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Extensions of Priest-da Costa Logic

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Abstract

In this paper, we look at applying the techniques from analyzing superintuitionistic logics to extensions of the cointuitionistic Priest-da Costa logic daC (introduced by Graham Priest as "da Costa logic"). The relationship between the superintuitionistic axioms- definable in daC- and extensions of Priest-da Costa logic (sdc-logics) is analyzed and applied to exploring the gap between the maximal si-logic SmL and classical logic in the class of sdc-logics. A sequence of strengthenings of Priest-da Costa logic is examined and employed to pinpoint the maximal non-classical extension of both daC and Heyting-Brouwer logic HB. Finally, the relationship between daC and Logics of Formal Inconsistency is examined.

1 Introduction and Semantics for Priest-da Costa Logic

In [12], Rauszer introduced an extension of intuitionistic logic (Int) called *Heyting-Brouwer logic* (HB), in which the language of Int was extended by connectives for *dual implication* (coimplication or preclusion in some quarters, symbolized by "$\leftarrow$") and *dual negation* (symbolized by "$\neg\!\!\!\!\!\!\;\!\!\!\!\!\urcorner$"). Kripke semantics for HB were offered by Rauszer in [13]. The Gödel-McKinsey Tarski embedding of intuitionistic logic Int into S4 demonstrated in [7] was extended by Łukowski in [6] to one from HB to $\mathsf{S4}^t$, the temporal analogue of S4. (If one is approaching this work from the standpoint of modal logic rather than intuitionistic logic, the sequel can be easily interpreted in the lights of this embedding.)

Fragments of HB are interesting in their own right. The $\{\wedge, \vee, \leftarrow, \neg\!\!\!\!\!\!\;\!\!\!\!\!\urcorner\}$-fragment of HB has been, *e.g.*, dubbed "anti-intuitionistic" logic and explored by Carnielli and Brunner in [2]. It is our aim here to explore extensions of the $\{\wedge, \vee, \rightarrow, \neg\!\!\!\rightarrow\}$-fragment of HB. In Priest [10] this fragment was called *da Costa logic* (daC), a name motivated by the observation that the da Costa system C_ω is contained in this fragment; in order to avoid confusion with systems falling under this title in the literature we will employ the term "Priest-da Costa logic" to denote the system described in that paper.[1] Good overviews of the philosophical motivations underlying the cointuitionistic negation of Priest-da Costa logic can be found in [10] and Wansing's [14]; the first section of the latter may be especially recommended to logicians sympathetic to the constructive bent yet not entirely sold on the usefulness of paraconsistency.

We may now rehearse the semantics for both Int and daC. A *Kripke frame* $\mathfrak{F} = (W, R)$ is a set of points W and a binary relation $R \subseteq W \times W$ such that R is reflexive, transitive, and antisymmetric. A *Kripke model* is a frame together with a valuation V mapping propositional parameters $A_0, A_1, ...$ to subsets of points such that for any parameter A, if $w \in V(A)$ and wRw', then $w' \in V(A)$. It is immediate that this *heredity constraint* yields the property that if $w \notin V(A)$ and $w'Rw$, then $w' \notin V(A)$. It should be noted that we drop the common assumption from the intuitionistic case that $\mathfrak{F}$ is a *tree*; as we shall see, the class of all frames $\mathfrak{F}$ such that R forms a tree on W corresponds to a stronger logic than daC, one we shall soon call PH_1.

The model determines a forcing relation, which we properly define as a relation holding between a point in a model and a formula, *e.g.*, $\mathfrak{F}, V, u \Vdash_\lambda A$, though when the model is clear from context, this will be abbreviated to $u \Vdash_\lambda A$. The subscript under the turnstile indicates the method of evaluating the relation rather than the logic employed; as the languages of Int, daC, and HB are distinct, some of the below conditions will not apply in some contexts. The forcing conditions where λ is Int, daC, or HB are defined recursively as follows:

- $u \Vdash_\lambda A$ *iff* $u \in V(A)$ for parameters A
- $u \Vdash_\lambda A \wedge B$ *iff* $u \Vdash_\lambda A$ and $u \Vdash_\lambda B$
- $u \Vdash_\lambda A \vee B$ *iff* $u \Vdash_\lambda A$ or $u \Vdash_\lambda B$

[1] A referee noted that the name "da Costa logic" has been employed to denote the logical systems of da Costa himself, in some cases preceding Priest's introduction of the present system by nearly three decades. I do not wish to introduce or reinforce confusion between these systems. I believe that the terminology employed in this paper ought to both be readily identifiable to readers of Priest's papers and yet distinct from the unqualified term "da Costa logic."

- $u \Vdash_\lambda A \to B$ *iff* at all v with uRv, if $v \Vdash_\lambda A$ then $v \Vdash_\lambda B$

These clauses are common for formulae in the languages of Int and daC. Two clauses for negation must be offered. One is for intuitionistic negation in the languages of Int and HB:

- $u \Vdash_\lambda \sim A$ *iff* for all v such that uRv, $v \nVdash_\lambda A$

The other is for cointuitionistic negation in the languages of daC and HB:

- $u \Vdash_\lambda \neg A$ *iff* for some v such that vRu, $v \nVdash_\lambda A$

It can be confirmed without difficulty that the heredity constraint extends to all formulae in the respective languages. This observation allows us to make a note concerning the inclusion of antisymmetry as a frame condition. While Priest's [10] does not include the antisymmetry constraint, we may justify inclusion of antisymmetry as a property of R.

First, we recall the canonical model for daC found in [10]. A set of formulae Δ is said to be *prime* if for any formula $A \vee B \in \Delta$, either $A \in \Delta$ or $B \in \Delta$ and is said to be *deductively closed* if for any formula A such that $\Delta \vdash_{\mathsf{daC}} A$, $A \in \Delta$. A set Δ is said to be *prime deductively closed* (pdc) if it has both properties. The canonical model for daC is defined as (W^C, R^C, V^C) where:

- $W^C = \{\Gamma : \Gamma \text{ is nontrivial and pdc}\}$
- $R^C = \{(\Gamma, \Delta) \in W^C \times W^C : \Gamma \subseteq \Delta\}$
- $V^C(A) = \{\Gamma \in W^C : A \in \Gamma\}$

Now we may justify the inclusion of antisymmetry.

Lemma 1. *The proof theories presented in [10] are sound and complete with respect to the class of reflexive, transitive, and antisymmetric frames.*

Proof. For soundness, as daC is sound with respect to reflexive and transitive frames *simpliciter*, if A is a theorem, then it is valid in all models on such frames. But the class of reflexive, transitive, and antisymmetric frames is a *subclass* of this class; *a fortiori*, A is valid in all such models.

For completeness, we appeal to the canonical model as described in [10]. As the accessibility relation R^C is defined so that $\Gamma R^C \Delta$ *iff* $\Gamma \subseteq \Delta$, antisymmetry may be immediately read off of R^C. As the canonical model is itself antisymmetric, we infer completeness. □

We make one more observation about the canonical model construction found in [10] that will be useful shortly.

Lemma 2. *For any family X of pdc sets of formulae in the language of* daC*, $\cap X$ is deductively closed.*

Proof. Suppose that $\cap X \vdash_{\mathsf{daC}} A$ for a formula A. Then for an arbitrary $\Delta \in X$, $\cap X \subseteq \Delta$ and by monotonicity, $\Delta \vdash_{\mathsf{daC}} A$. By deductive closure of Δ, $A \in \Delta$. As Δ was selected arbitrarily, this holds for any member of X; hence, $A \in \cap X$. □

We will say that a frame $\mathfrak{F}$ forces a formula A ($\mathfrak{F} \Vdash_{\mathsf{daC}} A$) if for every valuation on $\mathfrak{F}$ and every point $u \in W$, $\mathfrak{F}, V, u \Vdash_{\mathsf{daC}} A$. There at some points will be particular subsets of W with which we will concern ourselves; on a frame $\mathfrak{F} = (W, R)$ and a point $w \in W$, we let $w \uparrow = \{y \in W : wRy\}$ and $w \downarrow = \{y \in W : yRw\}$. These are the forwards and backwards "cones" of w, respectively.

We call the collections of extensions of daC and Int by the names $Ext\mathsf{daC}$ and $Ext\mathsf{Int}$, respectively, and likewise refer to a logic λ in one of these collections either a *sdc-logic* (super Priest-da Costa) or *si-logic* (superintuitionistic) following Chagrov and Zakharyaschev [3]. We moreover call a member of one of these classes a *proper* sdc-logic (*proper* si-logic) if $\mathsf{daC} \subsetneq \lambda \subsetneq \mathsf{CL}$ ($\mathsf{Int} \subsetneq \lambda \subsetneq \mathsf{CL}$).

A feature of Priest-da Costa logic that is convenient to our purposes is that intuitionistic negation is definable in daC.

Lemma 3. *Fix a formula A in the language of* daC *and define $\bot$ as $\neg\!\!\!\!\!\!\rightharpoondown (A \vee \neg\!\!\!\!\!\!\rightharpoondown A)$. Then for every $\lambda \in Ext\mathsf{daC}$, intuitionistic negation becomes definable as $\dot{\sim} B =_{df} B \to \bot$.*

Proof. We can quickly note that $\neg\!\!\!\!\!\!\rightharpoondown (A \vee \neg\!\!\!\!\!\!\rightharpoondown A)$ is unsatisfiable in any extension of daC and can thus treat some instance of this formula as a *de facto falsum* constant $\bot$. With a *falsum* constant, it is well-known that intuitionistic negation is definable in the above, prescribed manner. □

In the sequel, in order to keep clear that intuitionistic negation is not in the language but a defined notion, we use the symbol "$\dot{\sim}$." In general, when dealing with a superintutionistic axiom A, we will refer to the formulae in the language of daC by substituting $\dot{\sim}$ or $\neg\!\!\!\!\!\!\rightharpoondown$ for every instance of $\sim$ in A by $A^{\dot{\sim}}$ and $A^{\neg\!\!\!\!\!\!\rightharpoondown}$, respectively. A further observation concerning the interaction between intuitionistic negation and cointuitionistic negation is that, as Priest remarks in

[10], defined intuitionistic negation is stronger than cointuitionistic negation, that is, $\dot{\sim} A \rightarrow \neg A$ is a theorem of daC.

This motivates us to offer an additional lemma:

Lemma 4. *For any frame $\mathfrak{F}$, $\mathfrak{F} \Vdash_{\mathsf{Int}} A$ iff $\mathfrak{F} \Vdash_{\mathsf{daC}} A^{\dot{\sim}}$*

Proof. Right-to-left is trivial, so we will focus on the left-to-right half. Suppose that $\mathfrak{F} \nVdash_{\mathsf{daC}} A^{\dot{\sim}}$ and let V be the valuation on $\mathfrak{F}$ that fails to verify an instance A' of $A^{\dot{\sim}}$. Let $\{B_i : i \in I\}$ be an enumeration of the propositional parameters occurring in A; then for some $\{C_i : i \in I\}$ in the language of daC, $A' = A^{\dot{\sim}}[B_i/C_i]_{i \in I}$, that is, the substitution instance of $A^{\dot{\sim}}$ gotten by replacing every B_i with C_i, fails at some point u.

We can give a valuation V' on $\mathfrak{F}$ showing that A fails intuitionistically on $\mathfrak{F}, V'$. Take propositional parameters $\{D_i : i \in I\}$ and let $V'(D_i) = \{w \in W : w \Vdash_{\mathsf{daC}} C_i\}$. Since all C_i obey the heredity constraint, we can be assured that V' is a proper valuation. Moreover, this entails that $\mathfrak{F}, V, u \Vdash_{\mathsf{daC}} C_i$ *iff* $\mathfrak{F}, V', u \Vdash_{\mathsf{Int}} D_i$. As $A[B_i/D_i]_{i \in I}$ contains no instances of "$\neg$" except for instances in the *de facto falsum* constant, the reasoning that leads to the conclusion that $\mathfrak{F}, V \nVdash_{\mathsf{daC}} A[B_i/C_i]_{i \in I}$ will only appeal to the intuitionistically acceptable truth conditions and this will ensure that $\mathfrak{F}, V' \nVdash_{\mathsf{Int}} A[B_i/D_i]_{i \in I}$. Hence, we may conclude that $\mathfrak{F} \nVdash_{\mathsf{Int}} A$. □

We can use this lemma to analyze the strength of the traditional superintuitionistic axioms in this context by replacing every instance of $\sim$ with the above defined $\dot{\sim}$. For a set of formulae Γ, $\Gamma^{\dot{\sim}}$ is the set $\{A^{\dot{\sim}} : A \in \Gamma\}$. Finally, for a logic $\lambda = \mathsf{Int} + \Gamma$, where the operation $+$ denotes closure under *modus ponens* and uniform substitution, the defined analogue of λ will be called $\lambda^{\dot{\sim}} = \mathsf{daC} + \Gamma^{\dot{\sim}}$. We will retain this notation even if there are no instances of negation, *e.g.*, in the case of $\mathsf{LC} = \mathsf{Int} + (A \rightarrow B) \vee (B \rightarrow A)$, $\mathsf{LC}^{\dot{\sim}}$ will be defined as $\mathsf{daC} + (A \rightarrow B) \vee (B \rightarrow A)$ despite the additional axiom's being negation-free.

2 "Pigeonhole" Axioms in *Ext*daC

We here discuss a sequence of extensions of daC generated by the addition of salient axioms and demonstrate their correspondence to classes of frames. We will also show that the addition of any of several axioms to daC is sufficient to generate CL.

Theorem 1. *Call the formula* $(\neg A \wedge \neg\neg A) \to B$ weak ex contradictione quodlibet *(***WECQ***) and call the frame condition*

if tRv *and* uRv*, then there exists an* w *such that* wRt *and* wRu

"backwards convergence." Then $\mathsf{WECQ} = \mathsf{daC} + \mathbf{WECQ}$ *is the sdc-logic of backwards convergent frames.*

Proof. To show soundness, we must show that a frame forces **WECQ** *iff* it is backwards convergent.

· For right-to-left, suppose that a frame $\mathfrak{F}$ is backwards convergent. Consider an arbitrary formula A and point $w \in W$. First, we show that $w \nVdash_{\mathsf{daC}} \neg A \wedge \neg\neg A$. Suppose for contradiction that this formula is verified by w. Both $\neg A$ and $\neg\neg A$ are then verified by w and there hence exist points u, v with uRw and vRw such that $u \nVdash_{\mathsf{daC}} A$ and $v \nVdash_{\mathsf{daC}} \neg A$. Since $\mathfrak{F}$ is backwards convergent, we infer that there exists a point t such that tRu and tRv. As $v \nVdash_{\mathsf{daC}} \neg A$, we may reason that $t \Vdash_{\mathsf{daC}} A$. By heredity, that tRu implies that $u \Vdash_{\mathsf{daC}} A$ as well, contradicting the observation that $u \nVdash_{\mathsf{daC}} A$. Hence, for no point w and formula A is $\neg A \wedge \neg\neg A$ verified at w.

Looking to a further, arbitrary point w', it follows from this that $\neg A \wedge \neg\neg A \to B$ will be vacuously verified by w'. By the preceding observation, the antecedent will be verified nowhere in $w' \uparrow$, implying that for an arbitrary B, $w' \Vdash_{\mathsf{daC}} \neg A \wedge \neg\neg A \to B$. As the formulae A and B were selected arbitrarily, this will hold for any substitution instance of **WECQ**.

· For left-to-right, we prove the contrapositive. Suppose that $\mathfrak{F}$ is not backwards convergent. Then there exist points tRv and uRv such that for no w does wRt and wRu. By this, we can infer that $(t \downarrow) \cap (u \downarrow) = \varnothing$ and hence that $t \neq u$. We provide a valuation V on $\mathfrak{F}$ such that $\mathfrak{F}, V \nVdash_{\mathsf{daC}}$ **WECQ**.

Let $V(A) = (t \uparrow) \cup (t \downarrow)$ and $V(B) = \varnothing$. Then $u \nVdash_{\mathsf{daC}} A$ and so $v \Vdash_{\mathsf{daC}} \neg A$. Moreover, at every w such that wRt, $w \Vdash_{\mathsf{daC}} A$, and so $t \nVdash_{\mathsf{daC}} \neg A$, entailing that $v \Vdash_{\mathsf{daC}} \neg\neg A$. Hence, $v \Vdash_{\mathsf{daC}} \neg A \wedge \neg\neg A$, though $v \nVdash_{\mathsf{daC}} B$. By reflexivity, v itself witnesses the failure of $(\neg A \wedge \neg\neg A) \to B$. Hence, $\mathfrak{F} \nVdash$ **WECQ**.

For completeness, we employ the canonical model method as outlined in [10]. Let $\overline{\mathbf{WECQ}}$ be the set of all substitution instances of **WECQ**. The canonical model of WECQ comprises as points the nontrivial pdc sets of daC extending

$\overline{\textbf{WECQ}}$. Let the canonical model for WECQ be defined as before with the exception that :

- $W^C = \{\Gamma \supseteq \overline{\textbf{WECQ}} : \Gamma \text{ is nontrivial and pdc}\}$

Given pdc sets $\Delta, \Theta, \Gamma \in W^C$ such that $\Delta \subseteq \Gamma$ and $\Theta \subseteq \Gamma$, we are able to show the existence of a pdc set Λ such that $\Lambda \subseteq \Delta$ and $\Lambda \subseteq \Theta$.

Define Δ^* as $\cap\{\Delta' \in W^C : \Delta' \subseteq \Delta\}$ and consider the property that

$$\text{for all pdc sets } \Delta' \subseteq \Delta \text{ and } \Theta' \subseteq \Theta,\ \Delta'^* = \Theta'^*.$$

We first show that Δ and Θ have the property.

Suppose not; then there are pdc sets $\Delta' \subseteq \Delta$ and $\Theta' \subseteq \Theta$ such that either $\Delta'^* \setminus \Theta'^* \neq \varnothing$ or $\Theta'^* \setminus \Delta'^* \neq \varnothing$. Without loss of generality, assume the latter and let A be a formula found in Θ'^* but not in Δ'^*.

That $A \in \Theta'^*$ implies that A is a member of every pdc subset of Θ' and hence that $\neg\neg A \in \Theta'$. From $A \notin \Delta'^*$, we may infer the existence of a pdc set $\Delta'' \subseteq \Delta'$ such that $A \notin \Delta''$, whence $\neg A \in \Delta'$.

By transitivity, $\Delta' \subseteq \Gamma$ and $\Theta' \subseteq \Gamma$. Hence, $\neg A \in \Gamma$ and $\neg\neg A \in \Gamma$ and by closure under conjunction, $\neg A \wedge \neg\neg A \in \Gamma$. But as all formulae of the form $(\neg A \wedge \neg\neg A) \to B$ are members of Γ, closure under *modus ponens* means that Γ is trivial, contradicting the assumption that Γ is nontrivial.

So Δ and Θ have the above property. We make one further observation about the canonical model of WECQ before proceeding.

Suppose that Δ and Θ are pdc sets with the above property. Suppose also that $\Delta \cap \Theta$ is not prime with respect to a formula $A \vee B$; then there are $\Delta' \subseteq \Delta$ and $\Theta' \subseteq \Theta$ such that $A \vee B \notin \Delta'$ and $A \vee B \notin \Theta'$.

Since $A \vee B \in \Delta \cap \Theta$, the disjunction is a member of both Δ and Θ. Hence, Δ and Θ must each count as members at least one of the disjuncts (as they are each prime) and each disjunct can be a member of precisely one of these sets (otherwise it would be a member of $\Delta \cap \Theta$). This implies that either $A \in \Delta$ and $B \in \Theta$ or $B \in \Delta$ and $A \in \Theta$. Without loss of generality, suppose the former holds.

That $A \notin \Theta$ implies that $A \notin \Theta^*$; as $\Delta^* = \Theta^*$, there is a pdc $\Delta' \subset \Delta$ for which $A \notin \Delta'$. That $\Delta' \subset \Delta$ precludes B from being a member and by primeness of Δ', $A \vee B \notin \Delta'$ as well. Similar reasoning implies that we can find a pdc $\Theta' \subseteq \Theta$ such that neither A, B, or $A \vee B$ are members of Θ'. In the sequel, when using this procedure to choose such sets, we will call the chosen sets $\Delta^{\overline{A\vee B}}$ and $\Theta^{\overline{A\vee B}}$.

We can proceed to construct a pdc Λ. Enumerate the disjunctions of $\mathscr{L}_{\mathsf{daC}}$, denoting the nth formula in the enumeration by $[A \vee B]_n$ and recursively define the following sets:

- $\Delta_0 = \Delta$
- $\Delta_{\alpha+n+1} = \begin{cases} \Delta_{\alpha+n}^{\overline{[A\vee B]_n}} & \text{if } [A \vee B]_n \in \Delta_{\alpha+n} \cap \Theta_{\alpha+n} \text{ but} \\ & A, B \notin \Delta_{\alpha+n} \cap \Theta_{\alpha+n} \\ \Delta_{\alpha+n} & \text{otherwise} \end{cases}$
- $\Delta_\alpha = \cap_{\beta<\alpha}\Delta_\beta$, for limits α

We define the sets Θ_α in an analogous fashion. It ought to be shown that for a limit δ the set Δ_δ is pdc as intersections between pdc sets don't necessarily preserve primeness. Let δ be the least ordinal indexing a non-prime Δ_δ and let $A \vee B$ be the formula witnessing the failure of primeness. Then for $\alpha, \beta < \delta$, $A \notin \Delta_\alpha$ and $B \notin \Delta_\beta$; without loss of generality suppose that $\alpha < \beta$. But as defined, the sets constructed are linearly ordered under superset, so $\Delta_\alpha \supseteq \Delta_\beta$, whence neither A nor B is a member of Δ_β, entailing that Δ_β is not prime, contradicting the assumption that δ is the least index of a non-prime set.

Now, recursively define sets Λ_α as:

- $\Lambda_0 = \Delta_0 \cap \Theta_0$
- $\Lambda_{\alpha+n+1} = \Delta_{\alpha+n+1} \cap \Theta_{\alpha+n+1}$
- $\Lambda_\alpha = \cap_{\beta<\alpha}\Lambda_\beta$ for limits α

Then Λ_{ω^2} is a pdc subset of both Δ and Θ.

Lemma 2 entails that Λ_{ω^2}, as an intersection of pdc sets, is deductively closed.

For primeness, suppose that for some n, $[A \vee B]_n \in \Lambda_{\omega^2}$ but neither A nor B are members. Let $\alpha < \omega^2$ be the least ordinal at which $A \notin \Lambda_\alpha$, $\beta < \omega^2$ be the least ordinal at which $B \notin \Lambda_\beta$, and $\delta < \omega^2$ the first limit ordinal containing both α and β. That $[A \vee B]_n \in \Lambda_{\omega^2}$ implies that $[A \vee B]_n \in \Lambda_{\delta+n}$, but that $A \notin \Lambda_{\delta+n}$ and $B \notin \Lambda_{\delta+n}$ entails that $[A \vee B]_n \notin \Lambda_{\delta+n+1}$, contradicting that $A \vee B \in \Lambda_{\omega^2}$.

As Δ, Θ, and Γ were selected arbitrarily, we conclude that such a Λ exists for all such pdc sets, entailing that the frame of the canonical model is backwards convergent. $\square$

For every $n \in \omega$ such that $n \geq 1$ let $\mathbf{PH}_n$ stand in for the formula

$$[\bigwedge_{0\leq i\leq n} \rightharpoondown A_i] \rightarrow [\bigvee_{0\leq i<j\leq n} \rightharpoondown (A_i \vee A_j)]$$

and let the logic PH_n be $\mathsf{daC} + \mathbf{PH}_n$.

We will make a few observations about this axiom. For one, this is a "pigeonhole" axiom; $\mathbf{PH}_n$ "says" that for $m \geq n$, if $\rightharpoondown A_0, ..., \rightharpoondown A_m$ are all verified, then some $\rightharpoondown (A_i \vee A_j)$ must likewise be verified, that is, some point responsible for verifying one of the negated formulae must be responsible for verifying the negations of more than one of them. If we consider backwards closed subsets as possible "negation sinks," $\mathbf{PH}_n$ then is the statement that there exist n many such pigeonholes. Furthermore, we note that one direction of one of the De Morgan's laws, $(\rightharpoondown A_0 \wedge \rightharpoondown A_1) \rightarrow \rightharpoondown (A_0 \vee A_1)$, while failing in general for daC, is just $\mathbf{PH}_1$. We may think, then, of the approach of n to 1 as a "tightening down" towards the restoration of full De Morgan's laws.

In order to correctly characterize the frame conditions corresponding to the logics PH_n, we first extend the terminology of Chagrov and Zakharyaschev [3] by generalizing the notion of a "root." In Chagrov and Zakharyaschev's terminology, a point w is a root of a frame $\mathfrak{F}$ if w generates $\mathfrak{F}$. Let an *aerial root* (a.r.) in our context be a set of points $X \subseteq W$ such that $X\downarrow = X$. An a.r. of a point w will be an a.r. X such that $w \in X\uparrow$. Finally, a *root system* for a point w will be a collection of pairwise disjoint a.r.s of w $X_0, ..., X_{n-1}$ such that $w\downarrow \subseteq (\cup_{i=0}^{n-1} X_i)\uparrow$. Under our definition, every point has at least one aerial root, *i.e.*, the set W itself; moreover, the Chagrov and Zakharyaschev definition of a frame's being "rooted" is equivalent to the constraint that the cardinality of every root system for any point is 1.

Lemma 5. *(Aerial Root Splitting) If an aerial root $X \subseteq W$ contains two points w_0, w_1 such that $(w_0\downarrow) \cap (w_1\downarrow) = \varnothing$, then X can be split into two pairwise disjoint aerial roots X_0, X_1 such that $(X_0 \cup X_1)\uparrow = X\uparrow$.*

Proof. Suppose that the antecedent holds. Let $X_0 = w_0\downarrow$ and let $X_1 = X \setminus ((w_0\downarrow) \cup (w_0\uparrow))$. Then X_0, X_1 meet the criteria of the consequent. X_0 and X_1 are, by construction, pairwise disjoint, so we must check that the latter criterion is met. Since $(X_0 \cup X_1) \subseteq X$, that $(X_0 \cup X_1) \uparrow \subseteq X \uparrow$ is immediate. To show that $X \uparrow \subseteq (X_0 \cup X_1) \uparrow$ Suppose that a point $w \in X \uparrow$ and consider a $w' \in X$ such that $w'Rw$. Either $w_0 R w'$ or not. If so, then $w' \in X_0 \uparrow$ and by transitivity, $w' \in X_0 \uparrow$. If not, then there is a point $w'' \in X \smallsetminus w_0 \uparrow$ such that $w''Rw'$. As $w_0 R w'$ does not hold, $w'' \notin w_0 \downarrow$ and $w'' \notin w_0 \uparrow$; but this just means that $w'' \in X_1$, and hence, $w' \in X_1 \uparrow$. Either way, $x \in (X_0 \cup X_1) \uparrow$, so $X \uparrow \subseteq (X_0 \cup X_1) \uparrow$. This shows that $X \uparrow = (X_0 \cup X_1) \uparrow$ □

Theorem 2. *Call the frame condition that*

for all $w \in W$, w has no root system of cardinality greater than n

"n-rootedness." Then $\mathsf{PH}_n = \mathsf{daC} + \mathbf{PH}_n$ is the logic of n-rooted frames.

Proof. We must show that a frame $\mathfrak{F}$ forces $\mathbf{PH}_n$ *iff* for every point w, w has no root system of cardinality greater than n.

· For left-to-right, we prove the contrapositive. Suppose that $\mathfrak{F}$ does not have the property. Then there exists a point w with a root system of cardinality n, *i.e.*, there exist disjoint a.r.s $X_0, ..., X_{n-1}$ such that $w \uparrow \subseteq (\cup_{i<n} X_i) \uparrow$. We provide a valuation that fails to verify $\mathbf{PH}_n$.

Let $V(A_i) = W \smallsetminus X_i$ for $i < n$. Then for each A_i there exists a $u_i \in X_i$ such that $u_i R w$ and $u_i \nVdash_{\mathsf{daC}} A_i$. Hence, $w \Vdash_{\mathsf{daC}} \rightharpoondown A_i$ for all $i < n$, and $w \Vdash_{\mathsf{daC}} \bigwedge_{i<n} \rightharpoondown A_i$ as well.

Let us ask if $w \Vdash_{\mathsf{daC}} \rightharpoondown (A_j \vee A_k)$ for any $j, k < n$ such that $j \neq k$; the answer is negative. At any a.r. X_i and every $u_i \in X_i$, though $u_i \nVdash_{\mathsf{daC}} A_i$, for every $j \neq i$, $u_i \Vdash_{\mathsf{daC}} A_j$. Hence, for every $j \neq k$, $u_i \Vdash_{\mathsf{daC}} A_j \vee A_k$. Now suppose that for some distinct $j, k < n$, $w \Vdash_{\mathsf{daC}} \rightharpoondown (A_j \vee A_k)$; then there would be a $w'Rw$ such that $w' \nVdash_{\mathsf{daC}} A_j \vee A_k$, and by heredity, for some i and a $u_i \in X_i$, $u_i \nVdash_{\mathsf{daC}} A_j \vee A_k$, which we have seen is impossible. Since j, k were selected arbitrarily, we conclude that no disjunct of $\bigvee_{0 \leq i < j \leq n} \rightharpoondown (A_i \vee A_j)$ is verified at w, and so the disjunction itself is not verified. With the antecedent verified at w and the consequent not verified, we conclude that $w \nVdash_{\mathsf{daC}} \mathbf{PH}_n$ and thus, that $\mathfrak{F} \nVdash_{\mathsf{daC}} \mathbf{PH}_n$.

· For right-to-left, suppose that $\mathfrak{F}$ has no root system of cardinality n and that for some $u \in W$ and formulae $A_0, ..., A_{n-1}$, $u \Vdash_{\mathsf{daC}} \bigwedge_{i<n} \rightharpoondown A_i$. This entails that for all $i < n$, there exist points w_i such that $w_i R u$ and $w_i \nVdash_{\mathsf{daC}} A_i$.

Let $m < n$ be the greatest cardinality of a root system for u and consider one such root system $X_0, ..., X_{m-1}$. By the heredity of $\mathfrak{F}$, for each $i < n$ there are $w'_i R w_i$ such that $w'_i \in \cup_{l<m} X_l$. As $m < n$, by the pigeonhole principle there must be an X_h such that there are j, k with $w'_j, w'_k \in X_h$. We show that there exists a point w'' such that $w'' R u$ and $w'' \nVdash_{\mathsf{daC}} A_j \vee A_k$.

Suppose there were no w'' such that $w'' R w'_j$ and $w'' R w'_k$. Then $(w'_j \downarrow) \cap (w'_k \downarrow) = \varnothing$. But this implies that X_h can be refined into pairwise disjoint a.r.s X'_h, X''_h such that $w'_j \in X'_h$ and $w'_k \in X''_h$, and the root system $X_0, ..., X_{h-1}, X'_h, X''_h, X_{h+1}, ..., X_{m-1}$ is a root system for u of cardinality $m+1$, which contradicts the hypothesis. So there exists such a w'' and by heredity, $w'' \nVdash_{\mathsf{daC}} A_j$ and $w'' \nVdash_{\mathsf{daC}} A_k$, so $w'' \nVdash_{\mathsf{daC}} A_j \vee A_k$. It follows by transitivity of R that $u \Vdash_{\mathsf{daC}} \rightharpoondown (A_j \vee A_k)$ and hence $u \Vdash_{\mathsf{daC}} \bigvee_{0 \leq j < k \leq n} \rightharpoondown (A_j \vee A_k)$. ☐

For any $m, n \in \omega$ such that $0 < m < n$, any n-rooted frame will have countermodels to instances of $\mathbf{PH}_m$, and hence $\mathbf{PH}_m$ is not a theorem of PH_n for such a pair m and n. The closure of daC with any of the above axioms $\mathbf{PH}_n$ thus generates a distinct non-classical logic. One further comment on the above set of axioms should be made before proceeding:

Theorem 3. $\mathsf{WECQ} = \mathsf{PH}_1$

Proof. For left-to-right, suppose that in a backwards convergent model there is a point w with a root system of size greater than 1. Then there are distinct, pairwise disjoint a.r.s X_i, X_j. By backwards convergence, however, for arbitrary $u \in X_i$ and $v \in X_j$, there is a point t bearing R to both u and v. These two conditions by backwards closure imply that $t \in X_i$ and $t \in X_j$, respectively, contradicting pairwise disjointedness.

· For right-to-left, we note that $(\rightharpoondown A \wedge \rightharpoondown\rightharpoondown A) \to \rightharpoondown (A \vee \rightharpoondown A)$ is a substitution instance of $\mathbf{PH}_1$ and that $\rightharpoondown (A \vee \rightharpoondown A) \to B$ is a theorem of daC and *a fortiori* of PH_1. Hence, we may derive $(\rightharpoondown A \wedge \rightharpoondown\rightharpoondown A) \to B$ in PH_1, which is just **WECQ**. ☐

In essence, then, weak explosion is a side of the same medal as full De Morgan's laws in this context, which mirrors the equivalence of the addition of weak excluded middle $(\sim A \vee \sim\sim A)$ or full De Morgan's laws to Int; both generate the logic KC, which is indeed referred to at times as *De Morgan logic*.

Now, it is obvious that there exist axioms whose inclusion in daC suffices to generate CL in the same way in which the inclusion of the axiom $A \vee \sim A$ to Int generates CL. The axioms whose inclusion to daC has this effect are quite varied; though the below three axioms intuitively suggest distinct interpretations, they are all equivalent modulo daC.

Theorem 4. *The weakest logic* $\lambda \in Ext\mathsf{daC}$ *at which any of the formulae*

- $\rightharpoondown (A \to B) \to (\rightharpoondown A \to B)$
- $(\rightharpoondown B \to \rightharpoondown A) \to (A \to B)$
- $(A \wedge \rightharpoondown A) \to B$

becomes valid is CL*.*

Proof. That CL is a solution is trivial; the above formulae, when negation is construed as classical, are all theorems of classical logic.

To establish that CL is the *weakest* sdc-logic satisfying the above rules, consider an arbitrary frame $\mathfrak{F} = (W, R)$ such that the condition of single-pointedness fails for R. It then has distinct nodes u, v such that uRv but not vRu.

Let $v \uparrow$ be the collection of points accessible from v and $u \uparrow$ be the collection of points accessible from u. We can make some initial observations about $\mathfrak{F}$. For no $w \in v \uparrow$ does wRu; else, by transitivity, it would follow that vRu. Moreover, by transitivity, we observe that $v \uparrow \subseteq u \uparrow$. These general conditions are illustrated by Figure 2.

We provide a valuation V on $\mathfrak{F}$ making false each of the above formulae at u. Let $V(A) = v \uparrow$ and $V(B) = \varnothing$. Let us demonstrate that $\mathfrak{F}, V$ fails to confirm any of the above.

· $(\mathfrak{F}, V) \nVdash_{\mathsf{daC}} \rightharpoondown (A \to B) \to (\rightharpoondown A \to B)$: We observe that $v \Vdash_{\mathsf{daC}} A$ and $v \nVdash_{\mathsf{daC}} B$; hence $v \nVdash_{\mathsf{daC}} A \to B$ and as vRv, $v \Vdash_{\mathsf{daC}} \rightharpoondown (A \to B)$. Yet since $u \nVdash_{\mathsf{daC}} A$, $v \Vdash_{\mathsf{daC}} \rightharpoondown A$, and since $v \nVdash_{\mathsf{daC}} B$, it follows that $v \nVdash_{\mathsf{daC}} \rightharpoondown A \to B$. So $v \nVdash \rightharpoondown (A \to B) \to (\rightharpoondown A \to B)$ and with such a valuation, we conclude that $\mathfrak{F}$ fails to force this formula as well.

However, every point $w \in v \uparrow$ by transitivity bears R^{-1} to u, which does not force A, so at every such point $w \Vdash_{\mathsf{daC}} \rightharpoondown A$. Hence, $v \nVdash_{\mathsf{daC}} \dot{\sim} \rightharpoondown A$. So v witnesses the failure of the formula and $\mathfrak{F}$ has a valuation that fails to force the formula.

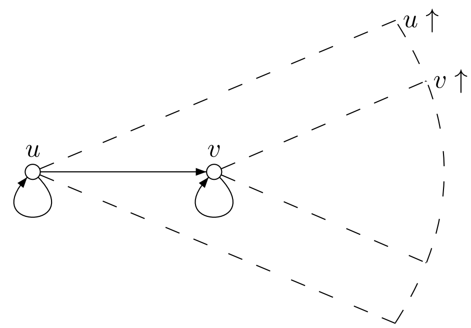

Figure 1: Condition on any non-singular frame $\mathfrak{F}$.

$\cdot$ $(\mathfrak{F}, V) \nVdash_{\mathsf{daC}} (\rightharpoondown B \to \rightharpoondown A) \to (A \to B)$: As $u \nVdash_{\mathsf{daC}} A$ and $u \nVdash_{\mathsf{daC}} B$, at every $w \in u\uparrow$, $w \Vdash_{\mathsf{daC}} \rightharpoondown A$ and $w \Vdash_{\mathsf{daC}} \rightharpoondown B$; hence, $u \Vdash_{\mathsf{daC}} \rightharpoondown B \to \rightharpoondown A$. Yet $v \Vdash_{\mathsf{daC}} A$ and $v \nVdash_{\mathsf{daC}} B$ so, as uRv, $u \nVdash_{\mathsf{daC}} A \to B$. Since uRu, $u \nVdash_{\mathsf{daC}} (\rightharpoondown B \to \rightharpoondown A) \to (A \to B)$ and as such a valuation exists, $\mathfrak{F} \nVdash_{\mathsf{daC}} (\rightharpoondown B \to \rightharpoondown A) \to (A \to B)$.

$\cdot$ $(\mathfrak{F}, V) \nVdash_{\mathsf{daC}} (A \wedge \rightharpoondown A) \to B$: Immediately, we note that $v \Vdash_{\mathsf{daC}} A$; moreover, we may observe that $u \nVdash_{\mathsf{daC}} A$ and hence, that $v \Vdash_{\mathsf{daC}} \rightharpoondown A$. From this, we infer that $v \Vdash_{\mathsf{daC}} A \wedge \rightharpoondown A$. But $v \nVdash_{\mathsf{daC}} B$, so v itself witnesses the failure of $(A \wedge \rightharpoondown A) \to B$. We conclude that $\mathfrak{F} \nVdash_{\mathsf{daC}} (A \wedge \rightharpoondown A) \to B$. □

Some notes about the above are in order; for one, we can substitute $\bot$ for B in the first two instances to draw conclusions concerning the relationship between intuitionistic negation and cointuitionistic negation in this context. We see, then, that CL is the first point in $Ext\mathsf{daC}$ at which $\rightharpoondown A \to \dot{\sim} A$ and $\rightharpoondown \dot{\sim} A \to \dot{\sim} \rightharpoondown A$ become theorems. As $\dot{\sim} A \to \rightharpoondown A$ and $\dot{\sim} \rightharpoondown A \to \rightharpoondown \dot{\sim} A$ are both theorems of daC (as can be easily checked), the commutativity of these negations amounts to their equivalence.

Furthermore, we may observe (but leave it to the reader to satisfy him- or herself) that the contraposition axioms $(\rightharpoondown B \to A) \to (\rightharpoondown A \to B)$ and $(B \to \rightharpoondown A) \to (A \to \rightharpoondown B)$, common in axiomatizations of relevant logics as described in [1], hold only at CL as well, while the logic PH_1 may be also seen as the addition of the principle $(\rightharpoondown \rightharpoondown B \to \rightharpoondown \rightharpoondown A) \to (\rightharpoondown A \to \rightharpoondown B)$.

Finally, if we restrict our attention to the lattermost above axiom- *ex con-*

tradictione quodlibet (**ECQ**)- we also may note that the only fully "explosive" sdc-logic is classical logic itself. This is interesting within the realm of paraconsistent base logics; note Meyer and Routley's work on so-called "classical" relevant logic in [8], in which the inclusion of **ECQ** to a positive relevant logic λ^+ generates a logic $\mathsf{K}\lambda$ such that $\lambda^+ \subsetneq \mathsf{K}\lambda \subsetneq \mathsf{CL}$.

3 The Maximal Proper sdc-Logic.

In this section, we wish to take a look at how we can define the strongest proper sdc-logic. In the intuitionistic case, there exists a maximal proper si-logic, the Smetanich logic SmL, the closure of Int with the axiom $\mathbf{SmL} = (\sim B \to A) \to (((A \to B) \to A) \to A)$. What we will show is that $\mathbf{SmL}^{\dot\sim}$ does *not* determine the maximal proper sdc-logic and that there exist infinitely many logics intermediate between $\mathsf{daC} + \mathbf{SmL}^{\dot\sim}$ and CL. We will show that the order type of the extensions of $\mathsf{SmL}^{\dot\sim}$ is $1 + \omega^*$, which will enable us to describe the maximal proper sdc-logic as well as the maximal proper extension of HB.

When discussing the si-logics Chagrov and Zakharyaschev [3] describe the frame condition associated with the Smetanich axiom **SmL** as "two-pointed, rooted frames." The traditional frame condition is stated is insufficient in this context; absent the assumption that all frames considered are trees, there are an abundance of frames forcing this axiom intuitionistically with more than two points. The class of frames forcing **SmL** in this case may be indiscernible with respect to Int but its members are varied enough that they have different theories in the setting of Priest-da Costa logic.

Lemma 6. *Let* $\mathsf{SmL}^{\dot\sim} = \mathsf{daC} + \mathbf{SmL}^{\dot\sim}$ *and let the frame condition*

$$\text{For all } w \in W,\ |\{x \in W : wRx\}| \leq 2$$

be called "2-sightedness." Then $\mathsf{SmL}^{\dot\sim}$ *is the logic of 2-sighted frames.*

Proof. We show that $\mathfrak{F} \Vdash_{\mathsf{daC}} \mathbf{SmL}^{\dot\sim}$ *iff* $\mathfrak{F}$ is 2-sighted.

· For left-to-right, suppose that $\mathfrak{F}$ is 2-sighted. Consider distinct u, v such that uRv; we can make the immediate observation that the only point accessible from v is v itself. Immediately, then, we may note that $v \Vdash_{\mathsf{daC}} \mathbf{SmL}^{\dot\sim}$, as v thinks of itself as a classical model.

Now, suppose for contradiction that some u and some valuation V on $\mathfrak{F}$, $u \nVdash_{\mathsf{daC}} \mathbf{SmL}^{\dot{\sim}}$. Then either $v \Vdash_{\mathsf{daC}} \dot{\sim}B \to A$ and $v \nVdash_{\mathsf{daC}} ((A \to B) \to A) \to A$ or $u \Vdash_{\mathsf{daC}} \dot{\sim}B \to A$ and $u \nVdash_{\mathsf{daC}} ((A \to B) \to A) \to A$. The former case, by the above, cannot obtain; hence, the latter must hold.

Similarly, this entails that either $v \Vdash_{\mathsf{daC}} (A \to B) \to A$ and $v \nVdash_{\mathsf{daC}} A$ or $u \Vdash_{\mathsf{daC}} (A \to B) \to A$ and $u \nVdash_{\mathsf{daC}} A$. The former, as we have seen, is impossible, so we infer the latter holds. As $u \Vdash_{\mathsf{daC}} (A \to B) \to A$, either $u \nVdash_{\mathsf{daC}} A \to B$ or $u \Vdash_{\mathsf{daC}} A$; as the latter contradicts our observation that $u \nVdash_{\mathsf{daC}} A$, we conclude the former.

As $u \nVdash_{\mathsf{daC}} A \to B$ either $v \Vdash_{\mathsf{daC}} A$ and $v \nVdash_{\mathsf{daC}} B$ or $u \Vdash_{\mathsf{daC}} A$ and $u \nVdash_{\mathsf{daC}} B$. As $u \nVdash_{\mathsf{daC}} A$, we accept the former. As $v \nVdash_{\mathsf{daC}} B$ and uRv, $u \Vdash_{\mathsf{daC}} \dot{\sim}B$. By reflexivity, there thus exists a point at which $\dot{\sim}B$ is forced but A is not, thus, $u \nVdash_{\mathsf{daC}} \dot{\sim}B \to A$. But by hypothesis u forces $\dot{\sim}B \to A$.

We conclude that there is no such V and hence, that $\mathfrak{F} \Vdash_{\mathsf{daC}} \mathbf{SmL}^{\dot{\sim}}$.

· For right-to-left, suppose that $\mathfrak{F}$ is not 2-sighted. Then there exist distinct points u, v, w such that uRv and uRw but $v \neq w$. If $\mathfrak{F}$ is linearly ordered, then assume without loss of generality that vRw. We provide a valuation V on $\mathfrak{F}$ showing that $\mathfrak{F} \nVdash_{\mathsf{daC}} \mathbf{SmL}^{\dot{\sim}}$. Let $V(A) = u\uparrow \setminus \{u\}$ and $V(B) = (v\uparrow \setminus \{v\}) \cup w\uparrow$. Now we may confirm that $\mathfrak{F}, V \nVdash \mathbf{SmL}^{\dot{\sim}}$.

For one, we can confirm that- as uRw and $w \Vdash B$- $u \nVdash \dot{\sim}B$; by the definition of V, we can moreover note that for all $w' \in u\uparrow \setminus \{u\}$, $w' \Vdash_{\mathsf{daC}} A$. Hence, at every point accessible from u forcing $\dot{\sim}B$, A is forced and so we can conclude that $u \Vdash_{\mathsf{daC}} \dot{\sim}B \to A$.

Now we can show that $u \nVdash_{\mathsf{daC}} ((A \to B) \to A) \to A$. Since uRv while $v \Vdash_{\mathsf{daC}} A$ and $v \nVdash_{\mathsf{daC}} B$, $u \nVdash_{\mathsf{daC}} A \to B$. By the above observation that all accessible points $w' \neq u$ force A, we can moreover conclude that $u \Vdash_{\mathsf{daC}} (A \to B) \to A$. But $u \nVdash_{\mathsf{daC}} A$, so $u \nVdash_{\mathsf{daC}} ((A \to B) \to A) \to A$. □

In order to fully describe the "gap" between $\mathsf{SmL}^{\dot{\sim}}$ and the trivial logic $\mathscr{L}_{\mathsf{daC}}$, we provide a series of additional lemmas:

Lemma 7. *For $\alpha \in \mathbf{Cn}$, let $\mathfrak{S}_\alpha = (W_\alpha, R_\alpha)$ with $W_\alpha = \{w_0, w_1, ..., w_\alpha\}$ and R_α the reflexive closure of the set $\{(w_\beta, w_\alpha) : \beta \leq \alpha\}$. Then the class $\{\mathfrak{S}_\alpha : \alpha \in \mathbf{Cn}\}$ exhaustively characterizes the $\mathsf{SmL}^{\dot{\sim}}$ frames.*

Proof. Immediate from the above; all $\mathfrak{S}_\alpha$ may be seen to be 2-sighted, and hence each forces $\mathbf{SmL}^{\dot{\sim}}$. Moreover, the only frames verifying 2-sightedness are frames with a point w and a class of points W' such that $R = W' \times \{w\} \cup \{(u,u) : u \in W' \text{ or } u = w\}$. Such a frame is obviously isomorphic to $\mathfrak{S}_{|W'|}$. □

Lemma 8. *For every $n \in \omega$ such that $1 \leq n$, $\mathfrak{S}_n \Vdash_{\mathsf{daC}} \mathbf{PH}_m$ for $m \geq n$ and $\mathfrak{S}_n \nVdash_{\mathsf{daC}} \mathbf{PH}_l$ for $l < n$.*

Proof. Immediately from the construction, we can observe that $\mathfrak{S}_n$ is m-rooted for all $m \geq n$ but not l-rooted for any $l < n$. Hence $\mathfrak{S}_n \Vdash_{\mathsf{daC}} \mathbf{PH}_m$ for all $m \geq n$ and $\mathfrak{S}_n \nVdash_{\mathsf{daC}} \mathbf{PH}_l$ for all $l < n$. □

Lemma 9. *The only subsets of $\{\mathfrak{S}_\alpha : \alpha \in \mathbf{Cn}\}$ that correspond to extensions of* daC *are either $\varnothing$, of the form $\{\mathfrak{S}_m : m \leq n\}$ for some $n \in \omega$, or is $\{\mathfrak{S}_\alpha : \alpha \in \mathbf{Cn}\}$ itself.*

Proof. To prove this, of a class $\mathbf{S} \subseteq \{\mathfrak{S}_\alpha : \alpha \in \mathbf{Cn}\}$ corresponding to an extension of $\mathsf{SmL}^{\dot\sim}$ we must show two things. For one, that if for some ordinal α, $\mathfrak{S}_\alpha \in \mathbf{S}$, then for all $\beta < \alpha$, $\mathfrak{S}_\beta \in \mathbf{S}$, that is, that every such $\mathbf{S}$ is backwards closed. Secondly, we must demonstrate that for an infinite cardinal α, if $\mathfrak{S}_\alpha \in \mathbf{S}$, then $\mathbf{S} = \{\mathfrak{S}_\alpha : \alpha \in \mathbf{Cn}\}$.

To show that any class corresponding to an extension of $\mathsf{SmL}^{\dot\sim}$ is backwards closed, we first show that for $\alpha > \beta$ and an arbitrary formula A, if $\mathfrak{S}_\alpha \Vdash_{\mathsf{daC}} A$ then $\mathfrak{S}_\beta \Vdash_{\mathsf{daC}} A$. We prove the contrapositive; suppose that $\mathfrak{S}_\beta \nVdash_{\mathsf{daC}} A$. Then there is a valuation V on $\mathfrak{S}_\beta$ such that $\mathfrak{S}_\beta, V \nVdash_{\mathsf{daC}} A'$ where A' is a substitution instance of A. For an arbitrary $\alpha > \beta$, we can extend V to a valuation V' on $\mathfrak{S}_\alpha$ so that $\mathfrak{S}_\alpha, V' \nVdash_{\mathsf{daC}} A'$. Call the sets of points associated with each frame W_β and W_α; by the definition of the frames, $W_\beta \subset W_\alpha$.

w_β is the point with the largest index in W_β; let $[w_{\geq\beta}]$ denote the set of points in W_α with indices greater than or equal to β. For all propositional parameters, define V' so that

$$V'(B) = \begin{cases} V(B) \cup [w_{\geq\beta}] & \text{if } w_\beta \in V(B) \\ V(B) & \text{otherwise} \end{cases}$$

We can prove by induction on complexity of formulae that $\mathfrak{S}_\alpha, V'$ forces precisely the same formulae as does $\mathfrak{S}_\beta, V$. We wish to show that the two properties hold for all formulae C:

- For all $\gamma < \beta$, $\mathfrak{S}_\beta, V, w_\gamma \Vdash_{\mathsf{daC}} C$ *iff* $\mathfrak{S}_\alpha, V', w_\gamma \Vdash_{\mathsf{daC}} C$ and

- For all $\delta \geq \beta$, $\mathfrak{S}_\beta, V, w_\beta \Vdash_{\mathsf{daC}} C$ *iff* $\mathfrak{S}_\alpha, V', w_\delta \Vdash_{\mathsf{daC}} C$.

We refer to these as properties 1 and 2, respectively. In the case of a propositional parameter C, the construction of V' guarantees that both properties hold immediately; we thus assume the induction hypothesis that for formulae C' of lesser complexity than C, both properties hold and move to the induction.

· C is $C' \wedge C''$ for some formulae C', C'':

Property 1: For $\gamma < \beta$, $\mathfrak{S}_\beta, V, w_\gamma \Vdash_{\mathsf{daC}} C' \wedge C''$ *iff* $\mathfrak{S}_\beta, V, w_\gamma \Vdash_{\mathsf{daC}} C'$ and $\mathfrak{S}_\beta, V, w_\gamma \Vdash_{\mathsf{daC}} C''$, which, by the induction hypothesis, holds *iff* $\mathfrak{S}_\alpha, V', w_\gamma \Vdash_{\mathsf{daC}} C'$ and $\mathfrak{S}_\alpha, V', w_\gamma \Vdash_{\mathsf{daC}} C''$ and hence, *iff* $\mathfrak{S}_\alpha, V', w_\gamma \Vdash_{\mathsf{daC}} C' \wedge C''$.

Property 2: For $\delta \geq \beta$, similar reasoning shows that $\mathfrak{S}_\beta, V, w_\beta \Vdash_{\mathsf{daC}} C' \wedge C''$ *iff* $\mathfrak{S}_\beta, V, w_\beta \Vdash_{\mathsf{daC}} C'$ and $\mathfrak{S}_\beta, V, w_\beta \Vdash_{\mathsf{daC}} C''$ *iff* $\mathfrak{S}_\alpha, V', w_\delta \Vdash_{\mathsf{daC}} C'$ and $\mathfrak{S}_\alpha, V', w_\delta \Vdash_{\mathsf{daC}} C''$ *iff* $\mathfrak{S}_\alpha, V', w_\delta \Vdash_{\mathsf{daC}} C' \wedge C''$. As δ was selected arbitrarily, this holds for all $w_\delta \in [w_{\geq\beta}]$.

· C is $C' \vee C''$ for some formulae C', C'':

Properties 1 and 2 run identically to the above, replacing "and" with "or" and "$\wedge$" with "$\vee$."

· C is $\neg C'$ for some formula C':

Property 1: For w_0, $\mathfrak{S}_\beta, V, w_0 \Vdash_{\mathsf{daC}} \neg C'$ *iff* $\mathfrak{S}_\beta, V, w_\gamma \nVdash_{\mathsf{daC}} C'$ for some $\gamma \leq \beta$ *iff* by induction hypothesis for some $\gamma' \leq \alpha$, $\mathfrak{S}_\alpha, V', w_{\gamma'} \nVdash_{\mathsf{daC}} C'$ *iff* $\mathfrak{S}_\alpha, V', w_0 \Vdash_{\mathsf{daC}} \neg C'$.

For $\gamma < \beta$ such that $\gamma > 0$, $\mathfrak{S}_\beta, V, w_\gamma \Vdash_{\mathsf{daC}} \neg C'$ *iff*, by 2-sightedness, $\mathfrak{S}_\beta, V, w_\gamma \nVdash_{\mathsf{daC}} C'$, which holds by hypothesis *iff* $\mathfrak{S}_\alpha, V', w_\gamma \nVdash_{\mathsf{daC}} C'$, holding *iff* $\mathfrak{S}_\alpha, V', w_\gamma \Vdash_{\mathsf{daC}} \neg C'$.

Property 2: For $\delta \geq \beta$, $\mathfrak{S}_\beta, V, w_\beta \Vdash_{\mathsf{daC}} \neg C'$ *iff* $\mathfrak{S}_\beta, V, w_\beta \nVdash_{\mathsf{daC}} C'$, holding by induction hypothesis *iff* $\mathfrak{S}_\alpha, V', w_\delta \nVdash_{\mathsf{daC}} C'$ *iff* $\mathfrak{S}_\alpha, V', w_\delta \Vdash_{\mathsf{daC}} \neg C'$. As δ was selected arbitrarily, this holds for all $w_\delta \in [w_{\geq\beta}]$.

· C is $C' \rightarrow C''$ for some formulae C', C'':

Property 1: For w_0, note that $\mathfrak{S}_\beta, V, w_0 \Vdash_{\mathsf{daC}} C' \rightarrow C''$ *iff* either $\mathfrak{S}_\beta, V, w_0 \nVdash_{\mathsf{daC}} C'$ or $\mathfrak{S}_\beta, V, w_0 \Vdash_{\mathsf{daC}} C''$, which we easily see holds *iff* $\mathfrak{S}_\alpha, V', w_0 \Vdash_{\mathsf{daC}} C' \rightarrow C''$.

In the case of w_γ such that $0 < \gamma < \beta$, $\mathfrak{S}_\beta, V, w_\gamma \Vdash_{\mathsf{daC}} C' \rightarrow C''$ *iff* each of the two disjunctions $\mathfrak{S}_\beta, V, w_\gamma \nVdash_{\mathsf{daC}} C'$ or $\mathfrak{S}_\beta, V, w_\gamma \Vdash_{\mathsf{daC}} C''$ and $\mathfrak{S}_\beta, V, w_0 \nVdash_{\mathsf{daC}} C'$ or $\mathfrak{S}_\beta, V, w_0 \Vdash_{\mathsf{daC}} C''$ hold. These, by the induction hypothesis, are equivalent to the disjunctions $\mathfrak{S}_\alpha, V', w_\gamma \nVdash_{\mathsf{daC}} C'$ or $\mathfrak{S}_\alpha, V', w_\gamma \Vdash_{\mathsf{daC}} C''$ and $\mathfrak{S}_\alpha, V', w_0 \nVdash_{\mathsf{daC}} C'$ or $\mathfrak{S}_\alpha, V', w_0 \Vdash_{\mathsf{daC}} C''$, respectively, which is in turn equivalent to $\mathfrak{S}_\alpha, V', w_\gamma \Vdash_{\mathsf{daC}} C' \to C''$.

Property 2: For $\delta \geq \beta$, $\mathfrak{S}_\beta, V, w_\beta \Vdash_{\mathsf{daC}} C' \to C''$ holds *iff* $\mathfrak{S}_\beta, V, w_\beta \nVdash_{\mathsf{daC}} C'$ or $\mathfrak{S}_\beta, V, w_\beta \Vdash_{\mathsf{daC}} C''$ and $\mathfrak{S}_\beta, V, w_0 \nVdash_{\mathsf{daC}} C'$ or $\mathfrak{S}_\beta, V, w_0 \Vdash_{\mathsf{daC}} C''$ hold. By the induction hypothesis, these conditions hold *mutatis mutandis* for w_δ and w_0 in the model $\mathfrak{S}_\alpha, V'$, and we reason that $\mathfrak{S}_\alpha, V', w_\delta \Vdash_{\mathsf{daC}} C' \to C''$. As δ was chosen arbitrarily, we conclude that this holds for all $w_\delta \in [w_{\geq \beta}]$.

The induction complete, we conclude that if $\mathfrak{S}_\alpha$ forces some set of formulae then $\mathfrak{S}_\beta$ likewise forces the set of formulae. Now suppose that $\mathbf{S}$ characterizes some $\lambda = \mathsf{SmL}^{\dot\sim} + \Gamma$ and that $\mathfrak{S}_\alpha \in \mathbf{S}$. Since the theory of $\mathfrak{S}_\alpha$ is by the above contained within the theory of $\mathfrak{S}_\beta$ (and *a fortiori* forces Γ) for all $\beta < \alpha$, $\mathfrak{S}_\beta \in \mathbf{S}$.

We now show that if a class of such frames corresponds to an extension of daC and contains $\mathfrak{S}_\omega$ then it contains $\mathfrak{S}_\alpha$ for all ordinals α. For $\alpha < \omega$, this has been shown from the above. For $\alpha > \omega$, suppose that $\mathfrak{S}_\alpha \nVdash_{\mathsf{daC}} A$ though $\mathfrak{S}_\omega \Vdash_{\mathsf{daC}} A$. Call the set of points corresponding to these frames W_α and W_ω, respectively; we note that $W_\omega \subset W_\alpha$.

Let V be the valuation on $\mathfrak{S}_\alpha$ that witnesses the failure of an instance of A- call it A'. We observe that A'- being a well-formed formula in the language of daC- contains finitely many propositional parameters so that we may enumerate these parameters by a set $\mathbf{B} = \{B_i : i < n\}$ for some n. It then follows that $\mathfrak{S}_\alpha, V \Vdash_{\mathsf{daC}} A'$ *iff* $\mathfrak{S}_\alpha, V \upharpoonright \mathbf{B} \Vdash_{\mathsf{daC}} A'$.

We use this feature to construct an equivalence relation on W_α. Define $\approx$ so that:

- $w_0 \approx w_0$
- For $w_\gamma, w_\delta \in W_\alpha$ with $\gamma, \delta \neq 0$, let $w_\gamma \approx w_\delta$ *iff* for all $B_i \in \mathbf{B}$, $w_\gamma \in V(B_i)$ *iff* $w_\delta \in V(B_i)$.

From this, we define the quotient frame $\mathfrak{S}_\alpha^\approx = (W_\alpha^\approx, R_\alpha^\approx)$, where $W_\alpha^\approx$ is the collection of equivalence classes modulo $\approx$ on W_α and $R_\alpha^\approx$ is defined by:

$$[w_\gamma]R^{\approx}_\alpha[w_\delta] \textit{ iff } \text{there exist } w'_\gamma \in [w_\gamma],\ w'_\delta \in [w_\delta] \text{ such that } w'_\gamma R_\alpha w'_\delta$$

The analogue of the valuation V modulo $\approx$ is immediate. Now we show by induction on complexity of formulae that for formulae C whose only propositional parameters are in $\mathbf{B}$, $\mathfrak{S}_\alpha, V \upharpoonright \mathbf{B}, w_\gamma \Vdash_{\mathsf{daC}} C$ *iff* $\mathfrak{S}^{\approx}_\alpha, (V \upharpoonright \mathbf{B})^{\approx}, [w_\gamma] \Vdash_{\mathsf{daC}} C$. (We will only reference the points on the left-hand side of the turnstile in the sequel; the notation will reveal in which model the valuation holds.)

As a basis step, we can note that for $C \in \mathbf{B}$, this follows by construction of $\approx$. For the induction step, C is not atomic; suppose that this result holds for all formulae of lesser complexity than C and for all points in W_α. Then we may examine the cases:

· C is $C' \wedge C''$ for some formulae C', C'': By the semantics for conjunction, $w_\gamma \Vdash_{\mathsf{daC}} C' \wedge C''$ *iff* $w_\gamma \Vdash_{\mathsf{daC}} C'$ and $w_\gamma \Vdash_{\mathsf{daC}} C''$, which by induction hypothesis holds *iff* $[w_\gamma] \Vdash_{\mathsf{daC}} C'$ and $[w_\gamma] \Vdash_{\mathsf{daC}} C''$. This moreover holds *iff* $[w_\gamma] \Vdash_{\mathsf{daC}} C' \wedge C''$.

· C is $C' \vee C''$ for some formulae C', C'': Immediate from the case of conjunction.

· C is $\rightharpoondown C'$ for some formula C': If $\gamma = 0$, then there is a w_δ with $w_\delta R w_0$ such that $w_\delta \nVdash_{\mathsf{daC}} C'$. By the induction hypothesis and definition of $R^{\approx}$, this occurs *iff* there is a $[w_\delta]$ such that $[w_\delta]R[w_0]$ such that $[w_\delta] \nVdash_{\mathsf{daC}} C'$, holding, in turn, *iff* $[w_0] \Vdash_{\mathsf{daC}} \rightharpoondown C'$. If $\gamma \neq 0$, then $w_\delta \Vdash_{\mathsf{daC}} \rightharpoondown C'$ *iff* $w_\delta \nVdash_{\mathsf{daC}} C'$ *iff* $[w_\delta] \nVdash_{\mathsf{daC}} C'$ *iff* $[w_\delta] \Vdash_{\mathsf{daC}} \rightharpoondown C'$

· C is $C' \to C''$ for some formulae C', C'': If $\gamma = 0$, then $w_0 \Vdash_{\mathsf{daC}} C' \to C''$ *iff* $w_0 \nVdash_{\mathsf{daC}} C'$ or $w_0 \Vdash_{\mathsf{daC}} C''$. By the induction hypothesis, this holds *iff* $[w_0] \nVdash_{\mathsf{daC}} C'$ or $[w_0] \Vdash_{\mathsf{daC}} C''$, which holds *iff* $[w_0] \Vdash_{\mathsf{daC}} C' \to C''$.

If $\gamma \neq 0$, then $w_\gamma \Vdash_{\mathsf{daC}} C' \to C''$ holds *iff*, of w_γ, $w_\gamma \nVdash_{\mathsf{daC}} C'$ or $w_\gamma \Vdash_{\mathsf{daC}} C''$ and, of w_0, $w_0 \nVdash_{\mathsf{daC}} C'$ or $w_0 \Vdash_{\mathsf{daC}} C''$. By the induction hypothesis, these conditions will be satisfied by $[w_\gamma]$ and $[w_0]$, respectively, and hence $[w_\gamma] \Vdash_{\mathsf{daC}} C' \to C''$.

With the induction complete, we now may observe that $\mathfrak{S}^{\approx}_\alpha$ in virtue of the finite length of A' is itself finite and hence isomorphic to a finite $\mathfrak{S}_m$ and hence there exists a valuation V' on $\mathfrak{S}_m$ such that $\mathfrak{S}^{\approx}_\alpha, (V \upharpoonright \mathbb{B})^{\approx}$ is identical to $\mathfrak{S}_m, V'$. By the earlier technique employed in showing backwards closure, we may extend this valuation to a valuation V'' on $\mathfrak{S}_\beta$ so that $\mathfrak{S}_\beta, V'' \nVdash_{\mathsf{daC}} A'$, which contradicts the hypothesis that $\mathfrak{S}_\beta \Vdash_{\mathsf{daC}} A$.

With these two properties in hand, the lemma is immediately confirmed. □

Recall that the *order type* of an ordered set $(A, <)$ is the isomorphism type of the set, that is, the class of all sets $(B, <)$ that are order isomorphic to $(A, <)$. For many linear orderings there are canonical representatives of such order types. In particular, the order type of $(\mathbb{N}, <)$ is naturally denoted ω, while order type of its reversed order- $(\mathbb{N}, >)$- is denoted ω^*. We may observe that the set of extensions of $\mathsf{SmL}^{\dot\sim}$ ordered under $\subset$ has order type $1 + \omega^*$.

Theorem 5. *The set* $Ext\mathsf{SmL}^{\dot\sim}$ *is of order type* $1 + \omega^*$ *under* $\subset$ *and can be exhaustively described as*

$$\mathsf{SmL}^{\dot\sim} \subsetneq ... \subsetneq \mathsf{PH}_n + \mathsf{SmL}^{\dot\sim} \subsetneq ... \subsetneq \mathsf{PH}_1 + \mathsf{SmL}^{\dot\sim} \subsetneq \mathsf{CL} \subsetneq \mathscr{L}_{\mathsf{daC}}$$

Proof. By Lemma 8, the only classes of $\mathsf{SmL}^{\dot\sim}$-frames that determine logics are the class of $\mathsf{SmL}^{\dot\sim}$-frames itself, those gotten by the inclusion of an instance of $\mathbf{PH}_n$ for some n, the CL-frame, or the empty class, which vacuously forces every formula. There are hence $\aleph_0$ many extensions. As every $\mathbf{PH}_n$ implies every $\mathbf{PH}_m$ for $m \geq n$, it follows that this is a linear order; that the proper extensions of $\mathsf{SmL}^{\dot\sim}$ are indexed by ω^*, followed by two points, the proper extensions have order type $\omega^* + 1 + 1 \cong \omega^*$. With the inclusion of SmL^* itself, it can be seen that the order type of $Ext\mathsf{SmL}^{\dot\sim}$ is simply $1 + \omega^*$.

□

From the above, we may infer that:

Corollary 1. $\mathsf{PH}_1 + \mathsf{SmL}^{\dot\sim}$ *is the maximal proper sdc-logic.*

Proof. Immediate from Theorem 5. □

We can from this also draw a corollary that holds in the case of Heyting-Brouwer logic. We can define HB by taking the conditions for Int and daC to hold for HB and adding truth conditions for the connective $\leftarrow$ by the following:

- $u \Vdash_{\mathsf{HB}} A \leftarrow B$ *iff* there is a w with wRu such that $w \Vdash_{\mathsf{HB}} A$ and $w \nVdash_{\mathsf{HB}} B$

As intuitionistic negation is a part of the language of HB, we may employ the **SmL** axiom without modification.

Corollary 2. $\mathsf{PH}_1 + \mathsf{SmL}$ *is the maximal non-classical extension of* HB*.*

Proof. By Lemma 4, the class of frames forcing **SmL** is precisely that of those forcing $\mathbf{SmL}^{\dot{\sim}}$. The inclusion of $\mathbf{PH}_1$ will hence "whittle away" all but two of these frames, by which observation we may conclude for geometrical reasons that this logic is maximal. □

4 The Relationship Between si-Axioms and sdc-Axioms

Given that intuitionistic logic is definable in Priest-da Costa logic, it is a reasonable question to ask how much influence superintuitionistic axioms have on the latter. As one ascends the lattice $Ext\mathsf{Int}$, one is met by increasingly greater restrictions on the frames; a näive assumption might be that the corresponding sdc-logics place correspondingly greater restrictions on, *e.g.*, the behavior of $\neg$.

This is, however, not the case. The superintuitionistic axioms, when translated into the language of daC, are, in a certain sense, quite inert with even the strongest of them having virtually no influence on how cointuitionistic negation behaves. We have axioms for sdc-logics that are wholly independent of the hierarchy of defined analogues of si-logics in $Ext\mathsf{daC}$, that is, force no superintuitionistic axiom and are forced by no superintuitionistic axiom. Indeed, we have already met infinitely many.

Lemma 10. *For every $n \geq 1$, for no A in the language of* Int *not a theorem of* Int *does $\vDash_{\mathsf{PH}_n} A^{\dot{\sim}}$ hold.*

Proof. We recall Corollary 2.20 of [3]:

$$\mathsf{Int} = \{A : \mathfrak{F} \Vdash_{\mathsf{Int}} A \text{ for every tree } \mathfrak{F}\}.$$

We may moreover note that every tree $\mathfrak{T}$ by definition is backwards convergent. Since Int is sound and complete with respect to a class of backwards convergent frames, the property of backwards convergence will force no superintuitionistic axiom. Since for all $n > 1$, PH_n is weaker than PH_1, this carries over to all such logics. □

Lemma 11. *For every $n \geq 1$, for no proper superintuitionistic logic λ does $\vDash_{\lambda^{\dot{\sim}}} \mathbf{PH}_n$ hold.*

Proof. For an arbitrary such n, suppose that some proper superintuitionistic logic λ, $\vDash_{\lambda^{\sim}} \mathbf{PH}_n$. Then, since $\lambda \subseteq \mathsf{SmL}$, it would follow that $\vDash_{\mathsf{SmL}^{\sim}} \mathbf{PH}_n$. As we have seen, however, frames such as $\mathfrak{S}_{n+1}$ are members of the class of $\mathsf{SmL}^{\dot{\sim}}$ frames at which $\mathbf{PH}_n$ fails. Hence $\mathbf{PH}_n$ is not a theorem of the logic $\mathsf{SmL}^{\dot{\sim}}$ and *a fortiori* for any defined analogue of a proper si-logic. □

These lemmas secure for us the next result.

Theorem 6. *Every logic* PH_n *is incommensurable with every logic* $\lambda^{\dot{\sim}}$ *where* λ *is a proper si-logic.*

Proof. Immediate from the foregoing lemmas. □

While there are sdc-logics wholly independent of any si-logical axioms, this is not to say that there is no influence. There are cases in which sdc-logical axioms and si-logical axioms can jointly exert influence where neither could independently.

To demonstrate such a case, we recall the Kreisel-Putnam axioms

$$\mathbf{KP} \qquad (\sim A \to (B \vee C)) \to ((\sim A \to B) \vee (\sim A \to C))$$
$$\mathbf{WKP} \qquad (\sim A \to (\sim B \vee \sim C)) \to ((\sim A \to \sim B) \vee (\sim A \to \sim C))$$

Let $\mathbf{KP}^{\rightharpoondown}$ and $\mathbf{WKP}^{\rightharpoondown}$ be the formulae in the language of daC gotten by substituting every instance of $\sim$ by $\rightharpoondown$ in the above two schemata. Inasmuch as $\mathbf{WKP}^{\rightharpoondown}$ is an instance of $\mathbf{KP}^{\rightharpoondown}$, it is clear that the latter implies the former in any $\lambda \in Ext\mathsf{daC}$.

First, we can give a partial analysis of the strength of $\mathsf{WKP}^{\rightharpoondown}$ by the following pair of results:

Theorem 7. $\vDash_{\mathsf{PH}_1} \mathbf{KP}^{\rightharpoondown}$ *and* $\nvDash_{\mathsf{PH}_2} \mathbf{WKP}^{\rightharpoondown}$

Proof. To show that $\vDash_{\mathsf{PH}_1} \mathbf{KP}^{\rightharpoondown}$, consider a frame $\mathfrak{F}$ such that $\mathfrak{F} \Vdash_{\mathsf{daC}} \mathbf{PH}_1$ but $\mathfrak{F} \nVdash_{\mathsf{daC}} \mathbf{KP}^{\rightharpoondown}$; then for some point u, both $u \Vdash_{\mathsf{daC}} \rightharpoondown A \to (B \vee C)$ and $u \nVdash_{\mathsf{daC}} (\rightharpoondown A \to B) \vee (\rightharpoondown A \to C)$. By the latter condition, there exist v, w such that uRv and uRw with $v, w \Vdash_{\mathsf{daC}} \rightharpoondown A$, $v \nVdash_{\mathsf{daC}} B$, and $w \nVdash_{\mathsf{daC}} C$.

Now, $u \downarrow \subseteq (v \downarrow) \cap (w \downarrow)$, and as $v \nVdash_{\mathsf{daC}} B$ and $w \nVdash_{\mathsf{daC}} C$, u likewise fails to force these formulae and hence, $u \nVdash_{\mathsf{daC}} B \vee C$. As $u \Vdash_{\mathsf{daC}} \rightharpoondown A \to (B \vee C)$ we infer that $u \nVdash_{\mathsf{daC}} \rightharpoondown A$, and so at every $u' \in u \downarrow$, $u' \Vdash_{\mathsf{daC}} A$. But $v \Vdash_{\mathsf{daC}} \rightharpoondown A$, so there must exist a v' such that both $v'Rv$ and at every point in $v' \downarrow$ does A fail. As such, $v' \downarrow$ and $u \downarrow$ must be pairwise disjoint and backwards closed; but this would imply that $\mathbf{PH}_1$ fails.

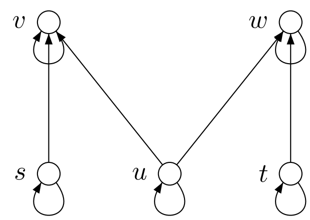

Figure 2: Frame for the countermodel to $\vDash_{\mathsf{PH}_2} \mathbf{WKP}^{\neg}$

We provide a general construction to show that $\nvDash_{\mathsf{PH}_2} \mathbf{WKP}^{\neg}$. For the initial setup, suppose there exists a point u such that $u \Vdash_{\mathsf{daC}} \neg A \to (\neg B \vee \neg C)$ but $u \nVdash_{\mathsf{daC}} \neg A \to \neg B$ and $u \nVdash_{\mathsf{daC}} \neg A \to \neg C$. Then there exist v, w accessible from u such that $v, w \Vdash_{\mathsf{daC}} \neg A$, $v \nVdash_{\mathsf{daC}} \neg B$, and $w \nVdash_{\mathsf{daC}} \neg C$. By the latter two conditions, we see that every point in $v \downarrow$ forces B and every point in $w \downarrow$ forces C; *a fortiori*, every point in $u \downarrow$ forces both of these formulae.

But as $u \Vdash_{\mathsf{daC}} \neg A \to (\neg B \vee \neg C)$ and both v, w force $\neg A$, we can easily see that $v \Vdash_{\mathsf{daC}} \neg C$ and $w \Vdash_{\mathsf{daC}} \neg B$, and hence there exist $v' \in v \downarrow$ and $w' \in w \downarrow$ such that at every point in $v' \downarrow$, C fails and that at every point in $w' \downarrow$, B fails.

Without loss of generality, we examine the backwards cone of v and note that $v' \downarrow$ and $w \downarrow$ are both backwards closed subsets, though at every point in the former C fails and at every point in the latter C holds. From such observations, we produce a countermodel. Let $\mathfrak{F}$ be represented by Figure 2.

Let valuation V on $\mathfrak{F}$ be $V(A) = \{u, v, w\}$, $V(B) = \{s, u, v, w\}$, and $V(C) = \{t, u, v, w\}$. We can confirm that $\mathfrak{F}, V \Vdash_{\mathsf{daC}} \mathbf{PH}_2$ but $\mathfrak{F}, V \nVdash_{\mathsf{daC}} \mathbf{WKP}^{\neg}$. The former task can be quickly attained by observing by inspection that no point in W has a root system of cardinality greater than 2.

For the latter, observe that $v, w \Vdash_{\mathsf{daC}} \neg A$ due to the failure of A at s and t, respectively. Moreover, as at every point in $v \downarrow$ does B hold and at every point in $w \downarrow$ does C hold, $v \nVdash_{\mathsf{daC}} \neg B$ and $w \nVdash_{\mathsf{daC}} \neg C$, though $v \Vdash_{\mathsf{daC}} \neg C$ and $w \Vdash_{\mathsf{daC}} \neg B$. This, in turn, implies that $\neg B \vee \neg C$ holds at both v and w.

We immediately see that $u \nVdash_{\mathsf{daC}} \neg A$, so at every point accessible from u

such that $\rightharpoondown A$ holds- namely, v and w, $\rightharpoondown B \vee \rightharpoondown C$ holds. Yet by the above observations, v witnesses the failure of $\rightharpoondown A \to \rightharpoondown B$ at u and w witnesses the failure of $\rightharpoondown A \to \rightharpoondown C$ at u. Hence, neither disjunct in the consequent of $\mathbf{WKP}^{\rightharpoondown}$ holds at this point; with the antecedent verified at u and the consequent not verified, this valuation reveals that $\nvDash_{\mathsf{PH}_2} \mathbf{WKP}^{\rightharpoondown}$. □

We now recall the si-logic KC, which is the closure of Int and $\mathbf{WEM} = \sim A \vee \sim\sim A$. Let $\mathsf{KC}^{\dot\sim}$ be the sdc-logic $\mathsf{daC} + \mathbf{WEM}^{\dot\sim}$. That $\mathsf{KC}^{\dot\sim}$ is a proper extension of daC may be seen by providing a countermodel to an instance of $\mathbf{WEM}^{\dot\sim}$. Let $W = \{w, w', w''\}$, let $R = \{(w, w'), (w, w')\} \cup \{(u, u) : u \in W\}$, and let the only point in W forcing A be w''. Then, fixing an unsatisfiable formula $\bot$, inasmuch as $\{w'\} = w' \uparrow$ and $w' \nVdash A$, we may infer that $w' \Vdash_{\mathsf{daC}} A \to \bot$, *i.e.*, $w' \Vdash_{\mathsf{daC}} \dot\sim A$. As $w'' \Vdash_{\mathsf{daC}} A$, it follows that $w'' \nVdash_{\mathsf{daC}} A \to \bot$. As wRw'' and $w'' \Vdash A$, we infer that $w \nVdash_{\mathsf{daC}} A \to \bot$; likewise that wRw' and $w' \Vdash_{\mathsf{daC}} \dot\sim A$ implies that $w \nVdash_{\mathsf{daC}} \dot\sim A \to \bot$. This is to say that $w \nVdash_{\mathsf{daC}} \dot\sim A$ and $w \nVdash_{\mathsf{daC}} \dot\sim\dot\sim A$, hence $\dot\sim A \vee \dot\sim\dot\sim A$ fails at w, *i.e.*, this is a daC model at which an instance of $\mathbf{WEM}^{\dot\sim}$ fails. Hence, the addition of this axiom scheme to daC filters out frames such as the present one and strictly extends the logic.

We first show that this logic is not in itself sufficient to secure as a theorem $\mathbf{WKP}^{\rightharpoondown}$.

Theorem 8. $\nvDash_{\mathsf{KC}^{\dot\sim}} \mathbf{WKP}^{\rightharpoondown}$

Proof. Simply add a point x to the above countermodel such that vRx and wRx and add x to the sets $V(A), V(B), V(C)$. It is easy to check that the frame is forwards convergent (as is required for KC and, by Lemma 4, for $\mathsf{KC}^{\dot\sim}$) but that the model does not verify $\mathbf{WKP}^{\rightharpoondown}$ at all points, as witnessed by u. □

Though Theorem 6 suggests that the si-logical axioms seem inert, in tandem with sdc-logical axioms, we can construct stronger logics. Let $\mathsf{KC}^{\dot\sim} + \mathsf{PH}_2$ be the logic $\mathsf{daC} + \mathbf{KC}^{\dot\sim} + \mathbf{PH}_2$. Then, despite the independence of $\mathbf{WKP}^{\rightharpoondown}$ from each logic alone, in tandem the axiom is indeed forced.

Theorem 9. $\vDash_{\mathsf{KC}^{\dot\sim} + \mathsf{PH}_2} \mathbf{WKP}^{\rightharpoondown}$

Proof. Recall the general construction from Theorem 7 and note that such requirements hold in any model at which $\mathbf{WKP}^{\rightharpoondown}$. We had noted that $v \downarrow$ contains a backwards cone $v' \downarrow$ such that every point in this set forces B but fails to force

C and that $w\downarrow$ contains a backwards cone $w'\downarrow$, every point of which forces C but fails to force B. We had also noted that while $u\downarrow$ is a subset of both $v\downarrow$ and $w\downarrow$, both B and C are forced at every point in $u\downarrow$. In light of these observations, we may furthermore note that $v'\downarrow$, $w'\downarrow$, and $u\downarrow$ are necessarily pairwise disjoint.

Suppose that we are dealing with a model of $\mathsf{KC}^{\dot{\sim}} + \mathsf{PH}_2$. Then the frame is forwards convergent and hence there exists a point x such that vRx and wRx. This implies that $v'\downarrow$, $w'\downarrow$, and $u\downarrow$ are pairwise disjoint, backwards closed subsets of $x\downarrow$; but this is prohibited by the assumption of 2-rootedness. Hence, the above, general conditions cannot obtain on such a frame. □

On their own, it appears, the defined analogues of superintuitionistic logics operate quite independently, though in conjunction with properly cointuitionistic axioms such "symbiotic" logics can become logically efficacious. While the above merely shows that there can be such an efficacy, how to properly characterize the relationship between si-logical axioms and sdc-logical axioms- possibly in $Ext\mathsf{HB}$ itself- deserves further thought.

5 daC as a Logic of Formal Inconsistency

We may make some observations concerning how daC relates to *Logics of Formal Inconsistency* (**LFI**s)[2]. We appeal to Carnielli, Coniglio, and Marcos' [**?**] and recall the definition of an **LFI** with respect to a negation $-$. Two conditions must hold for a logic to be an **LFI**. The first is that the logic must not be explosive, that is,

- there is a set of formulae Γ and formulae A, B such that $\Gamma, A, -A \nvdash B$.

The second condition is that the logic is *gently explosive*. This demands that

- there is a set of formulae $\bigcirc(C)$ depending exactly on parameter C with the property that
  - there are formulae A, B such that $\bigcirc(A), A \nvdash B$, and
  - there are formulae A', B' such that $\bigcirc(A'), -A' \nvdash B'$

with the property that

[2] I am grateful to a referee for directing my attention to the fact that daC is an **LFI**.

- for all sets of formulae Γ and formulae A, B, the following holds: $\Gamma, \bigcirc(A), A, -A \vdash B$.

We may observe a few features concerning daC *qua* **LFI**. Denote the set $\cup\{w' \downarrow: w' \in w \uparrow\}$ by the notation $w^{\searrow}$. Then if we extend the forcing relation to $\Vdash_{\mathsf{daC}^\circ}$ to include a consistency operator $\circ$ with forcing conditions:

- $u \Vdash_{\mathsf{daC}^\circ} \circ A$ *iff* either
  - $u^{\searrow} \subseteq V(A)$ or
  - $u^{\searrow} \cap V(A) = \varnothing$

we may observe that $\circ$ is definable in daC. The formula $\dot{\sim}(A \wedge \rightharpoondown A)$, *i.e.*, $(A \wedge \rightharpoondown A) \rightarrow \bot$, with $\bot$ an arbitrary, unsatisfiable formula, succeeds in the task of defining $\circ A$.

Observation 1. $u \Vdash_{\mathsf{daC}^\circ} \circ A$ *iff* $u \Vdash_{\mathsf{daC}} \dot{\sim}(A \wedge \rightharpoondown A)$

Proof. For right-to-left, suppose that in an arbitrary model $w \Vdash_{\mathsf{daC}} \dot{\sim}(A \wedge \rightharpoondown A)$; then at every point $w' \in w \uparrow R$, $w' \nVdash_{\mathsf{daC}} A \wedge \rightharpoondown A$. This entails that at every such point either $w' \nVdash_{\mathsf{daC}} A$ or $w' \nVdash_{\mathsf{daC}} \rightharpoondown A$. By reflexivity, w is such a point and hence either $w \nVdash_{\mathsf{daC}} A$ or $w \nVdash_{\mathsf{daC}} \rightharpoondown A$.

In the case in which $w \nVdash_{\mathsf{daC}} A$, for every $w' \in w \uparrow$, either $w' \nVdash_{\mathsf{daC}} A$ or $w' \nVdash_{\mathsf{daC}} \rightharpoondown A$. The latter can not hold, as this would imply that $w \Vdash_{\mathsf{daC}} A$; hence, for all such w', $w' \nVdash_{\mathsf{daC}} A$. By heredity, for every $w' \in w \uparrow$, at no point in $w' \downarrow$ is A forced, *i.e.*, $w^{\searrow} \cap V(A) = \varnothing$.

In the case in which $w \nVdash_{\mathsf{daC}} \rightharpoondown A$, we immediately may infer that $w \Vdash_{\mathsf{daC}} A$ and, by heredity, that for all $w' \in w \uparrow$, $w' \Vdash_{\mathsf{daC}} A$. As before, for all $w' \in w \uparrow$, either $w' \nVdash_{\mathsf{daC}} A$ or $w' \nVdash_{\mathsf{daC}} \rightharpoondown A$. In this case, the former cannot hold, hence at all such points $w' \nVdash_{\mathsf{daC}} \rightharpoondown A$. From this, we may infer that at every point in $w' \downarrow$, A is forced, *i.e.*, $w^{\searrow} \subseteq V(A)$.

· For left-to-right, we prove the contrapositive. Suppose that in an arbitrary model $w \nVdash_{\mathsf{daC}} \dot{\sim}(A \wedge \rightharpoondown A)$; as $\dot{\sim}$ behaves like intuitionistic negation, this entails that at some w' such that wRw', $w' \Vdash_{\mathsf{daC}} A \wedge \rightharpoondown A$. That $w' \Vdash_{\mathsf{daC}} A$ implies that $w^{\searrow} \cap V(A) \neq \varnothing$. That $w' \Vdash_{\mathsf{daC}} \rightharpoondown A$ implies the existence of a $w'' \in w'$ at which A fails, implying that $w^{\searrow} \nsubseteq V(A)$. This ensures that $w \nVdash_{\mathsf{daC}^\circ} \circ\varphi$. □

With this observation, we may employ the defined consistency connective without loss of generality in the sequel.

Observation 2. daC *is an* **LFI** *with respect to* $\rightharpoondown$

Proof. The failure of explosion *simpliciter* is immediate. For any propositional parameters A and B, paraconsistency of daC entails that $A, \rightharpoondown A \nvdash_{\mathsf{daC}} B$.

With respect to gentle explosion, we first set $\bigcirc(A) = \{\circ A\}$. We may observe that the selection of $\bigcirc$ is as required by considering a one point, reflexive frame with $W = \{w\}$. Consider valuations V_1 and V_2 such that $V_1(A) = W$, $V_2(A) = \varnothing$, and $V_1(B) = V_2(B) = \varnothing$. Clearly, w forces $\circ A$ and fails to verify B under each valuation, w forces A under V_1 and forces $\rightharpoondown A$ under V_2. By soundness, we conclude that both $\circ A, A \nvdash B$ and $\circ A, \rightharpoondown A \nvdash B$.

To show gentle explosion of daC, suppose for contradiction that at some point w in a model, A, $\rightharpoondown A$, and $\circ A$ are all forced. By the latter, either $w^{\searrow} \subseteq V(A)$ or $w^{\searrow} \cap V(A) = \varnothing$. By the first two conditions, however, there is a $w' \in w^{\searrow}$ at which A is false and a $w'' \in w^{\searrow}$ at which A is true. Inasmuch as there can be no model for this collection of formulae, at *all* models of this collection, B is true. □

Not only is daC an **LFI**, it is also a **dC**-system. To begin, we can rehearse the definition of a **C**-system found in [**?**]. A logic $\lambda 2$ is a **C**-system based on a logic $\lambda 1$ with respect to a negation $-$ if:

- $\lambda 2$ is a conservative extension of $\lambda 1$
- $\lambda 2$ is an **LFI** with respect to $-$ such that $\bigcirc(A)$ is a singleton
- The negation $-$ is not definable in $\lambda 1$
- $\lambda 1$ is non-trivial

Observation 3. daC *is a* **C***-system based on positive intuitionistic logic (*Int^+*) with respect to* $\rightharpoondown$

Proof. That Int^+ is non-trivial follows from the non-triviality of daC and that daC conservatively extends Int^+ is shown in [10]. Observation 2 demonstrates that daC is an **LFI** such that $\bigcirc(A)$ is a singleton.

As is well-known (and observed in [10]), Int is not definable in the positive intuitionistic logic Int^+. Suppose that the negation $\rightharpoondown$ were definable in Int^+; then daC would be definable in Int^+. But inasmuch as Int is definable in daC, this would entail that Int would be definable in Int^+. □

A **dC**-system is a **C**-system in which consistency of a formula is expressible without extending the signature, that is, a consistency connective $\circ$ is definable.

Corollary 3. daC *is a* **dC***-system*

Proof. Immediate from Observations 1 and 3. □

We may also make an observation concerning common "axioms of propogation." We recall from [**?**] that, in the Hilbert-style axiomatization of the **LFI** Cia, the following axioms govern the consistency operator $\circ$ (with a negation $-$):

bc1 $\circ A \rightarrow (A \rightarrow (-A \rightarrow B))$

ca1 $(\circ A \wedge \circ B) \rightarrow \circ(A \wedge B)$

ca2 $(\circ A \wedge \circ B) \rightarrow \circ(A \vee B)$

ca3 $(\circ A \wedge \circ B) \rightarrow \circ(A \rightarrow B)$

We can show that the majority of these axioms governing $\circ$ hold in daC and consider the points in $Ext\mathsf{daC}$ at which *all* these axioms hold.

Observation 4. **bc1** *is a theorem of* daC

Proof. Suppose that in some model, there is a point u such that $u \Vdash_{\mathsf{daC}} \circ A$ and $u \nVdash_{\mathsf{daC}} A \rightarrow (\rightharpoondown A \rightarrow B)$. By the latter, then, there exists a $u' \in u\uparrow$ such that $u' \Vdash_{\mathsf{daC}} A$ and $u' \nVdash \rightharpoondown A \rightarrow B$. The latter of these consequences implies the existence of a $u'' \in u'\uparrow$ such that $u'' \Vdash_{\mathsf{daC}} \rightharpoondown A$. This entails that there is a $u''' \in u''\downarrow$ at which A fails. But as uRu' and $u'Ru''$, $u'' \in u^{\searrow}$; likewise, as $u'''Ru''$, $u''' \in u^{\searrow}$. But there are hence points in $u^{\searrow}$ at which A holds and others at which A fails, contradicting the hypothesis that $u \Vdash_{\mathsf{daC}} \circ A$. □

Observation 5. **ca1** *and* **ca2** *are theorems of* daC

Proof. Suppose that a substitution instance of the common antecedent holds at a point u, *i.e.*, $u \Vdash_{\mathsf{daC}} \circ A \wedge \circ B$. Four cases emerge:

I $u^{\searrow} \subseteq V(A)$ and $u^{\searrow} \subseteq V(B)$

II $u^{\searrow} \subseteq V(A)$ and $u^{\searrow} \cap V(B) = \varnothing$

III $u^{\searrow} \cap V(A) = \varnothing$ and $u^{\searrow} \subseteq V(B)$

IV $u^{\searrow} \cap V(A) = \varnothing$ and $u^{\searrow} \cap V(B) = \varnothing$

We hence argue by cases, showing that in each case the consequents of **ca1** and **ca2** hold at u.

Case I: That A and B are forced at every point in $u^{\searrow}$ implies that $A \wedge B$ and $A \vee B$ are likewise forced at every point. Hence, $u^{\searrow}$ is a subset of both $V(A \wedge B)$ and $V(A \vee B)$, whence $u \Vdash_{\mathsf{daC}} \circ(A \wedge B)$ and $u \Vdash_{\mathsf{daC}} \circ(A \vee B)$.

Cases II and III: These cases are symmetric to one another. Suppose that $u^{\searrow} \subseteq V(A)$ and $u^{\searrow} \cap V(B) = \varnothing$. Then at every point $u' \in u^{\searrow}$, $u' \nVdash_{\mathsf{daC}} A \wedge B$ and $u' \Vdash_{\mathsf{daC}} A \vee B$. Hence, $u^{\searrow} \cap V(A \wedge B) = \varnothing$ and $u^{\searrow} \subseteq V(A \vee B)$; this implies that $u \Vdash_{\mathsf{daC}} \circ(A \wedge B)$ and $u \Vdash_{\mathsf{daC}} \circ(A \vee B)$.

Case IV: That $u^{\searrow} \cap V(A) = \varnothing$ and $u^{\searrow} \cap V(B) = \varnothing$ implies that at any point in $u^{\searrow}$, neither A nor B hold. This is to say that $u^{\searrow} \cap V(A \wedge B) = \varnothing$ and $u^{\searrow} \cap V(A \vee B)$, *i.e.*, $u \Vdash_{\mathsf{daC}} \circ(A \wedge B)$ and $u \Vdash_{\mathsf{daC}} \circ(A \vee B)$.

Thus, in any case in which $u \Vdash_{\mathsf{daC}} \circ A \wedge \circ B$, it follows that both $u \Vdash_{\mathsf{daC}} \circ(A \wedge B)$ and $u \Vdash_{\mathsf{daC}} \circ(A \vee B)$. □

Observation 6. **ca3** *is not valid in* daC

Proof. Suppose that $u \Vdash_{\mathsf{daC}} \circ A \wedge \circ B$ and recall the four cases from Observation 5. Before examining why **ca3** is invalid, we first examine the cases in which the consequent $\circ(A \to B)$ holds.

Cases I and III: From $u^{\searrow} \subseteq V(B)$ alone, we infer from heredity that for every $u' \in u^{\searrow}$, $u' \uparrow \subseteq V(B)$. Hence, every for $u' \in u^{\searrow}$, $u' \Vdash_{\mathsf{daC}} A \to B$; this implies that $u \Vdash_{\mathsf{daC}} \circ(A \to B)$.

Case II: As every $u' \in u^{\searrow}$ forces A and fails to force B, each u' itself serves as a witness to the failure of $A \to B$ at u'. Hence, $u^{\searrow} \cap V(A \to B) = \varnothing$ and $u \Vdash_{\mathsf{daC}} \circ(A \to B)$.

It is from Case IV that we may draw a counterexample. Suppose that a frame is not forwards convergent; then there are points $t, u, v \in W$ such that uRt, uRv, but $(t \uparrow) \cap (v \uparrow) = \varnothing$. Let $V(A) = v \uparrow$ and let $V(B) = \varnothing$. Then we have a countermodel to an instance of **ca3**.

As $V(B) = \varnothing$, that $t \Vdash_{\mathsf{daC}} \circ B$ is trivial. From $(t \uparrow) \cap (v \uparrow) = \varnothing$, we infer that at no point in $t \uparrow$ is A true and by heredity that at no point in $t^{\searrow}$ is A true. Hence, $t \Vdash_{\mathsf{daC}} \circ A$ and likewise for the conjunction $\circ A \wedge \circ B$.

Now, as at no point in $t\uparrow$ is A true, so $A \to B$ is forced at t vacuously. But at u, there exists a point v such that $v \Vdash_{\mathsf{daC}} A$ but $v \nVdash_{\mathsf{daC}} B$. Hence, $u \nVdash_{\mathsf{daC}} A \to B$. But both t and u are members of $t^{\searrow}$; as they disagree with respect to $A \to B$, it follows that $t \nVdash_{\mathsf{daC}} \circ(A \to B)$. $\square$

We are able to pinpoint the weakest point in $Ext\mathsf{daC}$ in which all the above axioms governing $\circ$ hold.

Observation 7. $\mathsf{KC}^{\dot{\sim}}$ *is the weakest sdc-logic including* **bc1** *and* **ca1** $-$ **3**

Proof. That $\mathsf{KC}^{\dot{\sim}}$ extends daC entails not only that **bc1**, **ca1**, and **ca2** are theorems, but that in Cases I-III, **ca3** holds. To show that **ca3** is a theorem of $\mathsf{KC}^{\dot{\sim}}$ it thus suffices to examine Case IV.

Suppose that $u \Vdash_{\mathsf{daC}} \circ(A \wedge B)$ and Case IV holds. Then for contradiction suppose that $u \nVdash_{\mathsf{daC}} \circ(A \to B)$. This implies that there exists a $v \in u^{\searrow}$ such that $v \nVdash_{\mathsf{daC}} A \to B$. As every point in $u\uparrow$ vacuously forces $A \to B$, v has to be in $u^{\searrow} \setminus u\uparrow$ but bears R to a $v' \in u\uparrow$. Since $v \nVdash_{\mathsf{daC}} A \to B$, there exists a point v'' such that vRv'' at which $v'' \Vdash_{\mathsf{daC}} A$. However, as A fails at all points in $u^{\searrow}$ A, we reason that v'' is not a member of $u^{\searrow}$.

That $v'' \Vdash_{\mathsf{daC}} A$ implies that $v \nVdash_{\mathsf{daC}} \dot{\sim}A$. Furthermore, as $v' \in u^{\uparrow}$, $v'\uparrow \subseteq u^{\searrow}$ and thus A fails at every point in $v'\uparrow$; that vRv' thus implies that $v \nVdash_{\mathsf{daC}} \dot{\sim}\dot{\sim}A$. But the model is presumed to be a model of $\mathsf{KC}^{\dot{\sim}}$ and hence we may infer that $v \Vdash_{\mathsf{daC}} \dot{\sim}A \vee \dot{\sim}\dot{\sim}A$.

To show that $\mathsf{KC}^{\dot{\sim}}$ is the *weakest* sdc-logic for which **ca3** is a theorem, consider a weaker logic λ. Then λ has model in which an instance of $\mathbf{WEM}^{\dot{\sim}}$ fails at a point u. The frame of the model is hence not forwards convergent and the counterexample in Observation 6 is general enough that it can be built on any such frame. Hence, **ca3** is not a theorem of λ. $\square$

It is important to note that despite the fact that these axioms of propogation hold in $\mathsf{KC}^{\dot{\sim}}$ a number of weak principles common to many **C**-systems generate classical logic when added to daC. For example, the addition of axiom **cl** of the presentation in [**?**], the formula $\neg(A \wedge \neg A) \to \circ A$, is not a theorem of $\mathsf{KC}^{\dot{\sim}}$. We observe that no proper sdc-logic has **cl** as a theorem.

Observation 8. *The weakest* $\lambda \in Ext\mathsf{daC}$ *at which* **cl** *holds is* CL

Proof. Recall the frame described in Theorem 4 and consider a valuation V in which $V(A) = v\uparrow$; immediately we observe that $u \nVdash_{\mathsf{daC}} A$ and *a fortiori*

$u \not\Vdash_{\mathsf{daC}} A \wedge \neg A$. This entails that $v \Vdash_{\mathsf{daC}} \neg (A \wedge \neg A)$. That $u \in v^{\searrow}$ and $v \in v^{\searrow}$ ensures that neither is $v^{\searrow} \subseteq V(A)$ nor is $v^{\searrow} \cap V(A) = \varnothing$. This entails that $v \not\Vdash_{\mathsf{daC}} \circ A$, whence we infer that $v \not\Vdash_{\mathsf{daC}} \neg (A \wedge \neg A) \to \circ A$. Hence, every non-singular frame admits a valuation contradicting an instance of **cl**, *i.e.*, its inclusion generates classical logic. □

Despite the observation that daC, we close with a negative observation that may attenuate the relevance of this logic to those primarily concerned with **LFI**s.

It is suggested in [**?**] that the logic C_{min}, generated by including the axiom $A \vee (A \to B)$ to C_{ω}, is a more appropriate lower bound to the da Costa hierarchy than C_{ω} itself. Indeed, most of the **LFI**s warranting discussion, *e.g.*, bC and Cila, are extensions of C_{min}. Yet C_{min} and daC cannot be meaningfully combined due to the following observation:

Observation 9. *The only sdc-logic extending* C_{min} *is* CL

Proof. Again recalling the details of Theorem 4, consider a valuation V in which $V(A) = W \setminus u \downarrow$ and $V(B) = \varnothing$; immediately we observe that $u \not\Vdash_{\mathsf{daC}} A$. Moreover that $v \Vdash_{\mathsf{daC}} A$ and $v \not\Vdash_{\mathsf{daC}} B$ implies that $u \not\Vdash_{\mathsf{daC}} A \to B$. Hence, $u \not\Vdash A \vee (A \to B)$. Thus, the only frame verifying C_{min} or its extensions is the singular frame and the only sdc-logic extending C_{ω} is CL. □

6 Future Work

A number of things come to mind that warrant further study.

For one, we have dealt only with the propositional Priest-da Costa logic and its extensions here and have omitted any discussion of quantification. Priest in [11] considers a first-order logic daCQ extending daC by adding intuitionistic quantifiers and notes that in general quantifier interchange (or De Morgan's laws for quantifiers) between the intuitionistic quantifiers and cointuitionistic negation fails in this logic; the only valid instance is $\neg \exists x \varphi(x) \vDash_{\mathsf{daCQ}} \forall x \neg \varphi(x)$ where $\varphi(x)$ is atomic. Whether frame conditions may be strengthened to support a greater number of instances of quantifier interchange rules in Priest's daCQ is an interesting question; moreover, whether an alternative approach to quantification in daC is possible warrants more thought.

Furthermore, the relationship between the lattices *Ext*Int, *Ext*HB, and *Ext*daC is worthy of exploration. A number of interesting, related systems- such as Brunner and Carnielli's hierarchy **AC** of dualized Gödel logics outlined in [2]- relate to a different fragment of HB than that determined by daC. It seems reasonable that such systems could be semantically explicated by further generalizing the si-logical approach to full HB.

Moreover, much of the interest in si-logics has historically been exploring whether, *e.g.*, the disjunction property holds in extensions of Int. Indeed, the genesis of the si-logical axioms **KP** and **WKP** is attributable to such a query. While the disjunction property fails in every $\lambda \in Ext\mathsf{daC}$, as can be seen by taking one of the axioms the addition of which to daC generates CL as a disjunct, a dual property may well be of interest. It seems possible that the *constructible falsity property*, in which $\models_{\mathsf{daC}} \neg (A \wedge B)$ implies that either $\models_{\mathsf{daC}} \neg A$ or $\models_{\mathsf{daC}} \neg B$, may hold for some sdc-logics.

Finally, there has been a great deal of work relating the strong (or *constructible*) negation of Nelson's logic N^3 described in [9] to intuitionistic logic, *e.g.*, exploring the consequences of adding strong negation to superintuitionistic logics in Kracht's [5] and Hasuo and Kashima's [4]. Exploring the relationship between the strong negation of N^3 (and its paraconsistent cousins) and cointuitionistic negation- already carried out by Wansing in [14]- is a quite interesting area; extending the analysis to extensions of daC and HB seems to be as worthy a project.

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