Against the Iterative Conception of *Set*

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Introduction

According to one account, the famous paradoxes of Cantor, Burali-Forti and Russell, are the result of an ill-conceived attempt to quantify without restriction over all sets.[1] The moral is that (meaningful) quantification over sets is necessarily restricted. However, this quantificational account has been poorly received. Perhaps the strongest reason for this is the discovery and development of consistent, axiomatic set theories, in which quantification is absolute.[2]

These set theories are philosophically motivated by the iterative conception of *set*, according to which sets are collections that are "formed out of" their members by means of an iterative process encoded in the axioms of the theory. Very roughly, this process goes as follows. One begins by forming collections—sets—out of some original things, called urelements. (If there are no urelements, one begins by forming the empty collection, $\emptyset$.) One then proceeds to form *new* collections out of these collections and the urelements (if there are any). Next, one forms more new collections out of the most recently formed collections, the earlier collections and the urelements. One goes on in this way, forming new collections out of previously formed collections and the urelements, forever.

Proponents of the iterative conception have represented this process of set formation as occurring in a series of primitive well-ordered "stages" (Shoenfield (1967), Boolos (1998, chaps. 1 and 6)) or as comprising a series of successively generated "levels" (Potter (2004)). Shoenfield and Boolos proceed to derive the traditional axioms of set theory from the set formation rule that at each stage all pluralities of sets formed at previous stages are formed into sets. Potter adopts an explicit, recursive definition of 'level' (originally formulated by Dana Scott (1974)), according to which (i) the first level, V_0, is the empty set and (ii) each successive level, V_α, is the set containing all members and sub-collections of all

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[1] This is Dummett's (1991) interpretation of Russell (1906, 144). See Uzquiano (2015) for a more recent defense of this account.

[2] The best known of these is Zermelo-Fraenkel set theory (ZF). Zermelo's original 1908 set theory did not include the axioms of Replacement or Foundation. The necessity of Replacement to derive large sets was discovered (independently) by Fraenkel (1922) and Skolem (1922). Foundation was subsequently added by von Neumann (1925). Zermelo incorporated both axioms in his revised 1930 set theory.

preceding levels V_β for $\beta < \alpha$.[3] He then proceeds to derive the traditional axioms from a version of the Separation axiom, according to which any members of a level form a set. All three accounts represent sets as being formed out of (or constituted by) their members: Shoenfield and Boolos's insofar as each application of the rule of set formation generates new sets whose members are previously formed sets; Potter's insofar as each level contains all members and sub-collections of members from previous levels (and sets are defined via Separation as sub-collections of levels).

The iterative process of set formation suggests the existence of a real relation of priority between members and the sets they form, which in turn provides a principled explanation for why the contradiction-inducing sets—sets such as the Russell set, the universal set and the set of all ordinals—do not exist. This explanatory power provides a powerful reason for adopting the iterative conception.[4]

Since this explanation is expressed in terms of priority, its success depends on our ability to make sense of a suitable priority relation. I will argue that attempts to do this have so far fallen short: understanding priority in a straightforwardly constructivist sense threatens the coherence of the empty set and raises serious epistemological concerns; but the leading realist interpretations—ontological and modal interpretations of priority—are deeply problematic as well. I conclude that the purported explanatory virtues of the iterative conception are, at present, unfounded.

1 Constructivist accounts

The constructivist account of priority is a metaphysical thesis about the nature of sets according to which sets are mental constructions. As such, it is not to be confused with *mathematical intuitionism*, which is not a metaphysical thesis about the nature of mathematical objects, but is rather a view about the correct logic for doing mathematics (famously, intuitionistic logic rejects the principle of excluded middle).[5]

Constructivism may be motivated by Cantor's (1895, 481) definition of *set* as "a collection into a whole of definite distinct objects of our intuition or of our thought." The

[3] In standard set theories, levels are recursively defined in terms of the operations of Powerset and Union as follows: (i) $V_0 = \emptyset$; (ii) for any successor ordinal, $\alpha+1$, $V_{\alpha+1} = \mathscr{P}(V_\alpha)$; (iii) for any limit ordinal, λ, $V_\lambda = \cup_{\gamma<\lambda} V_\gamma$. The novelty of Scott's definition is that it does not presuppose the operations of Powerset and Union.

[4] Gödel (1947, 518–519) praises the iterative conception for these reasons when he writes:

> This concept of set, however, according to which a set is something obtainable from the integers (or some other well-defined objects) by iterated application of the operation "set of", not something obtained by dividing the totality of all existing things into two categories, has never led to any antinomy whatsoever; that is, the perfectly "naive" and uncritical working with this concept of set has so far proved completely self-consistent.

[5] While mathematical intuitionism is often motivated by a general constructivist philosophy of mathematics, according to which all mathematical objects are mental constructions, it may also be (independently) motivated by more purely mathematical considerations. To give one example: the intuitionist (but not the classical mathematician) is able to adopt the Principle of Uniformity without logical inconsistency. This, in turn, allows her to express the difference between countable and uncountable sets in ways that elude her classical counterpart. See, for example, McCarty (2005, 357–360) and Koss (2013, ch. 6). Thanks to an anonymous referee for pushing me to clarify the relation between these two views.

identification of sets with *collections* whose members are objects "of our *intuition* or of our *thought*," suggests a mental process according to which new sets are formed by acts of mentally conceiving, or collecting, previously formed sets. This, in turn, suggests a temporal interpretation of the stages of set formation, according to which the priority of members to their set is to be understood as the temporal priority of mental construction: the members must be temporally *available* as objects of intuition or thought before it is possible to mentally collect them. If in addition, it is possible to mentally collect *any* elements that are temporally available in this sense, it follows that *any* available elements form a set. Availability may then be understood as equivalent to existence at a stage in the process of set formation. There are a number of objections that have been raised against this view.

The first concerns the empty set. It is based on the thought that if sets are formed by acts of collecting their members, then the act of collecting nothing ought to result in the failure to form any set at all, not in the formation of the empty set. Consequently, if we wish to maintain the coherence of an empty set, we must reject constructivism (Black, 1971, 618–622).[6]

The second objection concerns time. Since (presumably) mental acts take place in time, constructivism has the implication that the existence of infinite sets (at least, sets, such as ω, of infinite rank) depends on the existence of infinite sequences of temporally ordered mental acts. This is problematic for two reasons. First, it is doubtful whether such sequences are possible for finite creatures like us, and so constructivism may be committed to immortal minds. Second, even granting the existence of immortal minds, time itself may not be long enough for the construction of sets larger than the continuum.

Instead of an actual sequence of mental acts, the constructivist might try understanding set formation in terms of a more abstract, idealized possibility of mentally collecting (Wang, 1974). She might then reason as follows. Assume that $\emptyset$ is formed. Then it's possible to mentally collect $\emptyset$. So $\{\emptyset\}$ is formed. Then it's possible to mentally collect $\emptyset$ and $\{\emptyset\}$ (individually and jointly). So $\{\{\emptyset\}\}$ and $\{\emptyset, \{\emptyset\}\}$ are formed. And so on. Since none of these *possible* acts need be understood as actually occurring in time, there is no longer any reason to doubt the existence of sets that can be formed only after continuum-many steps.

However, this notion is quite obscure. Given the availability of a countable infinity of sets, e.g., the countable sequence $\emptyset$, $\{\emptyset\}$, $\{\emptyset, \{\emptyset\}\}$, ..., is it possible to mentally collect them in the relevant idealized sense, thereby forming an infinite set? What if we are given an uncountable infinity of sets? Is it possible to mentally collect these, thereby forming

[6]Related critical discussion of the empty set can be found in Oliver and Smiley (2013, 248–254) and Lewis (1991, 10–15). Oliver and Smiley argue that there are no good reasons for accepting the empty set, but, unlike Black, they do not view this as a threat to iterative set theory, which, they maintain can quite easily be founded on a basis of urelements instead. Lewis, on the other hand, seeks to preserve the empty set in the context of an eliminative structuralism. He points out that the eliminative structuralist can easily circumvent worries about the coherence of an empty collection by introducing an arbitrary object (which is not an empty collection) to play the empty set's role. Whatever the merits of this structuralist approach, it is not generally viewed as amenable to the constructivist's view of sets as mind-dependent. Even if it was, it cannot plausibly be viewed as a defense of the iterative conception of *set*, according to which sets have intrinsic natures: they are collections. I am grateful to an anonymous referee and to Sam Cowling for bringing these discussions to my attention.

an uncountably infinite set? Wang (1974) answers that it is possible in both cases; but it seems more likely that his answer is guided by prior commitments to standard set theory than by an independent grasp of what it is possible to mentally collect. It is of little help when he equates the ability to collect the sets falling under a concept with the ability to "look through" or "run through" them "in an *idealized* sense, ... in such a way that there are no surprises as to the objects which fall under the concept" (1974, 531).

Individual axioms of set theory raise additional questions. The axiom of Separation states that given a set, x, any set-theoretic predicate ϕ defines the subset $y = \{z|z \in x \wedge \phi(z)\}$. Separation is true, according to the constructivist, only if it is possible to mentally collect the members of each of these subsets. But is this possible? The constructivist wants to answer yes. What if ϕ involves quantification over *all* sets? Must it then be possible to mentally "look through" or "run through" every set in order to determine which sets belong to y? In order to answer these questions in a principled way, the constructivist must clarify her notion of an idealized (possible) act of mentally collecting available elements. The prospects of doing so are dim.

One might reply that the relevant notion of what can be constructed has already received a precise mathematical formulation in the (non-standard) axiom of Constructibility, which restricts set theory to the constructible sets (those that can be predicatively defined).[7] Unfortunately, the axiom provides the constructivist with both more and less than she requires. On the one hand, while the constructivist's view of sets as idealized mental constructions certainly does motivate the axiom's restriction to predicatively defined sets, it is unclear whether this is enough to capture the notion of what it is mentally possible to collect. On the other hand, the restriction goes too far: it is too restrictive for the iterative conception of *set* the constructivist is attempting to explain. I discuss each point in turn.

First, it is a theorem that the Axiom of Constructibility allows one to define all the ordinal numbers. However, it is unclear whether these definitions comport with the constructivist's strictures. Successor ordinals do not appear to be especially problematic. Each is defined in terms of its immediate predecessor in a way that makes it quite plausible that given the possibility of mentally collecting the elements of the predecessor, it should also be possible to mentally collect the elements of its successor. Limit ordinals are more worrisome since each is defined as the union of *all* preceding ordinals. Even if it is possible to construct each of these (individually, as it were), it is unclear whether it should also be possible to combine them *all* to form their limit.[8] One might reply by abandoning the attempt to provide a set-theoretic reduction of the ordinals, taking them instead as sui generis; but this conflicts with the spirit of the constructivist approach: if sets are mentally constructed, then why aren't the ordinals as well?

On the other hand, it would seem that the axiom of Constructibility must exceed

[7]The axiom was originally introduced and investigated by Gödel (1939, 1940). Like the standard set-theoretic universe, the constructible universe may be defined as the union of its levels, with the operation Definable Powerset, $\mathscr{P}_{\text{DEF}}$, replacing the full Powerset operation: (i) $L_0 = \emptyset$; (ii) for any successor ordinal, $\alpha + 1$, $L_{\alpha+1} = \mathscr{P}_{\text{DEF}}(V_\alpha)$; (iii) for any limit ordinal, λ, $V_\lambda = \cup_{\gamma<\lambda} V_\gamma$.

[8]I am indebted to an anonymous referee for encouraging me to clarify the relation between metaphysical constructivism and the axiom of Constructibility and for pushing me to distinguish between successor and limit ordinals in this argument.

the constructivist's strictures. For the constructivist is interested in using this notion to explicate the iterative conception of *set*, according to which any previously formed sets form a set. But it is not the case that any previously constructed sets can be singled out by a predicative definition. And so, in a theory of constructible sets, it is not the case that any previously formed sets form a set. Insofar as the axiom of Constructibility expresses a view of sets that differs from the view of sets expressed by the iterative conception, the former can be no help to the constructivist, who is trying to explicate the latter.

2 Dependence accounts

A second account of priority is in terms of the relation of ontological dependence. Intuitively, x depends on y if x is ontologically not self-sufficient, in the sense that x essentially involves y, but not vice versa (Correia, 2008).[9] This can be expressed as a condition on the existence of dependent objects: if x ontologically depends on y, then it is part of x's essence, or identity, to exist only if y exists but not vice versa. To illustrate, while it would appear that necessarily, Socrates exists iff {Socrates} exists, it is plausible that it is a part of {Socrates}'s essence, or identity, to exist only if Socrates exists, but not vice versa. The asymmetry here captures the sense in which Socrates is ontologically self-sufficient while {Socrates} is not.

Dependence accounts of priority may be motivated by reflection on the extensional identity conditions for sets. Viewed metaphysically, these identity conditions may be taken to suggest that sets depend (fully) on the plurality of their members.[10] One may then claim that the priority of the members to their set consists in the metaphysical fact that while their set depends fully on them (and partially on each one of them), they do not depend on it (either fully or partially).

There are two problems with this account. First, it legitimizes certain historical arguments from the late nineteenth and early twentieth centuries that purport to show the absurdity of an empty class. (In the discussion that follows, there is no real difference between 'class' and 'set'.) Such arguments were frequently based on claims such as, "sets are mere aggregates," or "sets consist entirely of their members," which appear to confuse sets with either mereological sums or pluralities. If either sort of confusion played an essential role in the historical arguments, it would not be unreasonable for philosophers to ignore them today. A famous example is from Frege (1895, 212):

> A class, in the sense in which we have so far used the word, consists of objects; it is an aggregate, a collective unity, of them; if so, it must vanish when these objects vanish. If we burn down all the trees of a wood, we thereby burn down the wood. Thus there can be no empty class.

[9]See Fine (1994), Fine (1995a), Fine (1995b), Lowe (1998, ch. 6) and Tahko and Lowe (2015) for more detail. For a somewhat different formulation, see Schaffer (2009) and Rosen (2010). I am grateful to an anonymous referee for encouraging me to clarify the notion of ontological dependence here.

[10]When an individual x depends on a (possibly infinite) number of distinct individuals, yy, perhaps related in some way, and on nothing else, x *fully depends* on yy so related. If x fully depends on yy, then x *partially depends* on each y among yy.

Context here is critical. This passage is taken from Frege's own critical review of Schröder's logical treatise (*Vorlesungen über die Algebra der Logik*). In that work, Schröder had endorsed a pseudo-mereological view of classes as aggregates and had also accepted the empty class. Frege is arguing that both views cannot be true: if classes really are aggregates then there can be no empty class, for there can be no aggregate of what is not. This argument is quite straightforward; however, it is not so clear what is needed to provide a satisfactory response. Donald Gillies (2011, 55) provides a typical gloss when he identifies the underlying problem as a failure by Schröder to adequately distinguish the membership relation $\in$ from the subset relation $\subseteq$ (a set can be thought of as the mereological sum of its subsets, but not its members). This evaluation leads Gillies to sum up his discussion of Frege's argument as follows:

> Once the distinction between '$\in$' and '$\subseteq$' had been clearly made (and this was no easy task historically), then the difficulties concerned with the empty set disappeared.

If Gillies is right, then Frege's objection poses no threat to the iterative conception of *set*, for surely the iterative conception invites no confusion between $\in$ and $\subseteq$. However, I think that Gillies's evaluation fails to register the full force of Frege's complaint. To see this, consider Frege's example once more. According to the iterative conception, sets are collections and so the *set* of trees that survive the fire is the *collection* of trees that survive the fire. It is certainly true that we must be careful to distinguish this collection from the mereological sum of surviving trees (as well as the plurality of surviving trees); but if collections depend fully on their members, then it would seem to follow that the collection of surviving trees, like their mereological sum, ceases to exist when no trees survive.

One might reply that in this case the collection of surviving trees, having no members, depends for its existence on nothing, in the sense that *nothing* is required for it to exist.[11] But this is to misunderstand the nature of a collection, under the present account. Collections, if they depend on their members, are to be understood as a type of dependent being. A collection may be said to depend on its specific members for its identity; but to belong to the kind of thing that it is, *collection*, it must have members. One might say that a collection is essentially a collection *of some things*. Understood in this way, 'a collection with no members' is not a trivial collection, but no collection at all and Frege's argument succeeds in establishing the absurdity of an empty set. Notice that this fits nicely with Cantor's own views on sets as collections. As Oliver and Smiley (2013, 248–250) attest, where modern theorists would speak of a set with no members, Cantor says that no set exists:

> For Cantor there was no such thing as an empty set. When describing a putative point-set that turns out not to contain any points, he says that strictly speaking it does not really exist at all. ... Two point-sets with no point in common do not have an empty intersection; rather they have *no* intersection.[12]

Russell (1903, ch. 6) presents a similar argument against the empty class; though, unlike Frege, he seems sympathetic to the view of classes he targets. He begins by observing

[11] I am grateful to an anonymous referee here.

[12] A "point-set" is simply a set whose members are points.

that finite classes can be defined by simply enumerating their members. Infinite classes, of course, are an exception, but Russell places little importance on this, since he attributes our inability to define infinite classes by enumeration to theoretically unimportant physical limitations, such as not having enough time. He concludes (p. 67) that classes simply *are* their members:

> Thus Brown and Jones are a class, and Brown singly is a class.

If classes are to be identified with their members, it follows that when there are no members there is no class (as Frege observed). Thus, Russell (1903, 68) notes that a consequence of his view is:

> there is no such thing as the null-class, though there are null class-concepts.

(A null class-concept is a property with no instances, e.g., non-self-identity.)

Insofar as this argument seems to depend entirely on the identification of classes with their members, it is bound to strike the contemporary reader as a non-starter. But, as with Frege's argument above, it would be overly hasty to simply dismiss the argument at this point. The crux of Russell's argument is the inference from no members to no class; and while this inference certainly holds if classes are identical to their members, it is not obvious that it holds *only if* classes are identical to their members. It may be enough if classes are fully dependent on their members. Later, Russell (1903, 74) expresses the objection in slightly different words:

> A class which has no terms fails to be anything at all: what is merely and solely a collection of terms cannot subsist when all the terms have been removed.

It's unlikely that Russell meant to distinguish the claim that a class is 'merely and solely a collection' of its members from his earlier identity claim; but it's noteworthy that his argument against the empty class seems to hold good if the former claim is interpreted as the weaker dependence claim that classes are distinct from but fully dependent on their members. In other words, it's possible to disentangle Russell's argument against the empty class from the identity claim that classes are their members. The critical inference—from no members to no class—holds even if a class (collection) is not identical to its members, provided that it is fully dependent on them.

The second problem with the dependence account of priority is that it fails to prevent the existence of the contradiction-inducing sets; indeed, it entails their existence! Applying the statement of ontological non-self-sufficiency above to sets, it follows that if a set ontologically depends on its members, then, it is part of the set's essence, or identity, that it exists only if its members do. But sets depend *fully* on their members. For the identity of the members is *all* that is required to determine the identity of a set. This would seem to license not only the claim (a) that the existence of its members is a necessary condition for the existence of a set, but also the claim (b) that the existence of its members is a sufficient condition for the existence of a set. All God must do to create a set is to create its members.

This move from the necessary condition (a) to the sufficient condition (b) may seem surprising, and for good reason. It is not generally valid. In many cases, an object x may depend fully on some other objects (in the sense that x depends on no other *objects*) and yet the existence of these objects may fail to provide a sufficient basis for x's existence. This occurs whenever x's existence requires that the objects on which it depends stand in some special relation(s). Thus, we might say that a physical object, such as a table, is fully dependent on its parts, but nevertheless, their mere existence is not enough to secure the table's existence. They must be put together in the right way. Sets are not like this: their identity conditions take into account only the identity of their members, any relations the members bear to one another are irrelevant. This strongly suggests that according to the dependence account of priority currently under consideration, all that is needed for the existence of a set is the existence of the sets on which it depends.

It follows that all that is needed for the existence of the contradiction-inducing universal set is the existence of the sets on which it depends.[13] But, according to iterative set theory, all these sets do exist. After all, the quantifiers in iterative set theory are unrestricted. So if priority is understood as dependence, iterative set theory is committed to the existence of a universal set. The same point may be made by a reductio argument. Suppose, for reductio, that there is no universal set. Then it must be the case that at least one of its members does not exist. (For it depends on these and on nothing else.) But the quantifiers of iterative set theory are not restricted: they range over all sets. So their set also exists.

In response, one might point out that the irreflexivity of the dependence relation motivates the axiom of Foundation, which explicitly prohibits self-membership. So there is one member of any purported universal set that (provably) does not exist: viz., itself! However, I believe that what this shows is not that the derivation of the universal set from the dependence account of priority fails, but rather that the dependence account is inconsistent, entailing both that the universal set exists and that it does not exist. On the one hand, according to the dependence account, there must be a collection of all the iterative sets. What can this be if not the universal set? (It will not do to contend that the dependence account entails only the existence of a proper class of all the sets, not a set of them. The iterative conception is supposed to be extensionally complete: it is intended to generate all and only sets. If the priority metaphor is to be understood in terms of ontological dependence, then every collection whose existence is entailed by the dependence relation must be a set.) On the other hand, according to the dependence account, there cannot be a set of all the iterative sets. For no iterative set can be a self-member. One may still be inclined to protest: even if all the sets exist, they are not all *available* to be formed into a set. For there is no stage in the iterative hierarchy at which all the sets are formed. But this is just to fall back on constructivism. We had hoped to avoid this by adopting the relation of ontological dependence. This strategy cannot succeed if the existence of the members of a set is sufficient for the set's existence.

[13] The contradictions of Cantor, Burali-Forti and Russell are each derivable in iterative set theory from the assumption that there exists a universal set, U. Cantor's paradox can be derived from Cantor's theorem, according to which U has more subsets than members (this is impossible, given that all of U's subsets are members). Burali-Forti's paradox and Russell's paradox are derived by applying the subset schema to define the set of all ordinals and the set of all non-self-membered sets as subsets of U.

3 Modal accounts

To date, the most developed accounts of the priority relation are modal accounts. These are presented by Parsons (1983, chaps. 10, 11), Linnebo (2010, 2013), Studd (2013) and to some extent Fine (2006a,b). The goal of these accounts is, in Parsons's words, to maintain the idea that "any available objects can be formed into a set," but "to replace the language of time and activity by the more bloodless language of potentiality and actuality" (1983, 293). More specifically, talk of an infinite process of set formation in stages is to be treated as a metaphor for the underlying modal facts: a potential hierarchy of sets in which each set "is an immediate possibility given its elements" (1983, 294). Linnebo describes the modal account similarly, as one "based on the idea that the hierarchy of sets is a potential one, not a completed or actual one," in which "the existence of a set is potential relative to its elements" (2010, 155). And Studd writes that the tensed language used to characterize the iterative conception of *set* is to be taken "seriously," but not "literally" (as describing an actual, temporal process of set construction); rather, the tense is to be "replaced with suitable modal operators governed by a tense-like logic," in which the potential nature of sets is expressed by the thesis that "any sets *can* form a set" (2013, 699–700). (This thesis may appear stronger than Parsons's qualified claim that any *available* objects can be formed into a set. However, a similar qualification is built into the modal context in which Studd's statement applies so that the occurrence of 'any sets' in this context has the force of 'any *actual* sets' outside of it.) In what follows, I shall refer to this as "the core thesis" of the modal account.[14]

In order to properly characterize the potential nature of sets, and to properly formulate the core thesis, we need a suitable modal logic. Recall that according to the iterative process of set formation—which will serve as our intuitive guide for the underlying modal facts—(a) new sets are formed at every stage and (b) sets, once formed at a stage s, exist at every later stage t. (a) and (b) can be represented in a modal logic by an ordering on possible worlds (of sets) under the relation $>$, where $w_j > w_i$ iff w_j's domain is a proper expansion of w_i's domain. Intuitively, the sets that exist at w_j are (b) all the sets that have been formed at previous stages and which therefore exist at every preceding world w_i, as well as (a) any (new) sets that *can* be immediately formed out of these. This ordering is captured by a directed, S4 logic, in which the accessibility relation is defined as $\geq$ (a relation that is transitive, reflexive, and anti-symmetric).[15]

The greatest challenge for the modal account is explicating the relevant notions of *possibility* and *necessity*. Linnebo represents these by means of the standard modal operators

[14]Studd (2013, 699) calls it "the maximality thesis." Linnebo (2010, 157) refers to it as "Collapse$^{\Diamond}$": a modal interpretation of the inconsistent "Collapse" principle, according to which any things *do* form a set. Parsons (1983, 280–297) traces this back to the claim, which he attributes to Cantor (1899), that any "consistent multiplicity" can form a set. That any sets *can* form a set is also a consequence of Kit Fine's (2006a; 2006b) modal formulation of restrictivism, according to which any mathematical domain can be "expanded" by appropriate procedural postulates.

[15]In the interests of deriving a modal set theory, Linnebo strengthens this logic to S4.2, in which accessibility is also *convergent*. (Accessibility between worlds is *convergent* if whenever a world w_1 accesses worlds w_2 and w_3, there is a world, w_4, that w_2 and w_3 both access.) Studd strengthens the logic to S4.3, in which accessibility is also *connected*. (Accessibility between worlds is *connected* if for any distinct worlds w_1 and w_2, either w_1 accesses w_2 or w_2 accesses w_1.)

$\Box$ and $\Diamond$, which are semantically modeled by quantification over possible worlds in the usual way:

- $\Box\phi$ is true at w iff ϕ is true at every w-accessible world.
- $\Diamond\phi$ is true at w iff ϕ is true at some w-accessible world.

In the interests of deriving the well-foundedness of sets, Studd complicates things slightly by employing a "bi-modal" logic, in which $\Box$ is replaced by a pair of operators: the (strictly) forwards looking operator $\Box_>$ and the (strictly) backwards looking operator $\Box_<$. Similarly, $\Diamond$ is replaced by the (strictly) forwards looking operator $\Diamond_>$ and the (strictly) backwards looking operator $\Diamond_<$. These four operators can be semantically modeled analogously to $\Box$ and $\Diamond$ as follows:

- $\Box_>\phi$ is true at w iff ϕ is true at every w-accessible world (other than w).
- $\Box_<\phi$ is true at w iff ϕ is true at every world that accesses w (other than w).
- $\Diamond_>\phi$ is true at w iff ϕ is true at some w-accessible world (other than w).
- $\Diamond_<\phi$ is true at w iff ϕ is true at some world that accesses w (other than w).

Using these operators in a suitable S4 modal logic, it's possible to formulate the core thesis that any sets can form a set in several ways: (i) as a single formula in a second-order language (Parsons); (ii) schematically in a singular, first-order language (Parsons and Studd); (iii) as a single formula in a plural, first-order language (Linnebo). I will briefly discuss each in turn.

(i): Parsons (1983) formulates the core thesis as:

FC$^\Diamond$ $\Box(\forall F)\Diamond(\exists y)(\forall x)(x \in y \leftrightarrow Fx)$.

Intuitively, FC$^\Diamond$ says that (necessarily) any property F determines the possible existence of the set of all Fs. To avoid paradox, 'all Fs' must be read in such a way that it refers to all the Fs there actually are (at w), not to all the Fs that there would be (at w) if these Fs formed a set.[16] Parsons enforces such a reading by "fully rigidifying" properties relative to worlds.

F is *rigidified* relative to w if anything that is F at w is necessarily F and anything that is not F at w is necessarily not F. F is *fully rigidified* relative to w if in addition,

[16] Without this qualification, a version of Russell's paradox can be derived from FC$^\Diamond$. This can be most clearly seen by applying the standard possible worlds interpretation to the modal operators as follows. Applying the standard possible worlds interpretation to $\Box$, FC$^\Diamond$ says that at any accessible world it is the case that $(\forall F)\Diamond(\exists y)(\forall x)(x \in y \leftrightarrow Fx)$. Instantiate to an arbitrary accessible world w and the property *non-self-membered set* to get:

(i) At w: $\Diamond(\exists y)(\forall x)(x \in y \leftrightarrow x \notin x)$.

Applying the standard possible worlds interpretation to $\Diamond$, (1.1) says that at w there is an accessible world v at which the Russell set $R_v = \{x | x \notin x\}$ of all the non-self-membered sets in v exists. Because R_v is a set in v, it follows that:

(ii) At v: $R_v \in R_v \leftrightarrow R_v \notin R_v$.

there are no Fs that do not exist at w, i.e., the Fs at w are all the possible Fs.[17] When F is fully rigidified relative to w, something counts as an F at a w-accessible world v iff it is an F at w. FC$^{\Diamond}$ is consistent if F is (fully) rigidified relative to the world w to which $\Box$ is instantiated. The result is that the occurrence of 'F' in 'Fx' is read as falling outside the scope of $\Diamond$ so that instead of referring to the things that are F at the w-accessible world v to which $\Diamond$ is instantiated, it refers to the things that are F at w.

(ii): The idea that (necessarily) any formula ϕ determines the possible existence of the set of all ϕs is naturally formalized as (the necessitation of):

(1) $\Diamond(\exists y)(\forall x)(x \in y \leftrightarrow \phi(x))$.

If ϕ is rigid, or as Studd calls it "modally invariant" (INV[ϕ]), so that at any stage in the process of set formation, everything is necessarily ϕ or necessarily not ϕ, this idea may be strengthened to the idea that any formula ϕ determines the possible existence of the set of all the ϕs *there could possibly be*. This is naturally formalized as:

(2) $\Diamond(\exists y)\Box(\forall x)(x \in y \leftrightarrow \phi(x))$.

Unfortunately, both (1) and (2) are false. Under a substitution of '$x \notin x$' for ϕ, they each imply the possible existence of the Russell set R from which the contradiction $R \in R \wedge R \notin R$ is derivable. Additional contradictions arise under substitutions of the formulas that define the other contradiction-inducing sets.

In an effort to diagnose the problem, Studd notes that these contradiction-inducing formulas share a property, which is intuitively characterized by the observation that at every stage, ϕ defines new sets that do not exist at earlier stages. Following Michael Dummett, Studd dubs this property "indefinite extensibility" which he defines as follows:

- A formula $\phi(x)$ is extensible ($\mathrm{EXT}_x[\phi(x)]$) $=_{df.}$ $\phi(x)$ is satisfied by sets that are merely possible (intuitively, they will be formed only at a later stage)
- A formula $\phi(x)$ is indefinitely extensible ($\Box\mathrm{EXT}_x[\phi(x)]$) $=_{df.}$ $\phi(x)$ is necessarily extensible (intuitively, $\phi(x)$ is extensible at every stage).[18]

He proceeds to modify (2) by restricting ϕ to those formulas that are not only invariant but also not indefinitely extensible. He dubs the modification "Max" for the core thesis (which he calls "the maximality thesis") that any sets can form a set.

Max $\mathrm{INV}[\phi] \wedge \neg\Box\mathrm{EXT}_x[\phi(x)] \rightarrow \Diamond(\exists y)\Box(\forall x)(x \in y \leftrightarrow \phi(x))$.[19]

[17]See Parsons (pp. 288, f.n. 29; 301–302) for a definition of 'the (full) rigidification of F'.

[18]Dummett (1963, 1981, 1991) famously attributed the set-theoretic paradoxes to the indefinite extensibility of concepts: a concept C is indefinitely extensible if given any "definite totality" of Cs and the ability to quantify over all members of this totality, it is possible to introduce new Cs which must lie outside of it.

[19]In the context of a modal set theory, statements of the form '$x = y$' and '$x \in y$' are only true if both x and y actually exist. In order to avoid treating instances of such statements in the language of standard set theory as being contingent (true or false depending on the stage at which they are evaluated), Studd translates them into the language of modal set theory by adding a $\Diamond$ operator out front: $\Diamond(x = y)$ and $\Diamond(x \in y)$. Taking this into account, his official statement of Max, is: $\mathrm{INV}[\phi] \wedge \neg\Box\mathrm{EXT}_x[\phi(x)] \rightarrow \Diamond(\exists y)\Box(\forall x)(\Diamond(x \in y) \leftrightarrow \phi(x))$.

Intuitively, the restriction to formulas that are not indefinitely extensible means that any sets that might be determined by such formulas, such as (all) the non-self-membered sets, are excluded from consideration: how then can Studd maintain that Max captures the core thesis that *any* things can form a set? His answer (p. 699) is that the proper interpretation of 'any sets' in the context of the iterative conception of set is 'any sets formed at some stage in the iterative process'.

> No matter how far we've proceeded up the hierarchy, the sets formed so far are *all* the sets there are: (otherwise) unrestricted quantification over sets ranges only over the ones formed so far.

One might wonder whether, given these remarks, Studd's modal account is ultimately just a variant of the quantificational account of the paradoxes that has been so poorly received. Correctly answering this question requires viewing things from the right perspective. From a reductionist perspective, the modal operators are quantifiers over the stages of the set-theoretic hierarchy and Studd's "proper interpretation" amounts to a quantificational restriction under which 'any sets' is understood as ranging over only those sets that are formed at some stage. It is clear, however, that this is not the perspective Studd means to defend. He wants us to view quantification over sets from *within* the modal perspective, according to which the set-theoretic hierarchy is (in some sense) genuinely potential. From this perspective, it would seem that Studd's "proper interpretation" does not impose a quantificational restriction. The fact that 'any sets' cannot take certain pluralities as its value—such as *absolutely all the sets*, i.e., all the sets that will ever be formed—is simply due to the modal fact that for any indefinitely extensible formula ϕ, there is no point at which the plurality of all ϕs is actual.

(iii): Linnebo formulates the core thesis in a plural, first-order modal logic as:

$\mathbf{FPC}^{\Diamond}$ $\Box(\forall zz)\Diamond(\exists y)\Box(\forall x)(x \in y \leftrightarrow x \prec zz)$.

$\mathrm{FPC}^{\Diamond}$ is a modalized version of the inconsistent full plural comprehension principle:

FPC $(\forall zz)(\exists y)(\forall x)(x \in y \leftrightarrow x \prec zz)$

and a "pluralization" of $\mathrm{FC}^{\Diamond}$:

$\mathbf{FC}^{\Diamond}$ $\Box(\forall F)\Diamond(\exists y)(\forall x)(x \in y \leftrightarrow Fx)$.

Like $\mathrm{FC}^{\Diamond}$, $\mathrm{FPC}^{\Diamond}$ involves quantification into the scope of the modal operator $\Diamond$ (though in this case the quantification is plural, not second-order). To avoid paradox, the plural variables must be "fully rigidified" so that they refer to the same things both inside and outside the scope of this operator (like the property variables in $\mathrm{FC}^{\Diamond}$). When the zz are (fully) rigidified relative to the world w to which the outermost $\Box$ in $\mathrm{FPC}^{\Diamond}$ is instantiated, the result is that the occurrence of 'zz' in '$x \prec zz$' is read as falling outside the scope of $\Diamond$. This blocks the derivation of Russell's paradox when '$\forall zz$' is instantiated to the non-self-membered sets (at w). Instead of referring to the non-self-membered sets that exist at the world v to which $\Diamond$ is instantiated, the occurrence of 'zz' in '$x \prec zz$' now refers to the non-self-membered sets that exist at w.

As with Studd's restriction to formulas that are not indefinitely extensible, Linnebo's rigidification of the plural variables excludes any "contradiction-inducing pluralities" from consideration. Like Studd, Linnebo argues that this is justified by the potential character of the set-theoretic hierarchy. He acknowledges that this has the unintuitive result that, for example, we cannot refer plurally to absolutely all the sets; however, such limitations are not quantificational in nature; rather, they are necessary consequences of the modal fact that certain pluralities are never actual. Thus, Linnebo (2010, 159–160) writes that "you cannot lose something you never had." He goes on to explain:

> A plurality consists of a fixed range of objects, but the set-theoretic hierarchy is inherently potential and thus resists being summed up by a fixed range of objects.

3.1 Modal set theory

The core thesis, together with supplemental modal principles, provides the basis for a modalized version of iterative set theory (MST) in which all quantified formulas of ordinary iterative set theory are rewritten according to the translation scheme:

- $(\forall x)\phi(x) \longmapsto \Box(\forall x)\phi(x)$
- $(\exists x)\phi(x) \longmapsto \Diamond(\exists x)\phi(x)$.[20]

When ϕ is a predicate satisfied by every set in the language of standard set theory, the modal theorist will say that ϕ is necessarily satisfied by every set in the language of modal set theory, i.e., ϕ is satisfied by every set at every stage. Intuitively, this allows the modal theorist to make general claims about "all sets" without presupposing their actual existence.

Of particular interest are the modal translations of the iterative existence axioms:

Empty Set$^{\Diamond}$	$\Diamond(\exists y)\Box(\forall x)(x \notin y)$
Pairing$^{\Diamond}$	$\Box(\forall x)\Box(\forall z)\Diamond(\exists y)(y = \{x, z\})$
Union$^{\Diamond}$	$\Box(\forall x)\Diamond(\exists y)\Box(\forall z)(z \in y \leftrightarrow \Diamond\exists w(w \in x \wedge z \in w))$
Power Set$^{\Diamond}$	$\Box(\forall x)\Diamond(\exists y)\Box(\forall z)(z \in y \leftrightarrow z \subseteq x)$
Infinity$^{\Diamond}$	$\Diamond(\exists y)(\emptyset \in y \ \wedge \ \Box(\forall x)(x \in y \rightarrow x \cup \{x\} \in y))$
Separation$^{\Diamond}$	$\Box(\forall x)\Diamond(\exists y)\Box(\forall z)(z \in y \leftrightarrow z \in x \wedge \phi z)$
Replacement$^{\Diamond}$	$\Box(\forall x)[\Box(\forall z \in x)\Diamond(\exists! w)(\phi(z) = w) \ \rightarrow$
	$\Diamond(\exists y)\Box(\forall w)(w \in y \leftrightarrow \Diamond(\exists z \in x)(\phi(z) = w))]$

The project of deriving these axioms from the core thesis, supplemented with additional axioms that express the iterative conception of *set* in the modal context parallels the project of earlier iterative theorists (most famously Boolos (1998)) of deriving the existence

[20]Putnam (1979, 56–59) may have been the first to suggest this strategy. See also Hellman (1989, 65–79).

axioms of standard (non-modal) iterative set theory from axioms that express the iterative conception in a non-modal context. Following upon earlier work by Parsons, Linnebo and Studd execute this project with great technical skill.[21] However, the significance of their achievement rests in large part upon the prior question of the intelligibility of the modal notions in play. To this question, I now turn.

3.2 Understanding the modal operators

Insofar as we think of worlds as stages of set formation and of the modal operators as quantifiers over worlds, we may be inclined to define the latter in terms of quantification over the stages of set formation as follows:

D1. $\Box\phi$ is true at stage $s =_{df.}$ ϕ is true at s and at every later stage t.

D2. $\Diamond\phi$ is true at stage $s =_{df.}$ ϕ is true at some later stage t.

Clearly, these definitions are inadequate for the modal theorist, who hopes to use modality to provide a non-constructivist interpretation of the priority relation. This is impossible if the modal operators themselves are being defined in terms of the stages in a constructivist process. Of course, D1 and D2 do not offer an analysis of stages (constructivist or not) and so they are compatible with a non-constructivist analysis of stage theory. But, at best, what this shows is that D1 and D2 do not go far enough.

The modal theorist might replace quantification over stages in D1 and D2 with a tensed language and use this to define the modal operators as temporal operators as follows:

D3. $\Box\phi$ is true (now) $=_{df.}$ ϕ is true (now) and will remain true, no matter how many sets are formed.

[21] Linnebo adds four axioms: the first two are (the necessitation of) the ordinary axioms of Extensionality and Foundation:

(Ext) $\Box(\forall x)(\forall y)[(\forall z)(z \in x \leftrightarrow z \in y) \to x = y]$

(F) $\Box(\forall x)(x \neq \emptyset \to (\exists y)(y \in x \wedge y \cap x = \emptyset))$.

The second two exploit the full rigidity of plurals to express the full rigidity—or, as he calls it, "extensional definiteness"—of membership and subsethood:

ED-$\in$ $\Box(\forall x)(\exists yy)\Box(\forall z)(z \prec yy \leftrightarrow z \in x)$

ED-$\subseteq$ $\Box(\forall x)(\exists yy)\Box(\forall z)(z \prec yy \leftrightarrow z \subseteq x)$.

Intuitively (ED-$\in$) says that any set x has the same members at any world in which it exists and (ED-$\subseteq$) says that any set x has the same subsets at any world in which it exists. From these, together with FPC$^{\Diamond}$, Linnebo is able to derive all the axioms above except Infinity$^{\Diamond}$ and Replacement$^{\Diamond}$. To derive Infinity$^{\Diamond}$, he adds a limitation of size principle for pluralities. To derive Replacement$^{\Diamond}$, he adds a modal variant of the principle of Reflection, according to which any statement of set theory is reflected down onto initial segments of the set-theoretic hierarchy.

Studd adds three axioms: a modalization of the axiom of Extensionality (intuitively: sets are identical if they have the same members at all stages) and axioms of Priority (intuitively: any set formed at a stage has its members formed at earlier stages) and Plentitude (intuitively: any sets ever formed will form a set at every later stage). From these, together with Max, Studd is able to derive all the axioms above except Infinity$^{\Diamond}$ and Replacement$^{\Diamond}$. To derive these, he adds a modal variant of Reflection (which is stronger than Linnebo's).

D4. $\Diamond\phi$ is true (now) $=_{df.}$ ϕ will become true when enough sets have been formed.

But this does not help. Rather than replacing "the language of time and activity" with "the language of potentiality and actuality," D3 and D4 proceed in the opposite direction, *reducing* "the language of potentiality and actuality" to "the language of time and activity".

The modal theorist might seek to preserve whatever insight D1–D2, or D3–D4, provide, by "downgrading" them from official definitions to unofficial glosses. Both Linnebo and Studd are attracted by this idea. For example, while Linnebo (2010, 155) acknowledges that "officially" the modal operators are "new primitives governed only by a modal logic" he suggests that—"for heuristic purposes"—'$\Box\phi$' may be read as: "no matter what sets we go on to form it will remain the case that ϕ," and '$\Diamond\phi$' may be read as "it is possible to go on to form sets so as to make it the case that ϕ." Studd (2013, 701) suggests a similar unofficial reading of '$\Box_>\phi$' and '$\Box_<\phi$' as: "it will be the case at every later stage that ϕ" and "it was the case at every earlier stage that ϕ". (Under the obvious extension to $\Diamond_>$ and $\Diamond_<$, '$\Diamond_>\phi$' is unofficially read as 'it will be the case at some later stage that ϕ' and '$\Diamond_<\phi$' as 'it was the case at some earlier stage that ϕ'.)

Such paraphrases mislead in certain cases. Consider the (non-modal) statement 'no sets include every set', which may be expressed in a plural first-order language as:

(3) $(\forall xx)(\exists y)(y \not\prec xx)$.

Following Linnebo, let $\phi^\Diamond$ denote the result of applying the modal translation scheme to the non-modal formula ϕ of (plural) iterative set theory. The modal translation of (3) is:

$(3)^\Diamond$ $\Box(\forall xx)\Diamond(\exists y)(y \not\prec xx)$.

Applying Linnebo's paraphrase, this may be expressed as: 'no matter what sets we go on to form, it will remain the case that, for any sets xx, it is possible to go on to form a set y such that $y \not\prec xx$'. The original sentence (3) is false in standard iterative set theory, since the open formula '$(\exists y)(y \not\prec xx)$' is not satisfied when 'xx' is assigned to all the sets. But $(3)^\Diamond$ is true in the corresponding modal set theory, in which $\text{FPC}^\Diamond$ holds (and so, intuitively, it's always possible to form new sets).[22]

How serious of an objection is this? It might be noted that (3) involves plural quantification, and consequently, even if it provides grounds for questioning Linnebo's gloss on modal operators, which is intended as a bridge between non-modal and modal formulations of set theory *in a plural language*, it does not apply to Studd's gloss on the modal operators, which is limited to singular formulations of set theory. On the other hand, it is now well known that iterative set theory can be formalized in a first-order plural language in which the use of schematic variables is replaced by plural quantification. Doing so has a number of advantages and has been subsequently adopted by a number of philosophers

[22] Note that the same result holds if we replace $(3)^\Diamond$ with its interpretation under a standard semantic model. The interpretation of $(3)^\Diamond$ (at a world w_0) may be expressed as: 'for any (w_0-accessible) world w, for any xx in w, there is a (w-accessible) world v at which there exists a set y such that $y \not\prec xx$'. The interpretation of $(3)^\Diamond$ is true since every world (in the model) accesses worlds containing additional sets.

of set theory.[23] Consequently, it would be unacceptably arbitrary to adopt a modal interpretation of singular set theory and refuse to extend this to plural set theory. Studd cannot evade the objection by simply ignoring plural set theory.

Moreover, given the equi-interpretability of plural first-order logic and monadic second-order logic, it is easy to show that this problem is not unique to formulations of set theory in plural languages. The very same problem arises for analogous glosses of the modal operators in a singular, second-order language. Intuitively, the interpretation of plural logic into second-order logic consists in replacing plural quantification with quantification over properties and the relation $\prec$ with the relation of property instantiation.[24] Thus, the second-order translation of (3) is:

(3*) $(\forall F)(\exists y)(\neg Fy)$,

where 'y' ranges over sets and 'F' ranges over properties of sets. The modal translation of (3*) is:

$(3^*)^{\Diamond}$ $\Box(\forall F)\Diamond(\exists y)(\neg Fy)$.

Applying a paraphrase analogous to Linnebo's above, $(3^*)^{\Diamond}$ may be expressed as: 'no matter what sets we go on to form, it will remain the case that, for any (fully-rigidified) property F, it is possible to go on to form a set y such that $\neg Fy$'. Just as before, the non-modal sentence (3*) is false in standard iterative set theory, since the open formula '$(\exists y)(\neg Fy)$' is not satisfied when 'F' is assigned to the property *set*, which is had by every set. But its modal translation is true in the corresponding modal set theory, in which $\text{FC}^{\Diamond}$ holds (and so, intuitively, it's always possible to form new sets, whose properties, by definition, cannot be the same as the (fully-rigidified) properties of previous sets).

Instead of explicating the operators by means of an official definition or an unofficial paraphrase, we might attempt to describe the intended modality more directly. For starters, we should note that this modality cannot be metaphysical. It is a widely held belief that mathematical objects are either metaphysically necessary or metaphysically impossible, and consequently that there is no middle ground which would allow for any sets to be metaphysically possible, but *not* metaphysically actual. Without such a middle ground, $\text{FPC}^{\Diamond}$ leads to contradictions. To see this, it suffices to note that relative to any sets, xx, considered to be actual, $\text{FPC}^{\Diamond}$ asserts the possible existence of their universal set U_{xx}. If U_{xx}'s possible existence implies its actual existence, then when xx are all sets considered to be actual, $\text{FPC}^{\Diamond}$ entails the actual existence of U_{xx}, which in this context

[23]These include Lewis (1991), Uzquiano (2003), Burgess (2008), Linnebo (2010, 2013) and Oliver and Smiley (2013).

[24]There is one complication. Because empty properties exist, though no things instantiate them, we cannot always render existential plural quantification as quantification over properties. To illustrate: the existential plural statement 'there are some self-membered sets' is false, but the corresponding second-order statement 'there is a property of self-membership' is not. Boolos's (1998, 68) solution is to translate existential second-order statements as disjunctions, whose second disjunct handles cases in which the second-order variables are empty. Thus, 'there is a property of self-membership' is translated as: 'either there are some self-membered sets or every set is such that it is not self-identical iff it is a self-member'. The first disjunct is false, but the second is true. Thanks to an anonymous referee both for pointing out the relevance of equi-interpretability to my argument and for pushing me to address this complication.

is the (actual) universal set. The paradoxes of Cantor, Burali-Forti and Russell are derivable from U_{xx} in the usual way. To restore consistency, we must understand $\Diamond$ in a way in which sets are (merely) possible relative to their elements. Doing so requires a special non-metaphysical modality.

A number of proposals have been made. Studd (2013, 706) suggests (but does not endorse) a linguistic interpretation, according to which the possible existence of a set relative to its members is to be understood in terms of a possible expansion of the set-theoretic vocabulary.[25] (Note: this is distinct from the reductionist perspective on the modal operators, mentioned above, according to which they are reducible to quantification over stages.) Since, under the intended interpretation, the set-theoretic vocabulary can *always* be expanded to accommodate new sets, there can be no interpretation of 'set' that is absolutely unrestricted. Consequently, this linguistic interpretation amounts to a modal form of the quantificational solution to the paradoxes mentioned at the outset of this paper. Unfortunately, this makes it unhelpful in the present context, in which discussion of the iterative conception, and hence the priority relation, is driven in part by the goal of providing an absolutist reply to the set-theoretic paradoxes. Such a reply—it would seem—should provide an explanation for why the paradoxes are blocked that is *not* based on the premise that quantification over sets is restricted.

Linnebo (2010, 158) writes that he favors understanding the modalities in terms of a process of "individuating mathematical objects." He describes this process as follows:

> To individuate a mathematical object is to provide it with clear and determinate identity conditions. This is done in a stepwise manner, where at any stage we can make use of any objects already individuated in our attempts to individuate further objects.

I find this description to be rather puzzling. Is the individuation of mathematical objects something that *we* do? If so, then the modal account under Linnebo's interpretation would appear to devolve into a constructivist account, according to which sets are formed (or individuated) by some sort of mental activity. As a philosophical account of the priority relation (and hence of the existence conditions for sets), this is objectionable for the reasons given in section 1 above: as a description of actual, human mental activity, or of any natural mental activity (which must take place in time), it is too weak to generate all the sets that exist according to iterative set theory. However, any suitable idealization of mental activity is both underdeveloped and poorly motivated: the notion of an idealized ability to mentally collect elements is not well understood, and any determinate answers to what it is possible to mentally collect in an idealized sense appear to come from the wrong direction. For surely it is fidelity to the axioms of iterative set theory, not an independent grasp of some idealized mental process of set construction, that determines the right modal logic and the right axioms for modal set theory.[26]

If the individuation of mathematical objects is not something that we do, then how might it be understood? Simon Hewitt (2015, 325) writes that Linnebo's process of individuating mathematical objects should be understood in terms of possible extensions of a language.

[25] See also Williamson (1998).

[26] I am grateful to an anonymous referee for clarity regarding the nature of this objection.

> What is meant by 'individuation'? Just this: I can individuate an x such that $\phi(x)$ iff there is a possible extension of my language such that there is a singular term 'a' such that '$\phi(a)$' is true.

Whatever its merits, Hewitt's interpretation reverts to the linguistic model for modality suggested by Studd, and we've already seen the inadequacy of this in the present context.

In related work on the question of unrestricted quantification in mathematics, Fine (2006a,b) recommends a special "postulational" mathematical modality according to which the possible existence of a set relative to its members is to be understood in terms of a rule, or "procedural postulate", for introducing it. An instance of such a rule is the procedural postulate

(4) Introduce the set of all sets (in the domain D).

Fine shows how (4) forces the expansion of D by introducing a universal set that (on pain of paradox) falls outside D. In Fine's terms, this universal set is a (mere) postulational possibility relative to D.

In his (2013), Linnebo also advocates for a special mathematical modality. He likens this to the modality employed in the historical distinction between the notions of potential and actual infinity. He goes on to remark (207–208):

> This modality is tied to a process of building up larger and larger domains of mathematical objects. A claim is possible in this sense, if it can be made to hold by a permissible extension of the mathematical ontology; and it is necessary if it holds under any permissible such extension.

(Notice that here he does not speak about individuating objects but of extending mathematical ontology.)

For both Fine and Linnebo, the intelligibility of mathematical modality appears to be tied to the ability to understand a particular process (for Fine, this is the process by which new sets are postulated; for Linnebo it is the process by which mathematical domains are extended). Taken literally, these views are constructivist. However, any constructivist process effects a change in the world and Fine, at least, denies that postulation has this consequence. By postulating objects, we do not change the world; rather we change which world is under consideration. Fine (2006b, 40) explains:

> On the present view, there is no such thing as *the* ontology, one that is privileged as genuinely being the sum-total of what there is. There are merely many different ontologies, all of which have the same right (or perhaps we should say no right) to be regarded as the sum-total of what there is.

This explanation is troubling insofar as it seems to involve a form of ontological relativism, according to which there are multiple (mathematical) ontologies and no principled way to privilege one over the others. In response, it might be argued that this is precisely how we should understand the modal account; that part of what it is for the set-theoretic hierarchy to be potential is for there to be no fact of the matter as to just how many sets are actual. Even if this is right, we are left with an interpretive dilemma: on the one hand, postulation is not to be understood in a constructivist sense, as effecting a genuine change in the world; on the other hand, it cannot be described in more familiar objectivist terms, such as ontological dependence or metaphysical possibility. Similar remarks apply to Linnebo's notion of the extension of mathematical domains. If this is not a constructivist process and it is not to be understood in terms of ontological dependence or metaphysical possibility, then how is it to be understood? Perhaps there is a way that I have not considered; but I cannot think of what this might be. I conclude that none of the attempts by Studd, Linnebo and Fine to elucidate the modal operators succeeds.

4 Conclusion

The iterative conception purports to provide an account of *set* that explains why none of the contradiction-inducing sets exists. This explanation depends on a relation of priority and I have argued that there is no satisfactory account of this relation that is consistent with a commitment to unrestricted quantification over all sets. Of course, iterative set theory is not the only candidate theory that purports to quantify without restrictions. Quine's NF is one alternative. Paraconsistent versions of naive set theory are another.[27] If we are to reject iterative set theory for the reasons I have provided, why not adopt one of these alternatives instead of a restrictivist set theory?[28]

Simply put, my answer is that each of these alternatives fails to articulate a *natural* conception of *set*. This answer invites further questions: What makes a conception of *set* natural? Why is naturalness required? Here, I can offer only a very brief sketch of how these should be answered. First, a view of sets is natural only if it can be motivated independently of the set-theoretic paradoxes. Second, naturalness is required for the sort of principled explanation of the consistency of set theory that we seek. In most cases, providing such an explanation amounts to answering the question: why don't the contradiction-inducing sets exist? In the case of paraconsistent set theories, it amounts to answering a different question: why doesn't the existence of the contradiction-inducing sets trivialize set theory?

If we accept this notion of naturalness, we can see why the alternatives mentioned above may not be viable. The syntactic restrictions on set existence employed by NF appear unnatural because they appear to be motivated only by the need to block the paradoxes. The resulting dissatisfaction with Quine's theory was eloquently stated by Russell (1959, 80):

> Professor Quine ... has produced systems which I admire greatly on account of their skill, but which I cannot feel to be satisfactory because they seem to be created *ad hoc* and not to be such as even the cleverest logician would have thought of if he had not known of the contradictions.

[27]Naive set theories include (some version of) the full comprehension scheme $(\exists y)(\forall x)(x \in y \leftrightarrow \phi x)$ *without* quantificational restrictions. A theory is paraconsistent if its logical consequence relation is not explosive. The argument for adopting a paraconsistent version of naive set theory is articulated in Priest (2002). More recently, paraconsistent set theories have been developed and defended by Weber (2010, 2012) and Verdee (2013).

[28]Thanks to an anonymous referee for raising this question.

Russell's critique is echoed by Michael Potter (1993, 178), who describes Quine's project as "essentially negative in character," a "one-step-back-from-disaster view." A similar feeling of dissatisfaction arises with regard to paraconsistent theories, insofar as the departures from classical logic appear to be motivated solely by the need to prevent the existence of the contradiction-inducing sets and the derivations of the resulting contradictions from trivializing set theory.

By contrast, I believe that there are independent, linguistic grounds for believing that quantification may be necessarily restricted.[29] If I am right, then it is reasonable to think of the restrictivist solution to the paradoxes as one which could been conceived without prior knowledge of them. This in turn suggests that quantificational restrictivism provides a solution to the paradoxes that is natural in a way that other alternatives to the iterative conception of *set* are not. I conclude that the time has come to take a second look at the quantificational account of the paradoxes.

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[29] See for example Glanzberg (2006).

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