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# Boolean filters of principal $p$-algebras

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**ABSTRACT**

*The concept of Boolean filters is introduced in principal p-algebras. Many properties of Boolean filters are studied. A set of equivalent conditions is given to characterize Boolean filters. For a closed element $a$ of a principal p-algebra $L$, we observed that the filter $[F_a)$ which is generated by the Glivenko congruence class $F_a$ is a Boolean filter of $L$. It is proved that the set $F_B(L) = \{[F_a) : a \in B(L)\}$ forms a Boolean algebra on its own. Finally, some properties of Boolean filters are investigated with respect to the direct products and homomorphisms.*

**Keywords:** p-algebras; principal p-algebras; Filters; Boolean filters; Glivenko congruence; Direct product; homomorphisms.

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## 1 Introduction

The study of pseudocomlemented lattices or shortly p-algebras has a long tradition in lattice theory(see [5] or [7]). The best known examples of p-algebras are the Boolean algebras and Stone algebras. The class of quasi-modular p-algebras was introduced by T. Katriňák [8]. M. Haviar [4] introduced the class of principal p-algebras which contains all quasi-modular p-algebras having a smallest dense element,i.e., it also generalizes the Boolean algebras. Recently T. Katriňák and J. Guričan [9] discussed the tight connection between the spectra and the Glivenko congruence of finite pseudocomplemented lattice.

Recently M. Sambasiva Rao and A. Badawy [10] introduced and characterized $\mu$-filters of distributive lattices. A. Badawy and M. Sambasiva Rao [1] introduced $\sigma$-ideals of distributive $p$-algebras. M. Sambasiva Rao and K.P. Shum [11] introduced the concept of Boolean filters of bounded pseudocomplemented distributive lattices. Also A. Badawy and K. P. Shum [2] studied the relationship between certain congruences and Boolean filters of a quasi-modular $p$-algebra.

After Preliminaries in section 2, the concept of Boolean filters is introduced in principal $p$-algebras and then many properties of Boolean filters are studied in section 3. It is observed that every maximal filter is a Boolean filter and the converse is not true. However, a set of equivalent conditions are derived for a Boolean filter to become a maximal filter. Also a characterization of Boolean filters of principal p-algebras is given. In section 4, we introduced a Boolean filter $[F_a)$ for each $a \in B(L)$, which is generated by the congruence class $F_a$ of the Glivenko congruence relation $\Phi$ on a principal p-algebra $L$. It is proved that the set $F_B(L) = \{[F_a) : a \in B(L)\}$ forms a Boolean algebra on its own. It is also observed that $F_B(L)$ is isomorphic to $B(L)$. Some properties of the direct products of Boolean filters are investigated in section 5. In the last section of this paper, the Boolean filters are characterized in terms of homomorphisms.

## 2 Preliminaries

In this section, we recall some definitions and results which are taken mostly from the papers [3], [4], [5], and [8] for the ready reference of the reader.

A pseudocomplemented lattice (or p-algebra) is an algebra $(L; \vee, \wedge, ^*, 0, 1)$ where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and $^*$ is the unary operation of pseudocomplementation, i.e.,

$$x \wedge a = 0 \Leftrightarrow x \leq a^*$$

A p-algebra $L$ is called *distributive (modular*) if the lattice $(L; \vee, \wedge, 0, 1)$ is a distributive (modular). The variety of modular p-algebras contains the variety of distributive p-algebras. A p-algebra satisfies the Stone identity

$$x^* \vee x^{**} = 1$$

is called an $S$-algebra. A distributive $S$-algebra is called a *Stone algebra*.

Let $L$ be a p-algebra. An element $a \in L$ is called *closed* if $a = a^{**}$. The set $B(L) = \{a \in L : a = a^{**}\}$ of all closed elements of $L$ forms a Boolean algebra $(B(L); \triangledown, \wedge, 0, 1)$, where the join $\triangledown$ is defined by the rule

$$a \triangledown b = (a^* \wedge b^*)^* = (a \vee b)^{**}.$$

In $S$-algebra, $B(L)$ is a subalgebra of $L$ where $a \triangledown b = a \vee b$. An element $d \in L$ is said to be *dense* if $d^* = 0$. The set $D(L) = \{x \in L : x^* = 0\} = \{x \vee x^* : x \in L\}$ of all dense elements of $L$ is a filter of $L$.

Besides distributive and modular p-algebras, a larger variety of quasi-modular p-algebras is interesting to investigate (see [5]). The variety of *quasi-modular p-algebras* is defined by the identity

$$((x \wedge y) \vee z^{**}) \wedge x = (x \wedge y) \vee (z^{**} \wedge x).$$

It is known (see [6.1, 8]) the quasi-modular p-algebras satisfy the identity

$$x = x^{**} \wedge (x \vee x^*)$$

which can be weakened to the equation $x = x^{**} \wedge (x \vee d_L)$ in the case the filter $D(L)$ is principal and $D(L) = [d_L)$.

For an arbitrary lattice $L$, the set $F(L)$ of all filters of $L$ ordered under set inclusion is a lattice. It is known that $F(L)$ is a distributive (modular) if and only if $L$ is distributive (modular). Let $a \in L, [a)$ denote the filter of $L$ generated by $a$.

The relation $\Phi$ of a p-algebra $L$ defined by

$$(x, y) \in \Phi \Leftrightarrow x^* = y^*$$

which is called the Glivenko congruence relation, is a congruence relation on $L$, and $L/\Phi \cong B(L)$ holds. Each congruence class $[x]\Phi$ contains exactly one element of $B(L)$ which is the largest element in the congruence class. The greatest element of $[x]\Phi$ is $x^{**}$. Hence $\Phi$ partitions $L$ into $\{F_a : a \in B(L)\}$, where $F_a = \{x \in L : x^{**} = a\} = [a]\Phi, a \in B(L)$. Clearly $F_0 = \{0\}$ and $F_1 = D(L)$. It is known that $[F_a) = \{x \in L : x^{**} \geq a\}$, for each $a \in B(L)$.

We shall frequently use the following rules of the computations in p-algebras.

For any two elements $a, b$ of a p-algebra $L$, we have (see [7],[9])

(1) $0^{**} = 0$ and $1^{**}$,

(2) $a \wedge a^* = 0$,

(3) $a \leq b$ implies $b^* \leq a^*$,

(4) $a \leq a^{**}$,

(5) $a^{***} = a^*$,

(6) $(a \vee b)^* = a^* \wedge b^*$,

(7) $(a \wedge b)^* \geq a^* \vee b^*$,

(8) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$,

(9) $(a \vee b)^{**} = (a^* \wedge b^*)^* = (a^{**} \vee b^{**})^{**}$.

M. Haviar [4] introduced the class of principal p-algebras which contains all quasi-modular p-algebras having a smallest dense element.

**Definition 2.1.** $[Definition\ 2.1, 4]$ A p-algebra $(L; \vee, \wedge, ^*, 0, 1)$ is called a *principal p-algebra*, if it satisfies the following conditions :

(i) the filter $D(L)$ is principal, i.e., there exists an element $d_L \in L$ such that $D(L) = [d_L)$;

(ii) the element $d_L$ is distributive, i.e., $(x \wedge y) \vee d_L = (x \vee d_L) \wedge (y \vee d_L)$ for all $x, y \in L$;

(iii) $x = x^{**} \wedge (x \vee d_L)$ for any $x \in L$.

If $L$ satisfies the identity $x^* \vee x^{**} = 1$, then it will be called a *principal S-algebra*. Throughout this paper, $d_L$ stands for a smallest dense element of a principal p-algebra $L$, unless otherwise mentioned.

## 3 Boolean filters of principal p-algebras

In this Section, the concept of Boolean filters is introduced in a principal p-algebra. Some properties of Boolean filters are investigated in a principal p-algebra. It is proved that the maximal filter and prime Boolean filter are equivalent. A characterization of Boolean filters of a principal p-algebra is given.

**Definition 3.1.** Let $L$ be a principal p-algebra with a smallest dense element $d_L$. A filter $F$ of $L$ is called a*Boolean filter* if $x \vee d_L \in F$ for each $x \in L$.

Now we give some Examples

(1) For any principal p-algebra $L$ with a smallest dense element $d_L$, the filter $D(L) = [d_L)$ is a Boolean filter of $L$ as $x \vee d_L \in D(L)$ for all $x \in L$. Moreover $D(L)$ is the smallest Boolean filter of $L$ and $L$ is the greatest Boolean filter of $L$.

(2) Let $L$ be a Boolean algebra. Then $D(L) = \{1\}$. Thus any filter $F$ of $L$ is a principal Boolean filter as $x \vee 1 = 1 \in F$ for each $x \in L$.

(3) Let $B_4 = \{0, a, b, c : 0 < a, b < c\}$ be a four elements Boolean lattice and $C_2 = \{d, 1 : d < 1\}$ a two element chain. Clearly $B_4 \bigoplus C_2$ is a principal p-algebra (where $\bigoplus$ stands for ordinal sum). The set of all Boolean filters of $L$ is $\{\{c, d, 1\}, \{a, c, d, 1\}, \{b, c, d, 1\}, L\}$. We observe that the filters $\{d, 1\}$ and $\{1\}$ are not Boolean.

**Lemma 3.1.** *Every maximal filter of a principal p-algebra $L$ is a Boolean filter.*

*Proof.* Let $M$ be a maximal filter of $L$. Suppose $x \vee d_L \notin M$ for some $x \in L$. Then $M \vee [x \vee d_L) = L$. Hence $a \wedge b = 0$ for some $a \in M, b \in [x \vee d_L)$. Then we have

$$\begin{aligned} a \wedge b = 0 &\Rightarrow 0 = a \wedge b \geq a \wedge (x \vee d_L) \geq (a \wedge x) \vee (a \wedge d_L) \\ &\Rightarrow a \wedge x = 0 \text{ and } a \wedge d_L = 0 \\ &\Rightarrow a \leq x^* \text{ and } a \leq d_L^* = 0 \\ &\Rightarrow a = 0 \end{aligned}$$

Then $0 = a \in M$ which is a contradiction. Hence $x \vee d_L \in M$ for all $x \in L$. Therefore, $M$ is a Boolean filter of $L$. □

It is not true that every Boolean filter is a maximal filter. For, in Example 3 above, the filter $\{c, d, 1\}$ is a Boolean filter but not a maximal filter.

**Lemma 3.2.** *A proper filter of a principal p-algebra $L$ which contains either $x$ or $x^*$ for all $x \in L$ is a Boolean filter.*

*Proof.* Let $F$ be a proper filter contains either $x$ or $x^*$ for all $x \in L$. Then $x \vee x^* \in F$ and $D(L) \subseteq F$. Since $L$ is a principal p-algebra, we have $D(L) = [d_L)$ for some $d_L \in L$. Then $x \vee d_L \in D(L)$ implies $x \vee d_L \in F$. Therefore $F$ is a Boolean filter. □

Now, we study some equivalent conditions for a Boolean filter of a principal p-algebra to became a maximal filter.

**Theorem 3.3.** *Let $F$ be a filter of a principal p-algebra $L$. Then the following conditions are equivalent*

*(1) $F$ is maximal,*

*(2) $x \notin F$ implies $x^* \in F$ for all $x \in L$,*

*(3) $F$ is prime Boolean.*

*Proof.* $(1) \Rightarrow (2)$ : Let $F$ is a maximal of $L$. Suppose $x \in L - F$. Then $F \vee [x) = L$. Thus $a \wedge x = 0$ for some $a \in F$. Hence $a \leq x^*$, which implies that $x^* \in F$.
$(2) \Rightarrow (3)$ : Suppose $F$ is not Boolean. Then $x \vee d_L \notin F$ for some $x \in L$. Then $x \notin F$ and $d_L \notin F$. Now $y \vee y^* = d_L \notin F$ for some $y \in L$. Hence $y \notin F$ and $y^* \notin F$, which is a contradiction to the condition (2). Then $F$ is a Boolean filter. Suppose $F$ is not prime. Let $x \vee y \in F$ such that $x \notin F$ and $y \notin F$. Then by the condition (2), we get $x^* \in F$ and $y^* \in F$. Hence $(x \vee y)^* = x^* \wedge y^* \in F$. Therefore $0 = (x \vee y) \wedge (x \vee y)^* \in F$, which is a contradiction. So $F$ is prime. Then $F$ is a prime Boolean filter.
$(3) \Rightarrow (1)$ : Let $F$ be a prime Boolean filter of $L$. Suppose $F$ is not maximal. There exists a proper filter $G$ of $L$ such that $F \subset G$. Choose $x \in G - F$. Since $F$ is Boolean, we get, $x \vee d_L \in F$. Then $x \vee x^* \geq x \vee d_L \in F$ implies $x \vee x^* \in F$. Since $F$ is prime and $x \notin F$, we get $x^* \in F \subset G$. Hence we have $0 = x \wedge x^* \in G$, which is a contradiction. Therefore $F$ is a maximal filter. □

The following lemma is obvious from the Definition 3.1 of Boolean filter.

**Lemma 3.4.** *Let $L$ be a principal p-algebra. Then we have the following :*

*(1) Any filter of $L$ containing $D(L)$ is a Boolean filter,*

*(2) Any filter of $L$ containing a Boolean filter is a Boolean filter,*

*(3) The set $BF(L)$ of all Boolean filters of $L$ is a $\{1\}$-sublattice of the lattice $F(L)$.*

Now, we characterize the Boolean filters on the following Theorem 3.5

**Theorem 3.5.** *Let $F$ be a proper filter of a principal p-algebra $L$. Then the following conditions are equivalent.*

*(1) $F$ is a Boolean filter,*

*(2) $x^{**} \in F$ implies $x \in F$,*

*(3) For $x, y \in L, x^* = y^*$ and $x \in F$ imply $y \in F$.*

*Proof.* $(1) \Rightarrow (2)$ : Assume that $F$ is a Boolean filter of $L$. Suppose $x^{**} \in F$. Since $F$ is a Boolean filter, we have $x \vee d_L \in F$ for all $x \in L$. Then $x^{**} \wedge (x \vee d_L) \in F$. Since $x = x^{**} \wedge (x \vee d_L)$ for every $x \in L$, then $x \in F$ and the condition (2) holds.
$(2) \Rightarrow (3)$ : Let $x, y \in L$ and $x^* = y^*$. Suppose $x \in F$. Then $y^{**} = x^{**} \in F$. Then by the condition (2), we get $y \in F$.
$(3) \Rightarrow (1)$ : Let $x \in D(L)$. So $x^* = 0 \leq a^*$ for all $a \in F$. Then $x^{**} \geq a^{**} \in F$. Hence $x^{**} \in F$. Since $x^* = x^{***}$ and $x^{**} \in F$, by the condition (3), we have $x \in F$. Then $D(L) \subseteq F$. Thus by Lemma 3.4(1), we get that $F$ is a Boolean filter of $L$. □

## 4 Boolean filters via Glivenko congruence classes

In this section, we show that for every closed element $a$ of a principal p-algebra $L$, the congruence class $F_a$ of the Glivenko congruence relation $\Phi$ on $L$ generates a Boolean filter $[F_a)$. Many properties of the Boolean filters $[F_a)$ for all $a \in B(L)$ are studied in a principal p-algebra $L$. Also, we derived that the set $F_B(L) = \{[F_a) : a \in B(L)\}$ forms a Boolean algebra. It is observed that $F_B(L)$ is isomorphic to $B(L)$.

**Theorem 4.1.** *Let $L$ be a principal p-algebra. Then for any two closed elements $a, b$ of $L$ we have the following conditions :*

*(1)* $[F_a) = [a \wedge d_L)$,

*(2)* $[F_a)$ *is a principal Boolean filter of* $L$,

*(3)* $a \leq b$ *in* $B(L)$ *if and only if* $[F_b) \subseteq [F_a)$ *in* $F_B(L)$,

*(4)* $[F_{a\wedge b}) = [F_a) \vee [F_b)$,

*(5)* $[F_{a\bigtriangledown b}) = [F_a) \cap [F_b)$,

*(6)* $[F_{a\vee b}) = [F_a) \cap [F_b)$ *whenever* $L$ *is a principal* $S$*-algebra.*

*Proof.* (1). Since $L$ is a principal p-algebra, then $x = x^{**} \wedge (x \vee d_L)$ for every $x \in L$. Now for all $a \in B(L)$, we get

$$\begin{aligned} [F_a) = \{x \in L : x^{**} \geq a\} &= \{x \in L : x = x^{**} \wedge (x \vee d_L) \geq a \wedge (x \vee d_L)\} \\ &= \{x \in L : x \geq a \wedge d_L\} \\ &= [a \wedge d_L) \end{aligned}$$

(2) Since $d_L \vee x \geq d_L \geq a \wedge d_L$, then $d_L \vee x \in [a \wedge d_L) = [F_a)$. Therefore $[F_a)$ is a principal Boolean filter of $L$.

(3) Let $a \leq b$ in $B(L)$. Assume $x \in [F_b)$. Then $x^{**} \geq b \geq a$. Hence $x \in [F_a)$ and $[F_b) \subseteq [F_a)$ Conversely, suppose $[F_b) \subseteq [F_a)$. Since $b \in F_b \subseteq [F_b) \subseteq [F_a)$. Then we get $b = b^{**} \geq a$.

(4) From (1) we have $[F_{a\wedge b}) = [a \wedge b \wedge d_L)$. Then

$$\begin{aligned} [F_{a\wedge b}) &= [a \wedge b \wedge d_L) \\ &= [(a \wedge d_L) \wedge (b \wedge d_L)) \\ &= [a \wedge d_L) \vee [b \wedge d_L) \\ &= [F_a) \vee [F_b) \end{aligned}$$

(5) Since $a, b \leq a \bigtriangledown b$ on $B(L)$, then by (2), we have $[F_{a\bigtriangledown b}) \subseteq [F_a), [F_b)$. Then $[F_{a\bigtriangledown b})$ is a lower bound of both $[F_a)$ and $[F_b)$ on $F_B(L)$. Assume $[F_z) \subseteq [F_a)$ and $[F_z) \subseteq [F_b)$ for some $z \in B(L)$. Then by (2) we have $z \geq a$ and $z \geq b$. Then $z = z^{**} \geq a \bigtriangledown b$ on $B(L)$. So $z \in [F_{a\bigtriangledown b})$. Hence $[F_z) \subseteq [F_{a\bigtriangledown b})$. Then $[F_{a\bigtriangledown b})$ the infimum of both $[F_a)$ and $[F_b)$ on $F_B(L)$.

(6) Since $L$ is a principal $S$-algebra, then $a \bigtriangledown b = a \vee b$. So $[F_{a\vee b}) = \{x \in L : x^{**} \geq a \vee b\} = \{x \in L : x^{**} \geq a, b\} = \{x \in L : x^{**} \geq a\} \cap \{x \in L : x^{**} \geq b\} = [F_a) \cap [F_b)$. □

**Theorem 4.2.** *Let $L$ be a principal p-algebra, the set $F_B(L)$ forms a Boolean algebra on its own. Moreover, $B(L) \cong F_B(L)$.*

*Proof.* Clearly $[F_1) = D(L)$ and $[F_0) = L$ are the smallest and the greatest elements of $F_B(L)$ respectively. For every $[F_a), [F_b) \in F_B(L)$, by Theorem 4.1(3),(4) we get $[F_{a\wedge b}) = [F_a) \vee [F_b)$ and $[F_{a\bigtriangledown b}) = [F_a) \cap [F_b)$. Then $(F_B(L), \vee, \cap, D(L), L)$ is a bounded lattice. For the distributivity of $F_B(L)$, let $[F_a), [F_b)$ and $[F_c)$ are three elements of $F_B(L)$. Using distributivity of $B(L)$ we get

$$\begin{aligned}
[F_a) \cap ([F_b) \vee [F_c)) &= [F_a) \cap [F_{b\wedge c}) \\
&= [F_{a\bigtriangledown(b\wedge c)}) \\
&= [F_{(a\bigtriangledown b)\wedge(a\bigtriangledown c)}) \\
&= [F_{a\bigtriangledown b}) \vee [F_{a\bigtriangledown c}) \\
&= ([F_a) \cap ([F_b)) \vee ([F_a) \cap ([F_c)).
\end{aligned}$$

Then $F_B(L)$ is a bounded distributive lattice. Define a unary operation $^-$ on $F_B(L)$ by $\overline{[F_a)} = [F_{a^*})$. Now

$$\begin{aligned}
\overline{[F_a)} \cap [F_a) = [F_{a^*}) \cap [F_a) = [F_{a^*\bigtriangledown a}) = [F_1) = D(L), \\
\overline{[F_a)} \vee [F_a) = [F_{a^*}) \vee [F_a) = [F_{a^*\wedge a}) = [F_0) = L.
\end{aligned}$$

Then $[F_{a^*})$ is the complement of $[F_a)$ in $F_B(L)$. Therefore $(F_B(L), \vee, \cap, ^-, D(L), L)$ is a Boolean algebra. Define $f : B(L) \to F_B(L)$ by $f(a) = [F_{a^*})$. Now

$$\begin{aligned}
f(a \wedge b) &= [F_{(a\wedge b)^*}) = [F_{a^*\bigtriangledown b^*}) = [F_{a^*}) \cap [F_{b^*}) = f(a) \cap f(b), \\
f(a \bigtriangledown b) &= [F_{(a\bigtriangledown b)^*}) = [F_{a^*\wedge b^*}) = [F_{a^*}) \vee [F_{b^*}) = f(a) \vee f(b), \\
f(a^*) &= [F_{a^{**}}) = \overline{[F_{a^*})} = \overline{f(a)}
\end{aligned}$$

Obviously, $f(0) = D(L)$ and $f(1) = L$. Then $f$ is a (0,1)-homomorphism. Let $f(a) = f(b)$, then $[F_{a^*}) = [F_{b^*})$. Then $a^* = b^*$ implies $a = a^{**} = b^{**} = b$. Hence $f$ is an injective homomorphism. Also $f$ is surjective as for every $[F_a) \in F_B(L)$, we have $[F_a) = [F_{a^{**}}) = f(a^*)$. Therefore $f$ is an isomorphism and $B(L) \cong F_B(L)$. □

**Lemma 4.3.** *Let $F = [x), x \in L$ be a principal Boolean filter of a principal p-algebra $L$. Then we have the following*

*(1) $F \cap B(L)$ is a principal filter of $B(L)$ generated by $x^{**}$,*

*(2) $F = [F_{x^{**}})$.*

*Proof.* (1). We prove that $[x) \cap B(L) = [x^{**})$. Obviously $[x^{**}) \subseteq [x) \cap B(L)$. Conversely, let $y \in [x) \cap B(L)$. Thus $y \geq x$ and $y \in B(L)$, which implies $y = y^{**} \geq x^{**}$. Hence $y \in [x^{**})$ and $[x) \cap B(L) \subseteq [x^{**})$. Therefore $F \cap B(L) = [x^{**})$.

(2) Since $[F_{x^{**}}) = [x^{**} \wedge d_L)$, $[d_L) = D(L) \subseteq F = [x)$ and $x = x^{**} \wedge (x \vee d_L)$, then

$$\begin{aligned}
[F_{x^{**}}) &= [x^{**} \wedge d_L) \\
&= [x^{**} \wedge (x \vee d_L) \wedge d_L) \text{ as } d_L \leq x \vee d_L \\
&= [x \wedge d_L) \\
&= [x) \\
&= F.
\end{aligned}$$

Therefore $F = [F_{x^{**}}) = [x^{**} \wedge d_L)$. □

**Corollary 4.4.** *Let $L$ be a finite p-algebra. Then we have*

*(1) Every Boolean filter can be expressed as $[F_a)$ for some $a \in B(L)$,*

*(2) $BF(L) = F_B(L)$.*

Now, we can represent any Boolean filter of a principal p-algebra $L$ as a union of certain elements of $F_B(L)$.

**Theorem 4.5.** *Let $F$ be a Boolean filter of a principal p-algebra $L$. Then $F = \bigcup_{x \in F}[F_{x^{**}})$.*

*Proof.* Let $x \in F$. Then $x^{**} \in F$ and $x \vee d_L \in D(L) \subseteq F$. Thus $x = x^{**} \wedge (x \vee d_L) \in [x^{**} \wedge d_L) = [F_{x^{**}}) \subseteq \bigcup_{x \in F}[F_{x^{**}})$. Then $F \subseteq \bigcup_{x \in F}[F_{x^{**}})$. Conversely, let $y \in \bigcup_{x \in F}[F_{x^{**}})$. Then $y \in [F_{z^{**}})$ for some $z \in F$. Hence $y^{**} \geq z^{**} \in F$. Then $y^{**} \in F$ implies $y \in F$ as $F$ is Boolean. Therefore $\bigcup_{x \in F}[F_{x^{**}}) \subseteq F$. □

## 5 Direct product of Boolean filters

Let $L_1$ and $L_2$ be two p-algebras. Then the direct product $L_1 \times L_2$ is also a p-algebra, where $^*$ is defined on $L_1 \times L_2$ by $(a, b)^* = (a^*, b^*)$. Firstly we study the following useful Lemma.

**Lemma 5.1.** *If $L_1$ and $L_2$ be principal $p$-algebras, then we have the following :*

*(1) $D(L_1 \times L_2) = D(L_1) \times D(L_2)$,*

*(2) $B(L_1 \times L_2) = B(L_1) \times B(L_2)$,*

*(3) $L_1 \times L_2$ is a principal p-algebra.*

*Proof.* (1). Let $(d, e) \in D(L_1 \times L_2)$. Then we get

$$\begin{aligned} (d, e) \in D(L_1 \times L_2) &\Leftrightarrow (d, e)^* = (0, 0) \\ &\Leftrightarrow (d^*, e^*) = (0, 0) \\ &\Leftrightarrow d \in D(L_1) \text{ and } e \in D(L_2). \\ &\Leftrightarrow (d, e) \in D(L_1) \times D(L_2). \end{aligned}$$

(2). For any $(a, b) \in B(L_1 \times L_2)$ we have

$$\begin{aligned} (a, b) \in B(L_1 \times L_2) &\Leftrightarrow (a, b)^{**} = (a, b) \\ &\Leftrightarrow (a^{**}, b^{**}) = (a, b) \\ &\Leftrightarrow a^{**} = a \text{ and } b^{**} = b \\ &\Leftrightarrow a \in B(L_1) \text{ and } b \in B(L_2) \\ &\Leftrightarrow (a, b) \in B(L_1) \times B(L_2). \end{aligned}$$

(3). Since $L_1$ and $L_2$ be principal $p$-algebras, then $D(L_1) = [d_{L_1})$ and $D(L_2) = [d_{L_2})$ for some $d_{L_1} \in L_1$ and $d_{L_2} \in L_2$. Thus by (1) we get

$$\begin{aligned} D(L_1 \times L_2) &= D(L_1) \times D(L_2) \\ &= [d_{L_1}) \times [d_{L_2}) \\ &= [(d_{L_1}, d_{L_2})). \end{aligned}$$

So $D(L_1 \times L_2)$ is a principal filter of $L_1 \times L_2$ and $(d_{L_1}, d_{L_2})$ is the smallest dense element of $L_1 \times L_2$. Since $x = x^{**} \wedge (x \vee d_{L_1})$ for all $x \in L_1$ and $y = y^{**} \wedge (y \vee d_{L_2})$ for all $y \in L_2$, then we get

$$\begin{aligned}(x,y)^{**} \wedge ((x,y) \vee (d_{L_1}, d_{L_2})) &= (x^{**}, y^{**}) \wedge (x \vee d_{L_1}, y \vee d_{L_2}) \\ &= (x^{**} \wedge (x \vee d_{L_1}), y^{**} \wedge (y \vee d_{L_2})) \\ &= (x,y).\end{aligned}$$

Consequently $L_1 \times L_2$ is a principal $p$-algebra. □

Now we study the direct product of Boolean filters of principal p-algebras.

**Theorem 5.2.** *If $F_1$ and $F_2$ are Boolean filters of principal p-algebras $L_1$ and $L_2$ respectively, then $F_1 \times F_2$ is a Boolean filter of $L_1 \times L_2$. Conversely, every Boolean filter $F$ of $L_1 \times L_2$ can be expressed as $F = F_1 \times F_2$ where $F_1$ and $F_2$ are Boolean filters of $L_1$ and $L_2$ respectively.*

*Proof.* Let $d_{L_1}, d_{L_2}$ be the smallest dense elements of $L_1, L_2$ respectively. Let $F_1$ and $F_2$ be Boolean filters of $L_1$ and $L_2$ respectively. Obviously $F_1 \times F_2$ is a filter of $L_1 \times L_2$. Since $F_1$ and $F_2$ are Boolean filters of $L_1$ and $L_2$ respectively, we get $a \vee d_{L_1} \in F_1$ for each $a \in L_1$ and $b \vee d_{L_2} \in F_2$ for each $b \in L_2$. So we have

$$(a,b) \vee (d_{L_1}, d_{L_2}) = (a \vee d_{L_1}, b \vee d_{L_2}) \in F_1 \times F_2$$

Then $F_1 \times F_2$ is a Boolean filter of $L_1 \times L_2$. Conversely, let $F$ be a Boolean filter of $L_1 \times L_2$. Consider $F_1$ and $F_2$ as follows :

$$F_1 = \{x \in L_1 : (x,1) \in F\} \text{ and } F_2 = \{y \in L_2 : (1,y) \in F\}$$

Clearly $F_1$ and $F_2$ are filters of $L_1$ and $L_2$ respectively. Now we prove that $F_1$ and $F_2$ are Boolean filters of $L_1$ and $L_2$ respectively. For each $x \in L_1, (x,1) \in L_1 \times L_2$. Since $F$ is Boolean, then $(x \vee d_L, 1) = (x,1) \vee (d_{L_1}, d_{L_2}) \in F$. Hence $x \vee d_{L_1} \in F_1$. Therefore $F_1$ is a Boolean filter of $L_1$. Similarly, we get $F_2$ is a Boolean filter of $L_2$. Now we prove that $F = F_1 \times F_2$. Let $(x,y) \in F$. Then we have

$$\begin{aligned}(x,y) \in F &\Rightarrow (x,1) \in F \text{ and } (1,y) \in F \\ &\Rightarrow x \in F_1 \text{ and } y \in F_2 \\ &\Rightarrow (x,y) \in F_1 \times F_2.\end{aligned}$$

Then $F \subseteq F_1 \times F_2$. Conversely, let $(x,y) \in F_1 \times F_2$. Now

$$\begin{aligned}(x,y) \in F_1 \times F_2 &\Rightarrow x \in F_1 \text{ and } y \in F_2 \\ &\Rightarrow (x,1) \in F \text{ and } (1,y) \in F \\ &\Rightarrow (x,y) = (x,1) \wedge (1,y) \in F.\end{aligned}$$

Then $F_1 \times F_2 \subseteq F$. Therefore $F_1 \times F_2 = F$. □

**Lemma 5.3.** *For any two Boolean filters $[F_a)$ and $[F_b)$ of principal p-algebras $L_1$ and $L_1$ respectively, $[F_a) \times [F_b) = [F_{(a,b)})$*

*Proof.* From the above Theorem 5.2, $[F_a) \times [F_b)$ is a Boolean filter of $L_1 \times L_2$. Now

$$\begin{aligned} (x,y) \in [F_a) \times [F_b) &\Leftrightarrow x \in [F_a) \text{ and } y \in [F_b) \\ &\Leftrightarrow x^{**} \geq a \text{ and } y^{**} \geq b \\ &\Leftrightarrow (x,y)^{**} = (x^{**}, y^{**}) \geq (a,b) \\ &\Leftrightarrow (x,y) \in [F_{(a,b)}). \end{aligned}$$

Therefore $[F_a) \times [F_b) = [F_{(a,b)})$. □

## 6 Boolean filters and homomorphisms

In this section, some properties of the homomorphic images and the inverse images of Boolean filters are studied. By a homomorphism on a p-algebra $L$, we mean a lattice homomorphism $h$ which preserves the pseudocomplementation, that is, $(h(x))^* = h(x^*)$ for all $x \in L$.

**Theorem 6.1.** *Let $L, L_1$ be principal p-algebras with smallest dense elements $d_L, d_{L_1}$ respectively and $h : L \to L_1$ an onto homomorphism. Then*

*(1)* $h(d_L) = d_{L_1}$

*(2)* $h([F_a)) = [F_{h(a)})$ *for all* $a \in B(L)$,

*(3)* $h(F)$ *is a Boolean filter of* $L_1$ *whenever* $F$ *is a Boolean filter of* $L$.

*Proof.* (1). We observe that $h(d_L) \in D(L_1)$ as $(h(d_L))^* = 0$. Then $d_{L_1} \leq h(d_L)$. Since $h$ is an onto homomorphism, then $d_{L_1} = h(x)$ for some $x \in L$. So $(h(x))^{**} = 1$. Now

$$\begin{aligned} d_{L_1} &= h(x) \\ &= h(x^{**} \wedge (x \vee d_L)) \\ &= ((h(x))^{**} \wedge (h(x) \vee h(d_L)) \\ &= h(x) \vee h(d_L) \geq h(d_L). \end{aligned}$$

Therefore $h(d_L) = d_{L_1}$.

(2). Let $a \in B(L)$. Let $t \in h([F_a)$. Then $t = h(x)$ for some $x \in ([F_a))$. Then $x^{**} \geq a$ implies $t^{**} = h(x^{**}) \geq h(a)$. It follows that $t \in [F_{h(a)})$. Conversely, let $y \in [F_{h(a)}$. Then $y^{**} \geq h(a)$. Hence $y^{**} \geq (h(a))^{**} = h(a^{**}) = h(a)$. Then $y \in h([F_a))$.

(3). Let $F$ is a Boolean filter of $L$. Clearly $h(F)$ is a filter of $L_1$. Since $F$ is Boolean, then $x \vee d_L \in F$ for all $x \in L$. Then by (1) we get $h(x) \vee d_{L_1} = h(x) \vee h(d_L) = h(x \vee d_L) \in h(F)$. Then $h(F)$ is a Boolean filter of $L_1$. □

**Theorem 6.2.** *Let $h : L \to L_1$ be a homomorphism of a principal p-algebra $(L; \vee, \wedge, ^*, 0_L, 1_L)$ onto a principal p-algebra $(L_1; \vee, \wedge, ^*, 0_{L_1}, 1_{L_1})$. Then $F_B(L)$ is homomorphic of $F_B(L_1)$.*

*Proof.* Define $g : F_B(L) \to F_B(L_1)$ by $g([F_a)) = [F_{h(a)})$ for all $a \in B(L)$. For every $a, b \in B(L)$, we get

$$\begin{aligned} h(a \bigtriangledown b) &= h(a^* \wedge b^*)^*) = h(a \vee b)^{**} \\ &= (h(a \vee b))^{**} = (h(a) \vee h(b))^{**} \\ &= ((h(a))^* \wedge (h(b))^*)^* = h(a) \bigtriangledown h(b). \end{aligned}$$

Consequently, we get

$$\begin{aligned} g([F_a) \vee [F_b)) &= g([F_{a \wedge b})) \\ &= [F_{h(a \wedge b)}) \\ &= [F_{h(a) \wedge h(b)}) \\ &= [F_{h(a)}) \vee [F_{h(b)}) \\ &= g([F_a)) \vee g([F_b)), \\ g([F_a) \cap [F_b)) &= g([F_{a \bigtriangledown b})) \\ &= [F_{h(a \bigtriangledown b)}) \\ &= [F_{h(a) \bigtriangledown h(b)}) \\ &= [F_{h(a)}) \cap [F_{h(b)}) \\ &= g([F_a)) \cap g([F_b)), \\ g(\overline{[F_a)}) &= [F_{h(a^*)}) \\ &= [F_{(h(a))^*}) \\ &= \overline{[F_{h(a)})} \\ &= \overline{g([F_a))}. \end{aligned}$$

Clearly $g([F_{1_L}) = [F_{1_{L_1}}))$ and $g(L) = L_1$. Therefore $g$ is a homomorphism of Boolean algebras $F_B(L)$ and $F_B(L_1)$. □

**Theorem 6.3.** *Let $h : L \to L_1$ be a homomorphism of a principal p-algebra $L$ with a smallest dense element $d_L$ into a principal p-algebra $L_1$ with a smallest dense element $d_{L_1}$. Then we have the following :*

*(1) $h^{-1}(G)$ is a Boolean filter of $L$ whenever $G$ is a Boolean filter of $L_1$,*

*(2) $Coker$ h is a Boolean filter of $L$ whenever $h(D(L)) = \{1_{L_1}\}$.*

*Proof.* (1). Let $G$ be a Boolean filter of $L_1$. Then $h^{-1}(G)$ is a filter of $L$. Let $x \in L$. Then $h(x) \in L_1$. Since $G$ is a Boolean filter of $L_1$, then $h(x) \vee d_{L_1} \in G$. Then $h(x \vee d_L) = h(x) \vee h(d_L) \geq h(x) \vee d_{L_1} \in G$ implies $h(x \vee d_L) \in G$. So $x \vee d_L \in h^{-1}(G)$. Therefore $h^{-1}(G)$ is a Boolean filter of $L$.

(2). Obviously $Coker\ h = \{x \in L : h(x) = 1_{L_1}\}$ is a filter of $L$. For every $x \in L$, $x \vee d_L \in D(L)$ as D(L) is a Boolean filter of $L$. Hence $h(x \vee d_L) = 1_{L_1}$ by hypothesis. Then $x \vee d_L \in Coker\ h$. Therefore $Coker\ h$ is a Boolean filter of $L$. □

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