THE FUNDAMENTAL COGNITIVE APPROACHES OF MATHEMATICS

A PREPRINT

Salvador D. Escobedo
University of Guadalajara
salvadec@live.com.mx

March 18, 2023

ABSTRACT

We propose a way to explain the diversification of the branches of mathematics, distinguishing the different approaches by means of which mathematical objects can be studied. In our philosophy of mathematics, there is a base object, which is the abstract multiplicity that comes from our empirical experience. However, due to our human condition, the analysis of such multiplicity is covered by other empirical cognitive attitudes (approaches), diversifying the ways in which it can be conceived, and consequently giving rise to different mathematical disciplines. This diversity of approaches is founded on the manifold categories that we find in physical reality. We also propose, grounded on this idea, the use of Aristotelian categories as a first model for this division, generating from it a classification of mathematical branches. Finally we make a history review to show that there is consistency between our classification, and the historical appearance of the different branches of mathematics.

Keywords Philosophy of mathematics · Categories · Classification of Science

1 Introduction

The act itself of knowing is differentiated by its object, which is that what the act is oriented to know. Philosophers of the School use to distinguish in this object what is formal and what is material. Formal object (*objectum formale*) is the aspect under which the object is related to the knowing faculty, while the material object (*objectum materiale*) is where that aspect underlies [1, 4]. Is said therefore in agreement with this, that science has a material object (the set of entities studied) and a formal object (its own point of view or particular approach).

Although sciences are differentiated by its object, as stated before, they are properly differentiated by their formal object, since material object can be common to many sciences. Consequently what is specific of every science is its formal object[1].

Therefore branches of the same science should have similar *point of view*, otherwise they would be no branches, but differentiated sciences. Simultaneously, they must have different objects, otherwise they would be the same branch. In other words, branches of a science must have same object with respect to the science they are branches of, but different object respect to the other branches.

So there are two ways a science can be divided into branches according to these two types of objects:

The division founded in the material object, consists simply in dividing the set of studied entities and naming a branch to the study of each part. The partition criteria may vary, but is always directly recognizable as a property of studied entities. For instance, in biology, whose *material* object we may say it is the set of living things, the partition can be made so that one branch studies animals (zoology), another one plants (botany), and so on. That is, we take the set of

[1] This vision is usually present in many *Logica Maior* treatises of scolastic and neo-scolastic tradition; see [2], [3], [4] and [5]

living beings, we divide it into subsets whose elements have properties in common, and we dedicate a branch to the study of each subset of the partition.

In mathematics this occurs in a similar fashion, so we have a mathematical branch that studies integers (number theory), graphs (graph theory), etc. This is the *material* way to dividing the branches of a science.

The other method in which a science can be divided is when its formal object ramifies, that is, when without abandoning the specific approach of that specific science, certain aspects are considered. To use our same example, in biology there are branches such as taxonomy, or evolutionary theory, that cannot be associated with a particular group of living beings. They are branches that attend to different aspects of the living being in general.

In a similar fashion, mathematics, within its essential approach, divides its formal object into several *sub-approaches*, with which different aspects of mathematical entities are studied. We try to identify these divisions from the mathematical point of view to better understand the principles that govern the separation of the different areas in the so called exact sciences.

As a context for our reasoning, we declare our position regarding the ontological *status* of mathematical objects; we endorse *moderate realism* (mathematical entities are entities of reason –*ens rationis*– lacking of own existence but founded on reality), and we conceive that mathematics is formally *the science that studies the inherent order in multiplicity, as multiplicity.*

There are however different theories on the nature of mathematical objects, and also different theories on distinction of sciences, some of them are not compatible with the foretold principles. The Bueno's materialistic philosophy proposes a alternative way for differentiation of sciences [6]. The unity of a science would be given by the *categorial closure*, not by it's object. Categorial closure takes place when terms in a science engender after operations more terms of the same category (therefore they stay into the same field of science). G. Bueno also proposed a way to divide branches of mathematics considering the different ways categorial closure can be produced. However should be noted that it implies that branches of matematics are not really branches of a science, but independent sciences. So, within categorial closure theory, we cannot talk about a single science called mathematics, because mathematics would be a collection of several independent sciences. Such a view is also present in the philosophy of Javier de Lorenzo [7], and others [8]. For a complete introduction to pluralism in mathematics see [9].

The discussion of ontological status of mathematical objects is not in the scope of this article. However we may say that the root proposition here is the one that states that mathematical thinking starts from abstraction of physical experience.

Starting from the idea that mathematics originates from abstraction, we have considered that different aspects of physical reality we perceive should originate differences in the way we conceive mathematical objects. Hence we are assuming that our perception of the physical world influences the way we approach mathematical problems and therefore the way we divide mathematics into different areas in agreement of physical categories.

So the method we propose begins with a system of physical categories, and use it as a criterion for branching the different mathematical approaches out.

For this purpose we consider the classic system of Aristotelian categories. We have also considered other systems of categories, such as the Kantian [10], the N. Hartman [11] system or other more recent ones; like the also Aristotelian system of Invar Jonhanson [12], that of Roderick Chisholm [13], that of Reinhardt Grossman [14] or that of Joshua Hoffman and Gary Rosenkrantz [15] or also that of EJ Lowe [16]. All in all, we have chosen the Aristotle's system because its importance and influence, also seems to us to be the most adequate for our objectives.

However, we should not rely on the peripatetic system without carefully reviewing it, and removing those elements that do not seem truly fundamental to us. Our goal is to have a catalog of elementary categories, to understand the origin of the ramification of the various mathematical disciplines at their most fundamental level.

The discussion in itself of a new categorial system supposes a separate study, but this is not the object of this work, so we will briefly expose the origin of our collection of categories.

According to the Stagirite (1b25-2a4) there are ten categories: *substance* and nine genera of accidents:

From that list we have to remove the categories that are not really fundamental to us. Since *substance* is not directly accessible to sensory perception (only accidents are), we will just take categories that are accidents. Consider first the *situs*, the posture. It is really about the location of the parts with respect to the whole, so it cannot be a fundamental category, since it emerges from the *ubi* of the parts. As for the relation (*relatio*), we could consider it as a *epicategory*, since really the relation is always said of another category: to be bigger than [quantity], to be better than [quality], etc. So instead of talking about relationship as a category, it would be more consistent to talk about relative categories. We will also exclude anthropomorphic categories, such as *habitus* (to be dressed, to be armed...), since it is only a concept

Where	[ubi]
When	[quando]
Posture	[situs]
relation	[relatio]
Quantity	[quantitas]
Quality	[qualitas]
Habit	[habitus]
Action	[actio]
Passion	[passio]

that has meaning for man by reason of his customs or social agreements, but not by reason of really characterized physical states. Aristotle himself has omitted this category in some of his listings (see [17, Category]) Thus, removing *situs*, *relatio*, and *habitus*, we are left with three groups, as shown below:

1. Space-time categories
   - *Where* [ubi]
   - *When* [quando]
2. Categories of the internal constitution.
   - *Quantity* [quantitas]
   - *Quality* [qualitas]
3. Categories of external causality.
   - *Action* [actio]
   - *Passion* [passio]

Note that this category paradigm is related to the elements that are needed so that physical mutation would be possible. The *space-time* is the "environment" required for physical changes to take place, or if you like, space and time are the "effects" of movement. The categories of the second group come from the distinction between matter and form (associating quantity with matter and quality with form); those are an internal articulation to the physical entity and are necessary to make possible the execution of any physical change. Finally, the categories of third group are related to the efficient causality of a change, since *action* is the act of producing a change, and *passion* is the act of being changed by the action of an agent.

According to our thesis, each element of this collection is related to some mathematical approach, not in the sense that mathematics studies physical issues, but because it has given rise to the abstractions that have naturally originated specific mathematical approaches. These approaches are listed below.

1. Spatial-variational approaches
   - Spatial or geometric approach [ubi]
   - Variational approach [quando]
2. Quantitative-formal approaches
   - Quantitative approach [quantitas]
   - Qualitative and structural approach [qualitas]
3. Logical causality approaches.
   - Solution approach (solution sets) [passio]
   - Algorithmic approach (algorithms and functions) [actio]

The relationship of the first four approaches to the respective categories does not seem to need an explanation. However, logical causality approaches require some clarification. Logical causality is said in analogy with ontological causality, and in this sense the premises and reasoning are *cause* of the conclusion. We say that algorithms and functions *do something*, hence they are related to the action category. While the solutions are *made* from some initial conditions and a procedure. Hence we associate the solution with the *passion* category. Note that when we talk about algorithms or solutions we are not referring to them as objects, but as a way of seeing objects. A mathematical object can be seen as a solution or not, it will depend on the approach. For example, we can consider number 2 as a solution of the equation $x^2 = 4$, but this is not the only way to think about number 2.

2 A brief survey: the different approaches in the historical development of mathematics

2.1 The material approach

This approach would be related to the category *substantia*, which does not refer to any accidental attribute of the thing, but to the thing itself. It is not a point of view, it is simply a matter of studying a certain set of mathematical objects but restricting our study to that set. For this reason we have to say that the *material approach* is in a sense a pseudo-approach. The oldest branches of mathematics were distinguished in this way. Thus arithmetic was restricted to the study of the *discrete quantity*, as the ancients said, while the *continuous quantity* was studied by Geometry. This distinction lasted a long time, and to this day it is the element that specifies modern *discrete mathematics*. In the opinion of some historians "the whole history of mathematics can be interpreted as the battle for supremacy between these two concepts"[18, p. 13].

As we will see, both ancient arithmetic and geometry applied certain points of view, which really specified them, but dividing a science by means of partitions in its object of study has always been the simplest. Hence, defining discrete mathematics is a relatively easy task, and it is correctly defined as "the mathematical study of countable sets".

Initially, geometry was a mathematical study of geometric figures, rather than a mathematical study of space. Much of its early development was aimed at studying the properties of triangles, circles, rectangles, polygons, spheres... etc. Arguably, in the beginning, geometry was an "object-oriented" science. However, this does not imply that there was no notion of space as such. Certainly such a notion existed, and Euclid's geometry, with its five postulates, establishes a general structure for space, regardless of the figures conceived in it. Euclidean geometry is the first theory of space that we have a record of.

At the time the brilliant Greek mathematician wrote the Elements, no other mathematical objects had been devised that were related to the "continuous quantity" than geometric objects. Keep in mind that the notion of real number would still take centuries to arrive.

Due to the above, the spatial approach distinguished geometry in the same way as the *material approach* that separated mathematics between the discrete and the continuous.

However, this equivalence has been lost, since non-geometric points of view have appeared in the study of continuous quantity, as well as geometric studies on discrete sets. This means that the *spatial* approach, characteristic of geometric science, does not equate the restriction of mathematics to the study of continuous quantity. All this leads us to conclude that to divide mathematics and understand what approaches its various branches engender, it is not correct (although it is the easiest) simply split the set of mathematical objects by assigning to each branch an object class. Such a classification procedure works for some cases (discrete mathematics or graph theory, to name two examples), but not for most.

2.2 The spatial approach in ancient mathematics

The spatial approach is to consider a multiplicity as distributed in space. Due to our physical experience, this approach is particularly useful, as it allows us to represent mathematical ideas graphically and visibly, thus being able to use imagination and fantasy to facilitate reasoning.

Actually, it is this approach that gave rise to geometry, for it is its characteristic way to think. Despite this and in a secondary way, other points of view were applied in the study of geometric objects.

The quantitative approach was taken when questioning about the distances, areas and volumes of the geometric figures; these magnitudes were a quantitative estimate of such figures.

The structural approach induced to investigate its qualities, its non-quantitative geometric properties, and to establish relations or *laws* for them.

The variational approach was hardly applied at that time, and we will not see it appear clearly until the 16th century with the arrival of calculus, and only until the appearance of differential geometry will we really have it as an approach applied to geometry.

2.3 Arithmetic

Arithmetic was born mainly with an algorithmic approach. Its task was to study the logical procedures that would allow a calculation to be carried out correctly. Thus, learning arithmetic in ancient times consisted of learning algorithms to perform basic operations, such as adding, subtracting, dividing, multiplying, and finding square roots. Also appeared the solution approach, and the notion of equation.

The ancients cleverly solved issues about solving equations. The structural approach joined the solution approach. Analyzing the structural properties of numbers, the associative, distributive, and commutative laws were discovered. The results achieved with the algorithmic approach were compiled in arithmetic, while the structural and solution approach applied to more abstract objects gave rise to Algebra.

2.4 Historical development of other approaches

During the 17th century René Descartes (1596-1650) managed to combine spatial and solution approaches, creating analytical geometry. The French philosopher conceived geometric figures as solutions to certain given conditions, namely algebraic equations. Thus, a circumference would be in Euclid's geometry the locus of the points that are equidistant from a center, while in Descartes's geometry, a circumference would be formed by the points that satisfy the equation

$$x^2 + y^2 = r^2$$

Descartes viewed geometric objects with a *solution* approach, fusing the three main approaches of ancient mathematics into a new geometry. "The construction of equations therefore plays a central role in Descartes"[19, p. 212]

Until Descartes, the variational approach had not been fully developed. With the arrival of the Differential Calculus it acquires its own appearance. I. Newton and G. Leibniz added such point of view to analytic geometry when conceiving their new mathematics.

The arrival of Calculus allowed the expression of a new type of equations, which included derivatives of functions. In this way the solution approach had new objects to apply to, engendering the theory of differential equations.

Differential calculus + solution approach $\rightarrow$ differential equation theory.

From its inception, calculus harbored a dual approach. The variational approach, which, as already mentioned, originated the differential version of calculus, and the quantitative approach, which is in the core of Integral Calculus.

Integration allows estimating areas and magnitudes, that is, it allows quantification, however, its close relationship with differentiation is striking; calculus establishes a beautiful relationship between the variational and the quantitative in the fundamental theorem of calculus. This relationship leads us to think about the close union that exists between matter and form, responsible according to the scholastics for accidents of quantity and quality, from which those two approaches arise.

We can think that the quantitative point of view sparked a new branch of mathematics, which is currently known as *theory of measurement*. Initially, this approach was tacitly applied in arithmetic and geometry; measuring and enumerating are essential activities in this approach. Calculation discovered an excellent tool for quantity measurement: integration; and this idea has since been generalized to conceive of any quantitative evaluation in a measurable space.

Geometry as such, even with Descartes contribution, remained in part as a *object-oriented* geometry, yet the basis for the geometric study of space *per se* was laid, and *vector spaces* and *non-Euclidean geometries* appeared in the 18th and 19th centuries, and finally, entering the 20th century, the *topology* and *tensor analysis*. Topology especially shows to us in its most abstract form the use of the spatial approach.

As we have seen, a second approach can be added to a given approach, to form a new study area. Thus, the spatial approach joins the variational to generate differential geometry.

Geometry + variational approach $\rightarrow$ differential geometry.

On the other hand, the mathematical algorithmic approach, inherent to the old arithmetic as it has been said, permeates almost all the areas of the current mathematics, where we need algorithms, to calculate or find something. The advent of electronic computers prompted the careful and systematic study of such procedures. Thus appear the computational number theory, computational geometry, etc. The most modern areas of mathematics are a direct consequence of this approach, as is the case of fractal theory and recursion.

There is no place here for a careful examination of how each area of mathematics has arisen and a justification of how the different approaches that we have proposed have been combined. However what has been said will be enough for the reader to understand our proposal and appreciate the convenience of it to explain the genesis of each area of mathematics. It is important to note that many areas have been divided for not mathematical reasons, but because pedagogical or pragmatic motives. The distinction, for example, of *applied mathematics* does not respond to a change of point of view towards mathematical objects, but to a segregation due to the utility for practical purposes. In spite of all this, we can find in the most conscientious classifications of branches of mathematics some similarity with our enumeration of approaches. In the AMS [20] *Mathematics Subject Classification*, a certain parity can be identified with our subject grouping.

3 Conclusions

As we have seen, the proposal of the six fundamental approaches (seven if material approach is considered) give us a good guideline to understand the historical development of mathematics. In addition it is also a useful tool to understand the internal structure of this beautiful science, and although it does not generate a library classification of the mathematical branches, it helps us understand the origin of the differences between some branches and others. Certainly other approaches may be added in later studies, however they do not necessarily have to be completely independent of the approaches listed here. It seems to be the internal ontological-psychological skeleton of the exact sciences, which should not be conceived as a tree, but as a network of combinations and connections.

References

[1] Thomas Aquinas. *De caritate*.

[2] Jacques Maritain. *Los grados del saber*. Tomo I. Buenos Aires: Ediciones Desclée de Brouwer, 1947.

[3] Vincentio Remer. *Logica maior: Iuxta editionem quintam emendatam et auctam a P. Geny*. Apud Aedes Universitatis Gregorianne, 1955.

[4] Tilmann Pesch. *Institutiones logicales secundum principle S. Thomae Aquinatis ad usum scholosticum accommodavit Tilmennus Pesch, 2.1*. sumptibus Herder, 1889.

[5] Leovigildo Salcedo and Clemente Fernandez. *Philosophiae scholasticae summa/1 Introductio in philosophiam. Logica. Critica. Metaphysica generalis/auctoribus Leovigildo Salcedo; Clemente Fernandez*. Biblioteca de Autores Cristianos, 1964.

[6] Gustavo Bueno. Teoría del cierre categorial. tomo 1: Introducción general, siete enfoques en el estudio de la ciencia, 1992.

[7] Javier De Lorenzo. Introducción al estilo matemático. 1980.

[8] Carlos M Madrid Casado. ¿qué son las matemáticas? la respuesta de la teoría del cierre categorial. *Berceo*, (175):163–184, 2018.

[9] Michèle Friend. *Pluralism in mathematics: A new position in philosophy of mathematics*. Springer, 2014.

[10] Oskar Fechner. *Das Verhältnis der Kategorienlehre zur formalen Logik*. Hinstorff, 1927.

[11] Nicolai Hartmann. *Der Aufbau der realen Welt: Grundriß der allgemeinen Kategorienlehre*. Walter de Gruyter, 2010.

[12] J. Ingvar. *Ontological Investigations*. New York: Routledge, 1989.

[13] Roderick M Chisholm. *A realistic theory of categories: An essay on ontology*. Cambridge University Press, 1996.

[14] Reinhardt Grossmann. *The categorial structure of the world*. Bloomington, Indiana: Indiana University Press, 1983.

[15] Joshua Hoffman and Gary S Rosenkrantz. *Substance among other Categories*. Cambridge University Press, 2007.

[16] E Jonathan Lowe. *More kinds of being: A further study of individuation, identity, and the logic of sortal terms*. John Wiley & Sons, 2015.

[17] José Ferrater Mora. Diccionario de filosofía, 4 vols. *Barcelona: Ariel*, 1994.

[18] Eric Temple Bell. *Historia de las matemáticas*. Fondo de cultura económica, 2016.

[19] Janet Broughton and John Carriero. *A companion to Descartes*, volume 91. John Wiley & Sons, 2010.

[20] American Mathematical Society. Mathematics subject classification. https://mathscinet.ams.org/mathscinet/msc/pdfs/classifications2020.pdf, 2020.