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Reading Conclusions Conjunctively

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Abstract

In philosophical logic and proof theory, we often find multiple-conclusion systems that induce a conjunctive reading of premises and a disjunctive reading of conclusions. In mathematical logic, in contrast, we often find multiple-conclusion systems that induce a conjunctive reading of both premises *and* conclusions. This paper studies some technical and philosophical aspects of this latter approach to multiple-conclusion consequence. The takeaway is that, while the importance of disjunctive multiple conclusions is beyond doubt, conjunctive multiple conclusions also have philosophical interest. First, because there is some evidence that there are arguments with conjunctive multiple conclusions in natural language. Second, because conjunctive multiple conclusions are compatible with the reflexivity and transitivity of logical consequence, and this allows them to cohere better with some of our best accounts of what logical consequence is.

1 Introduction

The received wisdom tells us that arguments from natural language can have several premises, but exactly one conclusion. Yet, in the actual practice of logicians we often find logical systems where arguments can have none, one, or many conclusions; we call them *multiple-conclusion* logical systems.

In philosophical logic and proof theory, multiple-conclusion systems typically induce what we call a conjunctive (or universal) reading of premises, and a disjunctive (or existential) reading of conclusions. By this we mean, roughly, that validity in these systems admits at least one of the following informal paraphrases[1]

> Γ entails Δ just in case the (perhaps infinite) conjunction of the things in Γ entails the (perhaps infinite) disjunction of the things in Δ

> Γ entails Δ just in case, whenever all things in Γ are true (or meet these and those conditions), some things in Δ are true (or meet these and those conditions)

where Γ and Δ are collections of the appropriate sort. Multiple-conclusion systems of this kind were first introduced by Gentzen [22, 23] and then studied by several other authors.[2] They are particularly popular in discussions related to logical inferentialism; one of the reasons—but not the

[1]The template is meant to be quite general. In the first paraphrase, the conjunction and the disjunction mentioned can but need not behave as whatever operations may be available in the relevant object language. In the second paraphrase, the conditions imposed on premises and conclusions need not match—which leaves room to so-called mixed consequence relations, see, e.g. Chemla et. al. [7].

[2]See, for instance, Carnap [5], Kneale [27], Scott [44] and Shoesmith and Smiley [45].

only one—is that several authors have claimed, and others have denied, that disjunctive multiple conclusions make classical logic acceptable from an inferentialist standpoint.[3]

In mathematical logic, in contrast, multiple-conclusion systems typically induce a conjunctive reading of both premises and conclusions. So, validity can be paraphrased

> Γ entails Δ just in case the (perhaps infinite) conjunction of the things in Γ entails the (perhaps infinite) conjunction of the things in Δ

> Γ entails Δ just in case, whenever all things in Γ are true (or meet these and those conditions), all things in Δ are true (or meet these and those conditions)

Multiple-conclusion systems of this kind have a long history, as they can be traced back at least to Bolzano [3]. They are particularly useful in algebraic and categorical logic;[4] the reason is that they enable notions of consequence that are reflexive and transitive, and such notions of consequence can be generalised to wider classes of structures.

Recently, conjunctive multiple conclusions also received philosophical application in the work of Cintula and Paoli [9]. The authors use them to answer a challenge faced by non-contractive logics. The challenge is that, given any single-conclusion consequence relation $\dashv$, we expect being able to associate it with some closure operation Cn in the following way: $\Gamma \dashv A$ just in case $A \in Cn(\Gamma)$; alas, there is an impossibility result saying that this cannot be done if $\dashv$ is non-contractive.[5] In response, Cintula and Paoli show that, once we move to a multiple-conclusion framework where conclusions are read conjunctively, non-contractive consequence relations and closure operations can be matched in the expected way. This leads the authors to conclude that non-contractive consequence relations are "intrinsecally" or "essentially" multiple-conclusioned (p. 753).

The purpose of this paper is to study some technical and philosophical aspects of the conjunctive approach to multiple-conclusion consequence, with a keen eye on the relationships with the disjunctive approach. I take classical propositional logic as my main test case—but many of my results and arguments apply to other systems as well. Section 2 is mostly technical. First, I give semantic presentations of classical logic with conjunctive and with disjunctive multiple conclusions, and compare the structural properties of the two systems; one of the most notable differences (that will be of philosophical importance in the sequel) is that the former system is, while the latter is not, reflexive and transitive in the usual, relation-theoretic sense of these notions. Second, I provide a sequent calculus for classical logic with conjunctive multiple conclusions—something that, as far as I know, is not yet to be found in the literature.

Sections 3 and 4 are philosophical. In Section 3, I provide some evidence to think that there are arguments with conjunctive multiple conclusions in natural language; I give examples of such arguments, and consider potential objections. In Section 4, I argue that the fact that conjunctive multiple conclusions are compatible with the reflexivity and transitivity of logical consequence makes them more satisfactory in a number of ways. On the one hand, they allow a more natural treatment of the notion of logical equivalence, and in particular, of the generalisation of this notion from sentences to collections thereof. On the other hand—and more importantly—they cohere better with some of our best accounts of what logical consequence is. The accounts I consider are the one that understands consequence in terms of preservation (of truth or some other property),

[3] For arguments broadly in favour of this claim see Hacking [24], Read [36], Cook [11] and Restall [38]. For arguments against it, see Tennant [50], Dummett [15] and Steinberger [47].

[4] See, e.g. Font [19], Galatos and Tsinakis [21], Novak [31] and Cintula et. al. [8].

[5] The objection was raised by Ripley [40].

the one that understands it in terms of content inclusion, and the one that understands it in terms of existence of a proof. In all these cases—I claim—disjunctive conclusions face certain challenges which conjunctive conclusions do not.

The paper by no means argues that multiple conclusions *should* be read one way or another. It does not seem reasonable to expect that there is 'one correct way' of interpreting multiple conclusions: the merits of each approach will be assessed relative to applications. Also, it is beyond doubt that disjunctive multiple conclusions have important applications in philosophy and proof theory. The more humble takeaway of the paper is that conjunctive multiple conclusions are more than a mere technical artefact, useful in some areas of (mostly mathematical) logic; they have philosophical interest on their own. First, because they do not seem entirely foreign to actual inferential practices. Second, because they harmonise well with some of our best ways of understanding the central object study of logic, that is, consequence.

2 Technical Exploration

As announced, we take good old classical logic as our main test case. I define a multiple-conclusion presentation of this logic where conclusions are read conjunctively, and compare it with the usual presentation where conclusions are read disjunctively. Sect. 2.1 studies our system from a semantic perspective. Sect. 2.2 addresses its proof theory.

Before going on, it pays to lay down some stipulations. We will treat languages as identical to their respective sets of well-formed formulas. We call our propositional language $\mathcal{L}$, and assume it has a denumerable stock of variables $p, q, r, ...$, and primitive constants $\bot$, $\wedge$, $\vee$ and $\to$ with their usual arities and interpretations. Negation $\neg A$ will be defined as $A \to \bot$, and logical truth $\top$ as $\neg\bot$. We will use capital Latin letters $A, B, C, ...$ for arbitrary formulas of $\mathcal{L}$, and capital Greek letters $\Gamma, \Delta, \Sigma, ...$ for collections of formulas that can be either sets or multisets—we will disambiguate the reference in each case. Lastly, we will keep using $\dashv\!\!\vdash$ as a neutral symbol for entailment.

2.1 Valuations

We start by defining the three logical systems that we will mainly focus on: single-conclusion classical logic, henceforth **CL**, classical logic with disjunctive multiple conclusions, **dCL**, and classical logic with conjunctive multiple conclusions, **cCL**.

Let $\mathcal{V}$ be the set of all classical (viz. Boolean bivalued) interpretations of $\mathcal{L}$. Throughout this section, $\Gamma, \Delta, \Sigma, ...$ will denote sets of formulas. We shall occasionally use a comma for set union, and omit the brackets in singleton sets; so, for instance, Γ, A stands for $\Gamma \cup \{A\}$. Given an interpretation v, we write $v[\Sigma]$ to denote the set $\{v(\sigma) : \sigma \in \Sigma\}$.

Definition 1. The set-to-formula relation $\models_{\mathbf{CL}} \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{L}$ and the set-to-set relations $\models_{\mathbf{dCL}}, \models_{\mathbf{cCL}} \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$ are defined by

$$\begin{array}{lll} \Gamma \models_{\mathbf{CL}} C & \text{iff} & \text{for each } v \in \mathcal{V}, \text{ if } v[\Gamma] \subseteq \{1\} \text{ then } v(C) = 1 \\ \Gamma \models_{\mathbf{dCL}} \Delta & \text{iff} & \text{for each } v \in \mathcal{V}, \text{ if } v[\Gamma] \subseteq \{1\} \text{ then } v[\Delta] \not\subseteq \{0\} \\ \Gamma \models_{\mathbf{cCL}} \Delta & \text{iff} & \text{for each } v \in \mathcal{V}, \text{ if } v[\Gamma] \subseteq \{1\} \text{ then } v[\Delta] \subseteq \{1\} \end{array}$$

So, reading 1 as 'true' and 0 as 'false', we have the following informal paraphrases: (a) an argument is valid in **CL** just in case whenever the premises are all true the conclusion is true; (b) an argument is valid in **dCL** just in case whenever the premises are all true, at least one of the conclusions is true; and (c) an argument is valid in **cCL** just in case whenever the premises are all true the conclusions are all true.

Next, I describe the behaviour of **cCL** in more detail, and analyse the structural properties that differentiate it from **dCL**. Along the way I record a number of simple but relevant facts; most of the the proofs are easy, and thus they are left to the reader.

To begin with, one set entails another in **cCL** just in case the former entails each of the sentences in the latter. In other words,

Fact 1. $\Gamma \models_{\mathbf{cCL}} \Delta$ if and only if $\Gamma \models_{\mathbf{cCL}} B$ for each $B \in \Delta$.

This justifies the idea that in **cCL** multiple conclusions should be read conjunctively.

Arguably, the most important difference between the conjunctive and the disjunctive approaches concerns the properties of reflexivity and transitivity. These notions come from the theory of relations: a dyadic relation R on a set $\mathcal{A}$ is *reflexive* if and only if, for every $a \in \mathcal{A}$, aRa; R is *transitive* if and only if, for every $a, b, c \in \mathcal{A}$, if aRb and bRc then aRc. At least since the work of Tarski [48], it is commonplace to say that logical consequence is both reflexive and transitive. Yet, single-conclusion consequence relations such as $\models_{\mathbf{CL}}$ are strictly speaking neither, because they are not relations on a single set, and so they are not even the kind of thing that can have these properties. (When we say that they are reflexive and/or transitive, we mean that they satisfy some principles resembling reflexivity and/or transitivity to a greater or lesser extent—for instance, the restrictions of these properties to sentences). Multiple-conclusion consequence relations, on the other hand, are relations on a single set, so they could be reflexive and transitive in principle. Now, typically, consequence relations inducing a disjunctive reading of conclusions are neither transitive nor reflexive, while consequence relations inducing a conjunctive reading of conclusions are both. In particular, we have:

Fact 2. The following properties hold for **cCL** and do not hold for **dCL**:

(i) $\Gamma \Yright \Gamma$, for every Γ.

(ii) If $\Gamma \Yright \Delta$ and $\Delta \Yright \Sigma$, then $\Gamma \Yright \Sigma$, for every Γ, Δ and Σ.

To exemplify the negative claims, we have that (i) $\varnothing \not\models_{\mathbf{dCL}} \varnothing$, and (ii) $\{p\} \models_{\mathbf{dCL}} \{p, q\}$ and $\{p, q\} \models_{\mathbf{dCL}} \{q\}$, but $\{p\} \not\models_{\mathbf{dCL}} \{q\}$. Certainly, **dCL** satisfies some properties resembling reflexivity and transitivity; for instance, reflexivity restricted to non-empty sets, and transitivity as encoded by the properties

(iii) If $\Gamma \Yright \Delta, A$ and $A, \Gamma \Yright \Delta$, then $\Gamma \Yright \Delta$, for every A, Γ, Δ

(iv) If $\Gamma \Yright \Delta, A$ and $A, \Sigma \Yright \Pi$, then $\Gamma, \Sigma \Yright \Delta, \Pi$, for every $A, \Gamma, \Sigma, \Delta, \Pi$

(These properties are the semantic counterparts of the sequent rules known as 'additive cut' (Cut+) and 'multiplicative cut' (Cut×) respectively.) Now, **cCL** satisfies (iii) and (iv) as well. Indeed, Ripley [41] distinguishes other eleven properties resembling transitivity that **dCL** satisfies, and **cCL** satisfies them all. Hence, even when we focus on non-relation-theoretic variations of transitivity, **cCL** is not any less transitive than **dCL**.

Another interesting difference between the conjunctive and the disjunctive approaches concerns the behaviour of the empty set, $\varnothing$. It is standard to assume that a disjunction with no disjuncts is always false (for it never has a true disjunct); in other words, letting $\bigvee(\Sigma)$ be the disjunction of all the things in Σ, we have that $\bigvee(\varnothing)$ is a logical falsehood. Dually, a conjunction with no conjuncts is always true (for it never has a false conjunct); letting $\bigwedge(\Sigma)$ be the conjunction of all things in Σ, we have that $\bigwedge(\varnothing)$ is a logical truth. Given these assumptions, under the disjunctive approach, $\varnothing$ turns out to be a kind of cyclothymic character: it plays different inferential roles depending on where it appears in the argument. When it is the set of premises, it works as a logical truth, in the sense that it only entails logical truths; when it is the set of conclusions, it behaves as a logical falsehood, in the sense that it only follows from logical falsehoods. Graphically, we have:

$$\varnothing \models_{\mathbf{dCL}} A \quad \text{iff} \quad \top \models_{\mathbf{dCL}} A$$
$$A \models_{\mathbf{dCL}} \varnothing \quad \text{iff} \quad A \models_{\mathbf{dCL}} \bot$$

Under the conjunctive approach, in contrast, $\varnothing$ is a more temperate fellow. It is read as $\bigwedge(\varnothing)$ no matter what; thus, it always behaves as a logical truth:

$$\varnothing \models_{\mathbf{cCL}} A \quad \text{iff} \quad \top \models_{\mathbf{cCL}} A$$
$$A \models_{\mathbf{cCL}} \varnothing \quad \text{iff} \quad A \models_{\mathbf{cCL}} \top$$

It follows that $A \models_{\mathbf{cCL}} \varnothing$ for every A. And the two claims above still hold when we replace 'A' with 'Γ'. Hence, **cCL** is a system where $\varnothing$ follows from any set whatsoever!

In view of this, the reader may perhaps wonder why **cCL** is not trivial. The answer concerns the property of monotonicity. A multiple-conclusion consequence relation $\dashv\vdash$ is *monotone* if and only if it satisfies the properties

(v) If $\Gamma \dashv\vdash \Delta$, then $\Sigma, \Gamma \dashv\vdash \Delta$, for every Γ, Δ, Σ

(vi) If $\Gamma \dashv\vdash \Delta$, then $\Gamma \dashv\vdash \Delta, \Sigma$, for every Γ, Δ, Σ

(These properties are the semantic counterparts of the sequent-rules of 'left weakening' (LW) and 'right weakening' (RW), respectively.) Monotonicity is often thought to encode the non-defeasible character of deductive reasoning. **dCL** satisfies both (v) and (vi), and so is monotone. **cCL** satisfies (v) but not (vi); to exemplify, $\{p\} \models_{\mathbf{cCL}} \{p\}$ but $\{p\} \not\models_{\mathbf{cCL}} \{p, q\}$. This explains why **cCL** is not trivial, even though $\Gamma \models_{\mathbf{cCL}} \varnothing$ for every Γ. The failure of (vi) should not be taken as a deductive weakness of the system, however, or as evidence that it models defeasible reasoning only. First and foremost, under the conjunctive approach, (vi) intuitively says that, whenever certain conjunction follows from our premises, adding some additional conjuncts delivers a conjunction that also follows. But this of course is not the case in general. Thus, (vi) is not reasonable in this context, and it should not be taken to encode the non-defeasibility of deductive reasoning. Secondly, there are some limitative results concerning the properties of reflexivity, transitivity and monotonicity:

Fact 3. Let $\dashv\vdash \subseteq \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$

(a) If $\dashv\vdash$ is reflexive and monotone, then it is trivial.

(b) If $\dashv\vdash$ is transitive, monotone, and there are at least two sets Δ and Γ such that $\varnothing \dashv\vdash \Delta$ and $\Gamma \dashv\vdash \varnothing$, then $\dashv\vdash$ is trivial.

Proof. (a) Reflexivity gives $\varnothing \dashv\vdash \varnothing$; by monotonicity we get $\Sigma \dashv\vdash \Pi$ for arbitrary Σ and Π. (b) By monotonicity, from $\varnothing \dashv\vdash \Delta$ and $\Gamma \dashv\vdash \varnothing$ we get $\varnothing \dashv\vdash \Gamma \cup \Delta$ and $\Gamma \cup \Delta \dashv\vdash \varnothing$, respectively; transitivity gives $\varnothing \dashv\vdash \varnothing$, and monotonicity again delivers $\Sigma \dashv\vdash \Pi$ for arbitrary Σ and Π. $\square$

Thus, having all these properties at once was never a realistic goal to start with. **dCL** gives up both reflexivity and transitivity. **cCL** only gives up (one of the sides of) monotonicity.

Related to monotonicity is the property

(vii) If $\Gamma \dashv\!\vdash \Delta, \Sigma$, then $\Gamma \dashv\!\vdash \Delta$, for every Γ, Δ, Σ

(This is the counterpart of a sequent-rule sometimes called 'right anti-weakening' (RaW).) This property is satisfied by **cCL**, but not so by **dCL**. This is perfectly reasonable, given the informal reading that validity receives in these systems.

What we already said implies that **cCL** and **dCL** are contralogics of one another, that is, there are arguments that are valid in **cCL** but not in **dCL** and vice versa:

Fact 4.
(a) $\models_{\mathbf{cCL}} \not\subseteq \models_{\mathbf{dCL}}$ (e.g. $\{p\} \models_{\mathbf{cCL}} \varnothing$ but $\{p\} \not\models_{\mathbf{dCL}} \varnothing$)
(b) $\models_{\mathbf{dCL}} \not\subseteq \models_{\mathbf{cCL}}$ (e.g. $\{p\} \not\models_{\mathbf{cCL}} \{p, q\}$ but $\{p\} \models_{\mathbf{dCL}} \{p, q\}$)

Nevertheless, **dCL** and **cCL** are both what is known as *counterparts* of **CL**; this means that they coincide with **CL** in single-conclusion arguments:

Fact 5. $\Gamma \models_{\mathbf{CL}} C$ iff $\Gamma \models_{\mathbf{dCL}} \{C\}$ iff $\Gamma \models_{\mathbf{cCL}} \{C\}$

Systems **CL** and **dCL** are compact, in the sense that whenever a set Γ entails something, there is a finite subset Γ' of Γ that entails that very thing. Now, **cCL** is not compact in this particular sense. To see this, consider the set PV of all propositional variables of $\mathcal{L}$. Clearly, PV $\models_{\mathbf{cCL}}$ PV. However, there is no finite subset PV$'$ of PV such that PV$' \models_{\mathbf{cCL}}$ PV. Luckily, **cCL** is compact in the following slightly amended way:

Fact 6. $\Gamma \models_{\mathbf{cCL}} \Delta$ just in case, for each finite subset Δ' of Δ there is some finite subset Γ' of Γ such that $\Gamma' \models_{\mathbf{cCL}} \Delta'$.

(The result follows easily by Fact 1, Fact 5 and the compactness of **CL**.) And this version of compactness seems entirely reasonable from the perspective of the conjunctive approach.

We have seen that conjunctive conclusions differ from disjunctive conclusions in that they are compatible with the common idea that logical consequence is reflexive and transitive. I will argue later that this has some relevant philosophical consequences. For the time being, it must be noticed that, as a counterpart of this difference (or perhaps, a price to be paid for it), disjunctive conclusions exhibit certain expressive richness that conjunctive conclusions lack.[6] Arguably, one of the major advantages of **dCL** is that its consequence relation displays some well-known and elegant symmetries. If Γ is a set of formulas, let $\neg\Gamma$ be the set $\{\neg A : A \in \Gamma\}$. Then, in **dCL** we have the following equivalence

$$\Gamma \dashv\!\vdash \Delta \qquad \text{if and only if} \qquad \neg\Delta \dashv\!\vdash \neg\Gamma \tag{1}$$

and more in general,

$$\begin{array}{lcl} \Gamma \dashv\!\vdash \Delta, A & \text{if and only if} & \neg A, \Gamma \dashv\!\vdash \Delta \\ A, \Gamma \dashv\!\vdash \Delta & \text{if and only if} & \Gamma \dashv\!\vdash \Delta, \neg A \end{array} \tag{2}$$

[6] I thank an anonymous referee of this journal for encouraging me to address this point.

Now, statement (1) seems to express a generalised form of contraposition. Statements in (2), in turn, seem to express the equivalence that exists in classical logic between proving (refuting) a sentence A and refuting (proving) its negation $\neg A$. All these claims appear to capture something central about the classical notion of consequence. Moreover, they make possible the formulation of certain proof systems such as, for instance, one sided sequent calculi. However, they all fail for **cCL**: for (1), we have $\{p, q\} \models_{\mathbf{cCL}} \{q\}$ but $\{\neg q\} \not\models_{\mathbf{cCL}} \{\neg q, \neg p\}$; for (2), we have, first, $\{\neg p\} \models_{\mathbf{cCL}} \{\neg p\}$ but $\varnothing \not\models \{\neg p, p\}$, and second, $\{p\} \models_{\mathbf{cCL}} \{p\}$ but $\varnothing \not\models \{p, \neg p\}$. Hence, one might say that the framework of conjunctive multiple conclusions is expressively weaker than the framework of disjunctive multiple conclusions, and in particular, it fails to express some central properties of classical consequence.

I think that the objection is correct, and that it pinpoints a potential limitation on the applications of the conjunctive reading of conclusions. Having said that, I point out that in **cCL** we still have *some* ways of expressing the facts about classical consequence mentioned in the objection. Let us officially stipulate that $\bigwedge(\varnothing) = \top$ and $\bigvee(\varnothing) = \bot$. Consider the following statements:

$$\Gamma \dashv\Delta, \text{ with } \Gamma, \Delta \text{ finite} \quad \text{if and only if} \quad \neg \bigwedge(\Delta) \dashv \neg \bigwedge(\Gamma) \tag{1*}$$

$$\begin{array}{rcl} \Gamma \dashv \{A \vee B : B \in \Delta\} & \text{if and only if} & \neg A, \Gamma \dashv \Delta \\ A, \Gamma \dashv \Delta & \text{if and only if} & \Gamma \dashv \{\neg A \vee B : B \in \Delta\} \end{array} \tag{2*}$$

It's easy to check that

Fact 7. Equations (1*) and (2*) hold for **cCL**.

When both premises and conclusions are read conjunctively, (1*) obviously expresses contraposition. With (2*) things are less obvious; but notice that it has the special cases

$$\begin{array}{rcl} \varnothing \dashv A \vee \bot & \text{if and only if} & \neg A \dashv \bot \\ A \dashv \bot & \text{if and only if} & \varnothing \dashv \neg A \vee \bot \end{array}$$

Modulo the meanings of $\bot$ and $\vee$, these statements arguably express the idea that to prove (refute) a statement and to refute (prove) its negation are similar businesses in classical logic.

One evident drawback of (1*) and (2*) is that, unlike (1) and (2), they use object linguistic conjunction and disjunction. In a way, these connectives make explicit an application of the De Morgan laws that remains implicit in (1) and (2) as read in **dCL**. To see this, take $\{p, q\} \dashv \{r, s\}$ as the left hand side of (1); the operational reading of this in **dCL** is $p \wedge q \dashv r \vee s$; contraposition gives us $\neg(r \vee s) \dashv \neg(p \wedge q)$, and then the De Morgan laws deliver $\neg r \wedge \neg s \dashv \neg p \vee \neg q$, which is the operational reading in **dCL** of $\{\neg r, \neg s\} \dashv \{\neg p, \neg q\}$. Either way, the appeal to these constants in (1*) is problematic, for two reasons; first, we might want to work in a language that lacks them; second, (1*) cannot be generalised to infinite Γ and Δ.

If we complicate things a bit, and allow ourselves to use *collections* of validity claims, we can avoid using conjunction and disjunction. Let $\Sigma^\top$ stand for Σ if Σ is non-empty, and for $\{\top\}$ otherwise. Consider the following statements:

$$\Gamma \dashv \Delta \quad \text{if and only if} \quad \begin{array}{l} \Gamma^\top/\{A\}, \neg B \dashv \neg A \\ \text{for } A \in \Gamma^\top \text{ and } B \in \Delta^\top \end{array} \tag{1**}$$

$$\begin{array}{rcl} \Gamma, \neg B \dashv A, \text{ for } B \in \Delta^\top & \text{if and only if} & \neg A, \Gamma \dashv \Delta \\ A, \Gamma \dashv \Delta & \text{if and only if} & \Gamma, \neg B \dashv \neg A, \text{ for } B \in \Delta^\top \end{array} \tag{2**}$$

While the proofs are less self-evident (and the reader can cheerfully skip them), we still have

Fact 8. Equations (1**) and (2**) hold for **cCL**

Proof. Equation (1**). *Left to right.* Suppose $\Gamma \models_{\mathbf{cCL}} \Delta$ and consider any $A \in \Gamma^{\top}$ and $B \in \Delta^{\top}$. Assume v assigns 1 to $\neg B$ and to everything in $\Gamma^{\top}/\{A\}$. Then v assigns 0 to B, and thus $B \neq \top$, and hence $B \in \Delta$. It follows that v assigns 0 to something in Γ. Hence, $\Gamma \neq \varnothing$, which implies $\Gamma = \Gamma^{\top}$. Thus, v assigns 0 to something in $\Gamma^{\top}$. But by assumption v assigns 1 to everything in $\Gamma^{\top}/\{A\}$. Hence, v assigns 0 to A, and thus 1 to $\neg A$. *Right to left.* Suppose that $\Gamma^{\top}/\{A\}, \neg B \models_{\mathbf{cCL}} \neg A$ for each $A \in \Gamma^{\top}$ and $B \in \Delta^{\top}$. Assume v assigns 1 to everything in Γ, and consider any $B \in \Delta$. Clearly, v assigns 1 to everything in $\Gamma^{\top}$. Since $\Gamma^{\top} \neq \varnothing$, there is at least one $A \in \Gamma^{\top}$ such that v assigns 1 to A, and thus 0 to $\neg A$. It follows that v assigns 0 to $\neg B$, and thus 1 to B. *Equation* (2**). We show the uppermost biconditional. *Left to right.* Suppose $\Gamma, \neg B \models_{\mathbf{cCL}} A$ for each $B \in \Delta^{\top}$. Suppose v assigns 1 to $\neg A$ and to all formulas in Γ. Then for each $B \in \Delta$, v assigns 0 to $\neg B$, and hence 1 to B. *Right to left.* Suppose $\neg A, \Gamma \models_{\mathbf{cCL}} \Delta$. Consider any $B \in \Delta^{\top}$. Assume v assigns 1 to $\neg B$ and to all formulas in Γ. Then v assigns 0 to B, and thus $B \neq \top$, and hence $B \in \Delta$. It follows that v assigns 0 to $\neg A$ or to some formula in Γ. But by assumption v assigns 1 to all formulas in Γ. Hence it assigns 0 to $\neg A$, and thus 1 to A. The lowermost biconditional is established by similar reasoning. ☐

Arguably, modulo the meanings of $\bot$ and $\top$, (1**) and (2**) still encode the target phenomena. However, now the evident drawback of these properties is that they appeal to the object linguistic expressions $\top$ and $\bot$. Which brings us directly to what might be the core, most basic expressive limitation of the framework with conjunctive conclusions: it has no means to say, without using object-linguistic resources, that a sentence A is refutable. In **dCL**, the fact that A is refutable is encoded in the claim

$$A \dashv \varnothing$$

but this claim tells us absolutely nothing about the logical status of A in **cCL**. Alongside with the behaviour with respect to reflexivity and transitivity, this is, arguably, the most conceptually relevant difference between the conjunctive and the disjunctive approaches to multiple conclusions, in general, and between **cCL** and **dCL** in particular.

2.2 Proofs

In this section I will provide a sequent calculus for **cCL**. In a nutshell, I will take a system for single-conclusion classical logic **CL**, and show how to extend it with appropriate rules for multiple conclusions. A disclaimer is in place: I do not contend that the resulting calculus is the best we can do, either in proof-theoretic or in philosophical terms. The goal is just to provide *one* possible calculus for classical logic where conclusions are read conjunctively, which is something that, as far as I know, is already new to the literature. The quest for nice(er) calculi with conjunctive multiple conclusions is an interesting enterprise, but one that must be left for future work.

Throughout this section, $\Gamma, \Delta, \Sigma, \ldots$ stand for multisets of formulas of $\mathcal{L}$.[7] Formally, they are functions from $\mathcal{L}$ to the set $\mathbb{N}$ of natural numbers. Intuitively, they are lists with multiplicity but

[7] We could have used sets in our proof-theory as well. But we choose to use multisets to avoid certain (solvable) complications that, as shown by Negri and von Plato [30], the use of sets brings about.

without order, and $\Gamma(A)$ is the amount of times that formula A occurs in Γ. The *root set* of Γ, denoted by $|\Gamma|$, is the set $\{A \in \mathcal{L} : \Gamma(A) > 0\}$; we write $A \sqsubseteq \Gamma$ as a shorthand for $A \in |\Gamma|$. A multiset is *finite* just in case its root set is finite. Γ is a *submultiset* of Δ just in case $\Gamma(A) \leq \Delta(A)$ for every A. The *multiset union* of Γ and Δ, denoted by Γ, Δ, is the multiset where each A occurs $\Gamma(A) + \Delta(A)$ times. We use $\oslash$ for the empty multiset—as opposed to $\varnothing$, which denotes the empty set. Officially, we use square brackets to describe multisets by extension; so, $[A, A, B]$ is the multiset containing two occurrences of A, one of B and nothing more. Occasionally, however, we omit the brackets on multisets with just one formula occurrence; so, Γ, A stands for $\Gamma, [A]$.

We define a *sequent* as a pair of finite multisets of formulas of $\mathcal{L}$, and we denote the sequent $\langle \Gamma, \Delta \rangle$ as $\Gamma \Rightarrow \Delta$. Below, we find the sequent-rule counterparts of the various semantic properties alluded to in the previous subsection:

$$\text{Ref}\ \frac{}{\Gamma \Rightarrow \Gamma} \qquad \text{Tr}\ \frac{\Gamma \Rightarrow \Delta \qquad \Delta \Rightarrow \Sigma}{\Gamma \Rightarrow \Sigma}$$

$$\text{Cut+}\ \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \qquad \text{Cut}\times\ \frac{\Gamma \Rightarrow \Delta, A \qquad A, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}$$

$$\text{LW}\ \frac{\Gamma \Rightarrow \Delta}{A, \Gamma \Rightarrow \Delta} \qquad \text{RW}\ \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, A} \qquad \text{RaW}\ \frac{\Gamma \Rightarrow \Delta, A}{\Gamma \Rightarrow \Delta}$$

We say that a sequent $\Gamma \Rightarrow \Delta$ is *valid* in **cCL** just in case $|\Gamma| \models_{\mathbf{cCL}} |\Delta|$. A rule is *sound* in **cCL** just in case, for each of its instances, if the premise-sequents are all valid, the conclusion-sequent is valid. We adopt similar definitions for **CL** and **dCL**. It is easy to check that, of the rules above, all but RW are sound in **cCL**, and all but RaW, Ref and Tr are sound in **dCL**.

Our single-conclusion calculus will be the one given by Negri and von Plato [29, p. 114]. We write A/B for a formula that is either A or B, and p for an arbitrary propositional variable.

Definition 2. The calculus $\mathcal{S}_{\mathbf{CL}}$ is determined by the following rules:

$$\text{Id-}at\ \frac{}{\mathsf{p}, \Gamma \Rightarrow \mathsf{p}}$$

$$\text{L}\vee\ \frac{A, \Gamma \Rightarrow C \qquad B, \Gamma \Rightarrow C}{A \vee B, \Gamma \Rightarrow C} \qquad \text{R}\vee\ \frac{\Gamma \Rightarrow A/B}{\Gamma \Rightarrow A \vee B}$$

$$\text{L}\wedge\ \frac{A, B, \Gamma \Rightarrow C}{A \wedge B, \Gamma \Rightarrow C} \qquad \text{R}\wedge\ \frac{\Gamma \Rightarrow A \qquad \Gamma \Rightarrow B}{\Gamma \Rightarrow A \wedge B}$$

$$\text{L}\to\ \frac{A \to B, \Gamma \Rightarrow A \qquad B, \Gamma \Rightarrow C}{A \to B, \Gamma \Rightarrow C} \qquad \text{R}\to\ \frac{A, \Gamma \Rightarrow B}{\Gamma \Rightarrow A \to B}$$

$$\text{L}\bot\ \frac{}{\bot, \Gamma \Rightarrow A} \qquad \text{Lem-}at\ \frac{\mathsf{p}, \Gamma \Rightarrow C \qquad \neg\mathsf{p}, \Gamma \Rightarrow C}{\Gamma \Rightarrow C}$$

Calculus $\mathcal{S}_{\mathbf{CL}}$ is sound and complete for **CL**:

Theorem 9 (Negri and von Plato, p. 119)**.** *A sequent $\Gamma \Rightarrow A$ is valid in* **CL** *if and only if it is provable in $\mathcal{S}_{\mathbf{CL}}$*

One important remark about the calculus is that applications of Lem-*at* can be restricted in derivations, as follows:

Fact 10 (Negri and von Plato, p. 120)**.** If a sequent $\Gamma \Rightarrow C$ is derivable in $\mathcal{S}_{\mathbf{CL}}$, then it has a derivation where Lem-*at* is applied only on subformulas of C.

Since Lem-*at* is the only elimination rule of the calculus, it follows that if a sequent is derivable in $\mathcal{S}_{\mathbf{CL}}$, then it has a derivation with the subformula property—that is, a derivation whose formulas are all subformulas of formulas in the end-sequent.

To obtain a calculus for **cCL**, we just add a pair of rules:

Definition 3. The calculus $\mathcal{S}_{\mathbf{cCL}}$ results from $\mathcal{S}_{\mathbf{CL}}$ by adding the rules

$$\text{R}\varnothing\ \frac{}{\Gamma \Rightarrow \varnothing} \qquad\qquad \text{SM}\ \frac{\Gamma \Rightarrow \Delta_1 \quad \dots \quad \Gamma \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta_1, \dots \Delta_n}$$

The intuitive readings of rules R$\varnothing$ and SM are quite straightforward: the former says that the empty multiset follows from any multiset whatsoever, and the latter says that, if one multiset entails several others, then it entails their union.

Theorem 11. *A sequent* $\Gamma \Rightarrow \Delta$ *is valid in* **cCL** *if and only if it is provable in* $\mathcal{S}_{\mathbf{cCL}}$.

Proof. We leave soundness as an exercise, and prove completeness. So, suppose $\Gamma \Rightarrow \Delta$ is valid in **cCL**. If $\Delta = \varnothing$, then $\Gamma \Rightarrow \Delta$ is provable by a single application of R$\varnothing$, and we are done. So, suppose $\Delta \neq \varnothing$. The fact that $\Gamma \Rightarrow \Delta$ is valid in **cCL** means that $|\Gamma| \models_{\mathbf{cCL}} |\Delta|$. From this it follows that, for every $C \in |\Delta|$, $|\Gamma| \models_{\mathbf{cCL}} \{C\}$ (Fact 1), and thus $|\Gamma| \models_{\mathbf{CL}} C$ (Fact 5), and thus the sequent $\Gamma \Rightarrow C$ is derivable in $\mathcal{S}_{\mathbf{CL}}$ (Theorem 9). Hence, suppose $\Delta = [C_1, ..., C_n]$. The above implies that there exists a sequence $\mathcal{D}_1, ..., \mathcal{D}_n$ such that each $\mathcal{D}_i$ is a derivation in $\mathcal{S}_{\mathbf{CL}}$ of the sequent $\Gamma \Rightarrow C_i$. Thus, we just merge together all these derivations as follows

$$\text{SM}\ \frac{\begin{matrix}\mathcal{D}_1 \\ \vdots \\ \Gamma \Rightarrow C_1\end{matrix} \qquad \begin{matrix}\cdots \\ \vdots \\ \cdots\end{matrix} \qquad \begin{matrix}\mathcal{D}_n \\ \vdots \\ \Gamma \Rightarrow C_n\end{matrix}}{\Gamma \Rightarrow [C_1, ..., C_n]}$$

and obtain a derivation in $\mathcal{S}_{\mathbf{cCL}}$ of $\Gamma \Rightarrow \Delta$. □

One consequence of soundness and completeness (together with Facts 1, 5 and 10) is that in $\mathcal{S}_{\mathbf{cCL}}$ applications of Lem-*at* can also be restricted:

Corollary 12. If a sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathcal{S}_{\mathbf{cCL}}$, then it has a derivation where Lem-*at* is applied only on subformulas of formulas occurring in Δ.

(We leave the proof as an exercise for the reader.) Again, this implies that if a sequent is derivable in $\mathcal{S}_{\mathbf{cCL}}$, then it has a derivation with the subformula property.

Another consequence of soundness and completeness is that all the rules that are sound in **cCL** are admissible in $\mathcal{S}_{\mathbf{cCL}}$.[8] Thus, for instance,

Corollary 13. Rules Ref, Tr, Cut+, Cut×, LW and RaW are admissible in $\mathcal{S}_{\mathbf{cCL}}$.

Proof. We just prove the case of Cut×. Suppose sequents $\Gamma \Rightarrow \Delta, A$ and $A, \Sigma \Rightarrow \Pi$ are both provable in $\mathcal{S}_{\mathbf{cCL}}$. By soundness, they are valid in **cCL**. By the fact that Cut× is sound in **cCL**, sequent $\Gamma, \Sigma \Rightarrow \Delta, \Pi$ is valid in **cCL** as well. Then, by completeness, it is provable in $\mathcal{S}_{\mathbf{cCL}}$. The remaining cases are analogous. □

[8] A sequent rule is *admissible* in a sequent calculus $\mathcal{S}$ if and only if, for each of its instances, if the premise-sequents are all provable in $\mathcal{S}$, the conclusion-sequent is provable in $\mathcal{S}$.

One worry that one might have about calculus $\mathcal{S}_{\mathbf{cCL}}$ concerns the fact that it only has two rules (namely R⦰ and SM) that feature multiple conclusions. On the one hand, this arguably endows $\mathcal{S}_{\mathbf{cCL}}$ with a certain elegance, as the calculus proves complete for **cCL** while extending $\mathcal{S}_{\mathbf{CL}}$ with rather minimal resources. On the other hand, however, it makes the calculus rather odd from a philosophical standpoint. If the consequence relation that the calculus encodes allows arguments with multiple conclusions, why would the rules for the logical constants not allow such arguments? It seems desirable to have a calculus for **cCL** where the rules for the logical constants allow multiple conclusions as well.

Luckily, such a calculus is at hand:

Corollary 14. The following rules are sound in **cCL**, and thus also admissible in $\mathcal{S}_{\mathbf{cCL}}$:

$$\text{Id-}at^{+}\ \frac{}{\mathsf{p}, \Gamma \Rightarrow \Gamma, \mathsf{p}}$$

$$\text{L}\vee^{+}\ \frac{A, \Gamma \Rightarrow \Delta \qquad B, \Gamma \Rightarrow \Delta}{A \vee B, \Gamma \Rightarrow \Delta} \qquad \text{R}\vee^{+}\ \frac{\Gamma \Rightarrow \Delta, A/B}{\Gamma \Rightarrow \Delta, A \vee B}$$

$$\text{L}\wedge^{+}\ \frac{A, B, \Gamma \Rightarrow \Delta}{A \wedge B, \Gamma \Rightarrow \Delta} \qquad \text{R}\wedge^{+}\ \frac{\Gamma \Rightarrow \Delta, A \qquad \Gamma \Rightarrow \Delta, B}{\Gamma \Rightarrow \Delta, A \wedge B}$$

$$\text{L}\rightarrow^{+}\ \frac{A \rightarrow B, \Gamma \Rightarrow \Delta, A \qquad B, \Gamma \Rightarrow \Delta}{A \rightarrow B, \Gamma \Rightarrow \Delta} \qquad \text{R}\rightarrow^{+}\ \frac{A, \Gamma \Rightarrow [C_1, ..., C_n]}{\Gamma \Rightarrow [A \rightarrow C_1, ..., A \rightarrow C_n]}$$

$$\text{L}\bot^{+}\ \frac{}{\bot, \Gamma \Rightarrow \Delta} \qquad \text{Lem-}at^{+}\ \frac{\mathsf{p}, \Gamma \Rightarrow \Delta \qquad \neg\mathsf{p}, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}$$

Let $\mathcal{S}^{+}_{\mathbf{cCL}}$ be the calculus that results from replacing, in $\mathcal{S}_{\mathbf{cCL}}$, each single-conclusioned rule $\mathcal{R}$ with its multiple-conclusioned counterpart $\mathcal{R}^{+}$. By the above corollary, calculus $\mathcal{S}^{+}_{\mathbf{cCL}}$ proves no more sequents than $\mathcal{S}_{\mathbf{cCL}}$. By the fact that each of the mentioned $\mathcal{R}$s is just a special case of the corresponding $\mathcal{R}^{+}$, calculus $\mathcal{S}^{+}_{\mathbf{cCL}}$ also proves no less sequents. That is to say, $\mathcal{S}^{+}_{\mathbf{cCL}}$ is also sound and complete for **cCL**.

Another source of worries has to do with the rules of cut, Cut+ and Cut×. Corollary (13) establishes the admissibility of these rules in $\mathcal{S}_{\mathbf{cCL}}$ by means of a semantic argument—which relies, remember, on the fact that these rules are sound in **cCL**. For many purposes, however, it is often useful to have a direct, syntactic proof of their admissibility. But calculus $\mathcal{S}_{\mathbf{cCL}}$ has two rules, R⦰ and SM, whose form is quite different from what we are used to see in the literature on sequent calculi; hence, it is not obvious that these rules do not 'mess' with the procedures that the usual proofs of cut admissibility employ. To dispel such a worry, in the Appendix I lay down purely syntactic proofs of the admissibility of Cut+ and Cut× in $\mathcal{S}_{\mathbf{cCL}}$.

One important application of cut-free sequent calculi is to enable purely syntactic proofs of consistency. However, proofs of this sort typically rely on the fact that sequent $⦰ \Rightarrow ⦰$ is invalid, and this sequent is valid in **cCL**. So, one may also worry that, in a framework with conjunctive multiple conclusions, purely syntactic proofs of consistency are not available. But the worry is unfounded. The consistency of $\mathcal{S}_{\mathbf{cCL}}$ can be proven by purely syntactic means. First, one shows that $\mathcal{S}_{\mathbf{cCL}}$ is consistent (viz. it does not prove any sequent of the form $\Rightarrow A \wedge \neg A$) if and only if sequent $\Rightarrow \bot$ is not derivable. Then, one goes on to show that in $\mathcal{S}_{\mathbf{cCL}}$ no derivation of $\Rightarrow \bot$ can exist. Intuitively, this is because $\Rightarrow \bot$ can only be obtained in one of the following two ways:

$$\text{Lem-}at\ \dfrac{\begin{array}{c}\vdots\\ p \Rightarrow \bot\end{array} \qquad \begin{array}{c}\vdots\\ \neg p \Rightarrow \bot\end{array}}{\Rightarrow \bot} \qquad\qquad \text{SM}\ \dfrac{\begin{array}{c}\vdots\\ \Rightarrow \bot\end{array} \qquad \dfrac{}{\Rightarrow}\,\text{R⦸} \quad \dots \quad \dfrac{}{\Rightarrow}\,\text{R⦸}}{\Rightarrow \bot}$$

But in each case, at least one of the premises can only be obtained using Lem-*at* or SM again, which leads to an infinite regress. Thus, purely syntactic proofs of consistency can still be obtained in our framework (although in a slightly different way).

Calculus $\mathcal{S}_{\mathbf{cCL}}$ does have one shortcoming, however. If we formulate it in a first order language and extend it with the usual rules for the quantifiers, then the resulting system, call it $\mathcal{S}^{\text{FO}}_{\mathbf{cCL}}$, is not complete for the first-order version of **cCL**; rather, it is a system where propositional connectives are classical but quantifiers are intuitionistic—so, for instance, we cannot prove $\exists x Ax \vee \neg\exists x Ax$ for arbitrary A.[9] There are a couple of possible reactions. One option would be to take $\mathcal{S}^{\text{FO}}_{\mathbf{cCL}}$ and, in the rule of excluded middle, drop the restriction that the formula being eliminated should be atomic. The problem with this strategy is that, although the resulting system is complete, it makes some variants of cut derivable,[10] and it is not clear if applications of excluded middle can be restricted in derivations in some meaningful way. Another, more radical option would be to start from some altogether different calculus for first-order, single-conclusion classical logic, and then add rules R⦸ and SM to that calculus. For instance, one could take the system given by Boolos [4, p. 183]. The issue with that particular system is that it is a natural deduction calculus in sequent style (viz. it has elimination rules for the connectives), and this could be less than satisfactory for some purposes. All in all, the proof theory of first-order classical logic with conjunctive multiple conclusions is to be developed, and I take it to be an interesting topic of research for future work.

3 Natural Language Arguments

Arguably, one of the most important applications of logical systems is to describe and/or prescribe the way in which we deductively reason and/or ought to reason in natural language; in other words, to model our everyday reasoning. In this section I claim that the conjunctive approach to multiple conclusions is useful in this respect.

In the philosophy of logic, most authors take sides with tradition and claim that arguments in natural language have exactly one conclusion:

> The vice of the idea of multiple conclusion arguments is that it seems completely foreign to the evidence of the arguments we see in practice. (Beall and Restall [1, p. 13].)

> The rarity, to the point of extinction, of naturally occurring multiple conclusion arguments has always been the reason why mainstream logicians have dismissed multiple-conclusion logic as little more than a curiosity. (Rumfitt [43, p. 79].)

(See also [15, 42, 47, 50].) Some have challenged this attitude, and argued that in natural language we sometimes find arguments with multiple conclusions [38, 45] or, at least, logical constants whose adequate formalisation requires a multiple conclusion framework [14]. Both sides in the debate, however, share a key implicit assumption, namely, that multiple conclusions are to be

[9] I thank an anonymous reviewer of this journal for bringing this matter to my attention.

[10] A sequent rule is *derivable* in a sequent system $\mathcal{S}$ just in case, for each of its instances, the conclusion-sequent is provable in the system that results from $\mathcal{S}$ by adding the premise-sequents as new axioms.

read disjunctively.[11] I drop that assumption, and argue that, at least when conclusions are read conjunctively, there are arguments in natural language whose most natural and simple formalisation involves multiple conclusions.

My starting observation is simple. In English, it makes perfect sense to say things as

(1) Such and such predictions follow from such and such hypothesis.
(2) What you said entails the following set of statements: $A_1, A_2, ..., A_n$.
(3) This theoretical standpoint has a series of undesirable consequences

It is clear that, were these fragments to be formalised using multiple conclusions, those conclusions should be read conjunctively. Hence, I next consider some objections against the idea that they can be plausibly formalised using multiple conclusions.

The most immediate objection is that the above fragments can be formalised just as well using a conjunction as the only conclusion; therefore, multiple conclusions are dispensable. For starters, it should be noted that, if this objection is convincing, then an analogous one applies to multiple conclusions under the disjunctive reading: they can be explained away by means of disjunctions. Setting this aside, there are several answers to the objection.

The first one is that, sometimes, to formalise using sets is more faithful to the speaker's intention than to formalise using a conjunction. For instance, let us precisify (3) as follows:

(3*) Hard determinism has a series of undesirable consequences, namely, the nonexistence of moral responsibility, the lack of individual autonomy, and a depressing fatalism.

and compare this with

(3**) Hard determinism has a series of undesirable consequences, namely, the nonexistence of moral responsibility, a depressing fatalism, and the lack of individual autonomy.

There are at least some contexts of utterance in which the difference between (3*) and (3**) seems entirely irrelevant. To avoid unnecessary logical manipulations of object linguistic constants, in such contexts it seems reasonable to formalise both fragments using a set of conclusions instead of a single conclusion of conjunctive form. While non-decisive, the point should not be very controversial: it is for similar considerations that, often, when we face an argument with prima facie many premises, we formalise it using a set of premises rather than a conjunction thereof.

Secondly, it is true that, in our setting, conjunction and multiple conclusions are equivalent in the following sense:

$$\Gamma \dashv\!\!\!\models A, B, \Delta \qquad \text{iff} \qquad \Gamma \dashv\!\!\!\models A \wedge B, \Delta$$

But this happens only because we work in classical logic; in many non-classical systems, the equivalence will break. For instance, consider any logic where conjunction has a non-standard behaviour in that it violates simplification ($A \wedge B \dashv\!\!\!\models A/B$) or adjunction ($A, B \dashv\!\!\!\models A \wedge B$).[12] Suppose also that in this logic validity is defined as preservation of designated value, following the general template that we can extract from Definition 1:

[11] *A note to avoid confusion.* In the terminology of Steinberger [47], those who favour the so-called bilateralist reading of multiple-conclusion consequence (where $\Gamma \Rightarrow \Delta$ is read as "It is incoherent to accept everything in Γ and deny everything in Δ") do not have a disjunctive reading of conclusions. But they do have such a reading in our usage of words, because they work with systems where validity *can* be paraphrased as per the disjunctive approach.

[12] There are many logics where conjunction is non-standard in this sense. Just to give a couple of examples, adjunction fails in Jaśkowski's [26] discussive logic as well as in non-falsity logic **NFL** (Shramko [46]), whereas simplification fails in paraconsistent weak Kleene logic **PWK** (Haldén [25]).

$\Gamma \multimap \Delta$ just in case, for every relevant interpretation, if all the things in Γ have designated value, all the things in Δ have designated value.

where Γ and Δ are sets. Then, conjunction and structural comma will come apart in this logic, for the definition of validity just given imposes on the comma the structural renderings of simplification ($A, B \multimap A/B$) and adjunction ($A, B \multimap A, B$). The point is that conjunctive multiple conclusions are not *in general* reducible to mere conjunctions.

Thirdly, infinite collections of conclusions are not expressible with ordinary conjunctions. And we sometimes use arguments with prima facie infinite conclusions, as when we say

(4) The Peano axioms entail all formulas of the form $t + 0 = t$.
(5) This theory of physic entails all the sentences of the language

Perhaps, the objector could insist by appealing to infinite conjunctions. But the formulation of infinite conjunctions requires set-theoretical vocabulary anyway; hence, it does not contribute to economise on expressive resources. For instance, let us try to formalise (5). Let T be the relevant theory, and $\mathcal{L}$ our language. With multiple conclusions, we can write

$$\mathrm{T} \models \{A : A \in \mathcal{L}\}$$

If we restrict ourselves to single conclusions, we have to write

$$\mathrm{T} \models \bigwedge\{A : A \in \mathcal{L}\}$$

In both cases we will need the machinery of set-theory. Thus, why don't we allow multiple conclusions from the outset?[13]

Fourth and last. If the above reasons do not convince the reader, then they may also lack good reasons to admit multiple *premises*. Now that our reading of multiple premises and conclusions is similar, reasons that justify the former tend also to justify the latter, and vice versa. Also arguments with prima facie many premises could be formalised with a conjunction as the only premise. If the reader is consequent, they should opt for such a formalisation. But I doubt that this would be pleasing for them.

The second possible objection to my proposal runs as follows. If an argument in natural language appears to have multiple conclusions, then it is just an abbreviation of multiple different arguments, one for each of the apparent conclusions. In particular, when a speaker asserts that Γ entails Δ, and Δ is understood conjunctively, what the speaker means is the universal statement "Γ entails C for each C in Δ". This picture builds on some suggestions by Cintula and Paoli [9].

This objection is more plausible, but its significance is rather limited. To begin with, if it is convincing, then again there is an analogous objection that affects multiple conclusions under a disjunctive reading; indeed, Cintula and Paoli's original argument is meant to give an eliminativist account of those. Very roughly, the assertion that Γ entails Δ, where Δ is read disjunctively, can be understood as expressing the universal statement according to which each C in Δ follows from Γ together with the negations of the remaining things in Δ. Regardless of this, the objection is again contestable.

First, it is true that, in our setting, the following 'reduction' holds:

[13] My last two answers develop some considerations made by Shoesmith and Smiley in their defence of disjunctive multiple conclusions [45, p. 2].

$$\Gamma \succ \Delta \qquad \text{iff} \qquad \Gamma \succ \delta \text{ for each } \delta \in \Delta$$

But again, whether some such result is available will depend on the details of the framework. For instance, if a consequence relation $\succ$ is defined on multisets rather than sets, then the obvious reformulation of the reduction would be the following

$$\Gamma \succ \Delta \qquad \text{iff} \qquad \Gamma \succ \delta \text{ for each } \delta \sqsubseteq \Delta$$

But this reduction could reasonably fail if $\succ$ is meant to formalise resource-sensitive reasoning: we could have, for example, $[A] \not\succ [A, A]$ but $[A] \succ [A]$ (where, notice, the latter implies that $[A] \succ [C]$ for each $C \sqsubseteq [A, A]$).[14] The point is, now, that conjunctive multiple conclusions do not in general merely abbreviate classes of single-conclusion arguments.

But even for those systems where the reduction *can* be done, it is hasty to conclude that conjunctive multiple conclusions are just dispensable. First and foremost, when a speaker utters, for instance, "Such and such predictions follow from such and such hypothesis", or even "This theory entails that one", they do not make explicit use of any quantifiers—and they are not aware of making any implicit use of quantifiers either. Hence, the most literal and simple way to model their claim is by means of two collections of statements, say Γ and Δ, of which they are saying that $\Gamma \succ \Delta$. If we relinquish from multiple conclusions, and model the speaker's utterance by means of a universal quantification over a certain class of single-conclusion arguments, we make a non-literal (or at least, a *less* literal) reading of what the speaker has said. Of course, non-literal readings can have their advantages sometimes. But, all other things being equal, the more literal reading is to be preferred.

For another thing, the reduction makes essential use of certain metatheoretical expressions such as the indicative biconditional and the universal quantifier. Since these expressions pertain to the metalanguage, they are preformal in that their usage is not regimented. Multiple conclusions allow us to dispense with these expressions and thus give a more rigorous account of the logic regulating fragments like (1) to (5)—and the interactions between these fragments. This increase in rigour arguably brings about some epistemic gains. For instance, it enables a proof-theoretic decision procedure for the validity of these fragments, and it allows a better analysis of the structural properties that these pieces of reasoning display. These epistemic gains, I take it, provide further reasons to formalise fragments (1) to (5) by means of multiple conclusions.

I conclude that, at least when conclusions are read conjunctively, we have good reasons to admit that there are multiple conclusions in natural language. This makes conjunctive multiple conclusions useful in the enterprise of modelling our everyday reasoning. To be clear, I have not argued that in natural language there are no arguments with *disjunctive* multiple conclusions. There may well be. For instance, one could try to mimic the reasoning from this section by appealing to examples like "These facts entail the following set of possible scenarios", "What you said leaves the following possibilities open", and so on. But following this line of thought escapes the subject of this paper.

[14] A concrete example of a system where this happens is logic **MŁ** in Cintula and Paoli [9, p. 753].

4 On Reflexivity and Transitivity

One of the major differences between the conjunctive and the disjunctive approaches has to do with their policies towards the properties of reflexivity and transitivity: the disjunctive approach induces failures of these properties, while the conjunctive one does not. In this section we will see that this difference is tightly related to a number of aspects in which the conjunctive approach seems to be more satisfactory than the disjunctive one. In Sect. 4.1, I claim that it allows a more natural generalisation of logical equivalence from sentences to collections thereof. In Sects. 4.2 to 4.4, I claim that it gets along better with some of our best accounts of logical consequence, namely, the ones based on preservation, on content inclusion, and on existence of a proof.

4.1 Generalising Logical Equivalence

The relation of logical equivalence is typically assumed to hold between sentences. But when we work in a multiple-conclusion framework, the relata of logical consequence are *collections* of sentences. Hence, it makes sense to ask what logical equivalence looks like when we generalise it to collections as well.

In the single-conclusion framework, two sentences A and B are said to be *logically equivalent* just in case they mutually entail each other, in symbols $A \dashv\vDash B$. So, let us extend this stipulation to the multiple-conclusion framework: two sets Γ and Δ are logically equivalent just in case $\Gamma \dashv\vDash \Delta$. Under this modest assumption, the disjunctive approach has some consequences that strike me as highly counter-intuitive:

> *Case 1:* Set $\{A, B\}$ is logically equivalent to $\{A \vee B\}$, but also to $\{A \wedge B\}$! (In other words, $\{A, B\} \dashv\vDash_{\mathbf{dCL}} \{A \vee B\}$ and $\{A, B\} \dashv\vDash_{\mathbf{dCL}} \{A \wedge B\}$.) How can the same set be logically equivalent to sentences that have different truth conditions (or, more precisely, to sets that do not have the same models)?[15]

> *Case 2:* The empty set, $\varnothing$, is logically equivalent to $\{A, \neg A\}$, but not to itself! (In other words, $\varnothing \dashv\vDash_{\mathbf{dCL}} \{A, \neg A\}$ but $\varnothing \not\dashv\nvDash_{\mathbf{dCL}} \varnothing$.) If something entails an inconsistent set, and classical logic is explosive, why does it not entail everything and, in particular, why does it not entail itself?

Of course, these questions have clear technical answers. As for *Case 1*, sets $\{A, B\}$ and $\{A \vee B\}$ are logically equivalent because, although they do not have the same models, every model of the latter assigns value 1 to at least one sentence in the former and vice versa; this is all we need to have validity in **dCL**. As for *Case 2*, $\varnothing$ is not equivalent to itself because, although every model assigns 1 to each of its sentences, no model assigns 1 to at least one of them; this precludes **dCL** validity. But notice that these answers *assume* a disjunctive reading of multiple conclusions; hence, they just beg the question in favour of this reading. Needless to say, none of the counter-intuitive examples affects the conjunctive approach: in **cCL**, $\varnothing \dashv\vDash \varnothing$ and $\{A, B\} \not\dashv\vDash \{A \vee B\}$.

Examples 1 and 2 have quite a bit in common. Indeed, they can both be explained by a single fact about **dCL**, namely, that the system invalidates the following principle:

$$\text{Ax-1}\ \frac{\Gamma \dashv\vDash \Sigma \qquad \Delta \dashv\vDash \Sigma}{\Gamma \dashv\vDash \Delta}$$

[15] It is noteworthy that, as a special case of this example, we have that $\{A, \neg A\}$ is logically equivalent both to $\{A \wedge \neg A\}$ and to $\{A \vee \neg A\}$, that is, both to a set without models and to a set without counter-models.

The informal reading is: 'If two things are logically equivalent to a third, they are logically equivalent to one another'. Label 'Ax-1' honours the clear similarity with the first axiom of Euclid. In *Case 1*, Γ is $\{A \vee B\}$, Δ is $\{A \wedge B\}$, and Σ is $\{A, B\}$. In *Case 2*, Γ and Δ are both $\varnothing$, and Σ is $\{A, \neg A\}$. Both examples show that Ax-1 does not hold in **dCL**.

I submit that Ax-1 throws light upon an important aspect in which **cCL** is more akin to **CL** than **dCL** is. In the single-conclusion framework, logical equivalence is defined for *sentences*, while in the multiple-conclusion framework, it is defined for *sets* of sentences. Thus, in the cases of both **CL** and **cCL**, logical equivalence is a genuine *equivalence* relation, that is, it is reflexive, transitive and symmetric. This is not so, however, in the case of **dCL**; the explanation is that Ax-1 is a necessary condition for a relation $\multimapboth$ to be an equivalence relation. The upshot is that there is a sense in which the conjunctive reading of multiple conclusions is more faithful to the spirit of single-conclusion consequence than the disjunctive reading is.

The sympathiser of the disjunctive approach could object. When we focus on sentences, the relations of 'having the same models' and 'entailing each other' are coextensive: A has the same models as B just in case $A \multimapboth B$. As a consequence, logical equivalence can be defined in terms of any of these relations, and the results will be the same. When we focus on sets, however, things are different. The relations mentioned now come apart: there are pairs of sets that have the same models, but do not entail each other (e.g. $\varnothing$ and $\varnothing$) and there are also pairs of sets that entail each other in spite of not having the same models (e.g. $\{A, B\}$ and $\{A \vee B\}$). Thus, logical equivalence cannot adjust to both relations at once. We must choose. And we have just seen that defining logical equivalence in terms of mutual entailment runs into troubles. Hence, we should define it in terms of sameness of models.[16] This keeps the counter-intuitive examples at bay: now, $\{A, B\}$ and $\{A \vee B\}$ are not logically equivalent, but $\varnothing$ and $\varnothing$ are.

The objection is relevant, since it provides a coherent picture where multiple conclusions are read disjunctively and, yet, undesirable consequences are avoided. However, I do not think that the position depicted is ultimately satisfactory; the reason is that it incurs in theoretical costs that can be avoided. Under the conjunctive approach, the relations of 'having the same models' and 'entailing each other' are coextensive *both* for sentences and for sets. Also, the approach is not prone to the counter-intuitive consequences that threaten the disjunctive reading. But then, why pay the cost that the disjunctive reading supposes? Why break the symmetry between the notions of logical equivalence for sentences and for sets, if this is not indispensable to save the data? I do not see good answers in favour of the disjunctive approach.

4.2 Validity as Preservation

One of the (if not *the*) most established analysis of logical consequence in the literature tells us that an argument is valid just in case it preserves certain property from premises to conclusion(s). The property that is assumed to be preserved varies across logical systems and philosophical views; it can be, e.g. truth or satisfaction, assertability, constructive provability or even evidence. For concreteness, in what follows I assume the relevant property to be truth. Not much hinges on this, however. My argument is quite general, and it aims to apply to most (if not all) explanations of logical consequence as preservation.

[16] This is not the only option, though. As an anonymous reviewer rightly observes, the sympathiser of the disjunctive approach could also say that two sets Γ and Δ are logically equivalent just in case they entail the same sets, viz. for every Σ, $\Gamma \multimap \Sigma$ if and only if $\Delta \multimap \Sigma$. My answer to this proposal is similar to the reply I give below to the proposal based on sameness of models.

The starting point I would like to make is that the idea of 'truth preservation' suggests that there is a pair of entities such that the first 'transfers' its truth to the second or, alternatively, the second 'inherits' the truth of the first. Of course, this is metaphoric. But that should not be a problem, since the very talk of 'truth preservation' is metaphoric as well. I am just positing further informal conditions that should intuitively hold for the metaphor to make sense. Our guiding question, then, will be the following: *What are the entities between which truth is preserved in valid arguments?* Let $\Yright$ be any logical consequence relation. We will consider three options: that $\Yright$ stands for consequence in **CL**, in **cCL**, and in **dCL**.

If $\Yright$ stands for consequence in **CL**, things seem quite straightforward. First, we stipulate that a set of sentences is *true* just in case all of its sentences are true. Then, we note that, under the usual reading of the semantics for classical logic (where 1 stands for 'true' and 0 for 'false'), the following obtains:

(Set-Fmla) $\Gamma \Yright C$ just in case, whenever Γ is true, C is true

Thus, we are justified in giving the next answer to our question: in valid arguments, truth is preserved *between the set of premises and the conclusion.* Label '(Set-Fmla)' stands for 'Set-Formula truth preservation'.

If $\Yright$ stands for consequence in **cCL**, no additional complications seem to arise. We stick to the above stipulation, and note that the following obtains:

(Set-Set) $\Gamma \Yright \Delta$ just in case, whenever Γ is true, Δ is true

Thus, we are justified in giving the answer: in valid arguments, truth is preserved *between the set of premises and the set of conclusions.* The meaning of label '(Set-Set)' is the expected one, namely 'Set-Set truth preservation'.

When $\Yright$ stands for consequence in **dCL**, however, things are way less obvious. To begin with, the idea that truth is preserved between sets seems bound to failure. The reason is that there seems to be no reasonable stipulation of what it means that a set of sentences is 'true' such that (Set-Set) obtains. Suppose that we stick to the stipulation we entertained so far: a set of sentences is true just in case all of its sentences are true. Then, (Set-Set) fails because $\{p\} \models_{\mathbf{dCL}} \{p, q\}$ but it is not the case that whenever al the sentences in $\{p\}$ are true, all the sentences in $\{p, q\}$ are true. Suppose, alternatively, that a set of sentences is true just in case at least one of its sentences is true. Then, (Set-Set) fails because $\varnothing \not\models_{\mathbf{dCL}} \{p \wedge \neg p\}$ even though, whenever some sentence in $\varnothing$ is true, some sentence in $\{p \wedge \neg p\}$ is true (namely, never). Maybe, one could try some more intricate stipulations; for instance, one could say something like this: "A set is true *in the premises of an argument* just in case all of its sentences are true, and a set is true *in the conclusions of an argument* just in case at least one of its sentences is true'. But this, I take it, makes no philosophical sense at all. In general, I see no stipulation that satisfies Set-Set without being horribly ambiguous or context-dependent.

Let us discard, then, the idea that in valid arguments truth is preserved between sets. Another option that could come to mind is that in valid arguments truth is preserved between the set of premises and *some sentence* in the set of conclusions. The problem with this proposal is that, for it to be justified, the following should obtain:

(Set-Set*) $\Gamma \Yright \Delta$ just in case there is a C in Δ such that, whenever Γ is true, C is true

Yet, this fact fails spectacularly in **dCL**; for instance, we have that $\varnothing \models_{\mathbf{dCL}} \{p, \neg p\}$ but it is neither the case that p is always true, nor that $\neg p$ is always true. Indeed, (Set-Set*) fails in most of the logical systems I know of.[17]

The last, and most plausible answer that I could come up with runs as follows: in valid arguments, truth is preserved between the *sentence-translations* of the sets of premises and conclusions. Let us define the premise-sentence-translation of a set Σ, denoted by $p(\Sigma)$, as $\bigwedge(\Sigma \cup \{\top\})$, and the conclusion-sentence-translation of Σ, denoted by $c(\Sigma)$, as $\bigvee(\Sigma \cup \{\bot\})$. Then, the justifying fact for this position would be the following:

(Set-Set**) $\Gamma \dashv\!\!\vdash \Delta$ just in case whenever $p(\Gamma)$ is true, $c(\Delta)$ is true.

Of course, this obtains for **dCL** as well as many non-classical systems. The problem I see with this proposal is that, if we take seriously the idea that the relata of logical consequence are sets and, moreover, we assume that logical consequence is to be explained in terms of truth preservation, then it seems odd, at the very least, that the relation of truth preservation does not have sets anywhere among its relata. In other words, under this proposal, sets can be arguably understood as mere abbreviations: the genuine relata of logical consequence are not sets anymore, but the sentences they abbreviate. But if this is the case, then the sympathiser of disjunctive multiple conclusions has lost multiple conclusions (as well as multiple premises) along the way.[18]

I conclude that the conjunctive reading allows a simpler and more reasonable specification of what are the entities between which truth is preserved in valid arguments. Arguably, the *reason* for this has to do with the properties of reflexivity and transitivity. The very notion of preservation seems to support these properties: any object $\mathfrak{a}$ preserves its own features, and for any objects $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$, if $\mathfrak{b}$ preservers a certain feature P of $\mathfrak{a}$, and $\mathfrak{c}$ preserves feature P of $\mathfrak{b}$, then $\mathfrak{c}$ preserves feature P of $\mathfrak{a}$. Since the disjunctive approach violates reflexivity and transitivity, it cannot account for this plausible fact about the notion of preservation. Notice that I nowhere appealed to specificities of the notion of *truth*. Hence, the above line of reasoning applies just as well to any other property that one may think that is preserved in valid arguments.

4.3 Validity as Content Inclusion

Another venerable explanation of logical consequence maintains that an argument is valid just in case the content of the conclusion is included in the content of its premises. The account can be traced back to Aristotle and passes through Sextus Empiricus and most prominently Kant.[19] In the early 20th century, some logicians such as Carnap [6] and Popper [33] thought that classical logic can be characterised in terms of content inclusion. As the discussion proceeded, however, a broad consensus was reached that this is not the case. The reason, in a nutshell, is that classical logic overgenerates valid arguments. In particular, it validates some arguments that allow the occurrence in the conclusion of a *subject matter* that was not present in the premises, and this is deemed incompatible with the idea that the content of the premises includes that of the conclusion. The point was famously made by Parry [32]:

> If a system contains the assertion that two points determine a straight line, does the theorem necessarily follow that either two points determine a straight line or the moon

[17] One should not confuse (Set-Set*) with the disjunction property of intuitionistic logic. The mentioned property holds for theorems (if $A \vee B$ is an intuitionistic theorem, then A is a theorem or B is a theorem) but not for valid arguments ($p \vee \neg p$ entails $p \vee \neg p$ in intuitionistic logic, but it neither entails p nor $\neg p$).

[18] One may perhaps understand the argument of this subsection as an elaboration of the complaint made by Gareth Evans (quoted in [45, 47]). One reading of Evans complaint is that, once disjunctive multiple conclusions are properly understood, they are nothing more than single-conclusions in disguise.

[19] See Ferguson [16] for a nice summary.

is made of green cheese? No, for the system may contain no terms from which 'moon,' etc., can be defined.

This is why, in the last decades, the literature on logics of content inclusion focuses mostly on non-classical systems. Accordingly, I shall not restrict my attention to classical logic in this subsection. Rather, I will make some general considerations that are relevant for (the multiple-conclusion counterparts of) many systems.

Logics of content inclusion are usually developed in a propositional language and a single conclusion framework. It is standard to impose on them a syntactic restriction that Parry called the *proscriptive principle:*

(PP) $\Gamma \multimap A$ only if $Var(A) \subseteq Var(\Gamma)$

where $Var(\Gamma)$ is the set of propositional variables occurring as subformulas in Γ, and likewise for A. The idea is that PP warrants that no novel subject matter appears in the conclusion of a valid argument, and thus avoids the kind of problems that affected classical logic. The question arising now is how we should extend PP to the multiple-conclusion framework.

I submit that, if the relata of logical consequence are assumed to be sets, then a natural answer to this question is:

(PP*) $\Gamma \multimap \Delta$ only if $Var(\Delta) \subseteq Var(\Gamma)$

This warrants that no conclusion introduces a subject matter that is absent in the premises, and thus allows us to read $\Gamma \multimap \Delta$ as saying that the content of Δ is included in that of Γ. Under this generalisation of PP, however, the disjunctive reading of conclusions runs into troubles. Arguably, the relation of content inclusion is like that of preservation in that it supports reflexivity and transitivity: any content $\mathfrak{a}$ is included in itself,[20] and for any contents $\mathfrak{a}$, $\mathfrak{b}$ and $\mathfrak{c}$, if $\mathfrak{a}$ is included in $\mathfrak{b}$ and $\mathfrak{b}$ is included in $\mathfrak{c}$, then $\mathfrak{a}$ is included in $\mathfrak{c}$. But the disjunctive approach will induce counterexamples to this. First, for any plausible choice of $\multimap$ we will have $\top \not\multimap \bot$, which, by the stipulation that $\bigvee(\varnothing)$ amounts to $\bot$ and $\bigwedge(\varnothing)$ amounts to $\top$, implies $\varnothing \not\multimap \varnothing$. Second, for many plausible choices of $\multimap$ we will have $\{p \vee q\} \multimap \{p, q\}$ and $\{p, q\} \multimap \{p \wedge q\}$ (these validities do not violate PP*); but of course we will also have $\{p \vee q\} \not\multimap \{p \wedge q\}$. For the conjunctive reading of conclusions, on the other hand, similar issues do not arise. For any reasonable choice of $\multimap$ we will have $\varnothing \multimap \varnothing$, avoiding the first problem, and $\{p \vee q\} \not\multimap \{p, q\}$, avoiding the second one. Notice also that since now $\varnothing$ behaves as $\top$ everywhere, we will also have that $\Gamma \multimap \varnothing$ for any Γ; this is coherent with a plausible interpretation of $\varnothing$, which is that it lacks any content.

One might perhaps object that, in a way, PP* begs the question in favour of the conjunctive approach. If the relation $\multimap$ is assumed to induce the disjunctive approach, then a different generalisation of the proscriptive principle should be imposed on it. For instance, Ciuni et. al. [10] work with logics with disjunctive multiple conclusions that satisfy the restriction

(PP**) $\Gamma \multimap \Delta$ only if $Var(\Delta') \subseteq Var(\Gamma)$ for some $\Delta' \subseteq \Delta$

This warrants that there is a subset of the conclusions that does not introduce a subject matter that is absent in the premises; thus, it allows us to read $\Gamma \multimap \Delta$ as saying that there is a subset of Δ whose content is included in that of Γ. However, I think that PP** is not a sufficiently demanding generalisation of PP. It allows for the reappearance, at the structural level, of the kind

[20] Of course, I am working with a *non-strict* notion of content inclusion here. Strict notions of content inclusion would justify non-reflexivity, and even require irreflexivity.

of phenomena that motivated the abandonment of classical logic in the first place. For many choices of $\dashv\!\vdash$ (for instance, the systems studied by Ciuni et al.) we will have validities such as $\{p\} \dashv\!\vdash \{p, q\}$. And I do not see why this is more innocuous than $\{p\} \dashv\!\vdash \{p \vee q\}$. If, following Parry's example, "Either two points determine a straight line or the moon is made of green cheese" does not follow from "Two points determine a straight line" because the language of geometry might not even have the means to talk about the moon, cheese and so on, then, by parity of reasoning, a set comprising the statements "Two points determine a straight line" and "The moon is made of green cheese" should not follow from the former of these two statements alone, for the very same reasons. In a way, PP** allows us to extend logics of content inclusion to a multiple-conclusion framework, but at the cost of giving up content inclusion.

4.4 Validity as Existence of a Proof

The philosophical standpoint known as *logical inferentialism* maintains that the meaning of logical constants is determined by the rules that govern their behaviour. These rules are assumed to be sound without further justification. Then, an argument is said to be valid if and only if there is a proof that goes from the premises to the conclusion(s) and only uses sound rules of inference.[21]

Steinberger [47] already provided a battery of reasons to think that typical multiple-conclusion systems (viz. systems inducing a disjunctive reading of conclusions) are not compatible with logical inferentialism. Here, however, I will rehearse an independent argument that I made elsewhere [17], which bears on the properties of reflexivity and transitivity. While there is no space to present the argument in full here, I offer a brief sketch of how it goes.

We focus on the metalinguistic comma that is used to aggregate premises and/or conclusions. In a nutshell, we present an analogy between, on the one hand, the comma as it behaves in systems with disjunctive multiple conclusions, and on the other, Prior's infamous connective TONK. As is well-known, Prior [35] presents TONK as an alleged counterexample to logical inferentialism; the idea is that the constant is meaningless or somehow illegitimate, and thus it is not the case that any set of rules determines a meaningful or legitimate constant. The analogy we present shows that TONK and the comma have much in common; indeed, the latter can be understood as nothing more a structural incarnation of the former. Arguably, then, whatever philosophical story one has to tell about TONK, there are good reasons to tell a similar story about the comma, and viceversa.

The first and most noticeable similarity between TONK and the comma stems from the rules governing these expressions. TONK can be characterised by means of the rules[22]

$$\text{L-TONK}\ \frac{A \dashv\!\vdash B}{C \ \text{TONK}\ A \dashv\!\vdash B} \qquad\qquad \text{R-TONK}\ \frac{A \dashv\!\vdash B}{A \dashv\!\vdash B \ \text{TONK}\ C}$$

The comma, in turn, can be characterised by means of the rules of left and right weakening, which for the sake of the analogy we restate as follows:

$$\text{L-SET}\ \frac{\Gamma \dashv\!\vdash \Delta}{\Sigma, \Gamma \dashv\!\vdash \Delta} \qquad\qquad \text{R-SET}\ \frac{\Gamma \dashv\!\vdash \Delta}{\Gamma \dashv\!\vdash \Delta, \Sigma}$$

where $\Gamma, \Delta, \ldots$ are sets of sentences. It is apparent that these two pairs of rules are formally identical: each rule for the comma results by taking the corresponding rule for TONK and uniformly

[21] I refer the reader to Murzi and Steinberger [28] for a gentle overview of inferentialism in general, and logical inferentialism in particular.

[22] These are the usual rules for TONK (see e.g. [18, 39]) with the only difference that we restrict them to a single-conclusion and single-premise framework.

replacing arbitrary formulas with sets and TONK with the comma. Both expressions are introduced as conjunctions on the left-hand side of the turnstile, and as disjunctions on the right-hand side—we could say that they are ambiguous in a similar way.

A second important similarity concerns the pathological character that both TONK and the comma display. It is well-known that TONK does not get along with the reflexivity and transitivity of consequence. If $\dashv$ satisfies reflexivity and transitivity for formulas, then the rules of TONK trivialise it:

$$\dfrac{\dfrac{\overline{A \dashv A}}{A \dashv A \textsc{ tonk } B} \qquad \dfrac{\overline{B \dashv B}}{A \textsc{ tonk } B \dashv B}}{A \dashv B}$$

Now, the comma does not get along with reflexivity and transitivity either. If $\dashv$ satisfies either reflexivity for sets, or reflexivity for formulas and transitivity for sets, then the rules for the comma make it trivial and quasi-trivial, respectively:

$$\dfrac{\dfrac{\overline{\varnothing \dashv \varnothing}}{\Sigma \dashv \varnothing}}{\Sigma \dashv \Pi} \qquad\qquad \dfrac{\dfrac{\dfrac{\overline{A \dashv A}}{\Gamma \dashv A}}{\Gamma \dashv \Gamma, \Delta} \qquad \dfrac{\dfrac{\overline{B \dashv B}}{B \dashv \Delta}}{\Gamma, \Delta \dashv \Delta}}{\Gamma \dashv \Delta}$$

(where $A \in \Gamma$ and $B \in \Delta$). Thus, with slight variations, both TONK and the comma are incompatible with the consequence relation being reflexive and transitive.

The last parallel we will highlight here is that, indeed, TONK and the comma have been treated likewise in the literature. A few attempts have been made to design logical systems where TONK is admissible without triviality. Cook, for instance [12], defined a non-transitive but reflexive system where the rules of TONK can be conservatively added. Fjellstad, however [18], convincingly argued that a system for TONK should be both non-transitive and non-reflexive; the main reason is that, in a sequent calculus containing an axiom of reflexivity, the rules of TONK fail to uniquely define a connective—which undermines the idea that the calculus admits the addition of *the* connective TONK, as opposed to a family of connectives.[23] Now, of course, reflexivity and transitivity are the key structural properties that fail in typical multiple-conclusion systems. Then, we could say that, since Gentzen, our sequent calculi avoid triviality by means of the same kind of trick that we do when we want to get away with TONK.

As we announced, the upshot of the analogy is that TONK and the comma are beasts of the same blood, and indeed, the latter can be seen as a structural incarnation of the former. The philosophical moral is that, whatever story we may have to tell about TONK, we should arguably tell a similar story about the comma, and viceversa. In particular: some inferentialist follow the trace of Belnap [2] and think that TONK is unacceptable only *relative* to certain background assumptions about the notion of logical consequence.[24] Those who follow this path *may* reject TONK and welcome multiple conclusions at the same time, as long as they claim that our notion of consequence is transitive and reflexive for formulas but not for sets. Many other inferentialist, however, follow the traces of Prawitz [34] and Dummett [15], and think that TONK is unacceptable in an inherent or *absolute* sense—the reason being that its rules are not in harmony.[25] Those who follow this

[23] Roughly, a connective is uniquely defined in a calculus just in case it is intersubstitutable in inference without loss of validity with any other connective that has formally identical rules. See Belnap [2] for the precise definition.
[24] See, for instance, Cook [12], Ripley [39] and Dicher [13].
[25] See, for instance, Read [37], Tennant [51] and Francez [20].

path will have a much harder time justifying why the comma of typical multiple-conclusion systems should not also be regarded as unacceptable.[26,27] Absent some such justification, they seem forced to part ways with typical multiple conclusions.

It goes without saying that conjunctive multiple conclusions are not subject to the kind of analogy we discussed, for they are governed by entirely different patterns of inference.

4.5 Takeaway

When we talk about the reflexivity and transitivity of logical consequence, we usually have in mind some non-relation-theoretic variants of these properties. In this section we have seen that *proper* reflexivity and transitivity can be of philosophical significance; indeed, they seem to be tightly related to some of our most entrenched ways of thinking about logical consequence.

5 Closing Remarks

In this paper, I explored some technical and philosophical aspects of an approach to multiple conclusions that is often employed in mathematical logic, has recently been shown to be useful in the conceptual justification of certain non-classical systems (namely, the non-contractive ones), and yet, has gone largely unnoticed by the philosophical community. I defined and analysed a presentation of multiple-conclusion classical logic where conclusions are read conjunctively. I argued that we can find arguments with conjunctive multiple conclusions in natural language. Lastly, I claimed that the fact that the disjunctive and the conjunctive approaches have different policies towards reflexivity and transitivity has philosophical consequences; in particular, it is related to various aspects in which the conjunctive reading seems to behave in a more satisfactory way than the disjunctive one. I hope that the previous pages awaken the reader's curiosity about the structure, informal reading and explanation of our claims of logical consequence. After all, following Tarski [49], "In considerations of a general theoretical nature, the proper concept of consequence must be placed in the foreground".

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[26] Of course, harmony, as usually defined, is a property of object-language constants (as TONK) and not of structural expressions (as the comma). So the point of the analogy cannot be that the comma is literally not harmonious. The point, rather, is that structural expressions can be ill behaved as well, and hence, we should impose on them similar meaning-theoretic constraints as the ones we impose on object-language constants. Thus, it is what we might call the structural counterpart of harmony what is at stake in the case of the comma.

[27] Note the following: the Dummettian cannot just say that, under the disjunctive reading of conclusions, reflexivity and transitivity for sets are invalid, the cut rules are valid and the rules of the comma are harmonious. Because the rules of the comma are formally identical to the rules of TONK, and since harmony is to be understood as an internal or intrinsic property of a pair rules (viz. it is not relative to whatever background structural rules we deem valid), it would follow that the rules of TONK are harmonious as well—unless the Dummettian gives, in addition, some intrinsic difference between the pairs of rules L-TONK and R-TONK, on the one hand, and L-SET and R-SET on the other: that is the challenge!

it is based) and making invaluable contributions. Lastly, I thank two anonymous reviewers of this journal, whose insightful comments helped to improve the piece.

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Appendix

We provide syntactic proofs of admissibility of rules Cut+ and Cut× in $\mathcal{S}_{\mathbf{cCL}}$.

Let us start with Cut+. The case of this rule is easy, because the move from its left premise-sequent alone to its conclusion-sequent is already licensed by rule RaW. So, to show that Cut+ is admissible in $\mathcal{S}_{\mathbf{cCL}}$, it suffices to show that RaW is admissible.

Lemma 15 (Admissibility of RaW in $\mathcal{S}_{\mathbf{cCL}}$)**.** If $\Gamma \Rightarrow \Delta, A$ is provable in $\mathcal{S}_{\mathbf{cCL}}$, then $\Gamma \Rightarrow \Delta$ is provable as well.

Proof. We define the *height* of a derivation as the number of nodes in its longest branch, minus 1. We proceed by induction on the height of the derivation of $\Gamma \Rightarrow \Delta, A$.

- *Base step.* The height is 0. Then $\Gamma \Rightarrow \Delta, A$ is an axiom. It follows that $\Delta = \obslash$, and thus, $\Gamma \Rightarrow \Delta$ is just an instance of R$\obslash$.
- *Inductive step.* The height is $n > 0$. There are two cases: (a) $\Delta = \obslash$ and (b) $\Delta \neq \obslash$. In case (a), $\Gamma \Rightarrow \Delta$ is again an instance of R$\obslash$. In case (b), we can assume w.l.o.g.[28] that the target application of RaW has the following form:

$$\cfrac{\cfrac{\Gamma \Rightarrow \Delta_1, A \qquad \Gamma \Rightarrow \Delta_2 \qquad \dots \qquad \Gamma \Rightarrow \Delta_m}{\Gamma \Rightarrow \Delta, A}\,\mathrm{SM}}{\Gamma \Rightarrow \Delta}$$

Then we just permute the applications of SM and RaW:

$$\cfrac{\cfrac{\Gamma \Rightarrow \Delta_1, A}{\Gamma \Rightarrow \Delta_1} \qquad \Gamma \Rightarrow \Delta_2 \qquad \dots \qquad \Gamma \Rightarrow \Delta_m}{\Gamma \Rightarrow \Delta}\,\mathrm{SM}$$

and since the derivation of $\Gamma \Rightarrow \Delta_1, A$ is of height at most $n-1$, the new application of RaW is admissible by our inductive hypothesis.

□

Fact 16 (Admissibility of Cut+ in $\mathcal{S}_{\mathbf{cCL}}$)**.** If $\Gamma \Rightarrow \Delta, A$ and $A, \Gamma \Rightarrow \Delta$ are both provable in $\mathcal{S}_{\mathbf{cCL}}$, $\Gamma \Rightarrow \Delta$ is provable as well.

Proof. If $\Gamma \Rightarrow \Delta, A$ is provable, then $\Gamma \Rightarrow \Delta$ is provable as well by the admissibility of RaW. □

Now, let us turn to Cut×. In the case of this rule, we will rely on Negri and von Plato's proof that the following single-conclusioned version of cut is admissible in $\mathcal{S}_{\mathbf{CL}}$:

$$\frac{\Gamma \Rightarrow A \qquad A, \Delta \Rightarrow C}{\Gamma, \Delta \Rightarrow C}\,\text{sc-Cut}\times$$

[28] Here and below, the 'w.l.o.g.' qualifications are because the respective derivations may have the premises of SM in a different order. But we can always rearrange them to match the form depicted.

Theorem 17 (Admissibility of sc-Cut× in $\mathcal{S}_{\mathbf{CL}}$)**.** *If sequents $\Gamma \Rightarrow A$ and $A, \Delta \Rightarrow C$ are provable in $\mathcal{S}_{\mathbf{CL}}$, then $\Gamma, \Delta \Rightarrow C$ is provable as well.*

(The proof can be found in [29, p. 117]). We will also use two more preliminary results about $\mathcal{S}_{\mathbf{cCL}}$. First, LW is admissible in the system:

Lemma 18 (Admissibility of LW in $\mathcal{S}_{\mathbf{cCL}}$)**.** If $\Gamma \Rightarrow \Delta$ is provable in $\mathcal{S}_{\mathbf{cCL}}$, then $A, \Gamma \Rightarrow \Delta$ is provable as well.

Proof. We proceed by induction on the height of the derivation of $\Gamma \Rightarrow \Delta$.

- *Base step.* The height is 0. Then $\Gamma \Rightarrow \Delta$ is an axiom. A quick inspection of the axioms of $\mathcal{S}_{\mathbf{cCL}}$ shows that $A, \Gamma \Rightarrow \Delta$ must be an axiom as well.
- *Inductive step.* The height is $n > 0$. There are eight cases, corresponding to the last rule applied in the derivation of $\Gamma \Rightarrow \Delta$. We just consider the case of SM—the other rules are dealt with similarly. In this case, the target application of LW has the form

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta_1 \qquad \dots \qquad \Gamma \Rightarrow \Delta_m}{\Gamma \Rightarrow \Delta}\,\text{SM}}{A, \Gamma \Rightarrow \Delta}$$

Then we just permute the applications of SM and LW:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta_1}{A, \Gamma \Rightarrow \Delta_1} \qquad \dots \qquad \dfrac{\Gamma \Rightarrow \Delta_m}{A, \Gamma \Rightarrow \Delta_m}}{A, \Gamma \Rightarrow \Delta}\,\text{SM}$$

and since the derivations of $\Gamma \Rightarrow \Delta_1, ..., \Gamma \Rightarrow \Delta_m$ have all heights at most $n - 1$, by our inductive hypothesis the new applications of LW are all admissible.

□

Second, system $\mathcal{S}_{\mathbf{cCL}}$ is conservative over $\mathcal{S}_{\mathbf{CL}}$, in the sense that if a single-conclusioned sequent is provable in the former, it was already provable in the latter.

Lemma 19 (Conservativity of $\mathcal{S}_{\mathbf{cCL}}$)**.** If a single-conclusioned sequent is provable in $\mathcal{S}_{\mathbf{cCL}}$, then it is also provable in $\mathcal{S}_{\mathbf{CL}}$.

Proof. Take any single-conclusioned sequent $\Gamma \Rightarrow C$ and suppose it is provable in $\mathcal{S}_{\mathbf{cCL}}$. We proceed by induction on the height of its derivation.

- *Base step.* The height is 0. Then $\Gamma \Rightarrow C$ is an axiom of $\mathcal{S}_{\mathbf{cCL}}$, and since it has a non-empty set of conclusions, it must be also an axiom of $\mathcal{S}_{\mathbf{CL}}$.
- *Inductive step.* The height is $n > 0$. There are two subcases.
  (a) The last rule applied is a rule $\mathcal{R}$ of $\mathcal{S}_{\mathbf{CL}}$. Then, the premises of this last rule application are also single-conclusioned. Since their derivations are of height at most $n - 1$, by inductive hypothesis it follows that they are provable in $\mathcal{S}_{\mathbf{CL}}$. So, take their respective derivations in $\mathcal{S}_{\mathbf{CL}}$, and extend these derivations by applying $\mathcal{R}$. We obtained a derivation of $\Gamma \Rightarrow C$ in $\mathcal{S}_{\mathbf{CL}}$.
  (b) The last rule applied is SM. Then, w.l.o.g. we can assume that the last step of the derivation has the following form:

$$\dfrac{\Gamma \Rightarrow C \qquad \dfrac{}{\Gamma \Rightarrow \varnothing}\,\text{R}\oslash \qquad \dots \qquad \dfrac{}{\Gamma \Rightarrow \varnothing}\,\text{R}\oslash}{\Gamma \Rightarrow C}$$

Since the sequent $\Gamma \Rightarrow C$ is single-conclusioned, and the height of its derivation is $n - 1$, by inductive hypothesis it follows that it is provable in $\mathcal{S}_{\mathbf{CL}}$. So, we are done.

□

With the above preliminaries at hand, we are ready to prove our target result:

Theorem 20 (Admissibility of Cut× in $\mathcal{S}_{\mathbf{cCL}}$)**.** *If sequents $\Gamma \Rightarrow \Delta, A$ and $A, \Sigma \Rightarrow \Pi$ are provable in $\mathcal{S}_{\mathbf{cCL}}$, then $\Gamma, \Sigma \Rightarrow \Delta, \Pi$ is provable as well.*

Proof. We call $\Gamma \Rightarrow \Delta, A$ and $A, \Sigma \Rightarrow \Pi$ the 'cut-premises', and $\Gamma, \Sigma \Rightarrow \Delta, \Pi$ the 'cut-conclusion'. We define the *cut-height* as the sum of the heights of the derivations of the cut-premises plus 1. The proof proceeds by induction on the cut-height.

- *Base step.* The cut-height is 1. This means that the cut-premises are both axioms. By inspection of the axioms of $\mathcal{S}_{\mathbf{cCL}}$ we know that the target application of Cut× has one the following forms:

$$\text{(a)}\ \dfrac{\dfrac{}{A, \Gamma' \Rightarrow A}\text{Id-}at \qquad A, \Sigma \Rightarrow \Pi}{A, \Gamma', \Sigma \Rightarrow \Pi} \qquad\qquad \text{(b)}\ \dfrac{\dfrac{}{\bot, \Gamma' \Rightarrow A}\text{L}\bot \qquad A, \Sigma \Rightarrow \Pi}{\bot, \Gamma', \Sigma \Rightarrow \Pi}$$

  In subcase (a), the cut-conclusion can be obtained from the right cut-premise alone by a number (perhaps 0) of applications of LW, which by Lemma 18 we know is admissible in $\mathcal{S}_{\mathbf{cCL}}$. In subcase (b), if the cut-premise $A, \Sigma \Rightarrow \Pi$ is an instance of R⦰, then the cut-conclusion is an instance of R⦰ as well; and if $A, \Sigma \Rightarrow \Pi$ is an instance of some other axiom, then $\Pi = \{\pi\}$ for some formula π, and the cut-conclusion $\bot, \Gamma', \Sigma \Rightarrow \pi$ is an instance of L⊥.
- *Inductive step.* The cut-height is $n > 1$. There are two cases:
  (a) Both cut-premises are single-conclusioned. In this case, by the conservativity of $\mathcal{S}_{\mathbf{cCL}}$ (Lemma 19) we know that they are both provable in $\mathcal{S}_{\mathbf{CL}}$. Then, by the admissibility of sc-Cut× in $\mathcal{S}_{\mathbf{CL}}$ (Theorem 17) we know that the cut-conclusion is provable in $\mathcal{S}_{\mathbf{CL}}$. Lastly, by the fact $\mathcal{S}_{\mathbf{CL}}$ is a subsystem of $\mathcal{S}_{\mathbf{cCL}}$, we know that the cut-conclusion is provable in $\mathcal{S}_{\mathbf{cCL}}$.
  (b) At least one cut-premise is multiple-conclusioned. We just cover the subcase where $A, \Sigma \Rightarrow \Pi$ is multiple-conclusioned, since the other subcase is symmetric. Inspection of the rules of $\mathcal{S}_{\mathbf{cCL}}$ shows that, if $A, \Sigma \Rightarrow \Pi$ is multiple-conclusioned, then we can assume that the target application of Cut× is of the form

$$\dfrac{\Gamma \Rightarrow \Delta, A \qquad \dfrac{A, \Sigma \Rightarrow \Pi_1 \quad \dots \quad A, \Sigma \Rightarrow \Pi_m}{A, \Sigma \Rightarrow \Pi}\text{SM}}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}$$

  Then, we proceed as follows, where the rule applications marked with dashed lines are to be performed only if $\Delta \neq ⦰$:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, A \quad A, \Sigma \Rightarrow \Pi_1}{\Gamma, \Sigma \Rightarrow \Delta, \Pi_1} \quad \dfrac{\dfrac{\Gamma \Rightarrow \Delta, A \quad A, \Sigma \Rightarrow \Pi_2}{\Gamma, \Sigma \Rightarrow \Delta, \Pi_2}}{\Gamma, \Sigma \Rightarrow \Pi_2}\text{RaW} \quad \dots \quad \dfrac{\dfrac{\Gamma \Rightarrow \Delta, A \quad A, \Sigma \Rightarrow \Pi_m}{\Gamma, \Sigma \Rightarrow \Delta, \Pi_m}}{\Gamma, \Sigma \Rightarrow \Pi_m}\text{RaW}}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\text{SM}$$

  The applications of RaW (if any) are warranted by the admissibility of this rule in $\mathcal{S}_{\mathbf{cCL}}$ (Lemma 15). Also, it is easy to check that the new applications of Cut× are all of cut-height at most $n - 1$; hence, they are admissible by inductive hypothesis.

□

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