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ON SAYING WHAT YOU REALLY WANT TO SAY: WITTGENSTEIN, GÖDEL, AND THE TRISECTION OF THE ANGLE

> For an answer which cannot be expressed the question too cannot be expressed.
> *The riddle* does not exist.
> If a question may be put at all, then it *can* also be answered.
> Scepticism is *not* irrefutable, but palpably nonsense, if it would doubt where a question cannot be asked.
> For a doubt can only exist where there is a question; a question only where there is an answer, and this only where something *can* be *said*.
> *Tractatus Logico-Philosophicus* 6.5–6.5.1

Wittgenstein's remarks on the first incompleteness theorem[1] have often been denounced, and mostly dismissed. Despite indirect historical evidence to the contrary,[2] it is a commonplace that Wittgenstein rejected Gödel's proof because he did not, or even could not, understand it.[3] Kreisel twice used the word "wild" when he reviewed Wittgenstein on Gödel.[4] Dummett, in many respects an admirer of Wittgenstein's philosophy, wrote that the remarks on Gödel and on the notion of consistency are "of poor quality or contain definite errors".[5] Gödel's own comments were damning (see Section III below).

Nevertheless, some have perceived in Wittgenstein's reactions to Gödel more than flat misunderstanding or elementary logical ignorance. Hintikka, who credits Wittgenstein with an intuitive, if insufficiently rigorous, insight into the philosophical basis of game-theoretic semantics, contends that because Wittgenstein subscribed, early and late, to the doctrine that "semantics is ineffable",

> What is primary for Wittgenstein is the unacceptability of Gödel's result. Much of Wittgenstein's efforts were devoted to a desperate effort to locate more specifically where Gödel went off the straight and narrow. In so far as these efforts focused on Gödel's actual proof methods, they were futile, and were doomed to remain so.[6]

Hintikka's view of what underlies Wittgenstein's philosophical resistance

Jaakko Hintikka (ed.), Essays on the Development of the Foundations of Mathematics, 373–425.

to Gödel is reminiscent of Carnap, who in 1934, in *The Logical Syntax of Language*, had claimed that for the young Wittgenstein, "syntax is not expressible".[7] But, Carnap further claimed, Gödel's arithmetization of syntax, by means of which "the syntax of [the] language can be formulated within this language itself", refutes this (alleged) Tractarian thesis about the limits of what can be said.[8]

Differing with all such analyses, Shanker has recently mounted a spirited defense of Wittgenstein, maintaining that Wittgenstein's criticisms are directed, not at the mathematics of Gödel's proof, but, rather, at Gödel's (and others') accompanying *philosophical* remarks.[9] Shanker argues that Wittgenstein's discussion must be seen against the background of his earlier criticisms of the Hilbert program, that is, his earlier rejection of metamathematics as an epistemically privileged philosophical foundation for mathematics. On Shanker's view, Wittgenstein held (correctly) that Gödel's theorem yields a *reductio ad absurdam* of the Hilbert program.[10]

In a similar spirit, Wang has recently tried "to find a perspective from which Wittgenstein's view becomes understandable"[11] Long engaged with the project of comparing, contrasting, and even in certain respects attempting to reconcile, the general philosophical approaches of Wittgenstein and Gödel,[12] Wang has recently written that despite Gödel's conviction that Wittgenstein did not understand the incompleteness theorem,

> At the same time, it is also clear that Wittgenstein did take Gödel's incompleteness theorems seriously and repeatedly wrote about them. They seem to me to be a more instructive topic (than set theory) to examine in trying to decompose the disagreements of Wittgenstein with Gödel (and, in this case, with most people who have made a serious effort to understand Gödel's result).[13]

I believe Wang is correct in holding that

> Even apart from the matter of proving Gödel's theorem, just to interpret the statements of it . . . becomes a complex task from Wittgenstein's perspective.[14]

Yet, as I shall argue in this paper, the "complexity" – for both Wittgenstein and for his interpreters – resides in the fact that Wittgenstein *had* no special animus with regard to the Gödel theorem. As he wrote, "My task is, not to talk about (e.g.) Gödel's proof, but to by-pass it."[15] For Wittgenstein Gödel's work has no more – and no less – significance for the nature of mathematics than any other strict impossibility

proof. Clearly Wittgenstein wishes to deflate the *apparent* significance of Gödel's theorem; for him it is neither a result concerning the nature of mathematical proof, nor a result concerning the nature of mathematics. It is simply one example among many of a proof *in* mathematics – albeit one which is more likely to mislead (some people) philosophically. Wittgenstein does not offer any criticisms of the incompleteness *proof*. Instead, I will suggest, the writings on Gödel amount primarily to a series of attempts by Wittgenstein to *appropriate* the theorem: to exercise, to indicate and, ultimately, to confirm his own philosophical attitudes. This is best seen when the texts mentioning Gödel are read, not only against the background of Wittgenstein's general philosophical aims and problems, but also in the context of his other, numerous discussions of impossibility proofs in mathematics. I shall initially focus on Wittgenstein's treatment of a striking, yet readily accessible achievement of nineteenth century algebra: the proof that it is impossible to trisect an angle with straightedge and compass alone.[16] This was one of Wittgenstein's most often invoked mathematical examples: he referred to it in every course of lectures for which we have records, and repeatedly in his post-Tractarian writing.[17] The trisection proof is subjected by Wittgenstein to the same idiosyncratic philosophical treatment as is the incompleteness theorem, yet no commentator has (so far as I know) suggested that he did not understand *it*.

I emphasize throughout this paper Wittgenstein's depiction of the situation and perspective of one who has not yet reached, but is seeking, an answer to a mathematical question. In the course of giving my reading, I hope to bring out the philosophical import of various alternative characterizations of such a situation. All too often Wittgenstein's readers forget – because they fail to grant – that philosophy is for Wittgenstein primarily an activity concerned with *understanding*; and that his philosophical discussions of Gödel and of the trisection of the angle (as well as his treatment of other classical impossibility proofs) informs – and is informed by – his lengthy scrutiny of the very concept of "understanding" in *Philosophical Investigations*.[18] Attention to the role of the trisection example in this text will show the continual interplay between general themes in the *Investigations* concerning the notions of understanding, thinking and meaning and what is usually conceived of as Wittgenstein's more specialized concerns with the philosophy of logic and mathematics. The common opinion, that Wittgenstein's discussions of mathematics and logic are inferior to, and separable from, the rest of his later philosophy is not, I think, tenable.

This does not mean that we should agree with what Wittgenstein has to say. But it is essential to stress that what is most difficult to understand (and to accept) in Wittgenstein's writings on Gödel is the radical nature of his philosophical conception, that is, his addressing himself primarily to methods of philosophical argument *as opposed to* specific philosophical conclusions. Making out this distinction is itself a central, if formidable, interpretive and philosophical task for Wittgenstein's readers,[19] made even more formidable by the fact that Wittgenstein's style of interlocutory writing, intrinsic to his conception of philosophy, allows (like Plato's) few words to be taken at face value alone.[20] In his remarks on Gödel, Wittgenstein is presupposing the usefulness of a challenge to our usual modes of applying and thinking about mathematical logic in philosophy. The difficulty of his remarks thus does not lie in a simple logical or mathematical error, or in a difference of opinion about only the incompleteness theorems. It lies deeper: at the heart of what twentieth century philosophers – especially philosophers of mathematics – have chosen to regard as fundamental to their own enterprise. For this reason alone, Wittgenstein's remarks are of special significance, whether or not one agrees with them.

The remainder of this paper is divided into three main sections. The detailed discussion of Wittgenstein's remarks on Gödel is left to Section III. Section I deals primarily with the place of the trisection example in the *Philosophical Investigations* and the general background of Wittgenstein's interest in Gödel, Section II with Wittgenstein's treatment of the proof of the impossibility of trisecting the angle and the status of mathematical conjectures in *Philosophical Grammar* and *Philosophical Remarks*. References to *Remarks on the Foundations of Mathematics* (and other writings of Wittgenstein) abound.

I

Why should Wittgenstein have concerned himself with Gödel's result at all? A superficially plausible answer would say that early and late Wittgenstein viewed the various – and differing – things a mathematician does under the rubric of finding and giving proofs to be central to our very understanding of mathematical statements. Thus he seems, at least on the surface, to be attacking any notion of mathematical truth given independently of modes of proof.[21] Prima facie, Gödel's incompleteness theorem poses a special difficulty for this point of view, since, as usually interpreted, it demolishes the hope of ever identifying the notion of mathematical truth with that of mathematical proof in a single recursively axiomatizable system. Furthermore, Gödel showed us how to construct what are usually called "true but unprovable" sentences of arithmetic for any given purported recursive axiomatization. This seems to validate the idea that our notions of truth and proof in mathematics do not and cannot coincide. And it is this idea which is often taken to be the object of Wittgenstein's criticisms in his discussions of Gödel.

However, from at least the time of the *Notebooks 1914–16* and the *Tractatus*, Wittgenstein had argued that *Principia Mathematica* – i.e., mathematical logic – does not give us the appropriate way to conceive of mathematics. In the *Tractatus* Wittgenstein explicitly rejected the "logistic reduction", in which Frege and Russell purported to use (what they called) "logic" to explicitly set forth the ultimate structure, the real nature of mathematics (i.e., for Frege and for Russell, "logic"). Wittgenstein's rejection of this reduction has two main aspects. Most generally, and most importantly, Wittgenstein attacked the whole conception of logic of Frege and Russell, labelling the so-called "propositions" of logic "tautologies", lacking in "sense", in information, altogether. Whatever the role of a *Begriffsschrift*, of formalization, in the *Tractatus*,[22] Wittgenstein never had in mind, and even argues against (in TLP 6.127ff) the interest of axiomatizing logic: if logical "propositions" are limiting cases of propositions, mere tautologies or empty redundancies, what point could there be in axiomatizing them? For Wittgenstein there simply *are* no general "laws" governing reality and/or our thought about it in the sense in which Russell and Frege maintained.

Secondly, however, Wittgenstein offered an equally radical recasting of mathematics. Although the *Tractatus*'s discussion of mathematics is notoriously difficult to understand, we can discern at least its general outline. Wittgenstein asserts that the essence of mathematical method is given through working with what he calls "equations" (TLP 6.2341). This extension of the usual notion of "equation" has much the same flavor, and much the same philosophical purpose, as Wittgenstein's extension of the term "tautology" to cover the so-called "propositions" of logic: mathematics is called a "method" of logic whose (pseudo) statements share with the "propositions" of logic the character of being "sinnlos", i.e., lacking in sense, not capable of truth (or falsity) in the same sense as genuine, "sinnvoll" propositions (such as those of (what Wittgenstein calls) "natural science").[23] Thus both proof in logic and proof in mathematics are labelled in the *Tractatus* (with another stretching of language) "calculation"[24] – a manipulation of symbols, as opposed to an activity of judging on the basis of empirical evidence, or comparing a sentence with reality. But at the same time, Wittgenstein insisted on the autonomy of mathematical technique from its presentation in logic, taking as basic for *his* "definition" of number the very notion Russell and Frege had claimed to define away in logical terms, *viz.*, a recursive process.[25] In the *Tractatus* the notion "b is a successor of a" is expressed in terms of a "formal series", and Wittgenstein writes (in 4.1273) that

The general term of a formal series can only be expressed by a variable, for the concept symbolized by "term of this formal series" is a *formal* concept. (This Frege and Russell overlooked; the way in which they express general propositions like the above is, therefore, false; it contains a vicious circle.)

We can determine the general term of the formal series by giving its first term and the general form of the operation, which generates the following term out of the preceding proposition.

Wittgenstein's alleging that the logistic reduction suffers from a "vicious circle" would appear to follow from the Tractarian show/say distinction, from the notion that the very language in which Russell and Frege couch their definition of natural number already presupposes, or exhibits, the form of a formal series. The question of whether this allegation does or does not have force against Frege's and Russell's logicism (and also Wittgenstein's (related) pronouncement (in 6.031) that "the theory of classes is altogether superfluous in mathematics") need not detain us here. The important point is that from a very early stage in his philosophical life, Wittgenstein characterized mathematics and logic differently, and regarded the mathematical logician's notion of (formal) "proof" as proof in only one sense of the word.

Similarly, numerous passages in *Remarks on the Foundations of Mathematics* are devoted to criticizing the idea that *Principia Mathematica* exhibits an underlying logical structure which forms the basis, or essence, of mathematics. This is a continuation of the *Tractatus* attitude, though by 1929 Wittgenstein had given up the idea that mathematics consists of linguistic expressions of a single general character. The later Wittgenstein is highly contextual in his treatment of mathematics: he insists over and over again that for philosophical purposes generalizations and theoretical structures in mathematics are not to be carved away from the particular theories, definitions and phenomena they may be said to mathematize. Roughly put, form cannot be completely

divorced from content in mathematics; "application," "employment" or "use" [Anwendung] – whether settled intratheoretically, within (pure) mathematics,[26] or settled outside it, in physics or ordinary ascriptions of number – cannot always be ignored by the philosopher:

The symbols "$(x).\phi x$" and "$(\exists x).\phi x$" are certainly useful in mathematics so long as one is acquainted with the technique of the proofs of the existence or non-existence to which the Russellian signs *here* refer. If however this is left open, then these concepts of the old logic are extremely misleading . . .[27]

(Notice Wittgenstein's persistence, as late as the 1940's, in referring to the logic of Frege and Russell as "the old (conception of) logic". This dismissive way of writing, lumping Frege and Russell in with the logical tradition since Aristotle, had begun by 1914.[28])

In a field that has been prepared in this way *this* is a proof of existence.

The harmful thing about logical technique is that it makes us forget the special mathematical technique. Whereas logical technique is only an auxiliary technique in mathematics. For example it sets up certain connexions between different techniques.

It is almost as if one tried to say that cabinet-making consisted in glueing.[29] . . . the danger is that one will think one is in possession of the complete explanation of the individual cases when one has this general way of talking . . .[30]

Note Wittgenstein's use of the demonstrative: it underscores his view that ultimately one cannot get away from *examples*, from *use* or *employment* – in logic, in philosophy, or in mathematics. "Generality", as he writes, "Does not stand to particularly in mathematics in the same way as the general to the particular elsewhere"[31]; "our mathematics is built up on . . . an unordered generality".[32] The later Wittgenstein's difficulty with the logistic reduction is not, we note, that it is *false*; but, rather, that it is *misleading*. That is, according to Wittgenstein the so-called "reduction" misleads by insisting upon a much too limited conception of mathematics. Wittgenstein will grant that one application of mathematical logic is to connect or assimilate different mathematical techniques, to present them in identical form. But Frege or Russell insisted that this form exposed the real underlying structure of mathematics, its essence. This is tantamount, in Wittgenstein's eyes, to insisting that the essence of cabinetry is glucing; it is not so much incorrect, as it is a kind of swindle, for such a philosophical use of logic overlooks the aims and purposes of mathematics, its history, its artistry.[33] ("If you use a trick in logic, whom can you be tricking other than yourself"?[34]) The Frege-Russell use of logic misleads us, Wittgenstein thinks, by forcing diverse mathematical techniques to wear, in the same Procrustean bed, the same guise:

The Russellian signs veil the important forms of proof as it were to the point of unrecognizability, as when a human form is wrapped up in a lot of cloth.[35]

And also:

"By means of suitable definitions, we can prove '25 × 25 = 625' in Russell's logic." – And can I define the ordinary technique of proof by means of Russell's? But how can one technique of proof be *defined* by means of another? How can one explain the *essence* of another? For if the one is an 'abbreviation' of the other, it must surely be a *systematic* abbreviation. Proof is surely required that I can systematically shorten the long proofs and thus once more get a system of proofs.

Long proofs at first always go along with the short ones and as it were tutor them. But in the end they can no longer follow the short ones and these shew their independence.

The consideration of *long* unsurveyable logical proofs is only a means of shewing how this technique – which is based on the geometry of proving – may collapse, and new techniques become necessary.

I should like to say: mathematics is a MOTLEY of techniques of proof: – And upon this is based its manifold applicability and its importance . . .

I should like to say: Russell's foundation of mathematics postpones the introduction of new techniques – until finally you believe that this is no longer necessary at all.[36]

Wittgenstein is insistent on emphasizing the importance of diverse mathematical techniques in order to question the philosophical significance of portraying mathematical arguments in logical form. Partly for this reason, in Wittgenstein's later discussions of mathematics, talk of "intensions" and "concepts" occurs throughout.[37]

All this implies that Gödel's theorem *could* not have been seen by Wittgenstein as decisive for our notions of "mathematically true" and "mathematically provable" – however one may choose to assess the general historical relation of Gödel's theorem to the logicist program.[38] Instead, for Wittgenstein the incompleteness theorem is just another impossibility proof. And the purpose of discussing the theorem in the particular way Wittgenstein does is to emphasize the importance of the *mathematical* (as opposed to the "logical") techniques at work in Gödel's argument. One thing to learn from Wittgenstein's scepticism about the general philosophical import of Gödel's theorem is that in order to view the theorem as crucially bearing on our notions of "mathematical truth" and "mathematical proof", one must *already* have invested the formalism (of, say, Frege and Russell) with a general philosophical significance.

Wittgenstein never did this; he was always concerned to question the relation between the formalism of Frege and Russell and mathematics.

Let us now turn to the proof of the impossibility of trisecting the angle in order to see how Wittgenstein treated impossibility proofs. Almost always Wittgenstein considered such proofs in connection with the question, What is it to search for, to try to produce, a mathematical proof? (See footnote 17.) This is a philosophical, not just a mathematical question, similar (and related) to the questions, What is a proof?, and What makes a proof a proof of *this* particular conjecture? These questions, part of the general inquiry into phenomena of intentionality and their expression in language – thinking, understanding, meaning, expectation, desire, belief, and so on – are of course woven through all of Wittgenstein's later work. Section 334 is the first passage in the *Investigations* which explicitly mentions trisection. I shall gloss this remark in some detail in order to show that mathematics is a central case in the *Investigations*, and to motivate a closer look at the actual proof of the impossibility of trisecting the angle:

334. "So you really wanted to say. . . ." ["Du wolltest also eigentlich sagen. . . ."] – We use this phrase in order to lead someone from one form of expression to another. One is tempted to use the following picture: what he really 'wanted to say', what he 'meant' was already *present somewhere* in his mind even before we gave it expression. Various kinds of thing may persuade us to give up one expression and to adopt another in its place. To understand this, it is useful to consider the relation in which the solutions of mathematical problems stand to the context and origin of their formulation [zum Anlass und Ursprung ihrer Fragestellung]. The concept 'trisection of the angle with ruler and compass', when people are trying to do it, and, on the other hand, when it has been proved that there is no such thing.[39]

The context of this remark is an exploration of the concept of thinking and its expression in language. The interlocutor asks in PI §327, "Can one think without speaking?" and Wittgenstein has remarked in PI §329 that "When I think in language, there aren't "meanings" ("Bedeutungen") going through my mind in addition to the verbal expressions: the language is itself the vehicle of thought." In §334 Wittgenstein grants, I take it, that some sense may be given to the notion that language is the "vehicle" of thought; but he is also suggesting that this notion can also mislead us. The phrase "So you really wanted to say. . . ." is a mark both of the fluency of linguistic communication *and* of its rupture in misunderstanding. Wittgenstein notes the naturalness, in some contexts, of supposing that "what is intended or meant" is present in the mind of the speaker. Examples are numerous: you make a transparent slip of the tongue, and I say, "So you really wanted to say . . ." in order to indicate or verify that you meant something other than what you actually uttered. Or again, in listening to a mathematics lecture, I might say, "So you really wanted to say . . ." in order to make sure that I understand precisely what it is that is being proved or claimed. We may hold before us in such cases the picture of a definite thought or content, more or less clearly expressed or communicated; the listener tries to express the thought in his or her own words; different ones, to settle communication. But the point – that is, the irony – of Wittgenstein's suggestion is that "So you really wanted to say . . ." is also often applied in *precisely* those contexts in which we believe that a speaker has not yet gotten hold of a clear thought. Here it is not that we see a thought clearly grasped but unclearly or misleadingly expressed; rather, we take the unclarity of the utterance to indicate that the speaker has not yet thought through a thought as clearly as he or she might. And so we propose another expression, and urge its adoption as an alternative. Teachers, editing their students' words (or computations, as in the case of the wayward pupil of PI §§143 and 185), are apt to use this sort of linguistic strategy to invite, or to argue, their students into behaving, from their point of view, "correctly". "So you really wanted to say . . ." can be used to secure the application of logic: in the course of presenting an argument, when one traces out the implications of a thought, one may be led from one step to the next by use of such a phrase. This is clearly exemplified when proofs are presented in mathematics, and especially vividly in impossibility proofs, where, *via* a *reductio* argument, one is brought to see that a thought once entertainable as mathematically true is not.

Wittgenstein's reference to the formulation and the resolution of the famous problem of trisecting the angle suggests that the "picture" of something clearly present to the mind ahead of (or apart from) its expression is both a useful picture, and at the same time one whose application is limited, appropriate only in a contextual sense, when it does apply.[40] Language is a vehicle of communication and thought; but it can also get in the way. What could be clearer, or more coherent – as a proposition, or an expression of a thought – than the conjecture that "There exists a general method of Euclidean trisection?" For over 2000 years people tried and failed to trisect the angle. We say that they had something definite "in mind" that they wanted, or were trying, to do. In fact, the (later discovered) impossibility proof can only function *as* such because we accept that it rules out exactly the construction trisectors were

seeking. And yet, in *accepting* the proof, we see that what they were trying to do was not only not done, but *could not possibly* (mathematically) be done. So that once the proof has been accepted, there can *seem* to be a conflict between wanting to grant full and determinate meaning to the (former) conjecture that "A trisection construction exists"; and yet wanting, as a result of the proof, to deny that this claim *really* makes any sense at all, to insist that no one *really*, ultimately, wants – or ever wanted – to say such a thing. For is not such insistence essential to accepting, to grasping, to applying, the *proof*? But then are we *forced* to say that for over 2000 years trisectors engaged in an inquiry which made no sense, an inquiry which they didn't *really* want to engage in?

The *Investigations* discussion of the trisection question and its proof bears traces of Wittgenstein's penchant (since at least 1913) for treating mathematical or logical "falsehoods" as *incoherent*, as opposed to simply *false*. In the *Tractatus* logical "falsehoods" were treated as "contradictions",[41] and mathematical "falsehoods" were viewed, implicitly, as having a similar character, reducing, presumably, to the form of an inequality, $A \neq A$. But the *Investigations* brings into question the notion that a general philosophical criterion to distinguish sense from nonsense is available or desirable. Wittgenstein is questioning philosophical (pre-)conceptions which are the source of debates about sense and senselessness. We might say, with justice, that one cannot really think or entertain or believe a contradiction; but we may with equal justice maintain that one *can* – as the long history of the trisection question illustrates. We can apparently inquire into something which is "contradictory". The point is raised in *Remarks on the Foundations of Mathematics*, where the interlocutor says,

> The difficulty which is felt in connexion with *reductio ad absurdum* in mathematics is this: what goes on in this proof? Something mathematically absurd, and hence unmathematical? How – one would like to ask – can one so much as assume the mathematically absurd at all? That I can assume what is physically false and reduce it *ad absurdum* gives me no difficulty. But how to think the – so to speak – unthinkable?

The suggestion here is that the possibility of entertaining something mathematically false or contradictory is somehow more difficult to make sense of than the possibility of entertaining something physically absurd. But elsewhere, Wittgenstein blurs this distinction (see, e.g., PI §§462–463, discussed below, p. 385). His aim is to bring out that the answer to any question about whether or not a question or a statement or an inquiry makes (rational) sense is, briefly put, "Yes and No". So Wittgenstein suggests, in response to his interlocutor, that

What an indirect proof says, however, is: "If you want *this* then you cannot assume *that*: for only the opposite of what you do not want to abandon would be combinable with *that*".[42]

Wittgenstein's italics on his demonstratives here are essential (compare RFM V §24, quoted above on page 379); he wishes to wean us away from a certain tempting conception of our idealizations of rationality. Only someone attached to a conception of proof, and of logic, according to which appreciation of the true logical basis of a judgment is essential to (fully) understanding it would worry that indirect argument in mathematics poses a special problem of coherence.[43] Wittgenstein warns us not to forget that our subscription to (what we, but not he, call) the "law" of contradiction, that "law"'s remarkable force with us, is a function of how we apply it in particular cases. We may express *respect* for certain practices in calling its application "inexorable"; but the "inexorability", the "compulsion" – as with any human law – finds its place at least in part in *our* inexorability in applying it, in the practices surrounding its statement and its application (cf. RFM I §118 for an investigation of the analogy with the "inexorability" of legal institutions).[44]

We may then grant that those who were searching for a trisection construction did not fully understand what they were asking for, what it was they "really wanted" to say. However, in so granting we do not place their contemporaries who were sceptical about the possibility of trisecting the angle on firmer ground. For neither those who attempted to trisect, nor those who conjectured the impossibility of trisection, were in possession of a proof. This is, of course, characteristic of a situation in which a *conjecture* is made in mathematics; like an expectation or an intention, it is, as Wittgenstein elsewhere writes (PI §337) "embedded in its situation, in human customs and institutions". (See Section II below.) Conjecturing falsely is analogous to playing a game of chess thinking that it is always possible to force a checkmate with only a king and a knight.[45] One suffers from a misunderstanding in not yet seeing what the rules of chess preclude; though in another sense, one may rightly be said to "understand" the rules of chess – indeed, it must be so, if one is ever to grasp the explanation of why such a checkmate is not possible.

Despite what some of his readers claim[46] – some on account of what they deem Wittgenstein's notion of an "internal" or "grammatical" relation, and some on account of the conviction that he *must* have been a verificationist – Wittgenstein is not insisting that conjectures in mathematics are meaningless, or that we do not understand a mathematical proposition until we possess its proof. (See section II below.) Whatever shift in understanding takes place as a result of the proof, it is not that we move from a situation in which there is *no* concept of trisecting – that is, a situation in which no meaningful statements may be made concerning trisection – to a situation in which we *now* have such a concept, can intelligibly talk. The point is illustrated by the only other occurrence of the trisection case in the *Investigations*, where it appears as a *counterexample* to such a view:

462. I can look for him when he is not there, but not hang him when he is not there.
One might want to say: "But he must be somewhere there if I am looking for him" – Then he must be somewhere there too if I don't find him and even if he doesn't exist at all.

463. "You were looking for *him*? You can't even have known if he was there!" – But this problem really does arise when one looks for something in mathematics. One can ask, for example, how was it possible so much as to *look for* the trisection of the angle?

This is a *reductio* of a misguided conception of the object of an intention or search: the trisection example shows that it makes sense to (systematically) "search for" something which not only does not exist, but *could* not exist. We can intelligibly entertain what Frege would have called "contradictory concepts":[47] we are not mathematically omniscient. (Note that here Wittgenstein both compares, and at the same time distinguishes, the empirical and the mathematical cases.)

Most generally, in the *Investigations* Wittgenstein uses the trisection example to try to complicate our idea of what it is to "really" understand, to fully mean or express, to "really" want to utter, a particular sentence. At stake is what he elsewhere calls the "very vague" quality of our (philosophical) concept of understanding a mathematical proposition[48]; and, hence, the inherent complexity of what it is to *really* understand a sentence or a language. The expression "So you really wanted to say . . ." has a multitude of legitimate and important applications in our language. But clearly Wittgenstein is sceptical that there is any systematic theoretical account which will informatively distinguish, in particular cases, between uttering or thinking a sentence with "real" meaning (that is, clearly and fully or completely expressing a thought, belief, desire or intention) and uttering or thinking a sentence which does not fully, clearly or completely express a thought, belief, desire or intention. The trisection example serves this scepticism concerning a general theoretical account of rational (logical) language use – at least if it is unattentive to our applications of logic in particular circumstances. The scepticism emerges in §334 of the *Investigations* through Wittgenstein's construing such a theory as a (purported) general account of our use of clarifying phrases such as "So you really wanted to say . . ."

II

As I have said, Wittgenstein's remarks on Gödel's theorem directly parallel his discussions of the classical nineteenth century impossibility proofs. A careful reading of the latter is essential for an understanding of the former.[49] A little bit of textual archaeology will illuminate this point, and will show why the trisection proof should have played such a significant role in Wittgenstein's later discussions.

Wittgenstein presupposes that his audience will have heard of this famous problem. In lectures given at Cambridge in Easter term of 1913, when Wittgenstein was in residence (we have no evidence that Wittgenstein attended) E. W. Hobson said:

> The popularity of the problem among non-Mathematicians may seem to require some explanation. No doubt, the fact of its comparative obviousness explains in part at least its popularity; unlike many Mathematical problems, its nature can in some sense be understood by anyone; although, as we shall presently see, the very terms in which it is usually stated tend to suggest an imperfect apprehension of its precise import. The accumulated celebrity which the problem attained, as one of proverbial difficulty, makes it an irresistible attraction to men with a certain kind of mentality. An exaggerated notion of the gain which would accrue to mankind by a solution of the problem has at various times been a factor in stimulating the efforts of men with more zeal than knowledge. The man of mystical tendencies has been attracted to the problem by a vague idea that its solution would, in some dimly discerned manner, prove a key to a knowledge of the inner connections of things far beyond those with which the problem is immediately connected.[50]

In a sense Wittgenstein was such a man; though he understood the impossibility proof perfectly. The following passage of the "middle Wittgenstein", from *Philosophical Grammar*, was penned some fifteen years earlier than the *Investigations*:

The trisection of an angle, etc.

> We might say: in Euclidean plane geometry we can't look for the trisection of an angle, because there is no such thing, and we can't look for the bisection of an angle, because there is such a thing.
>
> In the world of Euclid's Elements I can no more ask for the trisection of an angle than I can search for it. It just isn't mentioned.
>
> (I can locate the problem of the trisection of an angle within a larger system but can't ask within the system of Euclidean geometry whether it's soluble. In what *language* should I ask this? In the Euclideaen? – But neither can I ask in Euclidean language about the possibility of bisecting an angle within the Euclidean system. For in that language that would boil down to a question about absolute possibility, which is always nonsense.)
>
> . . . A question makes sense only in a calculus which gives us a method for its solution; and a calculus may well give us a method for answering the one question without giving us a method of answering the other. For instance, Euclid doesn't show us how to look for the solutions to his problems; he gives them to us and then proves that they are solutions. And this isn't a psychological or pedagogical matter, but a mathematical one. That is, the *calculus* (the one he gives us) doesn't enable us to look for the construction. A calculus which does enable us to do that is a *different* one. (Compare methods of integration with methods of differentiation, etc.)[51]

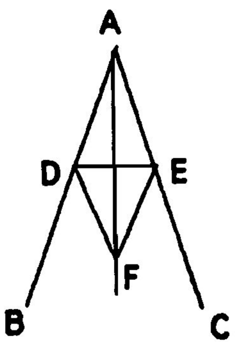
The passage has an odd sound, but there are some genuine mathematical points which Wittgenstein is exploiting. We need to look at some details of the proof of the impossibility of trisecting an angle in order to grasp Wittgenstein's intent.

Consider Proposition 9 and its proof in Euclid's *Elements*: to bisect a given rectilineal angle:

Proposition 9

To bisect a given rectilineal angle

Let the angle BAC be the given rectilineal angle.
Thus it is required to bisect it.
Let a point D be taken at random on AB;
  let AE be cut off from AC equal to AD; [I,3]
  let DE be joined, and on DE let the equilateral triangle DEF be constructed;
  let AF be joined.
I say that the angle BAC has been bisected by the straight line AF.
For, since AD is equal to AE, and AF is common,
The two sides DA, AF are equal to the two sides EA, AF respectively.

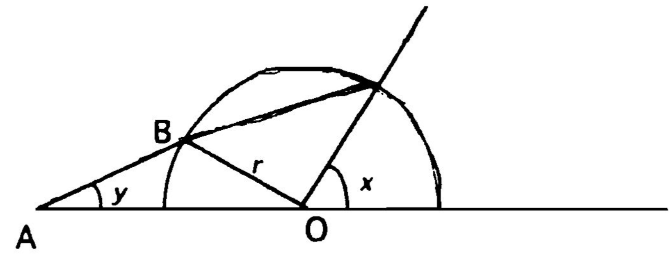


And the base DF is equal to the base EF; therefore the angle DAF is equal to the angle EAF. [I,8]
Therefore the given rectilineal angle BAC has been bisected by the straight line AF.

Notice that Euclid simply *exhibits* the construction, without any further preliminary remarks or summary. What is it to formulate a Euclidean problem: to *trisect* a given rectilineal angle? Wittgenstein's main concern is to show that the answer to this question is not as simple as it may at first appear to be; for to formulate the problem it does not suffice – or, suffices in only a very special sense – to simply form the sentence (in e.g., English, or Greek) "Is it possible to divide an angle into thirds in the same way?" Evidently, it is not enough to simply refuse to attempt to trisect the angle. (Behavioristic accounts of proof and understanding founder here.) *In asking a question, or in making a conjecture about trisection, one places constraints on possible answers or solutions insofar as one asks a mathematical question or makes a mathematical claim at all.* Let us see why.

The problem of giving a trisection construction is very ancient – older, certainly, than Euclid.[52] The idea that the problem ought to be determined or resolved in Euclidean terms represents a crucial step in its evolution.[53] For in any case Euclidean methods never exhausted the notion of a geometrical construction. For example, Archimedes offered a "trisection" of the following form:

Let an arbitrary angle x be given, as above. Extend the base of the angle to the left, and swing a semicircle with O as center and arbitrary radius r. Mark two points A and B on the edge of the ruler such that $AB = r$. Keeping the point B on the semicircle, slide the ruler into the position where A lies on the extended base of the angle x, while the

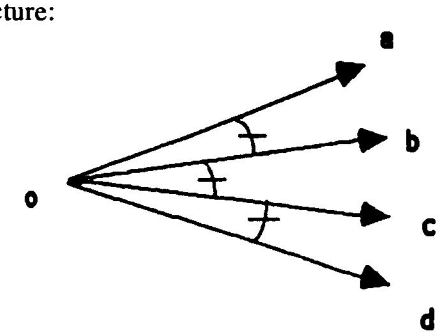


$y = {}^1/_3\, x$

edge of the ruler passes through the intersection of the terminal side of the angle x with the semicircle about O. With the ruler in this position draw a straight line, making an angle y with extended base of the original angle x.

This construction is *not* Euclidean, in that it calls for the use of a ruler not merely as a straightedge for drawing straight lines between given points, but also as a measure of distance (that is, in the course of the proof, you must slide the ruler, with a length marked on it, from one position to the other). Within Euclid's scheme, such uses are not discussed, much less permitted. Archimedes gives us a perfectly good geometrical construction; he does trisect the angle. Only not in what we *now* call "the relevant sense".

Furthermore, we must be able to distinguish *practical* from *theoretical* aspects of Archimedes' or Euclid's constructions. The thousands of people who occupy themselves with trying to trisect angles (or square circles, or double the volume of cubes) are often guilty of such confusion – creating a practice of giving what might be called "pseudo-constructions".[54] In fact, it is extremely easy to "trisect" an angle using protractor and pencil, or even using the naked eye. Just draw the following picture:



A trisected angle.

It is the ease with which one can "trisect" in a practical (or is this a theoretical?) sense which so naturally seduces one into thinking that it *cannot* be that Euclid's definitions preclude such an apparently simple construction. Despite what one might be tempted to insist, the above picture is no picture of a "trisected angle" in the relevant sense. Why not? – Because there *can be* no such construction. Hence there can be no picture which leads, step by step, by Euclidean means, *to* such a construction. Does this observation show that it is impossible to *imagine* trisecting the angle in the relevant sense? – Yes and No.[55] (Recall the discussion of *Investigations* §334 and §463 above in Section I.) In any case no image, picture, or diagram could *alone* give expression or meaning to a trisector's (or a conjecturer's) intention, much less to the mathematical problem itself. In the 2000 year quest for trisection we have, it seems, an example of what Wittgenstein means by a "picture holding us captive" (PI §115), or forcing itself on us (RFM I §14).[56] To free ourselves we require reflection on how we ask our own questions, on what we are inclined to *count* as a solution and what we are inclined to count as a non-solution. For example: we articulate the problem of trisection by drawing contrasts between trisecting the angle in a practical, or a physical, or an optical, or even a non-Euclidean geometrical way. This subtlety is essential because the ideal character of the construction shown to be impossible governs, not simply what we can draw or measure, but out attitude toward our own activities, our way of interpreting or conceptualizing them.[57]

Hobbes not only boasted that he had trisected the angle,[58] but also that he had squared the circle and doubled the cube: three equally impossible feats. What Hobbes really did was to give a method of construction *approximating* a solution. In fact it is possible to trisect in Euclid any arbitrary angle within close approximation. But this sort of "solution", however ingenious, was not (as we say) "what was wanted". We demanded (or wished to know about the possibility of) an *exact* solution. Again: it *is* possible to precisely trisect certain particular angles in Euclid (e.g., 90, 180 degrees). The trisection question eventually shown to be unsolvable is the *general* one: to give a single Euclidean method of construction which can be used to precisely trisect any arbitrary given angle.

Thus part of Wittgenstein's purpose in focussing on the formulation and the resolution of the trisection problem is to emphasize that there is no *absolute* requirement – mathematical or otherwise – that we restrict the conditions of "trisection" in the way we do. It is the *decision* to require that proofs be given within a particular setting, and that solutions take a particular form and be generally applicable which *generates* the unsolvable – this is, provably unsolvable, hence, resolvable – problem. Of course, this in no way renders the unsolvability of the task a matter of arbitrary human convention: God himself could not "trisect" an angle. But *we* can always, in Lakatos's words, bar – or create – what we *call* "monsters".[59]

The trisection proof is particularly well-suited to showing that a question is given the character it has by our appreciating the nature of the conditions we wish to place on an answer to it. As Wittgenstein well knew, it was long conjectured by mathematicians that the Euclidean trisection problem was unsolvable. (Indeed, by 1775, some sixty years before the proof was in hand, a resolution was passed by the Paris Academy that no more alleged solutions were even to be considered; apparently members of the Academy tired of reading through spurious "solutions".[60]) Before headway could be made on a *proof* of impossibility, an investigation was required into the abstract question, *How is it possible to prove that certain problems of construction can or cannot be solved?*[61] The answer, found toward the end of the eighteenth century and the beginning of the nineteenth century, relying on Descartes' analytic geometry, is to give a complete and rigorous algebraic characterization of all possible Euclidean constructions. Once one sees how to interpret in algebraic terms each legitimate Euclidean step, one has a setting within which to rethink or reinterpret the original notion of "constructible in Euclid". We use an equation to express the relation between a given set of line segments and the set of line segments needed as the solution to a particular construction problem. With the notion of a rational field, a class of so-called "constructible" numbers may be characterized which corresponds to the possible Euclidean steps with ruler and compass from a given point in the construction. To prove that it is impossible to give a general Euclidean procedure for trisecting any arbitrary angle, all one needs to do is to present an equation expressing a particular trisection problem which has no solution in certain extension fields of the rational numbers. Indeed, for an angle of 60 degrees, we need only show that the appropriate equation has no rational roots; and such an equation is surprisingly simple in appearance:[62]

$$8z^3 - 6z - 1 = 0.^{62}$$

The problem reduces to one in the theory of equations. It is solved by *mathematically* characterizing the constraints we place on any solution to a construction problem.

Elementary geometry alone does not hold, i.e., cannot express, a solution to the question about trisection. Of course, in higher algebra, we can clarify what is a possible Euclidean move; but, as we've just seen, in so clarifying, as Wittgenstein insisted, we also shift our question, i.e., we set ourselves *another* problem.

This is what Wittgenstein is getting at in the passage quoted above from *Philosophical Grammar* (see p. 387 above). Within the axiomatic system of Euclid, solutions are *exhibited*. This is the feel, the structure of Euclid (compare the bisection construction given above, p. 388). Questions about the *possibility* of asking and answering questions, or techniques of *searching* for solutions, aren't part of Euclid's system, in its original context. Indeed, as Wittgenstein emphasizes in his discussion of trisection in *Philosophical Grammar*, the absence of these possibilities is part of the *mathematical* – not merely the pedagogical or psychological – characterization of Euclid's system. We can contrast Euclid's "system" with mathematical contexts where we do have such an algorithm, or method (e.g., the computation of elementary sums, or the "calculus" of differentiating functions).[63] As Felix Klein wrote in his *Famous Problems of Elementary Geometry*, a renowned work it is quite likely Wittgenstein knew:

> We propose to treat of geometrical constructions, and our object will not be so much to find the solution suited to each case as to determine the *possibility* or *impossibility* of a solution . . .
>
> Our fundamental problem may be stated: *What geometrical constructions are, and what are not, theoretically possible*? To define sharply the meaning of the word "construction" . . .
>
> The singular thing [about the trisection problem] is that elementary geometry furnishes no answer to the question. We must fall back upon algebra and the higher analysis. The question then arises: How shall we use the language of these sciences to express the employment of straight edge and compasses? This new method of attack is rendered necessary because elementary geometry possesses no general method, no *algorithm*, as do the last two sciences.[64]

Wittgenstein's use of the word "calculus"[65] in *Philosophical Grammar* is loose. It would be an overstatement to hold that for Wittgenstein all mathematics is, as such, algorithmic or that only conjectures for which a method of resolution is in hand count as mathematical propositions.[66] By "calculus" or "system", I suggest, Wittgenstein means a practice of characteristic linguistic action involving (more or less) specific techniques. (Compare p. 402 below.) And by a "conjecture" or a "mathematical question", he means, one for which we make a *systematic* search

(in the above-named sense) *within* mathematics – in contrast, for example, to wishing or hoping to trisect the angle, or uttering the words "I wonder whether it's possible to trisect an angle" while doing nothing, or laying down a bet as to the theorem's outcome.

In this sense of "question" and "conjecture", Wittgenstein refuses, in the same passage from *Philosophical Grammar*, to treat either "Is it possible to *bisect* an angle in Euclid?" or "Is it possible to *trisect* an angle in Euclid?" as questions within Euclid's system – though each refusal is for a different reason. I can't ask about the possibility of *bisection*, because the proof exhibits the possibility in Euclid's *Elements*. That is: I cannot systematically search, in any sense of the notion, for something which I am already aware that I possess, which I can already locate. Once I accept the proof, I cannot conjecture its outcome. By contrast, my asking (before I know the proof) whether it's possible to *trisect* an angle in Euclid is really, as we've seen, a demand for further clarification of the notion of a possible construction. No techniques or methods given by Euclid help me to systematically search for this. I can play around with straightedge and compasses and tricks of construction I've already learned from Euclid; but this will take me only so far. What I require is a new way of interpreting the question.[67] In several places Wittgenstein likens a mathematical search of this kind to groping about; to trying to wiggle one's ears, without hands, if one doesn't yet know how to; or trying to will an object to move across the room.[68] Before we succeed (or fail), we have no clear understanding of what it would be like to succeed or fail. So we "grope around", we try to do, to formulate something. Like a philosophical problem, such searches, the contexts of conjecturing in mathematics, have the form, "I don't know my way about".[69] Conjectures about trisection in Euclid, even if they can be dressed up to look like propositions with truth-values, operate as much as linguistic stimuli, demands for clarification. What will satisfy those demands is doing something, i.e., exhibiting or producing a proof. And that requires coming to understand how something could possibly satisfy the conditions placed on a solution to the question. As Cora Diamond has emphasized,[70] searching for a proof, trying to prove something difficult, is in this sense more like searching for a solution to a riddle than searching for an object which already falls under a concept. Even given the correct thing we seek, we still need to be able to see how to fit our words, our notions, onto it if we are to recognize it *as* (if it is to *be*) an answer to our question, a *mathematical* solution. As Wittgenstein theatrically presented the point in his 1934–5 Cambridge lectures:

What one calls mathematical problems may be utterly different. There are the problems one gives a child, e.g., for which it gets an answer according to the rules it has been taught. But there are also those to which the mathematician tries to find an answer which are stated without a method of solution. They are like the problem set by the king in the fairy tale who told the princess to come neither naked or dressed, and she came wearing fish net. That might have been called not naked and yet not dressed either. He did not really know what he wanted her to do, but when she came thus he was forced to accept it. The problem was of the form, Do something which I shall be inclined to call neither naked nor dressed. It is the same with a mathematical problem. Do something which I shall be inclined to accept as a solution, though I do not know now what it will be like.[71]

Or again,

It is a genuine question if we ask whether it's possible to trisect an angle? And of what sort is the proposition and its proof that it's impossible with ruler and compasses alone?

We might say, since it's impossible, people could never even have tried to look for a construction.

Until I can *see* the larger system encompassing them both, I can't try to solve the higher problem.

I can't ask whether an angle can be trisected with ruler and compasses, until I can see the system "Ruler and Compasses" as embedded in a larger one, where the problem is soluble; or better, where the problem is a problem, where this question has a sense.

This is also shown by the fact that you *must* step outside the Euclidean system for a proof of the impossibility.

A system is, so to speak, a world.

Therefore we can't search for a system: What we *can* search for is the expression for a system that is given me in unwritten symbols.[72]

Does this amount to holding that the meaning of a mathematical statement is given by its proof? *No.* (See Section I.) Wittgenstein always denied that he held (unproved) conjectures to be meaningless.[73] Even in the early 1930's he wrote,

My explanation mustn't wipe out the existence of mathematical problems. That is to say, it isn't as if it were only certain that a mathematical proposition made sense when it (or its opposite) had been proved. (This would mean that its opposite would never have a sense (Weyl). On the other hand, it could be that certain apparent problems lose their character as problems – the question as to Yes or No.[74]

Sometimes, as we've seen, Wittgenstein will hold that he is simply distinguishing on mathematical grounds among different *sorts* of (mathematical) "questions":

Wouldn't all this lead to the paradox that there are no difficult problems in mathematics,

since if anything is difficult it isn't a problem? What follows is, that the "difficult mathematical problems", i.e., the problems for mathematical research, aren't in the same relationship to the problem "$25 \times 25 = ?$" as a feat of acrobatics is to a simple somersault. They aren't related, that is, just as very easy to very difficult; they are 'problems' in different meanings of the word.

"You say 'where there is a question, there is also a way to answer it', but in mathematics there are questions that we do not see any way to answer." Quite right, and all that follows from that is that in this case we are not using the word 'question' in the same sense as above. And perhaps I should have said "here there are two different forms and I want to use the word 'question' only for the first". But this latter point is a side-issue. What is important is that we are here concerned with two different forms. (And if you want to say they are just two different *kinds* of question you do not know your way about the grammar of the word "kind".

This amounts to asking: Does a mathematical proposition tie something down to a Yes or No answer? (i.e. precisely a sense.).[75]

Lecturing in 1932–33, Wittgenstein says explicitly that "the question has as much [mathematical] meaning as the messing about has."[76] And again:

Where you can ask you can look for an answer, and where you cannot look for an answer you cannot ask either. Nor can you find an answer.

Where there is no method of looking for an answer, there the question too cannot have any sense. – Only where there is a method of solution is there a question (of course that doesn't mean: "only where the solution has been found is there a question"). That is: where we can only expect the solution of the problem from some sort of revelation, there isn't even a question. To a revelation no question corresponds.[77]

III

In his discussions of Gödel Wittgenstein both relies on the lessons of his *Philosophical Investigations*, and at the same time attempts to reinforce those lessons by examining Gödel's proof. As always, a primary focus of Wittgenstein's investigation is the notion and role of "proposition". Indeed, given his insistence on the protean (if not illusory) character of the notion, Wittgenstein could not but have treated Gödel's theorem in the way he did.

Thus the first three sections of the Appendix on Gödel in RFM I do not explicitly discuss mathematics or logic at all; instead, they raise general points about our notions of *assertion* and *assumption*, closely paralleling *Investigations* §§21–24.[78] Indeed, it is striking that in their original context, in the early or *Frühversion* of *Philosophical Investigations*, this collection of remarks on Gödel grew out of an investigation of the circumstances in which we use the expressions "I *can* . . ." and "I *believe* I can . . ." when we are not doing . . . [79] (One such circumstance is, of course, when we are trying to prove something difficult in mathematics. Like his remarks about the impossibility of trisection, Wittgenstein's remarks on Gödel are intended to bring out the character of our concepts of "mathematical problem" and "mathematical solution".) In the first section of the Appendix Wittgenstein writes,

§1. It is easy to think of a language in which there is not a form for questions, or commands, but question and command are expressed in the form of statements, e.g. in forms corresponding to our: "I should like to know if . . ." and "My wish is that . . .".

No one would say of a question (e.g. whether it is raining outside) that it was true or false. Of course it is English to say so of such a sentence as "I want to know whether . . .". But suppose this form were always used instead of the question? – [80]

Wittgenstein asks us to reflect on how we apply "true" and "false" to sentences. "Of course" ("freilich"), Wittgenstein grants in the above-quoted remark, it is perfectly grammatical in one sense to apply "true" and "false" to a question formulated as a statement (for example, "I want to know whether it is possible to trisect the angle"). But the (perfectly grammatical) expression of the truth condition in this sort of case tells us no more than that the speaker has a desire to know something. And, as we have already seen (in Sections I and II) the very sense of that desire may itself be dependent on what turns out to be our understanding of the appropriate application of our expression of the truth condition. That is, if we construe a question, conjecture, or command in the grammatical form of a statement, then we must begin the statement with a clause such as "I should like to know . . ." or "I conjecture that . . ." or "My wish is that . . .". But we must not be misled by such declarative forms into thinking that every sentence of our language expresses a proposition, is true or false in the same way. Thus (for example) the sentence "It is (im)possible to give a general Euclidean method of trisecting the angle", *uttered before one accepts the proof*, can be construed, if one wishes, to be either true or false; but *saying* this, i.e., working with the truth-functions on such a "statement" (or is it a question?) amounts in effect to a demand for clarification, the announcement that one is going to *try* to prove something, to change the circumstances of the "statement"'s utterance. It says (like a command), "Go out and make a mathematical search!" Otherwise it is a kind of prophecy or empirical hypothesis whose connection to any subsequent mathematical activity is not *mathematical*. (Recall that in

the *Tractatus* mathematics and logic did not consist of propositions at all.) Wittgenstein's point in these sections is, of course, applicable beyond the mathematical cases:

§2. The great majority of sentences that we speak, write and read, are statement sentences.

And – you say – these sentences are true or false. Or, as I might also way, the game of truth-functions is played with them. For assertion is not something that gets added to the proposition, but an essential feature of the game we play with it. Comparable, say, to that characteristic of chess by which there is winning and losing in it, the winner being the one who takes the other's king. Of course, there could be a game in a certain sense very near akin to chess, consisting in making the chess moves, but without there being any winning and losing in it; or with different conditions for winning.[81]

"Assertion" is not something usefully separable from other characteristics of the practices ("games") in which assertions are made. The is a reiteration of the opening themes of the *Investigations*, where our very grasp of what it is to have a conception of language, or a theory of meaning, is brought into question. There Wittgenstein conveys, in myriad ways, how it is that pictures of language and thought and meaning tend to impose themselves full-blown on the phenomena. In particular, Wittgenstein tries to bring out that (and how) our grammatical taxonomies, such as our divisions of utterances into "kinds of sentences", are not ordered according to a pre-given structure or concept, but are only appropriate for particular (i.e., restricted) given purposes. In *Investigations* §23 he writes:

PI §23. But how many kind of sentences are there? Say assertion, question, and command? – There are *countless* [unzählige] kinds: countless different kinds of use of what we call "symbols", "words", "sentences". And this multiplicity is not something fixed, given once for all; but new types of language, new language-games, as we may say, come into existence, and others become obsolete and get forgotten. (We can get a *rough picture* [*ungefähres Bild*] of this from the changes in mathematics.)[82]

Now in §3 of the Appendix on Gödel Wittgenstein asks us to

§3. Imagine it were said: A command consists of a proposal ('assumption') and the commanding of the thing proposed.[83]

Here the imposition of the (Fregean) category of judgment-content or thought onto commands is depicted as overly artificial, insofar as it suggests that in every command there are really *two* logical categories at work: the judgment-content, or proposition, and the (utterly independent) willing of that content. Why not simply abandon the idea that the category of "proposition" (or "content" or "thought") is everywhere and generally applicable?

The subsequent sections of the Appendix in *Remarks on the Foundations of Mathematics* on Gödel revert to this theme in connection with logic and mathematics. The object – and presupposition – of Wittgenstein is that the so-called truth-values "true" and "false" have special characteristics in the practices of mathematics; and indeed, that these practices are themselves characterized by the ways in which "true" and "false" are applied in them. Wittgenstein's purpose is not to conceive of mathematics or language as *merely* formal games[84]; it is rather to bring to the forefront the significance of technique. It is not enough, he thinks, to merely insist that we *say* of logical and mathematical sentences that they are "true" and "false":

§4. Might we not do arithmetic without having the idea of uttering arithmetical *propositions*, and without ever having been struck by the similarity between a multiplication and a proposition?

Should we not shake our heads, though, when someone shewed us a multiplication done wrong, as we do when someone tells us it is raining, if it is not raining? – Yes; and here is a point of connexion. But we also make gestures to stop our dog., e.g., when he behaves as we do not wish.

We are used to saying "2 times 2 is 4", and the verb "is" makes this into a proposition, and apparently establishes a close kinship with everything that we call a 'proposition'. Whereas it is a matter only of a very superficial relationship.[85]

Wittgenstein's whole discussion of Gödel only takes on force if we are prepared to suppose that applying the notion of "proposition" in general is misleading, and particularly when we are talking about the sentences of arithmetic and logic. Once this is recognized, however, one has already broken away from the idea that we have a clear intuitive grasp of the concepts "mathematically true" and "mathematically provable". Wittgenstein used the trisection proof to raise general questions about both what is the role of mathematical conjecture, and what constitutes acceptance of a mathematical proof. Similarly,

However queer it sounds, my task as far as concerns Gödel's proof seems merely to consist in making clear what such a proposition as: "Suppose this could be proved" means in mathematics.[86]

In effect, Wittgenstein raises a series of questions about the very notion of a "*leading* problem of mathematical logic", which is, as he writes in the *Investigations*, "for us a problem of mathematics like any other."[87]

The discussion of changes in the concept of "provable in *Principia Mathematica*" in the Appendix to RFM I directly parallels Wittgenstein's treatment of the concept "constructible in Euclidean geometry". Section 380.(386.) of the early version of the *Investigations*, placed in context *between* what are now RFM I App. III sections 4 and 5, runs as follows:

Where in Euclid we are told: that such and such is to be *constructed* and in conclusion "q.e.c.", one could also put: it is to be *proved*, that this is the construction of this figure, and in conclusion to write "q.e.d.", thus, to bring the result of the *form* of the proved *sentence*.[88]

Wittgenstein wishes to compare the situation in which we make a conjecture with "the *situation*, into which such a proof brings us":

It might justly be asked what importance Gödel's proof has for our work. For a piece of mathematics cannot solve problems of the sort that trouble *us*. – The answer is that the *situation*, into which such a proof brings us, is of interest to us. 'What are we to say now?' – That is our theme.[89]

Thus what is presented in *Remarks on the Foundations of Mathematics* is not, as Goodstein (and many others) claimed, "an informal version of Gödel's nondemonstrable proposition"[90]; for it is neither a critique nor a rehearsal of Gödel's proof. Wittgenstein is attacking the very possibility of either by stressing, in his own idiosyncratic way, the importance of the precise *mathematical* character of Gödel's result. For example, in a letter to Schlick (July 31, 1935),[91] Wittgenstein insisted that there is no way to understand what Gödel means by "true but unprovable" propositions without looking at the details of the proof; he inveighs against the notion that there is anything essentially "surprising" or "mysterious" about Gödel's conclusion. He warns against confusing the "prose" surrounding the proof with the actual proof itself, and argues that the mere claim that "there are true but unprovable propositions of mathematics" says, on its own, absolutely nothing. Philosophy can say nothing about whether there can or cannot be "absolutely unprovable" truths of mathematics, or "absolutely undecidable propositions", and

if it attempts to, then you must certainly always fear that, like a false prophet, it will be reproved by reality.[92]

Wittgenstein's real enemy is the idea that as a result of (Frege's and Russell's) mathematical logic philosophy has been given clear concepts such as "provable", "unprovable", "mathematics", "proof" and "proposition". For those who take mathematical logic to clearly represent our concepts of "proof", "truth", and "mathematics", the Gödel result is a major step forward, definitively showing that we cannot simply identify "mathematical truth" and "mathematical proof". For Gödel showed, not only that *Principia Mathematica* was incomplete, he showed that it (or any analogous system) is essentially incomplete*able*. And, remarkably, he showed this by purely formal means, without assuming a notion of mathematical truth.[93] But Wittgenstein never granted that mathematical logic gave us a fully satisfactory way of viewing mathematics. So he refuses to grant the Gödel theorem a central philosophical place. I repeat: Gödel's incompleteness theorem is not a *threat* to Wittgenstein; for him it is simply an impossibility proof which when misconstrued gives rise to a way of talking about mathematics he abhors. The crucial point for Wittgenstein is that no symbolism is intrinsically self-applicable or meaningful. It is our *use* of the system (however complex) which brings it to life. And, similarly, it is our *use* (application) of our notions of "mathematical truth", "mathematical proposition" and "mathematical proof" which gives them the character they have (see p. 402 below).

Originally, according to Wittgenstein, all one could do to *show* provability in either Euclid's or in Russell and Whitehead's systems was to *exhibit* a proof or formal derivation from the given axioms. (This is not to say that there was no more to "provability" than just "has been proved" – see Section I above.) In the original situation, no general characterization of "possible proof" linking axioms to theorems was available, so without a proof in hand, a conjecture concerning the provability of a sentence or claim in either Euclidean geometry or in *Principia* amounted to looking for a "trick" to prove the sentence (See footnote 63); or else a demand for a general mathematical clarification of the notion of "provable" (or "constructible") itself. Question(s) about provability required, in short, a systematic mathematical search (in Wittgenstein's sense). Just as an ancient geometer might have asked, "Isn't it impossible to trisect an angle in Euclid?", so the interlocutor asks, *before* accepting Gödel's proof, in the original context of *Principia*:

§5. Are there true propositions in Russell's system, which cannot be proved in his system?[94]

This question is, for Wittgenstein, vague. One could always ask, of a given formula ("proposition") of *Principia Mathematica*, is it derivable from the axioms? Or one could ask, of each closed formula ("proposition") of *Principia Mathematica* whether it or its negation is derivable.

One could even ask whether the transcription in *Principia*'s symbolism of a particular theorem of mathematics is derivable in *Principia*. These are all meaningful mathematical questions (conjectures) for Wittgenstein. Do they say *anything* about absolute truth or provability in mathematics? For Wittgenstein, No. He is concerned that the interlocutor thinks that a system-independent notion of "true proposition" is not only available, but required for an understanding of the interlocutory question. So he asks the interlocutor,

– What is called a true proposition in Russell's system, then?[95]

Wittgenstein tries to show (with some irony) how important it is, and what it is like, to become clear about what will count as an answer to the interlocutor's question, and *how* Gödel's proof itself functions as a mathematical answer to it. The notion of a proposition's being "true" in Russell's system is, of course, not treated by Wittgenstein as appropriate or clear; he asks what it *means* to call a (logical) proposition "true":

§6. For what does a proposition's '*being true*' mean?
'*p*' *is true* = *p*. (That is the answer.)[96]

Read in context, this remark need not be read as an analysis of the truth of sentences generally, i.e., the so-called redundancy theory of truth, but rather perhaps only of those of logic, or, even more, specifically, of the sentences of *Principia* (see footnote 109). Yet in light of the more general discussion of assertion with which the Appendix on Gödel opens, the passage clearly continues Wittgenstein's attack on the availability of a general notion of "truth" or "proposition" for making sense of language, logic or mathematics. Wittgenstein goes on in the second paragraph of §6:

. . . So we want to ask something like: under what circumstances do we assert [behaupten] a proposition? Or: how is the assertion of the proposition used in the language-game? And the 'assertion of the proposition' is here contrasted with the utterance of the sentence e.g. as practice in elocution, – or as *part* of another proposition, and so on.[97]

Notice Wittgenstein's use of the notion of a "language-game" for what the interlocutor (in §5) called "Russell's system". Wittgenstein is trying to characterize how we employ *Principia Mathematica*. *Before* Gödel's proof, one circumstance in which we would claim a sentence of *Principia* to be "true" (i.e., "assert" it) is one in which it is formally derived (or taken as primitive). This is the third paragraph of §6:

. . . If, then, we ask in this sense: "Under what circumstances is a proposition asserted [behauptet] in Russell's game?" the answer is: at the end of his proofs, or as a 'fundamental law' (Pp.). There is no other way in this system of employing [Verwenden] asserted propositions in Russell's symbolism.[98]

What does Wittgenstein mean by "Russell's system"? Not, I think, what we would ordinarily mean by this term. (Compare the discussion of Wittgenstein's terms "calculus" and "system" above in Section II.) For Wittgenstein "Russell's system" – in contrast to what he calls "Russell's symbolism" – is not separable from (what we would call) the constructing of formal derivations according to the rules of *Principia Mathematica*. "Russell's system" is, for Wittgenstein, a particular activity or language-game, only one among many, albeit one in which our moves are relatively circumscribed in comparison to other "games" in our language. Thus in part I of RFM we read the following:

Now, what do we call 'inferences' in Russell or Euclid? Am I to say: the transitions [Übergänge] from one proposition to the next one in the proof? But where is the *passage* [Übergang] to be found? – I say that in Russell one proposition follows from another if the one can be derived from the other in conformity with the position of both in a proof and in the appended signs – when we read the book. For reading this book is a game that has to be learnt.[99]

In the context of playing Russell's game, we call this process of a human being writing down a sequence of signs "proving". But if "Russell's system" is to be a system of *proving*, then it must be used, employed, in a characteristic way, connected to the *sort* of conviction proof carries for us. This is precisely analogous, for Wittgenstein, to the way Euclid's *Elements* is used to "construct" geometrical figures. Hence "use" or "application" of a formalism [Verwendung, Anwendung] is *not* for Wittgenstein what we would ordinarily mean by "use", "interpretation" or "application" of a formalism to, say, the world, or to mathematical models.[100] Furthermore, "Russell's system" is to be distinguished from what Wittgenstein calls "Russell's symbolism", which he apparently views as a practice of transcribing sentences of German or English or mathematics with symbols of *Principia Mathematica*.

Now the interlocutor reacts to Wittgenstein's emphasis on use and assertion as if Wittgenstein is *denying* something; in an effort to clarify the question, he brings in the notion of "provable" [beweisbar]:

§7. But may there not be true propositions which are written in this symbolism, but are not provable in Russell's system?"[101]

Again, what is demanded by Wittgenstein is clarity about what is being asked *for*. How is our concept of "provable" *applied*? Wittgenstein remarks in §7 in response:

– 'True propositions', hence propositions which are true in *another* system, i.e. can rightly be asserted in another game. Certainly; why should there not be such propositions; or rather: why should not propositions – of physics, e.g., – be written in Russell's symbolism? The question is quite analogous to: Can there be true propositions in the language of Euclid, which are not provable in his system, but are true? – Why, there are even propositions which are provable in Euclid's system, but are *false* in another system. May not triangles be – in another system – similar (*very* similar) which do not have equal angles? – "But that's just a joke! For in that case they are not 'similar' to one another in the same sense!" – Of course not; and a proposition which cannot be proved in Russell's system is "true" or "false" in a different sense from a proposition of *Principia Mathematica*.[102]

The interlocutor's question only has sense within a particular practice. Russell's *symbolism* – and the interlocutor's conjecture about it – could be employed in various ways (e.g., to transcribe propositions of physics). Russell's *system* embodies a different practice. But not a more "general" theory – just another part of (what we may imprecisely call) mathematics. Wittgenstein is not merely insisting that "true and false" be relativised to sentences of Russell's *Principia*; his idea is much more radical, namely, that all the notions in which philosophers have been most interested – "truth", "provability", even "derivability" – find a home only within a specific, ongoing technique of use, and never within a formalism itself. A "formal system" does not, just by dint of its structure, "prove" anything – much less supermechanically. Our *handling* of the statement of its symbols and rules cannot be carved away from (our understanding of) what the structure is.[103] Thus, just as in Euclidean and non-Euclidean geometry there are alternative criteria of "similarity", in an analogous way in mathematics itself there is not one "system" of proof, nor is mathematics given by Russell's system. This much might have been taken, *before* Gödel's proof, to show the "incompleteness" of *Principia Mathematica*. Indeed, as I have repeatedly insisted, this had been for some time Wittgenstein's own view.

Wittgenstein is thus questioning the intuitive picture: for a system to be shown incomplete it must be shown incomplete with respect to some *independent* notion. Under this picture, Gödel's theorem shows that any formal system meeting certain basic requirements cannot completely capture mathematical truth (i.e., formally generate all and only the sentences which are (what we intuitively, uncritically call) the truths of mathematics). And the proof that this is so is, remarkably, formalizeable in the system itself – so that any such system is shown by Gödel to be sufficient to "prove" its own incompletability. But Wittgenstein renders this intuitive picture, these intuitive ways of speaking, suspect by treating Russell's mathematical logic as a "language-game", that is, by drawing an analogy between mathematical logic and (his conception of) Euclidean geometry. Before the nineteenth century, it appears to have been obvious that Euclidean geometry gave us knowledge of (the form of) space. Our intuitive notion, "space", was, one might have said, fully captured by Euclid's system, so that the very idea of a fact about space which was *false* in Euclid's theory was, at least, unlikely to be entertained (witness Kant's transcendental ideality of space). We knew, it might have been said, *what* Euclid's theory was a theory *of* (and this *apart* from whether we took that theory to be absolutely unreviseable, or certain, or not). But, Wittgenstein suggests, we didn't know (weren't clear about) any such thing. We never had a clear notion of "space" apart from how we mathematized space, or spoke about space in particular situations, i.e., apart from how we employed the concept of space. It is not that Wittgenstein is a conventionalist like Poincaré, or that he is insisting that the shift from a Euclidean theory of physical space to a non-Euclidean theory was merely a change in linguistic meaning.[104] Just the opposite: he is urging that that which is taken to be mathematicized not be separated from the mathematics (i.e., the techniques) employed in discussing it. Intuitive notions like "mathematical truth", "mathematical proposition" and "mathematical proof" are not independently and generally clear – any more than are the intuitive notions of "geometrical truth", "space", and "possible in space". So the very data of (certain) philosophical speculation and inquiry are unclear. And thus for Wittgenstein, the general semantic notions of truth and consistency play no determining role in Gödel's proof – just as, for Wittgenstein, they play no determining role in our acceptance of the trisection proof, or *any* indirect argument.[105]

In §8 Wittgenstein writes:

> §8. I imagine someone asking my advice; he says: "I have constructed a proposition (I will use 'P' to designate it) in Russell's symbolism, and by means of certain definitions and transformation it can be so interpreted (or clarified) [deuten] that it says: 'P is not provable in Russell's system'. Must I not say that this proposition on the one hand is true, and on the other hand is unprovable? For suppose it were false; then it is true that it is provable. And that surely cannot be! And if it is proved, then it is proved that it is not provable. Thus it can only be true, but unprovable."[106]

Wittgenstein is interested in seeing in what sense a sentence may be taken to say of itself "I am not provable". He feels the need to interpret the *reductio* which purports to summarize Gödel's reasoning, for the interlocutor makes it sound as if the concepts "provable in Russell's system", "proposition" and "true" have done the work of clarifying [deuten] the meaning of a well-formed formula in the *Principia* symbolism. But on Wittgenstein's view, there is no such (logical) "advice" or reasoning to be given independently of working through Gödel's proof:

> Jut as we ask: "'provable' in what system?", so we must also ask: "'true' in what system?" 'True in Russell's system' means, as was said: proved in Russell's system; and 'false in Russell's system' means: the opposite has been proved in Russell's system. – Now what does your "suppose it is false" mean? *In the Russell sense* it means 'suppose the opposite is proved in Russell's system'; *if that is your assumption*, you will now presumably give up the interpretation [Deutung] that it is unprovable. And by 'this interpretation' I understand the translation into this English sentence. – If you assume that the proposition is provable in Russell's system, that means it is true *in the Russell sense*, and the interpretation "P is not provable" again has to be given up. If you assume that the proposition is true in the Russell sense, *the same* thing follows. Further: if the proposition is supposed to be false in some other than the Russell sense, then it does not contradict this for it to be proved in Russell's system. (What is called "losing" in chess may constitute winning in another game.)[107]

The unclarities in the interlocutor's *reductio* vindicate, for Wittgenstein, his underlying idea that in a strict impossibility proof we clarify the terms in which a conjecture is originally posed. The sentence which may be interpreted to say of itself "I am not provable" says this only in a very particular context, i.e., that of Gödel's *proof*. In this case there is no application of an antecedently clear general notion of truth or provability or proposition which is not simultaneously a determining of what those notions themselves mean *here*.[108] As a result of Gödel's proof, for Wittgenstein there is a new sense in which we *now* call Gödel's sentences "unprovable in Russell's system" and yet at the same time "true". Wittgenstein suggests that, in accepting Gödel's theorem, we may be said to play a new game, *using* (i.e., applying) the words "true" and "provable in Russell's system" differently, with a *new* sense. Of course we will insist that the meanings of our words "provable" and "true", either in general, or in connection with *Principia Mathematica*, have not changed because of Gödel's proof – just as in the case where we insist that what trisectors meant by "construction" for 2000 years is precisely what we mean in ruling out as impossible a general Euclidean method of such "construction". But this does not show that our acceptance of Gödel's proof rests on an *antecedently* clear grasp of the concepts "true in mathematics" and "provable in mathematics" (or even "provable in *Principia Mathematica*"). In short, the interlocutory question ("But may there not be true propositions which are written in this symbolism, but are not provable in Russell's system?"), like any interesting mathematical question, requires for its answer a (re)interpretation. Rather than looking at the *Principia* as a faulty or inadequate structure for capturing "the truths of mathematics", Wittgenstein will argue that Gödel gave us a *new* way of understanding what it is to prove an arithmetical statement true;[109] and a *new* way of conceiving what we do when we construct formal derivations in the *Principia*. And thus the Gödel undecidable sentences – sentences in (what we call) "the language of *Principia Mathematica* (or, ultimately, "the language of arithmetic") have been placed in a new setting.

The result in this case is analogous to the situation when, after seeing the proof of the impossibility of trisecting an angle, we refuse to accept any purported construction as a "trisection construction" in the relevant sense:

14. A proof of unprovability is as it were a geometrical proof; a proof concerning the geometry of proofs. Quite analogous e.g. to a proof that such-and-such a construction is impossible with ruler and compass. Now such a proof contains an element of prediction, a physical element. For in consequence of such a proof we say to a man: "Don't exert yourself to find a construction (of the trisection of an angle, say) – it can be proved that it can't be done". That is to say: it is essential that the proof of unprovability should be capable of being applied [anwenden] in this way. It must – we might say – be a *forcible reason* [triftiger Grund] for giving up the search for a proof (i.e. for a construction of such- and-such a kind).

A contradiction is unusable as such a prediction.[110]

Wittgenstein's underlying view of Gödel's proof is that it demonstrates the impossibility of a certain construction – like a demonstration that it is impossible to trisect an angle with ruler and compass. The proof then – being a *proof* – contains "a physical element": we will not accept the goal of "constructing" a (formal) proof in *Principia* of a "Gödel sentence", and will warn people against trying to find derivations of these sentences. In fact we will *insist* that any purported derivation of such a sentence is not a derivation *in the relevant sense*. (We will also give up on trying to "codify mathematics" in a single recursively axiomatizable system, *if* we ever were inclined to try to do so.)

This may be interestingly compared with a remark by Gauss in his *Disquisitiones Arithmeticae* (1801), no. 365:

The limits of the present work exclude [a] demonstration here, but we issue this warning lest anyone attempt to achieve geometric constructions for sections other than the ones suggested by our theory (e.g. sections into 7, 11, 13, 19, etc. parts) and so spend his time uselessly . . .

Thus the "meta" mathematical method of Gödel is no "special *logic*",[111] it is a method of mathematics; and the Gödel sentences are simply like "propositions of a geometry which are actually applicable to themselves", i.e., to their own signs:

Could it be said: Gödel says that one must also be able to trust a mathematical proof when one wants to conceive it practically, as the proof that the propositional pattern can be constructed according to the rules of proof?

Or: a mathematical proposition must be capable of being conceived as a proposition of a geometry which is actually applicable [wirklich anwendbaren] to itself [to its own signs]. And if one does this it comes out that [in certain cases] it is not possible to rely on a proof.[112]

As when in the case of investigating the trisection of the angle we clarify what we take to be "constructible", so

15. Whether something is rightly called the proposition "X is unprovable" depends on how we prove this proposition. The proof alone shews what counts as the criterion of unprovability. The proof is part of the system of operations, of the game, in which the proposition is used, and shews us its 'sense'.

Thus the question is whether the 'proof of the unprovability of P' is here a forcible reason for the assumption that a proof of P will not be found.

16. The proposition "P is unprovable" has a different sense afterwards – from before it was proved.

If it is proved, then it is the terminal pattern in the proof of unprovability. – If it is unproved, then *what* is to count as a criterion of its truth is not yet *clear* – and, we can say – its sense is still veiled.[113]

Wittgenstein's idea is that Gödel's proof transforms the grounds of our willingness to "call" a certain sentence "unprovable" or "provable".

The acceptance of Gödel's proof *as a proof of the "unprovability" of "the Gödel sentences"* clarifies what we mean in calling something an "unprovable sentence". Wittgenstein's question is: What are we doing when we accept what we *call* "the proof of the unprovability of (the Gödel sentence) P" as a "forcible" or "cogent" reason (ein Triftiger Grund[114]) for ruling out the search for (what we *initially* call) "a proof of P"? What *is*, in short, "Gödel's proof"? Not just a string of sentences fulfilling a purely *formal* criterion of "being a derivation". The reasoning of Gödel is expressed in a series of sentences, which may in turn be formalized in the language of *Principia Mathematica*. But "Gödel's proof" is the employment of such sentences (i.e., their being taken as expressing *cogent* reasoning) in connection with our original (unclear) question (or conjecture) about "true but unprovable sentences of *Principia Mathematica*". Acceptance of a proof is a process, with a characteristic and *practically* visible result:

> Or: logic as the foundation of all mathematics does not work, and to shew this it is enough that the power of proof [Beweiskraft] of logical proof stands and falls with its geometrical proof-power.
>
> We incline to the belief that *logical* proof has a unique, absolute power of proof [Beweiskraft] deriving from the unconditional certainty in logic of the fundamental laws and the laws of inference. Whereas propositions proved in this way can after all not be more certain than is the correctness of the way those laws of inference are *applied* [Anwendung].[115]

In translating metastatements about *Principia* into sentences of *Principia* by way of the arithmetization of syntax, we clarify our metastatements mathematically – just as we clarified mathematically "possible construction" in the course of demonstrating the impossibility of trisecting the angle. In Gödel's proof, we are led to view both the constructing of formal derivations *and* certain number theoretic relations in a new light. This to Wittgenstein is a way of thinking about the situation which does far better justice to its complexity than the idea that Frege and Russell simply made a mistake in conflating one sharply expressible concept ("mathematical provability") with another sharply expressible concept ("mathematically true"); just as to Wittgenstein his treatment of the problem of trisecting the angle does far better justice to its complexity than the idea that lots of people simply make geometrical mistakes.

Of course there are important disanalogies between the proof of the impossibility of trisecting the angle and the Gödel proof. First, as we've seen, the evolution of concepts required to give the former proof took over 2,000 years, while Gödel was writing just 20 years after Russell and Whitehead. And (a related point) embedding the question in a whole new theory such as algebra was not necessary to obtain Gödel's result. Finally, it is not as if one constructed a triangle in Euclid which said of itself, "I am not constructible".[116] Still, the "self-applicability" of the Gödel sentences – and the strength of *Principia* to formalize a proof of its own incompletability – are not *grounds* for the view that mathematical logic analyzes or exhibits the underlying form of our notions of "mathematical proof" and "mathematical truth". To believe this (i.e., to take the Gödel result as informing us about these notions) we must *already* suppose that there is such an underlying form or structure to be presented. And as we have seen, Wittgenstein never presupposed this.

As a result of my reading I am inclined to regard Gödel's remarks on Wittgenstein's treatment of the incompleteness theorem as responding to the dialectical movement of Wittgenstein's writing, but confusing the interlocutor's voice, and Wittgenstein's answers to it, with Wittgenstein's understanding and his interpretation of the theorem and its proof. For Gödel wrote, in a letter to Menger, that

> As far as my theorem about undecidable propositions is concerned it is indeed clear from the passages you cite that Wittgenstein did *not* understand it (or pretended not to understand it). He interprets it as a kind of logical paradox, while in fact it is just the opposite, namely a mathematical theorem within an absolutely uncontroversial part of mathematics (finitary number theory or combinatorics). Incidentally, the whole passage you cite seems nonsense to me. See, e.g. the 'superstitious fear of mathematicians of contradictions'.[117]

Indeed, the interlocutory remarks are "nonsense", and Wittgenstein did, in the voice of his interlocutor, "pretend" not to understand the theorem; that is, he tried to depict the situation of someone making a mathematical search, preparing to accept the solution of a mathematical conjecture about incompleteness. But ultimately Wittgenstein's *interpretation* of (the mathematics of) Gödel's proof is really the same as Gödel's own: on Wittgenstein's view, Gödel's proof is *not* a logical paradox. Rather it is a piece of mathematics, the result of the application of mathematics *to* mathematics, yielding a clarification of the question about whether there are "true but unprovable" statements of *Principia Mathematica*.

The underlying point of Wittgenstein's remarks on Gödel is the underlying theme of the later Wittgenstein as a whole: our sentences do not carry their meaning with them intrinsically, or in virtue of something present to the mind ahead of, or apart from, how we give it expression in particular cases. Rather, what we can clearly say about what we mean or think can be made sense of only from within the context of some practice, or ongoing system of use. And sometimes, there is no one thing one *really* says or means with a grammatically well-formed expression. What at the beginning is apparently attempted, intended, expected, or reasoned about may, as a result of drawing a connection between notions, come to be seen as not thinkable, no goal at all. So that sometimes, what looks like a straightforward thought or proposition is perhaps less misleadingly seen as question or a demand for clarification, a function of a given situation. In this sense, there is no new thesis argued for in Wittgenstein's remarks on Gödel.[118]

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NOTES

[1] It is difficult to assess at the present time what percentage of Wittgenstein's writings on Gödel have been published, though we know it is not yet all of them (see footnote 20). For example, there are several passages relating to Gödel in MS 124 of Wittgenstein's *Nachlass*, the basis for RFM VII §§1–23, which do not appear in the published text. The published passages on Gödel are in RFM I App. III and RFM VII §§19ff; Letter to Schlick of 31 July 1935 and miscellaneous remarks published in Nedo and Ranchetti (eds.), *Wittgenstein: Sein Leben in Bildern and Texten* (Frankfurt am Main: Suhrkamp Verlag, 1983), p. 260; and LFM pp. 47, 56, 188–189. (References to Wittgenstein's works employ the usual abbbreviations; see the bibliography below.)

[2] For example, there is the paper "Mathematics and Its Foundations" (*Mind* 1938, pp. 440–451) by Alister Watson, a Cambridge physicist who attended Wittgenstein's 1939 lectures and apparently arranged a discussion group with Wittgenstein and Turing in the summer of 1937. Watson's paper shows a clear grasp of the Gödel theorem and related results, as well as the influence of both Turing and Wittgenstein. Watson writes (p. 445):

> The interpretation which I shall give of the famous example of Gödel owes much to lengthy discussions with a number of people, especially Mr. Turing and Dr. Wittgenstein of Cambridge.

Compare Andrew Hodges, *Alan Turing: The Enigma* (NY: Touchstone, 1983), pp. 109, 136; and Wittgenstein's RPP I §1096. Circumstantical evidence is also provided by Turing's presence at Wittgenstein's 1939 lectures: the two do not seem to differ radically over the Gödel theorem, and if Wittgenstein had been as ignorant as he is usually

claimed to be, it is difficult to see why Turning would not have raised the issue. Compare footnote 90.

[3] The *locus classicus* is A. R. Anderson, "Mathematics and the "Language Game" ", *The Review of Metaphysics* **11** (1958): pp. 446–458, reprinted in the first edition of Benacerraf and Putnam (eds.), *Philosophy of Mathematics* (Prentice Hall, 1964): pp. 481–490.

[4] See G. Kreisel, "Einige Erläuterungen zu Wittgensteins Kummer mit Hilbert und Gödel", P. Weingartner and J. Czermak (eds.), *Epistemology and Philosophy of Science. Proceedings of the 7th International Wittgenstein Symposium* (Vienna: Hölder-Pichler-Tempsky, 1983): pp. 295–303, esp. p. 295; and his review of "Wittgenstein's Remarks on the Foundations of Mathematics", *British Journal for the Philosophy of Science* **9** (1958): pp. 135–158, esp. p. 153.

[5] Dummett, "Wittgenstein's Philosophy of Mathematics", in *Truth and Other Enigmas* (Cambridge: Harvard University Press, 1978), p. 166.

[6] Hintikka, "The Original Sinn of Wittgenstein's Philosophy of Mathematics", Manuscript p. 8. Compare Jaakko Hintikka and Merrill B. Hintikka, *Investigating Wittgenstein* (New York: Blackwell, 1986), esp. chapter 1.

[7] According to Carnap, the *Tractatus* holds that there is only one language, and that no language can express its own syntax. Therefore, "syntax is not expressible". See Rudolf Carnap, *The Logical Syntax of Language* (trans. Amethe Smeaton; London: Routledge & Kegan Paul Ltd, 1937), pp. 53, 101, 282–284.

[8] Carnap, *The Logical Syntax of Language*, p. 53. As Dreben has stressed (see his "Quine", in *Perspectives on Quine*, eds. Robert B. Barrett and Roger F. Gibson (Cambridge, MA: Basil Blackwell, 1990): 81–95, esp. p. 85), Carnap "refutes" what he takes to be a Wittgenstein doctrine in the context of accepting and making precise what to him was the fundamental insight of the *Tractatus*, an "insight" on which he based all of his own philosophical work, namely, that logic, mathematics and (what remains of) philosophy are empty of content, are purely formal, i.e., tautological. The connection with Carnap is also mentioned by Goldfarb and Ricketts, who refer to one of Wittgenstein's remarks on Gödel in the context of assessing the impact of Gödel's incompleteness theorems on Carnap's program in *The Logical Syntax of Language* (see their "Carnap and the Philosophy of Mathematics", in D. Bell and W. Vossenkuhl (eds.), *Science and Subjectivity* (Berlin: Akademie Verlag 1992)).

[9] S. G. Shanker, "Wittgenstein's Remarks on the Significance of Gödel's Theorem", in the anthology he edited, *Gödel's Theorem in Focus* (London: Croom Helm, 1988): pp. 155–256.

[10] At first glance Shanker seems to attribute to even the later Wittgenstein the view that the syntax of a language cannot be expressed in that language itself, and to maintain, on behalf of this purported Wittgensteinian view, that Gödel's "Mirroring Lemmas", in which the meaning of metamathematical statements are "mirrored" by sentences of the formal language, are objectionable: "in no case can [meta-mathematical propositions] be construed as propositions about 'object' expressions, and, a fortiori, about the arithmetical relations between the corresponding expressions" ("Wittgenstein's Remarks on the Significance of Gödel's Theorem", p. 216). To what degree this first glance is misleading about Shanker is a matter I shall not discuss here.

[11] Hao Wang, "To and From Philosophy – Discussions with Gödel and Wittgenstein" (*Synthese* **88**, 2 (1991): pp. 229–277) section 6.1.

[12] See Wang's *Beyond Analytic Philosophy: Doing Justice to What We Know* (Cambridge: MIT Press, 1986), his *Reflections on Gödel* (Cambridge: MIT Press, 1987); and his address to the Kurt Gödel Society Meeting, "Imagined Discussions with Gödel and with Wittgenstein" at Kirchberg am Wechsel, Austria, 1991.
[13] Wang, "To and From Philosophy – Discussions with Wittgenstein and Gödel", section 6.2.
[14] Wang, "To and From Philosophy – Discussions with Wittgenstein and Gödel", section 6.2.
[15] RFM VII §19. The German may be read to mean: to evade in talking, to talk around, or at cross purposes to, Gödel's proof: "Meine Aufgabe ist es nicht, über den Gödelschen Bewis, z. B., zu reden; sondern an ihm vorbei zu reden."
[16] First laid out by Pierre Wanzel in his "Recherches sur les moyens de reconnaître si un Problème de géométrie peut se résoudre avec la règle et le compas," *Journal de Mathématiques Pure et Appliquées* **2** (1837), pp. 366–372.
[17] See, e.g., WVC, pp. 36f, 143f, 204ff; PR Section XIII, pp. 170–192; and PG pp. 387ff; RFM I App. III §14, RFM II §2; III §87; IV §30; VII §15; DL, p. 100; AL pp. 8–9, 185–186, 192–193; LFM pp. 56ff, 86–89; BB p. 41; PI §§334, 463 and G. E. Moore's "Wittgenstein's Lectures in 1930–33", in Moore's *Philosophical Papers* (New York: Humanities Press, Inc., 1959), pp. 304ff.
[18] The most carefully worked over of Wittgenstein's published remarks on Gödel, RFM I App. III, originally formed §§374. (380.) to 396.(402.) of the *Frühversion*, or Early Version, of *Philosophical Investigations* (G. H. von Wright and Heikki Nyman have compiled a critical edition of this material). These remarks were presumably composed in the fall of 1937, and in 1938 Wittgenstein wrote a preface to his manuscript, submitting it to the Cambridge Press for publication. Although the manuscript was accepted, Wittgenstein did not allow it to be published. (For details see von Wright's "The Origin and Composition of *Philosophical Investigations*", in his *Wittgenstein* (Minneapolis: University of Minnesota Press, 1983): pp. 111–136), pp. 125ff, and my "Wittgenstein on 2,2,2 . . .: On the Opening of *Remarks on the Founations of Mathematics*" (*Synthese* 87 1 (1991): pp. 143–180), p. 146.) Compare footnote 20.
[19] This point – along with its consequences for ways of reading Wittgenstein – I have learned primarily from lectures and conversations 1982–89 with Stanley Cavell, Burton Dreben, Warren Goldfarb and Hilary Putnam. Among published sources stressing and articulating the point are Cavell's *Must We Mean What We Say?* (Cambridge University Press, 1969); *The Claim of Reason: Wittgenstein, Skepticism, Morality, and Tragedy* (Oxford: Oxford University Press, 1979) and *This New Yet Unapproachable America: Lectures After Emerson After Wittgenstein* (Albuquerque: Living Batch Press, 1989) (see esp. pp. 30, 36, 45); Warren Goldfarb's "I Want You to Bring Me a Slab", *Synthese* 56 (1983): pp. 265–282, and "Wittgenstein on Understanding", *Midwest Studies in Philosophy XVII: The Wittgenstein Legacy*, eds. Peter A. French, Theodore E. Uehling, Jr. and Howard K. Wettstein: pp. 109–122; and Putnam's "Wittgenstein on Religious Belief", in *On Community*, ed. Leroy S. Rouner (Notre Dame: 1991): pp. 56–75. In my "Wittgenstein on 2,2,2. . .: On the Opening of *Remarks on the Foundations of Mathematics*" I have tried to bring this interpretive and philosophical method to bear on RFM.
[20] Given the interpretive weight I place on the form of Wittgenstein's writing, it must be admitted that none of Wittgenstein's later writings are, from a scholarly point of view, textually sacrosanct, since all have been published posthumously and only in part,

according to (sometimes strongly exercised) editorial discretion. This is especially true of Wittgenstein's discussions of mathematics. The editors of RFM wrote in their (1978) preface to the second edition that they felt that "the time had not yet come" to publish all of Wittgenstein's writings on mathematics in their original contexts. They make it clear that only a selection of remarks have been published; but chronology of entry, order of remarks, and status of manuscript (e.g., handwritten remarks or typescript) are obscured in the published version. The published passages relating to Gödel, encompassing only a portion of what Wittgenstein actually wrote on the topic, are a varied collection. While I recognize, and have indeed myself argued (in my "Wittgenstein on 2,2,2 . . . : The Opening of *Remarks on the Foundations of Mathematics*") for the need to bear in mind differences among texts of varying status, I still believe that, however rough Wittgenstein's drafts, he always at least aimed at the kind of finely honed self-reflexiveness in his writing I am convinced is crucial to his way of thinking. Meanwhile, we await a proper, scholarly respectable edition of Wittgenstein's *Nachlass*. For an illuminating discussion of the present state of affairs, see Hintikka's "An Impatient Man and His Papers", *Synthese* **87**, 2 (1991): pp. 183–202. And compare note 18.
[21] This, I believe, is Wang's suggestion in section 6.2 of his "To and From Philosophy – Discussions with Wittgenstein and Gödel". See section III below. Compare Wright, *Wittgenstein on the Foundations of Mathematics*, p. 28.
[22] An interpretation of the *Tractatus* has been offered by Eli Friedlander in his "Expressions of Judgment" (Harvard PhD Thesis, 1992), according to which formalization is itself a ladder to be thrown away and so plays, ultimately, more than a direct or straightforward role as a useful "concept-script". The suggestion is that the significance of the *Tractatus* as a whole is essentially autobiographical and ethical, a form of what Cavell has called (in, e.g., *Conditions Handsome and Unhandsome*, The Carus Lectures 1988 (University of Chicago: 1990)), "moral perfectionism". A different view of the point of formalizing logic and thinking about ethics according to the *Tractatus* is given by Cora Diamond in *The Realistic Spirit: Wittgenstein, Philosophy and the Mind* (Cambridge: M.I.T. Press, 1991).
[23] See TLP 6.2ff. The character of Wittgenstein's "analysis" is discussed in Dreben and Floyd's "Tautology: How Not to Use A Word" (*Synthese* **87**, 1 (1991): 23–50; and a discussion of the status of mathematics in the *Tractatus* in my "The Rule of the Mathematical: Wittgenstein's Later Discussions" (Harvard PhD Dissertation, 1990).
[24] TLP 6.126, 6.2331, 6.234.
[25] See TLP 6.02.
[26] Wittgenstein's use of "Anwendung" is not the notion of "application" traditionally associated by philosophers (such as Kant and Carnap) with the "application" of mathematics to the world. Cf., e.g., RFM III §§4, 30, 43; V §§5,8; VII §6, and compare p. 402 above.
[27] RFM V §13.
[28] See the "Notes Dictated to G. E. Moore in Norway, April 1914", Appendix II in NB, p. 109; and TLP 6.125. Compare the discussion in "Tautology: How Not to Use a Word", p. 35 and RFM V §48.
[29] RFM V §24.
[30] RFM V §40.
[31] RMF V §25.
[32] RFM V §38.

[33] No doubt Wittgenstein is drawing, at some level, a caricature of the philosophical positions of Frege and Russell. I must leave to the side the centrally important question of how much truth is contained in the cartoon.
[34] CV, p. 24.
[35] RFM III §25. The same figure of speech occurs, for different philosophical purposes, at TLP 4.002. Cf. PR XV, p. 206, RFM III §53. And at CV, p. 25:

> The queer resemblance between a philosophical investigation (perhaps especially in mathematics) and an aesthetic one. (E.g. what is bad about this garment, how it should be, etc.)

[36] RFM III §§45–46.
[37] See, e.g., RFM I §§106ff; III §§31ff, 41; V §§34ff; VII §§45ff.
[38] Historically speaking the role of the Gödel theorem with respect to Frege and Russell is complex. Frege did not live to see the publication of either of Gödel's two initial major results: the completeness theorem for first order validity, and the incompleteness theorem. While the Gödel proof may seem to justify or reinforce some aspects of the general philosophical conceptions of mathematics held by Frege and Russell, i.e., their so-called Platonism, at the same time Gödel undercuts the particular technical means by which Frege and Russell had attempted to present their view of logic as a maximally general (a priori) science. Not that if they had known Gödel's result, would either Frege or (the pre-1931) Russell have jettisoned their general philosophical accounts of sense, truth, objectivity and thought. In an interesting letter Russell wrote in 1963 to Leon Henkin, he confessed that Gödel's work seemed to him to be "of fundamental importance", but "puzzled" him, and made him "glad" to be no longer working on mathematical logic. (The letter is partially published on p. 96 of John Dawson, Jr. "The Reception of Gödel's Incompleteness Theorems", in Thomas Drucker, editor, *Perspectives on the History of Mathematical Logic* (Boston, Birkhäuser Boston, Inc., 1991, pp. 84–100.)) Peter Hylton, in *Russell, Idealism and the Emergence of Analytic Philosophy* (Oxford: The Clarendon Press, 1990), p. 287n writes that

> If logic is taken to be a formalism then Gödel's theorem shows at once that mathematics is not identical with logic, i.e. that logicism is false. From a Russellian point of view, however, there is no reason to identify logic with a formalism. Rather Gödel's theorem seems, from this point of view, to show that the theory of propositional functions – logic itself – cannot be completely formalized. This situation may be peculiar, but there is no evident reason why it should be fatal to the project of *Principia Mathematica*.

[39] PI §334.
[40] Compare *The Blue Book* (1933–34), p. 41:

> The phrase "to express an idea which is before our mind" suggests that what we are trying to express in words is already expressed, only in a different language; that this expression is before our mind's eye; and that what we do is to translate from the mental into the verbal language. In most cases which we call "expressing an idea, etc." something very different happens. Imagine what it is that happens in cases such as this: I am groping for a word. Several words are suggested and I reject them. Finally one is proposed and I say: "That is what I meant!"
>
> (We should be inclined to say that the proof of the impossibility of trisecting the angle with ruler and compasses analyses our idea of the trisection of angle. But the proof gives us a new idea of trisection, one which we didn't have before the proof constructed it. The proof led us down a road *which we were inclined to go*; but it led us away from where we were, and didn't just show us clearly the place where we had been all the time.)

Compare notes 55 and 56 below.
[41] TLP 4.46.
[42] RFM V §28.
[43] There is precedent in the history of philosophy. For example, in two articles concerning the history and philosophy of mathematics in the seventeenth century, Paolo Mancosu sets out the historical effects, within the pre- and post-Galilean period, of what was called the "Quaestio de Certitudine Mathematicarum", a debate over the question of whether mathematics counted as a true science in Aristotle's sense. Mathematicians came to focus on, and avoid, proofs by contradiction in the wake of this debate. See Mancosu's "Aristotelian Logic and Euclidean Mathematics: Seventeenth-Century Developments of the *Quaestio de Certitudine Mathematicarum*" (*Studies in the History and Philosophy of Science* **23**, 2 (1992): pp. 241–265); and his "On the Status of Proofs by Contradiction in the Seventeenth Century" (*Synthese* **88** (1991): pp. 15–41).
[44] Wittgenstein's concern with the status of Frege's and Russell's general conceptions of logic and understanding, rather than a subscription to an intuitionist style critique of the application of the law of the excluded middle, seems to govern his discussions of indirect argument. I cannot treat the complex question of Wittgenstein's relation to intuitionism here; but most discussions of his alleged (finitist) "worries" about the law of the excluded middle tend to gloss over the centrality of his concern with Russell and Frege, i.e., his never having granted their idea that logic consists of generally applicable laws. Mathieu Marion's "Wittgenstein and Finitism" (forthcoming, *Synthese*) usefully contrasts Wittgenstein's discussions of mathematics in the early 1930's with those of Weyl, Brouwer, Heyting, and others – though Marion still wishes to label Wittgenstein a "finitist". Compare Wang's suggestion that Wittgenstein subscribes to what Wang calls "variable-free finitism" ("To and From Philosophy – Discussions with Gödel and Wittgenstein", section 5).
[45] Cf. RFM V §28.
[46] See, e.g., Shanker, *Wittgenstein and the Turning Point in the Philosophy of Mathematics* (SUNY Press: 1987), chapter 3. In "Wittgenstein's Remarks on the Significance of Gödel's Theorem", Shanker reiterates his general view that for Wittgenstein there is (p. 185) "the distinction between *mathematical questions* – whose meaning is determined by the rules of the system in which they reside – and *mathematical conjectures*, which by definition inhabit no system". On this view, a mathematical conjecture is a "meaningless expression albeit one which may exercise a heuristic influence on the construction of some new proof-system" (p. 230).
[47] Cf. Frege's discussion in §76 of *The Foundations of Arithmetic* (2nd ed., trans. J. L. Austin (Evanston, IL: Northwestern University Press, 1980)), where Frege not only entertains the contradictory concept of "not identical with itself", but bases his definition of the number zero, hence, his logicist reduction, on this concept.
[48] RFM V §46.
[49] Shanker, in his "Wittgenstein's Remarks on the Significance of Gödel's Theorem", also draws on this parallel.

[50] E. W. Hobson. *Squaring the Circle: A History of the Problem* (Boston: Chelsea Publishing Co., 1953), p. 4.
[51] PG, pp. 387–388. Cf. PR XIII, esp. pp. 178–179.
[52] Carl B. Boyer (*A History of Mathematics* (Princeton University, 1968), p. 71) traces, according to legend, the formulation of the three "classical" problems of antiquity (the squaring of the circle, the doubling of the cube, and the trisection of the angle) to the time of Athenian Plague and the death of Pericles (428 B.C.E.). Euclid wrote around 300 B.C.E., presumably under the patronage of Ptolemy I at Alexandria.
[53] Cf. Hobson, *Squaring the Circle: A History of the Problem*, p. 16:

> From the time of Plato (429–348 B.C.), who emphasized the distinction between Geometry which deals with incorporeal things or images of pure thought and Mechanics which is concerned with things in the external world, the idea became prevalent that [such] problems . . . should be solved by Euclidean determination only, equivalent on the practical side to the use of two instruments only, the ruler and the compass.

[54] See Underwood Dudley's intriguing presentation of purported "trisections", *A Budget of Trisections* (NY: Springer Verlag, 1987).
[55] G. E. Moore, "Wittgenstein's Lectures in 1930–33", p. 304:

> 'How can we look for a method of trisecting an angle by rule and compasses, if there is no such thing?' [Wittgenstein] said that a man who had spent his life in trying to trisect an angle by rule and compasses would be inclined to say "If you understand both what is meant by "trisection" and what is meant by "bisection by rule and compasses", you must understand what is meant by "trisection by rule and compasses"' but that this was a mistake; that we can't imagine trisecting an angle by rule and compasses, whereas we can imagine dividing an angle into eight equal parts by rule and compasses; that 'looking for' a trisection by rule and compasses is not like 'looking for' a unicorn, since 'There are unicorns' has sense, although in fact there are no unicorns, whereas 'There are animals which show on their foreheads a construction by rule and compasses of the trisection of an angle' is just nonsense like 'There are animals with three horns, but also with only one horn': it does not give a description of any possible animal.

Compare PI §517.
[56] The appropriate way to characterize what it is to search for an answer to a difficult mathematical problem throws light, for Wittgenstein, on what it is to do philosophy. Moore reports ("Wittgenstein's Lectures in 1930–31", pp. 304–305) that

> Wittgenstein's answer to the original question ['How can we look for a method of trisecting an angle by rule and compasses, if there is no such thing?'] was that by proving that it is impossible to trisect an angle by rule and compasses 'we change a man's idea of trisection of an angle' but that we should say that what has been proved impossible is the very thing which he had been trying to do, because 'we are willingly led in this case to identify two different things'. He compared this case to the case of calling what he was doing 'philosophy', saying that it was not the same kind of thing as Plato or Berkeley had done, but that we may feel that what he was doing 'takes the place' of what Plato and Berkeley did, though it is really a different thing. He illustrated the same point in the case of the construction of a regular pentagon, by saying that if it were proved to man who had been trying to find such a construction that there isn't any such thing, he would say 'That's what I was trying to do' because 'his idea has shifted on a rail on which he is ready to shift it'. And he insisted here again that (a) to have an idea of a regular pentagon and (b) to know what is meant by constructing by rule and compasses, e.g. a square, do not in combination enable you to know what is meant by constructing, by rule and compasses, a regular pentagon.

Wittgenstein's remark on whether what he is doing is or is not continuous with what Plato or Berkeley would have called "philosophy" is itself an extraordinary interesting instance of his *doing* of (what we may call) philosophy.
[57] Cf. RFM IV §30:

> Do not look at the proof as a procedure that *compels* you, but as one that *guides* you. – And what it guides is your *conception* of a (particular) situation.
>
> . . . *In the course* of this proof we formed our way of looking at the trisection of the angle, which excludes a construction with ruler and compass.
>
> By accepting a proposition as self evident, we also release it from all responsibility in face of experience.
>
> In the course of the proof our way of seeing is changed – and it does not detract from this that it is connected with experience.
>
> Our way of seeing is remodelled.

[58] In his *Rosetum Geometricum* (1671) (cf. Dudley, *Budget of Trisections*, pp. 95–96.)
[59] See Lakatos, *Proofs and Refutations, the Logic of Mathematical Discovery*, eds. J. Worrall and E. Zahar (NY: Cambridge University Press, 1977), Ch. 1.
[60] Hobson, "*Squaring the Circle*": *A History of the Problem*, pp. 3–4.
[61] Cf. Richard Courant and Herbert Robbins, *What is Mathematics?* (New York: Oxford University Press, 1969), p. 118.
[62] Cf. Courant and Robbins, *What is Mathematics*?, p. 137, where the equation is derived from the trigonometric fact that $\cos \Theta = 4 \cos^3 (\Theta/3) - 3 \cos (\Theta/3)$. Compare I. N. Herstein's *Topics In Algebra* (2nd edition, Lexington, Mass: Xerox, 1975), pp. 230–231.
[63] Cf. AL, p. 116, where Wittgenstein argues for a "family resemblance" notion of proof:

> There are proofs in connection with which there is a rule for making up similar proofs, e.g., for proving that a certain number is a multiple of 2 others. But in Euclid there are no such rules; each proof is a sort of trick.

[64] Felix Klein, *Famous Problems of Elementary Geometry* (1895); trans. W. W. Beman, D. E. Smith, R. C. Archibald, 2nd ed.
[65] Wittgenstein relies on the notion of a "calculus" primarily during his so-called middle period. Gerrard and Hilmy have stressed that "calculus" comes to be supplanted by the notion of a "language-game" in Wittgenstein's later work. See Steve Gerrard, "Wittgenstein's Philosophies of Mathematics" (*Synthese* 87 No. 1 1991: pp. 125–142) and S. Stephen Hilmy, *The Later Wittgenstein* (Oxford: Basil Blackwell, 1987).
[66] The former is suggested by Mathieu Marion in his "Wittgenstein and the Dark Cellar of Platonism" (address to the XVth International Wittgenstein Symposium, Kirchberg am Wechsel, Austria, 1992); the latter by Shanker in his *Wittgenstein and the Turning*

Point in the Philosophy of Mathematics and in his "Wittgenstein's Remarks on the Significance of Gödel's Theorem".

67 Cf. PG pp. 389–390, where Wittgenstein investigates a trisection problem for the geometry in which *only* the Euclidean bisection construction is allowed, because the angle of the compass must remain fixed in all "constructions".

68 See WVC, pp. 34, 136, 144; PR XIII; PG p. 393.

69 PI 123. See footnote 87 below.

70 In her "Riddles and Anselm's Riddle", Chapter 10 of *The Realistic Spirit: Wittgenstein, Philosophy and the Mind.*

71 AL, pp. 185–186. Compare the reference to Busch's *Volksmärchen* in PG, p. 379, and the related PR, p. 179.

72 PR pp. 177–178.

73 RFM VI §13, VII §10; AL pp. 221–222.

74 PR, p. 170.

75 PG, p. 380. Cf. AL p. 185.

76 AL, p. 221.

77 PG, p. 377.

78 These sections should be compared with the *Tractatus*'s rejection of the Frege (and Russell) assertion-sign as "logically altogether meaningless [Bedeutungslos]" (TLP 4.442).

79 See FV §374.(380.), and compare footnote 18 above.

80 RFM I App. III §1.

81 RFM I App. III §2.

82 PI §23.

83 RFM I App. III §3.

84 Cf. RFM V §46.

85 RFM I App. III §4.

86 RFM VII §22.

87 PI §§123–125, whose meaning, in light of the other writings of Wittgenstein discussed above in Section II, is most complex:

> 123. A philosophical problem has the form: "I don't know my way about".
> 124. Philosophy may in no way interfere with the actual use of language; it can in the end only describe it.
> For it cannot give it any foundation either.
> It leaves everything as it is.
> It also leaves mathematics as it is, and no mathematical discovery can advance it. A "leading problem of mathematical logic" is for us a problem of mathematics like any other.
> 125. It is the business of philosophy, not to resolve a contradiction by means of a mathemaical or logical-mathematical discovery, but to make it possible for us to get a clear view of the state of mathematics that troubles us; the state of affairs *before* the contradiction is resolved.

88 FV. 380.(386.):

> Wo es bei Euklid heisst: das und das sei zu *konstruieren* und am Schluss "q.e.c.", könnte man auch setzen: es sei zu *beweisen*, dass das die Konstruktion dieser Figur sei und am Schluss schreiben "q.e.d.", also das Resultat auf die *Form* des bewiesenen *Satzes* bringen.

89 RFM VII §22.

90 R. L. Goodstein, "Critical Notice of *Remarks on the Foundations of Mathematics*", *Mind* 1957: pp. 549–553; see p. 551. Goodstein maintains (p. 551) that the Appendix on Gödel in RFM is "unimportant" and "throws no light on Gödel's work". But he also feels "faced with a mystery" at the date of composition of these remarks (1938), for, as he writes,

> . . . what Wittgenstein said on the subject in 1935 was far in advance of his standpoint three years later. For Wittgenstein with remarkable insight said in the early thirties that Gödel's results showed that the notion of a finite cardinal could not be expressed in an axiomatic system and that formal number variables must necessarily take values other than natural numbers; a view which, following Skolem's 1934 publication, of which Wittgenstein was unaware, is now generally accepted.

I believe Goodstein misread RFM (Compare Watson's 1938 paper, "Mathematics and Its Foundations", pp. 447–448, where a similar point about the notion of finite cardinal is made, and see footnote 2.)

91 Wittgenstein's letter to Schlick of 31.7.35 is printed in *Wittgenstein: Sein Leben in Bildern und Texten*, p. 260.

92 Letter to Schlick, 31.7.35.

93 Compare Gödel's remark in "On formally undecidable propositions of *Principia mathematica* and related systems I" (*Kurt Gödel: Collected Works Volume I*, eds. S. Feferman, J. W. Dawson, Jr., S. C. Kleene, G. H. Moore, R. M. Solovay, J. van Heijenoort (New York: Oxford University Press, 1986), p. 151:

> The purpose of carrying out the . . . proof with full precision in what follows is, among other things, to replace the [assumption that every provable formula is true in the interpretation considered] by a purely formal and much weaker one.

94 RFM I App. III §5.

95 RFM I App. III §5.

96 RFM I App. III §6.

97 RFM I App. III §6.

98 RFM I App. III §6.

99 RFM I §18. Wittgenstein is extending the point of Lewis Carroll, in his famous "What the Tortoise said to Achilles" (*Mind* New Series Vol. IV). See Russell's *Principles of Mathematics* (New York: W. W. Norton: 2nd ed. 1938), p. 35.

100 Compare footnote 26 and AL, p. 143 (a reworking of TLP 5.541):

> The attempt to build up a logic to cover all eventualities, e.g., Carnap's construction of a system of relations while leaving it open whether anything fits it so as to give it content, is an important absurdity. We must remember that if we feel the need of an instance of an n-termed relation we still have symbolism for n things not standing in relation. The need is for a *sample*, a paradigm, which is again *part of the language*, not part of the application.

101 RFM I App. III §7.

102 RFM I App. III §7.

[103] Something akin to this point has been emphasized by both Wright (*Wittgenstein on the Foundations of Mathematics* (Cambridge, MA: Harvard University, 1980)) and Kripke (*Wittgenstein on Rules and Private Language* (Cambridge, MA: Harvard University, 1982)), although neither of them applies their reading to Wittgenstein's discussions of Gödel. Both Wright and Kripke read Wittgenstein as, generally, a sceptic about rule-following necessity. Further discussion of their interpretations must be left for another place, but compare my "Wittgenstein on 2,2,2 . . .: On the Opening of *Remarks on the Foundations of Mathematics*".

[104] Compare Putnam's "Analyticity and Apriority: Beyond Wittgenstein and Quine" (in Putnam's *Realism and Reason*: Philosophical Papers Vol. III (Cambridge University Press, 1983)), ostensibly a critique of Wittgenstein, but perhaps better read as a critique of certain Wittgensteinians. See also Putnam's early discussions of the shift from Euclidean to Non-Euclidean geometry in his "Truth and Necessity in Mathematics" and "It Ain't Necessarily So", both reprinted in Putnam's *Mathematics, Matter and Method: Philosophical Papers Vol. I* (Cambridge University Press, 1975).

[105] Wittgenstein would view any attempt to give a precise definition of truth, such as Tarski's or Kripke's, as another piece of mathematics, and would treat the relevant proofs and definitions in the same way as he treats Gödel's incompleteness theorem. A full discussion of this point calls for another paper; but compare footnote 93.

[106] RFM I App. III §8.

[107] RFM I App. III §8.

[108] Wang ("To and From Philosophy – Discussions with Wittgenstein and Gödel", section 6.2) has raised a question about Wittgenstein's apparent equation in §8 of "true in Russell's system", "provable in Russell's system" and "proved in Russell's system", and ventures a guess that Wittgenstein's idea is that

> the sense of a proposition is determined by its proof (in the system), [so that] the provable or true 'propositions' of the system are just the 'proved' ones in the system, since their proofs are, by definition, in the system, which moreover, being our invention (creative), is transparent so that we know the proofs because we know all the properties of the system.
>
> . . . [Thus] one way to interpret Wittgenstein's idea is to take Russell's system to be made up of all its provable propositions and proofs. Since the proof determines for Wittgenstein the sense of the proved proposition, the proof is 'part' of the proposition so that the propositions are the *proved* propositions.

Wang's idea that for Wittgenstein "the sense of a proposition is determined by its proof" resonates with many passages Wittgenstein wrote, as does Wang's picture of logical truth (or a formal system) an in some sense potentially transparent to us because we have created or invented it. Nevertheless, I suggest that in the above quoted RFM I App. III §8 Wittgenstein is not simply restricting the application of "truth", "proof" and "provable" to the *Principia* formalism, and then equating them; rather, he is questioning the clarity of all three notions – and the notion of "proposition" itself – apart from our actual practices, i.e., apart from the question of whether we have (or have not) *accepted* Gödel's *proof*.

[109] This may be compared with Gödel's remark in his original paper "On formally undecidable propositions" (p. 151) that "the proposition that is undecidable *in the system PM* still was decided by metamathematical considerations."

In LFM, pp. 188–189 Wittgenstein seems to suggest that his own way of talking of changing the "sense of truth" of the Gödel sentence is misleading, and insists that

> the thing [in such cases] is to avoid the words "true" and "false" altogether, and to get clear that to say that *p* is true is simply to assert *p*; and to say that *p* is false is simply to deny *p* or to assert *-p*. It is not a question of whether *p* is "true in a different sense". It is a question of whether we assert *p*.

In this context Wittgenstein is discussing in what sense the law of contradiction may legitimately be said to be "true". We see evidence (as in RFM I App. III §6, discussed above) of the old Tractarian temptation to say that in logic (or mathematics) we do not utter "truths", but rather (empty) "tautologies" and "contradictions".

[110] RFM I Appendix III §14.

[111] Cf. RFM VII §22.

[112] RFM VII §21. I have interpolated in brackets from the original MS 124.

[113] RFM I App. III §§15–16.

[114] RFM I App. III §14 (quoted above, p. 406) uses the same notion of a "Triftiger Grund", as do RFM III §§86–87:

> The proof of consistency must give us reasons for a prediction; and that is its *practical purpose*. That does not mean that this proof is a proof from the physics of our technique of calculation – and so a proof from applied mathematics – but it does mean that that prediction is the application that first suggests itself to us, and the one for whose sake we have this proof at heart. The prediction is not: "No disorder will arise *in this way*" (for that would not be a prediction: it is the mathematical proposition) but: "no disorder will arise".
>
> I wanted to say: the consistency-proof can only set our minds at rest, if it is a cogent reason [Triftiger Grund] for this prediction.
>
> Where it is enough for me to get a proof that a contradiction or a trisection of the angle cannot be constructed in *this* way, the recursive proof achieves what is required of it. But if I had to fear that something somehow might at some time be interpreted as the construction of a contradiction, then no proof can take this indefinite fear from me.

[115] RFM III §43.

[116] This is vague, of course. That is, "self-reference" is a polymorphic notion. For example, in the sense in which, as Wittgenstein writes in RFM VII §21, "25 × 25 = 625" asserts something of itself, viz., that "the left-hand number is got by the multiplication of the numbers on the right", does the equation with no rational roots say "of itself" that it is not solvable – or that a certain angle it describes is not constructible?

[117] These remarks have been published and discussed by Wang in his *Reflections on Kurt Gödel*, p. 49.

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REFERENCES

Abbreviations of Wittgenstein's Works:

NB	*Notebooks 1914–16*
TLP	*Tractatus Logico-Philosophicus*
WVC	*Ludwig Wittgenstein and the Vienna Circle*
PR	*Philosophical Remarks*
DL	*Wittgenstein's Lectures, Cambridge 1930–32, from the notes of John King and Desmond Lee*
PG	*Philosophical Grammar*
AL	*Wittgenstein's Lectures, Cambridge 1932–35, from the notes of Alice Ambrose and Margaret MacDonald*
BB	*The Blue and Brown Books*
RFM	*Remarks on the Foundations of Mathematics*
LFM	*Wittgenstein's Lectures on the Foundations of Mathematics, Cambridge 1939*
PI	*Philosophical Investigations*
RPPI	*Remarks on the Philosophy of Psychology, Volume I*
RPPII	*Remarks on the Philosophy of Psychology, Volume II*
CV	*Culture and Value*
FV	The "Frühversion", or early version of *Philosophical Investigations*

All other references to unpublished materials use the system devised by von Wright in his "The Wittgenstein Papers" to number the manuscripts and typescripts from the Cornell Microfilm.

BIBLIOGRAPHY

Anderson, A. R.: 1958, 'Mathematics and the "Language-Game"', *Review of Metaphysics* **11**, 446–458.

Boyer, C. B.: 1968, *A History of Mathematics*, Princeton University Press, Princeton.

Benacerraf, P. and Putnam, H. (eds.): 1964, *Philosophy of Mathematics*, Prentice Hall, N.J., pp. 481–490.

Carnap, R.: 1937, *The Logical Syntax of Language*, Amethe Smeaton (trans.), Routledge & Kegan Paul Ltd., London.

Cavell, S.: 1969, *Must We Mean What We Say?*, Cambridge University Press.

Cavell, S.: 1979, *The Claim of Reason: Wittgenstein, Skepticism, Morality, and Tragedy*, Oxford University Press.

Cavell, S.: 1989, *This New Yet Unapproachable America: Lectures After Emerson After Wittgenstein*, Living Batch Press, Albuquerque, N.M.

Cavell, S.: 1990, *Conditions Handsome and Unhandsome, The Carus Lectures 1988*, University of Chicago.

Courant, R. and H. Robbins: 1978, *What Is Mathematics*? Oxford University Press.

Dawson, John W., Jr.: 1991, 'The Reception of Gödel's Incompleteness Theorems', in Thomas Drucker (ed.), *Perspectives on the History of Mathematical Logic*, Birkhäuser Boston, Inc., Boston, Mass., pp. 84–100.

Diamond, C.: 1991, *The Realistic Spirit: Wittgenstein, Philosophy and the Mind*, M.I.T. Press, Cambridge, Mass.

Dreben, B.: 1990, 'Quine', in R. B. Barrett and R. F. Gibson (eds.), *Perspectives on Quine*, Basil Blackwell, Cambridge, Mass., pp. 81–95.

Dreben, B. and J. Floyd: 1991, 'Tautology: How Not to Use A Word', *Synthese* **87**, 23–50.

Dudley, U.: 1987, *A Budget of Trisections*, Springer Verlag, New York.

Dummett, M.: 1959, 'Wittgenstein's Philosophy of Mathematics' in Dummett, M.: 1978, *Truth and Other Enigmas*, Harvard University Press, Cambridge Mass., pp. 166–185.

Floyd, J.: 1990, 'The Rule of the Mathematical: Wittgenstein's Later Discussions' (Harvard Ph.D. Dissertation).

Floyd, J.: 1991, 'Wittgenstein on 2,2,2 . . .: On the Opening of *Remarks on the Foundations of Mathematics*', *Synthese* **87** 1: 143–180.

Frege, G.: 1884, *The Foundations of Arithmetic*, first published as *Die Grundlagen der Arithmetik*, Verlag von Wilhelm Koebner, Breslau; published in English as *The Foundations of Arithmetic*, J. L. Austin (trans.), revised edition, Northwestern University Press, Evanston, IL, 1980.

Friedlander, E.: 1992, 'Expressions of Judgment', Harvard Ph.D. Thesis.

Gauss, C. F.: 1801, *Disquisitiones Arithmeticae*, A. A. Clarke (trans.), Yale University Press, 1966.

Gerrard, S.: 1991, 'Wittgenstein's Philosophies of Mathematics', *Synthese* **87**, 125–142.

Gödel, K.: 1931, 'On formally undecidable propositions of *Principia mathematica* and related systems I', reprinted in S. Feferman, J. W. Dawson, Jr., S. C. Kleene, G. H. Moore, R. M. Solovay and J. van Heijenoort (eds.): 1986, *Kurt Gödel: Collected Works Volume I*, Oxford University Press.

Goldfarb, W.: 1983, 'I Want You to Bring Me a Slab', *Synthese* **56**, 265–282.

Goldfarb, W.: 1992, 'Wittgenstein on Understanding', in P. A. French, T. E. Uehling, Jr. and H. K. Wettstein (eds.), *Midwest Studies in Philosophy XVII: The Wittgenstein Legacy*, pp. 109–122.

Goldfarb, W. and T. Ricketts: 1992, 'Carnap and the Philosophy of Mathematics', forthcoming in D. Bell and W. Vossenkuhl (eds.), *Science and Subjectivity*. Akademie Verlag, Berlin.

Goodstein, R. L.: 1957, 'Critical Notice of *Remarks on the Foundations of Mathematics*', *Mind*, 549–553.

Herstein, I. N.: 1975, *Topics in Algebra*, Xerox, 2nd edition, Lexington, Mass.

Hilmy, S. S.: 1987, *The Later Wittgenstein*, Basil Blackwell, Oxford.

Hintikka, J.: 1991, 'An Impatient Man and His Papers' *Synthese* **87** **2**, 183–202.

Hintikka, J.: 1993, 'The Original Sinn of Wittgenstein's Philosophy of Mathematics', in Klaus Puhl (ed.), *Wittgenstein's Philosophy of Mathematics* (Proceedings of the 15th International Wittgenstein Symposium), Hölder-Pichler-Tempsky, Vienna, pp. 24–51.

Hintikka, J. and M. B. Hintikka: 1986, *Investigating Wittgenstein*, Blackwell, Oxford.

Hobson, E. W.: 1953, *Squaring the Circle: A History of the Problem*, Chelsea Publishing Co., Boston.

Hodges, A.: 1983, *Alan Turing: The Enigma*, Simon & Schuster, New York.
Hylton, P.: 1990, *Russell, Idealism, and the Emergence of Analytic Philosophy*, Oxford University Press.
Klein, F.: 1895, *Famous Problems of Elementary Geometry*, W. W. Beman, D. E. Smith, R. C. Archibald (trans.), 2nd ed.
Kreisel, G.: 1958, 'Wittgenstein's Remarks on the Foundations of Mathematics', *British Journal for the Philosophy of Science* 9, 135–158.
Kreisel, G.: 1983, 'Einige Erläuterungen zu Wittgensteins Kummer mit Hilbert und Gödel', in P. Weingartner and J. Czermak (eds.), *Epistemology and Philosophy of Science*. Proceedings of the 7th International Wittgenstein Symposium, Hölder-Pichler-Tempsky, Vienna, pp. 295–303.
Kripke, S. (1982): *Wittgenstein on Rules and Private Language*, Harvard University Press, Cambridge, MA.
Lakatos, I.: 1977, *Proofs and Refutations, the Logic of Mathematical Discovery*, J. Worrall and E. Zahar (eds.), Cambridge University Press.
Mancosu, P.: 1991, 'On the Status of Proofs by contradiction in the Seventeenth Century', *Synthese* **88**, 15–41.
Mancosu, P.: 1992, 'Aristotelian Logic and Euclidean Mathematics: Seventeenth-Century Developments of the *Quaestio de Certitudine Mathematicarum*', *Studies in the History and Philosophy of Science* **23**(2), 241–265.
Marion, M.: 1992, 'Wittgenstein and the Dark Cellar of Platonism', Address to the XVth International Wittgenstein Symposium, Kirchberg am Wechsel, Austria.
Marion, M.: 'Wittgenstein and Finitism', forthcoming, *Synthese*.
Moore, G. E.: 1959, *Philosophical Papers*, Allen and Unwin, London.
Nedo and Ranchetti (eds.): 1983, *Wittgenstein: Sein Leben in Bildern und Texten*: Suhrkamp Verlag, Frankfurt am Main.
Putnam, H.: 1975, *Mathematics, Matter and Method: Philosophical Papers Volume I*, Cambridge University Press.
Putnam, H.: 1983, *Realism and Reason: Philosophical Papers Volume II*, Cambridge University Press.
Putnam, H.: 1991, 'Wittgenstein on Religious Belief', in *On Community*, Leroy S. Rouner (ed.), Notre Dame Press, pp. 56–75.
Russell, B.: 1938, *Principles of Mathematics*, 2nd ed., W. W. Norton Co., New York.
Shanker, S. G.: 1987, *Wittgenstein and the Turning Point in the Philosophy of Mathematics*, State University Press of New York, Albany.
Shanker, S. G.: 1988, 'Wittgenstein's Remarks on the Significance of Gödel's Theorem', in S. G. Shanker (ed.), *Gödel's Theorem in Focus*, Croom Helm, London, pp. 176–177.
von Wright, G. H.: 1983, *Wittgenstein*, University of Minnesota Press, Minneapolis.
Wang, H.: 1986, *Beyond Analytic Philosophy: Doing Justice to What We Know*, M.I.T. Press, Cambridge, Mass.
Wang, H.: 1987, *Reflections on Gödel*, M.I.T. Press, Cambridge, Mass.
Wang, H.: 1991a, 'Imagined Discussions with Gödel and with Wittgenstein', Address to the Kurt Gödel Society Meeting at Kirchberg am Wechsel, Austria.
Wang, H.: 1991b, 'To and From Philosophy – Discussions with Gödel and Wittgenstein', *Synthese* **88**(2), 229–277.
Wanzel, P.: 1837, 'Recherches sur les moyens de reconnaître si un Problème de géométrie peut se résoudre avec la règle et le compas,' *Journal de Mathématiques Pure et Appliquées* **2**, 366–372.

Watson, A.: 1938, 'Mathematics and its Foundations', *Mind* **47**, 440–451.
Wittgenstein, L.: 1921, *Logische-Philosophische Abhandlung*, final chapter, Ostwald's *Annalen der Naturphilosophie*; first published in English under the title *Tractatus Logico- Philosophicus*, 1922, C. K. Ogden (trans.), Routledge & Kegan Paul, Ltd., London; English translation reprinted with corrections 1933.
Wittgenstein, L.: 1958a, *Philosophical Investigations*, G. E. M. Anscombe and R. Rhees (eds.), G. E. M. Anscombe (trans.), second ed., Blackwell, Oxford.
Wittgenstein, L.: 1958b, *The Blue and Brown Books*, Blackwell, Oxford.
Wittgenstein, L.: 1961, *Notebooks 1914–16*, G. H. von Wright and G. E. M. Anscombe (eds.), G. E. M. Anscombe (trans.), Blackwell, Oxford.
Wittgenstein, L.: 1979, *Ludwig Wittgenstein and the Vienna Circle*, shorthand notes recorded by F. Waismann, B. McGuinness (ed.), J. Schulte and B. McGuiness (trans.), Blackwell, Oxford.
Wittgenstein, L.: 1974, *Philosophical Grammar*, R. Rhees (ed.), A. J. P. Kenny (trans.), Blackwell, Oxford.
Wittgenstein, L.: 1975, *Philosophical Remarks*, R. Rhees (ed.), R. Hargreaves and R. White (trans.), Blackwell, Oxford.
Wittgenstein, L.: 1978, *Remarks on the Foundations of Mathematics*, G. H. von Wright, R. Rhees, G. E. M. Anscombe (ed.), G. E. M. Anscombe (trans.) revised ed., Blackwell, Oxford.
Wittgenstein, L.: 1979, *Cambridge Lectures: 1932–1935*, from the Notes of Alice Ambrose and Margaret Macdonald, Alice Ambrose (ed.), Rowman and Littlefield, Totowa, N.J., reprinted by the University of Chicago Press.
Wittgenstein, L.: 1980a, *Remarks on the Philosophy of Psychology, Volumes I and II*, G. E. M. Anscombe and G. H. von Wright (eds.), G. E. M. Anscombe (trans.), University of Chicago Press.
Wittgenstein, L.: 1980b, *Culture and Value*, G. H. von Wright and H. Nyman (eds.), Peter Winch (trans.), University of Chicago Press.
Wittgenstein, L.: 1980c, *Wittgenstein's Lectures, Cambridge 1930–32, from the notes of John King and Desmond Lee*, ed. Desmond Lee, Blackwell, Oxford.
Wittgenstein, L.: 1984, *Werkausgabe in 8 Bänden*, Suhrkamp Verlag, Frankfurt am Main.
Wittgenstein, L.: 1989, *Wittgenstein's 1939 Lectures on the Foundations of Mathematics*, C. Diamond (ed.), The University of Chicago Press (Originally published by Cornell University Press, Ithaca, NY, 1976).
Wittgenstein, L.: The 'Frühversion', or early version of *Philosophical Investigations*, compiled by G. H. von Wright and Heikki Nyman (unpublished manuscript).
Wright C. (1980): *Wittgenstein on the Foundations of Mathematics*, Harvard University Press, Cambridge, MA.