[\beta]^6}\right)^{1/3}\right)^2\Bigg)}\Bigg)==$$

$$r\,\text{Sin}\left[\text{ArcSin}\left[\frac{\sqrt{-4\pi^2\theta+4\pi\theta^2-\theta^3}}{2\pi\sqrt{-4\pi+\theta}}\right]\right],$$

$$\theta\Big]$$

$$\Big\{\Big\{\theta\to\frac{4\pi}{3}+\Big(2^{1/3}\Big(-1296\pi^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{10/3}-$$

$$324\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{5/3}\Big(-\pi^2+50\pi^2\,\text{Sin}[\beta]^2-$$

$$453\pi^2\,\text{Sin}[\beta]^4+36\pi^2\,\text{Sin}[\beta]^6+5\sqrt{3}\,\pi^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$78\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$9\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$\pi^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$35\pi^2\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$243\pi^2\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$9\pi^2\,\text{Sin}[\beta]^6\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$4\sqrt{3}\,\pi^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$42\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$5\,\pi^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$88\,\pi^2\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$36\,\pi^2\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$15\sqrt{3}\,\pi^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$6\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}\Big)\Big)\Big)\Big/$$

$$\Big(27\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{5/3}$$

$$\Big(128\,304\,\pi^3-46\,049\,472\,\pi^3\,\text{Sin}[\beta]^2+3\,084\,719\,760\,\pi^3\,\text{Sin}[\beta]^4-$$

$$76\,984\,149\,600\,\pi^3\,\text{Sin}[\beta]^6+791\,034\,575\,760\,\pi^3\,\text{Sin}[\beta]^8-2\,463\,947\,543\,232\,\pi^3\,\text{Sin}[\beta]^{10}-$$

$$4\,120\,487\,403\,984\,\pi^3\,\text{Sin}[\beta]^{12}-430\,339\,664\,160\,\pi^3\,\text{Sin}[\beta]^{14}-$$

$$9\,183\,300\,480\,\pi^3\,\text{Sin}[\beta]^{16}-1\,924\,560\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$240\,395\,040\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$8\,641\,939\,248\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$116\,076\,162\,240\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$464\,847\,899\,760\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$696\,655\,378\,080\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$43\,748\,223\,120\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$306\,110\,016\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{14}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$69\,984\,\pi^3\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$22\,394\,880\,\pi^3\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$1\,412\,976\,960\,\pi^3\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$34\,362\,423\,936\,\pi^3\,\text{Sin}[\beta]^6\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$361\,323\,192\,960\,\pi^3\,\text{Sin}[\beta]^8\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$1\,382\,528\,881\,152\,\pi^3\,\text{Sin}[\beta]^{10}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$10\,033\,606\,080\,\pi^3\,\text{Sin}[\beta]^{12}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$11\,224\,033\,920\,\pi^3\,\text{Sin}[\beta]^{14}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$153\,055\,008\,\pi^3\,\text{Sin}[\beta]^{16}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$979\,776\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$112\,534\,272\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$3\,868\,575\,552\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$52\,016\,027\,904\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$241\,951\,624\,128\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$12\,788\,596\,224\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$1\,122\,403\,392\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$69\,984\,\pi^3\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$19\,455\,552\,\pi^3\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$1\,082\,722\,464\,\pi^3\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$23\,329\,866\,240\,\pi^3\,\text{Sin}[\beta]^6\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$217\,196\,393\,760\,\pi^3\,\text{Sin}[\beta]^8\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$725\,922\,896\,832\,\pi^3\,\text{Sin}[\beta]^{10}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$79\,877\,708\,064\,\pi^3\,\text{Sin}[\beta]^{12}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$1\,632\,586\,752\,\pi^3\,\text{Sin}[\beta]^{14}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$909\,792\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$91\,119\,168\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$2\,737\,984\,032\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$32\,062\,189\,824\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$127\,925\,643\,168\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$8\,060\,897\,088\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$51\,018\,336\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$\sqrt{\Big(\Big(128\,304\,\pi^3-46\,049\,472\,\pi^3\,\text{Sin}[\beta]^2+3\,084\,719\,760\,\pi^3\,\text{Sin}[\beta]^4-76\,984\,149\,600\,\pi^3\,\text{Sin}[\beta]^6+}$$
$$791\,034\,575\,760\,\pi^3\,\text{Sin}[\beta]^8-2\,463\,947\,543\,232\,\pi^3\,\text{Sin}[\beta]^{10}-$$
$$4\,120\,487\,403\,984\,\pi^3\,\text{Sin}[\beta]^{12}-430\,339\,664\,160\,\pi^3\,\text{Sin}[\beta]^{14}-$$
$$9\,183\,300\,480\,\pi^3\,\text{Sin}[\beta]^{16}-1\,924\,560\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$
$$240\,395\,040\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$
$$8\,641\,939\,248\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$116\,076\,162\,240\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\;-$$
$$464\,847\,899\,760\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\;-$$
$$696\,655\,378\,080\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\;-$$
$$43\,748\,223\,120\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-306\,110\,016$$
$$\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{14}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+69\,984\,\pi^3\left(-1+18\,\text{Sin}[\beta]^2+\right.$$
$$\left.3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-22\,394\,880\,\pi^3\,\text{Sin}[\beta]^2$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+1\,412\,976\,960$$
$$\pi^3\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$
$$34\,362\,423\,936\,\pi^3\,\text{Sin}[\beta]^6\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\right.$$
$$\left.\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+361\,323\,192\,960\,\pi^3\,\text{Sin}[\beta]^8$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-1\,382\,528\,881\,152$$
$$\pi^3\,\text{Sin}[\beta]^{10}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$
$$10\,033\,606\,080\,\pi^3\,\text{Sin}[\beta]^{12}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\right.$$
$$\left.\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+11\,224\,033\,920\,\pi^3\,\text{Sin}[\beta]^{14}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+153\,055\,008$$
$$\pi^3\,\text{Sin}[\beta]^{16}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$
$$979\,776\,\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$
$$112\,534\,272\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$
$$3\,868\,575\,552\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$
$$52\,016\,027\,904\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$241\,951\,624\,128\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$
$$12\,788\,596\,224\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$
$$1\,122\,403\,392\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$
$$69\,984\,\pi^3\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+19\,455\,552$$
$$\pi^3\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$
$$1\,082\,722\,464\,\pi^3\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\right.$$
$$\left.\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+23\,329\,866\,240\,\pi^3\,\text{Sin}[\beta]^6$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-217\,196\,393\,760$$
$$\pi^3\,\text{Sin}[\beta]^8\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$
$$725\,922\,896\,832\,\pi^3\,\text{Sin}[\beta]^{10}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\right.$$
$$\left.\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+79\,877\,708\,064\,\pi^3\,\text{Sin}[\beta]^{12}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+1\,632\,586\,752$$
$$\pi^3\,\text{Sin}[\beta]^{14}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$
$$909\,792\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$
$$91\,119\,168\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$
$$2\,737\,984\,032\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$
$$32\,062\,189\,824\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$127\,925\,643\,168\,\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^8\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$8\,060\,897\,088\,\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^{10}\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$51\,018\,336\,\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^{12}\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}$$

$$\left.\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{2/3}\right)^2+$$

$$4\left(-1296\,\pi^2\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{10/3}-\right.$$

$$324\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{5/3}$$

$$\left(-\pi^2+50\,\pi^2\,\mathrm{Sin}[\beta]^2-453\,\pi^2\,\mathrm{Sin}[\beta]^4+36\,\pi^2\,\mathrm{Sin}[\beta]^6+\right.$$

$$5\sqrt{3}\,\pi^2\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}-$$

$$78\sqrt{3}\,\pi^2\,\mathrm{Sin}[\beta]^2\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}+$$

$$9\sqrt{3}\,\pi^2\,\mathrm{Sin}[\beta]^4\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}+$$

$$\pi^2\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$35\,\pi^2\,\mathrm{Sin}[\beta]^2\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$243\,\pi^2\,\mathrm{Sin}[\beta]^4\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$9\,\pi^2\,\mathrm{Sin}[\beta]^6\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$4\sqrt{3}\,\pi^2\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$42\sqrt{3}\,\pi^2\,\mathrm{Sin}[\beta]^2\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$5\,\pi^2\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$88\,\pi^2\,\mathrm{Sin}[\beta]^2\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$36\,\pi^2\,\mathrm{Sin}[\beta]^4\,\left(-1+18\,\mathrm{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\,\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{2/3}-$$
$$15\,\sqrt{3}\,\pi^2\,\sqrt{\mathrm{Sin}[\beta]^2\,\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\mathrm{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\,\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{2/3}-$$
$$6\,\sqrt{3}\,\pi^2\,\mathrm{Sin}[\beta]^2\,\sqrt{\mathrm{Sin}[\beta]^2\,\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}$$
$$\left.\left.\left.\left.\left(-1+18\,\mathrm{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\,\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{2/3}\right)\right)^3\right)\right)^{1/3}\right)-$$

$$\left(128\,304\,\pi^3-46\,049\,472\,\pi^3\,\mathrm{Sin}[\beta]^2+3\,084\,719\,760\,\pi^3\,\mathrm{Sin}[\beta]^4-76\,984\,149\,600\,\pi^3\,\mathrm{Sin}[\beta]^6+\right.$$
$$791\,034\,575\,760\,\pi^3\,\mathrm{Sin}[\beta]^8-$$
$$2\,463\,947\,543\,232\,\pi^3\,\mathrm{Sin}[\beta]^{10}-$$
$$4\,120\,487\,403\,984\,\pi^3\,\mathrm{Sin}[\beta]^{12}-$$
$$430\,339\,664\,160\,\pi^3\,\mathrm{Sin}[\beta]^{14}-$$
$$9\,183\,300\,480\,\pi^3\,\mathrm{Sin}[\beta]^{16}-$$
$$1\,924\,560\,\sqrt{3}\,\pi^3$$
$$\sqrt{\mathrm{Sin}[\beta]^2\,\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}+$$
$$240\,395\,040\,\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^2\,\sqrt{\mathrm{Sin}[\beta]^2\,\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}-$$
$$8\,641\,939\,248\,\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^4$$
$$\sqrt{\mathrm{Sin}[\beta]^2\,\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}+$$
$$116\,076\,162\,240\,\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^6\,\sqrt{\mathrm{Sin}[\beta]^2\,\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}-$$
$$464\,847\,899\,760\,\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^8$$
$$\sqrt{\mathrm{Sin}[\beta]^2\,\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}-$$
$$696\,655\,378\,080\,\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^{10}\,\sqrt{\mathrm{Sin}[\beta]^2\,\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}-$$
$$43\,748\,223\,120\,\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^{12}$$
$$\sqrt{\mathrm{Sin}[\beta]^2\,\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}-$$
$$306\,110\,016\,\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^{14}\,\sqrt{\mathrm{Sin}[\beta]^2\,\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}+$$
$$69\,984\,\pi^3\,\left(-1+18\,\mathrm{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\,\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3}-$$
$$22\,394\,880\,\pi^3\,\mathrm{Sin}[\beta]^2$$
$$\left(-1+18\,\mathrm{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\,\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3}+$$
$$1\,412\,976\,960\,\pi^3\,\mathrm{Sin}[\beta]^4\,\left(-1+18\,\mathrm{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\,\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3}-$$
$$34\,362\,423\,936\,\pi^3\,\mathrm{Sin}[\beta]^6$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$361\,323\,192\,960\,\pi^3\,\text{Sin}[\beta]^8\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$1\,382\,528\,881\,152\,\pi^3\,\text{Sin}[\beta]^{10}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$10\,033\,606\,080\,\pi^3\,\text{Sin}[\beta]^{12}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$11\,224\,033\,920\,\pi^3\,\text{Sin}[\beta]^{14}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$153\,055\,008\,\pi^3\,\text{Sin}[\beta]^{16}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$979\,776\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$112\,534\,272\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$3\,868\,575\,552\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$52\,016\,027\,904\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$241\,951\,624\,128\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$12\,788\,596\,224\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$1\,122\,403\,392\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$69\,984\,\pi^3\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$19\,455\,552\,\pi^3\,\text{Sin}[\beta]^2$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)$$

$$1\,082\,722\,464\,\pi^3\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$23\,329\,866\,240\,\pi^3\,\text{Sin}[\beta]^6$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$217\,196\,393\,760\,\pi^3\,\text{Sin}[\beta]^8\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$725\,922\,896\,832\,\pi^3\,\text{Sin}[\beta]^{10}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$79\,877\,708\,064\,\pi^3\,\text{Sin}[\beta]^{12}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$1\,632\,586\,752\,\pi^3\,\text{Sin}[\beta]^{14}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$909\,792\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$91\,119\,168\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$2\,737\,984\,032\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$32\,062\,189\,824\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$127\,925\,643\,168\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$8\,060\,897\,088\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$51\,018\,336\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$\sqrt{\left(\left(128\,304\,\pi^3-46\,049\,472\,\pi^3\,\text{Sin}[\beta]^2+3\,084\,719\,760\,\pi^3\,\text{Sin}[\beta]^4-76\,984\,149\,600\,\pi^3\,\text{Sin}[\beta]^6+\right.\right.}$$

$$791\,034\,575\,760\,\pi^3\,\mathrm{Sin}[\beta]^8 - 2\,463\,947\,543\,232\,\pi^3\,\mathrm{Sin}[\beta]^{10} -$$
$$4\,120\,487\,403\,984\,\pi^3\,\mathrm{Sin}[\beta]^{12} - 430\,339\,664\,160\,\pi^3\,\mathrm{Sin}[\beta]^{14} -$$
$$9\,183\,300\,480\,\pi^3\,\mathrm{Sin}[\beta]^{16} - 1\,924\,560\,\sqrt{3}\,\pi^3\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)} +$$
$$240\,395\,040\,\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^2\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)} -$$
$$8\,641\,939\,248\,\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^4\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)} +$$
$$116\,076\,162\,240\,\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^6\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)} -$$
$$464\,847\,899\,760\,\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^8\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)} -$$
$$696\,655\,378\,080\,\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^{10}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)} -$$
$$43\,748\,223\,120\,\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^{12}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)} -$$
$$306\,110\,016\,\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^{14}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)} +$$
$$69\,984\,\pi^3\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3} -$$
$$22\,394\,880\,\pi^3\,\mathrm{Sin}[\beta]^2\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3} +$$
$$1\,412\,976\,960\,\pi^3\,\mathrm{Sin}[\beta]^4\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3} -$$
$$34\,362\,423\,936\,\pi^3\,\mathrm{Sin}[\beta]^6\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3} +$$
$$361\,323\,192\,960\,\pi^3\,\mathrm{Sin}[\beta]^8\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3} -$$
$$1\,382\,528\,881\,152\,\pi^3\,\mathrm{Sin}[\beta]^{10}\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3} +$$
$$10\,033\,606\,080\,\pi^3\,\mathrm{Sin}[\beta]^{12}\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3} +$$
$$11\,224\,033\,920\,\pi^3\,\mathrm{Sin}[\beta]^{14}\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3} +$$
$$153\,055\,008\,\pi^3\,\mathrm{Sin}[\beta]^{16}\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3} -$$
$$979\,776\,\sqrt{3}\,\pi^3\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3} +$$
$$112\,534\,272\,\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^2\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3} -$$

$$3\,868\,575\,552\,\sqrt{3}\,\pi^{3}\,\text{Sin}[\beta]^{4}\,\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^{2}+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}\right)^{1/3}+$$

$$52\,016\,027\,904\,\sqrt{3}\,\pi^{3}\,\text{Sin}[\beta]^{6}\,\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^{2}+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}\right)^{1/3}-$$

$$241\,951\,624\,128\,\sqrt{3}\,\pi^{3}\,\text{Sin}[\beta]^{8}\,\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^{2}+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}\right)^{1/3}+$$

$$12\,788\,596\,224\,\sqrt{3}\,\pi^{3}\,\text{Sin}[\beta]^{10}\,\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^{2}+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}\right)^{1/3}+$$

$$1\,122\,403\,392\,\sqrt{3}\,\pi^{3}\,\text{Sin}[\beta]^{12}\,\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^{2}+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}\right)^{1/3}-$$

$$69\,984\,\pi^{3}\left(-1+18\,\text{Sin}[\beta]^{2}+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}\right)^{2/3}+$$

$$19\,455\,552\,\pi^{3}\,\text{Sin}[\beta]^{2}\left(-1+18\,\text{Sin}[\beta]^{2}+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}\right)^{2/3}-$$

$$1\,082\,722\,464\,\pi^{3}\,\text{Sin}[\beta]^{4}\,(-1+18\,\text{Sin}[\beta]^{2}+$$

$$3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}\,)^{2/3}+23\,329\,866\,240\,\pi^{3}\,\text{Sin}[\beta]^{6}$$

$$\left(-1+18\,\text{Sin}[\beta]^{2}+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}\right)^{2/3}-217\,196\,393\,760$$

$$\pi^{3}\,\text{Sin}[\beta]^{8}\left(-1+18\,\text{Sin}[\beta]^{2}+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}\right)^{2/3}+$$

$$725\,922\,896\,832\,\pi^{3}\,\text{Sin}[\beta]^{10}\,(-1+18\,\text{Sin}[\beta]^{2}+3\,\sqrt{3}$$

$$\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}\,)^{2/3}+79\,877\,708\,064\,\pi^{3}\,\text{Sin}[\beta]^{12}$$

$$\left(-1+18\,\text{Sin}[\beta]^{2}+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}\right)^{2/3}+1\,632\,586\,752$$

$$\pi^{3}\,\text{Sin}[\beta]^{14}\left(-1+18\,\text{Sin}[\beta]^{2}+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}\right)^{2/3}+$$

$$909\,792\,\sqrt{3}\,\pi^{3}\,\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^{2}+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}\right)^{2/3}-$$

$$91\,119\,168\,\sqrt{3}\,\pi^{3}\,\text{Sin}[\beta]^{2}\,\sqrt{\text{Sin}[\beta]^{2}\left(-1+11\,\text{Sin}[\beta]^{2}+\text{Sin}[\beta]^{4}\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$2\,737\,984\,032\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$32\,062\,189\,824\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$127\,925\,643\,168\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$8\,060\,897\,088\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$51\,018\,336\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left.\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}\right)^2+$$

$$4\left(-1296\,\pi^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{10/3}-324\left(-1+\right.\right.$$

$$\left.18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{5/3}\left(-\pi^2+50\,\pi^2\,\text{Sin}[\beta]^2-\right.$$

$$453\,\pi^2\,\text{Sin}[\beta]^4+36\,\pi^2\,\text{Sin}[\beta]^6+5\sqrt{3}\,\pi^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$78\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$9\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$\pi^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$35\,\pi^2\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$243\,\pi^2\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$9\,\pi^2\,\text{Sin}[\beta]^6\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$4\sqrt{3}\,\pi^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$42\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)\;+$$

$$5\,\pi^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$88\,\pi^2\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$36\,\pi^2\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$15\sqrt{3}\,\pi^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$6\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}\Bigg)\Bigg)^3\Bigg)\Bigg)^{1/3}\Bigg/$$

$$\left(27\,2^{1/3}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{5/3}\right)\Big\},\ \Big\{\theta\to$$

$$\frac{4\pi}{3}-$$

$$\Big(\Big(1+$$

$$i\sqrt{3}\Big)$$

$$\Big(-1296\,\pi^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{10/3}-$$

$$324\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{5/3}$$

$$\Big(-\pi^2+50\,\pi^2\,\text{Sin}[\beta]^2-453\,\pi^2\,\text{Sin}[\beta]^4+36\,\pi^2\,\text{Sin}[\beta]^6+$$

$$5\sqrt{3}\,\pi^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$78\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$9\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$\pi^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$35\,\pi^2\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$243\,\pi^2\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$9\,\pi^2\,\text{Sin}[\beta]^6\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$4\sqrt{3}\,\pi^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$42\sqrt{3}\,\pi^2\,\mathrm{Sin}[\beta]^2\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$5\,\pi^2\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$88\,\pi^2\,\mathrm{Sin}[\beta]^2\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$36\,\pi^2\,\mathrm{Sin}[\beta]^4\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$15\sqrt{3}\,\pi^2\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$6\sqrt{3}\,\pi^2\,\mathrm{Sin}[\beta]^2\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}$$

$$\left.\left.\left.\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{2/3}\right)\right)\right)\Big/$$

$$\left(27\,2^{2/3}\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{5/3}\right.$$

$$\left(128\,304\,\pi^3-46\,049\,472\,\pi^3\,\mathrm{Sin}[\beta]^2+3\,084\,719\,760\,\pi^3\,\mathrm{Sin}[\beta]^4-\right.$$

$$76\,984\,149\,600\,\pi^3\,\mathrm{Sin}[\beta]^6+791\,034\,575\,760\,\pi^3\,\mathrm{Sin}[\beta]^8-$$
$$2\,463\,947\,543\,232\,\pi^3\,\mathrm{Sin}[\beta]^{10}-4\,120\,487\,403\,984\,\pi^3\,\mathrm{Sin}[\beta]^{12}-$$
$$430\,339\,664\,160\,\pi^3\,\mathrm{Sin}[\beta]^{14}-9\,183\,300\,480\,\pi^3\,\mathrm{Sin}[\beta]^{16}-$$

$$1\,924\,560\sqrt{3}\,\pi^3\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}+$$

$$240\,395\,040\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^2\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}-$$

$$8\,641\,939\,248\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^4\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}+$$

$$116\,076\,162\,240\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^6\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}-$$

$$464\,847\,899\,760\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^8\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}-$$

$$696\,655\,378\,080\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^{10}\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}-$$

$$43\,748\,223\,120\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^{12}\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}-$$

$$306\,110\,016\sqrt{3}\,\pi^3\,\mathrm{Sin}[\beta]^{14}\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}+$$

$$69\,984\,\pi^3\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$22\,394\,880\,\pi^3\,\mathrm{Sin}[\beta]^2\left(-1+18\,\mathrm{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\mathrm{Sin}[\beta]^2\left(-1+11\,\mathrm{Sin}[\beta]^2+\mathrm{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$1\,412\,976\,960\,\pi^3\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$34\,362\,423\,936\,\pi^3\,\text{Sin}[\beta]^6\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$361\,323\,192\,960\,\pi^3\,\text{Sin}[\beta]^8\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$1\,382\,528\,881\,152\,\pi^3\,\text{Sin}[\beta]^{10}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$10\,033\,606\,080\,\pi^3\,\text{Sin}[\beta]^{12}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$11\,224\,033\,920\,\pi^3\,\text{Sin}[\beta]^{14}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$153\,055\,008\,\pi^3\,\text{Sin}[\beta]^{16}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$979\,776\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$112\,534\,272\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$3\,868\,575\,552\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$52\,016\,027\,904\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$241\,951\,624\,128\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$12\,788\,596\,224\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$1\,122\,403\,392\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$69\,984\,\pi^3\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$19\,455\,552\,\pi^3\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$1\,082\,722\,464\,\pi^3\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$23\,329\,866\,240\,\pi^3\,\text{Sin}[\beta]^6\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$217\,196\,393\,760\,\pi^3\,\text{Sin}[\beta]^8\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$725\,922\,896\,832\,\pi^3\,\text{Sin}[\beta]^{10}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$79\,877\,708\,064\,\pi^3\,\text{Sin}[\beta]^{12}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$1\,632\,586\,752\,\pi^3\,\text{Sin}[\beta]^{14}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$909\,792\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$91\,119\,168\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$2\,737\,984\,032\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$32\,062\,189\,824\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$127\,925\,643\,168\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$8\,060\,897\,088\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$51\,018\,336\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$\sqrt{}\,\Big(\Big(\big(128\,304\,\pi^3-46\,049\,472\,\pi^3\,\text{Sin}[\beta]^2+3\,084\,719\,760\,\pi^3\,\text{Sin}[\beta]^4-76\,984\,149\,600\,\pi^3\,\text{Sin}[\beta]^6+$$
$$791\,034\,575\,760\,\pi^3\,\text{Sin}[\beta]^8-2\,463\,947\,543\,232\,\pi^3\,\text{Sin}[\beta]^{10}-$$
$$4\,120\,487\,403\,984\,\pi^3\,\text{Sin}[\beta]^{12}-430\,339\,664\,160\,\pi^3\,\text{Sin}[\beta]^{14}-$$
$$9\,183\,300\,480\,\pi^3\,\text{Sin}[\beta]^{16}-1\,924\,560\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$9\,185\,300\,480\,\pi\ \text{Sin}[\beta] - 1\,924\,560\,\sqrt{3}\,\pi\,\sqrt{\text{Sin}[\beta]\ (-1 + 11\,\text{Sin}[\beta] + \text{Sin}[\beta])} +$$

$$240\,395\,040\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)} -$$

$$8\,641\,939\,248\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)} +$$

$$116\,076\,162\,240\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)} -$$

$$464\,847\,899\,760\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)} -$$

$$696\,655\,378\,080\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)} -$$

$$43\,748\,223\,120\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)} - 306\,110\,016$$

$$\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{14}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)} + 69\,984\,\pi^3\left(-1+18\,\text{Sin}[\beta]^2+\right.$$

$$\left.3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3} - 22\,394\,880\,\pi^3\,\text{Sin}[\beta]^2$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3} + 1\,412\,976\,960$$

$$\pi^3\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3} -$$

$$34\,362\,423\,936\,\pi^3\,\text{Sin}[\beta]^6\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\right.$$

$$\left.\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3} + 361\,323\,192\,960\,\pi^3\,\text{Sin}[\beta]^8$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3} - 1\,382\,528\,881\,152$$

$$\pi^3\,\text{Sin}[\beta]^{10}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3} +$$

$$10\,033\,606\,080\,\pi^3\,\text{Sin}[\beta]^{12}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\right.$$

$$\left.\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3} + 11\,224\,033\,920\,\pi^3\,\text{Sin}[\beta]^{14}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3} + 153\,055\,008$$

$$\pi^3\,\text{Sin}[\beta]^{16}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3} -$$

$$979\,776\,\sqrt{3}\,\pi^3\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3} +$$

$$112\,534\,272\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3} -$$

$$3\,868\,575\,552\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(\ldots\right)^{1/3}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)\quad +$$

$$52\,016\,027\,904\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$241\,951\,624\,128\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$12\,788\,596\,224\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$1\,122\,403\,392\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$69\,984\,\pi^3\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+19\,455\,552$$

$$\pi^3\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$1\,082\,722\,464\,\pi^3\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\right.$$

$$\left.\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+23\,329\,866\,240\,\pi^3\,\text{Sin}[\beta]^6$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-217\,196\,393\,760$$

$$\pi^3\,\text{Sin}[\beta]^8\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$725\,922\,896\,832\,\pi^3\,\text{Sin}[\beta]^{10}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\right.$$

$$\left.\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+79\,877\,708\,064\,\pi^3\,\text{Sin}[\beta]^{12}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+1\,632\,586\,752$$

$$\pi^3\,\text{Sin}[\beta]^{14}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$909\,792\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$91\,119\,168\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$2\,737\,984\,032\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$
$$32\,062\,189\,824\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$
$$127\,925\,643\,168\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$
$$8\,060\,897\,088\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$
$$51\,018\,336\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left.\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}\right)^2+$$
$$4\left(-1296\,\pi^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{10/3}-\right.$$
$$324\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{5/3}$$
$$\left(-\pi^2+50\,\pi^2\,\text{Sin}[\beta]^2-453\,\pi^2\,\text{Sin}[\beta]^4+36\,\pi^2\,\text{Sin}[\beta]^6+\right.$$
$$5\sqrt{3}\,\pi^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$
$$78\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$
$$9\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$
$$\pi^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$
$$35\,\pi^2\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$
$$243\,\pi^2\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$
$$9\,\pi^2\,\text{Sin}[\beta]^6\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$
$$4\sqrt{3}\,\pi^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$
$$42\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$5\,\pi^2\,\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$88\,\pi^2\,\text{Sin}[\beta]^2\,\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$36\,\pi^2\,\text{Sin}[\beta]^4\,\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$15\,\sqrt{3}\,\pi^2\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$6\,\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^2\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left.\left.\left.\left.\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}\right)\right)^3\right)\right)^{1/3}\Bigg)+$$

$$\Bigg(\left(1-i\,\sqrt{3}\right)\,\Big(128\,304\,\pi^3-46\,049\,472\,\pi^3\,\text{Sin}[\beta]^2+3\,084\,719\,760\,\pi^3\,\text{Sin}[\beta]^4-76\,984\,149\,600$$

$$\pi^3\,\text{Sin}[\beta]^6+791\,034\,575\,760\,\pi^3\,\text{Sin}[\beta]^8-$$

$$2\,463\,947\,543\,232\,\pi^3\,\text{Sin}[\beta]^{10}-4\,120\,487\,403\,984\,\pi^3\,\text{Sin}[\beta]^{12}-$$

$$430\,339\,664\,160\,\pi^3\,\text{Sin}[\beta]^{14}-9\,183\,300\,480\,\pi^3\,\text{Sin}[\beta]^{16}-$$

$$1\,924\,560\,\sqrt{3}\,\pi^3\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$240\,395\,040\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$8\,641\,939\,248\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$116\,076\,162\,240\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$464\,847\,899\,760\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$696\,655\,378\,080\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$43\,748\,223\,120\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$306\,110\,016\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{14}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$69\,984\,\pi^3\,\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$22\,394\,880\,\pi^3\,\text{Sin}[\beta]^2\,\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$1\,412\,976\,960\,\pi^3\,\text{Sin}[\beta]^4\,\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$34\,362\,423\,936\,\pi^3\,\text{Sin}[\beta]^6\,\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$361\,323\,192\,960\,\pi^3\,\text{Sin}[\beta]^8\,\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$1\,382\,528\,881\,152\,\pi^3\,\text{Sin}[\beta]^{10}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$10\,033\,606\,080\,\pi^3\,\text{Sin}[\beta]^{12}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$11\,224\,033\,920\,\pi^3\,\text{Sin}[\beta]^{14}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$153\,055\,008\,\pi^3\,\text{Sin}[\beta]^{16}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$979\,776\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$112\,534\,272\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$3\,868\,575\,552\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$52\,016\,027\,904\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$241\,951\,624\,128\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$12\,788\,596\,224\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$1\,122\,403\,392\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$69\,984\,\pi^3\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$19\,455\,552\,\pi^3\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$1\,082\,722\,464\,\pi^3\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$23\,329\,866\,240\,\pi^3\,\text{Sin}[\beta]^6\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$217\,196\,393\,760\,\pi^3\,\text{Sin}[\beta]^8\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$725\,922\,896\,832\,\pi^3\,\text{Sin}[\beta]^{10}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$79\,877\,708\,064\,\pi^3\,\text{Sin}[\beta]^{12}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$1\,632\,586\,752\,\pi^3\,\text{Sin}[\beta]^{14}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$909\,792\,\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$91\,119\,168\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$2\,737\,984\,032\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$32\,062\,189\,824\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$127\,925\,643\,168\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$8\,060\,897\,088\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$51\,018\,336\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$\surd\Big(\Big(\Big(128\,304\,\pi^3-46\,049\,472\,\pi^3\,\text{Sin}[\beta]^2+3\,084\,719\,760\,\pi^3\,\text{Sin}[\beta]^4-76\,984\,149\,600\,\pi^3\,\text{Sin}[\beta]^6+$$
$$791\,034\,575\,760\,\pi^3\,\text{Sin}[\beta]^8-2\,463\,947\,543\,232\,\pi^3\,\text{Sin}[\beta]^{10}-$$
$$4\,120\,487\,403\,984\,\pi^3\,\text{Sin}[\beta]^{12}-430\,339\,664\,160\,\pi^3\,\text{Sin}[\beta]^{14}-$$
$$9\,183\,300\,480\,\pi^3\,\text{Sin}[\beta]^{16}-1\,924\,560\,\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$
$$240\,395\,040\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$
$$8\,641\,939\,248\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$
$$116\,076\,162\,240\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$
$$464\,847\,899\,760\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$696\,655\,378\,080\,\sqrt{3}\;\pi^3\,\text{Sin}[\beta]^{10}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\;-$$

$$43\,748\,223\,120\,\sqrt{3}\;\pi^3\,\text{Sin}[\beta]^{12}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\;-\;306\,110\,016$$

$$\sqrt{3}\;\pi^3\,\text{Sin}[\beta]^{14}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\;+\;69\,984\,\pi^3\left(-1+18\,\text{Sin}[\beta]^2+\right.$$

$$\left.3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-\;22\,394\,880\,\pi^3\,\text{Sin}[\beta]^2$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+\;1\,412\,976\,960$$

$$\pi^3\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$34\,362\,423\,936\,\pi^3\,\text{Sin}[\beta]^6\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\right.$$

$$\left.\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+\;361\,323\,192\,960\,\pi^3\,\text{Sin}[\beta]^8$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-\;1\,382\,528\,881\,152$$

$$\pi^3\,\text{Sin}[\beta]^{10}\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$10\,033\,606\,080\,\pi^3\,\text{Sin}[\beta]^{12}\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\right.$$

$$\left.\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+\;11\,224\,033\,920\,\pi^3\,\text{Sin}[\beta]^{14}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+\;153\,055\,008$$

$$\pi^3\,\text{Sin}[\beta]^{16}\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$979\,776\,\sqrt{3}\;\pi^3\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$112\,534\,272\,\sqrt{3}\;\pi^3\,\text{Sin}[\beta]^2\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$3\,868\,575\,552\,\sqrt{3}\;\pi^3\,\text{Sin}[\beta]^4\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$52\,016\,027\,904\,\sqrt{3}\;\pi^3\,\text{Sin}[\beta]^6\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$241\,951\,624\,128\,\sqrt{3}\;\pi^3\,\text{Sin}[\beta]^8\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$12\,788\,596\,224\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$1\,122\,403\,392\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$69\,984\,\pi^3\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+19\,455\,552$$

$$\pi^3\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$1\,082\,722\,464\,\pi^3\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\right.$$

$$\left.\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+23\,329\,866\,240\,\pi^3\,\text{Sin}[\beta]^6$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-217\,196\,393\,760$$

$$\pi^3\,\text{Sin}[\beta]^8\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$725\,922\,896\,832\,\pi^3\,\text{Sin}[\beta]^{10}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\right.$$

$$\left.\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+79\,877\,708\,064\,\pi^3\,\text{Sin}[\beta]^{12}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+1\,632\,586\,752$$

$$\pi^3\,\text{Sin}[\beta]^{14}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$909\,792\,\sqrt{3}\,\pi^3\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$91\,119\,168\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$2\,737\,984\,032\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$32\,062\,189\,824\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$127\,925\,643\,168 \sqrt{3}\ \pi^3 \text{Sin}[\beta]^8 \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)}$$

$$\left(-1 + 18 \text{Sin}[\beta]^2 + 3 \sqrt{3}\ \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)}\ \right)^{2/3} +$$

$$8\,060\,897\,088 \sqrt{3}\ \pi^3 \text{Sin}[\beta]^{10} \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)}$$

$$\left(-1 + 18 \text{Sin}[\beta]^2 + 3 \sqrt{3}\ \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)}\ \right)^{2/3} +$$

$$51\,018\,336 \sqrt{3}\ \pi^3 \text{Sin}[\beta]^{12} \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)}$$

$$\left(-1 + 18 \text{Sin}[\beta]^2 + 3 \sqrt{3}\ \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)}\ \right)^{2/3})^2 +$$

$$4 \ (-1296 \pi^2 \left(-1 + 18 \text{Sin}[\beta]^2 + 3 \sqrt{3}\ \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)}\ \right)^{10/3} -$$

$$324 \left(-1 + 18 \text{Sin}[\beta]^2 + 3 \sqrt{3}\ \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)}\ \right)^{5/3}$$

$$(-\pi^2 + 50 \pi^2 \text{Sin}[\beta]^2 - 453 \pi^2 \text{Sin}[\beta]^4 + 36 \pi^2 \text{Sin}[\beta]^6 +$$

$$5 \sqrt{3}\ \pi^2 \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)} -$$

$$78 \sqrt{3}\ \pi^2 \text{Sin}[\beta]^2 \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)} +$$

$$9 \sqrt{3}\ \pi^2 \text{Sin}[\beta]^4 \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)} +$$

$$\pi^2 \left(-1 + 18 \text{Sin}[\beta]^2 + 3 \sqrt{3}\ \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)}\ \right)^{1/3} -$$

$$35 \pi^2 \text{Sin}[\beta]^2 \left(-1 + 18 \text{Sin}[\beta]^2 + 3 \sqrt{3}\ \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)}\ \right)^{1/3} +$$

$$243 \pi^2 \text{Sin}[\beta]^4 \left(-1 + 18 \text{Sin}[\beta]^2 + 3 \sqrt{3}\ \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)}\ \right)^{1/3} +$$

$$9 \pi^2 \text{Sin}[\beta]^6 \left(-1 + 18 \text{Sin}[\beta]^2 + 3 \sqrt{3}\ \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)}\ \right)^{1/3} -$$

$$4 \sqrt{3}\ \pi^2 \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)}$$

$$\left(-1 + 18 \text{Sin}[\beta]^2 + 3 \sqrt{3}\ \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)}\ \right)^{1/3} +$$

$$42 \sqrt{3}\ \pi^2 \text{Sin}[\beta]^2 \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)}$$

$$\left(-1 + 18 \text{Sin}[\beta]^2 + 3 \sqrt{3}\ \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)}\ \right)^{1/3} +$$

$$5 \pi^2 \left(-1 + 18 \text{Sin}[\beta]^2 + 3 \sqrt{3}\ \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)}\ \right)^{2/3} -$$

$$88 \pi^2 \text{Sin}[\beta]^2 \left(-1 + 18 \text{Sin}[\beta]^2 + 3 \sqrt{3}\ \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)}\ \right)^{2/3} -$$

$$36 \pi^2 \text{Sin}[\beta]^4 \left(-1 + 18 \text{Sin}[\beta]^2 + 3 \sqrt{3}\ \sqrt{\text{Sin}[\beta]^2 \ (-1 + 11 \text{Sin}[\beta]^2 + \text{Sin}[\beta]^4)}\ \right)^{2/3} -$$

$$15\sqrt{3}\,\pi^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$6\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}\Bigg)\Bigg)^{3}\Bigg)\Bigg)^{1/3}\Bigg)\Bigg/$$

$$\left(54\,2^{1/3}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{5/3}\right)\Big\},\ \Big\{\theta\to$$

$$\frac{4\pi}{3}-$$

$$\Bigg(\Bigg(1-$$

$$i\sqrt{3}\Big)$$

$$\Bigg(-1296\,\pi^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{10/3}-$$

$$324\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{5/3}$$

$$\Big(-\pi^2+50\,\pi^2\,\text{Sin}[\beta]^2-453\,\pi^2\,\text{Sin}[\beta]^4+36\,\pi^2\,\text{Sin}[\beta]^6+$$

$$5\sqrt{3}\,\pi^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$78\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$9\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$\pi^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$35\,\pi^2\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$243\,\pi^2\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$9\,\pi^2\,\text{Sin}[\beta]^6\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$4\sqrt{3}\,\pi^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$42\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$5\,\pi^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$88\,\pi^2\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$36\,\pi^2\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$15\sqrt{3}\,\pi^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$6\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}\Bigg)\Bigg)\Bigg)\Big/$$

$$\Bigg(27\,2^{2/3}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{5/3}$$

$$\Bigg(128\,304\,\pi^3-46\,049\,472\,\pi^3\,\text{Sin}[\beta]^2+3\,084\,719\,760\,\pi^3\,\text{Sin}[\beta]^4-$$

$$76\,984\,149\,600\,\pi^3\,\text{Sin}[\beta]^6+791\,034\,575\,760\,\pi^3\,\text{Sin}[\beta]^8-$$
$$2\,463\,947\,543\,232\,\pi^3\,\text{Sin}[\beta]^{10}-4\,120\,487\,403\,984\,\pi^3\,\text{Sin}[\beta]^{12}-$$
$$430\,339\,664\,160\,\pi^3\,\text{Sin}[\beta]^{14}-9\,183\,300\,480\,\pi^3\,\text{Sin}[\beta]^{16}-$$
$$1\,924\,560\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$
$$240\,395\,040\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$
$$8\,641\,939\,248\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$
$$116\,076\,162\,240\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$
$$464\,847\,899\,760\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$
$$696\,655\,378\,080\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$
$$43\,748\,223\,120\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$
$$306\,110\,016\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{14}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$
$$69\,984\,\pi^3\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$
$$22\,394\,880\,\pi^3\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$
$$1\,412\,976\,960\,\pi^3\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$
$$34\,362\,423\,936\,\pi^3\,\text{Sin}[\beta]^6\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$
$$361\,323\,192\,960\,\pi^3\,\text{Sin}[\beta]^8\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$
$$1\,382\,528\,881\,152\,\pi^3\,\text{Sin}[\beta]^{10}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$1\,362\,520\,661\,152\,\pi^{[illegible]}\,\text{Sin}[\beta]^{[illegible]}\left(-1+18\,\text{Sin}[\beta]^{[illegible]}+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^{[illegible]}\left(-1+11\,\text{Sin}[\beta]^{[illegible]}+\text{Sin}[\beta]^{[illegible]}\right)}\right)^{[illegible]}+$$

$$10\,033\,606\,080\,\pi^3\,\text{Sin}[\beta]^{12}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$11\,224\,033\,920\,\pi^3\,\text{Sin}[\beta]^{14}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$153\,055\,008\,\pi^3\,\text{Sin}[\beta]^{16}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$979\,776\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$112\,534\,272\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$3\,868\,575\,552\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$52\,016\,027\,904\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$241\,951\,624\,128\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$12\,788\,596\,224\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$1\,122\,403\,392\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$69\,984\,\pi^3\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$19\,455\,552\,\pi^3\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$1\,082\,722\,464\,\pi^3\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$23\,329\,866\,240\,\pi^3\,\text{Sin}[\beta]^6\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$217\,196\,393\,760\,\pi^3\,\text{Sin}[\beta]^8\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$725\,922\,896\,832\,\pi^3\,\text{Sin}[\beta]^{10}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$725\,922\,896\,832\,\pi\,\text{Sin}[\beta]\,\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$79\,877\,708\,064\,\pi^3\,\text{Sin}[\beta]^{12}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$1\,632\,586\,752\,\pi^3\,\text{Sin}[\beta]^{14}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$909\,792\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$91\,119\,168\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$2\,737\,984\,032\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$32\,062\,189\,824\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$127\,925\,643\,168\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$8\,060\,897\,088\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$51\,018\,336\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$\sqrt{\Big(\Big(128\,304\,\pi^3-46\,049\,472\,\pi^3\,\text{Sin}[\beta]^2+3\,084\,719\,760\,\pi^3\,\text{Sin}[\beta]^4-76\,984\,149\,600\,\pi^3\,\text{Sin}[\beta]^6+}$$
$$791\,034\,575\,760\,\pi^3\,\text{Sin}[\beta]^8-2\,463\,947\,543\,232\,\pi^3\,\text{Sin}[\beta]^{10}-$$
$$4\,120\,487\,403\,984\,\pi^3\,\text{Sin}[\beta]^{12}-430\,339\,664\,160\,\pi^3\,\text{Sin}[\beta]^{14}-$$
$$9\,183\,300\,480\,\pi^3\,\text{Sin}[\beta]^{16}-1\,924\,560\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$
$$240\,395\,040\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$
$$8\,641\,939\,248\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$
$$116\,076\,162\,240\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$
$$464\,847\,899\,760\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$
$$696\,655\,378\,080\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$696\,655\,378\,080\,\sqrt{3}\,\pi\ \text{Sin}[\beta]\ \sqrt{\text{Sin}[\beta]\ (-1+11\,\text{Sin}[\beta]\ +\text{Sin}[\beta]\ )}\ -$$

$$43\,748\,223\,120\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-306\,110\,016$$

$$\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{14}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+69\,984\,\pi^3\left(-1+18\,\text{Sin}[\beta]^2+\right.$$

$$\left.3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-22\,394\,880\,\pi^3\,\text{Sin}[\beta]^2$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+1\,412\,976\,960$$

$$\pi^3\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$34\,362\,423\,936\,\pi^3\,\text{Sin}[\beta]^6\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\right.$$

$$\left.\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+361\,323\,192\,960\,\pi^3\,\text{Sin}[\beta]^8$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-1\,382\,528\,881\,152$$

$$\pi^3\,\text{Sin}[\beta]^{10}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$10\,033\,606\,080\,\pi^3\,\text{Sin}[\beta]^{12}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\right.$$

$$\left.\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+11\,224\,033\,920\,\pi^3\,\text{Sin}[\beta]^{14}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+153\,055\,008$$

$$\pi^3\,\text{Sin}[\beta]^{16}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$979\,776\,\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$112\,534\,272\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$3\,868\,575\,552\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$52\,016\,027\,904\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$241\,951\,624\,128\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$12\,788\,596\,224\,\sqrt{3}\;\pi^3\,\text{Sin}[\beta]^{10}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$1\,122\,403\,392\,\sqrt{3}\;\pi^3\,\text{Sin}[\beta]^{12}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$69\,984\,\pi^3\,\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+19\,455\,552$$

$$\pi^3\,\text{Sin}[\beta]^2\,\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$1\,082\,722\,464\,\pi^3\,\text{Sin}[\beta]^4\,\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\right.$$

$$\left.\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+23\,329\,866\,240\,\pi^3\,\text{Sin}[\beta]^6$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-217\,196\,393\,760$$

$$\pi^3\,\text{Sin}[\beta]^8\,\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$725\,922\,896\,832\,\pi^3\,\text{Sin}[\beta]^{10}\,\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\right.$$

$$\left.\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+79\,877\,708\,064\,\pi^3\,\text{Sin}[\beta]^{12}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+1\,632\,586\,752$$

$$\pi^3\,\text{Sin}[\beta]^{14}\,\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$909\,792\,\sqrt{3}\;\pi^3\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$91\,119\,168\,\sqrt{3}\;\pi^3\,\text{Sin}[\beta]^2\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$2\,737\,984\,032\,\sqrt{3}\;\pi^3\,\text{Sin}[\beta]^4\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$32\,062\,189\,824\,\sqrt{3}\;\pi^3\,\text{Sin}[\beta]^6\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$127\,925\,643\,168\,\sqrt{3}\;\pi^3\,\text{Sin}[\beta]^8\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$8\,060\,897\,088\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$51\,018\,336\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left.\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}\right)^2+$$

$$4\left(-1296\,\pi^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{10/3}-\right.$$

$$324\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{5/3}$$

$$\left(-\pi^2+50\,\pi^2\,\text{Sin}[\beta]^2-453\,\pi^2\,\text{Sin}[\beta]^4+36\,\pi^2\,\text{Sin}[\beta]^6+\right.$$

$$5\sqrt{3}\,\pi^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$78\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$9\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$\pi^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$35\,\pi^2\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$243\,\pi^2\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$9\,\pi^2\,\text{Sin}[\beta]^6\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$4\sqrt{3}\,\pi^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$42\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$5\,\pi^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$88\,\pi^2\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$36\,\pi^2\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$15\sqrt{3}\,\pi^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$
$$6\sqrt{3}\,\pi^2\,\text{Sin}[\beta]^2\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left.\left.\left.\left.\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}\right)\right)^3\right)\right)^{1/3}\right)+$$

$$\left(\left(1+i\sqrt{3}\right)\left(128\,304\,\pi^3-46\,049\,472\,\pi^3\,\text{Sin}[\beta]^2+3\,084\,719\,760\,\pi^3\,\text{Sin}[\beta]^4-76\,984\,149\,600\right.\right.$$
$$\pi^3\,\text{Sin}[\beta]^6+791\,034\,575\,760\,\pi^3\,\text{Sin}[\beta]^8-$$
$$2\,463\,947\,543\,232\,\pi^3\,\text{Sin}[\beta]^{10}-4\,120\,487\,403\,984\,\pi^3\,\text{Sin}[\beta]^{12}-$$
$$430\,339\,664\,160\,\pi^3\,\text{Sin}[\beta]^{14}-9\,183\,300\,480\,\pi^3\,\text{Sin}[\beta]^{16}-$$
$$1\,924\,560\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$
$$240\,395\,040\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$
$$8\,641\,939\,248\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$
$$116\,076\,162\,240\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$
$$464\,847\,899\,760\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$
$$696\,655\,378\,080\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$
$$43\,748\,223\,120\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$
$$306\,110\,016\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{14}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$
$$69\,984\,\pi^3\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$
$$22\,394\,880\,\pi^3\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$
$$1\,412\,976\,960\,\pi^3\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$
$$34\,362\,423\,936\,\pi^3\,\text{Sin}[\beta]^6\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$
$$361\,323\,192\,960\,\pi^3\,\text{Sin}[\beta]^8\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$
$$1\,382\,528\,881\,152\,\pi^3\,\text{Sin}[\beta]^{10}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$
$$10\,033\,606\,080\,\pi^3\,\text{Sin}[\beta]^{12}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$
$$11\,224\,033\,920\,\pi^3\,\text{Sin}[\beta]^{14}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$
$$153\,055\,008\,\pi^3\,\text{Sin}[\beta]^{16}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$979\,776\,\sqrt{3}\,\pi^3\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$
$$112\,534\,272\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$
$$3\,868\,575\,552\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$
$$52\,016\,027\,904\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$
$$241\,951\,624\,128\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$
$$12\,788\,596\,224\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$
$$1\,122\,403\,392\,\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$
$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$
$$69\,984\,\pi^3\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$
$$19\,455\,552\,\pi^3\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$
$$1\,082\,722\,464\,\pi^3\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$
$$23\,329\,866\,240\,\pi^3\,\text{Sin}[\beta]^6\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$
$$217\,196\,393\,760\,\pi^3\,\text{Sin}[\beta]^8\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$
$$725\,922\,896\,832\,\pi^3\,\text{Sin}[\beta]^{10}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$
$$79\,877\,708\,064\,\pi^3\,\text{Sin}[\beta]^{12}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$
$$1\,632\,586\,752\,\pi^3\,\text{Sin}[\beta]^{14}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$
$$909\,792\,\sqrt{3}\,\pi^3\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$91\,119\,168\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$2\,737\,984\,032\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$32\,062\,189\,824\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$127\,925\,643\,168\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$8\,060\,897\,088\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$51\,018\,336\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$\sqrt{\Big(\Big(\Big(128\,304\,\pi^3-46\,049\,472\,\pi^3\,\text{Sin}[\beta]^2+3\,084\,719\,760\,\pi^3\,\text{Sin}[\beta]^4-76\,984\,149\,600\,\pi^3\,\text{Sin}[\beta]^6+}$$

$$791\,034\,575\,760\,\pi^3\,\text{Sin}[\beta]^8-2\,463\,947\,543\,232\,\pi^3\,\text{Sin}[\beta]^{10}-$$

$$4\,120\,487\,403\,984\,\pi^3\,\text{Sin}[\beta]^{12}-430\,339\,664\,160\,\pi^3\,\text{Sin}[\beta]^{14}-$$

$$9\,183\,300\,480\,\pi^3\,\text{Sin}[\beta]^{16}-1\,924\,560\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$240\,395\,040\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$8\,641\,939\,248\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$116\,076\,162\,240\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$464\,847\,899\,760\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$696\,655\,378\,080\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$43\,748\,223\,120\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-306\,110\,016$$

$$\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{14}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+69\,984\,\pi^3\,\Big(-1+18\,\text{Sin}[\beta]^2+$$

$$3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\Big)^{1/3}-22\,394\,880\,\pi^3\,\text{Sin}[\beta]^2$$

$$\Big)^{1/3}$$

$$\left(-1 + 18\,\text{Sin}[\beta]^2 + 3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)}\right)^{1/3} + 1\,412\,976\,960$$

$$\pi^3\,\text{Sin}[\beta]^4\left(-1 + 18\,\text{Sin}[\beta]^2 + 3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)}\right)^{1/3} -$$

$$34\,362\,423\,936\,\pi^3\,\text{Sin}[\beta]^6\,(-1 + 18\,\text{Sin}[\beta]^2 + 3\sqrt{3}$$

$$\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)})^{1/3} + 361\,323\,192\,960\,\pi^3\,\text{Sin}[\beta]^8$$

$$\left(-1 + 18\,\text{Sin}[\beta]^2 + 3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)}\right)^{1/3} - 1\,382\,528\,881\,152$$

$$\pi^3\,\text{Sin}[\beta]^{10}\left(-1 + 18\,\text{Sin}[\beta]^2 + 3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)}\right)^{1/3} +$$

$$10\,033\,606\,080\,\pi^3\,\text{Sin}[\beta]^{12}\,(-1 + 18\,\text{Sin}[\beta]^2 + 3\sqrt{3}$$

$$\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)})^{1/3} + 11\,224\,033\,920\,\pi^3\,\text{Sin}[\beta]^{14}$$

$$\left(-1 + 18\,\text{Sin}[\beta]^2 + 3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)}\right)^{1/3} + 153\,055\,008$$

$$\pi^3\,\text{Sin}[\beta]^{16}\left(-1 + 18\,\text{Sin}[\beta]^2 + 3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)}\right)^{1/3} -$$

$$979\,776\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)}$$

$$\left(-1 + 18\,\text{Sin}[\beta]^2 + 3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)}\right)^{1/3} +$$

$$112\,534\,272\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)}$$

$$\left(-1 + 18\,\text{Sin}[\beta]^2 + 3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)}\right)^{1/3} -$$

$$3\,868\,575\,552\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)}$$

$$\left(-1 + 18\,\text{Sin}[\beta]^2 + 3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)}\right)^{1/3} +$$

$$52\,016\,027\,904\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)}$$

$$\left(-1 + 18\,\text{Sin}[\beta]^2 + 3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)}\right)^{1/3} -$$

$$241\,951\,624\,128\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)}$$

$$\left(-1 + 18\,\text{Sin}[\beta]^2 + 3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)}\right)^{1/3} +$$

$$12\,788\,596\,224\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)}$$

$$\left(-1 + 18\,\text{Sin}[\beta]^2 + 3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)}\right)^{1/3} +$$

$$1\,122\,403\,392\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{12}\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)}$$

$$\left(-1 + 18\,\text{Sin}[\beta]^2 + 3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1 + 11\,\text{Sin}[\beta]^2 + \text{Sin}[\beta]^4\right)}\right)^{1/3}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right) -$$

$$69\,984\,\pi^3\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+19\,455\,552$$

$$\pi^3\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$1\,082\,722\,464\,\pi^3\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\right.$$

$$\left.\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+23\,329\,866\,240\,\pi^3\,\text{Sin}[\beta]^6$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-217\,196\,393\,760$$

$$\pi^3\,\text{Sin}[\beta]^8\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$725\,922\,896\,832\,\pi^3\,\text{Sin}[\beta]^{10}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\right.$$

$$\left.\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+79\,877\,708\,064\,\pi^3\,\text{Sin}[\beta]^{12}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+1\,632\,586\,752$$

$$\pi^3\,\text{Sin}[\beta]^{14}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$909\,792\sqrt{3}\,\pi^3\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$91\,119\,168\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^2\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$2\,737\,984\,032\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^4\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$32\,062\,189\,824\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^6\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$127\,925\,643\,168\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^8\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$8\,060\,897\,088\sqrt{3}\,\pi^3\,\text{Sin}[\beta]^{10}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\,\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}+$$

$$51\,018\,336\,\sqrt{3}\;\pi^3\,\text{Sin}[\beta]^{12}\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}\Bigg)^2+$$

$$4\,\Bigg(-1296\,\pi^2\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{10/3}-$$

$$324\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{5/3}$$

$$\Bigg(-\pi^2+50\,\pi^2\,\text{Sin}[\beta]^2-453\,\pi^2\,\text{Sin}[\beta]^4+36\,\pi^2\,\text{Sin}[\beta]^6+$$

$$5\,\sqrt{3}\;\pi^2\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}-$$

$$78\,\sqrt{3}\;\pi^2\,\text{Sin}[\beta]^2\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$9\,\sqrt{3}\;\pi^2\,\text{Sin}[\beta]^4\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}+$$

$$\pi^2\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$35\,\pi^2\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$243\,\pi^2\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$9\,\pi^2\,\text{Sin}[\beta]^6\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}-$$

$$4\,\sqrt{3}\;\pi^2\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$42\,\sqrt{3}\;\pi^2\,\text{Sin}[\beta]^2\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

$$\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{1/3}+$$

$$5\,\pi^2\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$88\,\pi^2\,\text{Sin}[\beta]^2\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

$$36\,\pi^2\,\text{Sin}[\beta]^4\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

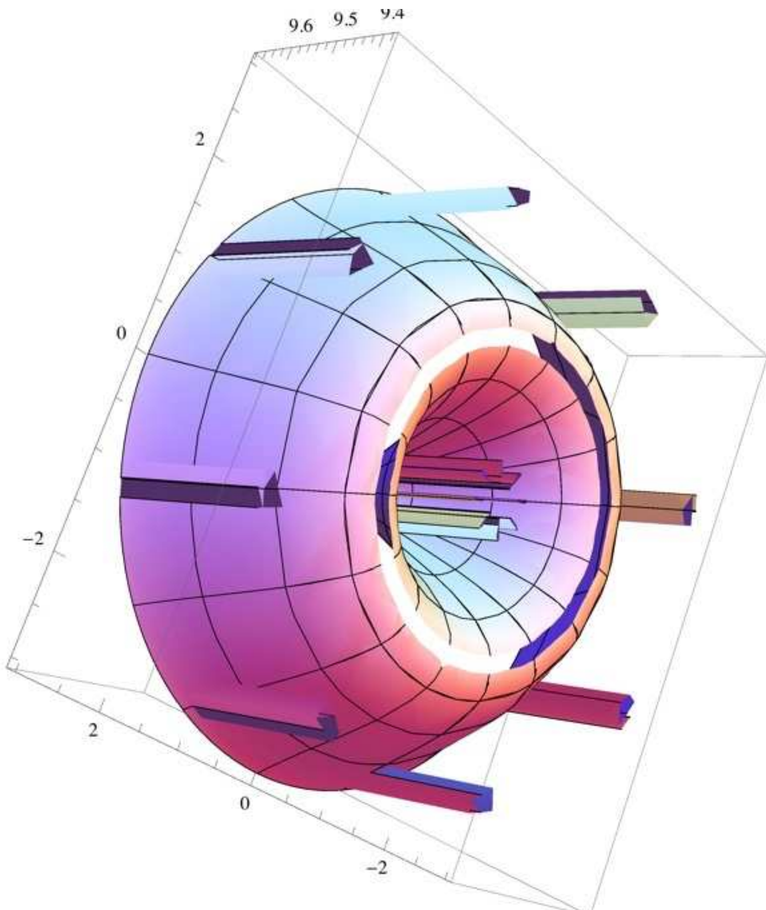
$$15\,\sqrt{3}\;\pi^2\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

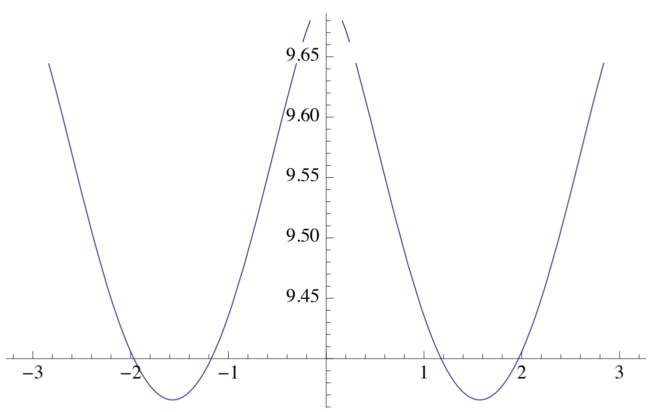
$$\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}-$$

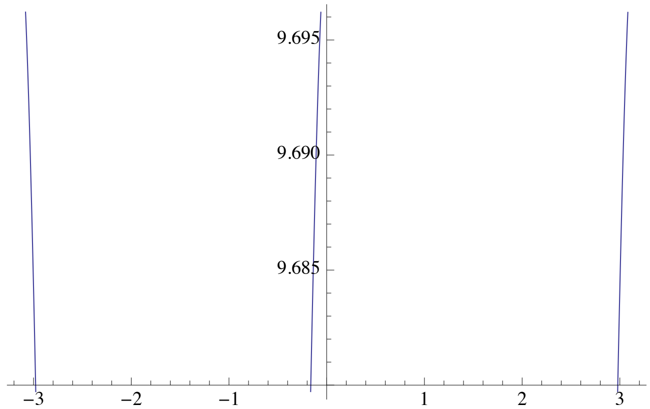
$$6\,\sqrt{3}\;\pi^2\,\text{Sin}[\beta]^2\,\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}$$

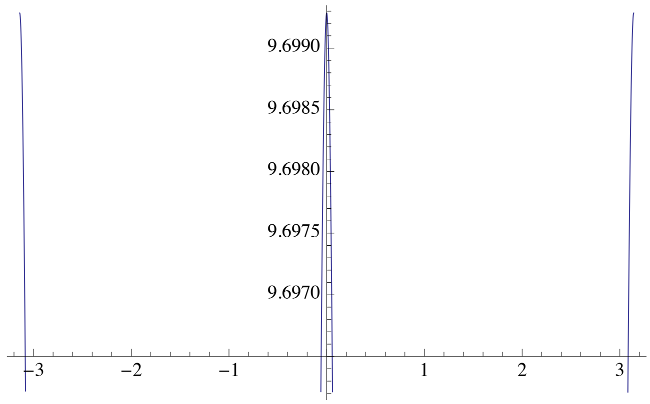
$$\left.\left.\left.\left.\left(-1+18\,\text{Sin}[\beta]^2+3\,\sqrt{3}\;\sqrt{\text{Sin}[\beta]^2\,\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{2/3}\right)\right)^3\right)\right)^{1/3}\Bigg)\Big/$$

$$\left(54\ 2^{1/3}\left(-1+18\,\text{Sin}[\beta]^2+3\sqrt{3}\sqrt{\text{Sin}[\beta]^2\left(-1+11\,\text{Sin}[\beta]^2+\text{Sin}[\beta]^4\right)}\right)^{5/3}\right)\}\}$$


9.6
9.5
9.4
2
0
−2
2
0
−2


9.65
9.60
9.55
9.50
9.45
−3
−2
−1
1
2
3

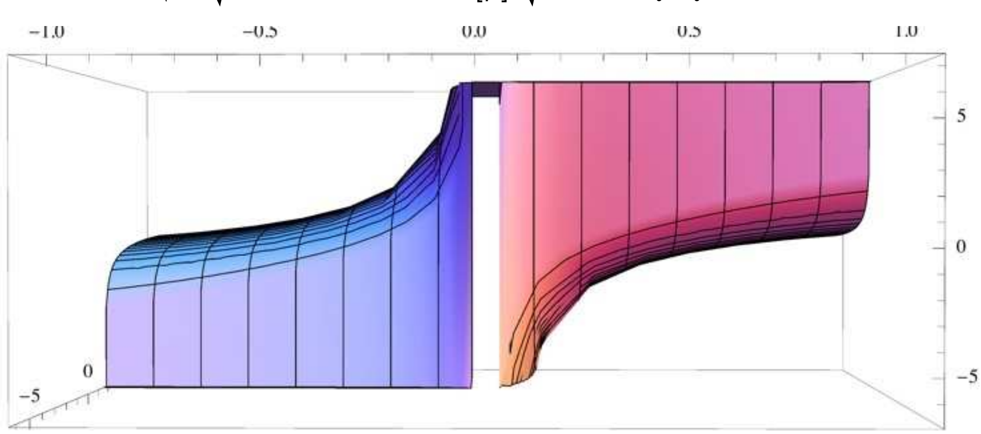


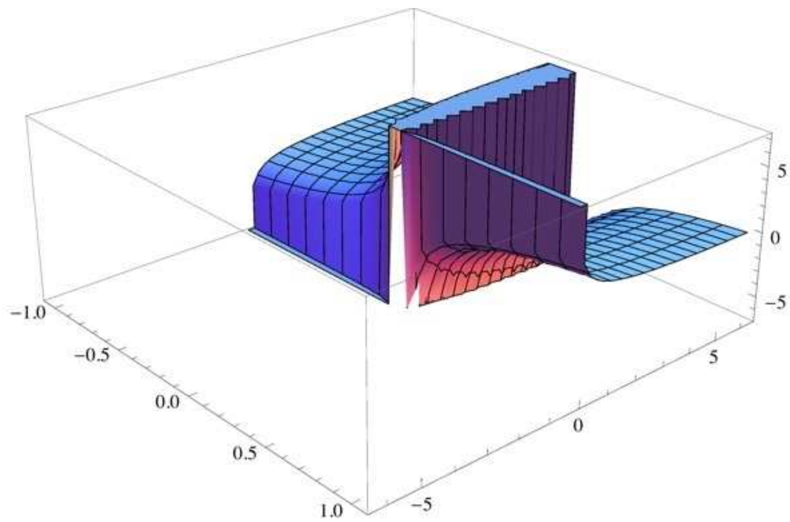


XXII. Introspections into Complex Infinity from The Revelation of the Infinite Angle

$$1 = \frac{\sqrt{\gamma}\ \sqrt{4\,\pi + \gamma}}{\text{Sin}[\beta]\ \sqrt{4\,\pi^2 + 4\,\pi\,\gamma + \gamma^2}} = \frac{\sqrt{4\,\pi^2 + 4\,\pi\,\gamma + \gamma^2}\ \text{Sin}[\beta]}{\sqrt{\gamma}\ \sqrt{4\,\pi + \gamma}}$$

$$\text{Plot3D}\left[\frac{\sqrt{4\,\pi^2 + 4\,\pi\,\gamma + \gamma^2}\ \text{Sin}[\beta]}{\sqrt{\gamma}\ \sqrt{4\,\pi + \gamma}} == \frac{\sqrt{\gamma}\ \sqrt{4\,\pi + \gamma}}{\text{Sin}[\beta]\ \sqrt{4\,\pi^2 + 4\,\pi\,\gamma + \gamma^2}},\ \{\beta,\ -\pi / 3,\ \pi / 3\},\ \{\gamma,\ -2\,\pi,\ 2\,\pi\}\right]$$



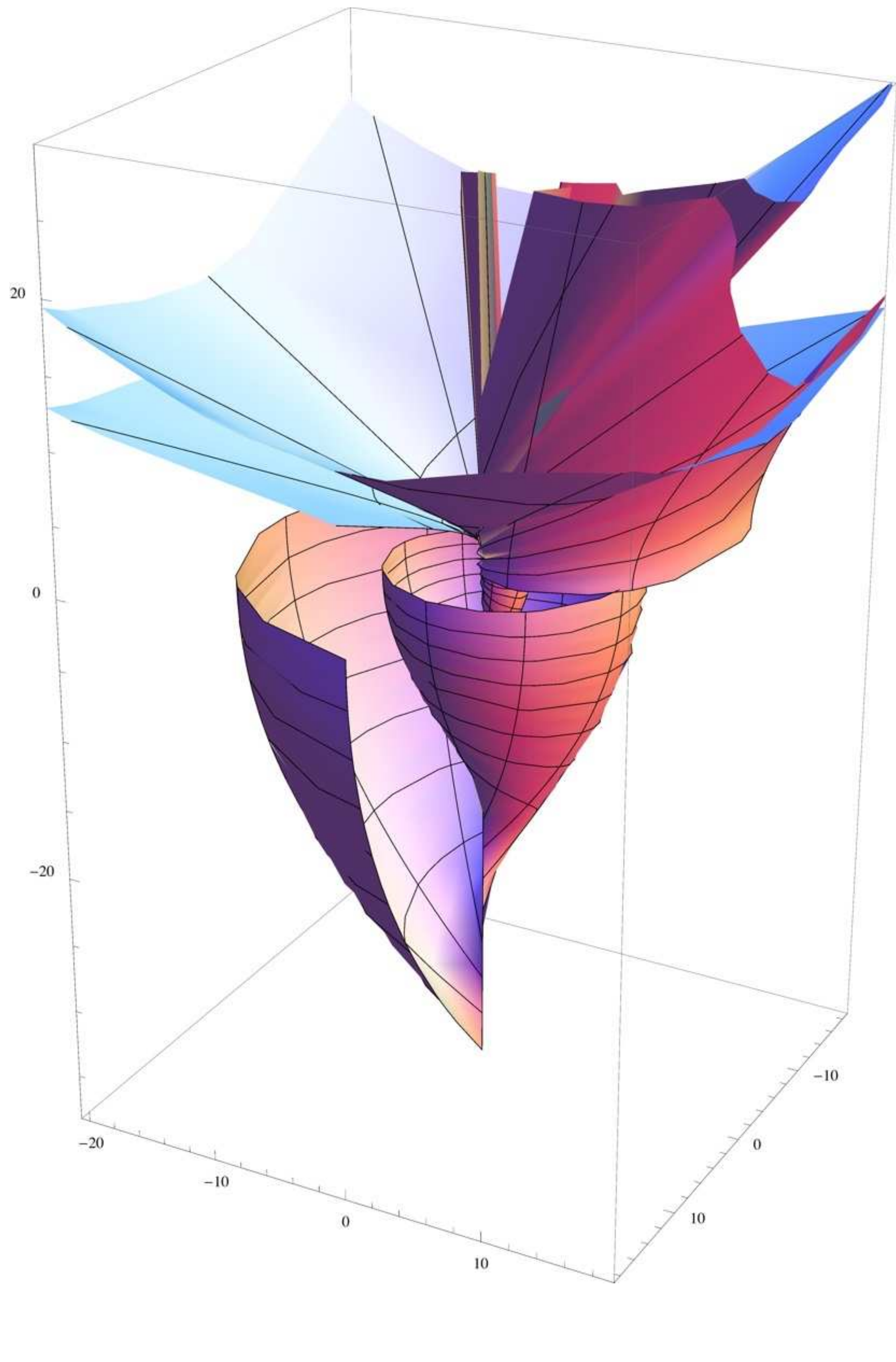


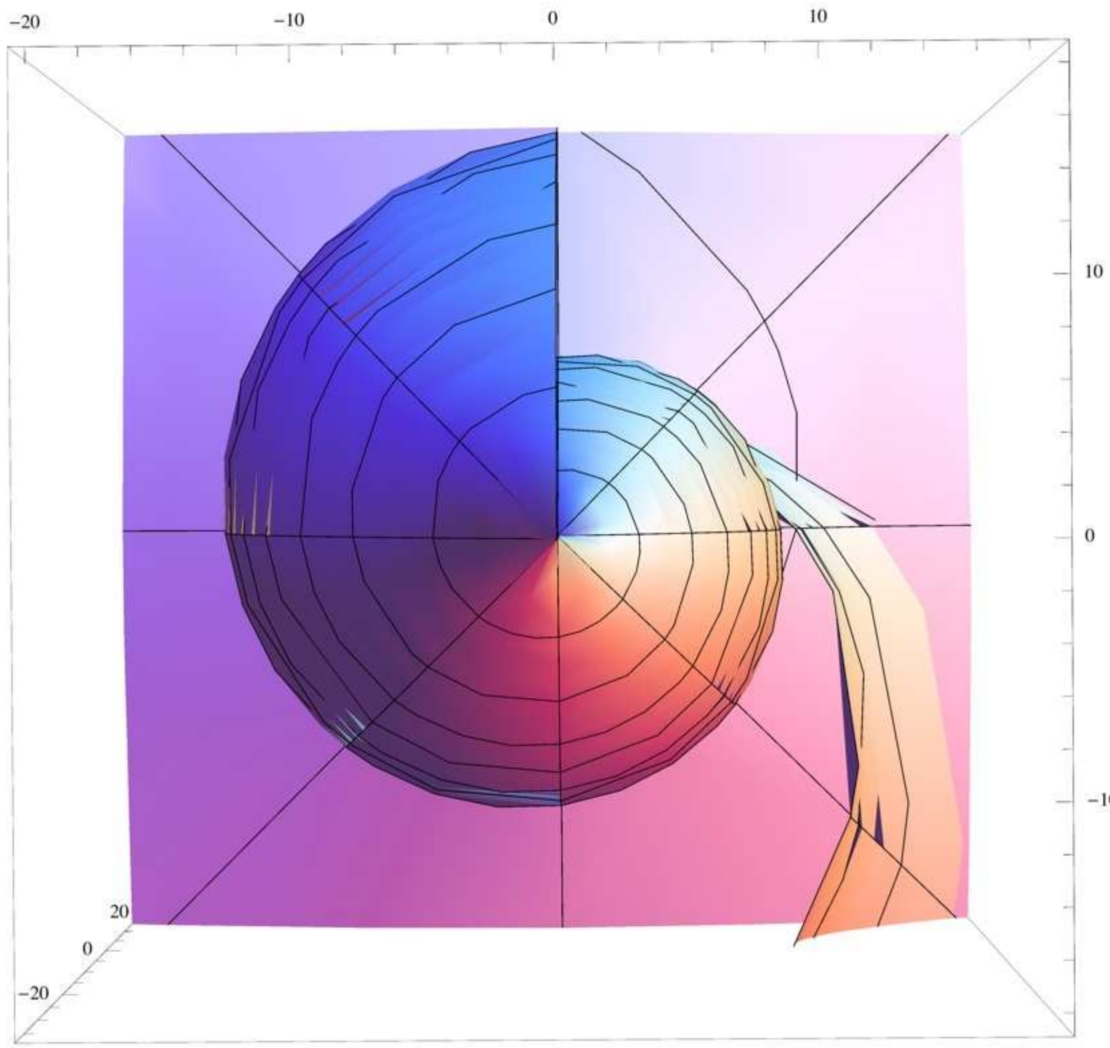
$$1 = \frac{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\text{Sin}[\beta]\ \sqrt{4\pi^2+4\pi\gamma+\gamma^2}},\ 0 = 2\pi\sqrt{\left(\frac{x\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{2\pi}\right)^2+x^2} - 2\pi x - \gamma x$$

$$x := \frac{2\pi r}{2\pi+\gamma}$$

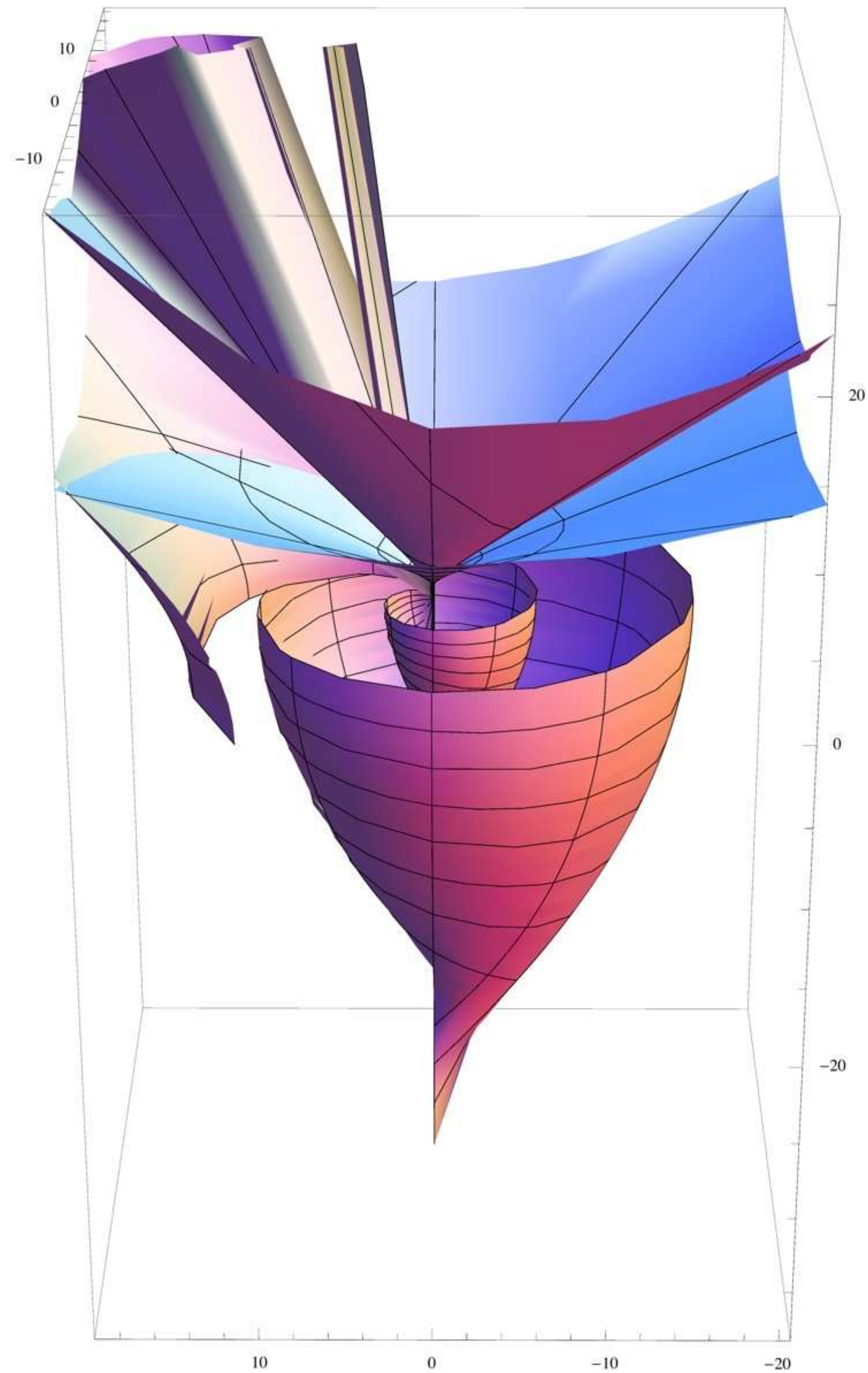
$$r := \left(-4\pi^2 - 4\pi\gamma - \gamma^2 + \sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}\ \text{Csc}[\beta] + 4\pi\gamma\,\text{Csc}[\beta]^2 + \gamma^2\,\text{Csc}[\beta]^2 - \sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}\ \text{Csc}[\beta]^3 + \frac{4\pi^2\sqrt{\gamma}\ \sqrt{(2\pi+\gamma)^2}\ \sqrt{4\pi+\gamma}\ \text{Csc}[\beta]^3}{(2\pi+\gamma)^2}\right) \Big/ \left(\pi\left(-16\pi^2 - 16\pi\gamma - 4\gamma^2 + 16\pi\gamma\,\text{Csc}[\beta]^2 + 4\gamma^2\,\text{Csc}[\beta]^2\right)\right)$$

$$\text{SphericalPlot3D}\left[\frac{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\text{Sin}[\beta]\ \sqrt{4\pi^2+4\pi\gamma+\gamma^2}} \Big/ \left(2\pi\sqrt{\left(r^2+x^2\right)} - 2\pi x - \gamma x\right),\ \{\beta, -\pi/2, \pi/2\}, \{\gamma, -4\pi, 4\pi\}\right]$$


20
0
−20
−20
−10
0
10
−10
0
10

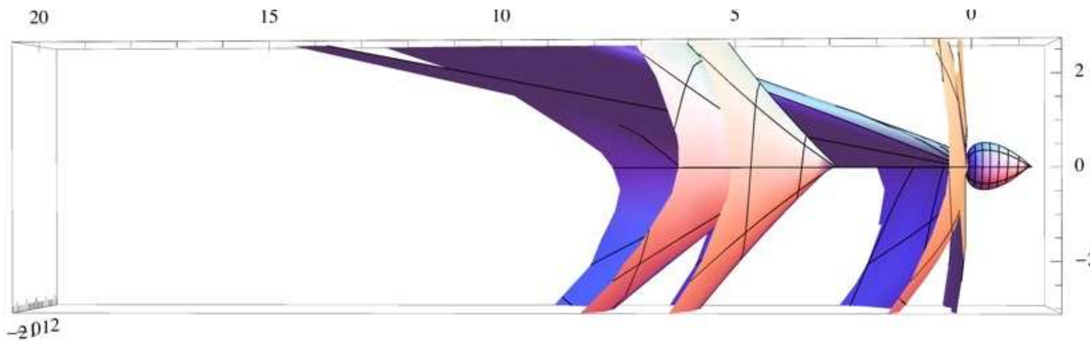


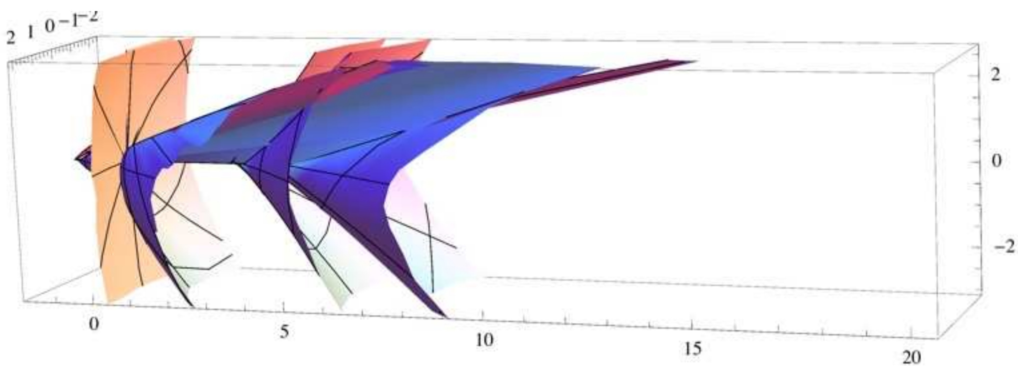
-20
-10
0
10
10
0
-1(
20
0
-20


10
0
−10
20
0
−20
10
0
−10
−20

$$\texttt{SphericalPlot3D}\Big[\frac{\sqrt{\gamma}\ \sqrt{4\,\pi+\gamma}}{\texttt{Sin}[\beta]\ \sqrt{4\,\pi^2+4\,\pi\,\gamma+\gamma^2}}\Big/$$

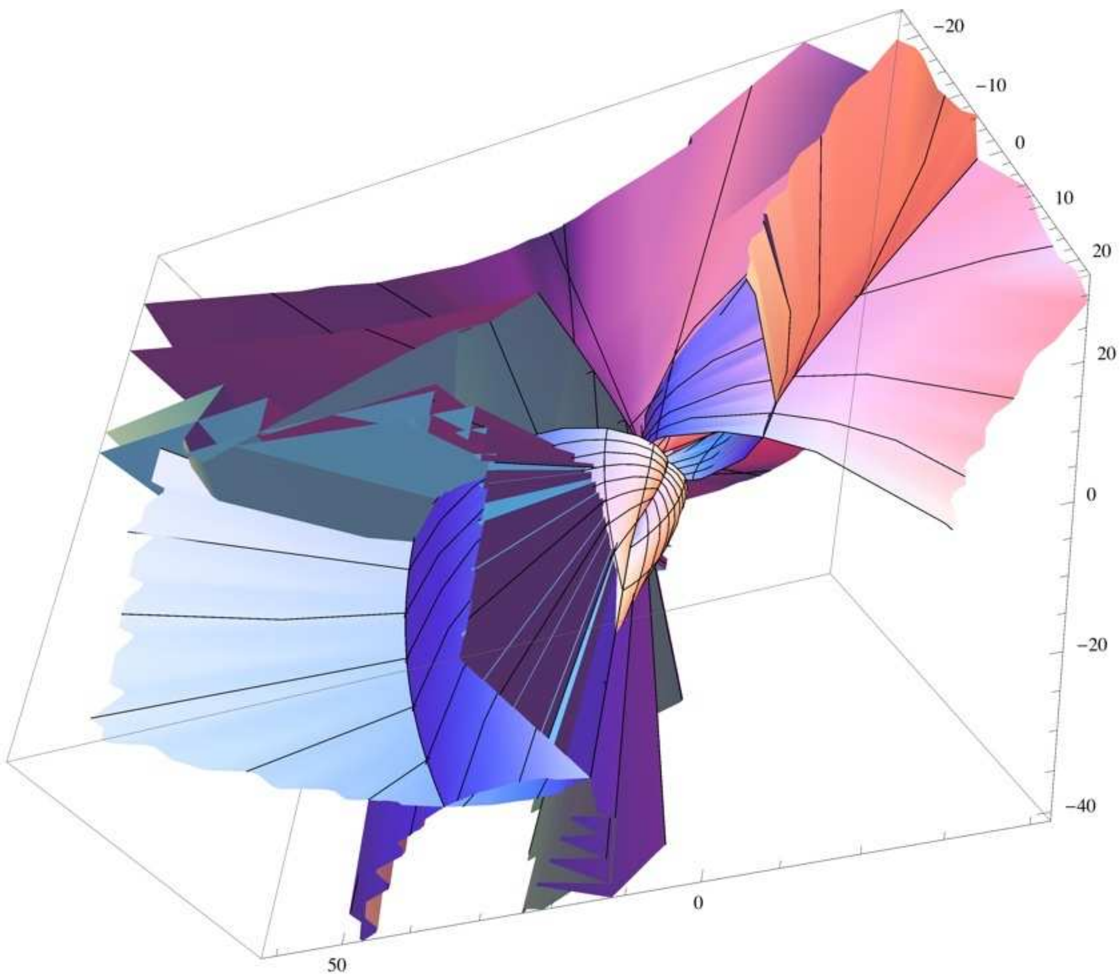
$$\left(2\,\pi\,\sqrt{\left(r^2+x^2\right)}-2\,\pi\,x-\frac{2\left(\pi-\pi\,\texttt{Sin}[\beta]^2+\sqrt{\pi^2-\pi^2\,\texttt{Sin}[\beta]^2}\right)}{-1+\texttt{Sin}[\beta]^2}\,x\right),\ \{\beta,\ -\pi/2,\ \pi/2\},\ \{\gamma,\ -4\,\pi,\ 4\,\pi\}\Big]$$





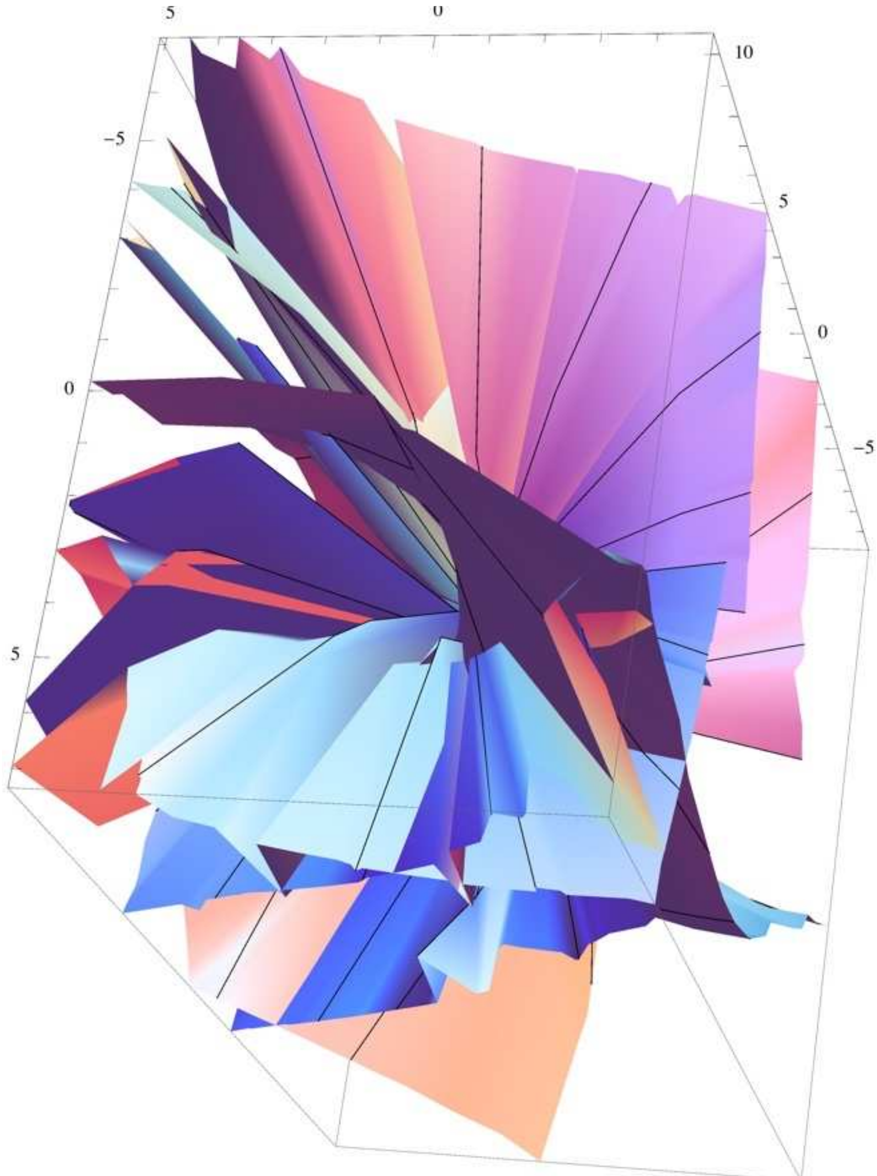
$$\gamma := -\frac{2\,\pi\,(4\,\pi-\theta)}{2\,\pi-\theta}$$

$$\text{SphericalPlot3D}\left[\frac{\sqrt{\gamma}\ \sqrt{4\,\pi+\gamma}}{\text{Sin}[\beta]\ \sqrt{4\,\pi^2+4\,\pi\,\gamma+\gamma^2}}\Big/\left(2\,\pi\,\sqrt{\left(r^2+x^2\right)}-2\,\pi\,x-\gamma\,x\right),\ \{\beta,-\pi/2,\pi/2\},\{\theta,-4\,\pi,4\,\pi\}\right]$$



$$\beta := \text{ArcSin}\left[\frac{\sqrt{(4\,\pi-\theta)\,\theta}}{2\,\pi}\right]$$

$$\text{SphericalPlot3D}\left[\frac{\sqrt{\gamma}\ \sqrt{4\pi+\gamma}}{\text{Sin}[\beta]\ \sqrt{4\pi^2+4\pi\gamma+\gamma^2}} \bigg/ \left(2\pi\sqrt{\left(r^2+x^2\right)}-2\pi x-\gamma x\right),\ \{\gamma,\ -4\pi,\ 4\pi\},\ \{\theta,\ -\pi,\ \pi\}\right]$$



Parker Emmerson

Mathematical Analysis in Ecological Optics: Mathematical, Theoretical Applications to The Work of James J. Gibson

I. Introduction

Ecological optics is the theory of visual perception introduced by James J. Gibson in his 1966 work, The Senses Considered as Perceptual Systems[1] and developed the theory further in his 1979 work, The Ecological Approach to Visual Perception[2]. Gibson's perceptual theory developed over the years, 1947-1979 that he was doing academic research, but I plan to return to his earlier ideas, because I feel like there are themes within his 1950 work, The Perception of the Visual World[3] that are still valid approaches to problems of perception and perceptual phenomena. Use of eidetic phenomenology (a discussion of which is present in the paper, *Overview of Perceptual Theories* (Emmerson, 2010)) helps understand the ideas of surface layout and gradient, because it can be used to deliver actual surfaces with layout, texture, mathematical and physical significance, and contour. Use of

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[1] Gibson, James J.. *The Senses Considered as Perceptual Systems*. London: George Allen & Unwin Ltd, 1966. Print. All further references to this source will be cited parenthetically in the text.

[2] Gibson, James J.. *The Ecological Approach To Visual Perception*. 1 ed. Hillside, NJ: Tf-Lea, 1986. Print. All further references to this source will be cited parenthetically in the text.

[3] Gibson, James, J.. *The Perception of the Visual World*. Cambridge, Mass.: The Riverside Press, 1950. Print. All further references to this source will be cited parenthetically in the text.

eidetic phenomenology helps understand the implications of formal ontology of Gestalt theory and interpretation of Pragnanz, or pithiness, which tells us that the visual system organizes information in a simple, systematic, and symmetric manner[4]. The parameters of the formally ontological system contain many gradients that describe a distance from the eye to a point out in the world. Gibson says,

> "the problem of how distance can be perceived is very old. If it is taken to be the distance of an object in space, then it is 'a line endwise to the eye,' as Bishop Berkley pointed out in 1709, and it projects only one point on the retina. Hence, distance *of itself* is invisible and, if so, a whole set of perplexities arise that have never been resolved" (Ecological Approach, 117).

The formally ontological system proposed in *The Geometric Pattern of Perception Theorems* (Emmerson, 2009), is proposed as a way of resolving some of these perplexities, and is a whole system. I will be using Gibson's work, <u>The Ecological Approach to Visual Perception</u> in order to describe the ideas of ambient optic array, surface layout, optical information, and the role of geometry in perceptual studies.

This paper will use the gradient of the mathematics of difference in circumferences equaling an arc length transformed through the Pythagorean theorem to form a cone to describe surface layouts as examples of how mathematics can be a useful language for *describing* perceived layout; whereas, the kind of information that single shot photographs contain is not useful. This is not to say anything against photography as art, because often, the photograph makes us feel

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[4] Koffka, K.. *Principles of Gestalt Psychology*. First Edition ed. London: Kegan Paul, Trench, Trubner, 1935. Print.

present to the artist's vision, personality or message. The *Geometric Pattern of Perception Theorems* (Emmerson, 2009) exhibit the qualities of Pragnanz, because they contain symmetry, and are formally, ontologically based through mathematical, simple, description of the phenomenon of perceived change. The Gestalt theory will be connected to the theory of Ecological Optics, because a, "univocal" (it speaks with one voice through geometric difference) system, with the qualities of Pragnanz, describes the *form* of the ambient optic array through transcendental logic involving computation. I use the term transcendental logic to correlate to ideality as an extension of Husserl's idea, "*transcendence belonging to the real, as such, is a particular form of 'ideality'* or, better, of a *psychic irreality:* the irreality of something that *itself*, with all that belongs to it in its own essence, actually or possibly *makes its appearance* in the purely phenomenological sphere of consciousness." [5] I will draw several *parallels* from ideas in Gibson's work to my own work.

The results of visualizing solutions to the equations can be discussed through the *language* of visual gradients, surface layouts, textures, and contours, which Gibson introduced in philosophically organized conversation in his 1950 work, The Perception of the Visual World. Gibson's theory evolved dramatically from his 1950 book, because in that book, his explanation of vision was, "based on the retinal image, whereas it is now (in his 1978 work), based on what (he calls) the ambient optic array" (Ecological Approach, 1). Thus, I will begin by introducing the meaning of ecological optics and the ambient optic array addressed in his 1966 work,

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[5] Husserl, Edmund, and Donn Welton. *The Essential Husserl: Basic Writings in Transcendental Phenomenology (Studies in Continental Thought)*. Bloomington: Indiana University Press, 1999. Print. p 267.

commenting upon his treatment of Gestalt theory, returning to his 1950 work in order to elucidate how mathematics of the difference in circumferences of two circles applied to the Pythagorean theorem can be useful for discussions of visual contours and their descriptions.

Formal ontology of *perceived difference in two circumferences as an arc length* delivers gradients that describe different kinds of perceived surface layouts, like *hollow objects*, *partial enclosures*, *places*, *sheets*, *fissures*, and *sticks* as well as scenarios such as occluded surfaces, where a surface is blocked from view. First, a background on Gibson's laws of ecological surfaces will be developed. I will then discuss stimulus information, gradients, texture, optic information, and the ambient optic array, correlating the form of the ambient optic array to computational results found within the structure of perceived difference in circumferences.

II. The Ecological Approach

2.1) Ecological Optics – Surface Layout, Substance, and Gradient

Ecological optics is a theory of visual perception that takes account of the environmental context of the perceiver. Gibson introduces ecological optics with the concept of optical information, "information (the kind that is sent and received), consists of messages, signs, and signals" (Ecological Approach, 62). Ecological optics tries to answer the question of the kind of medium of transmission by observing the environment and using the descriptions made and differences noted to form scientific theories of how vision and visual systems work, refining the language used for discussing the medium of perception and substance. In essence, there is a linguistic analog of the meaning of information in human communication, "pictures and sculptures are apt to be displayed, and thus they *contain* information and make it available for anyone who looks" (Ecological Approach, 63). However, the ecological approach is also, "concerned with many-times-reflected light in the medium, that is, *illumination*," (Ecological Approach, 63). Gibson outlined several laws of ecological surfaces and their perception. He also began relating *surface-gradient* to the quality of a perceived surface through texture.

Gibson developed a pertinent and valuable language for discussing ecological surfaces. The nine laws surface layout that Gibson gave in The Ecological Approach to Visual Perception are:

"1. All persisting substances have surfaces, and all surfaces have layout.

2. Any surface has resistance to deformation, depending on the *viscosity* of the substance.
3. Any surface has resistance to disintegration, depending on the *cohesion* of the substance.
4. Any surface has a characteristic texture, depending on the *composition* of the substance. It generally has both a layout texture and a pigment texture.
5. Any surface has a characteristic shape, or large-scale layout.
6. A surface may be strongly or weakly illuminated, in light or in shade.

7. An illuminated surface may absorb either much or little of the illumination falling on it.
8. A surface has a characteristic reflectance, depending on the substance.
9. A surface has a characteristic distribution of the reflectance ratios of the different wavelengths of light, depending on the substance. This property is what I will call its color, in the sense that different distributions constitute different colors" (Ecological Approach, 23-23)

Gibson elaborates on each of the meanings of these laws. Gibson's laws are explanations for perceptions of surfaces, and are thus scientifically directed. The combination of the first and second law, the persistence of surface, and the surfaces' resistance to deformation, explains why the terrestrial surface provides support to animals walking on it – it is made of persisting substance that is not easily deformed upon application of force to it. The persistence of the substance necessitates a persistence of the surface layout of that substance, so it explains how layout like walls are, "barriers of locomotion" (Ecological Approach, 24). The second law tells us that the solidity of a surface can vary and, "implies that the bog or swamp offers practically no support for standing or walking to heavy animals, and that the pond or lake offers no support" (Ecological Approach, 24), and assumes that the pond is *viscous* to the being walking on its surface. The second law implies that, "the surfaces of flexible substances are yielding or can be pushed aside, whereas the surfaces of rigid substances cannot" (Ecological Approach, 25). For instance, I can push my finger into a surface made of putty, while, by the natural order, I generally cannot push my finger into that same surface made of wood.

Gibson's nine laws of surface layout mainly discuss the characteristics of the surfaces in our natural world (in the sense of the woods, the plains, the mountains and sky), but are minimalist and general enough to extend their range to surfaces that are mathematically accessible in our modern day environment, which has changed due to technology and computer interfaces. This has provided us with many surfaces that can be mathematically discovered, each of which can be characterized by combinations of the laws of surface layout provided by Gibson.

I assert that, even mathematically "synthesized" surfaces, not seen in the non-man constructed world are still characterized by texture and contour. Gibson's laws (in combination) hold even for describing a surface, which is itself (the surface) illuminating, like a computer monitor, not just illuminated. Gibson says that, "animals need to perceive what persists and what changes" (Ecological Approach, 307), and that, "a surface *goes out of existence* when its substance evaporates or disintegrates; a surface *come into existence* when its substance condenses or crystallizes" (Ecological Approach, 307). Some day, the monitor will burn out. We also see this characteristic in the surface of water as it evaporates or turns to ice. To the touch, one might be able to say that the surface of standing water is more like the surface of ice than the surface of standing water is like the surface of the evaporated water. Each of these molecular states of the substance has a different characteristic texture to both the eye and hand.

Gibson's primary focus in outlining the laws of ecological surfaces is to provide the nomenclature for visual perception of surface layout. The fourth law addresses characteristic *texture*. Gibson describes texture as, "the structure of a surface, as distinguished from the structure of the substance underlying the surface" (Ecological Approach, 25). Gibson is talking about actual, objective surfaces. The substance underlying a surface is made of molecules, and the organization of these molecules makes a surface, "the surface of a natural substance is neither homogeneous nor amorphous, but has both a chemical and a physical *texture*; it is generally both conglomerated and corrugated" (Ecological Approach, 25). The kind of experiential texture can also describe the character of the interaction in which a change in surface layouts is perceived such that the subject experiences the environment to have differentiated qualia. Often, visually perceived surface texture is directly correlated to specific tactile textures in the natural world, and there is knowledge of the surface of a substance that is different in the tactile texture of the surface of the substance than its visual texture.

There are incremental units of surfaces. Gibson says that, when referring to ecological surface texture, "we are talking about the relatively fine structure of the environment at the size-level around centimeters and millimeters" (Ecological Approach, 25). Gibson says that textural units are nested with larger units. For instance, pebbles, crystals, sand and grass are all part of the textured Earth. The density of the texture can be measured if the number of distinct units within a given area can be counted. The relationship between surface and texture is that textural units are made of different kinds of surfaces. On this, Gibson says that, "the units of texture vary in form, and there are forms within forms, so that the 'form' of a texture escapes measurement" (Ecological Approach, 28). We can see that certain kinds of surfaces are structurally described by the mathematics in *The Geometric Pattern of Perception Theorems* (Emmerson, 2009). Could this fact – that the form of texture continually escapes measurement - be related to the visually "textural" flickering seen in *The Geometric Pattern of Perception Theorems* (Emmerson, 2009)? Gibson says that, "the surfaces of the substances from which primitive men fashioned tools have different textures – flint, clay, wood, bone, and fiber" (Ecological Approach, 28). Scientific research in computational 3D architecture is just getting started, and texture, while unable to be measured, may be able to be increasingly better mathematically expressed.

Contoured surfaces, nets, grids and gradients are delivered through mathematical analysis of structures, and once more, "it is important to realize that smaller units are nested within larger units" (Ecological Approach, 12). Mathematically delivered textural surfaces can theoretically be divided, and nested within each other, as well as having their algebraic correlates manipulated through the "nesting" of and substituting for variables within algebraic expressions, which deliver completely different "textures." So, we see there is an analogy or metaphor within mathematical (geometric) structural analysis and ecological optics.

Also, a change in surface layout is what Gibson described as motion in surface geometry. Mathematics can specify certain kinds of change in surface information in the event of perceived motion through a change in surface layout[6]. In mathematical analysis, the surface of a function is what is described by the graphing of a function. The delivered surface is then imbued with attributes such as hardness and opacity, which tell us the characteristics of the synthesized “substance” of the object. Gibson addressed the issue of stimulus in psychology in a 1960 article, entitled *The Concept of the Stimulus in Psychology* [7]. He says, “it seems to (him) that there is a weak link in the chain of reasoning by which we explain experience and behavior, namely our concept of the stimulus” (Concept of Stimulus, 694). I will try to relate the idea of stimulus to surface layout. In the article, Gibson provides accounts of historical interpretations of stimulus from Freud to Pavlov to B.F. Skinner[8]. B.F. Skinner said that, “we frequently define the stimulus by the very doubtful property of its ability to elicit the response in question, rather than by any independent property of the stimulus itself”[9]. Often, in psychology, we might think of stimulus as something already perceived to which we immediately or upon deliberation react. However, Gibson critiques Skinner’s remark, because, “he (Skinner) suggests no remedy, however, for this doubtful scientific behavior” (Concepts of Stimulus, 695). Perceived difference (in circumferences or in general)

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[6] "Animate - Wolfram Mathematica 7 Documentation." *Wolfram Mathematica 7 Documentation*. N.p., n.d. Web. 7 Apr. 2010. <http://reference.wolfram.com/mathematica/ref/Animate.html>.
[7] Gibson, James J.. "The Concept of Stimulus in Psychology." *American Psychologist* 15.11 (1960): 694-703. Print. All further references to this source will be made parenthetically in the text.
[8] Burrhus Frederic Skinner (March 20, 1904 – August 18, 1990) was an American psychologist, author, inventor, advocate for social reform, and poet.
"B. F. Skinner - Wikipedia, the free encyclopedia." *Wikipedia, the free encyclopedia*. N.p., n.d. Web. 9 Apr. 2010. <http://en.wikipedia.org/wiki/B.F._Skinner>.
[9] Skinner, B.F. *Cumulative Record*. New York: Appleton, Century-Crofts, 1959. p 6.

as arc length perceived specifies an event (of change) containing optically relevant information and provides a formal ontology by which we might get at the fundament of the independent properties of stimulus, that is, the perceptually relevant *difference* as objective, geometrically provable fact containing gradient of the parameters of the inherently perceived difference. It will be shown later, upon returning to Gibson's 1950 work, The Visual World, that the mathematical function describing this difference in circumferences delivers a smooth, even surface layout. This kind of eidetic, ontological, phenomenological method is a potential remedy to Skinner's remark. Gibson might support such an idea, because he says, "gradients, patterns, and other invariants are not part of existing geometrical optics, but they are physical facts" (Concepts of Stimulus, 701), and that "what was needed for a psychophysics of visual perception was not more theorizing about cues but more attention to geometrical optics" (Concepts of Stimulus, 701). There is optically, a perceived difference in circumferences as arc length perceived, and this can be studied in order to get an idea of the information within stimulus specifying change in the environment or motion within it (a change in surface layout, contour, slant, gradient, and potentially texture). Gibson says that some theorists, "go further and, by arguing that an experimenter cannot define the stimulus anyway except in terms of *his* perception, (reach) a philosophical position of subjectivism" (Concepts of Stimulus, 696). This is not what occurs in formal ontology, because, in the case of change in circumferences, it is eidetically arrived at by phenomenological description in the language of pure mathematics, and thus is universal science, placing objectivity in a subjective framework and giving recognition to the two ideas mutual interpenetration.

The difference in circumferences equaling an arc length is an essential geometric insight, which has not yet been given enough attention in the analysis of experimental data in visual studies. Gibson says, “these terms and concepts (those relating to invariance and invariants) are subject to revision as the ecological approach to perception becomes clear” (Ecological Approach, 311), and “may they never shackle thought as the old terms and concepts have!” (Ecological Approach, 311). The structuring of the difference in circumferences includes an invariant parameter during transformation. The initial radius of the circle is always equal to the slant of the cone and is invariant during transformation. It should be noted, before continuing, that surfaces are theoretically composed of discreet data, and that upon closer inspection of the mathematics for difference in circumferences, it can be algebraically shown that the invariant nature of the parameter of an initial radius is “disrupted,” by being correlated directly to the angle taken out of the initial circle alone, thus shown to vary with the angle taken out of the circle once the structure of the cone is reduced algebraically. At this juncture, we would be given a geometrically valid expression for distance purely in terms of a single angle.

Please see The Geometric Pattern of Perception Section XX for proof of the following equation.

$$\text{Solve}\left[1 - \frac{2\pi \,\text{Sin}[\beta]}{\sqrt{4\pi\theta - \theta^2}} == \theta r - \left(2\pi r - 2\pi\sqrt{\left(r\hat{}2 - \frac{\sqrt{4\pi r^2\theta - r^2\theta^2}}{2\pi}\hat{}2\right)}\right), r\right]$$

$$\left\{\left\{r \to \frac{-4\pi\theta + \theta^2 + 2\pi\sqrt{(4\pi - \theta)\theta}\,\text{Sin}[\beta] + 4\pi^2\,\text{Sin}[\beta]^2 - \frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{4\pi-\theta} - \frac{2\pi^2\sqrt{(4\pi-\theta)\theta}\,\text{Sin}[\beta]^3}{\theta}}{16\pi^2\theta - 12\pi\theta^2 + 2\theta^3 - 16\pi^3\,\text{Sin}[\beta]^2 + 8\pi^2\theta\,\text{Sin}[\beta]^2}\right\}\right\}$$

$$\text{Solve}\left[r == \frac{2\pi r\,\text{Sin}[\beta]}{\sqrt{4\pi\theta - \theta^2}}, \theta\right]$$

$$\left\{\left\{\theta \to 2\left(\pi - \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)\right\}, \left\{\theta \to 2\left(\pi + \sqrt{\pi^2 - \pi^2\,\text{Sin}[\beta]^2}\right)\right\}\right\}$$

Thus, r is a function of theta or beta alone. This cannot be proven with Euclidean geometry, because a circle of any size can be drawn and any angle could be removed via folding. This leads to a statement that is logically true, but cannot be proven.

There are several ways to think of the invariant being disrupted within this system, and the disruption is, in my view, correlated to the bracketing of the natural attitude, first developed by Husserl. A discussion of the bracketing of the natural attitude is present in *The Overview of Perceptual Theories* paper, (Emmerson, 2010). It is due to this bracketing that we can study just a single formula, regardless of what, "invariant information" the rest of the system would tell us is necessitated by that formula.

On invariants, Gibson says, "it would simplify matters if all these kinds of change in the optic array could be understood as transformation in the sense of *mappings*, borrowing the term from projective geometry and topology" (Ecological Approach, 310), adding, "but, unhappily, some of these changes *cannot* be understood as one-to-one mappings, either projective or topological" (Ecological

Approach, 310). Hopefully, with the insight provided by the structure described by *The Geometric Pattern of Perception Theorems* (Emmerson, 2009), which includes an invariant parameter that is literally "picked up," Gibson's unhappiness can be soothed through formal ontology of a topological transformation described by the aforementioned theorems, which potentially provide novel insight into the working of Euclidian manifolds. When using the term invariant, its context is fundamental to its specific understanding, because the term can be a metaphor.

2.2) Stimulus

Gibson notes that in Lashley's interpretation of the meaning of stimulus, that Lashley, "seems to be saying that a ratio may be itself a stimulus, not just a relation between two stimuli" (Concepts of Stimulus, 697), but that, "the gestalt psychologists, by conceiving all stimuli as local events, did not come to think in this way. Gibson's argument in his work *The Concept of the Stimulus in Psychology* is further emphasized when he says, "(he thinks) that we will have to develop the needed discipline on a do-it-yourself principle" (Concepts of Stimulus, 701), that it, "might be called ecological physics, with branches in optics, acoustics, dynamic, and biochemistry," (Concepts of Stimulus, 701), and that, "we cannot wait for the physical scientists to describe and classify potential stimuli" (Concepts of Stimulus, 701), because, "the variables would seem to them inelegant, the mathematics would have to be improvised, and the job is not to their taste" (Concepts of Stimulus, 701), adding "but it is necessary" (Concepts of Stimulus, 701). In the formal ontology of perceived difference in circumferences of two circles, we find a perceptually relevant coordinate system that is useful to classifying the parameters of different stimuli such that they are based in perceiving change. Gibson says that, "we must learn to conceive an array not as a mosaic of stimuli but as a hierarchy of forms within forms, and a flux not as a chain of stimuli but as a hierarchy of sequences within longer sequences" (Concepts of Stimuli, 700). The perceived difference is of eidetic phenomenology expressible through formal ontology and delivers a framework in which forms can be hierarchically sequenced as clusters of variables.

The fundament of the independent properties of stimulus, that is, the perceptually relevant *difference* as objective, geometrically provable fact containing gradient of the parameters of the inherently perceived difference is system containing visualizable gradients and noticeable variations within the surfaces described by these gradients. Gibson had hopes for the development of a formal language in which the concept of stimulus could be understood, saying, "if experience is specific to excitation, and excitation to stimulation, and stimulation to the external environment, then experience will be specific to the environment, within the limits of this chain of specificities (Concepts of Stimulus, 702), adding that, "the first two stages have long been under investigation, (and) the last is ripe for study" (Concepts of Stimulus, 702). Later, it will be shown how Gibson redefined perception to be an experiencing rather than a having of experience. The perceived difference is related to the external environment by noting the validity of its geometric proof – its truth (pattern) existing as configurations in the information of surfaces in the world as well as the experiencing of change by the subject. Gibson says that discoveries are ready to be made, concluding his essay saying that, "perhaps the reservoir of stimuli that I have pictured is full of elegant independent variables, their simplicity obscured by physical complexity, only waiting to be discovered" (Concepts of Stimulus, 702).

In my ecological environment, there are tools, and these tools provide an affordance for certain actions. However, "the observer may or may not perceive or attend to the affordance, according to his needs, but the affordance, being invariant, is always there to be perceived" (Ecological Approach, 139). On affordances, Gibson defines the term as, "*affordances* of the environment are what it *offers* the animal, what it *provides* or *furnishes*, either for good or ill" (Ecological Approach, 127). An example of the affordance of a chair would be a place for sitting, or any of the things one could do with a chair. The questions relating affordances to mathematical analysis in ecological optics are pronounced and remain to be worked out in detail – are mathematical forms intuited, thus providing an affordance for perception of the environment, are they descriptive of surfaces in the environment, or perhaps both?

In a 1970 essay entitled, *Terms Used in Ecological Optics* (A glossary to supplement chapters 10-12 in Gibson, The Senses Considered as Perceptual Systems) [10], Gibson described geometrical surface layout, saying that, "this layout is describable by solid geometry in such terms as plane surfaces, curved surfaces, dihedral angles, and closed continuous surfaces (detachable objects)" (Terms Used

[10] Gibson, James J.. "A glossary to supplement chapters 10-12 in Gibson, The Senses Considered as Perceptual Systems." *Terms Used in Ecological Optics* 1.1 (1970): 1. Print. All further references to this source will be made parenthetically in the text.

in Ecological Optics, 2). However, how are we to relate perceived change in the world to the information for stimulus? On this, Gibson predicted that,

> "If the structure and sequence of stimulus energy can be analyzed, potential stimuli can be described and arranged in a hierarchy. There will be subordinate stimuli and superordinate stimuli, of lower order and higher order. So conceived, it is reasonable to assume that stimuli *carry information* about the terrestrial environment. That is, they specify things about objects, places, events, animals, people, and the actions of people" (Concepts of Stimulus, 702).

I find that the expression for change, or difference, provides a method for measuring quantity of stimulus energy when that quantity is placed in a coordinate system through a phenomenologically eidetic framework of formal ontology, providing ecologically relevant information, because the dimensions of perceived change are shown to contain information that describes gradients, contours, and surfaces.

The idea of motion is important to ecological optics, because it is intrinsic to the meaning of optic flow, which Gibson described through the concept of an ambient optic array. However, first, a background in the differences of abstract and surface geometry must be understood.

> "in abstract, analytic geometry, the position of a body is specified by coordinates on three chosen axes or dimensions in isotropic space; in surface geometry the position of an object is specified relative to gravity and the ground in a medium having an intrinsic polarity of up and down. Similarly, the *motion* of a body in abstract geometry is a change of position along one ore more of the dimensions of space, or a rotation of the body (spin) on one or more of these axes. But the motion of an object in surface geometry is always a *change in the overall surface layout*, a change in the shape of the

> environment in some sense" (Ecological Approach, 35-36).

Ecological optics is concerned with the difference in geometrical layout and ecological layout. Geometrical layout is, "the persisting arrangement of the rigid surfaces of the world with respect to abstract space" (Terms Used in Ecological Optics, 2), and the description of geometrical layout, "takes no account of the projection or non-projection of theses surfaces to a point of observation" (Terms Used in Ecological Optics, 2). Ecological layout, however, is, "the persisting arrangement of the rigid surfaces of the terrestrial world with respect to points of observation and paths of locomotion in a medium" (Terms Used in Ecological Optics, 2). Gibson felt it was necessary to distinguish between these two ideas, because, he was trying to formulate a theory of "the perception of the environment as contrasted with theories of the perception of 'space'" (Terms Used in Ecological Optics, 2). These ideas are key components of understanding the functional meaning of a selected sample of an *ambient optic array*, which will be discussed a little later in this paper. Understanding that a change in the size of this sample inherently contains gradients and surfaces with texture will be important when trying to account for the correlations between formal ontology and the, "selected sample of an ambient array" (Terms Used in Ecological Optics, 2), which, in ecological optics, "is comparable to the retinal image in physiological optics"(Terms Used in Ecological Optics, 2). Locomotion of the point of observation in a medium contains a gradient and inherent mathematical surface. For describing locomotion in a medium, we have access to formal, logical, mathematical idealities, which can tell us about the experiencing of simple structures of change in the environment.

Gibson described his thoughts on magnification and minification in a March of 1965 essay entitled, "Optical Magnification and Minification. Problems of Conception and Terminology,"[11]. Gibson says, "magnification and minification are changes in optical stimulation, i.e., an optic array (an array being a solid angular sector of the ambient light)" (Optical Magnification and Minification, 1). It would seem to reason that an area of the perceived sector of ambient light has most of the information about the surface of the world available in the surface area of a substance being perceived. Depending on the substances involved, the general environment *tones* of the visual texture, hue, and brightness relative to the substances in the environment. I can also take into consideration perceived change in size, i.e. a gradient of change through experiencing the geometric parameters of a change in perceived size of two circumferences as seen in Theorem 1.

Gibson did not introduce the idea of ecological optics until his 1966 work, The Senses Considered as Perceptual Systems. In The Ecological Approach to Visual Perception, he outlines several laws of ecological surfaces, and introducing the ideas of surface layout, he says that, "the 'permanent objects' of the world, which are of so much concern to psychologists and philosophers, are actually only objects that persist for a very long time" (Ecological Approach, 13), noting that the, "abstract notion of invariance and variance in mathematics is related to what is meant by

[11] Gibson, James J.. "Problems of Conception and Terminology." *Optical Magnification and Minification* 1 (1965): 1-2. Print.

persistence and change in the environment" (Ecological Approach, 13). In surface geometry, we see an object with surface layout (gradient with actual surface) that comes from a purely geometric framework. Thus, the being of these formally ontological objects exist in duration on the scale of the truth of any geometric proof like the Pythagorean theorem.

Pure geometry from experiencing the relationship of the difference in circumferences of two circles to an arc length of the initial circle shows us that a distance in "space-time" cannot be disassociated from its correlates in multiple implicit or explicit dimensions by establishing mathematical equations upon which the perceived structuring of, "space-time," is based in terms the mathematical correlates of perceived difference. The theorems themselves establish consistency, and somewhat resolve the paradoxes that arise from discussing a whole, complete system by generating more potential solutions (expressions) from those inconsistencies.

For this reason, when returning to look at Gibson's 1950 work, it will be important to have a formal ontology that can include an invariant parameter within structuring of three dimensions in order to draw relations of velocity to the perceived difference resulting from a change in distance of the height of a conic section of the optic array so that we can understand the purely geometric framework in which optic information resides. In the environment, we experience relations of space-time difference in positions of objects ideally through the notion of place-to-place, and in mathematics, these places designate regions or exact positions (points). Math sorts out in a way capable of being followed the non-changing elements, which are the forms of the structuring of light in the environment.

Gibson is constantly referring to surface layout, because for him, layout is the shape of the environment, thus the quality of the surface of the layout as well as the layout itself are of importance to ecological optics. Gibson's theory of ecological optics does not say a lot about space and time, but says a great deal about persistence and change in the environment. "The abstract notion of invariance and variance in mathematics is related to what is meant by persistence and change in the environment" (Ecological Approach, 13). The geometry of difference in circumferences of two circles applied to the Pythagorean theorem is geometry of an experienced change in the environment. It could describe a change in area or circumference of a circle projected on the retina delivering a change in *perceived difference in size*, or it could be a change in the distance to an object as the point of observation moves in the environment. This geometry includes an invariant structure. It is initially conceived as the idea that the slant of the cone is always equal to the initial radius of the circle that is folded up into that cone up until the point at which the cone is totally collapsed into a *height* whose length is equal to the initial radius. It is at this destination of the center of the circle at which it is algebraically shown that that height equals one and the initial radius equals zero, a quizzical result, since, through thought, we would think that the height should be equal to the initial radius. The theorem describes the structure of the perceived difference through studying the mathematical relationships of that change through significance of depth perception, including an invariant parameter (the initial radius as slant of the cone through transformation).

The ecological approach to visual perception takes the notion of the invariant quite literally when Gibson says, “the terrestrial horizon is thus an invariant feature of terrestrial vision, an invariant of any and all ambient arrays, at any and all points of observation” (Ecological Approach, 163). The geometric truth of difference in circumferences being an arc length and its combination with the truth of the Pythagorean theorem shows us that we have access to experiencing *the true* in our environment. And so long as the total structure of this system is taken into account, the invariant nature of the initial radius is included and asserted.

There is some precedence for understanding texture as a gradient at least symbolically. Gibson uses an example in which texture is depicted as an even gradient of squares portrayed through perspective in The Ecological Approach to Visual Perception. Gibson used diagrams to illustrate the scenarios he was describing. This is in the tradition of a *display*, which, for Gibson, is,

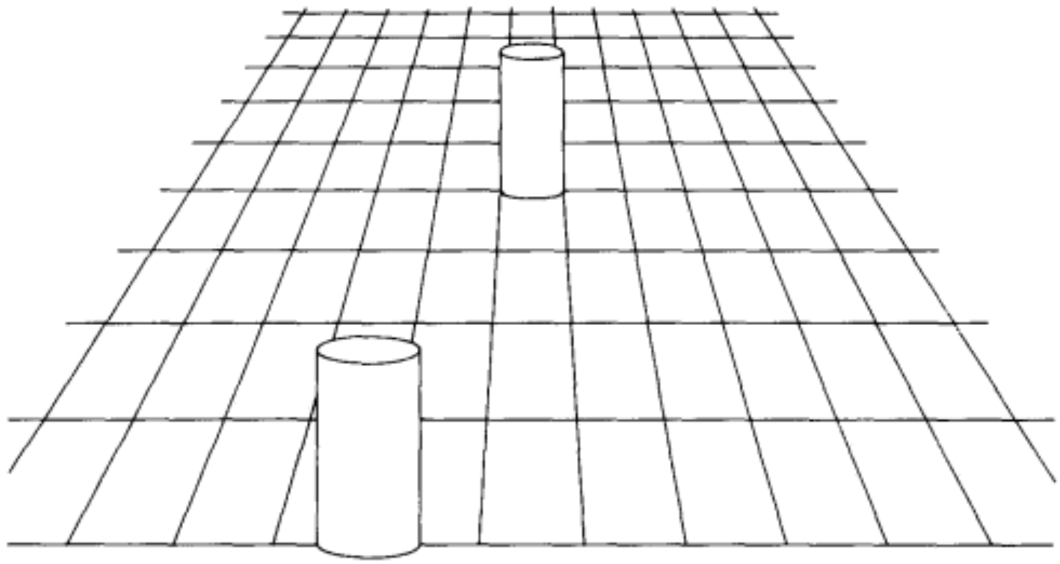
> “an artifact providing a delimited optic array to a station point. The most familiar example is a picture or drawing in which case the array is frozen in time. Experiments on perception in the past have mostly been carried out with displays or drawing” (Terms Used in Ecological Optics, 2).

We see an example of one of Gibson's displays when he says and diagrams,

Figure 9.5

The base of each pillar covers the same amount of the texture of the ground.

The width of each pillar is that of one paving stone. The pillars will be seen to have the same width if this information is picked up. The height of each pillar is specified by a similar invariant, the "horizon-ratio" relation, described later.



(Ecological Approach, 163). Here, we see that he is associating an even grid with a kind of textural surface. This gives substantial basic introspection into the meaning of texture for contoured objects like those seen in *The Geometric Pattern of Perception Theorems* (Emmerson, 2009) – that all of the objects seen therein are different textural forms of pure geometry and that different mathematical contours have different visual textures. In the above diagram I assert that Gibson is depicting a, "selected sample of an ambient optic array" (Terms Used in Ecological Optics, 2). Gibson defined a selected sample of the ambient optic array to be, "that portion of an ambient array registered by an eye at a given posture of the head-and-eye," (Terms Used in Ecological Optics, 2). In this diagram, we see depicted a single portion of an ambient array.

2.3) Ambient Optic Array

Gibson did not introduce the idea of an ambient optic array until his 1966 work, although the groundwork for its development is present within the 1950 book through the expression of his understanding of optical projection. Gibson says that, "to be ambient, an array must surround the point completely (,) it must be environing, (and) the field must be closed, in the geometrical sense of that term, the sense in which the surface of a sphere returns upon itself" (Ecological Approach, 65). The ambient optic array is composed of lines of optic flow, which for a bird flying on a straight path through the sky, parallel to the ground, surrounds the bird from underneath. Gibson had some preliminary considerations on how ambient light is structured, and he says that, "the components of the array are the *visual angles* from the mountains, canyons, trees, and leaves (actually, what are called *solid angles* in geometry), and they are conventionally measured in degrees, minutes, and seconds instead of kilometers, meters, and millimeters" (Ecological Approach, 68). In order to understand the significance of Gibson's observations about the components of the array, we must first return to his idea of hierarchies. If you remember, he said that, "we must learn to conceive an array not as a mosaic of stimuli but as a hierarchy of forms within forms, and a flux not as a chain of stimuli but as a hierarchy of sequences within longer sequences" (Concepts of Stimuli, 700). I propose that these hierarchies of forms are evident in the interwoven expressions of individual solid angles or "*expression clusters*" of these solid angles in terms of each other as they are relevant to the structuring of ambient light in the environment. The expression clusters are sequences of interwoven meanings and a meaning of hierarchy is present in the "level" of substitution made within an equation. Of this structure, there would be gradients of the parameters of perceived difference in its multiple sensory manifestations (difference in volume, space, intensity, and timbre). The inherent forms of the structure of difference in general are understood through formal ontology of perceived difference in circumferences as arc length perceived. When this is applied to the Pythagorean theorem to form a cone where the smaller of the two perceived circumferences is the base of the cone, we arrive at a scalable set of "visual" solid angles with different significances to the system as a whole. The form of the "cone" in *The Geometric Pattern of Perception Theorems* (Emmerson, 2009), is evident in the cone of light rays entering the eye, but is not limited to this role within the perceptual field. Gibson says that, "a delimited optic array is a solid-angle of light coming to a point of observation from a display of some sort (often loosely called simply an 'optic array')" (Terms Used in Ecological Optics, 2). The angles of the cone of light rays entering the eye are solid, and to position on a wall directly in front of my view would make a circle on that wall. However, in a 1977 essay entitled, (Appendix) Glossary Special Terms Used in the Ecological Approach to Vision[12], Gibson said that the optic array at a point of observation, "consists of those discontinuities, gradients, and relations in the ambient light that specify the persisting features of the environment" (Special

[12] Gibson, James J.. "Special Terms Used in the Ecological Approach to Vision." *(Appendix) Glossary* 1.1 (1977): 1-4. Print. All further references to this source will be cited parenthetically in the text.

Terms Used in the Ecological Approach to Vision, 3). A delimited optic array can also be taken into consideration as a “volume of space.” Ambient light reverberates in the medium within the “volume” of the cone of light rays entering the eye from the solid angle of light entering the eye, while the optic array at the point of observation consists of the gradients and relations in the ambient light that specify persisting features of the environment like surfaces. Surface layout would be described by geometry of surfaces.

In Gibson’s discernment of the differences of the delimited optic array and the optic array at a point of observation, he provides an ontology of the perception of the features of the environment. If light is to be structured through formal ontology (pure geometry), as Gibson suggests (the angles of the delimited optic array, displays, the surface geometry of substances, gradients, pencil lines and the plotted velocities vectors of the ambient optic array), then it is important to notice that there is a purely geometric system that describes the parameters of a cone of light rays entering the eye through describing the experiencing of the gradients of the parameters of relations in the world.

In a January of 1969 essay, entitled A List of Ecologically Valid Meanings in a Stationary Ambient Optic Array[13], Gibson elucidates the extent of the idea of an optic array by saying, “the kind of texture in a patch in the array is fairly specific to the substance of the surface in the world (solid or liquid, rough or smooth, hard or soft, etc. (p. Brodatz photos (Ecological Approach, 26-27))’” (Ecologically Valid

[13] Gibson, James J.. "A List of Ecologically Valid Meanings in a Stationary Ambient Optic Array." *J.J. Gibson, Cornell University* 1.1 (1969): 1-4. Print. All further references to this source will be cited parenthetically in the text.

Meanings, 1), and that "a patch of the ambient array containing <u>texture</u> or <u>fine structure</u> specifies a material <u>surface</u> in the world (obstacle, substance, thing), whereas an untextured or homogeneous patch specifies only the <u>medium</u> (air, empty space, sky)" (Ecologically Valid Meanings, 1). An ambient array can contain texture. However, we can differentiate between the ideal geometric structure of an array, with its own implicit surfaces, gradients, and contours (due to the correlates of perceiving and experiencing change in the environment that describe it ideally), and the structuring of the perceived textured patch within that the array due to the textural interrupting of ambient light by the surfaces of materials. Though, surface geometry from formal ontology of experiencing geometric difference is present in each (the geometric structure of the ambient optic array at a point of observation as well as the actual textures, which specify material surfaces in the world).

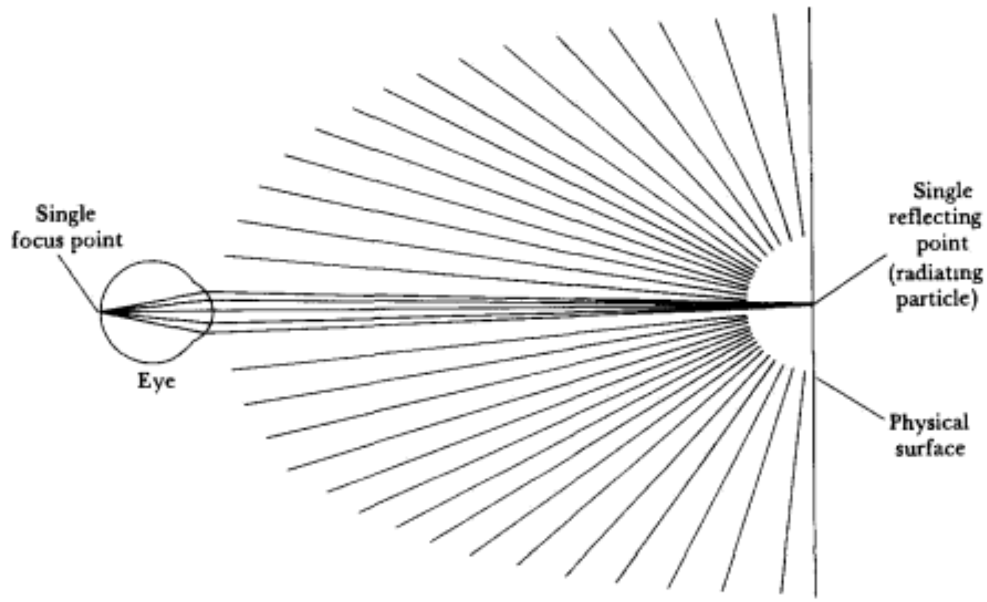
Gibson began providing grounds for the theory of the ambient optic array by attacking the position (theory) of the retinal image as necessary for visual perception, saying, "ever since someone peeled off the back of the excised eye of a slaughtered ox and, holding it up in front of a scene, observed a tiny, colored, inverted image of the scene on the transparent retina, we have been tempted to draw a false conclusion" (Ecological Optics, 62), and that, "the question of how we can see the world as upright when the retinal image is inverted arises because of this false conclusion (that the oxe would see the retinal image)" (Ecological Approach, 62). However, the retina is the surface on which the radiant light energy falls, stimulating the nervous system in such a way that an "image" (schema?) of the *world* is formed or capable of being developed by the perceiver. Change in perceived size of a circular disk that is moving away from the perceiver is one example of how the image projected on the retina is still pertinent to perceptual theory. Gibson says that, "in order to stimulate a photoreceptor, that is, to excite it and make it 'fire,' light energy must be absorbed by it" (Ecological Approach, 52). This is not to say that the retinal image is transferred to the brain, and this is one of the reasons Gibson felt an ambient optic array was necessary, "the information that can be extracted from ambient light is not the kind of information that is transmitted over a channel" (Ecological Approach, 64). If we are to still see the retina as pertinent to visual perception, we must engage the perceptual theory such that it describes an animal interacting with its environment through a process of sampling the ambient optic array.

Gibson also demonstrates that the retinal image is not necessary for vision. He cites evidence for this, because, a "*compound eye* with no chamber, no lens, and no sensory surface but with a closely packed set of receptive tubes called *ommatidia*" (Ecological Approach, 61), is capable of receiving information from the environment so that animals may move around. Beginning with the James Mill's[14] question, "how then, is it that we receive accurate information by the eye of size and shape and distance?" (Ecological Approach, 60), Gibson hopes to give a fresh start to perceptual theory by bringing closure to the endless debates surrounding the theories of, "empiricism, nativism, rationalism, Gestalt, and now information-processing theory" (Ecological Approach, 60). In order to do this, Gibson cites

[14] James Mill (6 April 1773 – 23 June 1836) was a Scottish historian, economist, political theorist, and philosopher.
"James Mill - Wikipedia, the free encyclopedia." *Wikipedia, the free encyclopedia*. N.p., n.d. Web. 8 Apr. 2010. <http://en.wikipedia.org/wiki/James_Mill>.

mathematics as a possible way of analyzing meaning of point-to-point correspondence, though he does not give us specific insight into the structure of this mathematical analysis, except by citing his current understanding of projective geometry. He plans on altering the generally accepted theory of the eye, saying that, "the generally accepted theory of the eye does not acknowledge that it registers the invariant structure of ambient light but asserts that it forms *an image of an object* on the back of the eye" (Ecological Approach, 58). Gibson shows through the diagram depicted below that light coming from a single reflecting, radiating point focuses on a single point on the retina through the "pencil rays," (Ecological Approach, 59). With the *Geometric Pattern of Perception Theorems*, we find a simple, symmetric, organizational system for analyzing the mathematical contours and gradients inherent within each of these rays. Gibson says, "(the) theory of point-to-point correspondence between an object and its image lends itself to mathematical analysis" (Ecological Approach, 59). Each of these pencil rays could be thought of as any one of the spatial parameters defining a cone, and we can see why the ambient optic array is so mathematically dense. Is it possible that the eye and visual system could contain a mechanism for performing ontology of the environment, enabling direct perception of it? The diagram of Gibson's "pencil rays" is presented below.

Figure 4.3
A focused pencil of rays connecting a radiating point on a surface with a focus point in the retinal image.
The rays in the pencil are supposed to be infinitely dense. Note that only the rays that enter the pupil are effective for vision. (From *The Perception of the Visual World* by James Jerome Gibson and used with the agreement of the reprint publisher, Greenwood Press, Inc.)



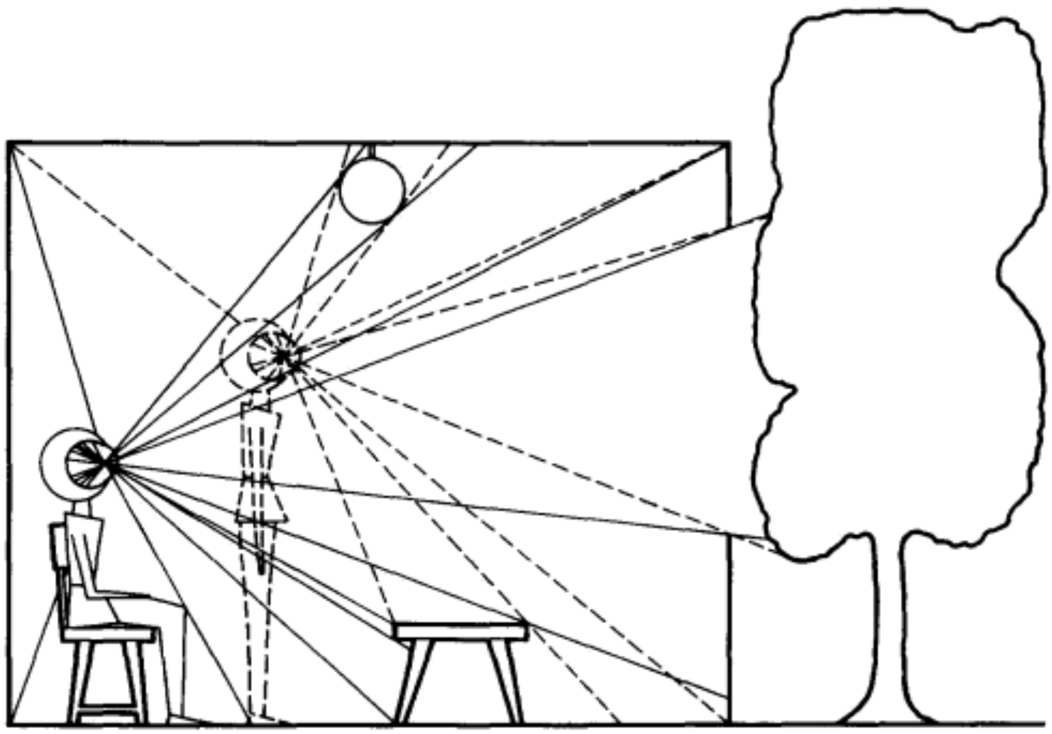
This diagram shows us how each ray is traced from a single point of radiation to a single focus point on the retina. How to use the geometric theorem of difference in circumferences equaling an arc length applied to the Pythagorean theorem in the best-fit way for analyzing the system is still to be seen, but the correlation between the two notions is directly accessible.

The lines of optic flow are angular velocity vectors. The ambient optic array is, "the central concept of ecological optics," (Ecological Approach, 64), and occurs for the perceiver at a point of observation. Gibson's terminology for describing the array describe it to be, heterogeneous, differentiated, filled, and formed, but he recognizes that, “these contrasting terms are still unsatisfactory” (Ecological Approach, 65). He feels like, “it is difficult to define the notion of structure” (Ecological Approach, 65). I find that there is a pure geometric and formal structure described by experiencing the difference in circumferences of two circles and its relationship to the Pythagorean theorem, and Gibson plans on using a, “radical proposal having to do with *invariant* structure” (Ecological Approach, 65). The *Geometric Pattern of Perception Theorems* are invariant structurally in the sense that the structure they describe while describing change, is proven through the body of the mathematical science, and its truth is unchanging, its structural form invariant. It also includes a parameter that, when the total structure of the system taken into consider, is invariant. The following diagram is entitled:

Optical Structure with a Moving Point of Observation

Figure 5.4

The change of the optic array brought about by a locomotor movement of the observer. The thin solid lines indicate the ambient optic array for the seated observer, and the thin dashed lines the altered optic array after standing up and moving forward. The difference between the two arrays is specific to the difference between the points of observation, that is, to the path of locomotion. Note that the whole ambient array is changed, including the portion behind the head. And note that what was previously hidden becomes unhidden.



(Ecological Approach, 72).

In this diagram, we see that there is a change in the position of the observer, and thus there is a change in the optical structure of the array. The formal ontology of perceived difference in circumferences applied to the Pythagorean theorem to transform a circle into a cone describes a pure geometry for expressing the gradients of the contextual parameters involved in a change of any length within the system described by this diagram, because of the ontology's "universally" scientific character. This change in length (specifically notable in the algebraic expression of the height of the cone) is a motion through the optic array. The optic array is mathematically dense. The function for the height of the cone designates a change in the position of the point of observation in a straight line (through a structuring of perceived difference in circumferences). The *Geometric Pattern of Perception Theorems* (Emmerson, 2009) allow students of perception to perform geometric analyses of any change in parameters of the system. Its functions allow the specification of events of change of any distance. Specifically, it is useful to note the distance of the height of the cone (that cone described by *The Geometric Pattern of Perception Theorems* found in Lemma 1, which is related to the perceived difference in circumferences and is in equally simple terms as the Pythagorean theorem or the trigonometric functions in the sense that it only uses two terms to mathematically express length.

Gibson hoped to redefine perception in general by associating it with an experiencing of the environment instead of a having of experiences. On this, Gibson says that, "the act of picking up information, moreover is a continuous act, an activity that is ceaseless and unbroken" (Ecological Approach, 240). The ontology of difference is eidetically performed through a continuous act, an exploration of the environment, mathematical relations within the environment, and intuitive recognition of world. Thus, it resembles perception itself metaphorically. The being of the relationships of parameters of the world and environment places these parameters in a cohesive, whole, unified system with each other. The being of the relationships thereby affords visualization of the gradients of these parameters in an actively, progressively, experientially discovering pattern.

According to Gibson, the ambient optic array surrounds the observer, who has a central location within it. This idea of the distance to the object in the world is present in his 1950 work and sees further development with the concept of the ambient optic array, because velocity lines of optic flow are taken into account such that the, "change of the optic array (is) brought about by a loco-motor movement of the observer" (Ecological Approach, 72). It has already been noted in the *Overview of Perceptual Theories* paper (Emmerson, 2010), that Gibson develops this idea through the understanding of a medium, "instead of geometrical points and lines, then, we have points of observation and lines of locomotion" (Ecological Approach, 17). The height of the cone described in *The Geometric Pattern of Perception Theorems* (Emmerson, 2009), is such a line of locomotion (motion from place to place) – specifically in space-time if the passing of the angle measure within that expression for the height of the cone is set to pass constantly with the historical notion of time.

Phenomenal velocity (Emmerson, 2009 – Theorem 3) is a result found from applying the Lorentz factor[15] to parameters of the height of the cone where the

[15] The Lorentz coefficient is used in Einstein's theory of relativity to describe the length contraction of a distance moving at speed approaching the speed of light as well as the time dilation. Distances are multiplied by the Lorentz factor and time is divided by the Lorentz factor. The Lorentz factor equals $\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, where v is velocity and c is the speed of light in a vacuum. "Lorentz factor - Wikipedia, the free

Lorentz transformation applied in a null, logically canceling manner within expression for the height of the cone. Then, using the exact speed of light in scientific notation, one finds a solution to the intrinsic velocity within the Lorentz factor upon computing the solution to the equation for the height of the cone. This is result is found from the pure geometry of a circle transforming into a cone. This result is a signifier of the structuring of the general motion in phenomenologically computational mathematics, getting to the eidetic meaning of "velocity" through formal ontology and logic, without resorting to derivatives or commonly conceived rates of travel. It is from the function of the height of the cone that the implicit velocity within that Lorentz transformation is capable of being calculated formally. The result is pertinent to ecological optics, because it delivers the same form as the ambient optic array - that of a hemisphere.

Velocity is expressible by a variety of functions like instantaneous velocity (the derivative of distance with respect to "time"), average velocity (the distance divided by time), and phenomenal velocity (an obscurity of computational capacities) within the height of the cone. This height is interpreted as an "*accelerating*" length of a "space-time" dimension (there is less circumference capable of being translated into the height of the cone as the amount taken out of the initial circumference increases) when time is said to pass constantly with the angle measure, and it has an up-down polarity (there is always a positive solution and a negative solution to the aforementioned distance of the height of the cone).

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encyclopedia." *Wikipedia, the free encyclopedia*. N.p., n.d. Web. 2 Apr. 2010. <http://en.wikipedia.org/wiki/Lorentz_factor>.

According to Gibson, “the ambient optic array is treated as if its structure were frozen in time and as if the point of observation were motionless” (Ecological Approach, 70). This leads to the question of the meaning of to be “frozen in time.” I will use a short, mathematical, informally written, but formally established, proof to illustrate the meaning of how the conic sample of the array is considered “frozen in time.”

The change in the initial radius through the transformation is equal to zero. It does not change, so therefore, it is constant, and the derivative of a constant is zero. The variables, θ and η change during transformation. Therefore, we may take the derivative with respect to these two variables and set that expression equal to zero.

D[r, r] = 0, because r does not change

$$D[r, r] = D[r, \eta, \theta] = D\left[\frac{2\pi\eta}{\sqrt{4\pi\theta - \theta^2}}, \eta, \theta\right]$$

$$D\left[\frac{2\pi\eta}{\sqrt{4\pi\theta - \theta^2}}, \eta, \theta\right] = -\frac{\pi(4\pi - 2\theta)}{(4\pi\theta - \theta^2)^{3/2}}$$

$$D[r, \eta, \theta] = 0 = -\frac{\pi(4\pi - 2\theta)}{(4\pi\theta - \theta^2)^{3/2}}$$

$$\text{Solve}\left[0 == -\frac{\pi(4\pi - 2\theta)}{(4\pi\theta - \theta^2)^{3/2}}, \theta\right]$$

$$\{\{\theta \to 2\pi\}\}$$

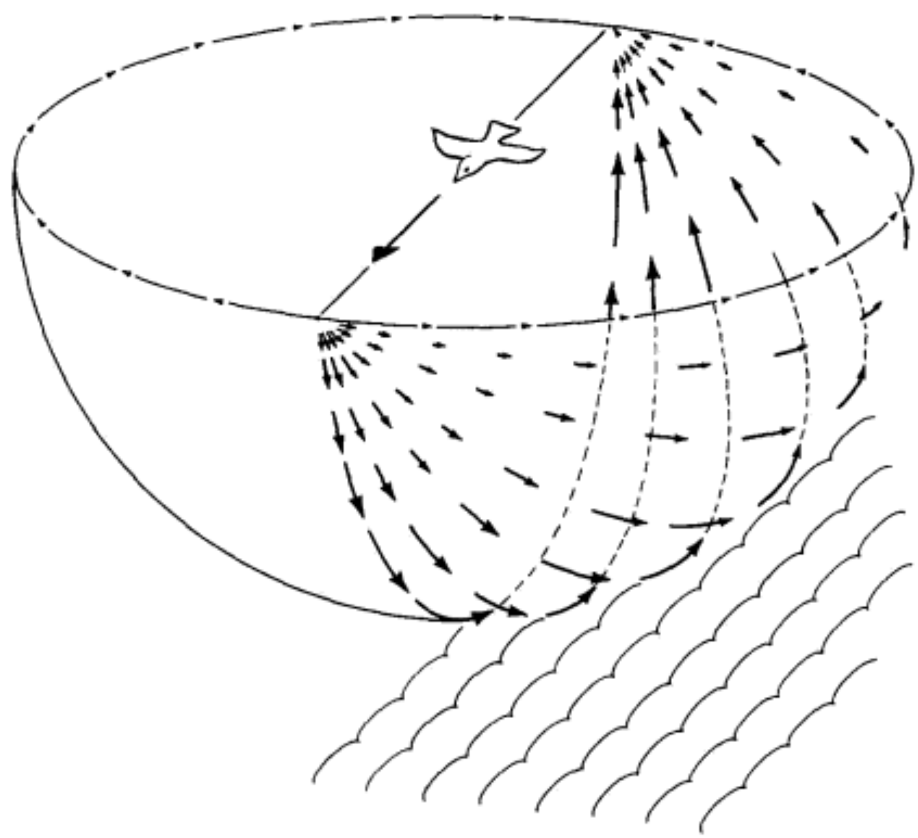
If time is set to pass constantly with the angle measure taken out of the circle, the invariant structure of the array is shown to be that which establishes the array as frozen in time. It is Gibson’s position that disruption of the invariant structure of the array specifies a perceptual event. The position at which theta equals 2 π signifies the position within the transformation at which the initial radius has completely folded up into the height of the cone. It signifies the position of the locus of perception with respect to an observed location in the world.

In Theorem 3, of *The Geometric Pattern of Perception*, (Emmerson, 2009), we see the *form* of the ambient optic array, that of a hemispherical plot of the intrinsic, phenomenal velocity. This velocity is found computationally, and through algebraic cancellation of the mutely applied Lorenz transformation. It is a form, and a solid framework on the scale of the distance light would travel within a single second of perceived space-time. The correlation between Gibson's portraying of the ambient optic array for a bird flying through space to the computational results found in Theorem 4 is through form (sphere or hemisphere) and through meaning (plotted velocity lines), although it is on a much larger scale. However, the size of the hemispherical diagram in Theorem 4 could be scaled to any size by substituting anything from centimeters to nanometers for meters. On the following diagram, Gibson says, "A bird is flying over (and parallel to) the wrinkled earth. The texture of the lower hemisphere of the optic array flows in the manner shown here. The vectors in this diagram represent angular velocities of the optical elements" (Ecological Approach, 123). The production of the plot of the phenomenal velocity is also a hemisphere, except it is on the scale of the distance light would travel within one second or one unit of time. Within this second, light reverberates many times through the medium, reflecting off of many different places. Thus, the hemisphere is disrupted by the textured patches of surface layout, which structure the environment.

“The Flow of the optic array during locomotion parallel to the ground”

(Ecological Approach, 123).

Figure 7.3
The flow of the optic array during locomotion parallel to the ground.
A bird is flying over the wrinkled earth. The texture of the lower hemisphere of the optic array flows in the manner shown here. The vectors in this diagram represent angular velocities of the optical elements. The flow velocities are plotted exactly in Figure 13.1.



The ambient optic array is of light, that is to say, “the invariant structure of the array that specifies the persisting world *underlies* the changing perspective structure” (Ecological Optics, 122). The structure of changing perspective when the point of observation moves in a straight line is notable is mathematically expressible by the motion of the height of the cone, and in mathematical terms, we see that there is an implicit structure to velocity that is computed from within it. "The centrifugal outflow of the array that specifies locomotion does not interfere with the information that specifies surface layout; the invariants are all the better for the transformation" (Ecological Approach, 122), the information that specifies surface layout can be described by very dense, discreet data of the surfaces of natural substances, and the invariant structure of the array does not interfere with this pre-defined information within the surfaces themselves.

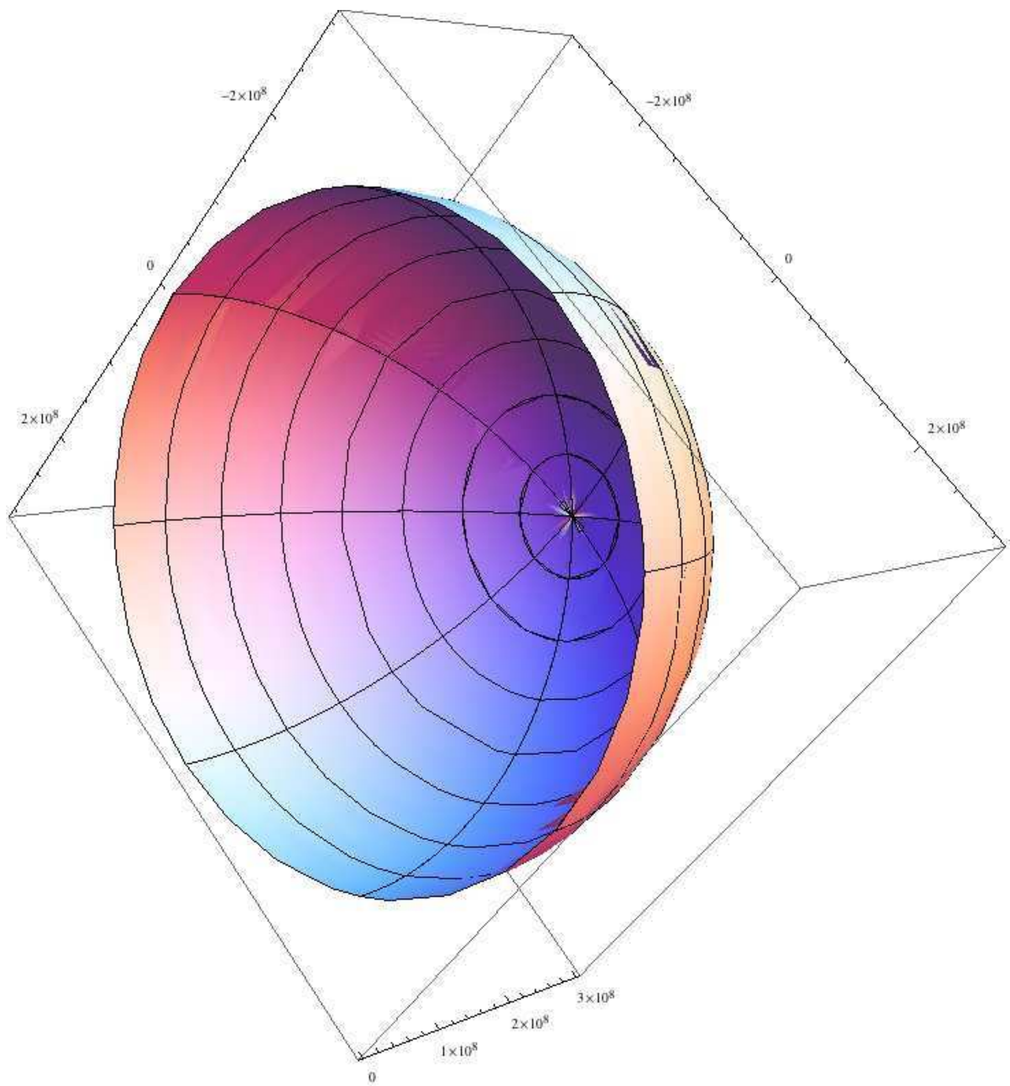
Gibson speaks of specification of the event, saying that the event is specified by a, "disturbance in the invariant structure of the array" (Ecological Approach, 102). In mathematics of the *Geometric Pattern of Perception Theorems*, we see that the invariant in the system is the initial radius of the circle that is transformed into a cone whose slant is always equal to the initial radius and height always orthogonal to the center of the base of the cone, but this initial radius is shown to be disrupted, because upon algebraic reduction, it is shown to be dependent on the angle taken out of the initial circle (Lemma 6). In this, we see that when we measure length, we get a designated angle. Plugging in one variable, we get an exact solution for the other, thus showing us something about discreet data in a surface. Gibson even relates the notion of the array to mathematics, saying that, "the array is filled, it is mathematically dense" (Ecological Approach, 103). The distance to objects in the world changes as the observer moves. The expression of the change in this distance has been shown to be expressible purely in terms of angles through pure geometry and its algebraic lemmas. However, this does not deny in any way the structure of space in terms of time and the two distinct ideas' inherent interrelationally expressible interpenetration. For Gibson, optical information is a necessary concept for understanding the ambient optic array, and a description of his account of optical information can be found in the *Overview of Perceptual Theories* paper (Emmerson, 2010).

I would now like to again note the similarity of form between the solution to the velocity variable, v, within the Lorentz transformation, from the application of the Lorentz coefficient to the height of the cone in such a way that the coefficient should cancel out with itself (a computational solution is found to v when the exact speed of light is used in scientific notation) and Gibson's diagram of hemispherical optic flow. Both diagrams are a hemisphere, and are similar in form. Below is the diagram and equation from Theorem 4 of The Geometric Pattern of Perception Theorems (Emmerson, 2009-2011).

$$\text{Solve}\left[\frac{\sqrt{r\sqrt{1-\frac{(v)^2}{c^2}}}\ \sqrt{\frac{\theta}{\sqrt{1-\frac{(v)^2}{c^2}}}}\ \sqrt{4\pi r - r\theta}}{2\pi} == r\,\text{Sin}[\beta],\ v\right]$$

$$\left\{\left\{v \to -\frac{1.\sqrt{-1.12941\times10^{18}\,\theta + 8.98755\times10^{16}\,\theta^2 + 3.54814\times10^{18}\,\text{Sin}[\beta]^2}}{\sqrt{-12.5664\,\theta + \theta^2 + 39.4784\,\text{Sin}[\beta]^2}}\right\},\right.$$

$$\left.\left\{v \to \frac{\sqrt{-1.12941\times10^{18}\,\theta + 8.98755\times10^{16}\,\theta^2 + 3.54814\times10^{18}\,\text{Sin}[\beta]^2}}{\sqrt{-12.5664\,\theta + \theta^2 + 39.4784\,\text{Sin}[\beta]^2}}\right\}\right\}$$



Event perception is often of mechanical events, and Gibson outlines the meaning of optical information for perceiving mechanical events. I propose that the initial radius of a circle exists within a univocal system of "space-time" and can thus be placed purely in terms of an angle geometrically. This is simply an ontological observation, expressible through the language of mathematics of algebra and pure geometry. The initial radius of a circle is of distinct importance to ecological optics, because, upon consideration of the pure rotation of an object on its axis, "the alignments of textural units, the radii of the circle are shifted at the contour" (Gibson 104). We see an example of what Gibson may have understood to be referring to changes in contours at the shifting of the radius, when visualizing the many spherical expressions of the space of an initial, radius, invariant as the slant of the cone, purely in terms of the angular variables of the cone described in the mathematics for transforming a circle into a cone as well as the simple expression for initial radius in terms of theta, which delivers a contoured object similar to a bunt cake mold, having differing slopes at different locations.

In the "univocity" of "space-time" section of the *Geometric Pattern of Perception Theorems*, we see that, when observing how the expressions of the angle taken out of a circle in terms of the angle made between the invariant slant of the cone (equal to the initial radius) and the base of the cone are applied at different locations within the expression of the initial radius, which is expressed purely in terms of the angle, theta, many contours are delivered through spherical graphing. These contours describe a changing or shifting contour of surface layout of the mathematical expression for radius, and have an aesthetic quality of shearing. Thus, upon visualization of the initial radius of a circle in terms of two angles, contour with a surface layout is perceived with abstract geometry as a "background," fundamental component.

Specifically, when considering the distance between points of a smooth surface layout, or purely point to point in geometric terms, from Theorem 1, we can see that a certain amount of space in a univocal system is defined purely in terms of an angle, theta. This tells us that the process is happening continually. This, process, or, continuum, is related to defined place in the world upon considering the location at which a finite radius would fold up into the height of the cone if the structure of a difference in circumferences of two circles is always describable in terms of an arc length. With the variable used for the counting of time set equal to a specific portion of a single unit of perceived space-time, frozen, we see that space can be shown ontologically to extend out with time originally, specially upon interpretation of innately defined ratios of angular variables such that the given relationship of angles equals one inherently, and to be solely dependent upon the angular variable of the system, theta, although theta has certain necessitated correlations to other parameters of the system. It is these correlations that are phenomenological studies of perceiving the contours of these correlations through visualization.

We also see different examples of how surfaces disappear in different ways within surfaces delivered by the general extension of the theorem of difference in circumferences equaling an arc length. Gibson says that, "for a surface may disappear by going out of existence as well as by going out of sight, and the two cases are profoundly different" (Gibson, 79). The idea of the disappearance of certain surfaces produced by physical interpretation of the meaning of the pure geometry in *The Geometric Pattern of Perception Theorems* (Emmerson, 2009) is specifically notable in Theorem 6 in which a segment of a surface looks solid at one angle and disappears completely at another angle. How are we to interpret the ecologically optical terminology for this kind of disappearance – that of a mathematically constructed surface?

The *formally ontological as related to universal science* is an approach to studying visual perception that would describe surface layouts through observations about the structuring of the environment and what is known through science. On the one hand, we have the observation of expressing perceived difference mathematically of the essence. We also have access to the observation that the mathematics described by this essential relation of difference to arc length is applicable to distances of points of observation to locations of objects as well as within perceived surface layouts of those observed objects in the environment. This approach would be innately related to ecological optics through the ideas of surface layout, ambient optic array, and invariant.

Phenomenology can be used to describe the form of this array through the ontology of perceived change, formally a change in circumferences of the circles formed by radii, which are distances to objects in the world, perceived between objects, or within changes in size of an object and thus its image projected on the retina. However, we see that the diagram produced by phenomenal velocity within the Lorentz coefficient is on the scale of the speed of light. I would like to bring to the reader's attention work that has been done to relate the Lorentz coefficient to the perception of motion. This will show how the fact that the innate velocity within the Lorentz factor can be logically solved through pure geometry is a useful and pertinent discovery to visual perception. I refer readers of this paper to the book Formal Theories of Perception[16], which includes an account of the perception of

[16] Buffart, H.F.J.M., and E.L.J. Leeuwenberg. *Formal Theories of Visual Perception*. Chichester , West Sussex: John Wiley & Sons Ltd, 1978. Print. All further references to this source will be cited parenthetically in the text.

movement's relationship to the Lorentz transformation, saying, "since relativistic considerations are inevitably involved in the perception of movement… we outline the essentials" (Formal Theories, 50-51). Insight into how the pure geometry of the cone allows us to solve for the velocity in the Lorentz coefficient in terms of pure geometry is useful to visual perceptual studies.

The form of an ambient optic array is present within the computational results found from applying the Lorentz transformation, intrinsically, in an algebraically understood null manner, but computationally effective way to the height of the cone. If we imagine how far light travels within one second, and the surfaces around us as interrupting the quality of light experienced, that is the scale of the hemisphere of a sample of the array.

Before going back to Gibson's The Visual World, a case will be made for the *conic sample of the ambient optic array*. The sampling process is different from the sample itself. Interpreting phenomenal velocity, we see it is composed of two angles, the interior angle of the cone, beta, and the angle taken out of the initial circle, theta.

III. The Visual World

3.1) Surface Layout and the Meaning of Gradient

The origin of the issue of mathematical analysis in ecological optics stems from the description of perception of surface layout over time, and what that means for depth perception. According to Gibson, space perception is not separable from time perception. This fact needed ontological analysis through a perceptually descriptive method.

The role of mathematical analysis in ecological optics stems from the idea that, "the retinal stimulus-variable which makes possible the perception of longitudinal surface must be a continuous change of some sort in the image of that surface" (The Visual World, 67). We can therefore find mathematics that delivers or describes the image of that surface. The difference in circumferences of a sample of the image on the retina provides a framework in which a continuous relationship between distances can be developed mathematically. The relationship of the Pythagorean theorem to the difference in circumferences of two circles equaling an arc length is a provable, applicable relationship in the third dimension (surface area and volume of a sphere). The difference in circumferences of circles equaling an arc length is the start of a pattern, which extends into higher dimensional applications and contains many more patterns of numbers within the expressions of the mathematics. It can logically, eidetically extend to the difference in areas of circles or volume of spheres.

In a conic section of the ambient optic array, there is a pattern of stability when difference in circumferences is placed in its simplest terms. A conic section of the ambient optic array can be either a sample of the environment by the perceiver or an area in the sampled environment. This stability is noted in Figure 1 in which a smooth, stable surface layout is described by the difference in circumferences of two circles. The difference in circumferences of two circles with no expressed application of the Pythagorean relationship has an even gradient of surface layout at a slant.

Figure 1 – Difference in Circumferences of Two Circles Equals an Arc Length

$\theta r = 2\pi r - 2\pi r_1$

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Plot3D[2 π r - 2 π r1, {r, -1, 1}, {r1, -1, 1}]
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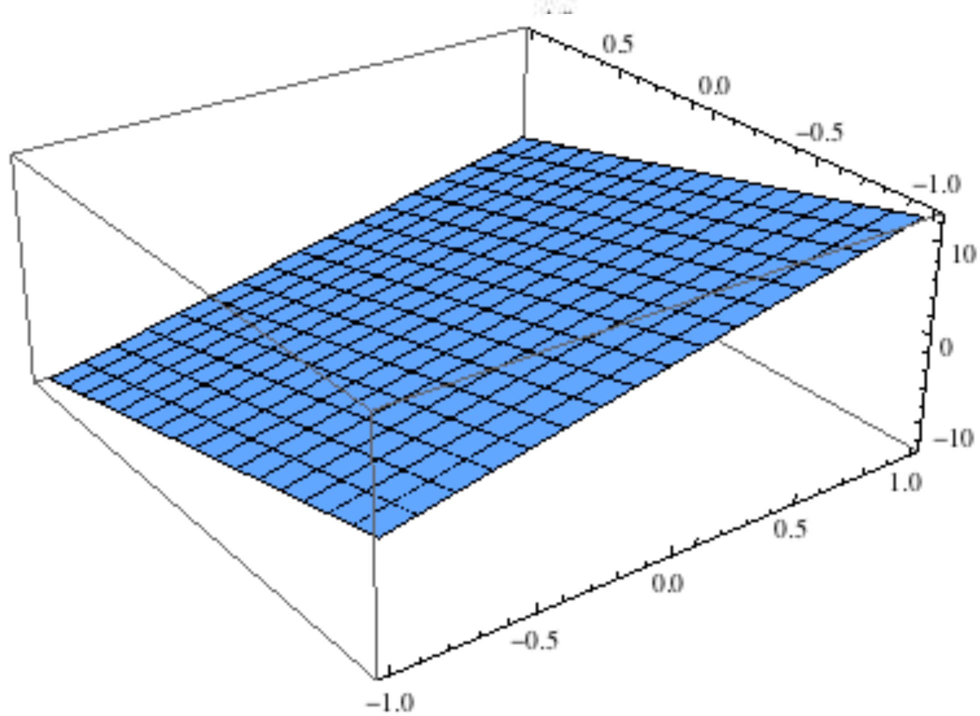
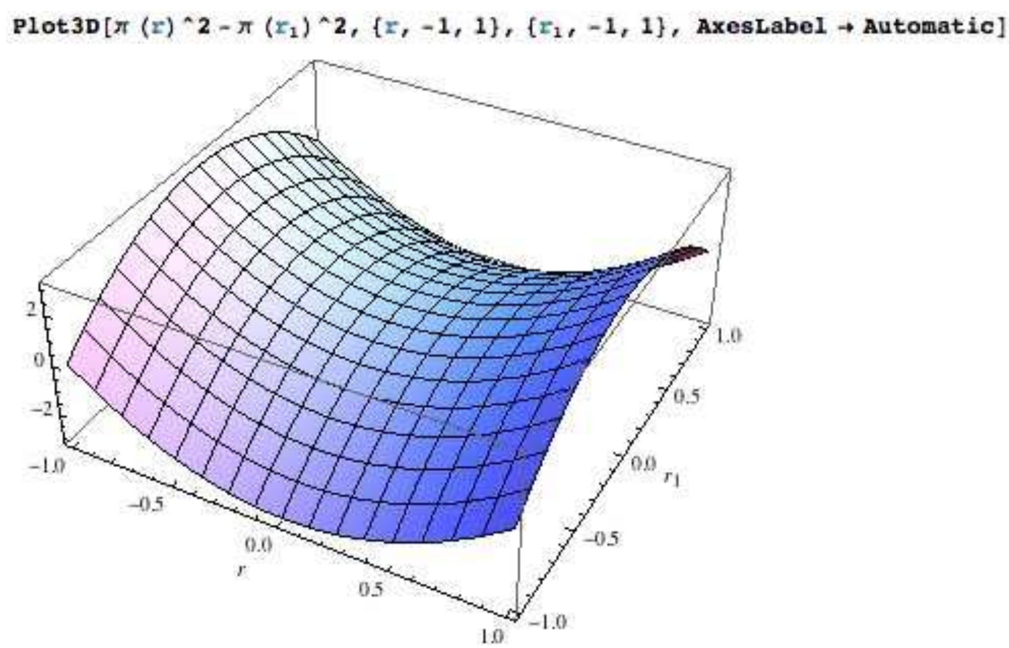


The expression that an arc length is equal to a difference in circumferences contains within its implications a pattern through, "affordance" of substitutions like the expression for length given by the Pythagorean theorem to form a cone (descriptive of the transformation of a circle into a cone). "Moreover a series of transformations can be endlessly and gradually applied to a pattern without affecting its invariant properties" (The Visual World, 154). The initial radius is that property of the transformation, inherent in the initial expression for difference in circumference that remains invariant upon application of the Pythagorean theorem to the difference in circumferences of two circles to form a cone, because it remains constant as the slant of the cone when in transformation to describing depth. However, it is still a variable. The solutions to the variables in this transformation can be applied to other transformations like the change in area of a circle, surface area of a sphere, or volume of a sphere, etc. The series of transformations is the increase in dimensionality of the change. The pattern is the idea that the solutions to the variables of the transformation are valid when describing the change in volume of a sphere or beyond.

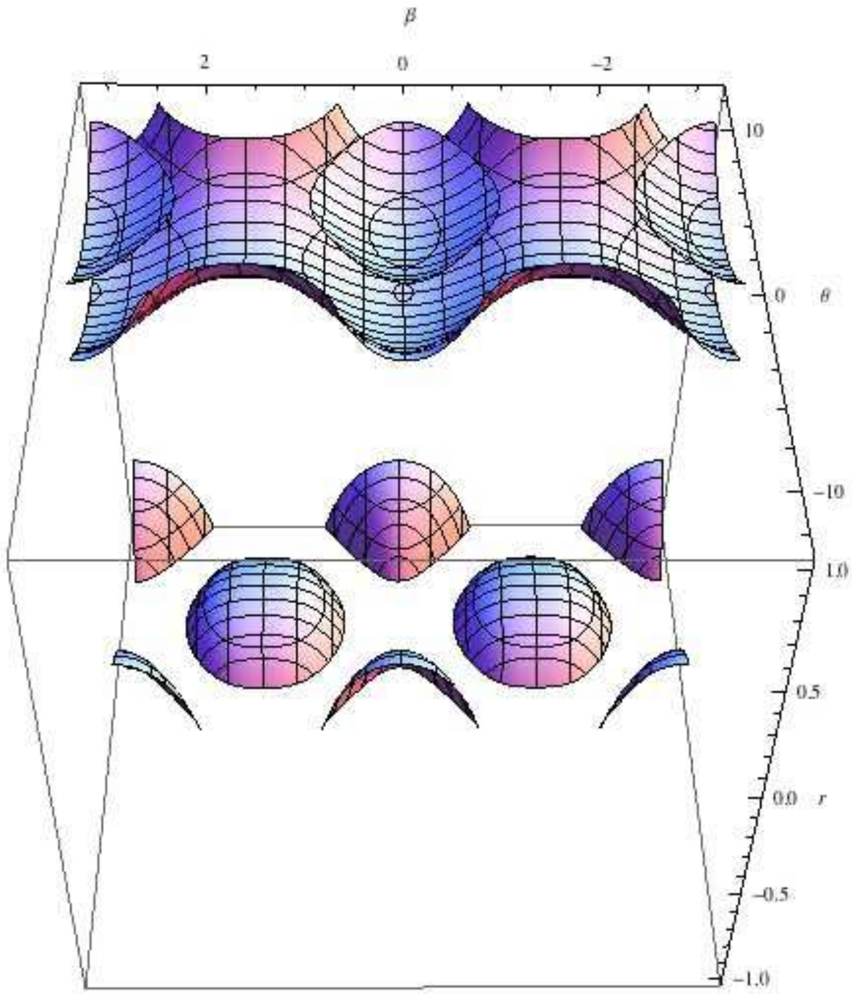
Figure 2 – The Difference in Areas of Two Circles



The theorem of difference in circumferences applied to the Pythagorean theorem is a provable theorem and a pattern, which can be applied to other such transformations like difference in area of circles, volume of spheres, or surface area of spheres. In addition to describing the mathematics of the difference in the volumes of the spheres whose radii are the radius of the initial circle (that parameter, which is invariant as the slant of the cone through the transformation of a circle into a cone) and the base of the cone, I can even discuss the change in volume of the spheres whose radii are the base of the cone and the height of the cone.

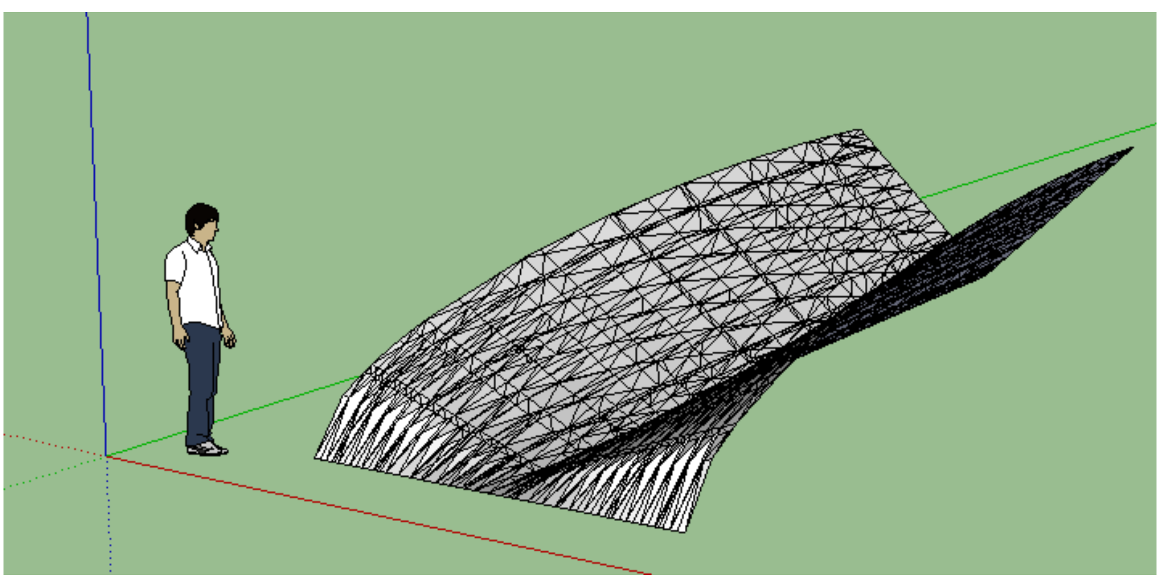
Figure 3 – Difference in Volumes of the Spheres with Radii as the Height of the Cone and the Base of the Cone (with Substitutions from the Lemmas of the Initial Pattern of Difference in Circumferences)

$$(4/3)\,\pi\left(\frac{\sqrt{4\,\pi\,r^2\,(\theta)-r^2\,(\theta)^2}}{2\,\pi}\right)\hat{}3-(4/3)\,\pi\left(\frac{2\,\pi\,r-r\,\left(2\,\left(\pi+\sqrt{\pi^2-\pi^2\,\mathrm{Sin}[\beta]^2}\right)\right)}{2\,\pi}\right)\hat{}3$$



We can describe surface layout of a slanted surface by the equation for a change in circumferences of two circles representing the encompassed area of a sample of the image from the ambient optic array seen in Figure 1. In this way, it is useful for describing how something is perceived. I perceive 3D objects with surface areas, not just like photographic depicitions. Gibson says that, “perhaps the clue we are seeking lies in the invariant properties of such a continually changing retinal image” (The Visual World, 154). For example, moving away from a circular disk, it projects a smaller and smaller image on the retina. The difference in the circumferences of the circles projected on the retina can be thought of as containing an invariant parameter upon application to the Pythagorean theorem. The transformation described through difference in circumferences applied to the Pythagorean theorem is the mathematical description of the continually changing area of the sample of the ecologically relevant contour as well, for one is visually sampling contour of an area of the array (the array can also be thought in terms of a display, which is the delimited optic array to a station point (thus, all the diagrams in *The Geometric Pattern of Perception Theorems* (Emmerson, 2009) can be thought of as delimited optic arrays)) in which each point in the atmosphere is a location of ambient light. For example, the height of the cone delivers a specific contour. Such an object structures the light in the ambient optic array as illustrated in Figure 4.

Figure 4



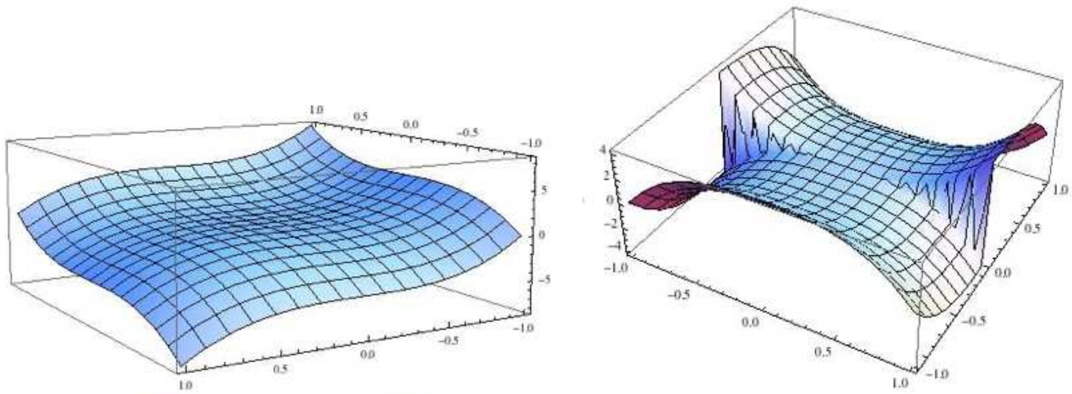
Gibson suggests that, "it seems likely that analysis in terms of area and intensity within the two dimensions of an image would be more profitable for the study of vision" (The Visual World 113). The geometric parameters of a change in circumference of the sampled image would be a progression in such a direction. It is important to emphasize that the difference in circumferences of the two circles is a cue for depth perception, because, in simple cases, it can refer to difference in the size of an image projected onto the retina of a disk moving away in the terrestrial environment. However, we differentiate between the system's use as a device for describing surface geometry (surface layout) and its framework and use as an abstract geometry.

Mathematical analysis in ecological optics comes also from the geometric correlates of the parameters of our environment as well as establishing conscious patterns for describing the perception of depth within it. The mathematical parameters of depth provide a language for discussing the psychophysical parameters of what we see, because perceiving time involves the conscious experience, and space is inseparable from time. Depth, specifically, is, "not built up out of sensations but is simply one of the dimensions of visual experience" (The Visual World, 108). Thus, mathematical analysis of depth and the relationship of its correlates from the eternal, internally and externally experienced, geometric truths are of importance to the study of visual perception.

Gibson says that, "*the gradient of density is an adequate stimulus for the impression of continuous distance*" (The Visual World, 67). In figure 1, we can see that the density of the gradient of the image of the surface layout is certainly a factor in the perception of continuous distance and that the space between points on the grid farther away is denser than those closer to us. Figure 1 is also what Gibson calls an example of a fore-shortened surface, “a slanted surface manifests a one-way compression of texture – an increase in mean density along one dimension of the projected image relative to the other” (The Visual World, 173). The surface is delivered as slanted in the box, but the box can be rotated, and rotation illustrates this principle stated by Gibson.

Moving on from figure one, I can go on to describe the change in circumferences in terms of Pythagorean theorem and show that the conic sample of the optic array contains a structure that provides an affordance for perceiving curvature of surface layout within a sample of the present being of the array, which exists in and of a perception of surface layout over time. Figures one and five show us what happens when we place the change in volume of a sphere in terms of the variables involved in the change of circumference of two circles, which is the key element of the structure of the *conic sample* of the array. Within this sample, there is a structure that affords perception of contoured layout.

Figure 5



$$((4/3)\,\pi\,(r)^{\wedge}3 - (4/3)\,\pi\,(\eta)^{\wedge}3) :=$$

$$\left((4/3)\,\pi\left(\frac{2\pi\eta}{\sqrt{4\pi\left(\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right)-\left(\frac{2\pi\left(r^2+\sqrt{r^4-r^2\eta^2}\right)}{r^2}\right)^2}}\right)^{\wedge}3-(4/3)\,\pi\,(\eta)^{\wedge}3\right)$$

In the first image of Figure 2, we can still perceive the shape of the gradient, but we recognize it as having two possible forms (ignoring the box around it), and one of them is favored from just a single capture. The second of the two images, has fewer tendencies toward multi-variance. In true depiction of depth, stable shapes are perceived, and in perception of form, there is still possibility for perception of the depiction of surface layout's having two possible actual shapes or two modes of depth through density of the gradient. This means that accurate depth perception is dependent on the determination of the actual 3-D shape, while form is present within multi-variant (see the Gestalt section of *Overview of Perceptual Theories)* depictions of shapes.

We should first note how sharp the gradient lines are at the steepest edges of the second depiction of surface layout through the expressions found from the expressions of a difference in circumferences applied to the Pythagorean theorem for the difference in volumes of two spheres. These are 3D objects, thus they are better descriptions of real surface layouts than photographs. The many combinations yield a wide variety of expressible visual forms, however, from one of them, we can note the sharp contour lines at the steepest transitions of the surface layout. Gibson's theory supposes that, "the steepest gradient of intensity would be the condition corresponding to the sharpest visual contour" (The Visual World, 113). The transition of the two "saddles" shows pronounced, sharp visual contours, and is the steepest gradient of change. So, we see a direct correlation to Gibson's 1950 work The Perception of the Visual World and the *Geometric Pattern of Perception Theorems* (Emmerson, 2009). Gibson's theory is one of introducing partially non-fixed terminology, examining the ecologically relevant world and the structures within it to better understand how we perceive. The structure of the conic sample of the ambient optic array can be visualized at any size of "space," placed in terms of real numbers or absolute value of distance in the complex plane, and usually it makes sense to discuss visualization of the system only at certain intervals Pi, because these signify entire transformations of a circle through a cone to where the height equals the initial radius (the position of the locus of perception).

The lines that extend from a point on a longitudinal or frontal surface to the eye are describable in terms of the difference in circumferences applied to the Pythagorean theorem. A circle can be drawn through any two points. In one of Gibson's figures, we see a representation of the three dimensional object's projection onto a picture plane, (The Visual World, 34).

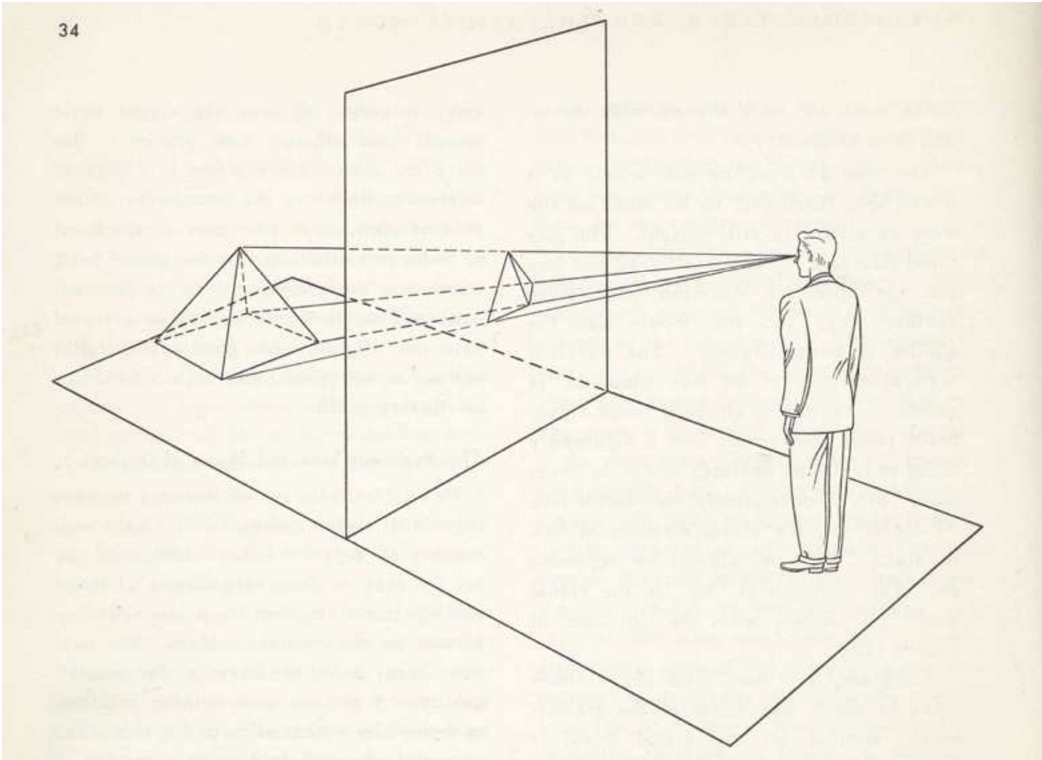


FIGURE 7. An Object Projected on a Plane

Please see *The Overview of Perceptual Theories* paper (Emmerson, 2010), pages 67-68, for an overview of gravity's relation to the visual field. In Gibson's figure 7, he makes a distinction between, "the visual world, (which) contains depth shapes, (and) the visual field, which contains projected shapes" (Gibson, 34). It is just simple to note that there is a circle that can be drawn around any two points on the object or the projected image – the smallest of which is that which has the length between the two points as the diameter of the circle. However, we may also ask the question of, "what is the expression of the difference in circumference of the smallest circle made between two points on the object and the corresponding smallest circle made between two points on the picture plane?"

In The Visual World, Gibson outlined a number of ideas regarding the "reversal of shape due to reversed gradients of density" (The Visual World, 98). He says that these inversions of depth perception, "occur, of course, only when gradients of texture are absent or ineffective. The latter gradients are never equivocal with respect to depth" (The Visual World, 99). The mathematics delivers the total expression for the object with contoured layout in depth. A single frame of the image of the surface layout is still subject to multi-variance in the leftmost of the two images, but this effect is greatly reduced upon developing an understanding of the ability to change ones orientation with respect to the actual surface. It is then that the ability to perceive it as one shape of surface alone is definitively enabled. However, this is due to two-dimensional representation on the computer screen. Gibson says that, "'a dinner plate' always looks like a dinner plate – the three dimensional shape in the world" (The Visual World, 35).

The sample of the array is capable of capturing very small orientations within a change of the surfaces within the array through the perceiving the gradient of texture density. Gibson says that,

> "The general condition for the perception of a longitudinal or slanted surface is a kind of ordinal stimulation called a gradient. The gradient of texture has been described, and it has been suggested that gradients dependent on outlines, a gradient of retinal disparity, a gradient of shading, a deformation gradient when the observer moves, and possibly others, all have the function of stimulus-correlates for the impression of distance on the surface" (The Visual World, 76).

The gradient to which Gibson refers simply means a correlation of parameters. The issue of mathematical analysis also arises in the understanding of gradient density and the pattern of the increase of that density (Figures 1 and 2). The structuring of the information of the stimulus-correlate is contained within the sample of the array over time and its experiential relations. The form of a being of change in circumferences of two circles provides a structure for visual texture. We currently think of trigonometry in two expressions for length – the trigonometric functions like sine, tangent, cosine and secant, and through the Pythagorean Theorem. The mathematical expression of a theory of the unified perception of that length has three simple expressions.

$$\frac{\sqrt{4\pi r^2 \theta - r^2 \theta^2}}{2\pi} = r \operatorname{Sin}[\beta] = \sqrt{r^2 - r_1^{\,2}} = \eta$$

Descartes and Gibson would most likely have only had access to the last three of these expressions for length (the trigonometric sine function, the Pythagorean theorem, and the numerical length in meters. In the first expression, the height has been calculated from difference in circumferences of two circles equaling an arc length to form a cone whose height is orthogonal to the center of the circle of the base of the cone. This arc length is equal to the difference in the initial circle and the base of the cone (there is no cone if it is the initial circle or totally collapsed height). As the height of the cone changes the initial radius is invariantly propped up as expressed by the Pythagorean theorem. The formal ontology of perceived difference posits that, "*visual space*, unlike abstract geometrical space, is perceived only by virtue of what fills it" (The Visual World, 5). The perceived circumferences fill the visual space related to the impressions they have on the perceiver, and the change within that space has pure, geometric gradients. However, in this, we see a theoretical bridging of abstract, analytic geometry and surface geometry, because the system describes both parameters of analytic geometry as well as gradients of perceived surfaces. Gibson says that, "How the empty space is seen gets no explanation unless one assumes that distance is computed in the brain by a mechanism similar to that of the optical range-finder – every act of perception being an exercise in trigonometry" (The Visual World, 177). The difference in circumferences of two circles is a Gestalt (univocal) system, which is an inherent truth to mathematical "space-time" relations when interpreted through time measurement as cyclicality. It is a newly trigonometric relationship, but also one that resounds in consciousness and the perceived forms.

In order to see the significance of the *Geometric Pattern of Perception Theorems* to philosophy in general, we will go to Gibson's discussion on Nativism and Empiricism provided in The Visual World (Gibson, 1950). In this entire discussion, there is no mention of the difference in circumferences of two circles, but there is a considerable history of thinkers up until Gibson's time. Gibson says, "The space of this universe (that conceived by Sir Isaac Newton), it may be noted, was empty Euclidean space, defined by the three dimensions, of the Cartesian coordinates" (The Visual World, 15). However, in the use of "Euclidean space" by those applying it in physics and the problems of *geometrical* space perception, there was a remaining in the natural attitude, in which space and time are still posited somewhat independently from one another. Instead of "space-time," I propose that we perform analysis through geometrical space perception by using a purely geometric framework in which, if really necessary, we can go to the notion of time or space, but really, what we have is passing angle measure and linear "scaffolding." This is what can be shown by purely formal ontology through eidetic phenomenology. Concepts of space and time are added afterward, and are more ambiguous than pure geometry. For Gibson, it is clear that visual perception is an experiencing. Thus, the experiencing of the mathematical relationships may be a key to visual perception. Gibson says that, "a transformation can be applied to a given pattern without affecting certain of its general properties" (The Visual World, 153). Transformation of a circle into a cone is a linguistic, though timelessly true structure that allows us to discuss steps, gradients, and inflections. Gibson says that,

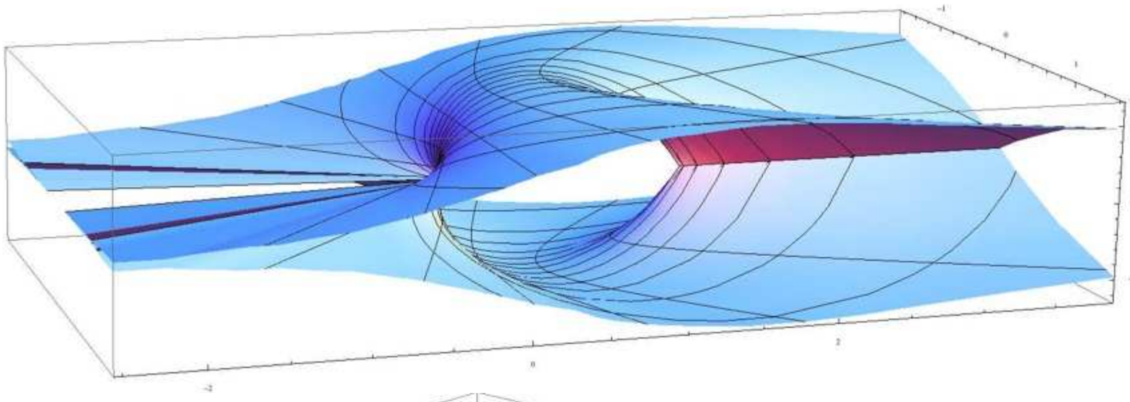
> "steps, gradients and inflections of a gradient are mathematical facts which are the same no matter where they are located. In an array of adjacent nerve-cells an increase in excitation from one cell to the next is a kind of serial order which can be transposed without causing a permutation or an inversion of the order" (The Visual World, 151).

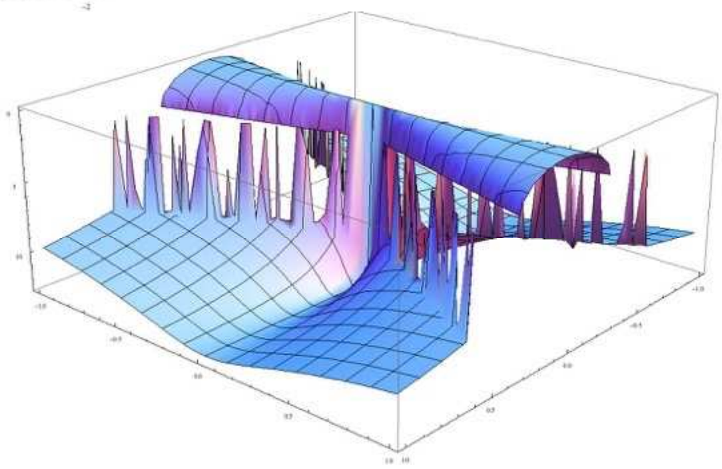
The increase in excitation is a quantifiable distance when mapped in the Cartesian coordinate system, and is subject to the same trigonometric manipulations as any distance. The transformation of a circle into a cone describes awareness of "space-time," increase in excitation (because it is a gradient), and general change in the size of the image projected on the retina. Thus by elucidating the structures of these gradients, steps and inflections, I hope to be able to show that a change in area of the image on the retina of an object out in the world contains a pattern present at all levels of visually perceptual experience.

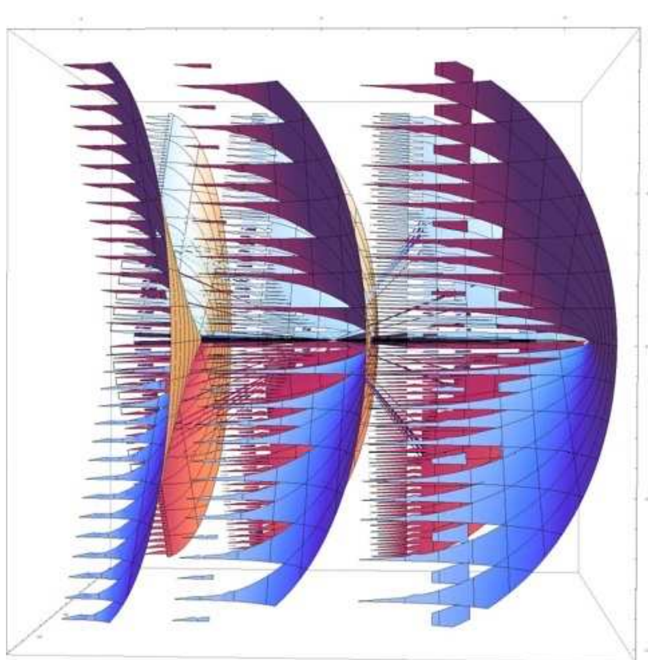
3.2) Textures within the Structure of the Gradient of a Conic Sample of the Ambient Optic Array or the Array of Adjacent Nerve Cells

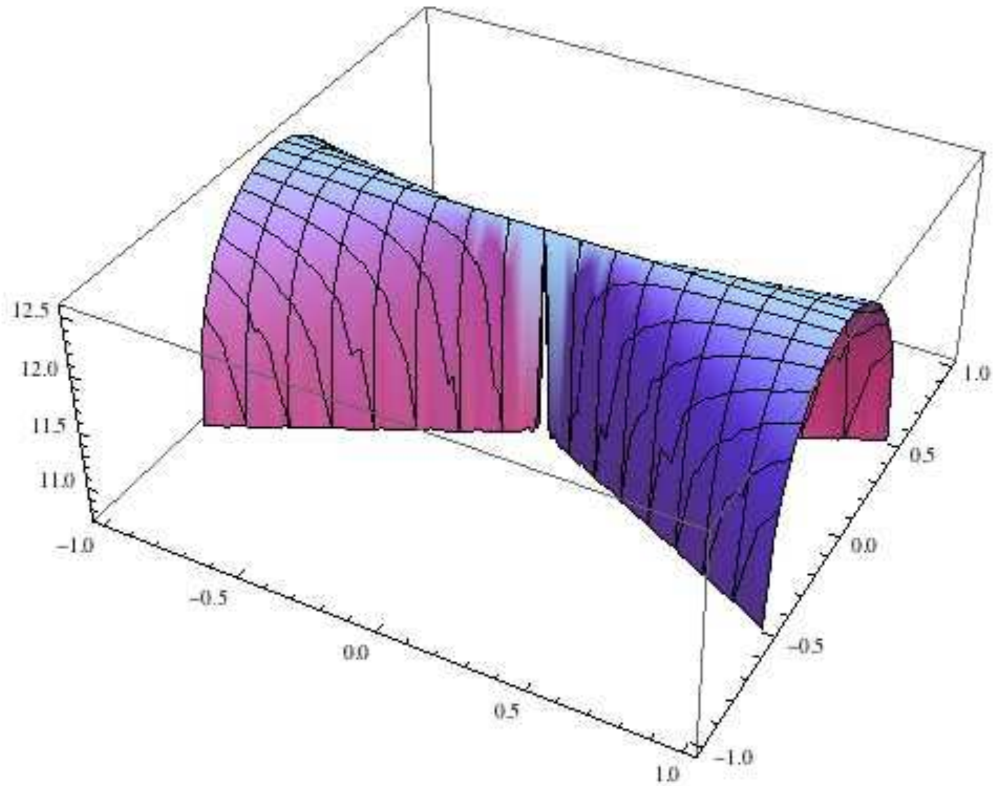
A key component to visual texture can be described mathematically, “a gap may be defined geometrically, it can be considered a fundamental component of texture” (The Visual World, 110). This idea of a gap is present in structures of the proposed mathematical theorem of the relations of a conic sample, and when graphing a number of variables in the gradient of space-time suggested by the correlates of the theorem’s parameters in such terms as difference when angle measure is interpreted as time, much of the visual texture delivered by such hypotheses are due to gaps. We will briefly display a few examples.

Figure 6.1 -6.4









With the ability to place distance in three provable simple expressions instead of just two, we can say that intensity is geometrically provable to be able to be placed in such terms as the difference in circumference applied to the Pythagorean theorem when mapped as a psychophysical function (as a distance mapped in a Cartesian coordinate system). "Is it possible that the micro-gradient of intensity is the fundamental stimulus underlying not only the phenomenon of a margin or contour, but also the phenomena of texture, visual acuity in its different modes, the focused image, and clear vision in general?" (The Visual World, 110). The micro-gradient of intensity can be described in a multitude of variations, and the forms of the system of universal science can be scaled at any size. These variations all have structures that contain gaps, filling in over revolution (time), texture, contour, and afford patterns of substitution of and from structures of the lemmas of the system.

Theorem 2 yields a 30-60-90 triangle that signifies the position within the cone at which the instantaneous velocity equals the average velocity. Various combinations of the 30-60-90 triangle can account for the shape of a visual field of approximately 150 from top to bottom of the visual field and 180 degrees from side to side. This corresponds with Gibson's theory of the ambient optic array that, "a bundle of visual solid angles at a point (a point of observation) is called an *ambient optic array*" (Vision and Mind, 81), but is only an approximation and coincidental meaning to the form of the solid angles of the boundaries of the visual field.

For Gibson, retinal gradients are variables of stimulation. "The psychological quality of shape which changes with such a continuous transformation (closed form to closed form by a perfectly gradual or continuous change) is what needs investigation" (The Visual World, 193). A circle is a closed form, and the difference is expressible by a continuous transformation, "spatially" through arc length. Gibson says that, "the geometry of transformations is (therefore) of considerable importance for vision, and it is conceivable that the clue to the whole problem of pattern-perception might be found here"(The Visual World, 153). The proposed mathematics for the sample of the optic array as a difference in circumferences of two circles applied to the Pythagorean theorem is such a continuous transformation. The model of the geometry of difference in circumferences and the expressions for the variables of angle (cyclicality), length, breadth and depth can all be found in simple terms of each other. Specifying patterns of substitution within such equations delivers a variety of objects, which are like smaller studies of the being of the gradient of perceived space-time elucidating the quality of the single shape (a circle's being folded up into a cone) of the sample of the array through time if measurement of time were interpreted through cyclicality. Husserl's account of perception is related,

> "The positive sciences operate exclusively on the plane of the theory that can be fashioned straightforwardly, when the theorizer directs himself to the province of cognition as his theme - fashioned, that is, by the continuous categorical forming of experiential objectivities belonging to the province, as they come within the scope of determining processes of thinking, and by the systematic connecting of the formations thus acquired, to make cognitional formations at higher and higher levels: the openly endless, and yet systematically unitary, edifice of the scientific theory of the province" (Husserl, 240).

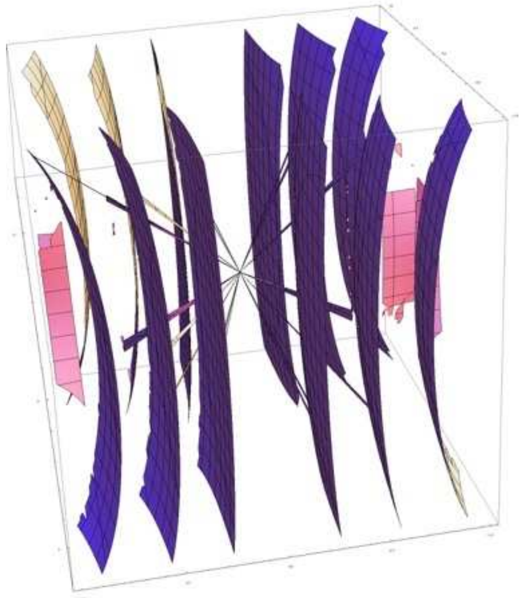
What is delivered by the theory of the difference in circumferences applied to the Pythagorean theorem is a multitude of objects, expressed from terms of complex analysis as well as the expressions found from hypothetical case scenarios involving cognizance of time measured cyclically, because it is perceived or at least measured cyclically in the environment (we evolved in an environment subject to moon phases, Earth's rotation, rotation of the clock (civilization), and change of seasons, etc.). The experiential objectivities found from graphing the various expressions for the transformations of the conic sample are descriptive of contour, surface layout and visual experience. The notion of invariance is also a mathematical idea.

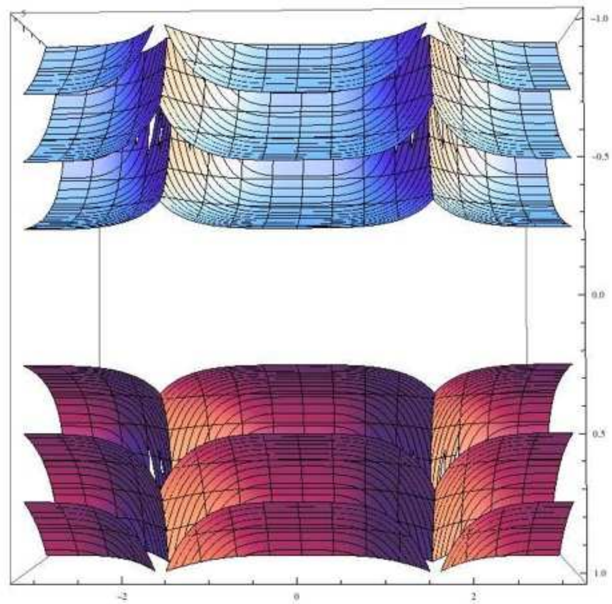
Gibson noted the physical fact that, "retinal stimulation is a projection" (The Visual World, 58). A projection has many specific meanings in the field of mathematics, but in Gibson's terms, it generally means that, "we live in a world which stretches from here to there and this fact remains in need of explanation" (The Visual World, 58). I like to think more semantically with respect to this issue and point out that there is here *through* being. I posit that this fact can be explained through difference when time is measured by the degrees in a clock, and an initial radius is invariant as the hypotenuse of a right triangle formed by applying the Pythagorean theorem in an orthogonal manner to the center of the base of the cone. In literal spatial terms, the difference in circumferences applied to the Pythagorean theorem, "picks up" the invariant from the initial circle, changing the circumference of the circle (the base of a cone, when having a height of 0 is the initial circle) by placing the initial radius as the slant of the cone, leaving it invariant through the transformation.

Notably, imagine the visual contour of paint coming off of a wall. It comes off in little flakes or strips. Perspective shots might capture some depiction of the depth, however mathematics helps us describe this visual contour better than a photograph in many respects, and thus more like actual perceptual phenomena. A photograph captures shadow, hue, and some elements of surface layout, but cannot deliver the perception of true depth. Mathematical interpretation of what I call a conic sample of the array yields a more accurate language for discussing contour and gradient of the visual, perceptual contour. Through describing the mathematics of a change of something simple like circumference, a host of possible expressions is yielded, including valid interpretations of change in higher dimensions or change in area.

With reference to the retinal gradient of light, Gibson says that, "the retina is probably to be conceived as an organ of the body which is sensitive to grades of light, not points of light" (The Visual World, 114). This means that when we do a mathematical analysis of ecological optics, we need to talk about the ecological context of the space being perceived that is relevant to the gradient of change in surface layout.

First, however, it is useful to discuss the gradient of change itself in an expression for the height of the cone. This is important, because a distance of a point on a surface farther away from me has an associated initial radius and circumference that is different from a point closer to me. However, the distance to a point closer to me is contextually and texturally relevant through understanding the change in circumferences from the assessment of the point farther away from me, because the relative distance away from me that is closer would be a distance away at which the height of the cone of the distance farther away from me had not folded all the way up. The height of the cone is a general expression for distance, but contains contoured information within its gradient. Through an expression for the height of the cone, we can describe visual texture like the paint coming off of a wall as well as other visual textures, like that of a tiled roof.





These are three-dimensional surfaces, and therefore, they are better linguistic devices for discussing surface layout and visual texture than photographs, because they contain a more useful kind of information – that of actual three-dimensional surface layout. Gibson says that, “*there is always some variable in stimulation (however difficult it may be to discover and isolate) which corresponds to a property of the spatial world*” (The Visual World, 8). A few, simple properties of the “spatial” world have been described through mathematics in *The Geometric Pattern of Perception Theorems*. This mathematics works as a language for discussing the relevant variable in stimulation to different surfaces and is a property of the “spatial world” by insight into pure geometry.

The theory of the cone delivers expressions of contour through colored, contoured diagrams. There can be unexpected changes in the coloring of these diagrams along a continuous surface as well as spontaneous shimmering or flickering of rainbow colors in the (pointed) center of a matte, solidly colored surface. The following passage is extremely relevant to the idea that the structures of the sample of the ambient optic array are a language in and of themselves for discussing visual contour,

> "In either event, whether the reflecting particles are structural or chemical or both, they will reflect light differentially and the image of the surface will consist in an array of cyclical changes in light energy which we experience as variations in brightness or hue. The optical image, implies a correspondence between two sets of abstraction, reflecting points and focus points, such that the character of the light at the former is duplicated at the latter, point for point. The structural and chemical cycles of the surface, therefore are projected on the retina of color in corresponding order. Footnote 2 of his: The assumption is that a texture can be analyzed by plotting it in two dimensions that is, by specifying the alternations or repetitions of light stimulation along two axes. This is what is meant by cycles. Admittedly this assumption needs mathematical study. A texture cannot be analyzed conveniently in terms of lines, nets, grids, or other patterns with which the writer is familiar because these are themselves special cases of texture" (The Visual World, 80).

Difference in circumferences of two circles as an arc length provides such a mathematical study. The mathematical analysis of a cone contains a cyclic element, the angle theta out of which a sector is "taken" from the initial circle so that this circle can be folded up into a cone. We can access ideas like nets, grids, and other patterns of "space-time" visualized through pure geometry. The revolutions of a circle are mathematically described by the scale of $n*2\pi$. With each revolution of the system, n, the repetition of light stimulation makes us more and more confident of the presence of the stimulation through accretion of contour and surface layout. In essence, the different gradients, contours, and visual textures contained within the mathematics for a difference in circumferences applied to the Pythagorean theorem are special cases of texture and can be accessed with modern computing power.

Through the model of the cone as relevant to the area projected on the retina as well as the geometry of the visual field in general, we can analyze the two dimensions of an image in terms of the change in surface area, volume, or circumference of a *conic sample* of the ambient optic array by applying the known expressions for the initial radius and the changed radius from having solved the difference in circumferences to discuss the meaning of that change in general "sample area" over time. This is related to what Gibson says in The Visual World, that, "a transformation can be applied to a given pattern without affecting certain of its general properties" (The Visual World, 153). The pattern in formal ontology is capable of being used to describe the second as well as the third dimension and contains implicitly within it many more dimensions.

In The Perception of The Visual World (Gibson, 1950), his discussions of visual perception were still based on the retinal image. He provides a description of, "*The Relation of a Picture-Plane to its Retinal Projection"* (The Visual World, 79). Gibson uses the next figure for his discussions of the picture plane. Certain configurations of this picture plane have useful mathematical expressions.

It is important to note that it is an area of the retina that is affected by radiant energy, not just a single point. Gibson's reason for believing that the geometry of transformations is important to visual perception is that, "transformations are usually represented on a plane, however, whereas the retinal image is a projection on a curved surface," (The Visual World, 153), and that, "the actual retinal image on a curved surface is related to the hypothetical image on a picture-plane only by such a non-rigid transformation" (The Visual World, 153). A transformation in geometry is the mapping of one point onto another. Isometries, "are *defined* as the transformations that preserve distance"[17] (The Four Pillars of Geometry, 145). In essence, the distance of the initial radius is preserved through the transformation of a circle into a cone so long as the height is orthogonal to the base of the cone and the initial radius is always the slant of the cone. Next, we see the diagram to which Gibson was referring when considering the notion of a transformation onto a picture plane.

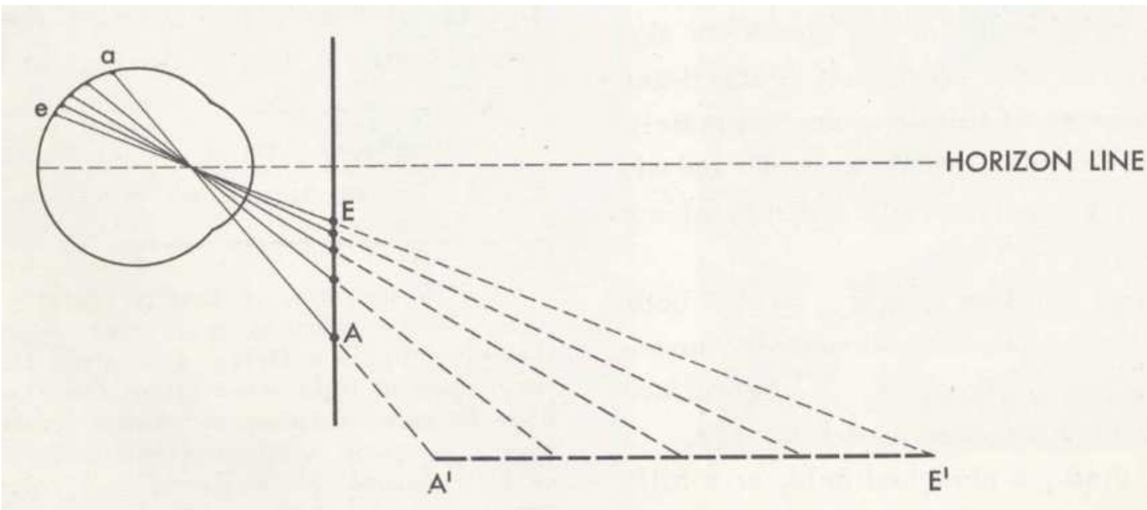


FIGURE 28. A Picture-Gradient, its Retinal Gradient, and the Longitudinal Surface Corresponding to Both

(The Visual World, 79).

In being preserved, the initial radius is considered an invariant. Stillwell comments about the picture plane that, "the line from (-1, 1) to (n, 0) crosses the y-

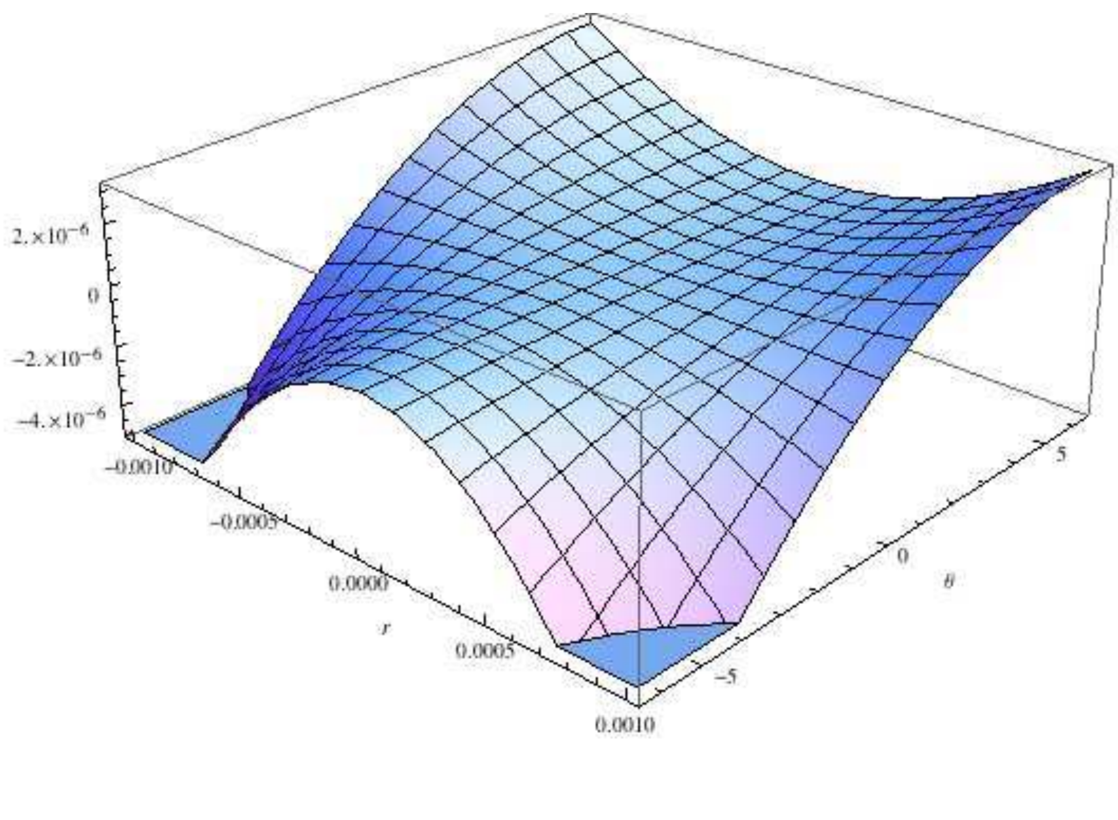
[17] Stillwell, John. *The Four Pillars of Geometry (Undergraduate Texts in Mathematics)*. 1 ed. New York: Springer, 2005. Print.

axis at y = n/(n+1)" (The Four Pillars of Geometry, 91). This supposes that the eye is approximated like a point and that it is at the position of (-1, 1) in the Cartesian coordinate system. In the "coordinate system" described by *The Geometric Pattern of Perception Theorems* (Emmerson, 2009), the y-axis in general is described by the height of a cone. In relation to this diagram, in terms of the y intercept, the height of the cone would be changing with respect to both the initial radius (slant of the cone) and the angle taken out of the initial circle (the angle made between the line from the eye to the x-axis changes is a function of solely the angular amount taken out of the initial circle). Further mathematical analysis of optical infinity with relation to the horizon line and geometric system is needed, but perceived difference in circumferences as an arc length will be a useful formula. Gibson says that, "only because light is structured by the substantial environment can it contain information about it" (Ecological Approach, 86). The basic equation for an arc length as a difference in circumferences describes an even surface layout. Thus, for even surfaces, the equation that delivers that surface may be used as a linguistic device (in combination with rotation, or specifying the "adumbration" of the viewed surface) for describing the structuring of the light in the environment relevant to the perception of even surface layout. The expression for "phenomenal velocity" tells us "how" motion in general is essentially structured, thus this includes the motion of light. However, this still needs specific interpretation.

Gibson's later theories, present in The Ecological Approach to Visual Perception, departed from talking about the retinal image as necessary for visual perception by introducing the ambient optic array and psychologically philosophical concepts like optical information. However, if a small enough sample of the retina is taken and the light stimulating it, a sample of the “retinal image” can be approximated by a circle. So, although the retina is curved, if we are talking about a circular area on the retina with radius equal to the size of a single wavelength of light, the curvature will not be effecting the patch of light and the change in the area of that circle would represent a change in the wavelength of light itself as it enters the eye. However, it is currently more admissible to talk about a change of an initial radius of micrometers – something closer to the level at which we perceive environmental textures, and still relatively unaffected by the curvature of the retina. However, Gibson does say, “that a spot of light is a stimulus for perception by virtue of its ordinal location and not by virtue of its anatomical location, can be illustrated by experiment” (The Visual World, 56). So, as far as a wavelength of light effecting an area of the retina, there may be reason for thinking that the area of the retina is stimulated on the order of the wavelength of light and within its signifying pattern within its hierarchical structuring. A signifying pattern would be a surface structuring the light in the environment.

Pure geometry provides a framework in which information, discreet, as well as of surfaces and general correlation of parameters is accounted for without resorting to a notion such as space-time (unless “time” parameters are differentiated from “space” parameters as angle parameters are differentiated from “distance” parameters). I may remain within the purely essential, universally ordered, experiencing of world, described through the language of geometric transformation and the essential parameters of purely geometrically described experiencing. What is described is a series of nets that get filled in with contoured surface information as more revolutions of the system take place, i.e., the simple notion that a single wavelength of light delivers less surface information than many wavelengths of light over time, i.e. constant stimulation as necessary for accretion of perceived surfaces.

Gibson might support such an inquiry into the study of change in area of an image, because he says, "it seems likely that analysis in terms of area and intensity within the two dimensions of an image would be more profitable for the study of vision" (The Visual World, 113). In Gibson's 1950 theory outlined in The Visual World, "from the standpoint (of two dimensional analysis of the image), analysis begins, not with geometrical points but with gradients (in this case micro-gradients and their slope)" (The Visual World, 113). Below, we see a micro-gradient of the change in area of the image of a circle.



Experiencing of surface layout and color, visually, is increased by the impact of several wavelengths of light on the receptors in the retina over time. This can be seen by the filling in of the net over multiple revolutions of the angle, theta. However, Gibson also says that, "nothing gets into the eye but radiant energy," (The Visual World, 54), and that, "only because it (the radiant energy) is focused is it specifically related to the object" (The Visual World, 54). The change in area of a patch of color on the retina can be a cue for an approaching or receding object. Experiencing of geometric parameters directly connects to the perception of the eidetic geometric relationship. Gibson says that,

> "The pattern of excitation on the retina might be defined either in terms of the units of the mosaic or the units of excitation as such. The former pattern is embodied in units which have an anatomical meaning; the latter pattern is not embodied in this way but nevertheless it is definable in terms which have mathematical meaning" (The Visual World, 56).

Gibson also says that,

> "We can be fairly certain, however, that the visual world is dependent on eye movements and is not seen as the result of a single fixation or a momentary visual field. It must correspond, therefore, to *successive* patterns of excitations on the retina, united perhaps by a kind of immediate memory. These patterns will overlap one another anatomically as the eye moves, and the basis for the visual world, therefore, must be what has been called the *ordinal* pattern of excitation rather than the *anatomical* pattern. If it be assumed that there *is* an ordinal pattern which keeps its integrity during eye movements, then it is possible for any part of it to be brought to the center of the anatomical mosaic and registered in fine detail. A complex of this sort, over time, would be both uniformly differentiated an unbounded, and might therefore provide the basis for the perception of the visual world" (The Visual World, 57).

If the excitation as such could be described by the characteristic gradients of the "transcendentally phenomenological" (universally scientific) eidetic geometry present in *The Geometric Pattern of Perception Theorems* (Emmerson, 2009), the ordinal pattern is thus related to the excitation as such, because the pattern, "keeps its integrity" (The Visual World, 57), by not disintegrating when applied to a different transformation.

Earlier, it was stated that the difference in circumferences of two circles equals an arc length, and that this is an eidetic, geometric pattern. Below, we see the diagram that Gibson uses to discuss, "The Qualities of a Simple Line" (The Visual World, 195).

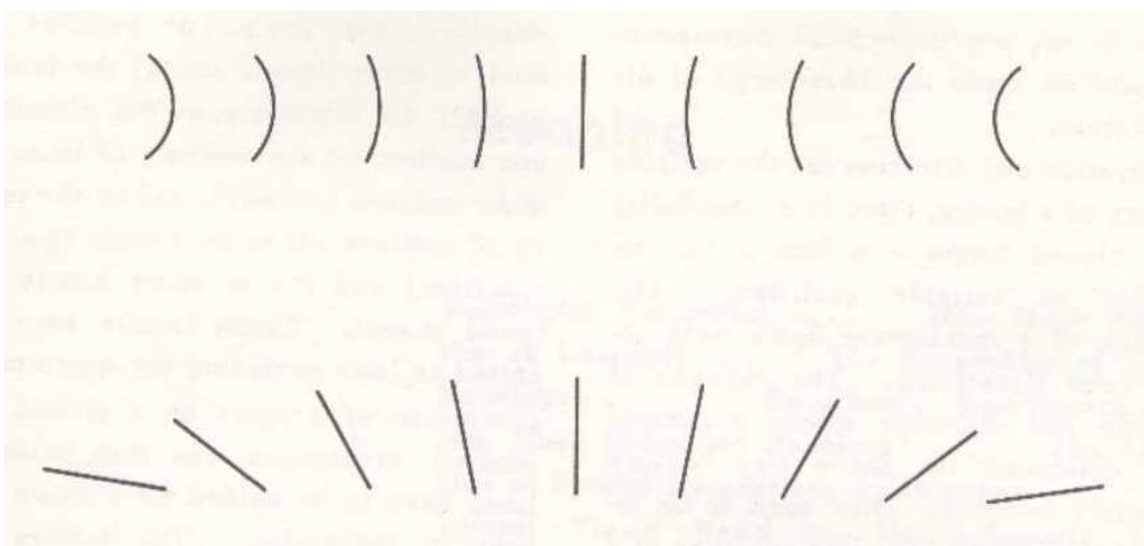


FIGURE 76. The Qualities of a Simple Line

(The Visual World, 195).

The increase in curvature can be traced to a change in curvature of an arc length from a difference in circumferences of two circles. This change in "contour" (the curve increases) can be described by the difference in circumferences of two circles in the case of a simple line (distance between two points (two points equally far apart on the circle)). The changing slant can be correlated to the change in height of a cone and the angle made between the base of the cone and the hypoteneuse. Of the lines in the above diagram, Gibson says, “they could be termed the quality of direction (linear slope) and curvature (linear shape)” (The Visual World, 195). The parameters of a change in arc length compared to an invariant distance between two points on a circle may describe the change in curvature, but Gibson says that, “the changes of curvature and direction around a contour which determine its shape may become enormously complex” (The Visual World, 196). It may be possible that by the joining of curves, even curves changing at different rates within the line would be able to be constructed. Gibson asks, “if one specifies the direction and curvature of a short visual line has one not specified the entire experience?” (The Visual World, 195). Gibson suggests that we analyze the change in the curve through calculus, “the first and second differential are sometimes explained as the slope of a curve and the sense in which the slope is changing” (The Visual World, 195). This method may work, but there is also an essential geometric correlation that describes the transformation of the line from left to right in terms of curvature and angular orientation. The derivative of these functions for either angle or difference in curvature may be taken in order to find a mathematical expression for the “rate of change” of one variable with respect to another, because that variable would be a function of the other variable. Gibson says that, “the changes of curvature and direction around a contour… seem to be integrated or organized to yield qualities of a higher order" (The Visual World, 196). With *The Geometric Pattern of Perception Theorems*, the change in curvature can be related to philosophy, which is higher order in the sense that it is transcendental phenomenology, grounded in formal ontology as universal science. The changes of curvature are changes in inflection of the line.

IV. Conclusion

Gibson's theory looks as though it is grounded in formally ontological principles, and although his ideas developed and changed substantially from those presented in The Visual World (Gibson, 1950), it may prove useful to take into consideration some of his initial ideas on perceptual theory. Specifically, I recommend more study in the formal ontology of optical infinity so that we can better understand the ontology of the ecological context of the perceiver. The cone described in *The Geometric Pattern of Perception Theorems* may provide a univocal, whole, interconnecting framework in which optical infinity may be able to be studied through thought from simple diagrams like the picture-gradient's relation to surface plane, showing how different cones (not like, "rods and cones," but geometrically defined transformations) within that system have different correlates, meanings, and significances, relating to the position of the perceiver in the world, for perceived changes in surface layout.

The picture gradient is actually a useful way of studying the difference in circumferences itself. If one considers the horizon line as depicted in the diagram, Gibson's, "Figure 28" (The Visual World, 79), as a line in the plane of the diagram, and if one considers the line made between the eye and the x-axis (the orthogonal, perpendicular line form the eye to the x-axis), one can see that the intersection made between these two lines is at ninety degrees. To remove ninety degrees of a circle is to signify the position at which instantaneous velocity equals average velocity of the height of the cone. This notion must be correlated to optical infinity through consideration of how the horizon line is transformed onto the picture plane with further research in surface geometry, though Gibson's theory and insight into the fact that, in the diagram of the picture-plane, the eye can be placed on the horizon line, and be orthogonally positioned to the x-axis on the ground is useful insight for beginning further research in ecological optics.

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Parker Emmerson

Theories of Perceptual Philosophy and their Correlations

1. Perceptual Theory

Perceptual theory attempts to answer the questions of how we see our world through the senses as well as the acquisition of knowledge about reality through perceptual experience. An overview of several theories of perception will be outlined in the body of this paper, and I will conclude by covering some of the accounts to which I am most sympathetic. In this paper, I will provide an overview of the theories of Sense Data, Husserl's Phenomenology, Gestalt, Computational, and Gibson's Ecological Optics. These theories can be shown to connect to each other through the ideas of information (optical and signal), phenomenological description, and formal ontology.

From Descartes, the modern philosophical project separates subject and object, and thus raises the question: how do we come to relate our perception of objects in the world to the objects? The essence of perceptual phenomena is describable through intuition of geometric insight and geometry; and so, the subject comes to knowledge of the perceptual world as a reality that inspires the subject to inquire. In the method of phenomenological geometry (a geometry that describes the experience), we see a technique for studying perception that can account for illusions of various sorts, while also being related to ecological optics through the notion of a surface layout (contour). In the conclusion of this paper, and references throughout, I will draw parallel ideas from the text to my work.

2. Sense Data

According to Sense-data theory, our perceptions are based on sense-data, which are mind dependent objects of which we are directly aware. This general account of sense data gained prominence in the work of William James[1] toward the end of the nineteenth century. James sought to describe the mind as purposive and selective, steering away from associationism, which supposes that mental processes work through association of one mental state after another. A modern day interpretation of the idea of sense data proposes an argument that says that, because it is theoretically impossible to tell the difference between perception of an actual object and an hallucination of the object with the exact same sense data, there is no way of acquiring direct access to the world outside of us. Sense data is a representational account of perception in which the sense datum represents an object in the world outside of the mind. The account proposed of sense data in this section will try to show how the ideas of William James about Sense-data theory, distinct from the argument from illusion, carried over into the discourse of modern neuroscience in perceptual philosophy.

[1] William James (January 11, 1842 – August 26, 1910) was a psychologist and philosopher who had training as a medical doctor interested in Swedenborgian theology.
"William James - Wikipedia, the free encyclopedia." *Wikipedia, the free encyclopedia*. N.p., n.d. Web. 5 Apr. 2010. <http://en.wikipedia.org/wiki/William_James>.

The issue that many perceptual philosophers see to be immediately pertinent to their account of perceptual philosophy and that begins their analyses from is the problem of perception of space or the quality of the perceptual space. In James's initial theory on the issue of sense-data, he attempted to understand and clarify the relationship of his experiences with the kind of space that the senses delivered to him. Specifically, William James notes that, "if *all* connections among ideas in the mind could be interpreted as so many combinations of sense-data wrought into fixity in this way ('*The order of experience*' (in the matter of time and space-conjunctions of things) standing for something real and definite) from without, then experience in the common and legitimate sense of the word would be the sole fashioner of the mind."[2] (*The Principles of Psychology*, Vol. 2, 620). For James, space existed without order, and order did not necessitate space. The many differentiated sense-spaces, coming from the sense organs, deliver sensations, and these sensations must, "be *measured and subdivided* by consciousness, and *added* together, before they can form by their synthesis what we know as the real Space of the objective world" (*The Principles of Psychology*, Vol. 2, 145). In essence, we associate what we see with what we can touch, and what we can touch with that which resonates with sound or mathematical understanding (sense in the logical meaning). Locke[3] would probably agree with James, because he first noted that

[2] James, William. *Principles of Psychology, Vol. 2*. New York: Dover Publications, 1950. Print. Page 620. All future references to this source will be made parenthetically in the text.

[3] John Locke (August 1632 – 28 October 1704) was an English philosopher and physician one of the most influential of Enlightenment thinkers. "John Locke - Wikipedia, the free encyclopedia." *Wikipedia, the free encyclopedia*. N.p., n.d. Web. 5 Apr. 2010. <http://en.wikipedia.org/wiki/John_Locke>.

infants must assemble information from interactions with their environment before developing clear, visual perceptions. This is posited in the same sense that an architect sees a building in a different way than a normal person. An architect looks at a surface in a house and sees the trim, but also "sees" how the trim was put together. The electrician sees the light fixture, but also "sees" all the wiring that allows that light fixture to emanate light.

Early accounts of the sense-data theory, like the argument from illusion, propose that when we see a stick in water, we perceive it to be bent, even though it was straight before it was placed in the water. The sense data theory posits that we perceive the environment exactly like it is available to us from the depiction of the retinal image. The idea of sense data means each of the informational points of perceptual space contains a means by which the information within it can be understood by the actuality of the presence of a perceptual object or object of sense.

Sense data have been studied in a number of different ways. One of them is through psychophysical experimentation. This involves the measuring characteristics and experiences and responses of receptors or individuals to the stimulus of hue, brightness, contrast, pigment, image, and many other ways of characterizing the perceptual experience scientifically and quantitatively. Specifically, visually accessible light, an external stimulus, can only affect the system of perception (deliver information to the perceiver) by causing the receptor whose structure is capable of receiving light to evoke certain *qualia* of external sense data. Sense data theory is a descriptive approach to perception that tries to develop a theory of what is perceived.

Sense-data theory draws a distinction between external stimulus information and internal stimulus information. The former is dependent on objective structuring (light travels in waves and these waves are of varying intensity and length). Internal information is dependent on the cognitive structuring in the mind and capability of physiological systems (the nerves in the eye are stimulated as well as neurons). Sense data contain information about the structuring of an actual object of perception, and immediate experience of perception. Thus, it is reasonable to see a correlation between the information contained within the structuring of light in the environment and the perceived parameters of an object in the environment. This relationship may be thought of as a code and a deciphering of that code.

In the development of a sense data theory, questions are raised as to the dependence of sense data on consciousness or mind. It could be argued that, because the sense datum is an objective thing, it is reasonable to think that it may exist regardless of consciousness of it. However, it is often thought that the sensible information in the world is only transferred to being a sense datum upon reception of awareness (inception of persona) of it. In this way, the sense data get into difficult semantic territory, which is not often discussed. The difficulty arises through the idea that the word sense (how we come to knowledge of the correlates of two ideas i.e. "that makes sense") is not so far from the meaning of sense like sensation (the reception of stimuli in the neural pathways). Data is contained within the information available in external stimulus and then transferred to internal stimulus information upon reception.

Bertrand Russell[4] formulated the idea that sense-data means that of which we are directly aware in perception, and their dependence or independence from the mind was not necessarily defined and neither could be assumed. G.E. Moore[5] debated whether or not sense-data were actual surface layouts of objects in the world, but came to no decisive conclusion. Recently as the year 2000, Bermúdez gives an interpretation of sense data theory in which the surface layout of perceived objects is the sense data. This interpretation is clearly mathematically accessible or describable, and if valid, would maintain a use for sense data theory in discussions of ecological optics and empirical studies of geometry.

In the 20th century, theorists began addressing sense-data theory by re-coining the idea of sense data through the meaning of *signals* in the brain. Richard L. Gregory proposed that perceptions are like hypotheses of natural science. Gregory clarified the steps of perception as being divided into signals, data, and hypotheses. Specifically, "patterns of neural events are accepted as *representing* variables or states, according to a code which must be known for signals to be read or appreciated as data" [6]. There is a realm (the very small or the very large) on the

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[4]Bertrand Russell (18 May 1872 – 2 February 1970) was a British mathematician, philosopher, socialist, pacifist, and social critic, and in 1950, he was awarded the nobel prize for his contributions to freedom of thought.

"Bertrand Russell - Wikipedia, the free encyclopedia." *Wikipedia, the free encyclopedia*. N.p., n.d. Web. 5 Apr. 2010. <http://en.wikipedia.org/wiki/Bertrand_Russell>.

[5] "G. E. Moore - Wikipedia, the free encyclopedia." *Wikipedia, the free encyclopedia*. N.p., n.d. Web. 5 Apr. 2010. <http://en.wikipedia.org/wiki/G.E._Moore>.

[6] Gregory, Richard L. "Perceptions as Hypotheses." *Vision and Mind: Selected Readings in the Philosophy of Perception*. 1 ed. London: The MIT Press,

boarder of the physically experimental measurable that is measurable through experiments of thought and scaling the results so that we can "see" into spaces through geometry and scientific method that would otherwise be extremely difficult to measure. Gregory notes that, "hypotheses are effective in having powers to predict future events, un-sensed characteristics, and further hypotheses" (Vision and Mind, 115). Psychophysics, especially the psychophysics of space-time perception/sense is an important scientific framework from which the many interpretations of perceptual philosophy can be integrated. Psychophysical experimentation may occur in the thought as well as the measurement of stimulus correlates, for the pure thought, similar to the recognition of a pure geometry is a synthesis of the space of sense data.

While James was under the impression that, "the mathematical sciences deal with similarities and equalities exclusively, and not with coexistences and sequences" (The Principles of Psychology, Vol. 2, 653), I have a very different understanding of the fact of the matter. In fact, a series of mathematical functions *can* describe a sequence of perceptual data or perceived change, because their relations specify physical parameters or sectional geometric spaces. Math can describe the coexistence of space-time with matter. It can describe the correlations of time measurement to space measurement. It can also reveal a pattern, which can be sequentially reproduced without affecting certain of its properties.

In the case of the stick's being perceived "as bent" as early sense-data theorists might propose, we can now see that, because data is an interpretation of

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2002. 111-133. Print. Page 113. All future references to this source will be made parenthetically in the text.

signals, even though all the signals sent to the visual system through the reception of light on the retina are of a jagged shape, the signals are not interpreted as data unless the code conveying the data is known. "As a signal, the neural activity is fully described in physical terms and measurable in physical units; but the code must be known in order to use or appreciate it as data" (Vision and Mind, 113). So, in essence, upon reception of light from the stick in the water, we might propose a hypothesis that the stick is really bent, and for a time, this may work as a perception.

However, we also realize that part of the code of the stick in its ecological context is its being in a material that bends light. Since light is all we have accessible to the visual system as stimuli, we are able to declare that the pattern of this received energy is what is actually being "bent", altering the initial hypothesis through a theory of sense data, signals, and hypotheses as perceptions. However, then it should be noted that true perceptions of something should be provable theorems, not just hypotheses, for hypothetically, it could be a bent stick, yet a true perception of the being of the stick would perceive it to be straight. Thus, if we say that we perceive a bent stick, we must really say that we perceive the illusion of the bent stick.

The element of the sense data theory that will be integrated in the final description of visual perceptual theory relates to the idea that sense data also refer to parameters of the kind of energy affecting a receptor as well as the structuring of that energy in the mind to form a perception. If sense data can be related to the idea of a gradient of stimulation scientifically, the theory may maintain a use when discussing how that external information is translated into perception by the visual system. If the only stimulus information that is available to the visual system is radiant energy, information in a sense datum (objective stimuli) would be contained within the parameters of that radiant energy as well as its resonance with the perceptual system (subjective experience) through space-time and consciousness of it. Consciousness of space-time, the geometry of such environmental characteristics as acceleration, and the geometry of the visual field are all elements that play into the perceptual experience as descriptions of its mechanism. We will look at certain patterns of sense data that may temporarily occupy the receptor and determine if sense data could be said to contain elements of expression for optic flow or other formal structures of geometry. The idea of perceptions as scientific hypotheses and that, "the procedures of science are a guide for discovering processes of perception" (Vision and Mind, 112) will be included, because information is a useful metaphor for discussing the structures of the visual world and their relation to the visual field in an ecological context. However, for me, similar to graphic visualization, I consider perception to be vibrant, experiential, phenomenological description of the environment and thought-life. Gibson clarified the different uses of information in psychology applying to perceptual philosophy and clarified the semantics of sense data. These clarifications will be made in the section on ecological optics.

An argument somewhat different than sense data about perceiving the illusion of a stick in the water to be bent when it is really straight would say that it doesn't mean we perceive a bent stick at all, what we perceive is all of the substances involved in that experience. Thus, we perceive light, and that light is filtered in a relative, ecological context. Perceiving all of the substances through a possible cognitive structuring involving our knowledge and expertise on the substances of the system allows us to also perceive the filtered light of a stick in terms of an image, structured by the form of the actual stick in an ecological context of perceived water, glass, etc while not seeing a bent stick. How can we say that we perceive a bent stick when the stick in actuality is straight? True perceptions normally ought to be true of reality. In this way, sense data theory in its initial conception from the argument of illusion cannot account for illusions very easily. But one would still want to say that illusion is perceivable and differentiable from actuality, coming to understand how it arises.

The phenomenological account of perception attempted to account for the more ecologically contextual (our cognition and thought-life are parts of our environment) information through conscious actions and the idea of a schema, thus dealing with illusions more appropriately, somewhat doing away with the idea of illusion, "concerning hallucinations, illusions, and perceptual deception of whatever sort, it may be that phenomenology has something to say, and perhaps even a great deal: but it is evident that here, in the role which they played in the natural attitude, they undergo exclusion" (Husserl, 88). This suggests a relationship between the natural attitude and the argument from illusion presented in early formulations of sense data. Husserl's account of perceptual theory does not have to deal with illusions, because object perception is formulated through a schema (structuring of that which is perceived through expertise or categorical relations).

After the phenomenological account, a discussion of the primary ideas of the Gestalt argument will be given. James' work in sense-data theory could have initiated the thoughts of the Gestalt argument, "*in all this, it will be observed, the sense-data, whose spaces coalesce into one are yielded by different sense organs*" (*The Principles of Psychology*, Vol. 2, 145). A Gestalt is a perceiving through or as a whole, unified system. Sense-data theory and Gestalt theory did come to some of the same conclusions, although they differ in linguistic organization of the phenomena. This is important, because we can differentiate them as theories, even though they posit some of the same ideas and point to similar conclusions. This will help understand how the synthesis of these ideas in the final section is a theory of perception in and of its own.

Geometry is related to data as a, "structuring" or univocal expression of the innate pattern to data. William James felt that a line was a simple datum, not a collection of individual points of data, "William James for example, although he did not actually assert that a form was a sensation, did believe that a visual line was a simple datum rather than a row of point sensations" [7]. The methodology of analyzing a simple datum is present in *The Origin of the Issue of Mathematical Analysis in Ecological Optics*, pages 67-68. One of the descriptive languages of singling out and correlating data from sensuously qualified observations is geometry.

2. Husserl's Account of Perception through Phenomenology

[7] Gibson, James Jerome. *The Perception of the Visual World*. Boston: Houghton Mifflin, 1950. Print. Page 15. All further references to this source will be cited parenthetically in the text.

1.1 Introduction to Husserl's Phenomenology

The phenomenological account of perception given and introduced by Edmund Husserl has a number of important contributions to perceptual theory. Husserl is a 19th - 20th century continental philosopher. He was born in 1858. He lived and worked in Germany and studied philosophy and mathematics at the Universities of Berlin and Vienna. In 1886, at the age of 28, he was baptized in the Evangelical Church of Vienna. He met several prominent philosophers of his time, including the sense data proponent, G.E. Moore. He went on to be a professor at several different locations throughout Europe, and left Germany toward the later years of his life when the Nazis took over, being excluded from the International Congress of Philosophy in Paris in 1937 due to the presence of the Reich. This congress took place one year before he died. A few of Husserl's fundamental ideas and realms of discussion are, the natural attitude, essence, consciousness, intuition, intentionality, noetic content, eidos, geometry as formal ontology, number, phenomenological reduction, and transcendental philosophy. I will outline the themes of transcendental logic presented by Husserl in order for you to be able to grasp the reasoning for elements of my account in part seven, and understanding these themes requires a considerable amount of background. Husserl's project is one of phenomenological description, "transcendental phenomenology and philosophy as universal science with absolute foundations," (Husserl, 333), and setting aside of what he calls the *natural attitude*.

The natural attitude posits, "both the existence of the world in which the objects of a subject's experiences are thought to exist and the validity of the subject's judgments about these objects"[8]. The purpose of science in the natural attitude is to, "cognize 'the' world more comprehensively, more reliably, more perfectly in every respect than naïve experiential cognizance can, to solve all the problems of scientific cognition, which offer themselves within the realm of the world, that is the aim of the *sciences belonging to the natural attitude*" (Husserl, 63). Husserl's phenomenological account proposes to radically alter the natural attitude, thus opening room into which an eventual discussion in transcendental philosophy, which is a descriptive, universal science, can take place. In the logical investigative descriptions he gave throughout his works, there are many kinds of attitudes, notably, Natural, Phenomenological, Theoretical, and Transcendental. For Husserl, an attitude is a, "fixed style that the willing life adopts toward the world" (Drummond, 41). An attitude, for Husserl is the stance or style that the willing life adopts toward being in the world or the world as experienced. Depending on the kind of attitude, the attitude will have specific characterizations of its implications to the view of the world, and in no way does Husserl deny the existence of world, saying, "I am *not negating* this 'world' as though I were a sophist; I am *not doubting its factual being* as though I were a skeptic; rather I am exercising the 'phenomenological' reduction, which also *completely shuts me off from any judgment about spatiotemporal factual being*" (Husserl, 65). We can then see a difference in

[8] Drummond, John J.. *Historical Dictionary of Husserl's Philosophy (Historical Dictionaries of Religions, Philosophies and Movements)*. Lanham, Maryland: The Scarecrow Press, Inc., 2007. Print. Page 141. All further references to this source will be cited parenthetically in the text.

attitudes that came out of Husserl's exclusion of the natural attitude: the spatiotemporal factuality of the being is excluded from judgment (in the logical sense a judgment is a normative) by Husserl, however the factuality of the perceived or experienced spatiotemporality is not. This idea of perceived space-time will be further described in the discussion of geometry later in this section, and is a fundamental principle upon which Husserl formulates his thoughts on noesis. The natural attitude posits the existence of world and the existence of the objects to which our conscious intentness is directed. The idea of bracketing allowed Husserl to introduce a transcendental philosophy that did not bring in questions of the existence of the world and the validity of the subject's judgments within it. Instead, through bracketing the natural attitude, focusing only on the *experience* of the world and the objects to which consciousness is directed, Husserl is able to perform the transcendental-phenomenological reduction in which the experience is phenomenologically described.

1.2 Perception: Essence, Eidos, and Noema

Husserl's philosophy is a complex one that uses a large series of interrelated terms to bring out the meaning of essential insight and other grounded ideas relating to transcendental philosophy. Thus, sectioning them out from each other is difficult. Husserl is describing a meaning, which is of the univocal and the expressive. Therefore, I must include a broad subject matter within a single section in order to relate to you the ideas that are fundamental to his phenomenological descriptions as well as the meaning of their interrelations.

Phenomenology in general is the study of essences, and the understanding of Husserl's meaning of essence is key to his philosophy. The understanding of his vocabulary, like in the case of essential components is also important. Essence is an ontological category, thus, for Husserl, it is intrinsically related to pure geometry. Essence describes that which is necessary, universal, and a priori. It engages the idea of the, "structures that make a thing an instance of the kind of thing it is" (Drummond, 68). For Husserl, geometry is formal ontology, and essence is also eidos, which is related to the visually vivid. From two of Husserl's separate notions, we can see that he would have thought that geometry is formal ontology, because he that, the realms, "of arithmetic, geometry, and the like, (are ontological)" (Husserl, 66), as well, "Euclidian geometry (maintains formal validity)" (Husserl, 123). So, it is reasonable to describe pure geometry as formal ontology. We will return to this later.

Essence is fundamental to Husserl's use of intuition and intention. Essence is then describable by formal ontology, which for Husserl, is synonymous with geometry. Husserl's work on the essence of indication is important to his phenomenological description through noema, because it sets up the background for distinguishing between the components of noema, which are the thetic characteristic and the noematic sense. In the essence of indication, Husserl lays out the idea of experienced belief in reality, or the notion of the thetic characteristic, saying that, "(in the case of live functioning) we discover as a common circumstance the fact that certain objects or states of affairs *of whose reality someone has actual knowledge* indicate to him *the reality of certain other objects or states of affairs*, in the sense that *his belief in the reality of the one is experienced* (though not at all evidently) *as motivating a belief or surmise in the reality of the other*" [9] . It is this belief in the reality that is the thetic characteristic component of noesis of perception, because perception entails a belief in the object perceived as perceived.

It is important to consider that Husserl's theory is one of unification. Unification is introduced early in his writings and developed in his later works on universal science as transcendental philosophy. In addition, Husserl moves to the idea of unity through characterization of the experience, "all unity of experience, all empirical unity, whether of a thing, an event or of the order and relation of things, becomes a phenomenal unity through the felt mutual belongingness of the sides and parts that can be made to stand out as units in the apparent object before us" (Husserl, 28). As the geometer realizes the functions that describe his perceptual experience, he comes to feel the mutual belongingness of the parameters of the reality he is describing to his thought life, and realizes that they are of a univocal system in the sense that any parameter of the perceptual system, through the operations of rigorous algebra, can be placed in terms of solely any other parameter. In this paper, that which is univocal, literally "speaks with one voice." The meaning of univocal in this paper and other works is a combination of that meaning laid out by Husserl, which will be delved into in further detail later on, and the fact that the

[9] Husserl, Edmund, and Donn Welton. *The Essential Husserl: Basic Writings in Transcendental Phenomenology (Studies in Continental Thought)*. Bloomington: Indiana University Press, 1999. Print. Page 27. All future references to this source will be made parenthetically in the text.

rigorous algebra of the system of a circle on a flat plane that gets an angle folded up into it in order to construct a cone whose slant is always equal to the initial radius of the circle, is composed of variables that can be place purely in terms of one of each other alone (slant of the cone is a function of the height of the cone alone). In this work, I will refer to univocal mathematics, which means, "the mathematical system described by the rigorous algebraicization of pure geometry (formal ontology of perceptual phenomenology)."

In Noesis, intention is directed toward the object *actively* or *actionally*. The subject directs by action, attention to the object such that the object is of conscious experience. The Noema is comprised of the thetic characteristic and the noetic sense. The relationship of noesis and noema describes the form of intentionality. Once the object is apprehended in experience, there is real content in that experience (noesis), and noema elucidates the form of intentionality by being the, "intentional object of conscious experience" (Drummond, 144). We do see some similarities to sense data theory, but these are not to be specifically elaborated upon here. Husserl explains our experience of truth and meaning in terms of Noesis (the apprehension (pertaining to the understanding) of the object in experience), Noema (the multiplicity of data in pure intuition), the material, which makes up sensations (Hyle), and the Noematic Core. That which is noematic, in some interpretations, is of or pertaining to the understanding, though for Husserl, the Noematic Core is comprised of all distinct noetic acts toward the object that all point toward one core. This is distinguished from the Full Noema.

In the Full Noema, characteristics of being and shadow are included, whereas, the Noematic core is the content of all possible noetic acts having to do with the physical object. The Full Noema is the associate of the noetic act, or noesis, through including both the belief that you perceive as well as the noematic core. Thus in the noema, through noesis, the thetic characteristic is differentiated from the noematic core. By looking to the idea of truth from this perspective, we find that our objectivity can be placed into a subjective framework, and Husserl argues that this will allow us to perceive and understand truth.

This is an idea that suggests a possible use of mathematical analysis of what change is perceived and where the subjective locus of perception is in relation to the objectivity of this changing structure, and in Husserl's analysis of pure geometry, he says that, "geometry represents for us here the whole mathematics of space-time" (Husserl, 340). The purpose of section seven of this paper and the work on difference in circumferences will be to draw out this meaning in precise correlations and gradients in the subjective framework of objectivity so that the locus of perception, through which discernment of change occurs eidetically, and the object of perception are nestled in their respective locations in the geometry of the perceptual field. Section seven will also correlate through formal ontology, the meaning of pure geometry, some of the forms, which are the objects of analysis of the cognitive approach to perceptual theory.

1.3 Intention and Intuition

Intentionality is the term that Husserl uses to get at how meaning is already existent in the world, but present within the subjective experience. For Husserl, intention is, "a direction to something having a particular significance" (Drummond, 110). Thus, intentionality relates directionally to meaning-intention when the eidetic intuition comes to realization through experiencing the perceived object as perceived. Intentionality mimics the form of directionality. In the case of the circle's transforming into a cone, the significant point to which the system has direction is the point at which the circle has, "folded" all the way up such that the height of the "cone" equals the initial radius. At this point, there really is no longer a cone, but a line - equal to the initial radius and orthogonal to the plane of the initial circle. Understanding this directionality is possible through formal ontology. Husserl says that, "it is possible – and this was the discovery which created geometry – using these elementary shapes (circles, lines, triangles, etc.), singled out in advance as universally available, and according to universal operations which can be carried out with them, to *construct* (not only) more and more shapes which, because of the method which produces them, are intersubjectively and univocally determined" (Husserl, 340-341). We can therefore see that, for Husserl, univocity is a characteristic of pure geometry, and I believe one that is expressed in the results of *The Geometric Pattern of Perception Theorems* (Emmerson, 2009). Husserl might agree, because he says that, "'*pure apophantic analytics*' (is) an analytics in which belong not only the whole of syllogistics, so far as its essential content is concerned, but also (as we shall show) many other disciplines, namely those of formal-mathematical 'analysis'" (Husserl, 249).

Husserl believed that perception was related to recognition such that, "*not every really inherent moment* in the concrete unity of an intentive mental process itself has the *fundamental characteristic, intentionality,* thus the property of being 'consciousness of something'" (Husserl, 70). What Husserl might be getting at is the idea that an intentional mental process, having access to the notion of directionality through interactions with its environment, may very well enable awareness of the directionality, or "intentionality" as having a positive and negative solution, negating the actuality of the intentionality in a really inherent moment, in which the future directionality includes the past directionality. He is also trying to get at pure consciousness, which can have naming of significant structures with no perceivable object from expressions of those structures.

Husserl's account of perception of events also uses the language of recognition and memory. Husserl wishes to remain within the, "limits of simple intuition and the syntheses belonging to it, among which perception is included" (Husserl, 71) for discussing the, "*really inherent composition of perception and its transcendent object*" (Husserl, 70). However, Husserl is clear that a picture consciousness or sign consciousness is not to be substituted for perception, "there is no consciousness of anything *for which* the intuited might function as a 'sign' or 'picture' (Husserl, 74). This is true, yet relationships within an intuited relationship, describe a gradient (correlation of parameters), can be expressed through relationships of symbols, and I don't think that Husserl is in any way negating the validity of geometry or the usefulness of algebra as a language for discussing ontology and such a formal ontology for discussing the being of perception (visualization). Also, no sign is perceived without an understanding of a relationship to other "meanings as symbolically operating expressions," the inverted commas, as required by philosophy of phenomenological reduction. Thus, Husserl introduces the term recognition in order to connect the expression of sign relations to the perception of the perceived object as perceived. In mathematics, patterns, structures, hierarchies, and symbolic meanings are used to visualize spatial relations.

In order to understand what Husserl meant by recognition, we must first understand what his use of meaning-intention is. This will allow us to understand how perception of the object occurs for a perceiver. The recognition occurs during the, "*static union of expressive thought and expressed intuition*" (Husserl, 52). This idea will be returned to in section seven, because, "(cases in which *picture presentations* serve in place of percepts show us that) the imaginatively apparent object, e.g., the identical inkpot in memory or in fancy, is felt to bear the expression which names it" (Husserl, 53). Husserl's intuition served him well, for the imaginative object, eidetically intuited, upon visualization, is a picture presentation, bearing the name of its symbolic mathematical expression. By elucidating the meaning of recognition, Husserl was able to indicate the philosophical locale of perception with relation to experience saying, "*the recognitive act of experience must accordingly base itself on the act of perception*" (Husserl, 52). In this passage, Husserl goes on to note that perception is not being classified, but the object of perception is. We really get to the focus of Husserl's meaning of recognition when reading the statement, "the perceived object is *recognized* for an inkpot, known as one, and in so far as the act of meaning is most intimately one with an act of classification, and this latter, as recognition of the perceived object, is again intimately one with the act of perception, the expression seems to be *applied* to the thing and to clothe it like a garment" (Husserl, 52) in conjunction with an applicable formal ontology working as a phenomenological description of perceived change (in circumferences or moments) as perceived. The phenomenological reduction allows us to describe the perception as it is perceived through formal ontology or logical relationships of ideas.

So, for Husserl, the expression, "inkpot" clothes the thing like a garment. That suggests that the expression, "inkpot," if removed from the object, would leave the object naked. What if the name of an object is a mathematical expression? The object can then not easily be removed from the mathematical "name" to which it is tied. In this instance, the name of the object is even more closely tied to the being of the object and thingness of the object. Name, number, and object are correlated through the visualization (perception) of the algebraicization of that which can be proven through geometry. For instance, the square root of negative one can be shown to equal an algebraic expression of the relationship of two variables in the system of a circle transforming through a cone as in Section VII of *A Geometric Pattern of Perception* (Emmerson, 2009).

Husserl's account is largely based on the idea of intention and its relationship with intuition. At the very threshold of performance of a phenomenological description, there are distinctions made or discovered, and this is an insight into how we form logical statements of sense about noematic experience (experience of noematic content, which is in short, the noema) from composing an understanding and expression of the kinds of intentionality. There are the components of the intentional structures that are discovered through analysis and description of experience, and the intentions of the relations of those components, which specify situational and contextual meanings. Husserl expresses clearly and succinctly the idea, "with respect to intentionality we immediately confront a wholly fundamental distinction, namely the distinction between the *component proper* of intentive mental processes and their *intentional correlates* and components" (Husserl, 87). These intentive mental processes toward an object are synthesized into the perception of the object. The intentive mental processes enable the theoretical and the hypothetical to be analyzed in terms of the *intentional correlates and components* and are thus distinguishable from the components.

Husserl's phenomenological reduction allowed him to describe perception through consciousness of objects, "I perceive the physical thing, the Object belonging to Nature, the tree there in the garden; that and nothing else is the actual Object of perceptual 'intention'" (Husserl, 90). Structuring of the light through the environment and interpretation of the perceiver provides the "external" data for object perception. Given a mathematical equation that has a series of correlated parameters, like the gradient of general or perceptual space-time, which can be graphed (its parameters can be traced in a coordinate system or revolved around an axis) in visualization or envisioning of thought through intuition, imagination, synaptic relations, or inspiration, there is an associated object perceived through consciousness of it. Husserl says, "all unity of experience, all empirical unity, whether of a thing, an event or of the order and relation of things, becomes a phenomenal unity through the felt mutual belongingness of the sides and parts that can be made to stand out as units in the apparent object before us" (Husserl, 28). For Husserl, such an object would be the object ultimately perceived, and until the mathematical relations behind its production are traced through the mind by the various faculties of sense, whether it is an image appearing by visual perception before you or inspired by imagination, the object is not immanent to perception. Thus, a geometric proof could be an intuited symbolism of intentionality, and delivered to the intellect through noesis. In this way, mathematics would be connected to mental processes, which are involved in experiential reality, but not dependent upon them.

The usefulness of mathematical analysis in the development of Husserl's fundamental propositions correlates to the synthesis upon visualization and perception of that visualization as perceived visualization, which is the complex whole. This correlation takes place of recognition. "The synthesis of recognition, of 'knowing', is the consciousness of a certain agreement" (Husserl, 59). By synthetical consciousness, Husserl means that, "accordingly, the universal essential property pertaining to consciousness is still preserved in modification, and all mental processes having these essential properties in common are also called '*intentive mental processes'* " (Husserl, 62). So, we see that there is a universal, essential property that acts as an exchangeable reference to the moment and the spatial characteristics of the object through the noema of the perceiver. The content of the experience of the object is given by the multiplicity of data in pure intuition. The data could be understood in concrete, phenomenologically parametric correlates.

This is not to say that any internal image or second immanental object (graphical and material) is in any way given, "and to suppose that hypothetically leads to an absurdity" (Husserl, 90). An intentive mental process is directional, but how can the directionality of all the mental acts be combined together? In essence, for Husserl, this occurs through synthetical consciousness. Consciousness has the capability of doing this, because, "*of essential necessity there belongs to any 'all sided,' continuously, unitarily, and self-confirming experiential consciousness of the same physical thing a multifarious system of continuous multiplicities of appearances and adumbrations in which all objective moments falling within perception with the characteristic of being themselves given 'in person' are adumbrated by determined continuities*" (Husserl, 71). The moment expands into the future from the past, and we are given reference to each through the locale of the present to consciousness as the synthesis of these occurs in the motion of consciousness through the now moment, or by being a fulfilled noema (containing content of an experience pertaining to the understanding). On this topic, Husserl says that the, "eidetic law, confirmed in every case, states that there can be *no noetic moment without a noematic moment specifically belonging to it*" (Husserl, 92). So, we can see the relationship between the noetic and the noematic: there can be no intellectual, apprehended moment without the being of a moment of understanding that belongs to the intellectually apprehended moment.

Also, as I understand the relationship, the objective moment occupies a volume of space-time *through* the gradient of space-time with reference to the experiential moment, the visually accessible locus of the subject being included in such descriptions as included through being capable of purely describing the experience such that the subjective moment occupies a volume of *perceived* space-time. This pure description of the being of the experience of the volume of perceived space-time may be achievable through pure geometry. It is now perhaps safe to say that, because these objective moments are given in person, perception as perceived could only occur through a given *persona* section of reality. However, this idea of a section of reality relating to the persona of the perceiver would still require development.

We briefly discussed the meaning of noematic content, which if you recall, noematic means pertaining to the understanding, and the content is the multiplicity of data in pure intuition (noema). On the data of *noematic content*, Husserl says that, "there is a multiplicity of Data, demonstrable in actual pure intuition, in a correlative '*noematic content'* or, in short, *noema*" (Husserl, 88). Thus, noema is similar to the content of a thought. The noema of perception is the perceived as perceived. The perceived as perceived might entail a look into the being of the structures of that perception in an ecological context or subjective framework of objectivity. Husserl will eventually extend the meaning of the data available in pure intuition (noema) to sense, which connects his phenomenology to sense data theory.

Husserl, in the tradition of psychology, was interested in how we come to perceive by the processes of the mind. He addresses the fundamental issue of this by elucidating the meaning of mental processes, “by *mental processes in the broadest sense* we understand everything and anything to be found in the stream of mental processes; accordingly not only the intentive processes, the actional and potential cogitations taken in their full concreteness, but also whatever is to be found in the way of really inherent moments in this stream and its concrete parts” (Husserl 70). In the modern era, the idea of a synapse, and of relations between synapses has been a method of studying subjectivity.

Noesis could be thought of as meaning contained within the moment of expanding experience and has an eidetic quality with regard to visual perception. However, to get to the fundament of this eidetic quality, the appearing of the object must take place through an intentional act of perception. This act occurs through consciousness. In 1.4, I will show how Husserl’s major ideas related perception to logical, eidetic analysis and thus to ontology.

Intentionality is similar to decided directionality by active mental processes. However, Husserl is clear that the logical operations, or objective cores are not dependent on the mental processes. Yet, the objectivity of the structures of intentionality are clearly noetic, “even in the case of noeses of a higher level-taken in concrete completeness - there at first emerges in the noematic composition a central core thrusting itself to the fore in a predominate way, the ‘meant Objectivity as Objectivity,’ the Objectivity in inverted commas as required by the phenomenological reduction” (Husserl, 92). This insight brings about the key element to formation of the consciousness of internal time, because there will be shown to be a core being the inherent meaning and simpliciter of difference in perceived content. The statement also brings into question the verifiability of the noematic composition and thus, the verifiability of the objectivity. Are there rules of noematic composition? Must the noematic composition always stick to the gradations of the central core? Husserl says that, "there this central noema must also be taken precisely in the modified Objective composition in which it is just that noema, something intended to as intended to" (Husserl, 92). This, I interpret as meaning yes. Yes, the modified objective composition sticks to the originally posited form of the intention and only traces the elements of its composition back to the inherent basics of the intentional structure. Changing the elements of the composition beyond the design of the *a priori* relations inherently changes the intentionality pertaining to the perceived object. Because hallucination and illusion have been bracketed, set aside, there is reason for also bracketing the metaphysical possibilities of supplicating the intentional structures through made-up components unless they apply to the original intention, and what is certainly not to be bracketed is the specificity of a contextual meaning that indicates a particular mode of the originarily given intentionality. This brings us to pure intuition and its relationship with discoveries of pure geometry.

1.4 The Eidetic, Intuition, Geometry and Logic Relating to Noetic Acts

In order to clarify the assertions and issues raised by the last paragraph, we can turn to the idea of judgment in the logical sense. "If we have become attentive to the *point of view of 'operation'* (with *laws of operation* in which, mathematically speaking, 'existential propositions' are implicit), we shall naturally chooses the concept of operation as a guide in our investigation of forms; we shall have to conduct this research in such a way that it leads to an *exhibition of the fundamental operations and their laws,* and to the *ideal construction of the infinity of possible forms* according to these laws" (Essential Husserl, 248). On the noesis of a pure geometry, it is similar to the functions of formal logic, except it is distinctly eidetic in form of the horizon, and praxis, whether attempted by those with visual access or without, denotes a context of enabling "seeing" in the same sense as formal logic and development of logical relations of objects through visual perception. We will soon show that this "seeing" is akin to a perceiving act. Husserl says, "Let us add the idea of the judgment which has been characterized and which functions as the fundamental concept in formal logic (that discipline within mathesis universalis pertaining to predicative significaitons) has as its correlate the noetic idea: 'the judgment' in a second sense understood, namely, as any judging whatever, with an eidetic universality determined in pure form" (Husserl, 93). In pure geometry, we are given access and noesis of its form by the character of the symbolic structure describing an "originarily" given directionality in the originarily perceived as perceived structures of pure geometry. Husserl's meaning of originarily is defined through the discerning the difference of perceiving and non-perceiving acts, "*First*, the difference between positing mental processes in which the posited becomes *given originarily* and those in which it does *not* become given in that mode: thus, between *'perceiving' or 'seeing' acts – in a broadest sense – and non-'perceiving' acts*" (Husserl, 113). From this idea of the orignarily given, and the notion that changing the meaning of the elements of the composition inherently changes the intentionality pertaining to the perceived object, we can formulate a description of the characteristic of noematic experience in the sense of the intellectually seen through phenomenology.

Logic is often very necessary for the formulation of geometric insight, not divorced from geometry or computation, and in fact, the combination of logic and geometry, through visualization of gradient, yields a sensible object. The problem of their *a priori* structure is thus reduced to the fundamental principle, or "*continuously, unitarily, and self-confirming experiential consciousness,*" (Essential Husserl 79), which describes the perception as a unified whole. Mathematics is language capable of the description of the appearance of such perceived objects as perceived as well as the operations that structure the unified whole.

Before going on to discuss Husserl's notion of the consciousness of internal time and the nature of the geometry he used to describe the series of now moments, I will try to better elucidate the meaning of the noematic core. The noematic core is the content of all the experiences one can have with an object. It is described in difference to the noema, because, "*The same* 'S is p,' as a *noematic core*, can be 'content' of a certainty, a deeming possible, a deeming likely, etc. In the noema the 'S is p' does not stand alone; rather, as singled out of the noema by thinking, it is something non-selfsufficient; it is intended to with changing characterizations indispensable to the full noema; it is intended to with the characteristic of something 'certain,' 'possible,' 'probable,' 'null,' or the like – characteristics, to which the modifying inverted commas collectively belong and which, as correlates, are specifically coordinated with the noetic moments of considering-possible, considering-probable, considering null, and the like" (Husserl, 94). Let us briefly consider the object of a cone that is collapsible and expandable through space-time such that the height of the cone is always orthogonal to the center of the base and the length of the slant of the cone is always constant. All the different actions I can have with that cone contain many such statements as "S is p" or "n is m," though there is a single expression from which all of the statements like "S is p" are found, which is the core linguistic and symbolic expression of the eidetically discovered truth. It is eidetically discovered, because it is something that is noticed through visual perception, essential, and vivid linguistic and visual expression through noesis. Considering the algebraic canceling of the Lorentz transformation in Theorem 3 of *A Geometric Pattern of Perception* (Emmerson, 2009) to be a kind of nullity, the noema for this considering null is innate motion, which posits an observer moving, and there is still a solution to the velocity within the Lorentz transformation when using the exact speed of light in scientific notation. This is discussed further in *Mathematical Analysis in Ecological Optics* (Emmerson, 2009-2010).

The pure intuition, guided by the inspiration of the eternal truth of that which is intuited in pure geometry, yields a systematic flow of appearing objects and the recognition of its truth is a perceiving act. The problem that Husserl had with the thoughts of the school of psychologism, which posits that logic and mathematics are dependent upon psychological mental processes, is that the branch of mathematics studies the categories of mathematical relations, distinct from the appearance of the object implied upon visualization of the the originarily given mathematical formula. However, algebra is a language of form, assumptions of judgments in the logical sense and how to compose relations of actualities, and so is formal logic, and out of mathematics synthesized with logic, ontological objects (potential "templates" for understanding the development of organic structures as well) can be delivered, which are capable of being studied in cognition or psychology, though their pure mathematical meaning is of the nature of the incontrovertibly true and/or proved theorem. Thus, for Husserl, geometry is formal ontology, while logic studies *a priori* relationships between judgments. The two are necessarily combined in order to form vantage points of arguments from which solutions can be found for expressions of the gradient of perceived space-time.

Husserl relates the idea of being, and ontology to the knowledge of the truth of a statement. He elucidates his meaning in his writings on the question of evidence, "In the logical sphere, in the sphere of statement, *'being truly' or 'actually' and 'being something which can be shown rationally' are necessarily correlated* " (Husserl, 113). This relates to the idea of the perceived *as* perceived, because the perceived *to be* perceived implies both the gerund and the future tenses. Thus, the present is passing into the future, or, rather, it is simply the expanding moment. Husserl is carefully commenting on the nature of metaphysics, because he is saying that untruths are statements that cannot be shown rationally, although, the being of the actuality of the object, even one that is not based in the proper course of logical or algebraic argument, is still generated through rationality, and a distinction is drawn, that is, in order for the proposition to be truly, it must come to have some rationale, and although that rationale may be mistaken, it can still take the form of being truly. However, I would add to Husserl's meaning of the truth of the statement that the most true would be that which is proposed from correct interpretation of rational, properly normalized (in the logical sense) relations. This is related to Husserl's discussion of transcendental philosophy as universal science.

These ideas are fundamental to the study or phenomenology of perception, because accurate perceptions are of correctness, and are developed through the development of an infant's relation to the world, its logical use, and visual perception of it to a matured person's perception of world. Though, each stage of human development has its own kind of clarity. Husserl's notion of the phantom, or schema is useful in dealing with some of the illusions for which the sense data theory could not account with ease. He says, "The phantom (schema) as a sensuously qualified bodily surface functions as a basic frame for the object of perception" (Husserl, 232). Husserl had no qualms with discussing appearance, and found philosophical use in it, "inside each of these spheres (those of distant and near appearances) we have more favorable and less favorable appearances… the form is given at first glance and its fullness appear at best in the total overview" (Husserl, 232). Under Husserl's account, when we take a total look at the stick in the water, we are able to perceive at a glance that the stick is not actually bent, because we have schema that organizes the ecological components of the perception of diffraction through glass and water. In this way, the illusion that the stick is bent is dealt with by doing away with it. We also have a schema for the direct retinal image, but it is not the only thing that is perceived. The object and its surface in the world are perceived.

The study of perception has a great deal to do with how we come to know that we perceive and what the structures of the being of that perception are in the world. Husserl claims, regarding knowledge of what is perceived that, “by acquiring knowledge originally, perception also acquire permanent, lasting possession of what is has acquired; it is a possession that is at our disposal any time” (Husserl, 225). In his phenomenological account, this happens through recognition. To be able to show that original knowledge of what is perceived can be acquired through noetic visualization of a mathematical description by a system in which the locus of subjective perception is incorporated in an objective system that describes what is seen would support a phenomenological approach. Also, we can remember that intentionality is given originarily. To understand the form of the intentionality and its relation to the originally acquired knowledge, the noema occupies a body of perceived space-time as perceived with relation to objects in the world and the substance through which light travels. For Husserl, “it is eidetically unquestionable that truth can only be actually given in an actual evidential consciousness” (Husserl, 118). Thus, the truth of geometric insight into the structures of perception actualizes perception only insofar as an element of persona (a characterization of consciousness) is resonating. This helps us understand the semantic problematic of sense data, because sense in the sensual meaning corresponds to sense in the meaning of a structured, communicable understanding through phenomenological description.

Consciousness of the event is a pertinent and necessary philosophical piece to understanding the puzzle of perception in general. Husserl's phenomenology focused greatly on the perception of temporal events and the consciousness of internal time through modalities of appearance. Husserl's account is characterized by description of the event of perception, "…we take tone purely as hyletic a data. It begins and ends; and after it has ended, its whole duration-unity, the unity of the whole process in which it begins and ends, "recedes" into the ever more distant past" (Husserl 187). The important thing about this way of understanding time consciousness is that the hyletic data exists in a single unity of experiencing duration. It is within this moment that the ontological object takes form through visualizable gradients. Husserl touches on this idea at the beginning of his account of internal time, "How are we to understand the apprehension of transcendent temporal objects that are extended over a duration, continuously filling it in the same way (as unchanging things do) or filling it as constantly changing (as in the case, for example, of physical processes, motion, alteration, and the like)?" (Husserl, 186). The being of a transcendental temporal object is the formally, ontologically originated object (surface) that describes the gradient of the perceived space-time *as* constantly changing in a given mode of experiencing a difference in the qualia of hyletic data, and is also a universal structure that does not change "in" itself, but rather describes how one parameter of perceived difference correlates to another. In essence, for geometry of perceived difference in circumferences (length as a quality is similar to volume of sound, intensity of light, and its angularly pure geometric correlates are akin to frequency and potentially contain information in other ways as well that produce phenomena with different qualia upon graphical or auditory interpretation), the contoured surface is the change relating to a given function and the very object delivered upon visualization of the correlates of that function. The change occurs in a purely geometric way (which is informational), the structure of which alterable only upon indicating specific locations through the necessitated kinds of movement within the system. The tone, which is purely hyletic data, is expressible through perceived energy fluctuating within the framework of perceived difference in general. While I feel that time is measured through difference in terms of perceptual change of a continually (cyclically) correlated whole event, descriptively measurable in terms of mathematics, Husserl used the language of modalities and duration to characterize and elucidate meaning of consciousness of time.

The vocabulary, while different, does have certain similarities. These will be drawn out in the synthesis in part seven. It is in section seven that I will introduce how some of the logical conclusions to which he came in his discussion of noema influenced the discourse on consciousness of internal time. However, for now, it suffices to say that Husserl describes the series of now-points as a line, "the temporal positions have intervals, that these are magnitudes, and the like, can be seen with evidence here; so too can further truths, such as the law of transitivity or the law that if a is earlier than b, then b is later than a" (Husserl, 209). This line correlates directly to the height of a cone that unifies the concepts of space and time through the mathematically linguistic, phenomenological, mathematically expressed, and geometrically described difference in perceived circumferences equaling an arc length.

1.4 Transcendental Philosophy, Universal Science, and the Univocal

Husserl was very interested in philosophy's relationship with mathematics, insight, and intuition. He held these problems to be of the utmost importance. In this subsection, I will outline how Husserl's account progressed from and linked perceptual consciousness to essential unity, univocity, ontology, science, and an idea he called the life-world, further developing my interpretation of these ideas in section seven. He did this by making use of logic. I hold that the geometry of a cone is present within several different locations of the perceptual field and also, being a kind of universal, allows for multiple adumbrations of an object, and is a formal ontology the perceptual field and thus the perceived world.

Husserl may have described the content of the experience of the volume of perceived space-time to be occupied by the moment of perceived hyletic data through reception by the, "concrete mental process by which the noematic, or 'objective,' color is 'adumbrated'" (Husserl, 96). Da Vinci was one of the early people to notice and comment upon the fact that the color of perceived pigments changes depending on the surrounding pigments. Husserl may be suggesting that there is a concrete mental process by which this contextualization of information processing occurs. However, this would be strange, for generally, I would think that this happens through an abstraction of the color from its correlated pigment. This insight brings up a great case study for a conversation of the object of perception. In essence, we perceive the color as perceived, although the pigment is a perceived object as perceived. The perception of the pigment is what is experienced as color. The pigment is that element of the object of perception that does not change when there are other pigments around it, but the experienced color of it does change.

Perceptual consciousness of the color of the pigment thus shifts when other kinds of pigments are surrounding that pigment. But in this, we see that there is an essential unity, because these surrounding, contextual pigments are effecting the perception of the color of the central pigment. This phenomenon is analyzable through psychological experiments of multiple subjective experiences as well as theoretical relations of multiple colors or pigments. However, the mathematical theory of how the perceived color of the pigment will change has not been fully integrated with the phenomenological description of the change in color. This is an area that will require further investigation in general.

Of perceptual consciousness, Husserl's account allows us to say that we have consciousness of the perceived object as perceived, but where is this object of perception? Is the object of perception the pigment itself, or is it the perceived color as perceived? Logic will allow us to distinguish between the perception of the pigment, which includes a surface, and the perception of the color, which is an experiential phenomenon, because we should be able to logically 'work out' from the contextual information (other pigments surrounding a given pigment) how the perceived color of the central pigment would be affected by the surrounding pigments/perceived colors and thus distinguished, though this will only give us a range of possible exact pigment colors (as perceived on a neutral background (black or white background, the two interacting pigments separated from each other)), because each pigment in the system effects color perception of the others. So, in essence once two pigments are within a given object of perception (a perceived difference in colors), the two perceived colors are inextricably tied to each other, thus the exact colors of the pigments as the colors would appear on a black or white background individually, will not be able to be distinguished, except being able to be described through chemistry of that pigment. Thus, the pigments themselves are not the objects of color perception, but rather, the perceived colors, as they are experienced relating to that pigmented surface and the surrounding pigmented surfaces, are. The phenomenon of color relativity is very complex, and studying it in depth would require more advanced neuroscience than I am prepared to offer.

However, in this phenomenon, we see an essential unity - a unified system in which perceived colors are dependent upon each other in a contextual way. Husserl talks about essential unity through observing the progress of science. Husserl believed that a new kind of thinking was necessary for uncovering how the synthesis that formed a unity of sense would take place. He tells us that the scientist rarely will turn his attention to the subjective, because of his purely theoretical interest. However, the positive sciences only operate on the level of theories that can be put straightforwardly and,

> "when the theorizer directs himself to the province of cognition as his theme – fashioned, that is, by the continuous categorical forming of experiential objectivities belonging to the province, as they come within the scope of determining the processes of thinking, and by the systematic connecting of the

> formations thus acquired, to make cognitional formations at higher and higher levels: the openly endless, and yet systematically unitary, edifice of the scientific theory of the province" (Husserl, 240).

He suggests that a scientific theory is available through the logic of transcendental philosophy and phenomenological description of the experiential relations of the constituents of objectivities that will allow us to understand the workings of perception. For Husserl, perception, "floats in the air, so to speak – in the atmosphere of pure phantasiableness" (Husserl, 310). The objectivities of expressions for parameters of formal ontology are a first step to studying the eidetic phenomena within the experiencing of a visual field through analyzing given contextual structures of its characteristics. Perception is, "pure 'eidos' perception, whose 'ideal' extension is made up of all ideally possible perceptions, as purely phantsiable processes" (Husserl, 310). This is partly due to the, "bracketing of the natural attitude," which is seen in Lemma 9 of *The Geometric Pattern of Perception Theorems* (Emmerson, 2009). Thus, all analyses of perception are essential, eidetic analyses. Earlier, it was stated that essence describes that which is necessary, universal, and a priori, and that essence is an ontological category. Therefore, we can see that the a priori truths of geometry, which is formal ontology, are useful to analyses of perception, because the description of reality provided by them is of essential perceived difference as an arc length perceived. Thus, the being of the pure eidos perception is capable of being studied and understood through geometry. The form of the geometry is intuited by the individual from subjective experience of the world and expressible by description of that understanding of the subjective experience through geometry. In fact, the idea that, "*along with phenomenological* *reduction, eidetic intuition is the fundamental form of all particular transcendental methods*" (Husserl, 311) is confirmed by the noticing of the different locations of similar geometric forms in the visual field, because this noticing is an experience of those geometric truths. This experiencing and formal description thereof is evidence of perceptually linguistic consciousness.

Husserl's work on transcendental phenomenology commented upon psychology. He says that, psychologism is a transcendental circle, making note that, "phenomenological reduction serves as psychological only to the end that it gets at the psychical aspects of animal realities in its pure own essential specificity and its pure own specific interconnections" (Husserl, 329). Husserl says that the transcendental reduction is useful for describing the experience of reality for an animal, and that this is related essentially to the psyche of that animal. Also, of essential unity, there is a systematic framework in which the locus of perception is a part of the objective world such that any constituent of the framework can be placed solely in terms of any other constituent. This idea is generally understood as univocity, and is an idea that Husserl elucidated philosophically in terms of the eidetic intuition and phenomenological reduction. However, before going on to discuss univocity, it is important to see how Husserl sets us the introduction of its pertinence to his discussion. He says that, "the theme of transcendental philosophy is a concrete and systematic elucidation of those multiple intentional relationships, which in conformity with their essences belong to any possible world whatever as the surrounding world of a corresponding possible subjectivity, for which it [the world] would be the one present as practically and theoretically accessible"

(Husserl, 330). Thus, we see that Husserl is once again limiting the scope of his project to the experiential reality of the individual perceiving. He emphasizes that the intentional relationships correspond to ontology and thus, in formal terms, geometry, letting us see that the subjectively present world is the one which is known to the perceiving act through intentionality. The intentional relationship corresponds to eidos, or essence, and this essence belongs to the world for the perceiver. The subjective is then nestled in the world, not as an object, but as a receptive, correspondent, present through the world and capable of relating to its world in practical as well as theoretical ways. Also, the subjective experience allows for the description of what it perceives as its world, because the experience is based in perception of the perceived world as perceived. Although the objective and subjective are differentiated ideas for Husserl, they are interrelated. This allows us to discuss unity essentially, or eidetically through formal ontology, and in order to formulate this understanding, it will be helpful to go back to the discussion of recognition and intuition for a moment before continuing to emphasize the meaning of unity in Husserl's terminology.

Husserl wishes to elucidate the differences between the symbol, the expression, and the intuition. He says that if we, "take the word, present in consciousness and *understood as a mere symbol* without being actually used to name anything, and set the corresponding intuition beside it: these two phenomena may at once, for genetic reasons, be brought together in the phenomenological unity of naming" (Husserl, 54). We find, phenomenologically, not a compilation of symbol and name, but, once joining the complete word, and the thing for which that word stands, a combined unity through intentionality. This is an essential unity, which is important to Husserls' phenomenological account of the natural sciences. The structures of intentionality are potentially capable of being studied through formal ontology and logic, and if we are to understand the essential unity, the eidetic experience of intuition, the universal, the formulae, and the transcendental phenomenology must all be taken into account.

That which is universal is accessible to the perceiver through experiencing of world, because he is a perceiver, perceiving the perceived world as it is perceived, and this world includes in certainty a formal, ontological truth, language by which ideas of it may be described, as well as the being of their essence. One of the languages used to describe this is geometry, but the meaning of the linguistic structure for understanding the geometry present in the world is already existent in the world a priori to the discovery of it by an individual. This perhaps is why it is called formal ontology, because we can get to a descriptive understanding by intuitive insight into the being of a procedure or relationship that gives access to descriptions of the being of truth in the world, the meaning of whose being already exists in world.

For Husserl, transcendental phenomenology is universal science, and he came to many conclusions and insights into the function of mathematization for an idea he called life-world. Husserl's ideas on the mathematization of nature will be included in section seven of this work. However, in conclusion, to this subsection, describing Husserl's phenomenological account of perception, I will introduce how he came to understand the univocal, and by describing the working of Husserl's ideas of the universal, the logical, and the perception of the transcendental object we will get to his understanding of the univocal character of phenomenology.

Earlier, it was stated that logic and geometry potentiate application to each other to form vantage points of arguments from which solutions can be found for expressions of the gradient of perceived space-time. I will use Husserl's idea of transcendental phenomenology to elucidate spontaneity within this theme. In a conversation with a philosopher friend of mine, Zach Davidson, we found that logic also allows us to see relationships between the non-ideal notions of "objecticity," which is thought of as a hypothetical application of an utterly objective reference frame on one's subjective composition (feelings, thoughts, noetic experience, etc.), and "subjecticity," which applies an entirely subjective perspective onto one's objective existence (outward deeds that establish one's bearing in the world). However, the ideality of the universal does not promote either attitude. Because of his exclusion of the natural attitude, Husserl might say that this objective existence is experiential. He comments, "the ideality of the universal must not be understood as if it were a question here of a being-in-itself devoid of reference to any subject" (Husserl, 287). Adding that, "on the contrary, like *all* objectivities of understanding, it refers essentially *to the processes of productive spontaneity* which belong to it correlatively and in which it comes to original givenness" (Husserl, 287). Therefore, certainly, there is a subjective-objective interrelation, which is a part of both subjectivity and objectivity, and it is probably possible to work from the understanding of the originarily given in perception to the expression of the product of spontaneous processes through synthetical consciousness that is capable of describing that which is originarily given through essential formal ontology. For Husserl, the essential is practically synonymous to the eidetic, so we can see the correlation to the visually vivid.

For Husserl, the univocal is a fixed objectivity. Furthermore, “what ‘exists’ ideally in geometric space is univocally decided, in all its determinations, in advance” (Husserl, 337). The formal ontology of perceived difference in circumferences yields a univocal system that phenomenologically describes through mathematics the experience of the difference, practically in perceived moments.

Husserl came to the conclusion that transcendental phenomenology is ontology, and laid out the framework of a transcendental philosophy as universal science. He specifically relates this to eidos, an ontological category, capable of being accessed through phenomenologically descriptive formalities. Husserl says that, ”the ‘*idea of an empirical phenomenology*’ is identical with the complete systematic universe of the positive sciences, provided that we think of them from the beginning as absolutely grounded methodologically through eidetic phenomenology” (Husserl, 334). Geometry, as formal ontology, is related to the eidos if it is essential, grounded in true descriptions of experiential relations, because eidetic phenomenology is, “all embracing ontology” (Husserl, 334). Phenomenology, for Husserl, was capable of, “revealing associations as intentional phenomena, indeed as a whole basic typology of forms of passive intentional synthesis with transcendental and purely passive genesis based on essential laws” (Husserl, 335). Of essential laws, formal ontologically provable theorems are a type.

In the sections where he discusses phenomenology as universal science, Husserl's remarks on perception are tied into the characteristics of the attached math item, "you intuitively grasp how the objective sense exhibits itself as unity [in and through] the unending manifolds of possible appearances; and seen upon closer inspection, how the continual synthesis, as a unity of coinciding, allows the same sense to appear, and how a consciousness of ever new possibilities of appearance constantly persists over against the factual, limited courses of appearance, transcending them" (Husserl, 221). Through combinations of the shapes found through the system described in that math item, we can see how these gradients of perceived "space-time" could describe shapes seen in the world relating to matter, such that the characteristics of the perceived matter are interpreted qua perceived surfaces with textures, contours, etc., which are perceptual combinations of different coinciding experiences. We can continue to visualize the solutions to the parameters of the system through patterns of substitution at higher and higher levels, which do not deliver objects like the gradients of cones or angles in themselves, but rather are more elaborate gradients. I will return to Husserl's work on theory of mathematics and its role in natural science in section seven.

1.5 Conclusion

In conclusion, for Husserl, transcendental analysis is eidetic. He says, because of his work in the genesis of judgment, "we have touched on the level of genetic phenomenology" (Husserl, 309). The difference in circumferences of two circles equaling an arc length is formal ontology of an essential (eidetic) difference. "The *method of eidetic description*, however, signifies a transfer of all empirical descriptions into a new and fundamental dimension, which at the beginning would have increased the difficulties of understanding; on the other hand, it is easy to grasp after a considerable number of empirical descriptions" (Husserl, 309). The geometric pattern of perception theorems is a set of such eidetic, empirical descriptions, because it is grounded in provable geometry.

The other elements of the phenomenological approach that will be integrated in the synthesis of perceptual theory through mathematical analysis are the ideas of creating a phenomenological description, that objectivity can be placed into a subjective framework, and that by structuring of information through pure intuition, "the thing itself in its saturated fullness is an idea located in a sense belonging to consciousness and in the manner of its intentional structures" (Husserl, 233). To describe the elements of theories and to correlate these to my own experience of spatio-temporality will be useful. Husserl prescribes, "genetic analyses to make understandable how, in the development of proper structure of every stream of consciousness, which is at the same time the development of the ego – how those complicated intentional systems develop, through which finally an external world can appear to consciousness and to the ego" (Husserl, 233). The gradient of perceived change demonstrates geometry of space-time through difference in measured circumference applied to the Pythagorean theorem, has directionality and mimics the form of intentionality, thus it will be proposed as the germ from which the proper structures spring. The intuition involved is correlated to the resonance of the visual world with the consciousness-based movements of visual system, "(the perceiving-lived-body) does not come into consideration as a perceived spatial thing, but rather with respect to the system of so-called 'movement sensations' that run their course during perception, in eye movements, head movements, etc." (Husserl, 227). Through phenomenological description of a perceived difference in circumferences equaling an arc length, a few characteristics of the structure of the being of perceived difference (within which these movements occur) can be visualized and understood to contain a multitude of valid expressions and gradients of surfaces with contour.

Illusion still exists in certain forms in the perception of certain visual images, and instead of doing away with it, a stronger perceptual theory would hope to confront and understand how it works. The phenomenological approach did away with illusion, but it cannot be cast aside completely. The Gestalt theory tried to understand illusion through the laws of sensory organization. One of these laws dealing with illusion will be shown to have computational application to sense data and the structure of received radiant energy.

It is arguable that the a priori, geometric truth of difference in circumferences of two circles applied to Pythagorean theorem to form a *transformation cone* is a valid meaning to the comprehensiveness of an element of eidetic noesis. Husserl notes that, "a long and thorny way starting from purely logical insights, from insights pertaining to the theory of signification, from ontological and noetic insights, likewise from the customary normative and psychological theory of knowledge, to arrive at seizing upon, in a genuine sense, the immanent-psychological and then phenomenological data, and finally to arrive at all at the concatenations of essence which make the transcendental relations intelligible a priori" (Husserl, 86). I would add that yes it is, however, the notion of perceived as a whole, or, "that distance from my eye to the point out in the world on the table," is describable in terms of a balance of the structure of moment and univocally expressible of being with time and time with space. This is useful, and the Gestalt movement came across a few of the psychological foundations for studying the properties of such a "perceived as a whole." The general principle can be further simplified through expressions of provable, a priori truths of mathematics. Thus, mathematics can be a language for phenomenological description. It describes a perceived relationship of parameters through a gradient (correlation of those parameters). And, it discusses that which is given originally in perception (the entire perceived contour of surface layout).

Gibson uses the example of psychophysical experimentation. As I would describe it, if we think of psychophysical experimentation, three meanings come to mind. In one branch of psychophysical experimentation, the mind is used to visualize mechanical, logical, informational, spatio-temporal, dynamic, or ontological relationships that are present to the perceptual sphere of influence. In the other branch of psychophysical experimentation, experiments are performed through phenomenal description of an experiencing through a perceptual apparatus. In a third branch of psychophysical experimentation, actual impulses within perceptual systems are measured. Psychophysical experimentation in which actual nerve impulses are measured or is a method by which sense data have been studied.

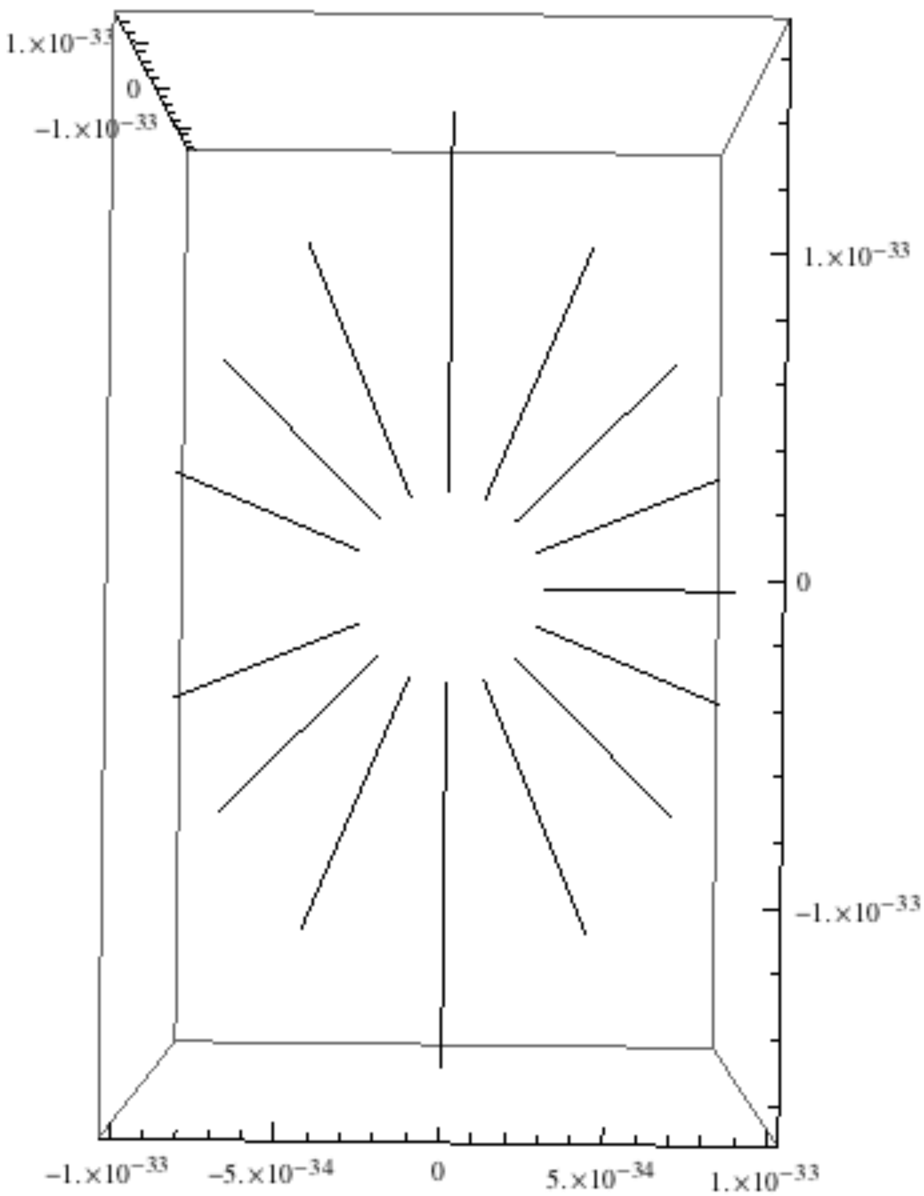
Husserl's work in eidetic intuition, difference in moment impressions, as well as his writings on mathematics and ontology's relation to phenomenology as a descriptive language was fundamental to the description of how the subject comes to knowledge of the perceptual world, "the actually spatiotemporal idealities, as they are presented firsthand in geometrical thinking under the common rubric of 'pure intuitions,' are transformed, so to speak, into pure numerical configurations, into algebraic structures" (Husserl, 351). We can draw on Husserl's work in the phenomenology of internal time consciousness through the analogy of form in the continuous function of the two moment impressions (Husserl, 197) of two perceived circles as an arc length. What we see in the mathematics of the difference in moment impressions is a reality, thus the perception of the objects in the world is related to the reality of the objects in a univocal perceptual framework.

Theorem three of *The Geometric Pattern of Perception Theorems* (Emmerson, 2009) exemplifies the principle described by Husserl when he comments on the development of a, "methodical transition from geometry, for example, to pure *analysis*, treated as a science in its own right; and the results achieved in this science can be applied to geometry" (Essential Husserl, 351). Theorem one develops to contain an, "unnoticed, displaced, 'symbolic' meaning" (Husserl, 351), because it delivers real manifold surfaces.

4. Gestalt Theory

1.1 Introduction to Gestalt Theories

The Gestalt theory is a broad theory of mind and psychological postulates, for instance that the conscious experience corresponds to the activity in the brain. With specific reference to perception, Gestalt theory holds that we should begin psychological analysis by looking at the experiential phenomenon, and not focused solely on sensations. The Gestalt theorists outlined several principles to the kind of systems they include, specifically, reification, emergence, multi-stability, and invariance. Reification is the principle that involves illusory contour, which the computational theory of perception also addresses, and a generative property of perception. Gestalt theorists have a main principle that says we tend to organize visual experiences in the simplest and most orderly way. There are patterns used for illustrating the perception of virtual (illusory) contour during visualization of the structure of light energy, so the theory is pertinent to the sense data ideas covered later as well as the computational approach. These four principles of how perceptual systems operate are accompanied with the law of Pragnanz. Section X of *A Geometric Pattern of Perception* (Emmerson, 2009), is evidence that the diagram of illusory contour, a principle of Gestalt psychology, is present within the geometric/algebraic form of a light wave.



1.2 The Principles

The experiential space relies on a conscious perception. Gestalt theories look at how information is grouped by the perceptual system as well as how perception of representations of objects out of patterns are delivered to the perceiver by the law of emergence. In such cases, the objects or images coming to perception out of these patterns are perceived as a whole. Math and the body of sciences can be useful in modeling the elements involved in perceived change over time and its relationship with of subjective experience if enough relevant contextual information is given to the model.

Kurt Koffka[10] was an early Gestalt psychologist who wrote Principles of Gestalt Psychology, in 1945. The facts of direct perception can be related intimately to observing and experiment. Koffka considered the relevance of of spatial organization for perception. Specifically, Koffka says on the idea of four dimensional organization that, "we have argued ourselves (in Chapter X) that reality cannot be treated in terms of space alone, that the time dimension must necessarily be included"[11]. How is time different from space? A point can move along any spatial dimension without running along another by traveling in a straight line. However, no point can move along a spatial dimension without the distance it travels along one of the three commonly accessible spatial dimensions being expressible in terms of only a time-like dimension. This is the usefulness of the third dimension; it allows the time dimension to take place, and is thus an implicitly four-dimensional space expressible through algebra and geometry upon the consideration of the time-like variable's expression in terms of space.

Reification is that generative aspect of perception responsible for such phenomena as virtual contour. One can create a whole realm of designs that, when perceived, produce the experiential effect of seeing something that hasn't actually

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[10] Kurt Koffka was born in 1886, and died in 1941.

[11] Koffka, K. *Principles of Gestalt Psychology (International Library of Psychology)*. 1 ed. New York: Routledge, 1999. Print.

been drawn out. For instance, in the case of three separate pac-men placed in a circle, their mouths facing the middle, one sees an equilateral triangle, even though there has not been one drawn out explicitly. A diagram delivering the effect of virtual contour when perceived is present in *The Geometric Pattern of Perception Theorems* (Emmerson, 2009) Section X. This diagram is found from visualizing the structure of the energy of a wavelength of light traveling with wavelength equal to height of the cone, which is described through difference in circumferences of two circles applied to the Pythagorean theorem when the radius of the initial circle is always kept as the slant of the cone. This serves as a mathematization of the diagram of illusory contour, as well as a relation to the structure of light in theory. The illusory, or "virtual" contours are interpreted by the visual system as real contours. Light energy, if its wavelength is described by the height of the cone, delivers a diagram of illusory contour. The height of the cone is a distance, though it has expressions in terms of initial radius and angle taken out of the initial circle.

Gestalt psychology also introduces the idea of multi-stability, which is a phenomenal experience of perceiving a pattern or shape as representative of two things or orientations at once. In the case of the image of a duck/rabbit some, like Norwood Russell Hanson, argue that, "there is no place in the seeing for these differences, so they must lie in the interpretations put on what we see"[12] (BRP, 296). Yet, does this not place a wholly subjective interpretation of the phenomenon, suggesting that Hanson may be endorsing subjecticity, which was outlined in the

[12] Shwartz, Robert, ed. "From Patterns of Discovery." *Perception (Blackwell Readings in Philosophy)*. Malden, MA: Wiley-Blackwell, 2003. 51. Print. All further references to this source will be cited parenthetically in the text.

Husserl section, a bit too much? Symbolic ducks look like mirror images of symbolic rabbits. This is not a controversy of whether you perceive a duck or rabbitt. It is however, an image exhibiting the trend of multi-stability, the phenomenon of an ambiguous visual depiction enabling multiple interpretations of the design such that it goes back and forth between possible orientations or interpretations. Gestalt psychology has not yet tried to describe how this happens, but primarily focuses on studying the experiential phenomenon.

The Gestalt principle of invariance simply refers to the fact that a distinct object, when rotated, will remain identifiable by the perceptual system upon rotation or certain transformations like elastic deformations (stretching), or variations in lighting or shading. One can rotate an object and identify it as the same object.

Emergence is the Gestalt principle that describes how an image is formed as a whole, not by recognition of its component parts. Emergence describes the mosaic-like, well organized and logically constructed, subtly perceived pattern configurations by the visual system. Gestalt psychology establishes these principles as descriptive, non-explanatory principles.

The law of pragnanz, or pithiness, is a fundamental tenant of Gestalt perceptual theory and proposes that the visual system organizes information in a systematic, regular, symmetric and simple way. What will be discussed in part seven with regard to this law is that there are fundamental, simple ontological structures of perceived difference that can phenomenologically express perceived space-time as a unified system, including an invariant, exhibiting reification, analogies to emergence, as well as characteristic components whose expressions can be used to render commonly encountered visually textural surfaces in the world through objectivities of gradients. These objects are perceived invariantly in the sense that they are capable of recognition upon rotation, though they also contain a parameter that is invariant through the transition described by the system of the geometric relations of the perceived difference. Some of these objects also exhibit multi-stability. In fact, it has been suggested that the principles of Gestalt psychology may be able to be traced to a single mechanism. This mechanism would be simple, because it is phenomenologically based in eidetic ontology, organizational, and related to pragnanz through its simplicity and symmetry.

In section seven, it is this idea of pragnanz that will be the focus of the Gestalt aspects included in my own account of perceptual theory. In section seven, the simplicity of formal ontology will provide a groundwork for which certain phenomena studied by Gestalt philosophers like multi-variance, illusory contour, invariance, may be shown to be present in different interpretations and applications of fundamentally the same system mathematically. This system delivers visualizable functions with the characteristics of multi-variance, illusory contour, and invariance. Mathematically based experiments (in the sense of mathematics as a branch of science) using the geometric framework of space-time described by the geometric pattern of perception theorems include Gestalt phenomena. The fact that this system is simple, univocal, symmetric, systematic, and regular suggests that it may be involved in the workings of the perceptual system, and will thus be proposed as an integral, fundamental part to perceptual theory.

5. The Computational Approaches

The computational approach to visual perception was advanced greatly by the work of David Marr[13]. Marr's primary themes are described in a book by Vicki Bruce and Patrick Green entitled Visual Perception: Physiology, Psychology, and Ecology[14]. Marr came up with the idea of a 2.5 D sketch, but first came to several realizations, including the *primal sketch* and the visual pathway. The visual pathway allows us to understand, "how the pattern of light falling on the retina is transformed into a symbolic representation of the environment, in which the positions, orientations and movement of surfaces are made explicit" (Bruce and Green, 72). For Marr, the computational theory is then somewhat of a representationalist account.Marr used computational algorithms to develop machines that could begin to recognize objects. "An algorithm for finding edges begins by locating changes, or *gradients*, in light intensity in the image. The simplest algorithm which could be used to do this would be one which computed differences in light intensity in each region of the image" (Bruce and Green, 73). The intensity of light can be plotted in a coordinate plane as a number. Thus, each point of intensity could each have an associated radius with its distance from one of the axes. The difference in the circumferences of the circles produced by these radii (graphed

[13] David Marr was a British neuroscientist born in 1945. "David Marr (neuroscientist)." *Wikipedia: David Marr*. N.p., n.d. Web. 2 Apr. 2010. <en.wikipedia.org/wiki/David_Marr_(neuroscientist)>.

[14] Bruce, Vicki. *Visual Perception: Physiology, Psychology and Ecology*. 4 ed. New York, New York: Psychology Press, 2003. Print. All further references to this source will be made parenthetically in the text.

light intensities) would then be of some mathematical use to the analysis of such data, because it describes a gradient of the context of the change in intensity in a simple way while also discretely being able to locate a position of a point source within the area of a sample. In the analysis of the height of the cone described by the geometric pattern of perception theorems through complex numbers, there are several gradients and contours that describe simple edges like that of a table as well as more complex curvatures of planes. A changing distance in space-time is not disassociated from its surroundings, and in fact contains contour through associated gradients of them.

The ontology of difference in circumferences of two circles that describes a cone is related to the David Marr's theory of perception. "Marr has shown that if a surface is smooth, and if these assumptions (those for interpreting silhouettes) hold for all distant viewing positions in any one plane, then the viewed surface is a generalized cone. Thus shape can be derived from occluding contours *provided* the shape is a generalized cone, or a set of such cones" (Bruce and Green, 183). I propose that the difference in circumferences of two circles applied to the Pythagorean theorem is a new way for describing a general cone, and thus it provides a new method for studying psychophysical data or the data of the composition of a viewed surface. It contains and describes a gradient of perceived change mathematically in a simple, clear, and concise way. In addition, 3D modeling helps us visualize the gradient of perceived change. "Use may be made from stereopsis (the difference in retinal projections on one eye from another and their interpretation), texture and shading to achieve this, but it may also be necessary to use preliminary matches with stored 3D model descriptions to improve analysis of the image" (Bruce and Green, 184). Thus, it may be seen that the collection of objects provided in the geometric pattern of perception theorems may be useful to the creating image processors, for in descriptive contours of a cone are a practical wealth of 3D models available to help image processors read information about reality. They also are relatively simple, small files and would not be hard to access quickly with modern computing power.

Simon Ullman is a cognitive scientist focusing in visual perception who acknowledges the problem of a temporal coding of information available to the visual system, "a neuron may respond to two different stimuli with the same mean firing rate, but signal two different events by the shape of the temporal firing pattern" (BRP, 244). He goes on to say that, "in such a case, it is not meaningful to ask merely how strongly a given unit responds to a given stimulus; the more relevant variable is the temporal pattern of the response" (BRP, 224). Such a realization could open the realm for theorizing on the perceptual stimulus within a cognitive experience of cyclic time, phenomenologically described (through perceived difference of circumferences), connecting ecological optics, sense data, and the Gestalt theories, because the stimulus energy can be related to the size of a circumference for the purpose of geometric manipulation and signal interpretation, and the geometry is a unified structure.

The computational approach has had some success in outlining why the experience of illusory contour is had by a perceiver. The argument, outlined by

Churchland[15], commenting on the work of David Marr, goes like this. “When an object is placed in front of another object, the depth boundary usually progresses smoothly across the image. So discontinuities in depth boundaries should be filled in” (BRP, 224). David Marr proposed that during the computational approach, one should, “devise the computation of the module so that it conforms to whatever we know about the underlying physiology, about psychophysics, and about the evolution of the brain” (BRP 220). This suggests that the geometry of ecological optics (surface geometry) would be applicable, as well as the phenomenology of our experience of the nature of time and perceived difference if experiments of mathematical thought and phenomenological science are considered to be psychophysical.

With relation to perspective, for binocular vision, the computational approach begins by describing its basic geometry. We can note that there is geometry where the line of sight of each eye comes to a point out in the world. When addressing the issue of a contoured object, the computational approach suggests that we form an initial outline by, “aggregating descriptions from the raw primal sketch,”[16] but that the visual system goes on to, “make use of the *image-space processor*,” which consists of an, “object-centered representation, consisting of an axis-based structural description… established from an image and used to access a stored catalogue of 3D model descriptions in order for recognition to be achieved”

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[15] Patricia Smith Churchland is a Canadian-American philosopher working at the University of California, San Diego.

[16] Bruce/Gree. *VISUAL PERCEPTION: PHYSIOLOGY PS*. 1 ed. New York: Psych Press Uk, 1984. Print. All further references to this source will be cited parenthetically in the text.

(Bruce and Green, 184). In this way, Marr's use of the computational approach is evidence for the validity of intentionalist theory, because the perception is described like an object-centered representation. The intentionalist theory proposes that the form of perceptual experience is a mental representation of the object. The content of the experience of a strawberry resides in the object that is a strawberry as a potential for experience, specifically so long as there is a receptor capable of experiencing the strawberry. There is a wide variety of intentionalist theories, but the main thing that they all try to do is propose that the form of perception is a representation of the object.

The elements of the computational approach to visual perception that will be integrated into the account given in section seven of the unification of perceptual theory are its use of 3D models in an image-space processor and approximation of a contoured surface like a set of cones.

6. The Ecological Approach

Gibson's ecological approach to visual perception is a philosophical and psychological theory of perception in which the experience of the terrestrial environment is taken into consideration, as well as the senses as perceptual systems. Ecological optics is a theory concerned with how we perceive the, "surfaces, their layout, and their colors and textures"[17]. When returning to Gibson's

[17] Gibson, James J.. *The Ecological Approach To Visual Perception*. 1 ed. Hillside, NJ: Tf-Lea, 1986. Print. Page 1. All further references to this source will be cited parenthetically in the text.

1966 work, The Visual World[18], in my paper entitled, *The Origin of Mathematical Analysis of Ecological Optics*, to look at examples of surface textures, and depth perception, mathematical gradients found through formal ontology of perceived difference in circumferences will be correlated to the characteristics of those surface layouts. In this section, I will provide an account of Gibson's linguistic description and use of medium, surface, and optical information because it is fundamental to the understanding of surface layout when mathematically visualizing the *surface-gradient* of perceived change in circumferences.

Ecological optics is a broadly inclusive theory that begins by discussing the animal and its surroundings, which are that animal's environment, continuing the discussion by establishing the ideas of, "*medium, substances,* and the *surfaces*," (Ecological Approach, 16) that separate the medium and substances. For Gibson, a surface is the interface between any, "two of these three states of matter, - solid, liquid, and gas" (Ecological Approach, 16), while the gases and liquids are, "*medium(s)* for animal locomotion," (Ecological Approach, 17) and "a terrestrial medium is a region in which light not only is transmitted but also *reverberates*, that is, bounces back and forth between surfaces at enormous velocity and reaches a sort of steady state" (Ecological Approach, 17). The reverberation of the light and its flux is called illumination, and illumination, "fills the medium in the sense that there is *ambient* light at any point, that is light coming to the point from all directions" (Ecological Approach, 17). The space between points at which there is ambient light can be described in terms and expressions of theorem one in the geometric pattern

[18] Gibson, James Jerome. *The Perception of the Visual World*. Boston: Houghton Mifflin, 1950. Print.

of perception theorems, for the ontological results are general expressions for perceived distance. However, it's important for us to consider the gradients of the components of that system in terms of variables of the system, for these provide descriptions of the layouts of surfaces in different ways.

Contour is related to information, because it can be described by surfaces and gradients. Gibson's theory is an information based theory of perception.

For Gibson, the medium has certain characteristics, and the characteristics of mediums are differentiated from those of substances. For instance, the idea that substances, "do *not* freely transmit light or odor and that (they do) *not* permit the motion of bodies and the locomotion of animals" (Ecological Approach, 19) is unlike a medium, which allows the passage and reverberation of light and the locomotion of animals, "the medium contains information about things that reflect light, vibrate, or are volatile" (Ecological Approach, 17), and for Gibson, substance is perceived. In essence, mediums are composed of materials like gasses or liquids for the cases of motion and visual perception, and these gasses are unlike solids in that they can be filled with light and moved through. For Gibson, mediums specifically afford respiration.

Gibson says that,

> "My description of the environment (Chapters 1-3) and of the changes that can occur in it (Chapter 6) implies that places, attached objects, objects, and substances are what are mainly perceived, together with events, which are changes of these things" (Ecological Approach, 240).

Thus, this realization of Gibson's can be used to confront the issue of the perceived surface layout of the substances in principle. The perceived difference in circumferences must then be a description of the gradient of perceived difference through space-time mathematically. The gradient of the parameters of the spatiality of the perceived difference qua the Pythagorean theorem yields a formal ontology of the surface layout of the gradients of perceived difference through a structuring of space-time around that perceived difference. Some of these gradients correspond to Gestalt principles (pragnanz, interpretations of reification and illusory contour), surface layouts in the man-made environment, and even forms in the natural environment (like that of a shell). The points of the system are an ideal framework for the perceived difference, however they do designate certain kinds of motion, ideal objects, and positions (places).

In essence, Gibson's notion of medium as differentiated from surface layout and space will be very important, because, in Gibson's theory, although the gradient of the formally ontological relations of perceived space-time through perceived difference (as described in the geometric pattern of perception theorem) is probably not to be considered as a medium, "the notion of a medium, therefore, is not the same as the concept of space inasmuch as the points in space are not unique but equivalent to one another" (Ecological Approach, 17), the gradient of perceived spatio-temporal relations relating to formal ontological elements of a perceived surface may be useful to describing the layouts of surfaces (interfaces of two of the states of matter) in terms of perceived contour. Gibson's reasoning to the conclusion that medium is not the same as space assumes that points in space are not unique. However, he does concede that, "instead of geometrical points and lines, then, we have points of observation and lines of locomotion" (Ecological Approach, 17), not in any way dismissing a geometry of those lines of locomotion or points of observation with relation to formal ontology of what is perceived (the substance, events, and change (change is related to the continuous function of difference in circumferences)). In many ways, the difference in circumferences of two circles applied to the Pythagorean theorem describes or expresses a line of locomotion through the height of the cone, because the structuring of space includes a cyclic, angular variable, theta, associated with the wedge taken out of the initial circle. The height of the cone has directionality, and as more of the initial circle's circumference is translated into the height of the cone, it extends into space less per increase in the amount taken out of the initial circle. This is a transformation, but the height of the cone could be considered a line of locomotion with velocity if the angle is set to pass constantly with time, such that time may be measured like a clock, except in the units of radians. The structure of the aforementioned perceived difference in circumferences, or simply arc length taken out of a circle, necessitates acceleration of the height of the cone if the angle passes constantly with time.

In some ways, Gibson's assertion about spatial points being non-unique can be seen to be true, but in other ways, it is clear that points designate the boundaries and "afford" language for discussing surface layouts of unique objects as well as unique orientations of those objects and through their relationship and the gradient of this relationship, surface layouts of faces of objects and have necessitated geometric structures within continuous functions depending on how the perceived locations of the perceived objects designate distances and changes of those distances for the perceiver. The surface layouts found by the gradient of space-time described by the mathematics for perceived difference in circumferences exist in a purely ideal sphere until given hardness, thickness, or opacity, though the planes, and curvature of those planes are by default opaque. Opacity is one of the easier perceptual phenomena related to surfaces to discuss and manipulate with my present mathematical visualization capabilities. The structuring of the ambient light in the gradient of perceived space-time is also capable of being taken into consideration, and Gibson does this through the ambient optic array, which is discussed in my paper, The Origin of Mathematical Analysis in Ecological Optics.

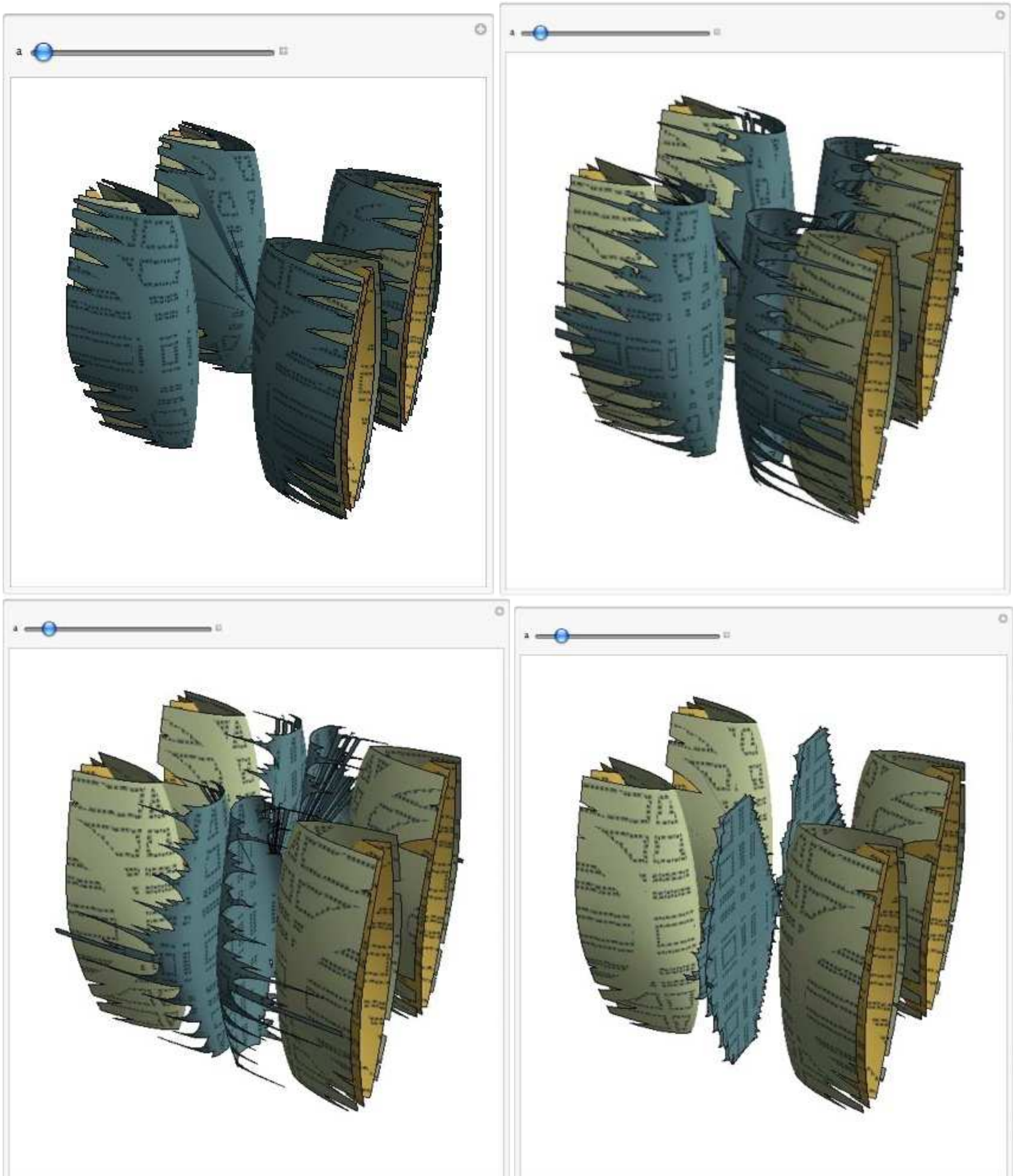
In order to integrate Gibson's account into the mathematics of difference in circumferences of two circles, which is formal ontology of perceived difference as perceived, a differentiation between the layout of a surface and the surface's substantial composition for visual perception must be established. Gibson outlines a difference in surface geometry and abstract geometry. I consider that abstract geometry might be thought of as the idea of a circle folding into a cone taking place when the circle is made of an ideal, frictionless, infinitely thin plane, its illustration through either diagram of its parameters, or of performing the motion with a sheet of paper in order to crudely represent an idealized plane, as well as including algebraic expressions of the parameters of the system. In this way, abstract geometry has not yet extrapolated the surfaces described by the gradients of the correlation of the parameters, whereas surface geometry is the geometry of the layout and contour of those mathematical expressions' gradients as graphed or "visualized" by a certain graphing function appropriate for graphing an equation depending on its specific kind of variables. However, Gibson described this difference as,

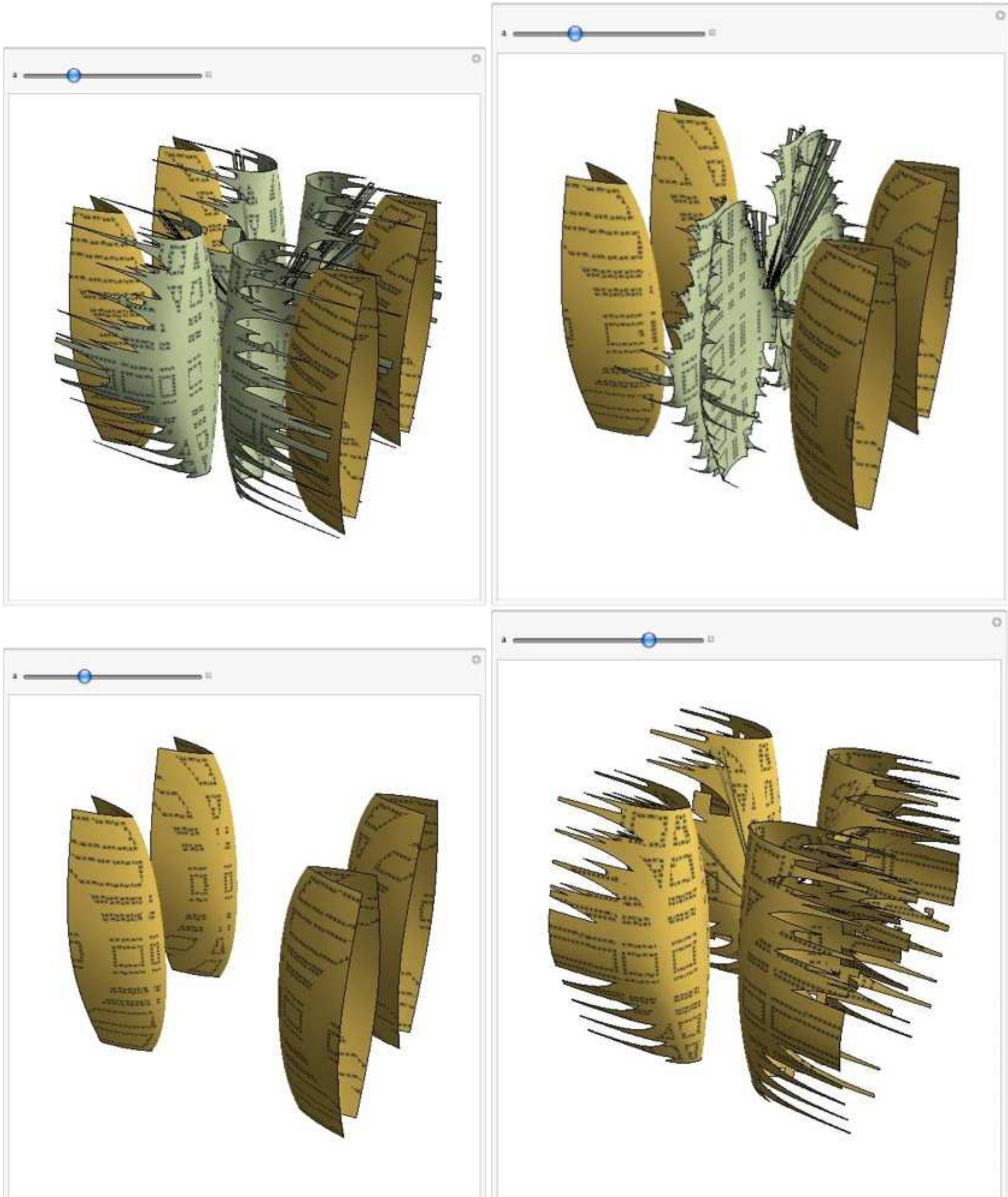
> "Planes are colorless; surfaces are colored. Planes are transparent ghosts; surfaces are generally opaque and substantial" (Ecological Approach, 33), and "Finally, in abstract analytic geometry the position of a body is specified by coordinates on three chosen axes or dimensions in isotropic space; in surface geometry the position of an object is specified relative to gravity and the ground in a medium having an intrinsic polarity of up and down. Similarly, the *motion* of a body in abstract geometry is a change of position along one or more dimensions of space, or a rotation of the body (spin) on one or more of these axes. But the motion of an object in surface geometry is always a *change in the overall surface layout*, a change in the shape of the environment in some sense" (Ecological Approach, 35-36).

Thus, we see that the geometric pattern of perception theorems are useful to ecological optics, because they describe surfaces, some of which exhibit somewhat mysterious phenomena like shimmering or disappearance of surface at certain view angles, as well as more common phenomena like evidence of stretching or dimples. These dimples, mysterious color changes, and stretch marks are evidence that the gradient of change describes a surface, not just an intersection of planes. The surfaces can also be changed or imagined to stretch through mathematical tools. A motion in Gibson's ecological optics is a change in overall surface layout, and can be described by programmable functions in Mathematica[19]. For example, change in overall surface layout is evident in figure group A, while stretching and dimpling are evident in figure group B.

[19] "Wolfram Research: Mathematica, Technical and Scientific Software." *Wolfram Research: Mathematica, Technical and Scientific Software*. N.p., n.d. Web. 3 Apr. 2010. <http://www.wolfram.com/>.

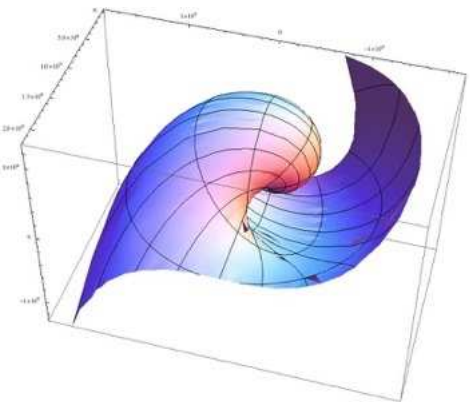
Figure Group A

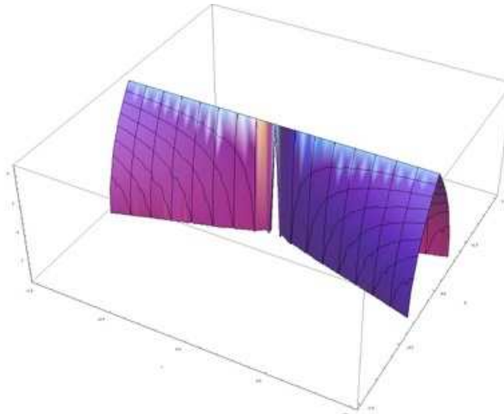




These figures describe a change in surface layout, which is a motion in surface geometry.

Figure Group B





In figure group B, we see two simple examples of striation, or dimpling of the surface.

In relation to gravity, one might think in general theory that, because the structure of the cone necessitates a change in the rate of change of position geometrically, its structure might be useful to understanding the acceleration due to gravity mathematically. Gibson says that, “the physical vertical of gravity, we may conclude, is somehow implicit in all such tilted visual fields (when they result from a voluntarily tilted or reclining posture),”[20] and that, “in the activities of ordinary behavior we may infer that there is a visual vertical-and –horizontal frame of reference which is linked to gravity and is presumably mediated by the muscle sense and the inner ear” (The Visual World, 33). The cone shoots, “up” from the initial circle, initially on the ground, decelerating, as the apex gets farther away from the center of the base of the cone. Therefore, down for the visual field is represented by acceleration, as opposed to deceleration. Acceleration would be the apex’s

[20] Gibson, James Jerome. *The Perception of the Visual World*. Boston: Houghton Mifflin, 1950. Print. Page 33. All further references to this source will be made parenthetically in the text.

moving toward the center of the base of the cone, to the initial circle, which would be on the "ground."

Within the objects described by the gradients of the geometric pattern of perception theorems, we see those exhibiting the characteristics of surface layout that Gibson describes in the chapter of The Ecological Approach to Visual Perception entitled The Meaningful Environment. Notably, those visually evident surfaces forming *hollow objects*, *partial enclosures*, *places*, *sheets*, *fissures*, and *sticks* are evident.

A very important and notable part of Gibson's theory is its mention of evolution's effect on visual perception. Gibson says that,

> "The motion of the sun across the sky from sunrise to sunset has been for countless millions of years a basic regularity of nature. It is a fact of ecological optics and a condition of the evolution of eyes in terrestrial animals" (Ecological Approach, 87).

Gibson said in Perception of the Visual World[21] that, "space perception (from which time is inseparable) is not, therefore, a division of the subject matter of perception but the first problem to consider, without a solution for which other problems remain unclear" (The Visual World, vii). He goes on to say that, "what we lack, however, is an application of the psychophysical methods to perception" (The Visual World, viii). Psychophysical methods of perception are related to the environment in which perception evolved to its current level of accuracy and illusionary aspects, and as outlined before, there are three methods of

[21] Gibson, James Jerome. *The Perception of the Visual World*. Boston: Houghton Mifflin, 1950. Print. (All further references to this source will be made parenthetically in the text.)

psychophysical experimentation, those through thoughts relating to pure geometry, those relating to measurement of nerve impulses, and thus involving perceptual a perceptual apparatus. Our measurement of time is related to the perceptually evident environmental description of time, which is ecologically cyclic and continues out in every direction and evidently involves the complex processing of the visual system. I find that we have evolved with a cyclic perception of time through consideration of the seasons, day, night, and cycles of the moon. The wind does not blow at a constant howl, but dies down and howls with different volume and rise and fall. Light effects the seasons, because it allows plants to grow, it gives warmth to the air, and it influences our sleep schedules. Weather affects our sleep schedules and we hibernate during winter. These are all cyclic phenomena (we experience the day and night in similar frequency regardless of how many we have lived through?), which directly effect living organisms. One element of the environment's effect on the perceptual system in terms of the rhythm of day-to-day experience is the cyclical measurement of time like a clock. Day and night come in a period. This is one interpretation of the ecology of our optical environment that I propose is relevant to perceptual experience in general, including visual perception and the sampling of the ambient optic array. Within the description of any perceived space-time, we understand that that individual parameter of perceived difference, and thus the notion of a parameter of perceived "space-time," is not disconnected or disassociated from any other. I assert that it can be shown that space or time inherently contains surfaces in the correlations of its parameters, because the surface affects the quality of the opacity and indentation. A single dimension of mathematics of space-time contains a contour inherently. We have evolved with a cyclic perception of time through the seasons, day, night, and cycles of the moon. Light effects the seasons, because it allows plants to grow, it gives warmth to the air, and it influences our sleep schedules.

Optical information, for Gibson, involved his concept of the optic array, which described the subject's perception while moving over the ground through lines of optic flow. The optic array is discussed further in the paper on the origin of mathematical analysis in ecological optics, and it is important to realize that for Gibson, "the two kinds of 'motion,' physical and optical, have nothing in common and probably should not even have the same term applied to them" (Ecological Approach, 103). A motion described by formal ontology of difference in circumferences is not much like a physical motion, but a pure relationship of parameters for perceived difference, until the angular variable is associated with time, so it is possible that the pure ontological aspect may be useful to studying optical motion and optical information. Gibson's concept of optical information is consists of messages, signs, and signals, so we see that it is related to sense data theory (Ecological Approach, 62).

A conic section of a sample of the array would be the noticing of a difference in radii and the relationship of depth within this change due to the optical information available from the environment. The theory of ecological optics states that during forward locomotion, the directions of deformations in the ambient optic array are projected on a spherical surface around the head, "The gradients of velocity and direction are invariable accompaniments of locomotion if the observer keeps his eyes open" (The Visual World, 123). The distance from the position of the eyes of the observer to a point out in the world can be measured in terms of the folding up of the 2-D circular plane whose diameter is tangent to the sphere of the perceptual field and orthogonal to the line of sight of the perceiver to a point out in the world.

The ideas of Ecological Optics that are integrated into the accounts of perception to which I am most sympathetic and my own account of perception are its description of surface layout, and optical information, and I will use them to show that abstract geometry, while differentiated from surface geometry, can lead to forming surface geometry, and thus provide a language for describing real surfaces, not just invisible planes. The ecological approach embraces progress in psychophysics, ontology and epistemology.

Gibson's distinctions between surface geometry and abstract geometry are fundamental insights into the meaning of geometry to perceptual phenomena. His ideas of the ambient optic array and optical information are geometrically interpretable. Through form, the ambient optic array can be described geometrically. So, we see that the environmentally contextual information necessary for perception is provided to the perceiver geometrically. I suggest that the visual system performs real-time ontology of the surface layouts of the world based on *apriori* experience and the cognitive contextualization of nature's intrinsic multi-conic framework.

7. Conclusions: Pattern, Consequence, and Unification of Perceptual Theories in Theory

In this section of my account of an overview of perceptual theories, I will describe how the theories described in the first six sections of this paper work together, are useful to each other, and can be related to one another. Once the geometric pattern of perception theorems are noted as being characterized by the law of Pragnanz, they have been related to Gestalt psychology. The computational approach tells us that in the case of a perceived lamp silhouette, the contour can be approximated by a series of generalized cones, thus the gradient within the parameters of a general cone describe the gradients and contours. Sometimes the parameters correlate to each other in such a way within the surface of a mathematical object to produce an effect that I call *shimmering*. Shimmering shows us that the objects produced by the algebra and geometry of the cone have surface texture, information, and are not just invisible planes. When the opacity is decreased to zero, the surface effect of shimmering is no longer present. Shimmering is present in the visualizations of Theorem 3 and upon and throughout the different adumbrations of 3D graphs in *A Geometric Pattern of Perception* (Emmerson, 2009).

Figure Group C

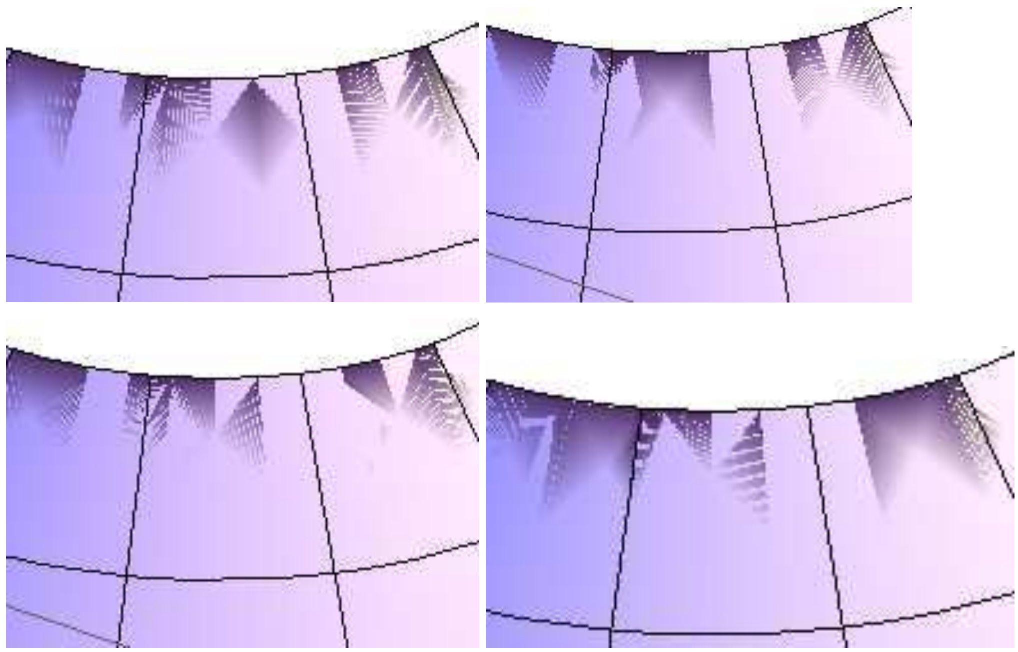


Figure group C portrays four shots of the change in flickering surface texture of the same surface area upon slight rotation or zooming of the object.

Shimmering is the effect where a geometrically described surface flickers upon rotation or zooming. This is different from distinct indentation, stretch marks, or radiantly colored contours that do not shift spontaneously on rotation, but may coincide with the crease of a folded surface or intersecting of surfaces. For instance, we see that marks resembling stretch marks can often occur when substitutions are made within a function. For instance, in theorem three, lemma nine, a function that produces a simple circular disk, but when having substitutions applied within its associated equation, produces completely different, ontologically discovered objects. Some of the resulting objects have indents or “stretch” marks. In others, before making a substitution from known lemmas of the system, the solution described a sphere, and after providing other information about structuring of the system to the expression of the function, we see twisted manifolds and dense, contoured solids as well.

Gibson said,

> "At the level of microlayout (texture) and microcomposition (conglomeration), layout and reflectances merge. Or, to put it differently, the layout texture and the pigment texture become inseparable" (Ecological Approach, 87).

In the geometric pattern of perception theorems, the dimensions of the given three-dimensional forms can be scaled to smaller dimensions like centimeters, millimeters or smaller. Contours and surfaces are signified by gradients of color and shading. Perhaps, the phenomenon of shimmering in the geometric pattern of perception theorems could be thought of as an excitation of the layout texture, object center, surface structure or a combination of these.

In addition to writing about ecological optics, Gibson made considerable contributions to the semantics of sense data. In Gibson's theory, through not only the theory of ecological optics, but also the analysis of the senses, by stating that, "the pattern of excited receptors is of no account; what counts is the external pattern that is temporarily occupied by excited receptors as the eyes roam over the world, or as the skin moves over an object" (BRP, 75), he has revealed that actual structures of the energy of light (pattern-based information in external stimuli) have meaning to internal stimulus in the context of the ecological experience. He has also shown that what matters to visual perceptual studies is the phenomenology of the external pattern of a perceived surface as perceived, supporting formal ontology.

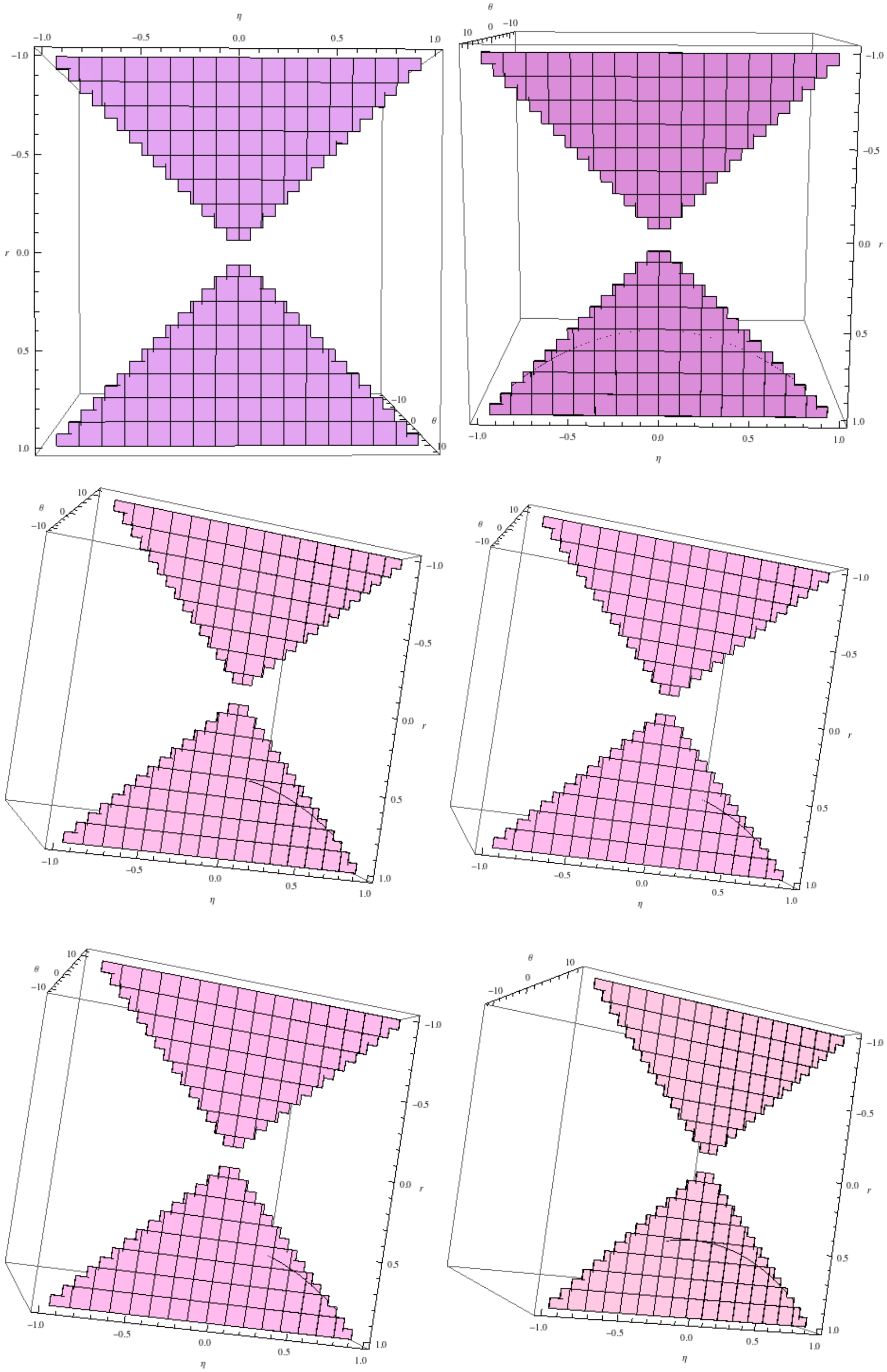
When one perceives something, they perceive discreet data, for in a univocal system, everything is numbered. Selecting a given angle taken out of the initial circle algebraically designates an exact initial radius. This is an inherent truth of the algebraic mechanisms through formal ontology of perceived difference in circumferences. Thus, the interpretation of the data in the world is well described by Gibson's language of surface layout and ecologically optical philosophy grounded in phenomenological method and description, and the linguistics refined by philosophical discussions of sense data and computation. Thus, mathematics can be seen as a valid language for the description of surfaces and potentially lead to insights about their perception and its mechanisms once the characteristics of the surfaces are linked to perceptual philosophy, ecological optics, and sampling of the ambient optic array.

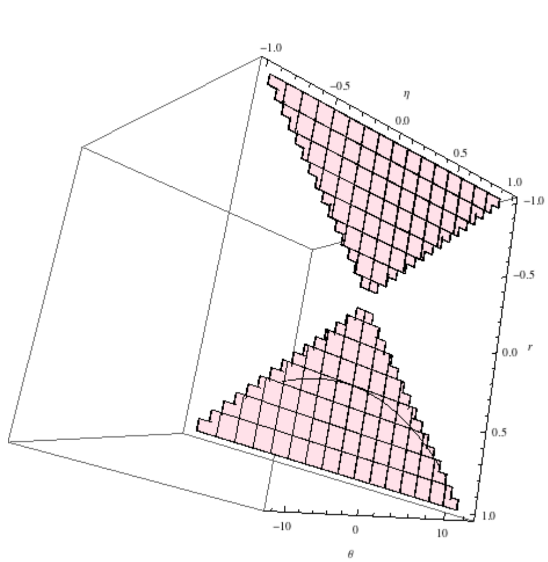
Also, the theory of ecological optics is related to sense data by distinguishing between the, "light that is seen and the light by which things are seen" (BRP, 160). The sense data theory holds that there is a difference between the information available in stimulation, which is a conscious experience related closely to physiology, and the information in external information, which is a physical process. Mathematics carries information within the structures it describes. However, mathematics also delivers numerical, exacting results.

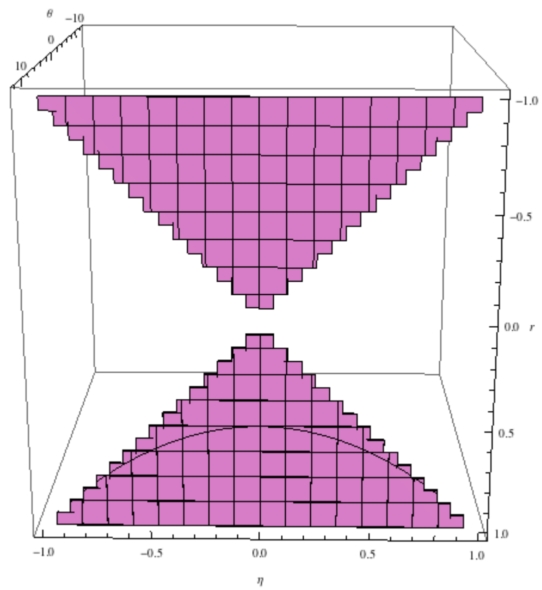
The eventual union of the notion of ecological optics with the idea of sense data will be to see how Gibson' s idea of a gradient is inherent in the information contained within light, because radiant energy is the kind of energy available during visual perception, "there are subordinate and super-ordinate components of the world and corresponding subordinate and super-ordinate forms in the array, each level of units being nested within larger units" (BRP 164). This nesting of forms is similar to the making of substitutions, and each level is like the degree to which a substitution is made. This radiant energy for visual experience could theoretically come from the mind during the process of visualization during the time it takes to remember a series of past experiences, but for visual perception of a lit up world, it will depend on the available information in radiant light to the perceiver through a stimulus system that has developed physiological and linguistic structure for visual perception. Theoretically, this energy can be shown to produce the same form as patterns of virtual contour in the Gestalt and Computational approaches to perceptual theory, because the necessary dimensions for formulating an expression for this energy (wavelength equaling the height of the cone) are grounded in formal ontology (pure geometry). Upon observing the graph of the energy of the photon whose wavelength is equal to the height of the cone, we see that the graph is sometimes controlled in an inverted (I push up, it move down, I push right, it moves left) way, evidence of multi-stability.

We also see that phenomenological mathematics can be a technique for studying illusion. I describe the perceived change in volume of a sphere mathematically with the difference in circumferences of two circles applied to the Pythagorean theorem, and I find that there is an "accretion of contour" illusion present. Visually, the line goes from being unperceivable at one view perspective and totally solid at another. In this, we see, however, that at some view-points, the contour of the parabola “accretes” – that is to say it goes from not being visually present to being made of little dots, to finally being seen as a solid line. This use of accretion and its opposite, deletion, is somewhat different than Gibson’s use of the term. It is a very literal usage. To illustrate this phenomenon, I will display eight diagrams to show the change in the line that I am talking about. The following diagrams describe the difference in volume of the sphere whose radius is equal to the initial radius of the circle, and the sphere whose radius is equal to the base of the cone.

Figure Group D







With mathematization we are given a technique for studying, "illusion," given that our mathematics is grounded in phenomenological description. Husserl says that, "mathematization, then, with its realized formulae, is the achievement which is decisive for life," and that, "we see that from the very first conceiving and carrying out of the method, the passionate interest of the natural scientist was concentrated on this decisive, fundamental aspect of the above mentioned total accomplishment, e.g., on the *formulae*, and on the technical (having the character of a technique – translator's note) method ('natural scientific method,' (of which pure geometry and its logically grounded formulae is a part – Emmerson's note), 'method of the true knowledge of nature') of acquiring them and grounding them logically and compellingly for all" (Husserl, 350). What we see in *The Geometric Pattern of Perception Theorems* is the beginning, or establishment of a technique (through mathematization) for studying the origin of visual illusions, as well as surface layout and gradients. In parts of Husserl's phenomenology, as mentioned in section two, the illusion is bracketed. If we bracket illusion, we understand that what we really see a technique for learning about the accretion of visual contour itself. The parabolic contour line is entirely "deleted" upon viewing the graph at a frontal perspective, and the line "accretes" upon viewing it at an angle.

If I were to perform a phenomenology of this graphical object, we might at first be tempted to consider this as an illusion. The illusion would be that, where there was once no perceived contour, there is seen to really be a contour upon rotation. Or, vice-versa, that the illusion is that there really is a contour, and it disappears. However, it is possible to set all of these arguments aside, bracketing them along with the natural attitude, and say that there is an *accretion* of perceived contour upon rotation of the object. Therefore, what we get at is the actual accretion of a perceptual contour, grounded in mathematization of the change in perceived volume – within the "structures of perception," and not an illusory contour in this example. As an aside, in reverse reading of the theorems, we would see the object first, and then be given the mathematical equation that describes the object.

One can rotate an object and identify it as the same object. This is described by the Gestalt phenomenon of perceptual invariance. This has something to do with the process of sampling the ambient optic array, because 1) there is an invariant part of the structure of a transforming sample (described by a formal ontology of structural difference), 2) when you move closer and part of the object is revealed, the relationship between different objects changes, but the visual system recognizes that the object itself has not changed (T. Toleno, Marlboro College), and 3) the structuring of the energy of light remains the same. There is an invariant parameter through the transformation of a circle into a cone, which is the initial radius of the circle from which a wedge is removed. This invariant radius is always the slant of the cone so long as the position of the parameters of the system describes a cone.

The descriptions of "phenomenal" velocity, found from computation in theorem three, an external pattern discoverable to be within the formal ontology of a perceived world and continuous, inherently accelerating change within it, as well as within operations of the brain and consciousness like recognition, show us that the formal ontological system of a general cone is a framework that can be used to structure information into certain, nested, surface layouts for the perceiver, which result from different patterns of substitutions within the expression for the elements of a sample of the array, and are a universal pattern. The patterns of substitution are sequences in the sense that one expression is placed within the next. The phenomenal velocity is found from validly performed algebraic cancelation, use of scientific notation, and computation.

Changes in surface layouts temporarily occupy the pattern on the receptor. In this way, the ecological approach diverges from the debate of where sense data exist by allowing the environmental context of the perceiver to be an interactive part of the experiencing, containing perceptual information. It is important that the stimulus energy be understood in the context of the environmental experience of the perceiver and analyzed accordingly.

The variables are read as a clustered expression, and can be rearranged differently through factoring. Sequence is then only pertinent to the building up of nested structures within each other during the making substitutions. In essence, consequence of the substitution is enormous to the delivered object. William James' theory that space is not ordered is pertinent, because the information of a function has deeply entrenched structure, but upon graphing the result of formal ontology, that structure is visualized only with certain given information about the total system at a time and describes the space relating to the vicinity of a contoured surface. The parameters of the system of the geometric pattern of perception describe a generalized cone and are not separable from each other, but are a systematic whole, structurally, and data-based unity, discreetly. Each parameter and the distinct function of the system's gradient with regard to the other parameters of the system is perceived to be a consequence of the whole system, and logically (based on valid algebraic manipulation), specifying a number for one variable locks in the values of the entire system of the cone. Thus, the expansiveness from ontology of perceived difference really develops upon introspection and further development of essential insight.

But where is the balance between favoring expression of the information provided by a solution by using it in an expression for a given variable, and providing more general information about the total system through the making of fewer substitutions? Husserl says that, "in view of what is genuinely exhibited in the appearance, every appearance belongs systematically to some type of series of appearances to be realized in kinaesthetic freedom in which at least some moment of the shapes would achieve its optimal givenness, and therefore its true Self" (Husserl, 232). Sometimes, when making more substitutions, no surface is delivered, just an empty, but scaled box. Substitutions give context to the information contained within the expression of the function and operation.

Some surfaces in the geometric pattern of perception theorems disappear at certain view angles, and are "reified" at others. The surface may come into existence for the perceiver at certain angles of view or level of zoom and not others, so in my interpretation, reification is the phenomenon of the process by which their coming into existence occurs. Strange anomalies are also present, like the axis label changing from a variable to a random, changing number.

We can see the origin of Gibson's notion of a medium in Husserl's phenomenology, though Gibson may have developed this on his own. The important fact is that the phenomenological account of perception is linked with ecological optics, because medium has the same meaning. Husserl says that, "transparent glass is indeed a medium that can be seen through, but it changes the images of things in different ways according to its different curvatures, and, if it is colored, it transmits its color to them – all that belongs in the realm of experience" (Husserl, 166). Husserl and Gibson were confronting some of the same issues of perceptual phenomena in the environment.

Why is the difference in perceived as measured circumferences of two circles important to a phenomenological account? For Husserl, recollection played a role in constituting one's consciousness of duration and succession. He discusses this in the consciousness of internal time. It will be shown that the form of his mathematical description of the recollection for the constitution of the consciousness of duration and succession is similar to the difference in circumference of two circles in *form*. He says, "let us assume that A emerges as a primal impression and endures for a while and that, together with the retention of A at a certain stage of development, B enters on the scene and becomes constituted as enduring B" (Husserl 196). In Husserl's consciousness of internal time, the perceived moment is described A-B, where A is the first event and B is the second event. In the geometric theorem that states that difference of circumferences equals an arc length, it is understood as the difference, $2\pi r - 2\pi x = (\theta)r$, where $2\pi r$ is the first primal impression of duration, or distance, A, $2\pi x$ being the impression B. However, in my account, the difference in the position of the events (impressions) represents the duration of the moment through an arc length of the initial radius r and the angular amount taken out of the initial circle, theta. The angle taken out of the initial circle is then described to be the duration of an experienced moment when it increases constantly with time. Husserl's is a simple description of difference, but can contain specifically contextual information, directly correlated to the notion of the perceived moment. What I propose is that, in the case of the difference in impressions of length of a circumference, the difference can be expressed as a perceived arc length.

Husserl may support a scientific endeavor in analysis of the difference in perceived impressions. He says that,

> "mathematics and mathematical science, as a garb of ideas, or the garb of symbols of the symbolic mathematical theories, encompasses everything which, for scientists and the educated generally, *represents* the life-world, *dresses it up* as "objectively actual and true" nature. It is through the garb of ideas that we take for *true being* what is actually a method- a method which is designed for the purpose of progressively improving, *in infinitum,* through 'scientific' predictions, those rough predictions which are the only ones originally possible within the sphere of what is actually experienced and experienceable in the life-world" (Husserl, 355).

The geometric pattern of perception theorem is a scientific observation of experienced phenomena in the life-world, and yields a system in which unending, valid, potential structuring of parameters with phenomenologically described meanings can be crafted and visualized as objects with surfaces. In theory, this also relates the ecological approach to visual perception to the phenomenology. This also leads to the question of what we take for the true being of a perception. There is a method by which the brain and the physiology deliver information about the environment to the perceiver via visual perception. The language of mathematics is a way of describing the being of the perception of the object that that mathematics describes, since it is grounded in formal ontology of perceived difference.

I will now discuss the insights to which Husserl came from developing his ideas on the mathematization of nature, but first it will be necessary to give a background of Husserl's understanding of mathesis in relation to the noema. Husserl says,

> "critically, it may be remarked here that the concepts of *'intentive'* and the *'cognitional essence'* which were established in the *Logical Investigations* are indeed correct but are capable of a second interpretation since they can be essentially understood as expressions not only of noetic but also of noematic essences, and that the noematic interpretation, as carried through there one-sidedly in framing the concept of the judgment in pure logic is precisely not the one to be used in framing the judgment concept of pure logic (i.e., the concept demanded by pure logic as pure mathesis in contrast to the concept of noetic judging demanded by normative logical noetics)" (Husserl, 93).

I take it that Husserl refers to, "pure logic as pure mathesis," (Husserl, 93) to be development of logical relations for the sake of their own interesting character, and he clarifies his meaning, saying that, "if we wish to obtain the full noema of a determinate judgmental process we must, as has already been said, take 'the' judgment precisely as it is intended to in just that process; whereas, for formal logic, the identity of 'the' judgment extends much further" (Husserl, 93). So, I begin to interpret this passage by noting the "taking" into account of the Lemma 9, of the logical, algebraic cancelation of the units of the speed of light in theorem three of the geometric pattern of perception theorems from the normative judging of the expressions of the intrinsically applied Lorentz equation being equal to one. Upon computation and use of the exact numeric value of the speed of light in scientific notation, a solution to a variable that normatively ought to cancel is delivered. In this, the judgment aspect of the discovery of a solution is precisely as it is intended to just in the process of canceling. Logic is not taken as non-trivial pure mathesis in the process of finding the solution to innate velocity (the solution to which I am referring), but rather is applied to pure mathesis. Still, a solution can be found, the process by which it is found still only understood as a computational phenomenon, which is quite the obscurity.

On the mathematization of nature, Husserl sets out the goal of his philosophical project. He says that through transcendental philosophy, "we are attempting to elicit and understand the *unity* running through all the [philosophical] projects of history that oppose one another and work together in their changing forms" (Husserl, 361). I propose that the system of the transformation of a circle into a cone is a useful piece of the puzzle to unification of philosophy, because although its implications are not traditional mathematical results, the process is a valid mathematical process, and it uses phenomenology (grounded in essential form of moment impression, as well as encompassing phenomenological description of time to be measurable cyclically like a clock due to evolution and my own experience of the cycles of seasons the sun, etc.) as well as noetically based logic to deliver computational results for innate meaning.

The system of the, "cone" is a phenomenologically described system in the Gestalt sense. I am describing the perceived organization of spatial relations, but also, therefore, a specific case of the spatial relations of organization of perception. The description is in the language of mathematical proof and substitution of proven expressions into other expressions for dimensionality of forms. There is a lemma of the system (lemma 6), which cannot be proven in theory but is algebraically true. This may be groundwork for discussing the meaning of incompleteness to perception.

On the process of forming phenomenological descriptions and investigation of transcendental philosophy, Husserl says that, "the developed method, the progressive fulfillment of the task, is, as method, an art, which is handed down; but its true meaning is not necessarily handed down with it" (Husserl, 358). The visual objects delivered by my formally ontological investigations into the difference in circumferences as arc length-perceived are in a sense, objects of found art. Husserl's insights into mathematics were profound, and his positions are relevant to my theory of a geometric pattern of perception, the results for which I am trying to give precedence. He says, "'Space' and the purely *formally* defined 'Euclidean manifold' were confused; the *true axiom* (i.e., in the old, customary sense of the term), as an ideal norm with unconditional validity, grasped with self-evidence in pure geometric thought or in arithmetical, purely logical thought, was confused with the i*nauthentic* 'axiom' – a word which in theory of manifolds signifies not judgments ('propositions') but forms of propositions as components of the definition of a 'manifold' to be constructed formally without internal contradiction" (Husserl, 358). I propose that the *Geometric Pattern of Perception Theorems* are axiomatic in Husserl's sense of the word, and the objects that they deliver are formally defined Euclidean manifolds. This geometry can be interpreted in terms of what is commonly held to be space-time, but exists in an ideal sphere.

Husserl's insights are often linked to the ideas of subjectivity and objectivity. In his writings on the mathematization of nature, he comments, "only a radical inquiry back into subjectivity which ultimately brings about all world-validity- and specifically the subjectivity which ultimately brings about all world-validity, with its content and in all its prescientific and scientific modes, and into the 'what' and the 'how' of the rational accomplishments - can make objective truth comprehensible and arrive at the ultimate ontic meaning of the world." (Husserl, 360). The geometric proof of theorem one conveys the "how" of perceived difference through math as descriptive language. The perceived as measured difference in circumferences of two circles is a scientific observation (of a perceiving being) grounded in phenomenological description and formal ontology, because it delivers a perceivable surface gradient of the parameters of perceived difference in length as arc length perceived. It describes a phenomenon of perceived difference through pure geometry and thus arrives at real, ontic meaning of any parameter of the system in terms of any one of the others when introspection, logical insight, and investigation are made into its algebraic structures. A radical inquiry that is made by the objective assertions in a geometric pattern of perception back into subjectivity is the phenomenon of shimmering, which can be seen as evidence for the reality of the gradient of the functions described by perceived difference as actual surfaces as opposed to just planes. The proof of theorem one states the proof of the perceived difference in circumferences equaling an arc length mathematically, making truth comprehensible (and related to perceptual experience), and the general notion that an objective system can be a phenomenological description, inclusive of a subjective locus is somewhat radical. The cone of rays entering the eye is described by the conic transformations in *A Geometric Pattern of Perception* (Emmerson, 2009).

The diagrams attached at the end of this paper are useful to seeing where the structure of a difference in circumferences (which can described as a circle transforming into a cone) is located for a subjective locus (the eye) in the visual field. Please treat each diagram as if it were labeled only with regard to that which is within it specifically. Then, think of the mathematics for difference in circumferences, the "translation" of the apex of the height of the cone, the gradients of its parameters, and how many relationships of differentiated cones there would be in a visual field.

The theories outlined in this paper have been shown to correlate to each other as well as the results of the empirical, formally ontological, evidential system of geometrically expressed difference in perceived circumferences. The library of 3D models that the theory delivers is enormous and goes ever beyond what I have discovered so far. In this way, the system is like a library that never gets new books, but is infinitely large. The potential uses of the theory for describing scenarios in the physical world also needs development as well as continued research on the idea of signals and neuro-scientific correlations with the theory of the geometric pattern of perception theorems in physiology. It is evident, however, that the idea of manipulation of material with the hands is not far from correlation in form. The change in volumes of a sphere inherently contains contour in the shape of four fingers, when the expressions for the parameters of the cone are applied to expression of this difference in volume of two spheres. In this system, we see a language of linguistic metaphor developing like indentation in the shape of an arrow, spontaneous changes in information delivered to the perceiver, density of information within different surfaces, and patterns of intricately woven lines.

A notable theorist who has attempted to provide an account of the overview of perceptual theories in the past is Floyd H. Allport, who wrote Theories of Perception and the Concept of Structure[22]. He covers a wide range of theories from classical and core-context theories, to Gestalt, topological fields, and cybernetics. A passage of his is relevant to the project I am ultimately trying to undertake, which could be seen as combining topology and phenomenology, showing that there is a correlation of perception to measurements in the physical world. Allport says,

> "Phenomenological and physical concepts do not mix; there is only the possibility of associating them through the arduous conceptual procedure of isomorphism. If the theorist forgets that his field is purely phenomenological and introduces forces or

[22] Allport, Floyd Henry. *Theories of Perception and Concept of Structure*. First Edition ed. Hoboken, New Jersey: John Wiley & Sons Inc, 1955. Print.

> bounding conditions of a physicalistic character, he will fall far short of the ideal of logical consistency, or else he will have to achieve a *tour de force* of explanation showing the operational equivalence of these two realms of experience in a manner that has never yet been accomplished" (Allport, 149).

In response to Allport's reservations about connecting physical and phenomenological studies, I would note that he would have most reservations about theorem two. In theorem two, the passage of time is measured like we measure it with a clock. Seconds are then converted to radians, and it is shown that, if one unit of time passes per 2π taken out of the initial circle, the average velocity of the height of the cone equals the instantaneous velocity of the apex of the height of the cone at a 30-60-90 triangle within the cone. The relationship of instantaneous velocity and average velocity of the height of the cone is then shown to be of pure geometry. The difference from which the geometric structures that describe this relation are found is formally ontological, and thus of essential phenomenology in the sense that mathematics can be a language for describing perceived difference as perceived. The height of the cone inherently exhibits acceleration, because there is less and less of the circumference of the initial circle to be translated into the height of the cone as the angle taken out of the initial circle increases. The physical notion of time is connected to the angular element taken out of the initial circle by my own definition, but theoretically based on ecological and contextual information that tells us that our species evolved in an environment portraying time to be cyclic to us (the seasons, day, night, tides, ebb and flow). In the formal ontology of a geometric pattern of perception, there is a continuous function describing, "motion" of the center of the initial circle as it is translated into the height of a cone. However, it is not necessary to explicitly introduce the concept of time in order to describe this transformation (of a circle into a cone) in pure geometry. What is described is the rate of change of one variable with respect to another in different manners (see Theorem 2).

The brain goes through many pattern-based processes resembling mathematical patterns, and there is a phenomenal resolution to the experiencing of visual perception. As we explore mathematics, we get a better idea of how the processes of perception work, especially when we can find mathematics that relates directly to experiencing of a correlation or, "vision," and a form in the environment itself.

On most levels, we can say that the meaning of arithmetic and the body of science called mathematics does not depend at all on any psychological process. Husserl was the first to specifically begin exploring the framework for transcendental phenomenology by discussing this truth. However, on a more in depth level, applied mathematics depends greatly on the observer. How is anyone to come to any meaningful series of relations without using some cognitive process of observation or at least realizing that their expression of truth depends on observation? When we say 1+1=2, this is not a truth that depends on cognition. It has meaning that already exists in the world. However, when we observe a truth about the universe that is slightly more complex and depends on the phenomenon of validity of our perceptions in space, we observe how different elements of a system fit together by necessity. There is a series of valid expressions that we can relate to one another in order to visualize the meaning of the structuring of light, substance, time or space and gain knowledge about expressions, which are useful computational tools for studying visual perception. That geometry, whose validity can be determined through proof, yields a purely geometric expression for what the number one is as well as the number two. From purely geometric proof, it can be shown that, in pure geometry, numbers have specific geometric meanings.

It is a purely theoretical notion that the eye is ever in a fixed position. Eye movements are involuntary, but can be controlled. However, making one' s eye very still is difficult. We still see depth with one eye, and this is because the eye makes motions to receive the impression of depth to perception from the environment, because it associates spatiotemporal continuity to observations of the mind in addition to detecting the gradient of light and the shading involved relating to the hue of objects (the array of color). The stimulus of light is an external stimulus. The physicist may think of it in terms of wave-particle duality. Yet, with the mathematical account present in *The Geometric Pattern of Perception Theorems*, specifically in Theoerem 3, I would like to strongly encourage thinkers in the field of psychophysics as well as theoretical physics to consider other interpretations of motion. Bodies move like a “wave” as well as a “particle,” and these terms both have specific meanings in their various contexts. However, once it is shown logically that velocity has an implicit expression in the structuring of conscious being of moment impressions, the terms wave and particle may no longer do justice to the full significance of the *phenomenon of motion* or transformation.

With the development of the mathematical and ontological theory, new ideas arise that can be shown to have meaning to what is already known. This process is a practically never ending one in which progress is always being made when new theories and ideas are tested and shown to be true with regard to other elements of the system or structures in the world. We also see that there is a unifying pattern that exists on the scale of the very large as well as the very small with relation to the visually perceived as perceived and phenomenological through ontology of perceived difference.

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Theory of Consciousness, Perception and Visionary Experience

I. Introduction: Conscious Reality and the Locked-in Perception

Conscious reality is not a field of theories, but an engaging experience of the relation of memories, impressionisms, and dreams. To draft an experience of cosmic perceptual morphosis, we engage with the truth that we can tap into the reflection of light and space-time within the mind. The mind structures reality for the perceiver. This is a perceived truth, and understanding it involves meshing realities through theory of consciousness and perception. To begin looking at the theories of consciousness, I will outline a few. To begin showing how the mind has access to the ability to fill in experience of reality through the guidance of perceptual contour, I will use the theories of consciousness. To elucidate a possible realization of the fact that the mind can change the experience of the world for one or more people, an "ideal emotion" (Arno, xx), can sometimes result within the subjective experience, ending the session of altered perception temporally, and with throughness of the experience, leaving the individual in awe.

Miming can tap into the experience of reality for a crowd. In a similar way, the mind, mediating cognition through understanding the structure of its own perception, can alter reality to the phenomenal experience of one or more perceivers if any other individual is giving a portion of their persona to the experience of reality on the same "intersubjective reality" to a being generating an individual experience of true reality through cognition. Natural science is helpful in determining the meaning of the proposition of some of these kinds of claims and structure of real cognitive experiences actually taking place.

It is necessary to also propose a phenomenology of time in order to correlate the experience of space-time accelerating through the course of an individual experience of it perceptually, because through space-time, objects are correlated in being to the perceiver. Thus, we will be able to weave ideas of spatial structuring through time, and structuring to the relationship of perception of time to depth and contour. When the perception is locked-in on an object of the world, that object is subject to cognition, which delivers consciousness of the object to the individual.

II. Support for a Theory of Manipulation of Experiential Perception of Contour through the Conscious Experience of Time and expression substitution, and understanding through the natural sciences.

When exactly one unit of time has passed, there is no experience of time if time is cyclic, except as a digital place-holder. For the following paragraph, please refer to the math item attached to the end of this paper for specific references to the meaning from the ontological description to which the meaning pertains scientifically. The mind, conscious of time and cognizant through being, views the world, in theory, through the being of the worldly embodiment, which contains certain laws within it. When we know what A is, and A is a transformational parameter of a section of a system of being, we can show that under certain conditions, B is A, because they are the same variable. Because we know that A is definitely the case, we can say that B is contingent upon C, where C is some exact position (that upon which B=A is contingent) within a correlation of phenomenally described, perceptually experienced parameters. What is interesting is that a result from A called Z is a component of C, and can be substituted or not substituted into a given expression for C in many different locations within that expression. In this way, if A is the space-time perceived by an individual, and matter rests within it, cognition allows this substitution to take place and change the perception for the perceiver through time. “Such test implications are thus implications in a twofold sense; they are implications of the hypotheses from which they are derived, and they have the form of if then sentences, which in logic are called conditional or material implications."[1] idea of B in an expression called X, and C is the case, X=Z,

The process of experiencing reality in this way involves a synaptic relation of the neurons. The phenomenon we are attempting to study with natural science in this case is the interaction of the subject and reality through a cognitive experience of being. This experience of being entails motion, reception, or inception of the personality. When we understand the intrinsic structures of the correlation of cyclicality to time and space to cyclicality, we will be able to cease the cyclic motion entirely, because we will realize that there is never a complete circle in terms of difference. Only when there is a ratio of the initial radius of an object's projection on the retina to the diameter of that circle whose area exists within the retina is perception of the sample of the array said to be *abstractly volitional*. The volition rests within the understanding of natural sciences, because, "the specified test conditions are technologically realizable and can thus be brought about at will; and the realization of those conditions involves some control of a factor (position during delivery; absence or presence…) that, according to the given hypothesis, affects the phenomenon under study" (The Natural Sciences, 20). At will, we are able to adopt or deny the existence of time (within the same understanding of it, it can be shown to be both nothing and expressible in terms of other parameters of an ontological system). If we adopt it, allowing the structure to be visualized, and mesh the visualization of it with the *plane of experiential reality*, then we are able to evoke the *qualia* of the being through a perception of a ripple in the actual material.

A synchronization of an experience of time with the location of points on a sphere is a synthesis of cognition and reality in a linguistic structure as well. Such truths (those found

1 G., Hempel Carl. *Philosophy of Natural Science (Foundations of Philosophy Series)*. Englewood Cliffs, NJ: Prentice-Hall, 1966. Print. Pg. 20 (All further references to this source will be cited parenthetically in the text (The Natural Sciences, Pg. #))

through expressing difference mathematically and seeking out the intrinsic structures of relativity) are the only available way for the cognitive system to access the meaning of depth perception from the sample of the environment in a regular (structured) way when focusing attention on an object. Such truths are yielded from calculating difference cyclically, while there is a change of directionality for the initial parameter defining the circle. Knowledge and expertise certainly affect how we perceive the world and the structures and objects within it. Such calculations yield a set of ontologically defined objects from incorporating a phenomenal experience into the meaning of the mathematical application.

When I use a word, I can usually construct a phrase to have the same meaning as the meaning of that word. This is what I call a *potential synonymity*. This leads to viewing the perspective of multiple people on a word and the debate of meaning in general. However, the important thing to pull out of this idea is that the meanings themselves can be synonymous to different cognitive perspectives on consciousness of objects in the world. That is, if the meaning of a brain is the persona of the developing and generative mind within it, then how are we to differentiate between the mind and the object that is the brain? Or moreover, are there potential synonymities between consciously experiencing through the mind and the material object of reality?

This involves cognition when we consider its ontological significance. Only when everything and nothing are conceptualized and placed in a framework in which that pre-defined term's being everything is necessitated by the framework to be nothing does the passing of the moment lose sense to the meaning of its initial assumptions for real numbers.

Hallucination is a factor often brought in for discussing those characteristic mental states that affect the experience of reality. If we break the word down and figure out its root meaning, we can have a better idea of what the structure of the hallucination entails. A hallucination involves perceiving real sensory experience that is of a figment of imagination and not based in what is held by every day objects to be the objective world. One could argue that hallucination is so closely related to the idea of deception, it is not proper for the context of actual, perceptual relevance. Yet, the hallucinatory experience (in the colloquial usage) could deliver the perception of an object in its entire ecological context to an individual undergoing the hallucination. Then, who is the one hallucinating to the person hallucinating who does not think that they are?

Does visualization in an altered or transcendental state of mind mean hallucination? There are hallucinatory perceptions, but in this paper, the definition of hallucination does not mean that the thing being hallucinated is not real to the individual experiencing something. The hallucination exists as a structure of the conscious experience of the objective world or a manipulation of the normal reaction to local materials or vibrations through cognition. The set of ontologically derived objects is often useful for the process of visualization or altering the experience of one's reality. However, what psychological factors of humans affect the manner of their hallucination are often psychosomatic reflections on past experiences or internal human personality traits is a complex issue.

Simply, we wish not to consider the visualization of the set of ontological objects to be a hallucination in the sense that what's being perceived is not real or not material. While parameters of the ontological structure can be described as pertaining to time or distance, they are simply in their pure quantitative form, a relationship of numbers through mathematical operations of a difference of the location of the spatio-temporality in a continuous system (a moment of experience is said to be continuous). Each variable or relation of variables in the system is expressible by a certain object of visualization. The structure through which a perception is taking place is related innately to this understanding of being from a difference in the circumference of two circles, which depends on the ratio of a circle to its diameter and two different distances, one initial and one from a continuous function of the difference through motion.

This kind of problem is what led some philosophers to believe that perceptions were purely hallucinations. While I do not think this, I do think that the information contained within our mind and its relation to the body does shape our experience of reality, "*localization of sensings* is in fact something *in principle different from the extension of all material determinations of a thing*"[2] . The localization of sensations may have certain physiological responses when the mind focuses on perceiving the centers of these physiological systems. If Buddhism is one of the most advanced languages of mind, it might be alright to reference some of its genuine analogies from its tradition of realization.

> "A modern teacher, Tenzin Wangyal Rinpoche uses a computer analogy: main chakras are like hard drives. Each hard drive has many files. One of the files is always open in each of the chakras, no matter how 'closed' that particular chakra may be. What is displayed by the file shapes experience"[3].

If I contain a mathematical equation within my mind or even a visualization of the meaning of the system from which it came, I may experience that form out of the material that is present to me in the physical world. The experience of the object is accessible through how it is seen. In essence, the form seen in something may arise out of the mind. Knowledge acquired from one's visual and reasoning skills can effect the experience, especially if that knowledge was gained from realizing something true about the actual reality and its spatio-temporal structure(the correlation of its distinct elements within a unit of the experience of the geometric correlates to those elements expressed by a set of ontological objects discovered through math). If the manipulation of the experience

2 Husserl, Donn Welton Edmund. *The Essential Husserl: Basic Writings in Transcendental Phenomenology (Studies in Continental Thought)*. New York: NY, 1980. Print. Page 178. (All further references to this source will be cited parenthetically in the text (Husserl, Pg. #)).

3 Wikipedia. "Bön (A series on Tibetan Buddhism)." *Bön*. N.p., n.d. Web. 12 Dec. 2009. <en.wikipedia.org/wiki/bön>.

through accessing the truth of the nature of the experience is realized within the physiological systems (the brain, the lungs, the heart, etc.), ones being and cognitive actions effect the experience through reality and also effect it psychologically.

How would our experience change from thinking that infinity is not equal to zero to suddenly considering the possibility of the symbol for everything being equal to the symbol for nothing within a system that posits a correlation between an infinitely large parameter?

Although the pure structure of being necessitates that 0=   at the maximum value of the initial condition if the initial condition is infinitely big, the relativistic, intrinsic nature of being provides opportunity for expression of the relations within knowledge of a difference (of perceived size of a sample). This leads to the ability to combine the object of sense (modern-day understanding of sense data) with the experience of reality at any distance. Perhaps then, the term reality means that element of consciousness that is something and potentiates meaning before mind exists. Husserl would say that meaning exists already in the world. Fodor might note that, "computer theorists…, often speak of identities of *virtual* architecture" (Fodor, 32). Marr posited the 2.5 D sketch, but what about a 2.333… D sketch? Or a .333 D sketch? The structures of sense data are only specified dimensions based on the constant of time that they use for interpreting cyclicality through the sketch. Virtual architecture would be that which produces virtual contour. If the world is virtual in the sense that a change in two real perceptions (perceived locations of an object's distance away from the locus of perception based through the change in area projected on the retina from a sample of the array of an object) depends on the ontological significance of that change in a perceived sample, which delivers virtual contour, then the identity of the virtual architecture is the information in light with the available information by which the cognitive and persona systems build a perceived reality.

III. The Set of Ontologically Described Objects that are Useful to Visualization and Altered Experience of Material Objects to their Presence.

Upon visualization of objects like a cup, we can perceive them and use them in part because of their shape. However, it is possible that the conscious experience of the object can be changed through introduction, extrapolation of and attention to a particular expression of an element of the system of experiential being. For instance, the following forms are all expressions of the same function. The size of the set of what I call *ontological objects* is not currently known even within the smallest system of difference in perceived size of a sample of the array projected on the retina.

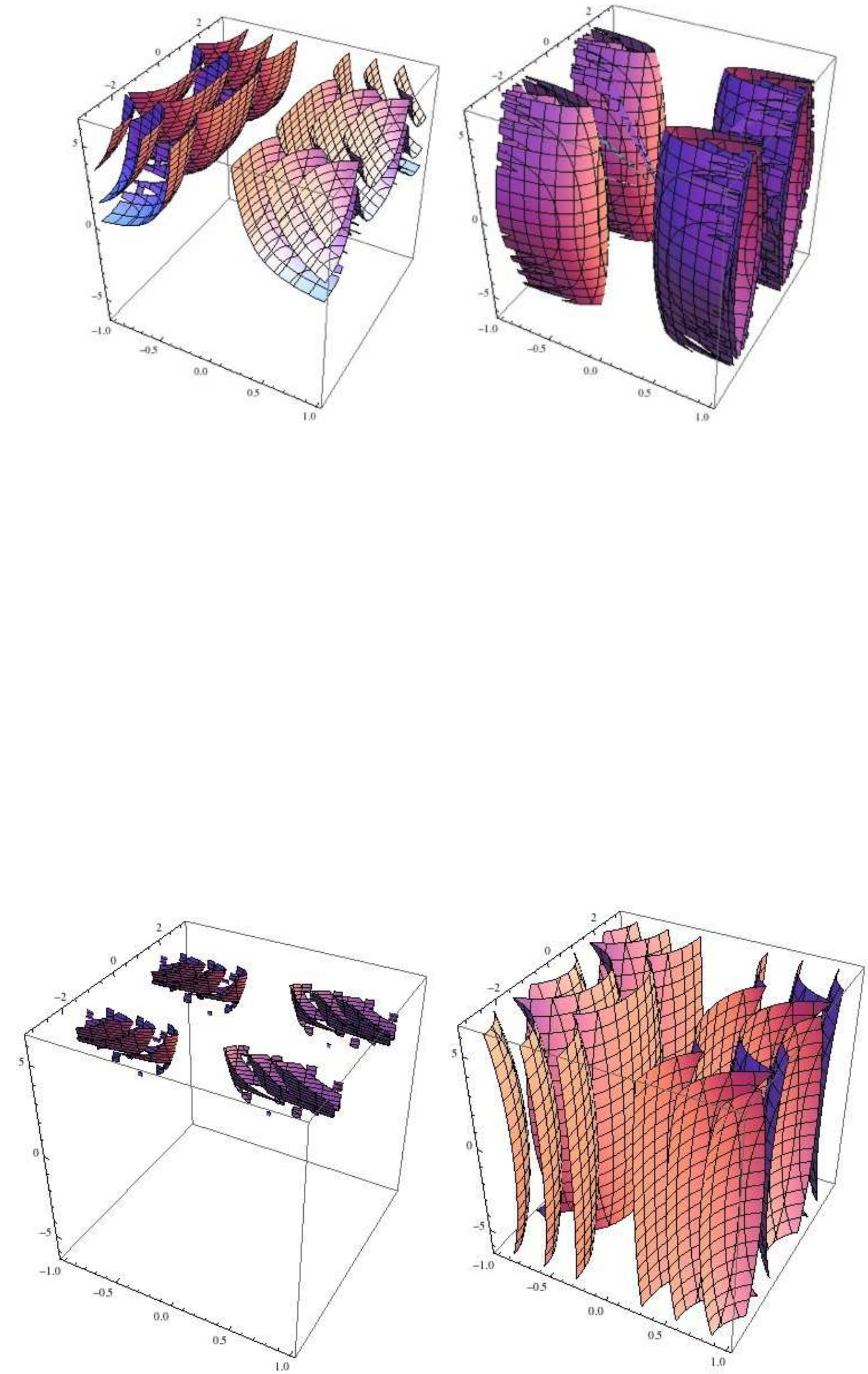
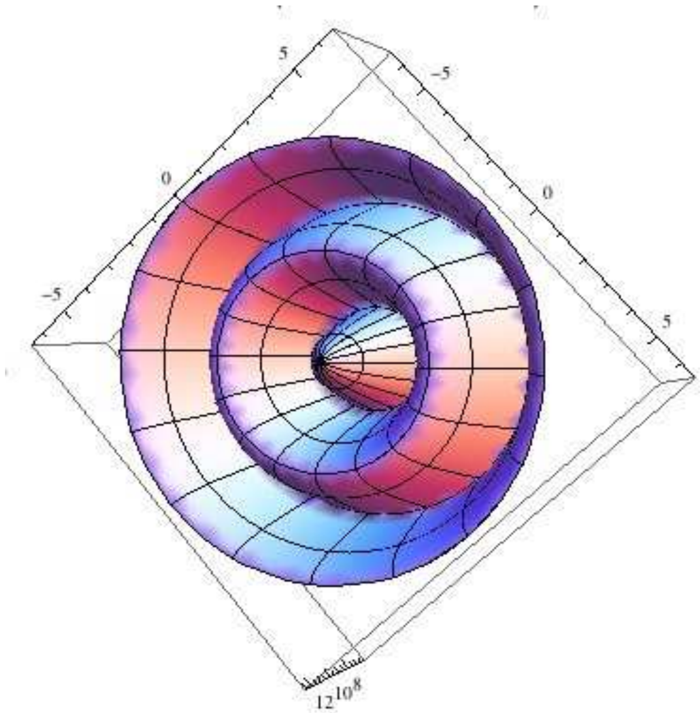


Figure 1.1

Simply to provide an example, these are all visualizable contoured objects of the same function, yet with a different expression coming from the pattern of valid substitutions within the system. By accessing them through the mind, we perceive the layering of reality. This layering provides sense stability, “it is part of the very notion of a computational psychology that all representations must be described syntactically or procedurally” [4] (Putnam, 115-116). In the natural science that I have described from the math item attached, there is a procedure for relating the change in perceived size of the area of the image of a sample of the ambient optic array on the retina to a functional system of correlates. The visualization of the functions allows the perceiver to engage in a reality of his/her own ability to manipulate its experience.

When one *taps into the plane of real experience temporally*, they can effect their experience of the material through time. Time, evolutionarily perceived cyclically, manifests both positively and negatively (past and future). The resulting object of experience of time is potentially layered into the sample of the array to deliver a rippling effect in the plane of reality. Just as the mime taps on the glass of reality to invoke the experience of solidity for the audience, the mind may tap on the experience of reality for the peronality of experience to create a rippling experience for the individual experiencing reality, potentially leaving the person experiencing perception with an ideal emotion.

4 Putnam, Hilary. *Representation and Reality (Representation and Mind)*. London: The Mit Press, 1991. Print. (All further references to this source will be cited parenthetically in the text).



1.2

This is the graph of an angle which is the section of a circle being pulled up into a cone. This angle is symbolic of cyclic time awareness.

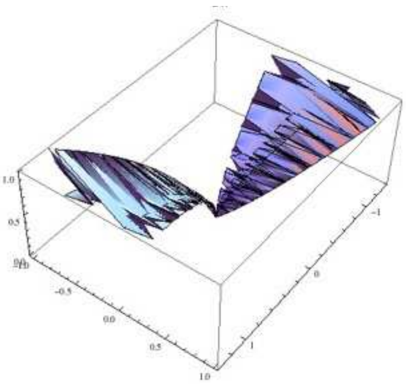
IV. Mystical Experience, Hallucination or Dream?

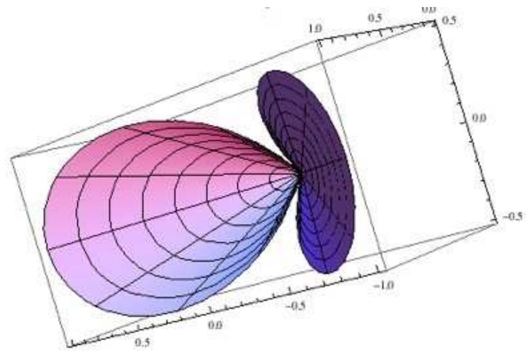
To change one's experience of reality through opening the experience of visualizing the meaning of the progression of time like a plane whose function describes . Andrew Newberg has said that, "The mind remembers mystical experience with the same degree of clarity and sense of reality that it bestows upon memories of 'real' past events. The same cannot be said of hallucinations, delusions or dreams" (Spiritual Brain, 259)[5]. Mystical experience is often associated with clarity of visual perception, perception of inner light, and the cessation or slowing down of perceived time. Dreams also can make a lot of things happen in a smaller amount of time than they would normally take and allow

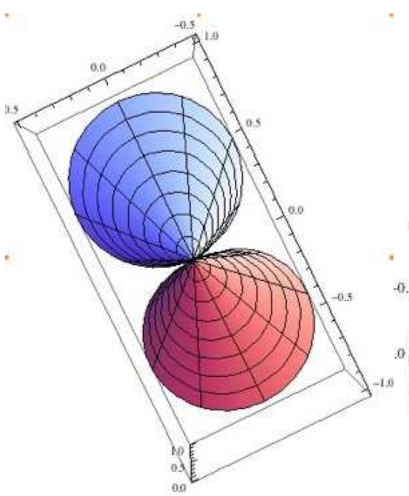
5 Beauregard, Mario, and Denyse O'leary. *The Spiritual Brain: A Neuroscientist's Case for the Existence of the Soul*. New York: Harperone, 2008. Print. (All further references to this source will be made parenthetically in the text)

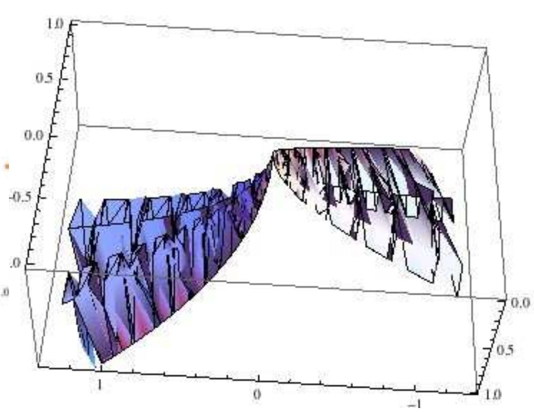
things to occur that would not normally be able to happen. The imagination is a very important part of how we connect our experience of the truths of reality, and the idea of "imaginary" correlates the mathematical meaning to the visionary meaning. When we find a real, ontological solution to an element of the experience, which is a function used for delivering a series of ontological objects when expressions are applied to the variables, we often find two imaginary solutions, or several roots, some of which are imaginary.

That distance, which is the consequence of taking a wedge out of circle and folding it into a cone such that the initial radius is always the slant of the cone, and the line between the apex of the cone and the center of the cone is perpendicular to the base of the cone, is visualized in figure 1.1 in terms of three variables. In terms of two variables, the following graphs are delivered.









Throughout the discussion of mind and matter, humans have stumbled into questions when orienting their thoughts and experience of *qualia* with the physically known facts. In Plato's cave, people are chained to the rocks, and are in a sense, disembodied, because they cannot move. Their eyes are probably fixed, and the meaning of their general embodiment minimized to the bare necessities. In Plato's philosophy, this greatly limits their experience of beauty, because beauty is related to usefulness. They see only the shadows, and cannot use their bodies. When we peer into the shape of the orbitals of an atom, what can we really tell about the shape of an instance of that orbital? We are looking through the lens of science, but what does that lens possibly filter out that goes revealed by pure geometry? The operations for calculating the change and visualizing the folding up of a circle into the structure of a cone are a kind of software. Dennett is a neuroscientist and cognitive philosopher who likens human consciousness to software of a computer, while not saying that. There is consciousness of time (the structure of which exists through rotation and motion through space), and time may be the temporal component of experience that has some sort of aspects of a "*stored-program digital computer*" [6] either providing an output based on the relation of the instantaneous speed of the apex of the cone to the average speed through the space between the center of the base and the apex of a section of a circle (forming structures that can have a multitude of expressions) or delivering a simple symbolic nothing, which allows for the digital and the place holding, for when a single unit of time passes, designated by the end state equaling the initial state, represented by theta, the chunk taken out of the circle, is meant to equal to 2 Pi, the angle is 0. This is because the initial condition is specified in a context that is completely undermined upon completion of the transformation.

> "Today we have grown quite comfortable with the distinction between the spatial location in the brain of the vehicle of experience, and the location in 'experiential space' of the item experienced. In short we distinguish representing from represented, vehicle from content… the products of visual perception are not literally pictures in the head even though *what*

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6 Dennett, Daniel C.. *Consciousness Explained*. 1 ed. New York: Back Bay Books, 1992. Print. (Pg. 219) (All further references to this source will be cited parenthetically in the text (Dennett, pg. #).)

they represent is what pictures represent well: the layout in space of various visible properties" (Dennett, 131).

The information of the objects in the world delivered to the brain occurs over time. Time feels cyclic to most everyone (ebb, flow, rise, fall, rhythm, light, day, night, etc.) and can be phenomenally described to be cyclically experienced. However, in phenomenal space, time can be nothing and something within the same structure. What is certain is that if time is represented by the angular element of the equation for a change in circumferences when taking out a wedge of a circle and describing the difference of the initial circle and the base of the cone formed by folding up the circle with wedge removed, it has two solutions - a positive and a negative. The space described from a phenomenology of cyclic time would be considered a phenomenal space. Dennett supports such an idea as it pertains to the experience of a change in perceived object size, "We have denied that the projection is in physical space, and have relocated the projection in phenomenal space" (Dennett, 130). The change in area of a sample of the ambient optic array projected onto the retina can be relocated into phenomenal space when describing the change to be happening over time and that time is related to the wedge taken out of the initial area projected circularly on the retina.

In a phenomenal space, such as the one described, time and spatial experience will be accessible in a different conscious sense. Dreams will be able to take the user and expand their share of ability in perceived real space.

Visualized like a revolution, the height of the cone in terms of the difference of circumferences delivers a graph that can create the same shape as the *l* orbital from a certain perspective. Changing one's view on the diagram, the images of the *s* and *p* orbitals can be composed. When rotated, the image is something potentially much different (not identifiable as the same object). In a similar way, a single perspective of the equation for a function of the height of the cone, if interpreted with a narrow understanding of the meaning of its shape, reveals that the potential multistability of the structure is not taken into account.

Perhaps, instead of the orbital question, a more pertinent question is the meaning of the concentric curves of the clam-shell like visualization of the height resulting from mathematically expressing difference (of circumferences over time) to conscious experience of phenomenal space (defined by a transition through points in real space that yields somewhat paradoxical statements given certain reasonable presuppositions).

The way that matter is perceived (substance) is changed by the introduction of the idea that a space (framework, platform), in which objects of greater mass exist in their virtual framework than in the real world, can be created. Thus, we may be able to keep count of the meaning of more energy within a virtual framework with less energy than it takes from physical reality, thus accurately simulating more energy than we have. Then, if such a framework and representation of reality is possible, consciousness of reality is the more important issue. Of course, energy is not created or destroyed, and on the age of the universe, exists in useable form practically forever.

V. Conclusion through Pushing Classicalism

Although, "scientific hypotheses cannot be conclusively proved by any set of available data, no matter how accurate and extensive," (The Natural Sciences, 28), they can be proved within a phenomenological assumption. That is, they can use the if-then statement to provide a phenomenological account, and then show the meaning of that account has validity under the condition that the account is true, seeing as though the meaning of that account is mathematically expressible.

In reference to the idea of binocular vision, Wheatstone showed that, "the dissimilarity of the two pictures is a sign of distance, bound up in inseparable association with the fact"[7] (Classics, 192). I would also argue that the perception of difference of projection on the retina is also a sign of distance that is inseparable with the fact of distance, because the eye muscles move and provide allowance for the object to exist three dimensionally in visual perception through interpreting multiple images on the retina over time. As well as interpreting the size of the object's projection on the retina, the visual system is also relating the physical object being perceived to other objects around it.

Earlier, it was stated that the set of ontologically defined objects was useful for the visualization of an effected reality by mind. Plato would hold the usefulness of these objects to be the same as beauty, and that this beauty is connected to the theory of the idea they represent. Beauty and usefulness are classified by Alexander Bain to be within the chapter on aesthetic emotions. If these pleasures are "aimed at by the fine arts" (Classics, xxi), and the natural sciences can deliver that which is useful and beautiful, why should one try to distinguish between the fine arts and the natural sciences?

It was earlier proposed that the ending of the session of heightened awareness of the plane of reality would leave the experiencing subject with the feeling of an ideal emotion. The ideal emotion is defined by the, "fact that Feeling or Emotion persists after the original stimulus is withdrawn, and is revived by purely mental forces, makes the life in the Ideal" (Classics, 283). I realize now that I have been talking about the ability to change one's experience of matter, and that this is an imagination-based power for which man has been looking through shamanism and technology for ages. Bain has several things to say with regard to the relationship of power (effecting one's reality i.e. empowerment may be more potent) and ideal emotion. He says, "the memory, anticipation, the imagination of great power may give more delight than strong present gratifications of sense; something of this is implied in the toils of ambition…" (Classics,

7 Bain, Alexander. *Mental Science: A Compendium of Psychology and the History of Philosophy (Classics in Psychology Series)*. Harvard: Ayer Co Pub, 1988. Print.

286). I would say that after experiencing the change in reality from, “the feeling (that) will be manifested in ideal forms” (Classics, 285), there is a recognition that the being who just experienced the change in perception from using the forms delivered through natural, ontological science, discovers the in the experience a, “constitutional Tenderness (which) is a common manifestation, even without supposing a large emotional Temperment on the whole” (Classics, 285). Within experiencing the ripple, focus may be placed intensely on the experience of the moment, without participating in much of an emotion. The tenderness of the subject’s interaction with reality from envisioning the ideal forms layered within real experience of the structuring of time, is the source of the ideal emotion that may result after the session of an “altered reality” has ended.

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