SYMPOSIUM: THE NOTION OF INFINITY.

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I.—*By* J. N. FINDLAY.

I AM about to introduce a symposium on infinity. I do so, not because I can claim any special intimacy with the infinite, nor yet because I feel myself specially competent to unravel its intricacies, but because I think it all-important that a notion so fundamental should be rescued from the grip of the experts, and should be brought back into general circulation. It is a notion so common and so clear as to lie behind practically every use of the ordinary phrases " and so on " or " and so forth," but it is none the less capable of giving rise to vertiginous bewilderments, which may lead, on the one hand, to the mystical multiplication of contradictions, as also, on the other hand, to that voluntary curtailment of our talk and thought on certain matters, which is as ruinous to our ordered thinking. A notion which is at once so tantalising and so ordinary plainly deserves the perpetual notice of philosophers. Throughout the history of human reflection the fogs of an interesting, and often interested obscurity have surrounded the infinite ; they were dispersed for a brief period by the sense-making genius of Cantor, but have since gathered about it with an added, because more wilful, impenetrability. In the growing illiteracy of our time, when the lamp of memory barely sheds its beams beyond the past two decades, and the controversies or discoveries of 1890 or 1910 have been allowed to become as stale and as irrelevant as those of Anselm or Xenocrates, it is well that someone should at times seek to recapture and to revivify some of the positive illuminations of the past. It is no doubt regrettable that my own personal grasp of mathematical formulations should so often halt and stumble ; I pursue symbolic intricacy in the way of duty, my taste in philosophy being for the gnomically simple. If I make mistakes, there will, however, be many to correct me, and I may hope, also, that, here as elsewhere, sheer myopia and symbolic clumsiness, may at times prove the mother of philosophical invention. I must attempt, at any rate, to do what others, better qualified than myself, have so entirely neglected ; it is better that someone should discuss this topic with the freedom of philosophy, than that all talk about it should be allowed to flow along those technical channels which, whatever else they may do, never enrich our philosophical understanding.

I shall divide my treatment of infinity into two sections. In the first I shall skim briefly over the historical terrain of western thought about the infinite, so that it may be plain where I propose to come down in this field, and precisely what drifts of thought I intend to reinforce or to combat. In the second part I shall let you have a few of my own personal intuitions on the matter, backed up with an amplifying commentary, which will try to show how our thought about the infinite may be fitted into the general pattern of our thought about number.

I begin, therefore, with my historical perspective, foreshortened as to its remoter phases, of western thought about the infinite. The notion of something so comprehensive (in some respect or another) as to be at least equal to anything we can build up in thought, either by the putting together of parts, or the successive running through of elements or aspects, and which is yet *not* such as to be contained or exhausted in such constructions or resumptions, is a notion of no peculiar difficulty or obscurity, and it is also one that made an early appearance in the clear thought of the Ionians. Here we have successively brought before us, as the " nature " of the things in our world, a number of august, embracing media, sometimes identified with the homely substances of everyday experience, and sometimes hedged about with negations, which are all infinite in extent, and which are also such that out of their bosom an infinity of worlds can be successively or simultaneously generated. There is nothing, in such a straightforward picture, to suggest that the infinite is *itself* in some manner unexhausted or incomplete, or engaged in some perpetual, restless process of trying to run through, or to sum up, the totality of its parts and phases ; it is all there, in majestic, fully-realised plenitude and repose, and it is only we, or the hurrying series of worlds of which we form a part, who are trying vainly to exhaust whatever may be in it. There is also nothing, in such a straightforward picture, which demands that the infinite should itself be in any way exhaustive ; the worlds which arise out of its bosom are themselves infinite in number, yet there remains always an infinitude of *other* stuff around them, and outside of them.

What we have called a straightforward picture of the infinite is not, however, one that western thought has found easy to hold. While it may not involve anything intrinsically difficult, and while the only real questions connected with it may be those concerning its precise implications and applications, it has none the less always seemed to abound in contradictions, and to render absurd and self-contradictory any idea in which it could be shown to be present. To speak of it seems to involve exhausting the inexhaustible, than which nothing more absurd could be conceivable ; it also gives difficulty in that, without being accessible to our imagination, it lacks the happy circumscribability which would render it acceptable to our thought. Hence there has arisen, at an early stage of thinking, a tradition which has tried to substitute for the infinite what we may call the variable finite, the finite which can always be pushed a stage further, without ever achieving *all* the stages or values of which it is capable. It is this sort of variable finite, always pushing out beyond every bounded unit, or breaking out within it, which confronts us in the Platonic-Pythagorean Unlimited, or, as it was finally called, the Great and Small. As such it is the irrational, evil, essentially formless or flowing principle, which has to be dominated by " the One " or " the Limit " in order to give rise to everything intelligible or good. It is the indefinite element in the ideal world, which calls for a first bounding by the One, in order to give rise to that whole range of precise, quantitative patterns, with which the Platonic Forms were ultimately identified. It is also the indefinite element in the world of the senses, whose bounding by precise, numerical ratios gives rise to whatever is healthful, strong, musical or visibly excellent. In itself, however, it admits only of a contradictious characterisation ; it is the "Others," the Great and Small, without any principle of unity or definition in itself. As soon as one seizes a part of it, and attempts to treat it as something definite and single, it will at once begin to evanesce into other parts, and these into other parts without end. And as soon as one imagines it as endowed with beginning, middle and end, one will find another beginning emerging before the first beginning, another end emerging after the first end, and another truer middle showing itself within the first middle. I need here only refer to the superb seventh hypothesis of the Platonic *Parmenides* for a full account of the tantalising and elusive behaviour of the Platonic infinite, whose complex, contradictious characterisation is only equalled by the still more complex and contradictious characterisation of his indefinite, super-essential One or Good.

The Platonic mysticism passes away in the Aristotelian treatment of the infinite, but the essential features of the Platonic treatment remain. The infinite only exists *after a fashion* ; after another fashion it does not exist at all. It never exists in the sense that we ever actually *have* an infinitely large number of parts or elements of anything, nor anything which exceeds all things of limited bigness in its number or its size. The presence of infinity in a field is always something facultative ; it means that we *can* go on stretching sizes and numbers in that field as much and as far as we like, that we can always go outside of any given size or number to the one lying beyond it. Whatever else may be true of this infinite, it is wholly incompatible with totality ; for something to be infinite means that one *never* can have all of it, only more and more of it. For Aristotle as for Quine it simply makes no sense to speak of an infinite magnitude in connection with our whole cosmos ; infinite magnitude applies at best to the *potential* divisibility of a spatial stretch or of a temporal lapse, or to the endless potential augmentation of temporal lapses or numerical aggregates. Even this potential being of the infinite is, however, a rather queer sort of potentiality, for it is not a potentiality that can *ever* be translated into the actual.

The Aristotelian treatment of the infinite is in effect a reductive treatment ; our words suggest that infinity is a straightforward qualification of certain actual magnitudes, but those words only really make sense, if we can give them some complex and less obvious restatement. It is this reductive treatment of the infinite which has, in the main, prevailed in post-Renaissance thought. It prevails in Locke when he speaks of our idea of the infinite as an " endless growing idea," and when he denies that we ever have a positive and distinct idea of infinite magnitude. It prevails in Kant when he circumvents his antinomies by holding that, while there may very well be a *regressus in indefinitum* from one state to previous states, or from a whole to its parts, or from an event to its prior conditions, there cannot ever really be a *regressus in infinitum* which covers all the conditions or all the presuppositions of some actual stage of affairs. It also prevails in Hegel when he cries down his so-called " bad infinite " as the mere would-be negation of the finite, which latter always crops up again and again, and is never truly superseded; it prevails as much when he cries up the virtues of his " true infinite," which is merely his bad infinite grown staid and self-complacent, inasmuch as it has come to realise that there can be no other outcome of its whole vain effort at self-transcendence, but itself and itself alone. Nor is it possible to find any straightforward, whole-hearted espousal of the infinite in either Leibniz or Spinoza. The former may *say* that he believes in an actual infinite, but a man whose main reason for denying the reality of the extended world consists in the fact that it always melts away into parts within parts, and into parts outside parts, can hardly be said to show a very robust faith in the actual infinite. And Spinoza altered the whole meaning of infinity by his identification of it with all-embracing, exhaustive totality, which is also indivisible unity.

The first attempt to give a non-reductive account of the infinite, in harmony with what our untutored verbal instincts might lead us to put forward, is to be found in the work of Bolzano, and, more fully and finally, in the work of Cantor. The latter managed to show that we can talk consistently about infinity, without needing to translate our talk into talk about the variable finite. He disposed effectively of the many apparent contradictions in straightforward accounts of the infinite which had made such translations seem necessary. He made short work of the supposed contradictions involved in exhausting the inexhaustible, or in completing the everlastingly incomplete, for he showed that while it would be self-contradictory to speak of exhausting an infinite series or assemblage, in the sense of finding some last term to it, yet the very fact that we cannot *thus* exhaust it, means also that we can and do exhaust it in another manner. For if we ask what the number may be of the whole set of terms in some series lacking a final term—and it would be highly unnatural to say that such an assemblage had no number at all—we should be forced to say that it was a number different from, and exceeding the number of any set reachable in the ordered running through of such a class or series. It would be different from any such number, for the very reason that no instance of it could be reached in any such ordered running through, and it would exceed any such reachable number in that a group or whole having this new sort of number would contain parts exemplifying all previous ordinary numbers, while none of these latter would contain parts which exemplified it. It is, in short, the number *of* an ordered progression lacking a final term, but it isn't a number anywhere to be found *in* such an ordered progression. It may therefore be said to lie outside of, and beyond the numbers reached in our advance along a progression which lacks a final term, and it may also be called the limit towards which such a series perpetually aspires. But in another sense it does not really lie outside of the whole of such a progressive series for, in having the whole of that series, one automatically *has* an instance of the number in question. One cannot arrive at such a number by the steady stepwise addition of units which Cantor called his *first* principle of the formation of numbers, but one can and does arrive at it by considering *all* the terms reached by such a stepwise procedure, which Cantor called his *second* principle of the formation of numbers.

At the new conceptual level thus attained, the paradoxes attending our former conceptions of the infinite become the truisms stating its essential properties. It becomes plain that an infinite whole can't be increased or reduced by the addition or subtraction of a unit, nor yet of a finite number of such units ; it becomes plain, too, that an infinite whole can have *infinite* parts added to it, or taken away from it, while remaining just as large as ever. The most paradoxical property of an infinite whole, that it is possible to take parts from it, which are just as infinite as itself, and which are therefore (on a natural interpretation) equal to itself, was in fact adopted by Dedekind as the defining property of an infinite assemblage. The work of Cantor took away the awe and mystery of the infinite ; he taught us to do sums with it ; he showed in fact, that there was nothing difficult about it. He himself seemed to enjoy the same sort of hob-nobbing acquaintance with the infinite that some of his Jewish forbears had with the Ancient of Days. And not only did he acquaint us with a single infinite, but he brought into our ken a whole family of infinites, each living above the next on an entirely different floor of the family mansion. In all this he was but developing and giving sense to a notion put forward in a mistaken form by Bolzano : that it is possible for one infinite aggregate to be larger than another.

Hardly, however, had the infinite thus begun to be brought into focus, than the whole picture of it was again blurred. This new blurring was not connected with supposed contradictions in the notion of infinity, but with the whole difficulty of being sure that anything really *was* infinite, or even that we could be clear in our minds what it would be like for something *to be* infinite. Russell, who had done so much to disseminate Cantor's notions, confused them by his extensional account of number, according to which numbers were to be identified with sets of classes which could be brought into one-one correlation with some chosen class. On such an account it became doubtful, not merely whether the notion of the infinite had an application, but even whether it had a distinguishable content. For unless there were at least *one* actual assemblage in our world having infinitely many members, the class of classes, membership of which would be tantamount to being of such an infinite number, would be simply null, and to be infinite in number would be quite the same as being both five and six in number, or as being anything else which has no application whatever. At first Russell tried to prove that there *were* certain infinite assemblages, but most of his proofs involved the lumping together of things best treated as of radically different type or level, and so not properly mentionable in a single breath. We were therefore obliged to *assume* the existence of such infinite classes, in order that the whole ordered, mathematical system could be rounded off and made to work. And Russell cast doubt on our power to carry out certain quite commonplace operations upon infinite aggregates ; we could not find the product of the numbers of such aggregates, unless we could select terms from them according to a definite and discoverable principle, and it wasn't clear that there always would be such a definite principle. And so while the square of the number of some simply infinite aggregate would at times be simply infinite, there would be times when it would have no assignable value whatsoever. There might also, on account of similar selective difficulties, be certain transfinite assemblages concerning which it was impossible to say whether one was or was not larger than the other. And the whole set of possible rearrangements of finite and transfinite series, to which Cantor had assigned numbers of his " second class," had to be thrown back into a conceptual melting pot, since it was quite doubtful whether there always would be rules or principles on which such rearrangements could be carried out.

The difficulties raised by Russell were carried a stage further by the verificationists and the intuitionists. For them the infinite had no meaning at all except where there was a rule or principle guiding us through its labyrinth ; an assemblage such as that of the points of space, or that of all real numbers, to which no definite principle of arrangement corresponded, should not be said to be infinite in number. Nor would they see sense in a mathematical question to which no *general* method of solution corresponded ; a problem that we could solve only by carrying out an infinite number of steps, or by running through an infinity of cases, could not, on their ruling, be mathematically significant. To ask whether there are or are not three successive sevens in the development of π, or whether all numbers of the form $2^{2^{n+9}}+1$ are or are not factorable, is to pose a wholly senseless enquiry, even if, embarrassingly, it suddenly acquires significance when someone hits on an instance which verifies or refutes it. Here we are back, after a long series of unprofitable windings, at the facultative infinite or variable finite of Aristotle ; the infinite can be said to exist only so far as we can apply a rule, or can carry out a procedure, over and over again without let or hindrance ; it never exists as something actual and complete, to which our precise route of approach, whether haphazard or systematic, must be indifferent.

The last stage in this gradual obscuration of the infinite is to be found in the interesting article of Quine and Goodman entitled " Steps toward a Constructive Nominalism." Having tried at first to prove the existence of certain infinite classes, by methods which involved the relaxation of type-restrictions, and having found such methods treacherous and questionable, Quine was led to conceive a general loathing for the infinite, which was part and parcel of his wider disgust for " entities of higher order." For Quine it became supremely dangerous to depart in the smallest degree from a purely syncategorematic or contextual use of predicates and common nouns, so as to make them even *appear* to be the names of peculiar entities. This deep danger existed, to Quine's fine perceptions, even when we merely spoke in *general* fashion about the character of some object, when we said that it was of *a certain* shade of brown, or that it would be beautiful *whatever* its colour. Not only must we never seriously employ abstract nouns as the subjects of sentences ; we must never even make use of variables for which predicates would serve as values. To say that a thing is of *some* character or other, is to say that there *are* characters, or that characters exist, and from such a dangerous flight of hypostatisation one must shrink intransigently. In a style of speech thus puritanically restricted, there can of course be small room for that freely ranging talk about *all* numbers having certain properties, or about *some* number having a certain property, in which classical arithmetic principally consists; a formula like (n) $(n + n = 2n)$ will at best be a set of meaningless marks from which meaningful formulae can be derived by appropriate substitution and translation. There can, in such a linguistic scheme, be little place for the infinite, whose very definition normally involves the mention of *all* finite or ordinary numbers.

There is yet another reason for Quine's thoroughgoing renunciation of the infinite. He takes with extraordinary seriousness all that the physicists have told us about the limited divisibility of natural things and processes, or about the limited extent of all space or all time. In a cosmos severely limited as our own is thought to be, we can, according to Quine, only manage to *talk* about infinite aggregates in so far as we suppose the existence somewhere of an infinite number of abstract objects, in addition to that finite number of concrete objects actually present in our cosmos. Nor can we hope to translate statements about the infinitely large, or even about what is merely *very* large, into long conjunctions, or long sets of statements about their individual components : not only would this take us too long, but we could not actually find *the room* in our world for such statements or sets of statements. Logicians, already so much harassed and badgered, must now submit to yet further restrictions ; their notions must not be so many as to require the symbolic service of more than the total number of objects, or empty spaces in our universe. Nor must their sentences ever be so long as to be incapable of being inscribed within our cosmic boundaries ; there is no place at all for that indefinite concatenation of expressions into ever longer expressions which is both demanded and permitted in classical syntax. It seems plain that, by a long and circuitous route, we have ended up in a finitism which surpasses even that of Aristotle in its narrow rigour.

II.

I have now hurried over the whole historical terrain of thought about the infinite ; I descend into the field to make certain personal observations. Some of these are expressions of my own persuasion or intuition, for which I can indeed give reasons, but never wholly adequate or convincing ones. But if Quine and others can be permitted their intuitions, which may exact from them such important sacrifices as that of all abstract objects and with them the infinite, I too may be allowed my own internal promptings, which may lead me to jettison much accepted doctrine.

The first point that I wish to make is that I think that the work of Cantor, and that of his many developers and elucidators, has effectively removed all the apparent contradictions in our notion of the actual infinite. We are now past the point at which we could find it absurd to speak about the *whole* of some series lacking a final term, or to say that its number exceeds the numbers that can be reached from zero by a stepwise increase by unity. We are also past the point at which we find it shocking to say that a whole can at times be equal to its proper part, or that it may be impossible at times to reduce or increase a whole by the subtraction or addition of a unit. All such statements obviously involve new and stretched uses of the terms "whole," "part," "number," "equal" and so forth, and it would be open to a determined finitist to deny them application beyond the realm of countable, and therefore finite aggregates. But if we do decide so to employ them, the work of Cantor shows that we can do so without any contradiction, and without coming into conflict with our former principles, as long as we allow these latter to apply only to things finite. What we still must show is that this whole new extension and enrichment of our talk is in any way profitable, and that we genuinely have something in mind when we round off our language in this manner.

The second point I wish to make is that I don't think we have the smallest reason for thinking that there *is* some actual set of things in our world which is infinitely numerous. There may be such a set, but again there may not. I find no obscurity in the notion that there should be physically minimal objects and happenings in our world, any more than I find it obscure that there should be minimal portions of *experienced* space and time ; on such suppositions sets of parts of any finite thing or happening will be finite, not infinite in number. I also find no difficulty in believing that our world may have that closed, re-entrant structure, and that consequent finite extent, with which modern physics credits it ; I have, in fact, found it an amusing exercise to determine the optics, and to work out the visual appearances, in a very small world of this kind. But I do not think the question whether the things in our world are finite or infinite in number, is of any philosophical importance whatsoever ; the philosopher has to ask what it would be for things *to be* infinite in number, and what might be the consequences and the possible applications of this notion, not whether it *has* any actual applications. Philosophers who ignore these questions, because they allow themselves to be bemused by the latest findings of the physicists, are in no better case than their predecessors, who let all their analyses be distorted by the findings of the evolutionary biologists.

This brings me to my third obvious, but important point ; that what we mean by infinity, or by any other numerical concept, can never be bound up with the actual contents of our world, nor affected by their numbers, their orderings or the relational bonds that connect them. Being three in number, or being infinitely numerous, would be exactly what they now are, and would carry with them the same body of implied properties, whether our world contained many or few objects, and however these might be connected or varied. This means that there is something radically unsatisfactory and misleading in an extensional account of number on the lines put forward by Russell ; classes or sets may in some sense be the *subjects* of number, but we cannot profitably identify numbers with sets, or with sets of sets of objects that are anywhere present in our world. Though the application of numbers may be to the most purely extensional aspect of things—we may say, with Hegel, to things in their more otherness and mutual externality—yet it is none the less impossible to talk of them satisfactorily except in an intensional idiom. Russell maintained that his extensional treatment had the great merit of making the existence of numbers indubitable ; the class of trios certainly existed if we could but find a single instance of three objects, whereas the property of being three in number was a metaphysical or Platonic entity, which it was hard for us to lay hold of or to track down. He therefore identified each number with what we should ordinarily call the class of its instances, instances not, however, brought together by virtue of showing forth a common kind or character, but solely by virtue of the possibility of pairing their members, one for one, with the members of other similar instances, a procedure which succeeded, surprisingly, in sorting them out into mutually exclusive classes, while yet involving nothing peculiar in the case of each such class. A set of things does not, on such an account, belong to one such class *because* it is of a certain number ; to be of a certain number simply *is* to belong to one such class. But this ingenious analysis has the monstrous consequence that if our world contained no more than 728 objects—and we refused to augment their numbers either by the free manufacture of fictions, or by the introduction of objects from higher levels of discourse—all numbers above 728 would become confounded in a common nothingness, and it would be quite the same to say that certain things were 728, or 1029, or infinite in number. And we could only talk significantly about things having certain high or transfinite numbers, if there *were* things actually exemplifying the numbers in question, a thing which, in the case of the infinite, we could never know or prove. Here we have the old Anselmian situation of essence entailing existence, with the queer added corollary that non-existence entails the annihilation of essence. If this sort of analysis is the only way to steer clear of the difficulties of a metaphysical Platonism, then it is possible to evade such difficulties at too great a price. The same may be said of all those ingenious nominalistic analyses of such simple statements as " There are more dogs than cats," which have been put forward by Quine and Goodman.

The sort of treatment that we call " intensional " will not, however, profit us much, unless the things said about the infinite, or about other numbers taken in intension, differ profoundly from the things said about the infinite, and about other numbers taken in extension, and that not merely in the comparatively trivial respect mentioned by Quine: that there may be more than a single intension corresponding to a given extension. We cannot bring out the full difference between an intensional and a purely extensional treatment of some subject-matter without pointing to *modal* differences ; intensional accounts cover what *could* be the case even in circumstances remote from the actual, whereas extensional treatments confine themselves to what actually is the case. Now it is plain that we should not hesitate to apply any of our notions of number to things not in any way actual ; there were seven recognised sages in antiquity and four recognised cardinal virtues, but there might very well have been four sages and seven virtues. We should even go further and say that *being a recognised sage* would maintain its difference from *being a recognised cardinal virtue* even if there were no recognised sages or virtues, and no things at all which were collectively four or seven. Varying sets of objects will all be reduced to a common nothingness when their constituent objects vanish, but it doesn't make sense to say the same of *being of this sort* or *being of that sort*, or of any other attribute or characterising feature. If we like, in this connection, to make use of the highly confusing term "existence," then we may say that the existence-conditions of things taken in intension differ profoundly from the existence-conditions of things taken in extension; a collection of things may be said to exist when there are at least two or (by a stretch of charity) a single object in that set, whereas a *kind* of thing may be said to exist even when it is only *possible* for there to be things of that sort. It will not, in short, be actual embodiment, nor yet the mere significance of its verbal counterpart, which will constitute existence for an attribute; its existence will consist in the logically possible existence of its instances. Such a manner of speech is, in fact, the one we most frequently adopt; we do not say that a set of salmon-pink objects exist, unless there actually are objects of this colour, whereas we should not hesitate to say that there *was* a shade of colour called salmon-pink, if it were merely possible that there should be objects of that shade. In this sense there is a colour between red and blue, and no colour at all between red and green, even though both can be talked of with equal significance. The only axiom of infinity that can therefore be tolerated in a philosophical account of number is one which defines the significance, and which asserts the possibility, that certain things should (in their collective capacity) be infinite in number; this is the only sense in which the whole series of natural numbers can be said to exist, and it is in this sense, therefore, that there can be transfinite numbers beyond them. All this is in effect acknowledged by Russell when he makes use of his axiom of infinity only as a protasis in other theorems, and never as a proposition independently asserted; his whole approach is, however, such as to belie this limitation.

The intensional treatment we are recommending may be Platonic in that it allows us to speak of "being infinite in number" just as it allows us to speak of "being red in colour," but it need not be Platonic in some dubious or noxious sense. To speak in this manner is not really to suppose that, in addition to the ordinary objects in our world, there also are an indefinite number of extraordinary objects. To be red or to be infinitely numerous are not, properly speaking, entities, nor are the words which express them, properly speaking, names; they are, if one so likes to put it, merely sorts of things, of which things may, in their individual or their collective capacity, be. Their verbal expressions merely tell us what sort of thing we have before us, and whether of the same, or of a different sort from other things. And if we *do* sometimes speak of them as of things enjoying a certain style or manner of existence, we can also make plain that we intend no more by such a style or manner of existence than we intend by the merely possible existence of ordinary things of certain sorts. We may here advert to Quine's queer doctrine that to speak *generally* of the character of some object is to commit ourselves to an unwarrantable ontology, that quantification necessarily goes hand in hand with reification. Because the statement "Some men are amorous" can, with a certain amount of creaking, be transformed into the statement "There are amorous men," it is therefore assumed that any and every use of "some," "every," "a certain" "any" and so forth, must necessarily involve an assumption of existence, and that not in an innocent and translatable, but in some dubious and noxious sense. There is only confusion in this doctrine. A man who says that something is of some colour or other is not reifying colours, any more than a man who says that he will find his way *somehow* to London need be reifying manners. Nor is one practising reification if one defines *being infinite in number* in terms of having proper parts which are of every natural number, and if one then goes on to define the latter in some form of the accepted rigmarole which amounts to saying that they are the total progeny which can be generated out of *being nought in number*, through repeated fecundation of this notion and its offspring, by the relation of *being one more in number*.

We may note further that not only are the phrases "being three in number" or "being infinitely numerous" not necessarily to be taken as names of peculiar entities; it is not even necessary to take them as *applying* to such entities. We may here repudiate with peculiar vigour Frege's doctrine that it is to class-concepts alone that numerical predicates can be properly said to pertain, that it is *being a recognised sage in antiquity* that is seven in number, and not Thales and his confrères. This view is mistaken for the double reason that, if there is a sense in which *being an ancient sage* can have number, then it will be one in number and not seven, and also because the exemplification of number by a set of things does not at all depend upon their being of a common sort or character. Of *some* sort each of a number of things must undoubtedly be, since it is only as exemplifying a sort that a thing can be said to be a thing at all, but it is by no means necessary that the things which collectively are of a given number should be homogeneous rather than heterogeneous. My arm, your toothbrush and Quavam es Sultaneh are three in number, and they are as much three in number as are the three Norns or the three Persons of the Trinity, but it is mere artificiality to demand that they should therefore be reduced to some common denomination. And it is merely *our* difficulty that we cannot refer to things having certain large or transfinite numbers, except as being all the cases of a common sort of kind. We may note, further, that we have no need to say that it is to classes or sets that numbers pertain, if by "classes" or "sets" are meant, not things in the plural, but mysterious higher-order compartments into which things may be herded or concentrated. Quite obviously it is to *things in the plural* that numbers are normally applicable, even if, in our charity, and for verbal convenience, we also extend this privilege to things taken in the singular, or even to nothing whatever. It is not my pen singly, nor your arm singly, nor Quavam es Sultaneh singly, that is three in number, nor yet any higher order unity that these objects may form. They are so *collectively* only in the sense that they are so co-operatively; we must not, except for mere verbal convenience, confuse the collective possession of a property by things in the plural, with the non-collective possession of a property by some single entity called a collection. In uttering these forgotten truisms I am not, of course saying that, in addition to being many, things may not also be multiple many, or multiply multiply many, and so on indefinitely, and that, from the standpoint of such multiplied manifoldness, what is simply manifold may not rightly be treated as unitary. Nor am I casting scorn on any of the convenient, if misleading, devices and distinctions of the logic of classes, for which my respect is immeasurable. I would not, further, wish to deny that there is a legitimate higher-order application of numbers to *sorts of things*, and to *pluralities of things*, and *to sorts of pluralities* and *pluralities of sorts*, and to any complication of sorts and pluralities one might care to elaborate. I am only denying that, in all this collective and abstract treatment of things abstractly and collectively treated, and in all the resultant piling of adverb upon prior adverb, our talk ever loses its ultimate touch with ordinary things (whether actual or possible), however cumbrous it might sometimes be to express this.

There are, no doubt, certain perplexing peculiarities in the grammer of numerical statements—the *joint* ownership of numerical predicates is particularly teasing—which suggest a recourse to further reductions. We find it illuminating to hold that numerical predicates " aren't really predicates at all," any more than existence " really " is a predicate, or diversity " really " is a relation between objects. To say that something of a certain sort exists is not " really " to say that this object is of a certain other sort ; it is merely to say that one *has* an object of the first sort in question. In the same way to say that an object and an object are diverse, is not " really " to point to a relation between them, but merely to say that one has objects, as distinct from *an* object, to deal with. In much the same way to attribute numbers to objects is, in a sense, merely to give an exact specification of diversity, to indicate with precision *what* objects we have before us for characterization, without saying *anything* about their kind or character. Quite plainly to say that one has four things before one, is not different from saying that one has something of an unspecified sort before one, and something else of an unspecified sort before one, and something else of an unspecified sort before one, and yet again something else of an unspecified sort before one. And if one possessed a set of coordinated pens, such as those with which it was fabled President Roosevelt used to sign cheques under the New Deal, and if the dimensions of our universe allowed such a set of pens to be infinite, rather than finite in number, then it would be perfectly easy to write down a specification of diversity which would amount to the attribution of a transfinite number. All this is brought out clearly in Tarski's illuminating notation, where we express number by special variants of the existential operator, under whose inverted *E* various numerical subscripts are written. But even if numbers are thereby shown " not really " to be properties, any more than they really can be considered as classes, our treatment of them remains intensional, since their " existence " will amount to no more than the mere possibility of a certain specified diversity, of which their " instances " will be actual realizations.

We have strayed long and far into the general philosophy of number, a procedure necessitated by the technical barnacles which have been allowed to encrust every inch of the subject ; we may now return to the narrower limits of infinity. We have disposed of the general, threshold difficulties which concern the notion, but we have yet to find sufficient motive for framing it, or an adequate guarantee that we can attach more than an empty, syntactical meaning to the terms that seem to stand for it. We have said that the only sense in which the existence of infinite numbers can be philosophically significant, is the sense in which it is logically *possible* for things to be infinitely numerous, in other words, the sense in which it is logically possible for them to be of a number greater than, and different from any that can be elicited out of the directly showable numbers, through repeated use of the notion "greater by unity." We here come face to face with a doctrine that has acquired much recent authority ; that modal distinctions merely reflect arbitrary linguistic choices, and that it is *we* who, by determining what we will, or won't, or may say in certain circumstances, also fix the bounds of the necessary and the possible. On such a view it would be good to replace all ordinary, first-order talk about what might or must or couldn't be the case, by a corresponding metalinguistic set of verbal prescriptions, permissions and prohibitions ; our straightforward talk about objects could then be left in extensional purity. On this view it would merely depend on a linguistic fiat whether it was or was not possible for there to be infinitely many objects. This whole approach to modality may, however, be described as unrealistic, if not frivolous. It ignores the fact that our decisions to speak in one way rather than another are by no means arbitrary, but profoundly motivated, and that they depend not merely on personal habit or inclination, but on our deep intercourse with, and repeated turning to, the matter on hand. We may, if we like, say that there is a " lie on the land " in the realm of essence, just as there undoubtedly is a " lie of the land " in the realm of existence ; this " lie of the land " never *forces* us to do linguistic road-making or bridge-building in a given manner, but it none the less makes it easier and more " natural " to proceed in one manner rather than another. It is not unlike Hume's " gentle force " of association, which while it does not compel us to say one thing to the exclusion of others, none the less sets bounds to our linguistic liberty, and in the end " everywhere prevails." Applying these thoughts in the field of numbers, it would no doubt have been possible for us to have called a halt in our formation of numerical notions after the number Ten (as Plato and the Pythagoreans are reported to have done) ; we could then have refused to predicate number of the Apostles, or the States of the American Union. Such a decision would, however, be arbitrary in a vicious sense ; it would have involved a wanton refusal, not based on differences in the material on hand, to carry on with a certain general procedure. We are subject to a rational obligation, not at all minatory and coercive, but insistently, if mildly hortatory, which urges us always to leave room in our thought and language for sets of things exemplifying an unending series of natural numbers, each arising out of its predecessor when an aggregate is increased by a unit. And being obliged to admit all these natural numbers, we are also obliged to admit the possibility of aggregates which have parts such that each of the natural numbers will be exemplified in some of them, and which can't therefore themselves be of any of the ordinary natural numbers. And since it would be highly unnatural to say that they weren't of any number at all, we are urged to say that they are of a number different from, and greater than, any of the natural numbers. We might indeed come to admit the existence of such transfinite numbers by mere reflection on the existence of the ordinary natural number series ; this latter series exists in the only sense in which numbers can be said to exist at all, and hence the former also exists, even if at a higher level of discourse, as the number *of* the latter. We may then be led, by precisely similar considerations, to concede numbers to aggregates of aggregates, or to aggregates of ordered aggregates, much more intricately organized than are simple progressions ; we shall then also be led, by Cantor's irrefragable arguments, to accord other, higher kinds of infinite number to such aggregates. In all this process of extending our notions, we might stop short where we wished, but it would be highly unnatural to stop short anywhere. We were, in a sense, committed to the whole indefinitely ascending hierarchy, on the occasion when we first passed in thought or experience from a unit to a couple, or from a couple to a triad. Those who, like Quine, seek to check the natural increase of the numerical population so as never to exceed the resources of our actual universe, commit a fault worse than that strangulation of births condemned by the Church ; they commit the fault practised by all unjust judges and idolaters since the beginning of the world : that of undue deference to the powers that be.

We have not, however, given a satisfactory justification of our introduction of the infinite, as long as it merely serves as some great gilt cupola rounding off a conceptual edifice, not itself made of solid materials nor resting on solid underpinning, so as to permit of circumambulation or closer examination. The standing objection to the infinite is, after all, that it is impossible to produce an instance of it, as one can very well produce instances of lesser numbers. And even if such an instance were forthcoming, we could never be sure that it *was* infinite ; we should need all time to correlate its members, one for one with the finite inductive numerals, and should therefore, have neither occasion nor need to speak of a number lying beyond all natural numbers. These considerations, as well as a large number of fallacious ones, have led philosophers in all ages to attempt reductive analyses of the infinite in terms of the variable finite, and to condemn other ways of speaking as metaphysical *flatus vocis*. Against all such tendencies and difficulties I should like to take a stand ; I want to maintain, on grounds that are themselves intuitive, that though the infinite may be for us no more than a concept, it might also very well have been an intuition, that it is, in fact, no more than an accidental infirmity that we have to grope and gesture after it as we actually do. I also wish to maintain that our difficulty in exhibiting the infinite isn't really the difficulty of exhibiting something quite unlike anything we have ever seen and known ; in a sense, its exhibition would involve nothing novel, and we know exactly what it would be like. It is, in fact, no more inaccessible to imagination or sense-perception than any other highly complex object or property. We may here point out that not all instances of number are known for what they are by that step-by-step procedure known as counting ; there are many lower degrees of number that can be recognized and exhibited *non*-successively. We can see at a glance, and can plainly recognize, such inferior grades of number as unity, duality, triplicity and so forth ; shepherds, company commanders and other practised persons carry this sort of immediate discrimination much further. We can also sometimes see at a glance how a certain quartet is made up out of two couples, or how a group of eight consists of parts which are respectively five and three in number. What then is the fundamental difficulty in supposing that such a non-counting, non-successive apprehension of number might not be indefinitely extended, so that one might see at a glance how each of the natural numbers was exemplified in a certain part of an assemblage before us, and could therefore also recognize at a glance the presence of that first simple infinity which sums up them all ? I can see no difficulty in the supposition. It is plain, further, that, even where our apprehension of number *is* successive, it may none the less fall wholly within the present (which won't deserve any qualification of "specious," since it is the only present we can understand or know). What then is the difficulty in supposing that our present faculties of time-discrimination might be so indefinitely refined and sharpened, as to take in, within the limits of the present, one of those infinitely numerous, indefinitely diminishing Zenonian series, to which our present poor capacities give an air of paradox and absurdity ? And what further difficulty could there be in supposing that our sense of what is actual and present might not be so infinitely extended, as *never* to break up into any succession of disjoined phases, only linked to each other by memory or anticipation ? If this were the case, we might certainly enjoy an unending counting apprehension, which would also be, throughout its *undivided* extent, an apprehension of the actual infinite. If anyone doubts whether I really know what I mean by the alternatives I am putting forward, I confess I can't convince him that I do know it ; I am in the position of Hume when he tells us that he knows what a certain intermediate shade of grey is like, of which he cannot produce an instance.

I shall conclude my contribution by saying that I see no difficulty, apart from the purely human difficulties I have mentioned, in the making of an infinite number of arbitrary selections from some aggregate before us, or in the carrying out of an infinite set of arbitrary pairings among the terms of two such aggregates. There can therefore be no reason, apart from human infirmity, why we should not be able to refer to some infinite set of objects except as being of a common sort or kind, or why we should not be able to predicate equality between two such aggregates except by virtue of a specifiable one-one relation among their members. Nor is there any but an accidental human reason why we should not be able to order a transfinite aggregate except by means of a specifiable serial relation. The bearing of all this on the Theorem of Zermelo, on the Multiplicative Axiom, and on the equality or inequality of certain transfinite numbers, may be left to others more competent to determine. But I see no reason why an infinite aggregate, arranged as are the points in the continuum, or consisting of all the series belonging to Cantor's second class, should not, like some simply infinite progression, become a direct object of intuitive apprehension. I also have no reason to doubt that it might be perfectly possible to carry out in a single flash of vision, some calculation involving a transfinite number of steps, and that it would then be possible to write the result down by means of an infinite series of simultaneously functioning pens. It would then be quite definite whether there were or were not three successive sevens in the development of π, and the Law of Excluded Middle would apply in this field without any restriction whatever.

II.—*By* C. Lewy.

Professor Findlay's paper is concerned with a large number of questions, and I cannot hope to discuss them all ; I shall therefore select a few which seem to me to be important and shall try to say something about them. I must emphasize, however, that I find the whole subject extremely difficult, and that I do not feel at all certain of the truth of my remarks.

I.

I shall not comment in any detail on the first, largely historical, section of Findlay's paper, although I find some of his statements very puzzling. For instance, to mention only one point, he says that Russell "cast doubt on our power to carry out certain quite commonplace operations upon infinite aggregates," and seems to imply that Russell was somehow responsible for the difficulties associated with the axiom of choice. But the truth is surely that certain mathematical proofs involved tacit application of the principle, or assumption, that given a class K whose members are mutually exclusive classes none of which is empty, there exists a class which has exactly one member in common with each member of K[1] ; and, as Russell says, it was Zermelo's merit to have made the assumption explicit.[2] Now, this principle, which is equivalent to the principle that every class can be well-ordered, has been challenged (1) on the ground that it does not seem to be logically necessary and hence should not be accepted as a logical axiom, and (2) on the ground that it is "non-effective." I gather from Findlay's paper that he attaches no weight to the second objection ; but he does not discuss the first objection and I do not know what he would say about it. In any case, my point is that it is very misleading on Findlay's part to talk as if the realization of the fact that the axiom of choice is involved in certain proofs in

[1] Cf., *e.g.*, B. Russell, " On Some Difficulties in the Theory of Transfinite Numbers and Order Types," *Proc. of the London Mathematical Soc.*, 2nd series, vol. 4 (1906), pp. 29–53. Cf. also *Introduction to Mathematical Philosophy* (London, 1919, 2nd ed., 1920), chapter XII.

[2] Zermelo's original axiom was different but equivalent.

the theory of sets, and hence of the fact that these proofs are invalid if the axiom be rejected, were a stage in the "gradual obscuration of the infinite."

Another point which I wish to mention briefly concerns Findlay's criticism of Frege. Findlay says that he wishes to repudiate with "peculiar vigour" Frege's doctrine that it is to class-concepts alone that numerical predicates can be properly said to pertain. But in fact Findlay's criticism of Frege's doctrine, far from being "peculiarly vigorous," seems to me to be peculiarly weak. Findlay says that it is "quite obviously" to things in the plural that numbers are normally applicable ; but what does he mean here by "things in the plural"? What does it mean to suppose, for instance, that I am attributing the number four to "things in the plural" when I say "The number of armchairs in my study is four"? The only meaning I can attach to the phrase is that, when I make the statement, I am attributing the number four to a certain physical collection of objects ; as Frege, however, points out, a number cannot be uniquely attributed to a physical collection. To use one of his examples,[3] we can say with equal truth "Here are four companies," and "Here are 500 men." And the reason why both these propositions are true is precisely that a numerical statement contains an assertion about a concept. I can find nothing in Findlay's arguments which invalidates this doctrine or which gives an answer to the question as to how, if a numerical statement is *not* about a concept, we can truly attribute different numbers to the same physical collection. My arm, your toothbrush and Quavam es Sultaneh can be anything in number unless reference to a concept is specified ; and I think it is entirely wrong to say that if there is a sense in which "being an ancient age" can have number, it will be one in number and not seven. Although we do not normally say that the concept "being an ancient sage" has the number seven, Frege clearly explains what he means by this, I fully understand the explanation, and so, I feel sure, does Findlay.

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[3] *The Foundations of Arithmetic*, tr. by J. L. Austin (Oxford, 1950), p. 59[e].

Let us turn, however, to what is, after all, the subject of our symposium, namely, to the concept of the infinite. What exactly is it that Findlay wishes to say about it? Although I have read his paper several times, I am not at all clear what his views actually are. He says, referring to the work of the verificationists and the intuitionists, " Here we are back, after a long series of unprofitable windings, at the facultative infinite or variable finite of Aristotle ; the infinite can be said to exist only so far as we can apply a rule, or carry out a procedure, over and over again without let or hindrance ; it never exists as something actual and complete. . . ." It would seem that Findlay wishes to maintain that the infinite does exist as something " actual and complete " ; but I can find in his paper neither any clear statement as to what the words " actual " and " complete " here mean, nor any clear argument in support of his claim. The only part of his paper which seems to me to bear on the problem is the last one in which he points out that we can sometimes see at a glance how a certain quartet is made up of two couples, or how a given group of eight consists of parts which are respectively five and three in number, and then asks " What . . . is the fundamental difficulty in supposing that such a non-counting, non-successive apprehension of number might not be indefinitely extended, so that one might see at a glance how each of the natural numbers was exemplified in a certain part of an assemblage before us, and could therefore also recognize at a glance the presence of that first simple infinity which sums up them all?" He answers his own question by saying that he sees no difficulty in the supposition. I, on the other hand, see a very great difficulty in it, which is as follows. Findlay talks here as if an infinite series, say the series of natural numbers, were a series in the same sense of the word " series " as that in which a finite series is a series. And he talks as if aleph-nought were the sum of the series of natural numbers in the same sense of the word " sum " as that in which a finite number may be said to be the sum of other finite numbers. But this is not so ; aleph-nought is the *limit* of the series of natural numbers ; and a limit is not a sum in the sense in which we use the word " sum " in connexion with *finite* series. I think, therefore, that Findlay's analogy between seeing at a glance how a quartet is made up of two couples and how aleph-nought is " made up " of the natural numbers which it " sums up " simply breaks down.

Let me try to put the matter in a different way. Findlay claims (and this seems to me to be the core of his paper so far as the problem of infinity is concerned) that it is logically possible that an infinite series should become a direct object of intuitive apprehension. But this supposition makes no sense unless it makes sense to speak of an infinite series as something " complete " ; now Findlay realizes, of course, that an infinite series cannot (logically) be " complete " in the sense in which to say that a series is complete entails that it has a last member. So he must attach to the word " complete," when he talks of the infinite as being actual and complete, some other meaning. But what is that meaning ? I am afraid I do not see that Findlay has done anything towards explaining it ; he may perhaps say that he is so using the word that *if* it is logically possible for an infinite series to become a direct object of intuitive apprehension, it will follow that the infinite is not a name for a rule but exists as something actual and complete. But he has neither given nor attempted to give any argument for thinking that the supposition *is* logically possible. Those who maintain that the infinite is a name for a rule would of course say that Findlay's supposition is meaningless. I think he has certainly failed to explain its meaning (if it has any), and I can't therefore see that he has done much to advance our understanding of the subject.

II.

I should like now to discuss what seems to me to be one of the main difficulties connected with our problem. G. H. Hardy has said " Pure mathematics . . . seems to me a rock on which all idealism founders ; 317 is a prime, not because we think so, or because our minds are shaped in one way rather than another, but *because it is so*, because mathematical reality is built that way."[4] And also " I believe that mathematical reality lies outside us, that our function is to discover or *observe* it, and that the theorems which we prove, and which we describe grandiloquently as our 'creations,' are simply our notes of our observations."[5] I believe this idea is one of the main sources of the temptation to talk about the infinite in the sort of way that Findlay does. Let us take an example which arises naturally out of the last paragraph of Findlay's paper. Suppose Findlay says " Surely, the continuum hypothesis is either true or false." What can this mean ? It may mean that the continuum hypothesis is *decidable* : in other words, that the continuum hypothesis can be either proved or disproved in an axiomatic system (Zermelo's or a similar one) of set theory. Gödel has shown, however, that the continuum hypothesis is consistent with the axioms of set theory (if they are consistent) ; that is to say, that the hypothesis cannot be disproved in a system based on those axioms.[6] This still leaves open the possibility that the hypothesis can be proved in the system ; and if it were proved, we could say that it was true, and hence of course either true or false. But let us suppose that it has also been shown that the continuum hypothesis cannot be proved on the basis of the existing axioms (though this, so far as I know, has not been shown). In this case, one might construct alternative systems of set theory, and the position would be similar to that of alternative systems of geometry. I do not see what could be meant, in such circumstances, by asking whether the continuum hypothesis was true or false, or saying that it must be either true or false, any more than I can see what could be meant by asking whether the parallel postulate was true or false. Perhaps I am wrong in drawing this analogy ; if so, I hope I shall be corrected ; but it seems to me that, in the circumstances I am imagin-

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[4] *A Mathematician's Apology* (Cambridge, 1940), p. 70.
[5] Op. cit., pp. 63–64.
[6] K. Gödel, *The Consistency of the Axiom of Choice and of the Generalized Continuum-Hypothesis with the Axioms of Set Theory* (Princeton, 1940).

ing, the analogy would be quite close. Does Findlay wish to maintain that the question would make sense?

It may however be said[7] that even if the continuum hypothesis were shown to be undecidable, it would still be either true or false, and one may support the statement by pointing out that it is conceivable that a new set of axioms for set theory should be constructed, in which the continuum hypothesis might be provable (or disprovable). Let us suppose that such a new system of axioms has been constructed; and let us suppose that in this new axiomatic system a formula which we say expresses the continuum hypothesis has been disproved. How should we interpret such a result? It seems to me that in this case one would be presented with a new mathematical theory, different from the existing set theory, and defining a new concept of "set." In this new theory all sorts of theorems which have no meaning in the existing theory might be provable. On the other hand, there would be a large number of analogies between the new theory and the present one, and it is in virtue of those analogies that we should call the new mathematical system a "new theory of sets." Now the fact that in such a new theory what I may call the "continuum hypothesis formula" was disprovable, would not, so far as I can see, entitle us to say that Cantor's conjecture has been shown to be false; for in the new theory the formula would have a different meaning. I realize that this will not convince Findlay since he seems to think that a mathematical formula has a well-determined meaning quite independently of any calculus to which it belongs, and that in fact the problem of the continuum is logically capable of solution by "intuition." As I have already pointed out, however, he has given no arguments for his views; and if I am right in what I have said above, it follows that the supposition that one might "see at a glance" whether the continuum hypothesis is true or false, is devoid of meaning.

[7] Cf. K. Gödel, "What is Cantor's Continuum Problem?" *The American Mathematical Monthly* vol. 54 (1947), pp. 515–525. Cf. also K. Gödel, "Russell's Mathematical Logic," *The Philosophy of Bertrand Russell* ed. by P. A. Schilpp (Evanston and Chicago, 1944), pp. 125–153.

Hardy and Gödel talk of a "mathematical reality" which mathematical theorems "describe," and seem to *infer* from this that a mathematical theorem must be true or false even if it is undecidable in any existing calculus. But is this an inference? Does the claim that there is a mathematical reality which is described by mathematical theorems amount to anything more than the claim that every well-formed mathematical sentence must express something true or false? So far as I can see, one is adding nothing to the latter claim, and one is not justifying it, by bringing in the reference to a "mathematical reality." And I can see no good reason for supposing the claim to be true.

At this point, however, there is a danger of confusion. I am not maintaining that a mathematical formula cannot be said to express something true or false merely because it has not, as a matter of fact, been proved or disproved: I am talking only of undecidable formulas. It seems to me incorrect to suppose that just because a formula has not yet been proved (or disproved), it does not express something true or false, and that it only comes to do so when someone happens to prove it (or to disprove it). But the case of undecidable formulas is different. For in the former case by asking whether the formula expresses something true or false, I am asking whether it is provable or disprovable in a certain mathematical system. This I cannot, *ex hypothesi*, be asking in the latter case; and there is no clear meaning I can attach to the question. Now, the following objection may be made to this view. It may be said that in order to show that a formula is undecidable one must first understand it, and hence that it must have meaning; and that it cannot cease to have meaning by being proved to be undecidable. I do not see, however, that the reference to "meaning" and "understanding" is helpful in this connexion. Of course, there is a sense in which I understand an undecidable mathematical formula, and in which I do *not* understand a collection of nonsense words. But from the fact that I understand it in *some* sense, it does not seem to me to follow that the formula has meaning in the sense of expressing something true or false.

III.

There is another point arising out of Findlay's paper which has puzzled me a good deal, and which I should like to mention. By claiming that there is no logical impossibility in the supposition that one's "non-counting, non-successive apprehension of number" should be so extended that one would recognize at a glance the presence of Cantor's "first simple infinity," Findlay clearly implies that there is no logical impossibility in the supposition that one should be able to see at a glance whether, for instance, a sheet of paper in front of one contains a million and eleven dots or a million and twelve. For if the latter supposition is logically impossible, it seems to follow that the former is also logically impossible. (Although, of course, the latter supposition may be logically possible without the former being logically possible.) But is it logically possible that one should see at a glance whether there are a million and eleven dots on a sheet of paper or a million and twelve? Let us imagine that whenever Findlay says, just after glancing on a sheet of paper, "There are a million and eleven dots here," and we then count the dots, we find him to be right. We might naturally describe the situation by saying "Findlay can recognize at a glance the presence of a million and eleven dots." If by saying that it is logically possible that a man's non-counting, non-successive discrimination of number should be extended up to a million, or two million, one means that we can imagine cases, like the one given above, which we might naturally describe by using some such words, then the supposition is logically possible. But is this what Findlay means? I hardly think so for such cases could also be described by saying that a man always *guesses correctly* the number of dots which he sees, say up to two million. Findlay would probably say that what he means is that it is logically possible for a man to recognize at a glance the presence of a million and eleven dots *in the same way* in which one normally recognizes at a glance the presence of three dots. But what exactly does the expression "in the same way" mean?

III.—*By* S. Körner.

I propose to divide my contribution into two parts. In part I, I shall, within the limits set by the previous papers, sketch my own view on the topic of this symposium. In part II, I shall deal specifically with the points made by Professor Findlay and Dr. Lewy. By proceeding in this manner I hope to do justice to their arguments without undue repetition and to avoid the appearance of deliberate quibbling. It would be practically impossible to treat of all the issues raised by my predecessors in the discusssion and I have therefore selected those on which their disagreement seems strongest.

I.

1. *Some necessary preliminaries.*—To clarify the role of a term (a statement-part which itself neither is nor contains a statement) is to exhibit more or less completely its logical content, *i.e.*, the deducibility-relations, if any, which it bears to other terms ; its range of applicability or reference, *i.e.*, the particulars, if any, to which it refers ; and, lastly, its interconnection with other terms, *i.e.*, the relations *other than logical deducibility* which it bears to them. The distinction between the logical content, the reference, and the interconnection of terms is in one form or another generally recognised and is at any rate easily understood. It is clear that every term has all sorts of interconnections with other terms and that not all of them are of equal philosophical interest. It is similarly clear that predicates are the only terms which have logical content. To prepare the ground for our main undertaking it will be useful to consider briefly and quite generally first the differences in the applicability to their instances, between empirical and non-empirical predicates ; and secondly a fairly frequent type of interconnection of empirical and non-empirical predicates

With regard to some predicates it is logically possible that they should apply to particulars which are located in time or space or both and are accessible to perception. I shall call such predicates, the particulars to which they apply, and the reference of the former to the latter " empirical." The best examples of empirical predicates are ostensively definable or, briefly, ostensive predicates : for to give an ostensive definition is to indicate empirical particulars.

Although non-empirical predicates such as " mathematical point " or " mathematical circle " do not refer to empirical particulars, they may nevertheless by postulation be provided with a non-empirical reference. A reason for doing this may, for example, be a need to speak not only of intersecting physical circles but also of intersecting mathematical circles. It is important to notice that the reference of " mathematical circle " or any non-empirical predicate, unlike the reference of " physical circle " or any empirical predicate, is *a part of its logical content.* In other words, whether " being a mathematical circle " does or does not have a reference depends on whether it entails or does not entail " having a reference " (to non-empirical or postulated particulars of some more or less specified sort).

An empirical and a non-empirical predicate, while differing fundamentally in their reference, may yet be similar enough in their logical content to permit their being used interchangeably in certain contexts. The predicates " physical circle " and " mathematical circle " are a case in point. Even without going more deeply into the nature of their resemblance we find that the two predicates have a closely similar logical content owing to which the results of reasoning involving " mathematical circle " can often profitably be used in the characterisation of physical circles. We might describe this interconnection between the two predicates by saying that the predicate " mathematical circle," although without empirical reference, nevertheless applies, as it were, *by proxy* or *via* the predicate " physical circle " to empirical particulars and thus acquires a quasi-empirical reference.

It is easy to confuse (i) empirical reference (*e.g.*, of " physical circle " to empirical particulars) with (ii) non-empirical reference (*e.g.*, of " mathematical circle " to postulated particulars) or with (iii) quasi-empirical reference (*e.g.*, of " mathematical circle " *via* " physical circle " to empirical particulars). To the clarification of the notion of infinity these confusions are fatal.

2. *Empirical and non-empirical aggregates and natural numbers.*—The words " one thing," " couple," " triad," etc., are often used as empirical predicates, *i.e.*, as referring directly to what is or can be perceived in space or time. Indeed what is meant by these empirical aggregate-notions can, and perhaps must, be explained with the help of ostensive definitions. The same is true of the notions of adding one empirical aggregate to another and of the resulting emergence of a new such aggregate.

It is necessary to distinguish between empirical and non-empirical aggregate-notions. In order to do this we note the following two points about the empirical aggregate-notions. *First*, their reference, like that of all empirical predicates, is imprecise in the sense that with respect to some particulars we may not be able to decide whether or not they are instances of a certain empirical aggregate-notion. For example the rules governing " couple " and " triad " in their empirical use, or the rules governing, as I shall say, " empirical couple " and " empirical triad," may not permit us to decide whether something we happen to perceive on a wet window-pane is an empirical couple or an empirical triad. The rules may indeed permit us to say quite properly that it is rather more of a couple than of a triad. This has the corollary that we cannot properly speak of the notion of an empirical aggregate as having an extension. When logicians and mathematicians speak of the extension of a predicate they imply that everything quite unambiguously either is or is not characterised by the predicate. In this fundamental sense of

" extension " empirical predicates and therefore empirical aggregate-notions have an empirical reference but have no extension.

The *second* point concerns the logical content of empirical aggregate-notions and may be made by means of an example. If we observe the physical addition of an empirical triad to an empirical couple, an empirical quintet may, but does not necessarily, emerge. In other words, " being the immediate result of the physical addition of an empirical triad to an empirical couple " does not entail " being the emergence of an empirical quintet " but overlaps with it.

Now, it is for many purposes convenient to ignore the overlapping cases and even to define new notions of couple, triad, etc., which are like the empirical ones except that, *e.g.*, " being the result of the addition of a triad to a couple " does not overlap with but entails " being the emergence of a quintet." More generally while the successive physical addition of an empirical unit to an empirical unit does not necessarily yield the sequence " empirical unit," " empirical couple," *etc.*, but may be full of surprises, we can through modifying the logical content of these predicates arrive at the well-behaved sequence " non-empirical unit," " non-empirical couple," etc. (where the meaning of " etc." is not yet clearly determined). By modifying the logical content of empirical aggregate-notions in this manner we define non-empirical aggregate-notions which, roughly speaking, are idealisations of the empirical ones but as such have lost their empirical reference. " Being the successor of a non-empirical couple " entails " being a non-empirical triad," but does not refer to what we perceive on window-panes or anywhere else.

Whether our non-empirical aggregate-notions already deserve the name of number-predicates or whether for that purpose further changes in their logical content are necessary, is a question of little importance. We shall be approaching more closely to the notion of natural number if we provide the non-empirical aggregate-notions, by postulation, with a non-empirical reference. This, as we have seen (section 1) again involves only a change of logical content. More precisely, we replace "non-empirical couple," etc., which so far do not entail "having a non-empirical reference" by otherwise similar predicates for which this entailment does hold

The non-empirical reference of, *e.g.*, "non-empirical couple" may, of course, be specified in many ways. We may in particular postulate that the non-empirical particulars to which the predicate refers are in some way distinguishable or that they are not distinguishable. In the latter case the predicate would refer to a single individual. (This follows from the principle of the identity of indiscernibles which for *non-empirical* particulars is, I believe, generally accepted.)

3. *On the infinite totality of natural numbers.*—The non-empirical aggregate-notions and the notion of natural number are thus not abstractions from experience, as are the notions of a green patch or a physical circle. They are idealisations of experience or, more precisely, the result of modifying the logical content of empirical predicates. They are not "made by God" but "the work of man." *If* it is a sin to transcend experience by the use of non-empirical predicates then it has already been committed by postulating that aggregate-notions have a precise reference and that the sequence generated by the successive addition of units be free of surprises. Objections against further modifications in the logical content of number-predicates on the ground that they are thereby deprived of their intuitively clear reference can thus have little force. This is in particular true of objections to the introduction of various notions of infinity into mathematics.

The statement that physical addition of an empirical couple to an empirical triad yields a physical quintet is an empirical statement. So are the statement that the process of forming successively greater physical aggregates by successive physical addition can be completed, and the statement that this process cannot be completed (*e.g.*, because we have not " world enough, and time " or because some more elaborate physical theory happens to be true). It follows that the statement to the effect that the process is incompletable and that it is completed is an internally inconsistent conjunction of two incompatible empirical statements.

On the other hand, in speaking of a successive non-empirical addition of non-empirical aggregates, of the completability and completion of this process we are no longer making empirical statements. We are using the words " successive," " completable," " completed " metaphorically for non-empirical predicates. Indeed, the successive addition of a non-empirical unit to a non-empirical unit takes no time or does not take place in time since only empirical particulars are located in time. The generation of the sequence of natural numbers by successive addition may thus be both incompletable in the sense that no natural number is the greatest and yet in some other sense of the term complete. Whether or not this is so depends on the logical content and in particular on the non-empirical reference of " natural number."

By postulating different kinds of non-empirical reference we may endow the words " actually and completely given infinite extension of ' natural number ' " with different and precise meanings. Thus we may postulate that " natural number " has an infinite and complete extension in the sense that with respect to every (or almost every) property any natural number either does or does not possess it. We may, moreover, postulate that with the set of all natural numbers the set of all its sub-sets is given ; and even that the last-mentioned set can be well ordered. Such postulations, their logical consequences and mutual compatibility are the proper concern of mathematicians. Although the philosopher has to deal with different questions concerning number and infinity he does well to remember that in defining " natural number " as having one or another kind of infinite extension one modifies the logical content not of an empirical predicate but of a non-empirical one.

4. *Non-empirical continua and other notions of infinity.*— Other notions of infinity which occur within and outside mathematics can be tackled on the lines of the preceding discussion: we compare the non-empirical predicates having infinite extensions with empirical predicates of which they are idealisations and consider the nature of their postulated reference. If, as is convenient but not essential, we compare these predicates in terms of a step-by-step modification of logical content, then we must note that different routes leading from empirical predicates to their non-empirical counterparts are possible.

In order to clarify the notion of a line consisting of an infinite number of points we must distinguish between an empirical and a non-empirical notion of line. The distinction need not be worked out in any detail. It is analogous to that between empirical and non-empirical aggregate-notions and almost generally accepted.

To speak of a successive physical division of an empirical line as both incompletable and complete is to be guilty of a contradiction in terms. The division of a non-empirical line may, however, in a metaphorical sense of the terms, be both incompletable and complete. Whether or not this is so depends on the logical content of "non-empirical line," "non-empirical division," etc., and in particular on the sort of extension which is postulated for those predicates. In this context it may be worth pointing out that Brouwer's "incompletable set of parts of a continuous line" as a "medium of free becoming" is not an empirical predicate but at best a less radical idealisation than the notion of a line according to the classical theory of sets. It is not necessary here to consider the relations between various conceptions of natural number on the one hand and of the mathematical continuum on the other.

The notions of the infinitely small, the infinitely large, of infinite space, time or power have not yet received as subtle and complex an elaboration as the set-theoretical notions of infinity. They too are non-empirical predicates which are the result of modifying the logical content of empirical ones and do not from that point of view present any essentially new features.

5. *Questions of existence.*—A predicate which refers to empirical particulars is necessarily self-consistent. The internal consistency of non-empirical predicates must be shown in other ways. For some non-empirical predicates, which include predicates with infinite extensions, this has been recognised by Kant and Hilbert. The former attempted consistency proofs in order " to make room for faith " in certain moral and religious beliefs, the latter in order to make room for faith in classical mathematics.

If a mathematician asks whether an infinity, say the infinite extension of " natural number," exists he is, as a rule, merely asking whether this predicate, which *inter alia* entails " having an infinite extension " (of a certain kind), is internally consistent, and consistent with other predicates. If a philosopher asks this question he is rarely, if ever, concerned merely with questions of consistency. At the very least he will also wish to know whether the actual use of the notion of natural number by a group of thinkers, especially of mathematicians, does in fact involve the postulation of an infinite extension of one kind or another and perhaps whether such postulation is from the point of view of a specific metaphysical outlook required or permitted.

However, the heart of this philosophical existence problem is the question how the predicate " natural number," or any other non-empirical predicate with an infinite extension, is characteristic of or related to empirical fact. This very old philosophical question is answered neither by the true statement that " natural number " has no empirical reference nor by the equally true statement that it refers by definition to non-empirical particulars. It is answered only by showing, on the lines which I have indicated, in which respects the non-empirical predicate is a modification of an empirical one and how in consequence of this relation the two predicates can in many contexts be used interchangeably. In other words the answer consists in explaining the quasi-empirical reference of non-empirical aggregate-notions *via* empirical ones to instances of the latter.

II.

1. *The bearers of natural numbers.*—While I agree with Professor Findlay and Dr. Lewy that the question as to the kind of entities, if any, to which natural numbers apply is highly relevant to our subject, I cannot agree with the answers which they propose.

According to Findlay natural numbers apply to what he calls " things in the plural " and contrasts both with Russell's classes and Frege's concepts. It is one of his principal contentions that the bearers of natural numbers may, but need not, be accessible to perception. This view seems to me mistaken and based on the failure to distinguish between empirical aggregate-notions which have perceivable instances and non-empirical aggregate-notions which cannot have such instances. The difference between, say, " empirical couple " and " non-empirical couple " is, we have seen, analogous to that between " physical circle " and " geometrical circle." To replace the empirical by the non-empirical predicate is in both cases to pay for the achievement of precision with the loss of empirical reference.

It is strange that philosophers who would never confuse the notions of empirical and geometrical figures do not hesitate to confuse the notions of empirical and non-empirical aggregates. Yet Peano's rules for the logical behaviour of natural numbers are as remote from the rules governing empirical aggregate-notions as are Euclid's rules for the logical behaviour of geometical predicates from the rules governing the notions of perceivable shapes.

If the distinction between empirical and non-empirical aggregate-notions is justified, then Findlay should hold not that natural numbers may but need not apply to what can be perceived, but that empirical aggregate-notions do apply to perceivable instances and that non-empirical aggregate-notions, including the various notions of natural number, do not. His emphatic insistence that we sometimes perceive, say, a couple of things as a couple, without judging its elements to fall under a common concept, seems to me justified and important if the word " couple " is used in the sense of " empirical couple."

Lewy, like Findlay, does not distinguish between empirical and non-empirical aggregate-notions. His thesis that a physical collection or anything perceivable cannot be the bearer of a number can, therefore, be accepted only with the qualification that it concerns non-empirical aggregate-notions. He adopts Frege's doctrine that numbers apply to concepts and Frege's argument that unless numerical statements contained assertions about concepts we could not explain why we can truly attribute different numbers to the same external phenomenon ; *e.g.*, by saying with respect to what we see with equal truth " Here is one copse " and " Here are five trees." I do not find this argument at all convincing and believe that one can reject Frege's theory and yet provide the required explanation.

We often truly attribute different numbers to the same external phenomenon in the sense that we judge it to be an instance of different empirical aggregate-notions. This it may be, for at least two reasons. First, the same external phenomenon can be perceived in different groupings and the possibility of its being so perceived does not or, at least, does not necessarily depend on a preliminary application to it of a concept. The application of a concept, say, " copse " or " tree," may merely serve the purpose of identifying, for others or even the percipient, one of the many perceptually different groupings of the same external phenomenon. If, however, the application of " copse " or " tree " serves only to identify a grouping then it is the identified grouping and not the identifying concept which is an empirical unit or quintet.

It is, secondly, quite possible that even after we have identified a perceptual grouping, say of clouds in the sky, as clearly as we can wish, it may be an instance of different empirical aggregate-notions. A clearly identified grouping may be an empirical couple *and* an empirical triad in exactly the same sense in which a clearly identified shape may be both an instance of " physical circle " and of " physical ellipse." That the overlap between the notions of physical shapes has nothing to do with an insufficiently sharp identification of their bearers was made painfully clear to some of Pavlov's dogs ; and if not to the dogs then at least to the experimenters. Its root lies in the imprecise reference which is characteristic of all empirical notions including those of empirical aggregates. It is not characteristic of non-empirical aggregate-notions because these apply neither to empirical particulars nor to concepts whose instances are such particulars but only to postulated particulars whose nature is a matter of definition.

I have argued that no account of numerical statements can be successful which ignores the distinction between empirical and non-empirical aggregate-notions. It is just possible that exception might be taken to this contention on the ground that the distinction is not made in ordinary talk. It might be pointed out that we say there are five persons in a room without indicating that we are speaking of an empirical quintet. The statement is, however, ambiguous and its meaning depends on its context. It may merely refer to an empirical quintet ; but it may, as, *e.g.*, in the context of a statistical enquiry, express the speaker's intention to replace the predicate by a non-empirical one which is governed by altogether different rules. An analogous ambiguity attaches to the statement that there is a circle on the black-board.

2. *The actual versus the potential infinite.*—I am not fully satisfied with Professor Findlay's spirited defence of the notion of actual infinity. My main difficulty concerns the sense in which in his view the notion of, say, natural number *refers* to an infinite set of objects. In some passages he seems to imply that the reference is not wholly a matter of definition and postulation. If, as he says, a super-human being " could . . . recognise at a glance the presence of that first simple infinity " then this being's report on its perceptions might tell us something about the reference of " natural number " which does not follow from considering its logical content.

In other passages, which on the whole seem more important to me, he implies that our question can be fully answered by considering the logical content of " natural number." He says, " the only sense in which the existence of infinite numbers can be philosophically significant, is the sense in which it is logically *possible* for things to be infinitely numerous. . . ." He holds that a finite or transfinite number is a *kind* of thing whose existence therefore " will consist in the logically possible existence of its instances." To state, however, with regard to a predicate that the existence of its instances is logically possible is to state no more than that the predicate is internally consistent. Once this question is answered by considering the logical content of the predicate there remains nothing to be discovered by any being with whatever powers of perception.

I have argued that " natural number " is a non-empirical predicate and that, therefore, in speaking about its (non-empirical) reference we are exhibiting its logical content. If this is so then we can without inconsistency adopt and use two different predicates : " natural number_1" and " natural number_2" of which the first but not also the second entails " referring to an actual and complete infinity." There is no need to choose between them or defend one against the other, but only not to confuse them.

As Lewy emphasises, the " completeness " of an infinite set differs from that of a finite set ; and, I may add, both these senses differ from the sense in which an empirical aggregate, such as a group of five trees, can be *perceived* as complete. The completeness of the infinite set of natural numbers has been *defined* in different ways, *e.g.*, by postulating that the set of all its sub-sets can be well ordered or by making some weaker postulation.

Since the notion of natural number is in any case non-empirical we cannot by referring to a direct object of intuition or to the possibility of such intuition explain the meaning of " actual and complete set of natural numbers " or prove that " the infinite is not a name for a rule but exists as something actual and complete." I agree with Lewy that any attempt to do this must fail, but I do not think, as he does, that such an attempt constitutes the core of Findlay's paper. In spite of some impressions to the contrary Findlay means, I believe, by " the existence of a transfinite number," by " the logically possible existence of the instances of a transfinite number " and lastly by " the logical possibility of a direct apprehension of an instance of a transfinite number " no more than the internal consistency of a predicate " transfinite number " as defined by postulates.

In that case, however, his somewhat light-hearted dismissal of " supposed " contradictions in Cantor's theory is difficult to understand. One would like to know whether he thinks that no antinomies (such as the Russell paradox for example) arise within the classical theory of sets ; or whether he thinks that they can be removed from the theory without seriously affecting its structure. This is not idle curiosity : for whoever wishes to justify the use of a notion of actual infinity or save it from being blurred or obscured by other notions must assume that it is internally consistent.

3. *The reality of infinite extensions.*—Findlay claims for his account of finite and transfinite numbers as kinds or sorts of things that it is not " bound up with the actual contents of our world " and that it yet " never loses its ultimate touch with ordinary things (whether actual or possible)."

Thus the kind " being a couple " happens to have actual instances, while the kinds " being of number *n*," where *n* is very large, and " being infinitely numerous " may for all we know have no actual instance. Nevertheless, we can significantly talk about all these kinds. I have rejected this view on the ground that it confuses the essentially different notions of empirical and non-empirical aggregates.

Lewy in his contribution criticises the belief in a special mathematical reality because he thinks that it lies at the back of Findlay's paper. Although I am not at all sure about this I shall nevertheless briefly consider Lewy's criticism. He holds that to assume a mathematical reality implies the belief that every well-formed mathematical formula expresses a true or false proposition. Since against the latter belief serious objections can be raised which are connected with the undecidability of some formulae, Lewy argues that the same objections also discredit the assumption of a mathematical reality.

I am not convinced by Lewy's argument, because the two positions seem to me logically independent. A person may hold that every mathematical proposition is true or false but that no true mathematical proposition is true *of* anything, arguing, *e.g.*, that it is a true (bilateral) hypothetical proposition without existential import. Such a person would not believe in any special mathematical reality. On the other hand a person does believe in a special mathematical reality if he believes that *some* mathematical propositions are true *of* particulars *sui generis* which are neither empirical nor postulated. This belief, however, will not commit him to believing that every well-formed formula expresses a true or false proposition. The real difficulty about the doctrine of a special mathematical reality lies in the obscure nature of the particulars of which, it asks us to suppose, some mathematical statements are true.

The considerations which militate against the doctrine of a special mathematical reality seem to favour Russell's account of finite and transfinite numbers. According to this account, quite roughly speaking, every number is a class of classes ; and every class consists either immediately or mediately (through the hierarchy of types) of empirical particulars. This doctrine must be rejected if only for the absurd consequences which Findlay has shown to flow from it.

All the theories which I have so far mentioned in the present section are monistic in the sense that they do not recognise the difference between empirical and non-empirical aggregate-notions. A dualistic theory which does recognise this difference has been developed by Plato but gives rise to insuperable difficulties. These are mainly connected with the thesis that the mathematical Forms, among others, describe or constitute a special non-empirical reality ; with the distinction between the mathematical Forms and their instances (τὰ μαθηματικά) ; and, most important of all, with the nature of the relation (μέθεξις) of empirical aggregates to the mathematical Forms.

None of these difficulties arises in the dualistic theory which I have outlined in part I. The non-empirical aggregate-notions do not describe anything, but are merely modifications of the logical content of empirical aggregate-notions. The nature of their instances is determined by postulations and forms part of their logical content. Lastly, the connection between non-empirical aggregate-notions (with or without infinite extensions) on the one hand, and empirical aggregates on the other, consists in what I have called the quasi-empirical reference of the former to the latter.

In the beginning of his contribution Findlay suggests that " the only real questions " connected with the infinite " may be those concerning its precise implications and possible applications". This assumption does not, I think, bear close examination. On the one hand it misleadingly suggests an opposition between exhibiting the logical content and exhibiting the possible applicability of non-empirical aggregate-notions whereas, as we have seen, the latter activity is included in the former. On the other hand it ignores the problem of the non-deductive interconnection between non-empirical aggregate-notions and empirical ones, and consequently the eminently philosophical problem of the relation between mathematical theory and empirical fact.