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MATHEMATICAL SKEPTICISM: A SKETCH WITH HISTORIAN IN FOREGROUND

A skeptical look should be cast upon mathematical theories...
(Philip J. Davis and Reuben Hersh[1])

HISTORICAL AMNESIA

We know very little about mathematical skepticism in modern times.[2] Imre Lakatos once remarked that "in discussing modern efforts to establish foundations for mathematical knowledge one tends to forget that these are but a chapter in the great effort to overcome skepticism by establishing foundations for knowledge in general."[3] And in a sense he was clearly right: modern thought—with its new discoveries in mathematical sciences, the mathematization of physics, the spreading of Pyrrhonist doctrines, the centrality of epistemological foundationalism and the diffusion of the geometrical method in philosophy— was the most natural arena in

[1] Philip J. Davis and Reuben Hersh, *Descartes' Dream: The World according to Mathematics* (London, 1986), 58.

[2] An issue of central importance in sixteenth-century philosophy of mathematics was whether mathematics could be correctly interpreted as a science from an Aristotelian perspective. Known as the "debate on the certainty of mathematics" (*Quaestio de certitudine mathematicarum*), the problem has been amply studied, but it has to be stressed that it has got very little to do with skepticism. It develops before the publication of Sextus Empiricus' works and within a thoroughly Aristotelian framework; see the series of articles published by Giulio C. Giacobbe in *Physis-Rivista Internazionale di Storia della Scienza*: "Il *Commentarium de certitudine mathematicarum disciplinarum* di Alessandro Piccolomini," 14 (1972), 162-93; "Francesco Barozzi e la *Quaestio de certitudine mathematicarum*," 14 (1972), 357-74; "Epigoni nel seicento della *Quaestio de certitudine mathematicarum*: Giuseppe Biancani," 18 (1976), 5-40; "A progressive Jesuit in the Renaissance *Quaestio de certitudine mathematicarum*: Benito Pereyra," 19.1-4 (1977), 51-86; William A. Wallace "The Certitude of Science in Late Medieval and Renaissance Thought," *History of Philosophy Quarterly*, 3 (1986), 281-91; and Paolo Mancosu "Aristotelian Logic and Euclidean Mathematics: Seventeenth-Century Developments of the *Quaestio de certitudine mathematicarum*," *Studies in History and Philosophy of Science*, 23 (1992), 241-65.

[3] Imre Lakatos, "Infinite Regress and Foundations of Mathematics," Lakatos, *Philosophical Papers*, ed. John Worrall and Gregory Currie, 2 vols. (Cambridge, UK, 1978), 2:3-23, here 4.

which skepticism and mathematics could confront each other.[4] The problem remains, however, that no investigation of the whole topic has yet been attempted. Thus, as far as we know, mathematical certainties should have clashed with skeptical doubts, but whether and to what extent there was indeed a historical debate on mathematical skepticism in modern thought remains to be ascertained.

The worst way to cope with the conceptual amnesia highlighted by Lakatos would be to implant an utterly new memory in our system of knowledge. Luckily, there is no need to fabricate an ideal history of the Zeitgeist. We can be more moderately Platonist and work toward a recollection of our intellectual past by uncovering the archeological origins of our knowledge. There are few primary sources containing an explicit and extensive discussion of mathematical skepticism in modern times,[5] but one of them is the *Histoire des mathématiques* by Jean-Etienne Montucla, the monumental work that marks the beginning of a truly scientific approach to the historiography of mathematics.[6] It is a rich mine that I intend to exploit in this paper.

[4] Cf. Enrico de Angelis, *Il metodo geometrico nella filosofia del seicento* (Pisa, 1964) for an interesting historical reconstruction of the use of geometrical methods in seventeenth- century philosophy.

[5] To my knowledge, none of the subsequent histories of mathematics has ever again dedicated so much space to mathematical skepticism, and this is not by chance, if my interpretation of the foundationalist role of mathematical skepticism is correct. Before Montucla, I know of only a few other texts which discuss Sextus' objections at some length, among which are Gianfrancesco Pico della Mirandola's *Examen ... vanitatis doctrinae gentium, et veritatis Christianae disciplinae* (Mirandulae [Mirandola], 1520), about which see below; the *De Veritatibus geometricis libri II prior contra scepticos et Sextum Empiricum* (Hafniae [Copenhagen], 1656) by Wilhelmus Langius (Villum Lange, 1624-1682); Pierre-Daniel Huet, *Demonstratio evangelica* (Paris, 1679; Amsterdam, 1680; Paris, 1690; and Leipzig, 1703), about which see below; and a long section in Jean-Pierre de Crousaz, *Examen du Pirrhonisme ancien et moderne* (The Hague, 1733) dedicated to the relation between skeptical doubts and mathematical certainty. Of course, to these texts one must add Descartes' discussion of mathematical skepticism in the *Meditations* and the debate he engendered, Bayle's *Dictionary*, Hobbes' polemic against analytic geometry and Hume's remarks on the nature of mathematics. I hope to study these sources in my future research.

[6] Edouard Doublet, "Montucla: l'historien des mathématiques," *Bulletin de l'Observatoire de Lyon*, 5 (1913), 3-8: "L' *Histoire des Mathématiques* de Montucla est un livre précieux pour tous ceux qui s'intéressent à l'histoire des sciences. A leurs yeux, cet ouvrage n'a qu'un tort: - il est fort difficile de se le procurer. ... D'autres Histoires des Mathématiques ont paru au dix-neuvième siècle. Leurs auteurs ont assurément trouvé de grands secours dans le travail de Jean-Etienne Montucla." On Montucla see Auguste-Savinien le Blond, *Notice historique sur la vie et les ouvrages de Jean-Etienne Montucla* (Paris, 1800), rpt. in Jean-Etienne Montucla, *Histoire des mathématiques*, ed. Charles Naux (Paris, 1968), 4:662-72: "Sur la vie et les ouvrages de Montucla, Extrait de la Notice historique lue par Auguste-Savinien le Blond à la Société de Versailles, le 15 janvier 1800, avec des additions par Jérôme de Lande;" George Sarton, "Montucla (1725-1799): His Life and Works," *Osiris*, 1 (1936), 519-67; Kurt Vogel, "L'historiographie mathématique avant Montucla," *Actes du XIe Congrès International d'Histoire des Sciences (1965)*, vol. 2 (Wroclaw, 1968), 179-84; Dirk J. Struik, "The historiography of mathematics from Proklos to Cantor," *Schriftenreihe für Geschichte der Naturwissenschaften, Technik und Medizin*, 17 (1980), 1-22; and Noel M. Swerdlow, "Montucla's Legacy: The History of the Exact Sciences," *Journal of the History of Ideas*, 54 (1993), 299-328.

When Montucla died, in 1799, he had already published the first two volumes of the new edition of his *Histoire des mathématiques*.[7] Montucla had dedicated ample space to the discussion of mathematical skepticism in the first and much shorter edition, which had appeared in 1758. The second edition differed from the first in several ways: there were a number of new details that enriched the original version;[8] many remarks previously placed in the footnotes were now inserted in the main text; and there was a new section, number 9, which illustrated the roots of our conception of mathematical certainty. The part on mathematical skepticism, however, remained unmodified. After discussing the nature and internal organization of mathematics in general, Montucla shows in section 6 of the first volume how all great philosophers have always admired mathematics. He then introduces the skeptical attack, which becomes the central subject of section 7, which is dedicated to Sextus Empiricus and Epicurus, while section 8 discusses the importance of mathematics with respect to the other sciences.

Montucla approaches the topic of mathematical skepticism in several stages which, for the sake of simplicity, may be organized into a broad premise, five restricting steps and two further assumptions with a clearly anti-skeptical purport.

Premise) The premise concerns the nature of mathematics itself. Geometry and arithmetic, the most fundamental branches of mathematics dealing with the physical continuum and the discrete manifold, are *abstract*, *true*, *intuitive* and *certain*. They have what Montucla calls an *origin métaphysique*: their elements are the result of a process of intellectual abstraction from everyday experience. Because of its abstract nature, mathematics provides effective knowledge about reality, while possessing at the same time a self-evident basis. Its statements are true descriptions of states of the world, and its basic notions are obvious and epistemologically intuitive. Finally, because of its deductive structure, mathematics is a body of knowledge which enjoys the highest degree of logical certainty.

i) In the past, there have been several types of detractors of mathematics. However, the only ones deserving a theoretical discussion are the Pyrrhonists, who attempted to undermine mathematical knowledge by means of epistemological arguments.

ii) Not every skeptical argument is of interest. In general, Pyrrhonism is a ridiculous philosophy. Skeptics can employ their subtle sophisms and paradoxes

[7] Jean-Etienne Montucla, *Histoire des mathématiques*, 2 vols. (Paris, 1758). Of the second new edition, revised and augmented in four volumes, only the first two were edited by Montucla (1799). After his death, the remaining two volumes were completed by Jérôme Lalande (1802). The entire set was reprinted with a preface by Charles Naux (Paris, 1968).

[8] Ibid., 2nd ed. (Paris, 1799-1802), 1:20-24, concerning mixed mathematics and the theory of the point are identical to the original version's 1:23-28, whereas the following discussion of Epicurean philosophy is enlarged. On 1:29 of the new edition we find a new, long paragraph on Pico della Mirandola, which replaces a shorter note in the first edition. According to Montucla, *le célèbre Pic del la Mirandole* believed that theology and mathematics were incompatible. Montucla agrees with the view but concludes that *tant pis pour la théologie*. On 1:33 of the new edition we find a new section on mathematical certainty which is fundamentally Cartesian in its nature.

only to engender confusion among the simple-minded, but the real outcome of their attacks is a confutation of themselves.[9]

iii) Only some skeptical arguments deserve to be discussed, namely those put forward against geometry by Sextus Empiricus in *Contra Geometras*. Note that no reference is made to Sextus Empiricus' anti-arithmetical attack contained in *Contra Arithmeticos*.

iv) Not all anti-geometrical arguments are of equal importance, though. Only the epistemological arguments against the nature of elementary geometrical objects, that is point, line, and surface, are worth a reply, for the latter can help the mathematician to cast further light on the solid foundation of the discipline. No reference is made to the first half of Sextus Empiricus' attack, which in *Contra Geometras* attempts to undermine the value of the process of postulating premises from which necessary conclusions can then be inferred.[10]

v) Sextus' arguments against the possibility of geometrical objects are all alike, so in order to have a clear grasp of their nature and shortcomings it is sufficient to analyze and refute just a sample of them.

Having finally reached the point he wishes to discuss, Montucla makes explicit his two anti-skeptical premises:

a) Geometrical objects are of such an abstract nature that they are bound to raise questions and uncertainties in those who do not understand them properly. This assumption obviously cuts the ground from under the skeptical challenge: casting doubt on the nature of geometrical entities becomes now tantamount to showing how little one has understood them.

b) Before discussing mathematical skepticism, Montucla invites the reader to endorse what he calls "a necessary rule in the search for truth:" even if the skeptical difficulties concerning the initial principles were insuperable, they should not affect our trust in the validity of the mathematical consequences established on the basis of such principles and reasoning, whose evidence cannot be questioned. An obvious anti-foundationalist claim which purports to safeguard Montucla's pragmatic approach to the utility and effectiveness of mathematics.

[9] Ibid., 2nd ed., 1:21: "Il suffiroit presque, pour répondre à ses objections, de remarquer le ridicule d'un pyrrhonisme qui va jusques à prétendre qu'il n'y a aucune démonstration, aucun moyen de se procurer la moindre certitude, pour qui les axiomes du sens commun sont de moindre poids que le témoignage des sens si souvent exposés à l'erreur; qui prétend enfin détruire et anéantir la science du raisonnement."

[10] The point is developed by Pierre-Daniel Huet in his *Demonstratio evangelica*. This is an impressive work of systematic erudition, in which Huet attempts to prove the principles of Christian religion by means of an axiomatic apparatus. The *Praefatio*, section III, and *Axiomata* IV, sections II and III, contain interesting if occasional remarks on a skeptical philosophy of mathematics. Euclidean geometry, as the fundamental branch of mathematics, is based on definitions and axioms that are widely accepted, but cannot be demonstrated. According to Huet, the probatory force of geometrical demonstrations depends, therefore, on conventions and universal consensus, and the certainty of our conclusions cannot be absolute, but remains constantly, if only hypothetically, open to falsification.

Montucla's position, just sketched, is based upon a number of interesting presuppositions, which can be grouped under three main headings.

1. The mathematization of physics.

Montucla writes in a century when mathematics enjoyed one of the less troublesome of all its great periods of development.[11] The new techniques of the infinitesimal calculus led to the development of several major branches of mathematics, which turned out to be particularly suited for the description of natural phenomena. Analysis made possible a full mathematization of physics, and it is not by chance that serious historiographical work on mathematics appeared only in the middle of the eighteenth century, after many mathematicians had started to believe that all the major discoveries in the field had been made. The great scientific fertility shown by mathematical physics provided a practical justification for a number of mathematical procedures and results which, although still found wanting in logical rigor and a clear conceptual foundation, *de facto* vindicated the epistemological practice of the mathematicians of the time who, in their turn, were inclined to value practical results more than mathematical means. Mathematical knowledge was simply supposed to provide true information about nature, while the mathematization of physics was thought to be possible and productive because reality in itself was supposed to be intrinsically mathematical. As a result, the eighteenth century saw the triumph of mechanics and the Galilean vision of nature as a book written in mathematical language, despite the gradual disappearance of the Cartesian-Newtonian God, the source of ontological stability and epistemological correspondence between certainty and truth. In *The Analyst* (1734), Berkeley had already exposed the logically unsatisfactory status of the calculus, despite its practical accomplishments. But the lack of an adequate conceptualization and systematization of the new mathematical field was not felt to be a genuine problem. Mathematical theorems and their applications did not float in the empty space of free axiomatic constructions, as it were, but were thought to repose directly on the essential nature of reality which, given their success, obviously granted them full justification. Coherence and consistency of mathematical knowledge were *semantic* concepts: they depended on, and were thought to follow from the coherence and consistency of the model provided by Nature. We must wait until the following century and the work of Bolzano, Cauchy, Abel, Dirichlet and Weierstrass, among others, for the development of a satisfactory, fully rigorous analysis that removed from geometrical concepts all appeals to spatial intuition, by means of an interpretation based upon number theory. In Montucla's time, foundationalist problems were not yet crucial.

[11] See Morris Kline, *Mathematical Thought from Ancient to Modern Times* (New York, 1972 and rpt.) for a very instructive overview of the history of mathematics in the period. I largely agree with most of what Kline says in the two more philosophical chapters entitled "Mathematics as of 1700" and "Mathematics as of 1800," but for reasons I shall clarify in the conclusion I cannot share his later criticism of the axiomatic movement and the foundationalist programs.

They became central only in the nineteenth century, when the discovery of non-Euclidean geometries and the paradoxes of set theory generated a radical dissatisfaction with the realist and intuitive interpretation of mathematics, and caused geometry to lose its centrality.

The mathematization of physics, with its applied fields and problem-solving mentality, pervades Montucla's work and provides a favorable context that probably engendered in him a certain optimism about the possibility of providing a final refutation of skeptical problems. It also goes some way toward explaining why he suggested the assumption of the "necessary rule in the search for truth," with its clearly anti-foundationalist character.

2. *The supremacy of geometry*

For centuries, Euclidean geometry represented the best model of logical and systematic thought and hence of mathematical certainty. The Greeks left a rigorous and systematic body of geometrical theorems, but only a heuristic, empirical practice of arithmetical computation. It is tempting to see the prevalence of geometrical methods in Euclid as the result of the first foundationalist crisis brought about by the discovery of the incommensurables and the consequent breakdown of Pythagoreism. But whether one endorses such an interpretation or objects to it, the point remains that geometrical methods, both in algebra and in number theory, represented for centuries the only widespread approach to mathematics.[12] Only in the seventeenth century were the conditions established for a fundamental process of algebraization of the theory of space, thanks to the introduction of algebraic methods in geometry by Viète, Fermat and Descartes[13] and then the development of the infinitesimal calculus by Newton and Leibniz. It was the beginning of a process that led to the foundation of mathematical knowledge based on arithmetic and then on set theory and mathematical logic, and in the end deprived geometry of its role as the Queen of all mathematical sciences. But it was a slow process. In number theory the acceptance of negative and imaginary numbers, two essential steps toward the de-physicalization of mathematics, was not immediate, even in Descartes. Leibniz thought that metaphysics was related to all other sciences, including geometry, as the latter was to all the other mathematical disciplines. And although Euler had rejected geometry as the basis for the calculus and tried to work only with functions, that is by means of algebraic formulae, Montucla still believed that the greatness of analytic geometry consisted in its geometrical method, not in its translation of curves into algebraic equations. As for the calculus itself, he could still refer to Newton as the greatest geometrician of Europe. No wonder he perceived *Contra Geometras*

[12] For a critical analysis of the evolution of Euclid's *Elements* see Wilbur Richard Knorr, *The Evolution of the Euclidean Elements: A Study of the Theory of Incommensurable Magnitudes and Its Significance for Early Greek Geometry* (Dordrecht, 1975), 306-13, who argues convincingly against the "foundationalist crisis" hypothesis.

[13] See Michael S. Mahoney, "The Beginning of Algebraic Thought in the Seventeenth Century," *Descartes: Philosophy, Mathematics, and Physics*, ed. Stephen Gaukroger (Sussex, 1980), 141-55, for a very clear presentation of the issue.

so close to his interests. One must wait until the first half of the nineteenth century to find Gauss describing arithmetic as the most fundamental of all mathematical disciplines. Whitehead and Russell did not even trouble to publish the fourth and last volume of *Principia Mathematica*, which was planned to provide the logical foundation of geometry, but in the eighteenth century mathematicians still referred to themselves as "geometers." In a way, Plato had been vindicated (geometrization of the world) but not yet Pythagoras (arithmetization of geometry): rigorous mathematics meant geometry, so the philosophy of mathematics meant the philosophy of geometry. The definitive replacement of the synthetic approach to the description of geometrical figures as intuitively representable (visual thinking) by a logical and purely analytic approach, made possible by algebraic thought-handling relations, rather than properties, via an operative symbolism that is ontologically free, was still to come.

3. Mathematical skepticism

Implicit in what Montucla has to say on the utility of skepticism in mathematics are three ways of interpreting Sextus' anti-mathematical arguments.
a) There is a philological approach, interested in linguistic matters or in what Sextus says on other authors, Euclid included. In different ways, this is the case in Vossius,[14] Bochner,[15] or, more interestingly, scholars such as Heiberg or Heath.
b) There is an anti-intellectualist approach, adopted for example by Gianfrancesco Pico della Mirandola. Mathematics is attacked on the basis of Sextus' arguments, with the intention of undermining the dogmatist's excessive faith in human knowledge. During the sixteenth and seventeenth centuries the main polemic target was usually Aristotelianism.[16]
c) Finally, there is a foundationalist interpretation, which employs skepticism in order to investigate the solidity and reliability of mathematical knowledge. This is the approach taken by Montucla, who explicitly connects it to Descartes.

The difference between the anti-intellectualist and the foundationalist interpretation is easily clarified once we realize that there are a number of problems Sextus

[14] Gerardii Ioannis Vossii, *De Universae Mathesios natura & constitutione liber cuis subjungitur chronologia mathematicorum* (Amsterdam, 1650). On p. 1, introducing the topic of the *scientiae mathematicae*, their nature and number, Vossius writes: "Sic voce matematon utitur Sextus Pyrrhonius, cùm libros x inscribit adversus Mathematicos. Nec enim disputat adversus Arithmeticen, & Geometriam; quàm Grammaticem, Historiam, Poëticen, Rhetoricen, Astrologiam judiciarim, Musicen, Logicen, Physicen, Ethicen." He then refers to Sextus a few more times in order to explain some linguistic matters, but never actually discusses his skeptical arguments, even when he deals critically with Epicurus and Ramus.

[15] See Salomon Bochner, *The Role of Mathematics in the Rise of Science* (Princeton, 1966), 363: "... [Sextus Empiricus'] works are boring, but important. For instance, the proemium in the poem of Parmenides comes from Sextus."

[16] Socrates, for example, objected to the utility of the study of mathematics on ethical grounds, and philosophers such as Gianfrancesco Pico della Mirandola employed skeptical arguments for anti-intellectualist and theological purposes, see his *Examen ... vanitatis doctrinae gentium, et veritatis Christianae disciplinae*, lib. I, cap. 7, 750-51, which contain a brief summary of the Sextian issues with definitions of point, line, and plane, and lib. III, cap. 5-6 against geometry, and cap. 7 against arithmetic.

never mentions in *Contra Geometras*. Although it would have been in keeping with the skeptical strategy of accumulating any sort of arguments in order to undermine the dogmatic position, Sextus never rejects what is stated by the postulates or the common notions—possibly because, whoever is the source of *Contra Geometras*, he did not mean to be cut off from discussion with other geometricians—; he does not question either the fifth postulate or the use of superposition and disregards each of the three classic problems of Greek geometry, which were so well known in his time: the duplication of the cube (how to construct the edge of a cube having twice the volume of a given cube), the trisection of an angle (how to divide a given arbitrary angle into three equal angles), and the famous quadrature of the circle (how to construct a square having an area equal to that of a given circle). We know nowadays that none of these problems can be solved, except by approximation, with an unmarked straight edge and compasses, that is by means of algebraic methods. But the three problems were still discussed as open questions in the seventeenth century. Now, a foundationalist attack against the roots of geometry, an epistemological challenge, had no great interest in investigating such issues. Considered simply as difficulties that had yet to be solved because they were particularly complex, their destiny would depend on the status of geometry as a science of space, not vice versa. However, a general denunciation of the intellectual ambitions of mathematicians could obviously take advantage of such clear cases of failure by presenting them as a reminder of the limits of human knowledge. In line with this interpretation, we observe Agrippa,[17] Sanchez, and Guy de Brués all making use of the geometrician's incapacity to square the circle to stress the limits of mathematical knowledge. When Sextus (or his source) criticized geometry he had a more scientific aim in mind.

THE SKEPTIC PORTRAYED BY THE HISTORIAN

We can now turn to Montucla's discussion of the anti-geometrical arguments. His sample consists of three skeptical paradoxes:

1) Let us assume a circle with circumference C, center O and radius r, and let us draw a radius r_n from O to every point p_n belonging to C. The sum of the sequence

[17] Heinrich Cornelius Agrippa von Nettesheim, *Of the Vanitie and Uncertaintie of Artes and Sciences* (1531), ed. and tr. Catherine M. Dunn (Northridge, Calif., 1974). On p.58 (ch. 11: "Of Mathematical sciences in general"), Agrippa writes that the mathematical sciences, thought to be the most certain, consist only of the opinions of teachers to whom great credit is given. Their objects, like a perfect sphere or a circle, do not and cannot exist. And even if mathematical theories have never been the cause of heresies, Augustine wrote that they do not further salvation but lead men into error and separate them from God, while Jerome says that they are not "sciences of Godlinesse." A note by the editor suggests Augustine's *De actis cum Felice Manichaeo*, I.10, or *Confessions*, V.3 as possible sources. But on p.75 (ch. 22: "Of Geometry"), we read that Geometry is the Princess and mother of all learnings, as Philo Judaes has called it (the source is possibly *De Agricultura*, 13). Geometricians agree on everything and discuss only points, lines, and other things. However, no geometrician has ever been able to discover how to square the circle, in spite of Archimedes' claims to the contrary.

of all the radii $r_1 + r_2 + \ldots$, that is $\sum_{i=1}^{\infty} r_i$ covers the entire surface of the circle.

Let us consider now all the concentric circles (CC), including the innermost and the outermost: they will be crossed (K) by the same number of radii (R(x), that is

$$\forall x\ \forall y\ ((R(x)\ \wedge\ CC(y,C)) \rightarrow K(x,y)).$$

But then all circles will be equal to each other, for they will all contain the same number of points, that is

$$\forall x\ \forall y\ \forall z\ (((R(x)\ \wedge\ CC(y,C)\ \wedge\ CC(z,C)) \rightarrow (K(x,y)\ \wedge\ K(x,z))) \rightarrow (y = z))$$

and this, according to Sextus, is an obvious *reductio ad absurdum* (see Fig. 1 at the end of this paper).

2) A perfect sphere touches a perfect plane at one geometrical point, which is unextended by definition. By rolling forward, the sphere draws a perfect line, which is made of a series of geometrical points. The absurd result is that now a set of unextended points gives rise to an extended line (see Fig. 2).

3) Let us assume a circle with circumference C, center O and radius r. Through every point p_n of r let us draw a circle γ_n concentric with respect to C, that is

$$\forall p\ ((p \in r) \rightarrow \exists\gamma(K(\gamma, p)\ \wedge\ CC(\gamma, C)).$$

The sum of the sequence of all concentric circles $\gamma_1 + \gamma_2 + \ldots$, that is $\sum_{i=1}^{\infty} \gamma_i$

covers the entire surface of the circle, but this is absurd, since each circle γ_n is only a line, which is supposed to have length but not breadth (see Fig. 3).

Although the first paradox follows an obvious Sextian pattern, I have not been able to trace it to its original source, whereas the other two belong to the set of objections constructed in *Contra Geometras* (*Contra Mathematicos*, III, 27 and 66ff.) in order to show that, even if the geometricians are allowed to use their hypothetico-deductive methods—and this has been already questioned by Sextus in the first half of the book—they cannot rely on their starting points, since none of the three elementary objects, i.e., the point, the line, and the surface, are free from contradictions.

The Geometrician Portrayed by the Skeptic

The skeptical objections are based on a thoroughly empiricist view. Sextus treats geometrical entities as if they should maintain some resemblance to material objects in order to be meaningful at all. He adopts this line of reasoning on the basis of an empiricist epistemology according to which:

i) the logical possibility of an object is equivalent to the possibility of conceiving it;

ii) no conceivable object can be a purely mental object;[18] therefore
iii) a conceivable object cannot be completely void of empirical content but must preserve some mimetic feature.

The empiricist tone of Sextus' criticism suggests that the epistemological turn, i.e., a clear focus on what the mind can know about the mathematical realm, has already occurred. It is a criticism justified by the fairly concrete approach adopted by Euclid himself in his *Elements*.

I remarked above that, for more than two millennia, the *Elements* have been the most popular and influential paradigm of a deductive body of knowledge, very often the only one with which most educated people were acquainted. Like the Bible, it is one of those texts that have shaped Western culture. In it, we encounter the classic metaphor of the building as a model for the structure of knowledge, a metaphor that, together with the image of the tree of knowledge, will become common currency within any foundationalist project. The unique style of the work, which contributed so significantly to its popularity throughout the centuries, is the result of an admirable balance between empirical intuition and logical postulates, visual imagination and purely rational deductions. The overall structure of the thirteen books bears witness to a remarkable effort made toward the systematic construction of an abstract, universal, and logically rigorous body of mathematical knowledge, in which 465 theorems are logically inferred from a limited number of first principles explicitly stated at the outset.[19] And yet, a fundamental empiricism still pervades the entire work. For example, the criterion of existence, provided by the notion of geometrical constructability, is justified by an empiricist approach which became too limited in the nineteenth century. And one needs to mention only the first proposition of Book One, which requires an equilateral triangle to be constructed on a given finite straight line, to recall that the very notion of demonstration often relies on the visualization of the theorem in question (*δέκνυμι*

[18] On Greek philosophy of mathematics and the skeptical attack see Ian Mueller, "Geometry and Skepticism," *Science and Speculation: Studies in Hellenistic Theory and Practice*, ed. Jonathan Barnes et al. (Cambridge, UK, 1982), 69-95. In *Coping with Mathematics (The Greek Way)* ([Chicago], 1980), Mueller comments upon M III, 37 writing that "the force of this sceptical argument derives from the representation of mental apprehension as imagining or picturing and the imposition of severe limits on imagination" (13). The point is that nothing can be apprehended unless it can somehow be imagined. See also p.17: "For Proclus the mathematical imagination is quite like what later philosophers called intuition. Its images are produced by reason itself as a necessary condition of its mathematical knowledge: the images are a "projection" (*probole*) of concepts and principles contained in reason but not fully grasped by it."

[19] Euclid's geometry can be presented as a formal organization, not axiomatic and not thoroughly syllogistic, of material previously accumulated; cf. Ian Mueller "Greek Mathematics and Greek Logic," *Ancient Logic and Its Modern Interpretations*, ed. John Corcoran (Dordrecht, 1974), 35-70. "Euclid shows no awareness of syllogistic or even of the basic idea of logic, that validity of argument depends on its form" (37). "... In his systematic presentation of the categorical syllogism in the first twenty-two chapters of the Prior Analytics, Aristotle never invokes mathematics" (48). "... Stoic propositional logic, investigated most thoroughly by Chrysippus in the third century, shows no real connection with mathematical proof" (66). For a full analysis of Euclid's mathematical methods and a comparison with Hilbert's axiomatic approach see Ian Mueller, *Philosophy of Mathematics and Deductive Structure in Euclid's Elements* (Cambridge, Mass., 1981).

means in Greek "to bring to light," hence "to show," and also "to prove").

As the limited use of the fourth ("Things which coincide with one another are equal to one another") and fifth ("The whole is greater than the part") postulate shows, Euclid was at least partially aware of, and perhaps not entirely happy with such "realistic" features of his geometry.[20] If he could still regard them as unproblematic it was because of a more fundamental assumption underlying the *Elements*. Classic geometry was thought to be the result of the correct idealization of the properties of physical space and the corresponding behavior of extended bodies in that space. And since "Euclidean" geometry was the abstract grammar of physical space, until the nineteenth century space was understood as intrinsically Euclidean, that is, as Poincaré clearly put it, three-dimensional, (at least potentially) infinite, continuous (no gaps), homogeneous (no privileged points), isotropic (no privileged directions through any point, i.e., equal in every direction), and such that any discrete object in it would satisfy the theorems of Euclidean geometry.[21] Given such a strict relation between space and geometry, theorems were supposed to be true descriptions of actual features of physical space (*physicalization of geometry*). The truth of geometrical statements (*alethization of geometry*) eclipsed the need for a tight verification of the consistency of the system and hence of the independence of its set of axioms. Sound proofs (valid inferences from true premises) rather than logically correct deductions (valid inferences in which it is never the case that the premise is affirmed and the conclusion is negated) represented the backbone of Euclidean geometry. On the basis of such a moderate alethization and physicalization of geometry, it is obvious that empirical factors could not only be tolerated as useful aids to the understanding, but also appreciated as the semantic links whereby the geometrical system was tied to the natural world of empirical intuition. The interpretation of Euclidean geometry as the idealized model of physical space was explicitly conveyed by the characterization of the most elementary geometrical objects in terms of abstract (in the strong sense of *abstracted*) entities: insofar as points, lines, and surfaces have unique properties they are no longer physical objects, but insofar as they are the result of an evident process of refinement and generalization from particular objects of intuition they are not "mere" logical constructs either, which may or may not be amenable to physical (let alone visual) interpretation. Such abstract objects seem to enjoy a peculiar ontological status. They are not like other physical objects, but they are linked to perception and the real world via the criterion of conceivability in imagination, which is precisely the criterion exploited by Sextus Empiricus in the construction of

[20] The fourth axiom, another clear case of empirical influence in the *Elements*, states that "things which coincide with one another are equal to one another," and this implies superposition, which in turn is necessary to prove congruence of figures. It is significant that Euclid tries to avoid its use whenever possible. Likewise, the fact that all Euclidean geometry is based on the avoidance of geometrical objects with actually infinite dimensions may not necessarily be due to the fact that the Elements present a geometry of touch or are even a tactile-muscular study of metric space; cf. William M. Ivins, Jr., *Art & Geometry: A Study in Space Intuitions* (Cambridge, Mass., 1946).

[21] Cf. Henri Poincaré, *Science and Hypothesis*, tr. William J. Greenstreet (1905; rpt. New York, 1952).

his paradoxes. In Book One, Euclid provides five geometrical postulates and five more general "common notions" without any further justification. They are left unproved on the basis of their self-evidence. Before this, Euclid lists twenty-three definitions which are supposed to clarify in more intuitive terms his technical vocabulary. The logical utility of such definitions is dubious, since they use other undefined terms, yet Euclid seems to believe they could serve to interpret the geometrical objects as referring to physical entities. They are not organized into primitive and derivate terms, but there is a tendency to accept this implicit distinction, and Sextus Empiricus attacks precisely those three terms that appear to be the most primitive in Euclid, namely (1st) "A point is that which has no part," (2nd) "A line is length without breadth" and (5th) "A surface is that which has length and breadth only."

The aim of the skeptical challenge is sufficiently straightforward: to show that geometrical and arithmetical statements cannot be claimed to provide actual knowledge about the world. In order to achieve such an end, Sextus relies on the usual weapon of logical possibility: against the empirical truth of geometrical statements he sets consistent counterfactuals leading to contradictions. In this way, he can highlight the physical content still pervading Euclidean geometry. Because of such empiricist criticism, two radically different interpretations of the skeptical strategy become possible, one slightly superficial and the other somewhat incorrect.

The skeptic may be supposed simply to have failed to grasp the abstract nature of geometrical objects, and hence to have misunderstood Euclidean geometry. Sextus is incapable of seeing that the geometrical objects discussed by Euclid are not physical points and physical lines, so no further attention should be wasted on Pyrrhonian arguments. A champion of this position was Sir Henry Savile. Savile owned a manuscript, now in the Bodleian Library, containing a late sixteenth or early seventeenth-century copy of *Contra Mathematicos*.[22] The text is in perfect condition apart from the book of *Contra Geometras*, which is underlined throughout and seems to have been studied by Savile. Savile does not appreciate Sextus' criticism. He mentions him only very briefly in one of his manuscripts,[23] and in his *Lectures on Euclid*, after having discussed the nature of geometrical definitions and axioms, he dedicates a few paragraphs to Epicureans and Pyrrhonists, but only to dismiss them because "their arguments against the principles of Geometry are thoroughly insignificant and indeed completely sophistic."[24] This does not mean

[22] Ms Savilianus Gr. 1, f. 10v: "Extant Sexti Empirici libri decem pros mathematikos, adversus mathematicos, hoc est universam dogmaticorum nationem. Nec enim illis in libris tam Geometriae et Arithmeticae, quam Grammaticae Poeticae, Historiae, Rhetoricae, Astrologiae divinatricis, Musicae, Logicae, Physicae et Ethicae fundamenta conbellantus" [*conbello* means literally "uproot"].

[23] Cf. Ms. Savile 37, f. 11, where Savile gives a reference to Sextus Empiricus' work without any further remark.

[24] Henry Savile, *Praelectiones* (Oxford, 1621). 157: "... contra quae [i.e., Geometriae], totamque, adeò Geometriam acriter insurgunt duae philosophorum sectae, Pyrrhoniorum dico (qui sceptici & ephectici) & Epicureorum. Ac Ephecticorum quidem, qui quasi hostium more ex philosophiae agris fertilis cumprimis & foecundae frumenta populantes, & tanquam solem è mundo, sic ex animis nostris omnes scientiae non ramos modò, sed radicum fibras evellentes, totam evertunt philosophiam: horum, inquam,

that Savile himself failed to recognize that geometry faced the major problem represented by the lack of full evidence, for example in the case of the fifth postulate.[25] But it is interesting to notice that, when Riemann in his famous lecture "On the Hypothesis on Which Geometry Ultimately Lies" introduced his version of non-Euclidean geometry, he started by explaining the problems arising from the definitions of point and line in Euclid, the very issue Savile had been unable to grasp when reading Sextus.

We come in this way to the second perspective from which mathematical skepticism can be interpreted. The glass of Euclidean geometry is only half empty of empirical presuppositions, as it were. Thus the skeptical challenge can also be seen as a radical attempt to eliminate all the intuitive and physical residues in the geometrical system, that is as a *reductio ad absurdum* of the empirical elements still present in classic geometry. This was Leibniz' position. In a letter to Varignon, he wrote that:

> I even find that it means much in establishing sound foundations for a science that it should have such critics. It is thus that the skeptics, with as much reason, fought the principles of geometry; that father Gotignies, a Jesuit scholar, tried to throw out the best foundations of algebra; and that Mr. Cluver and Mr. Nieuwentijt have recently attacked our infinitesimal calculus, though on different grounds. Geometry and algebra have survived, and I hope that our science of infinites will survive too. ... I have often thought that a reply by a geometrician to the objections of Sextus Empiricus and to the things which Francis Sanchez ... sent to Clavius, or to similar critics, would be something more useful than we can imagine. This is why we have no reason to regret the pains which are necessary to justify our analysis for all kinds of minds capable of understanding it.[26]

Leibniz appreciated the anti-empirical impact of Sextus' arguments. He certainly knew very well that "all the difficulties raised by the Pyrrhonians concern only the empirical truths" (*veritez sensibles*),[27] and correctly understood that the skeptical challenge had a foundationalist nature.

argumenta contra principia Geometriae perquàm levia sanè aut planè sophistica videre licet apud Sextum Empiricum lib. I, cap. 19." See also the original manuscript in the Bodleian, Ms Savile 37f., 99v-100, which contains a brief, erased sentence not included in the printed text.

[25] Ibid., 140. Savile also mentions a second problem, the theory of proportion, which was discussed by Leibniz.

[26] Leibniz to Varignon, Hanover, 2 February 1702, *Leibnizens mathematische Schriften*, ed. Carl Immanuel Gerhardt, 1st pt., Vol. 4 (Halle, 1859), 94-95 (*Gesammelte Werke*, ed. Georg Heinrich Pertz, 3rd Series); see also Gottfried Wilhelm Leibniz, *Philosophical Papers and Letters*, ed., tr. and intr. Leroy E. Loemker, 2nd ed. (Dordrecht, 1969), 544. Leibniz was not alone in appreciating Sextus Empiricus. Walther von Tschirnhaus wrote to him that: "Sexti Philosophi Pyrrhoniarum hypotheseon libri tres, Parisiis 1569 in folio, habe mitt delectation gelesen." See *Der Briefwechsel von Gottfried Wilhelm Leibniz mit Mathematikern*, ed. Carl Immanuel Gerhardt (Berlin, 1899), 1:397.

[27] Leibniz to Edmonde Mariotte, mid-1676, *G. W. Leibniz: Philosophischer Briefwechsel*, vol. 1: 1663-1685 (Darmstadt, 1926), 268-69. In the letter Leibniz presents geometry as the most fundamental of all mathematical branches.

As I have suggested, such a foundationalist interpretation of the Pyrrhonist challenge can be slightly incorrect. The main difficulty is that for Sextus Empiricus a non-empirical geometry was impossible. Insofar as Euclidean geometry provides information about physical space and the behavior of objects in it, it must be true of the world and rely on a physicalization of its primitive notions. One can then demonstrate that geometry provides no direct knowledge about the real nature of the world by determining what contradictions must necessarily arise from a physical interpretation of its elements. The process of de-physicalization thus amounts to a process of de-alethization of mathematics—mathematical statements in themselves are not necessarily true of the world. Many contemporary mathematicians and certainly Poincaré would have found little to object to this position. But according to Sextus, a full de-alethization of geometry amounts to showing that no geometry is possible at all. There is no conceptual space between a de-alethizated geometry and a non-geometry at all because no distinction between syntactic consistency and semantic truth is clearly in view. This is why, contrary to Huet's or Hume's, Sextus' mathematical skepticism should be interpreted as a radical use of empiricism only, without further implications for the foundation of axiomatic geometry. For the Pyrrhonist, either mathematics counts as knowledge of the world, or it is nothing at all, but not the former, therefore the latter. No appreciation of a purely a priori, hypothetico-deductive approach is envisaged by Sextus. This appears very clearly in the first part of *Contra Geometras*, where the use of axioms is criticized because postulating does not amount to a justification of the hypotheses, i.e., cannot provide the hypothesis in question with a truth-content, whilst no notice is taken of the possibility of interpreting geometry as a purely consistent system of hypothetico-deductive, ontologically non-informative statements. Once again, it is worth remarking that Leibniz, in his correspondence with Simon Foucher on mathematical skepticism, defended an interpretation of geometrical statements as only conditionally true.[28]

THE HISTORIAN'S REPLY TO THE SKEPTIC

Unlike Leibniz, Montucla maintains that the skeptical objections miss the point, yet he does not simply dismiss them as useless sophisms, like Savile. He attempts to show that geometry does have an empirical content, which explains its applicability and usefulness, but in a sense different from the one criticized by the skeptics. This leads him to endorse an ambiguous position half way between Euclidean "abstractionism" and modern "structuralism."

In order to overcome Sextus' extreme form of empiricism, Montucla defends the abstract nature of the most elementary of geometrical entities. Like Aristotle, he denies that points, lines, and surfaces are actual bodies.[29] But when he comes to

[28] Hidé Ishiguro "Les vérités hypothétiques: Un examen de la lettre de Leibniz à Foucher de 1675," *Leibniz à Paris (1672-1676)*, Vol. 2, *Studia Leibnitiana, Supplementa*, 18 (1978), 33-42.

[29] Cf. Aristotle, *De Caelo* II, 13, 296 a 17.

specify more positively what such objects may be, Montucla hesitates between two alternatives. On the one hand, he insists on defending the classic abstractionist view: the elements are the result of an intellectual decomposition of physical bodies. A surface is the limit (*terme*) of a volume, a line is the limit of a surface, and a point is the limit of a line. This, however, implies only that geometrical elements still have an empirical value. It does not explain why we should not treat them as Sextus Empiricus does, that is, as conceivable objects, possessing only a carefully selected number of physical properties but still somewhat resembling empirical objects in some of their features. The fact that no perceived entities exactly correspond to the basic geometrical concepts has always been a problem for any empiricist philosophy of geometry, from Mill to Lotze and Wundt, a difficulty usually tackled by speaking of abstraction (i.e., elimination) and idealization (i.e., improvement) of certain properties starting from empirical perceptions. In Montucla's case, the attempted solution consists in introducing a functionalist interpretation: elementary objects in themselves are only divisors of more complex objects. Surfaces are what volumes can be divided into, lines are the divisors of surfaces, points of lines, and a point is the geometrical entity that has no divisor. Their hierarchy is established by the asymmetric nature of the relation of division implemented. Volumes, surfaces, and lines are not assemblages of more elementary components, for a limit-divisor of something is not part of that something.[30] Thanks to such a functionalist interpretation, Montucla can maintain that the first and the third arguments put forward by Sextus are misleading. In both cases, the lines in question are not constitutive elements of the surface, but terms of division (*termes des divisions*) of it, and no collection of lines-as-divisors will ever give rise to a surface. Likewise, two segments of different length can be said to contain the same number of points in the sense that they can be divided into an equal number of parts, but no conclusion can be drawn about their corresponding magnitude. In both cases, the problem is solved by eliminating one of its conditions of possibility. Note, however, that Sextus' second argument is not directly affected by the reply, and indeed Montucla simply leaves it unexplained.

Montucla's functionalist interpretation of geometrical entities could lead him to endorse a purely structuralist view about the reality of geometrical constructs. A point or a line in a geometrical system would be like the bishop in a chess game: a set of axioms or rules establishes a finite number of relations among a limited number of arbitrarily chosen primitive terms that remain undefined, thus determining, implicitly, what consistent functional properties the latter must satisfy. Hence the entities in themselves are not self-subsisting objects, but discrete *hypostatizations* of function-bearers, logical constants of the axiomatic system standing for abstract classes of properties-relations with no intrinsic ontological status over and above the role they play within the system. In this way, a set of axioms or rules can be said to provide the meanings of some basic terms only in the sense that it specifies how those terms have to be employed, not in the sense that it makes explicit what

[30] Cf. Aristotle, *Physics*, IV, 8, 215 b 19.

"things" they are or stand for. This is how we think mathematics works today.[31] And Montucla seems to go some way toward this view, for he maintains that: a) geometricians do not care whether or not the perfect objects (either elementary or complex constituents of a geometrical body) they speak about exist in reality; and b) geometrical statements are only hypothetically true, since they depend on non-existing elementary objects: if one assumes that p then q must follow, yet there may be nothing exactly like p in reality.

But all this must have appeared to him insufficient to explain the applicability of geometry to the physical world, for in order to counterbalance his anti-empiricism, endorsed as a defense against the Pyrrhonist attack, we have seen that Montucla re-asserts his interpretation of perfect geometrical bodies as intellectual limits (*limites intellectuelles*) of material objects, idealizations which become increasingly useful the closer reality approximates them. Insofar as points, lines, and surfaces are *abstracted* and not just theoretical entities, they depend, for their existence, on the existence of such perfect bodies. Whether such idealizations can exist in reality will be a matter of physical discovery, according to Montucla. As long as the physical nature of space remains unknown, it is sufficient that the "metaphysical ideas" of such perfect bodies are clear and evident. This defense of geometrical abstractionism allows Montucla to rely on the notion of truth to justify the value of mathematical knowledge and applied mathematics, and hence to disregard the purely internal feature of logical consistency. The crisis begun by non-Euclidean geometries is still far in the future, and there is no reason why a purely axiomatic approach should be appreciated or even employed to replace the intuitive correspondence between physical space and Euclidean geometry. The imagistic interpretation of clear and distinct ideas, rather than the algebraic construction of a consistent and economic system of axioms and theorems, can still lead geometrical research.

THE MISSING FIGURE IN THE HISTORIAN'S CANVAS

We have seen that there are a number of interesting issues Sextus never mentions in his objections, including the three Geometrical problems. There is a significant omission in Montucla that deserves equal attention. Although he refers to many people whom he himself recognizes as very little known, Montucla never mentions David Hume. Of course, there may be several explanations for this fact, ranging

[31] Roberto Torretti, *Philosophy of Geometry from Riemann to Poincaré*, 2nd ed. (Dordrecht, 1984), 141: "... structure i.e., relation nets is all that geometers really care for. It is not the nature of points and lines (which nobody has ever been able to explain) but how they stand to one another in a system of relations of incidence and order which is the concern of projective geometry, and this is sufficiently known once we know the group which preserves this system. [The modern axiomatic method] is based on the assumption that the objects of a mathematical theory need not be ascribed more than what is strictly necessary for them to sustain the relations we require them to have to one another. The basic objects of such a theory are determined just by its basic propositions, the axioms, that lay out the relational net into which those objects are inserted. Such a determination is as much as a mathematical theory requires."

from the diffusion of Hume's writings in France[32] to the kind of sources Montucla relied upon for his knowledge of mathematical skepticism (possibly Bayle's *Dictionary*, see Montucla's *Histoire*, 1:21, or Proclus[33]). From a purely theoretical perspective, however, there remains the fact that Hume's absence is perfectly explicable within the development of mathematical thought in the eighteenth century.

I have remarked above that the modern history of mathematical theories appears to have followed two fundamental directions: a progressive mathematization of our knowledge of the natural world, and an equally impressive, if slightly later, de-physicalization of mathematics, which led, between the end of the last and the beginning of our century, to the structuralist approach and a full axiomatization of the foundations of the discipline. Hume wrote at a moment when the mathematization of physics was increasing dramatically but the de-physicalization of mathematics had not yet become a major trend. So his philosophy of mathematics went largely unnoticed. The odd fact is that by the time the de-physicalization of mathematics became a central issue, Hume's position had been forgotten, for Kant had become the central figure in the philosophy of mathematics. Between the end of the nineteenth and the beginning of the twentieth century, when Musil describes the young Törless's bewilderment about the nature of numbers—no longer geometrical entities, mind—he makes his professor of mathematics refer him to the *Critique of Pure Reason*, even though Kant had devoted much more attention to geometry than to arithmetic. Discussing the fortune of Euclidean geometry, Hans Reichenbach remarked once that:

> Unless one was a [Pyrrhonian] skeptic, one was content with the fact that certain assumptions had to be believed axiomatically [indeed, Sextus would say "dogmatically"]; analytical philosophy has learned through Kant's critical philosophy to discover genuine problems in questions previously utilized only by skeptics in order to deny the possibility of knowledge.[34]

But it is only when a reaction against Kant takes place that neopositivists such as A.J. Ayer rediscover Hume as one of their main reference points, thus transfiguring what should probably count as a skeptical view of mathematical knowledge into one of the most common philosophies of mathematics of our time. Mathematics becomes a hypothetico-deductive study of logical structures, consisting of analytic statements, true a priori because ontologically content-empty. After the foundationalist crisis, mathematics shifts its balance from truth to coherence, thus losing its

[32] See the paper by Laurence L. Bongie "Hume and Skepticism in Late Eighteenth-Century France" in this volume.

[33] Proclus, *A Commentary on the First Book of Euclid's Elements*, tr. with intr. and notes by Glenn R. Morrow (Princeton, NJ, 1970), 199: "Up to this point we have been dealing with the principles, and it is against them that most critics of geometry have raised objections, endeavoring to show that these parts are not firmly established. Of those in this group whose arguments have become notorious some, such as the Sceptics, would do away with all knowledge, like enemy troops destroying the crops of a foreign country, in this case a country that has produced philosophy ...."

[34] Hans Reichenbach, *The Philosophy of Space and Time*, tr. Maria Reichenbach and John Freund, with intr. remarks by Rudolf Carnap (New York, 1958).

strongest anti-skeptical peculiarity.

AN END WITH A VIEW

The interpretation of mathematical skepticism I have offered in the previous pages could be summarized in a fistful of famous quotations: for Plato "God eternally geometrizes," and at the end of the sixteenth century Kepler still agreed with that. Yet the algebraization of geometry made Kronecker believe that "God made the integers; all else is the work of man." The discovery of non-Euclidean geometries and the development of Cantor's treatment of infinite sets convinced Hilbert that "No one shall expel us from the paradise which Cantor created for us." God, and above all geometry, had been replaced by the human construction of set theory, but the former was going to reappear in the mathematical imagination. For after Gödel's proof that consistency of number theory cannot be established by the narrow logic permissible in metamathematics, Weyl suggested that "God exists since mathematics is consistent, and the devil exists since we cannot prove its consistency." By the time geometry had been replaced by set theory and the de-physicalization and the corresponding de-alethization of mathematics had been completed, Russell wrote that "mathematics is the subject in which we never know what we are talking about, nor whether what we are saying is true," while Einstein believed that "As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality." I suppose Sextus Empiricus would have found the escalation implicit in this series of remarks very reassuring.

The history of reason is one of a constant striving of the mind for the achievement of intellectual freedom from reality. In the course of the history of thought, the distance between mind and being widens and, through such a constant process of detachment, reflection becomes epistemologically ever more responsible for its own constructs, while at the same time increasingly self-referential in its activities. From Proclus' invitation "to free geometry from Kalypso's embrace," to Cantor's suggestion that "Mathematics is entirely free in its development and its concepts are restricted only by the necessity of being noncontradictory and coordinated to concepts previously introduced by precise definitions. ... The essence of mathematics lies in its freedom,"[35] the history of mathematical theories appears to be perfectly coherent with the previous view, which I acknowledge to be unashamedly metaphysical. A progressive mathematization of our knowledge of the world in its most diverse aspects, from physical to social phenomena, and an equally impressive, if somewhat later, de-physicalization of mathematics, which led, between the end of the last and the beginning of our century, to a full axiomatization of the foundations of the discipline: these two movements are aspects of the same phenomenon. It was precisely the detachment of mathematics from its empirical models that made it possible to interpret and dominate more and more aspects of reality with the same

[35] Georg Cantor, *Gesammelte Abhandlungen*, ed. Ernst Zermelo (Berlin, 1932), 182, quoted by Kline, *Mathematical Thought*.

mathematical theories. When Euclidean geometry disengaged itself from empirical interpretations via its arithmetization and then axiomatization, geometries only locally isomorphic to it became conceivable, geometries that could replace the fifth postulate with a different axiom and hence become capable of handling non-Euclidean spaces. Only a purely algebraic approach allows us to provide rigorous definitions of Weierstrass' curve, which is nowhere differentiable, or of Peano's curve, which is capable of covering a whole surface. The same non-empirical approach to set theory makes it possible to understand how the part may not necessarily be smaller than the whole. Geometry has moved from the abstraction and idealization of selected properties of physical objects to the hypostatization of logical relations. The loss of intuitive certainty has been repaid by the acquisition of certain universality. As thought increasingly detached from what common sense offers up as apparently undisputable in ordinary experience, a kind of constructive skepticism has often been a fundamental driving force. Radical questioning is made possible by the capacity of the mind to conceive what is logically consistent but not actual, and the presentation of the conceivable is usually the best conceptual tool whereby thought can disengage itself from its momentary forms of more or less dogmatic realism, and hence move toward a better appreciation of its theoretical responsibilities. I hope my discussion of Montucla's anti-skeptical arguments has helped to provide such a view with sufficient cogency.

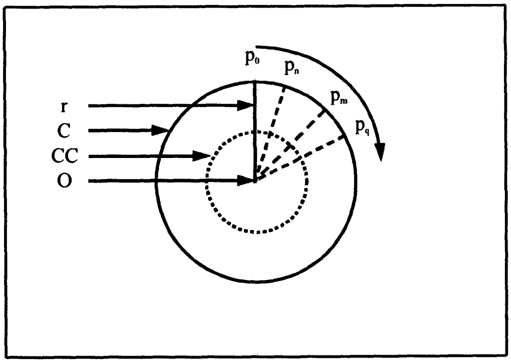

Figure 1

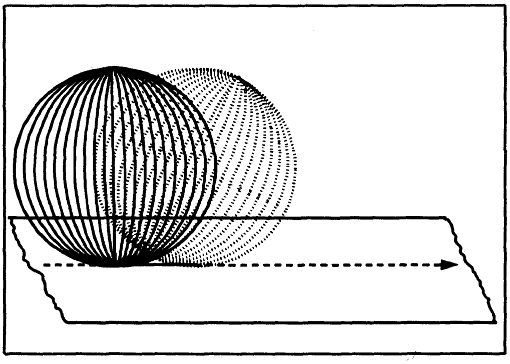

Figure 2

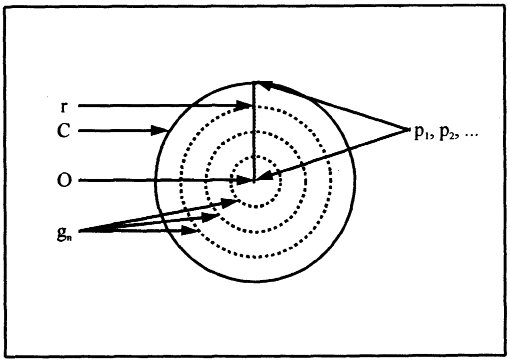

Figure 3