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a philosophical study of plural logic

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SALVATORE FLORIO & ØYSTEIN LINNEBO

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A Philosophical Study of Plural Logic

SALVATORE FLORIO AND ØYSTEIN LINNEBO

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To Aneta and Laurel

Contents

Preface xi
Acknowledgments xiii

1. Introduction 1

I. PRIMITIVE PLURALS

2. Taking Plurals at Face Value 9
   2.1 Some prominent views of plural sentences 9
   2.2 Taking plurals at face value 14
   2.3 The language of plural logic 15
   2.4 The traditional theory of plural logic 19
   2.5 The philosophical significance of plural logic 20
   2.6 Applications of plural logic 24
   2.7 Our methodology 29

3. The Refutation of Singularism? 31
   3.1 Regimentation and singularism 31
   3.2 Substitution argument 34
   3.3 Incorrect existential consequences 36
   3.4 The paradox of plurality 38
   3.5 Plural Cantor: its significance 41
   3.6 Plural Cantor: its statement and proof 42
   3.7 Conclusion 45
   3.A Alternative formulations of Plural Cantor 47

II. COMPARISONS

4. Plurals and Set Theory 55
   4.1 A simple two-sorted set theory 55
   4.2 Plural logic and the simple set theory compared 57
   4.3 Plural logic vs. set theory: classifying the options 59
   4.4 Against the elimination of pluralities in favor of sets 60
   4.5 Against the elimination of sets in favor of pluralities 64
   4.6 The iterative conception of set 66
   4.7 Zermelo-Fraenkel set theory 68

4.8 Proper classes as pluralities 70
4.9 Are two applications of plural logic compatible? 72
4.A Defining the translations 73
4.B Defining the interpretation 74

5. Plurals and Mereology 76
5.1 Mereology 76
5.2 Can mereology represent the plural? 77
5.3 One-sorted plural logic 81
5.4 Classifying some ways to talk about the many 83
5.5 Mereological singularism in linguistic semantics 85
5.6 Assessment of singularism in linguistic semantics 88
5.7 The elimination of mereology in favor of plural logic 90
5.8 Keeping both plural logic and mereology 92
5.A Partial orders and principles of decomposition 96
5.B Some notions of sum 98
5.C Atomicity 101
5.D One- and two-sorted plural logic compared 102

6. Plurals and Second-Order Logic 104
6.1 Second-order logic 104
6.2 Plural logic and second-order logic compared 107
6.3 The elimination of pluralities in favor of concepts 109
6.4 The elimination of concepts in favor of pluralities 116
6.5 Conclusion 118

III. PLURALS AND SEMANTICS

7. The Semantics of Plurals 123
7.1 Regimentation vs. semantics 124
7.2 Set-based model theory 126
7.3 Plurality-based model theory 130
7.4 Criticisms of the set-based model theory 135
7.5 The semantics of plural predication 139
7.6 The problem of choice 144
7.7 Absolute generality as a constraint 146
7.8 Parity constraints 147
7.9 Conclusion 150

8. On the Innocence and Determinacy of Plural Quantification 151
8.1 Introduction 151
8.2 A plurality-based Henkin semantics 154
8.3 The legitimacy of ascending one order 155

8.4 Does ontological innocence ensure determinacy? 157
8.5 The semantic determinacy of plural quantification 160
8.6 The metaphysical determinacy of plural quantification 162
8.7 A generalized notion of ontological commitment 163
8.8 Applications reconsidered 168
8.A Henkin semantics 170
8.B Completeness of the Henkin semantics 172

9. Superplurals 174
9.1 Introduction 174
9.2 What superplural reference would be 174
9.3 Grades of superplural involvement 178
9.4 Possible examples from natural language 179
9.5 The possible examples scrutinized 182
9.6 The multigrade analysis 185
9.7 Covers 188
9.8 Mixed-level predications 193
9.9 Mixed-level terms, order, and repetition 195
9.10 Conclusion 198
9.A The notion of upwards closure 200

IV. THE LOGIC AND METAPHYSICS OF PLURALS

10. Plurals and Modals 205
10.1 Introduction 205
10.2 Why plural rigidity matters 207
10.3 Challenges to plural rigidity 208
10.4 An argument for the rigidity of sets 210
10.5 An argument for plural rigidity 216
10.6 Towards formal arguments for plural rigidity 219
10.7 The argument from uniform adjunction 221
10.8 The argument from partial rigidification 223
10.9 The argument from uniform traversability 224
10.10 Pluralities as extensionally definite 226
10.11 The status of plural comprehension 229
10.A Traversability and quasi-combinatorial reasoning 231
10.B Proofs 232

11. Absolute Generality and Singularization 240
11.1 Absolute generality 240
11.2 A challenge to absolute generality 241
11.3 A trilemma 244
11.4 Relativism and inexpressibility 246

11.5 Traditional absolutism and ascent 249
11.6 Ascent and inexpressibility 253
11.7 Lifting the veil of type distinctions 256
11.A The Ascent Theorem 262

12. Critical Plural Logic 268
12.1 Introduction 268
12.2 The extendability argument 269
12.3 Our liberal view of definitions 272
12.4 Why plural comprehension has to be restricted 276
12.5 The principles of critical plural logic 278
12.6 Extensions of critical plural logic 284
12.7 Critical plural logic and set theory 286
12.8 Generalized semantics without a universal plurality? 290
12.9 What we have learnt 293

References 297
Index 307

Preface

Plural logic has become a well-established subject, especially in philosophical logic. We want to explore its broader significance for philosophy, logic, and linguistics. What can plural logic do for us? Are the bold claims made on its behalf correct?

Different readers may want to follow different threads running through the book. Readers interested in the application of plural logic in philosophy will find Chapters 1, 2, and 8 especially relevant. We argue that plural logic has useful applications, though not all those it is commonly thought to have. Next, questions about the correct logic of plurals are discussed in Chapters 1, 2, 4, and 9–12, where we defend an unconventional view. We reject traditional plural logic in favor of a "critical" alternative. The most striking feature of this alternative is that there is no universal plurality. Chapters 1–3, 5, 7, and 9 discuss the significance of plural logic to linguistics. Advocates of plural logic often claim that linguistic semantics should avoid "singularist" prejudices and be formulated taking plurals at face value. We contest this claim.

A few words about the origin of the project may be appropriate. Both authors have for a number of years been interested in questions about the logic, meaning, and metaphysics of plurals. Many of the ideas in the book were first conceived during long runs along the River Thames in the period 2010–12. A first glimpse of the book project arose in connection with the course "Plurals in Semantics and Philosophical Logic" taught at ESSLLI 2012 in Opole, Poland.

There are a lot of people to thank. This book has benefited enormously from extensive comments given by Peter Fritz, Simon Hewitt, David Nicolas, Alex Oliver, Agustín Rayo, Sam Roberts, Timothy Smiley, Eric Snyder, Hans Robin Solberg, and Gabriel Uzquiano. For useful feedback and discussion, we are also indebted to Colin Caret, Aistė Čelkytė, Eyjólfur Emilsson, Vera Flocke, Olav Gjelsvik, Nicholas Jones, Jönne Kriener, Dan Marshall, Ian Rumfitt, Stewart Shapiro, Sean Walsh, Tim Williamson, the students in our course at ESSLLI, and the audiences of numerous talks where material from the book has been presented. Peter Momtchiloff has provided invaluable help as an editor.

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1
Introduction

English and other natural languages contain plural expressions, which allow us to talk about many objects simultaneously, for example:

(1.1) The students cooperate.

(1.2) The natural numbers are infinite.

How should such sentences be analyzed? In recent years, there has been a surge of interest in plural logic, a logical system that takes plurals at face value. When analyzing language, there is thus no need to eliminate the plural resources of English in favor of singular resources. Rather, the plural resources can be retained as primitive, not understood in terms of anything else.

Plural logic has emerged as a new tool of great potential significance in logic, philosophy, linguistics, and beyond. What is this new tool, and what is it good for? We wish to provide a more nuanced discussion than has been given so far.

Three questions run through our discussion. First:

> The legitimacy of primitive plurals
>
> Should the plural resources of English and other natural languages be taken at face value or be eliminated in favor of the singular?

Different considerations pull in different directions. On the one hand, there is the tremendous success of set theory, which shows how to represent many objects by means of a single complex object, namely their set. This is a powerful theory, which has proven to be of great theoretical value. Why bother with the many when we have a supremely successful theory of complex "ones"? On the other hand, there is a strong case for taking plurals at face value. English and many other natural languages allow us to talk about the many, apparently without any detour via complex "ones" such as

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sets. Why not utilize these expressive resources in our systematic theorizing? Moreover, attempts to eliminate the plural in favor of the singular appear to lead to paradoxes. We are all familiar with Russell's paradox of the set of all sets that are not elements of themselves. While this set leads to paradox, its many elements—considered as many, not one—do not. It thus appears to follow that talk about the many elements cannot be eliminated in favor of talk about their set. These considerations encourage the view encapsulated in Bertrand Russell's trenchant remark that "the many are only many, and are not also one" (Russell 1903, Section 74).

While we end up favoring a "pluralist" view, which takes plural resources at face value, this book tries to give the opposing "singularist" view a fair hearing. Our reasons for endorsing pluralism are somewhat unconventional. We reject many of the usual arguments against singularism and, in particular, argue that linguists are often entitled to their predominantly singularist approach. We place greater weight on a less familiar argument for pluralism, namely that primitive plurals are of great value for the explanation of sets and set theory.

Suppose we accept primitive plurals. This gives rise to our second overarching question.

> How primitive plurals relate to the singular
>
> What is the relation between the plural and the singular? We are particularly interested in the circumstances under which many objects correspond to a single, complex "one" and whether any such correspondence can shed light on the complex "ones".

Consider all the students at the nearest university. Presumably, they are very many. It is natural, however, to think that they also correspond to various single objects, such as a single group, or set, of students. The question thus arises what kinds of singularizing transformations there are and whether such transformations might be used to shed light on the resulting "ones". Following Georg Cantor and others, we find it illuminating to explain a set as an object that is somehow "constituted" by its many elements. This suggests a non-eliminative reduction of certain "ones" to the corresponding "many"; that is, we retain the "ones" as objects in good standing but seek an account of them in terms of the corresponding "many". It is important to notice that this non-eliminative reduction would proceed in the opposite direction of the singularists' proposed elimination of the plural in favor of the singular.

Thus, our proposal is not to eliminate the many but, on the contrary, to put them to use in explaining certain complex "ones".

As is well known, however, singularizing transformations are fraught with danger. If you know Cantor's theorem, you won't be surprised to learn that traditional plural logic enables us to prove that there are more pluralities of objects than single objects. (If you don't know the theorem, don't worry—it will be explained in due course.) This generalization of Cantor's theorem appears to show that it is impossible for every "many" to correspond to a unique "one". For there are more "manys" than there are "ones"! This result appears to limit severely what singularizing transformations can exist—and thus also to threaten the explanatory value that such transformations might have.

When examining the relation between the plural and the singular, we face conflicting logical and metaphysical pressures. On the one hand, the traditional and most intuitive plural logic severely restricts what singularizing transformations there can be. On the other hand, such transformations are intuitively plausible in their own right and (more importantly) promise to be of great theoretical value. How are we to negotiate these conflicting pressures? Following an approach recently defended by Timothy Williamson (2013, 2014), we reject a "logic first" orientation according to which we *first* choose a plural logic and *then* require every other theory to conform to this logic. Instead, we argue that the choice of a plural logic is entangled with commitments in metaphysics, semantics, and the philosophy of mathematics. We must therefore choose between various "package deals" that include not only a plural logic but also commitments far beyond.

Three such package deals will be examined. One is based on *generality relativism*, which rejects the possibility of quantification over absolutely everything. This surprising rejection of absolute generality has the benefit of reconciling traditional plural logic with the availability of singularizing transformations. When we apply such transformations, the range of our quantifiers expands in a way that enables us to avoid paradox. The other two package deals hold on to absolute generality but differ on how to address the conflicting pressures identified above. The more familiar version of absolute generality retains traditional plural logic and therefore limits what singularizing transformations there can be. We also explore a less familiar version of absolute generality which is more liberal concerning singularizing transformations and instead restores consistency by developing a more "critical" plural logic. In the final part of the book, we argue that the first two package deals suffer from analogous expressibility problems and should therefore be rejected in favor of the third package deal.

Finally, there is our third overarching question.

> The significance of primitive plurals
>
> What are the philosophical and (more broadly) scientific consequences of taking plurals at face value?

The very fact that primitive plural resources are available in thought and language is itself highly significant. Many recent writers on this subject, especially philosophers, have claimed that there are major further consequences as well. For example, we encounter claims to the effect that primitive plurals: (i) help us eschew problematic ontological commitments, thus greatly aiding metaphysics and the philosophy of mathematics; (ii) ensure the determinacy of higher-order quantification; and (iii) require us to reformulate the semantics of natural language using primitive plurals not only in the object language but also in the metalanguage. We argue that these claims are severely exaggerated. While primitive plurals are indeed legitimate and often very useful (especially for the explanation of sets), many other debates are unaffected by our choice of whether or not to accept primitive plurals. In particular, we argue that (i) the use of plural quantifiers incurs a form of commitment analogous to ontological commitment as traditionally understood; (ii) primitive plurals provide no additional assurance of the determinacy of higher-order quantification; and (iii) linguists are, for the most part, fully within their rights to continue in their old "singularizing" ways.

The title of our book might entice some readers who ponder the ancient question of whether reality is fundamentally a unity or a multiplicity. Parmenides famously views reality as a unity, asserting of it:

> Nor is it divisible, since it is all alike, and there is no more of it in one place than in another, to hinder it from holding together, nor less of it, but everything is full of what is. Wherefore it is wholly continuous; for what is, is in contact with what is. (Fragment 8, translated in Burnet 1920, 262)

Russell vehemently disagrees:

> Academic philosophers, ever since the time of Parmenides, have believed that the world is a unity. [...] The most fundamental of my intellectual beliefs is that this is rubbish. I think the universe is all spots and jumps,

> without unity, without continuity, without coherence or orderliness or any of the other properties that governesses love. (Russell 1949, 98)

We shall not take a stand on Parmenides's question about the fundamental nature of reality. But we fully endorse the ancient view that the relation between the many and the one is of profound philosophical importance. As Russell observes, there are many objects (whether fundamental or not). Our discussion—and book title—therefore start with the many. But as we shall see, there are some surprisingly hard puzzles and problems concerning the relation between the many and the one. Our analysis of these puzzles and problems leads us to propose an unconventional solution, namely to replace the traditional plural logic with a more "critical" alternative.

I

PRIMITIVE PLURALS

2

Taking Plurals at Face Value

2.1 Some prominent views of plural sentences

Many natural languages contain a grammatical distinction between singular and plural expressions.* Consider these examples:

(2.1) John is hunting.

(2.2) The gnus are gathering.

When available, plural expressions can play a critical role in thought and language. On the one hand, by grasping their meaning and deploying them, we are able to think and speak about many as well as about one. For instance, we are able to sort objects into collections and communicate important information about such collections. On the other hand, plural expressions have logical properties that generate valid patterns of reasoning through which we organize and extend our knowledge about collections of objects, for example:

(2.3) (a) The gnus are gathering.
(b) The gnus are the animals being hunted.
(c) The animals being hunted are gathering.

These patterns of reasoning go beyond those studied and systematized in traditional first-order logic, forming the subject matter of a new branch of logic known as *plural logic*.

Following the lead of George Boolos's seminal work, research on plural logic has flourished in recent decades.[1] It has also begun to influence

* Sections 2.1–2.5 draw from Florio and Linnebo 2018.

[1] See, e.g., Boolos 1984b, Boolos 1985a, Yi 1999, Oliver and Smiley 2001, Rayo 2002, Linnebo 2003, Yi 2005 and Yi 2006, McKay 2006, and Oliver and Smiley 2016.

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linguistic semantics, where plurals have received considerable attention since the 1980s.[2]

Although this focus on plurals is a relatively recent phenomenon, semantic questions concerning plurals were already entertained by the founders of modern logic.[3] Gottlob Frege, for instance, addressed the question of the proper logical analysis of sentences with a plural subject, such as:

(2.4) Socrates and Plato are philosophers.

He writes:

> Here we have two thoughts: Socrates is a philosopher and Plato is a philosopher, which are only strung together linguistically for the sake of convenience. Logically, Socrates and Plato is not to be conceived as the subject of which being a philosopher is predicated.
>
> (Letter to Russell of 28 July 1902, in Frege 1980, 140)

In effect, Frege proposes to eliminate plurals and analyze (2.4) as:

(2.5) Socrates is a philosopher and Plato is a philosopher.

However, he realizes that this strategy isn't always available. Sentences such as (2.6) and (2.7) are not amenable to the conjunctive analysis proposed for (2.4).

(2.6) Bunsen and Kirchhoff laid the foundations of spectral analysis.

(2.7) The Romans conquered Gaul.

Frege remarks:

> Here we must regard *Bunsen and Kirchhoff* as a whole. 'The Romans conquered Gaul' must be conceived in the same way. The Romans here are the Roman people, held together by custom, institutions, and laws.
>
> (Frege, ibidem)

[2] See Nicolas 2008 and Moltmann 2016 for applications of plural logic to linguistic semantics. For some research in linguistic semantics particularly relevant to our project, see Link 1983, Link 1998, Schein 1993, Schwarzschild 1996, Moltmann 1997, and Landman 2000.

[3] For historical details, see Oliver and Smiley 2016, Chapter 2.

Elsewhere he explains that, in (2.7), 'the Romans' must be regarded as a proper name whose logical function is to stand for an object (Frege 1980, 95).

While Frege understands "wholes" in broadly mereological terms—an approach to which we will return shortly—various alternatives, such as sets and groups, have been suggested in the subsequent literature. Let us briefly consider the appeal to sets.

The most famous advocate of this approach is Willard Van Orman Quine. One of the sentences he grapples with is known as the *Geach-Kaplan sentence*:[4]

(2.8) Some critics admire only one another.

According to Quine, by "invoking classes and membership, we can do justice to [the Geach-Kaplan sentence]" (Quine 1982, 293). He proposes what amounts to the following analysis, or, as he puts it, "regimentation":[5]

(2.9) There is a non-empty set such that every element of the set is a critic who admires someone and everyone she admires is an element of the set other than herself.

(2.10) $\exists s[\exists x\, x \in s \wedge \forall x(x \in s \rightarrow (x \text{ is a critic} \wedge \exists y\, x \text{ admires } y \wedge \forall y(x \text{ admires } y \rightarrow (y \in s \wedge x \neq y))))]$

Quine's use of set theory to eliminate plurals exposes him to an objection (see Boolos 1984b, 440). Consider the following sentence, which appears to be a set-theoretic truism:

(2.11) There are some sets such that any set is one of them if and only if that set is not an element of itself.

[4] As shown by Boolos, who credits David Kaplan, there is no correct paraphrase of this sentences comprising only singular vocabulary and the predicates occurring in it (Boolos 1984b, 432–3).

[5] We return in Sections 2.7 and 3.1 to a discussion of the important Quinean notion of regimentation, which differs from the familiar philosophical notion of analysis.

It is reasonable to demand that no proper regimentation of this sentence render it obviously false.[6] However, a strict application of Quine's set-theoretic paraphrase would turn (2.11) into (2.12), which is inconsistent:

(2.12) There is a non-empty set *x* such that, for every set *y*, *y* is an element of *x* if and only if *y* is a not an element of *y*.

Can Quine's strategy be salvaged by using a different paraphrase? Perhaps Quine is right that plural terms should be understood as "wholes" that are set-theoretic in character. But such "wholes" need not be sets; they can be collections of a more general sort. This provides a response to the objection presented above, since it licences this consistent regimentation of (2.11):

(2.13) There is a non-empty collection *c* such that, for every set *y*, *y* is a member of *c* if and only if *y* is not an element of itself.

However, this approach faces an immediate "revenge problem". How should we analyze the following variant of (2.11)?

(2.14) There are some collections such that any collection is one of them if and only if that collection is not a member of itself.

James Higginbotham aptly labels this style of objection *the paradox of plurality* (1998, 17). We provide a detailed discussion in Section 3.4.

In linguistics, an influential analysis of plurals is that of Godehard Link, who invokes mereological sums. Central to his analysis is a special mereological relation ($\leq$), corresponding to the notion of *individual parthood*. This notion is not to be confused with that of material parthood. For example, in the individual sense of the mereological vocabulary, Annie is an atomic part of the mereological sum of Annie and Bonnie. Here Annie is an atom, namely an individual with no other individual as part. In the material sense, by contrast, Annie is obviously not an atomic part of the sum of Annie and Bonnie, as she has proper material parts.

Link's proposal is to use mereology in this individual sense and analyze a plurality in terms of the mereological sum of its members. For example, the plurality of Annie and Bonnie would be analyzed in terms of the mereological sum of the two girls. In this setting, the relation of 'being one of' is

[6] According to the view defended in Linnebo 2010, a natural reading of (2.11) is false, but only for the *non-obvious* reason that every plurality must be extensionally definite, or properly circumscribed, which contrasts with the extensional indefiniteness of the notion of a self-identical set. This approach will be explored in Chapter 12.

best analyzed as 'being an atomic part of' ($\leq_{\mathrm{At}}$). Let '+' stand for the binary operation of mereological sum in the individual sense. And let $\sigma x.\varphi(x)$ be the mereological sum, again in the individual sense, of the objects satisfying the formula $\varphi(x)$.[7] Then some of the plural sentences we have encountered may be analyzed as follows:[8]

(2.15) Bunsen and Kirchhoff laid the foundations of spectral analysis.

(2.16) $F(b + k)$

(2.17) The Romans conquered Gaul.

(2.18) $C(\sigma x.R(x), g)$

(2.19) There are some sets such that any set is one of them if and only if that set is not an element of itself.

(2.20) $\exists x\,[\forall y(y \leq_{\mathrm{At}} x \rightarrow \mathrm{Set}(y)) \wedge \forall y(\mathrm{Set}(y) \rightarrow (y \leq_{\mathrm{At}} x \leftrightarrow y \notin y))]$

Finally, let us mention a singularist strategy based on a neo-Davidsonian analysis of predication in terms of events (broadly understood to include states).[9] This strategy eliminates a plural subject by reducing it either to a collection serving as agent of the underlying event or to the single co-agents of that event, where a co-agent is any object that participates in the event as a subject. Here is how the second version of the strategy may be applied to one of Frege's examples.

(2.15) Bunsen and Kirchhoff laid the foundations of spectral analysis.

(2.21) There is an event *e* of laying the foundations of spectral analysis such that Bunsen is a co-agent of *e*, Kirchhoff is a co-agent of *e*, and there is no other co-agent of *e*.

Are any of these singularist analyses of plurals successful? This question is discussed in Chapter 3, which provides a detailed assessment of the prospects for singularism. Whether singularism is a viable option, we argue, depends on some hard theoretical questions concerning absolute generality and the correct plural logic. Now, we would like to consider an altogether different approach to plurals.

[7] If desired, the notion of sum can be defined in terms of the parthood relation by exploiting the fact that a sum is the minimal object whose parts include the things to be summed.

[8] For more details and applications of the mereological framework, see Link 1983, Link 1998, Moltmann 1997, Champollion and Krifka 2016, and Champollion 2017. We explore the relation between plurals and mereology in Chapter 5.

[9] See, e.g., Higginbotham and Schein 1989, and, for more recent implementations, Landman 2000 (especially Lecture Six) and Champollion 2017 (Chapter 2).

2.2 Taking plurals at face value

Boolos rejects all the singularist strategies, favoring instead an approach that takes plurals at face value. Thus he completely rejects Quine's attempt to analyze plural discourse in terms of sets. He writes:

> Abandon, if one ever had it, the idea that use of plural forms must always be understood to commit one to the existence of sets [. . .] of those things to which the corresponding singular forms apply.
>
> There are, of course, quite a lot of Cheerios in that bowl, well over two hundred of them. But is there, in addition to the Cheerios, also a set of them all? [. . .]
>
> It is haywire to think that when you have some Cheerios, you are eating a set [. . .]. [I]t doesn't follow just from the fact that there are some Cheerios in the bowl that, as some who theorize about the semantics of plurals would have it, there is also a set of them all. (Boolos 1984b, 448–9)

In fact, Boolos's rejection of singularism has a distinguished pedigree featuring, most prominently, Russell (1903).[10] Russell distinguished between a *class as one* and a *class as many*. A class as one is a single object that may have a multiplicity of members. Objects of this kind are the subject matter of traditional first-order set or class theory. By contrast, a class as many is a multiplicity of objects *as such*: there need not be a single entity that represents, collects, or goes proxy for the objects that make up the multiplicity. Russell emphasized the usefulness of this second way of thinking about multiplicities. More recently, Max Black (1971) and Peter Simons (1982, 1997) have advocated a treatment of plurals in the spirit of classes as many.[11]

What is the broader significance of Boolos's attack on singularist analyses and of Russell's earlier pluralist approach based on the notion of classes as many? At the heart of their remarks is the simple idea that plurals should be taken at face value. That is, we should allow certain forms of plural discourse in our regimentation. Frege, Quine, and others were simply wrong to think that plurals should be paraphrased away. Rather, plurals deserve to be understood in their own terms by allowing the use of plural expressions in

[10] See Klement 2014 for a recent discussion of Russell's view.
[11] Again, see Oliver and Smiley 2016, Chapter 2, for more historical details.

our regimenting language. This is the key idea behind plural logic. We outline some basic aspects of plural logic in the next two sections. First, we introduce a formal language for plural logic. Then, we provide a basic deductive system that characterizes correct reasoning in this language. The semantics for this system is discussed in Part III of this book. Some of the most interesting and philosophically significant questions concerning plural logic arise, as we will see, in connection with its semantics.

Let us be clear about what is at stake in the debate about whether plurals should be taken at face value. It is one thing to observe that primitive plural resources are found in many natural languages and quite another to accept these expressive resources as legitimate, or even indispensable, for scientific purposes. While no one disputes the former, the latter is controversial, as the views of Frege and Quinean on plurals illustrate. Speaking for ourselves, we grant that there is a presumption in favor of taking resources available in natural language to be scientifically legitimate. But there may be exceptions to this general rule. Quine thought that metaphysical modality provided an example. Another possible example is linguistic tense, which appears to presuppose a standard of absolute simultaneity, in conflict with the special theory of relativity. How, then, can we bridge this gap between availability in natural language and scientific legitimacy or even indispensability? Some arguments purporting to bridge the gap will be discussed in Chapter 3 and in Part II. Although we will find many of these arguments to be weak, we will develop and defend one argument having to do with the role of plurals in the explanation of set theory.

2.3 The language of plural logic

We now describe a language that may be used to regiment a wide range of natural language uses of plurals. It captures Boolos's and Russell's suggestion and enables us to represent many valid patterns of reasoning that essentially involve plural expressions. This language is associated with what is known in the philosophical literature as PFO+, which is short for *plural first-order logic plus plural predicates*. In one variant or another, it is the most common regimenting language for plurals in philosophical logic.[12]

We start with the the standard language of first-order logic and expand it by making the following additions.

[12] We adopt the notation for variables used in Rayo 2002 and Linnebo 2003. An ancestor of this notation is found in Burgess and Rosen 1997. Other authors represent plural variables by

A. Plural terms, comprising plural variables ($vv, xx, yy, \ldots$, and variously indexed variants thereof) and plural constants ($aa, bb, \ldots$, and variants thereof), roughly corresponding to the natural language pronoun 'they' and to plural proper names, respectively.
B. Quantifiers that bind plural variables ($\forall vv, \exists xx, \ldots$).
C. A binary predicate $\prec$ for plural membership, corresponding to the natural language 'is one of' or 'is among'. This predicate is treated as logical.
D. Symbols for collective plural predicates with numerical superscripts representing the predicate's arity ($P^1, P^2, \ldots, Q^1, \ldots$, and variously indexed variants thereof). Examples of collective plural predicates are '... cooperate', '... gather', '... surround ...', '... outnumber ...'. For economy, we leave the arity unmarked.

Let $\mathcal{L}_{\text{PFO+}}$ be the language just introduced. The fragment of this language containing items A-C, that is, $\mathcal{L}_{\text{PFO+}}$ minus plural predicates, is the language of the subsystem of PFO+ known as PFO. The following chart summarizes which linguistic items are added to the standard language of first-order logic to obtain PFO+.

type of expression	natural language equivalent	symbolization
plural variables	they$_1$, they$_2$, ...	$vv, vv_0, \ldots, xx, \ldots$
plural constants	the Hebrides, the Channel Islands[13]	$aa, bb, \ldots, aa_1, \ldots$
plural quantifiers	there are some (things)	$\exists vv, \exists xx, \ldots$
	whenever there are some (things)	$\forall vv, \forall xx, \ldots$
plural membership	is one of, is among	$\prec$
collective plural predicates	cooperate, gather, surround, outnumber	$C(xx), G(vv), S(xx, y), O(xx, yy)$

The recursive clauses defining a well-formed formula are the obvious ones. However, some clarifications about the language are in order.

means of different typographical conventions: boldface letters (Oliver and Smiley), capitalized letters (McKay), or singular variables pluralized with an 's' (Yi).

[13] These purported examples of plural terms are controversial; for an argument that they are best treated as semantically singular, see Rumfitt 2005, 88. Additional examples can be found in Oliver and Smiley 2016, 78–80.

First, our language has two types of variable: singular and plural. It is also possible to use plural variables only and regard the singular as a limiting case of the plural. (See Section 5.3 for discussion.)

Second, one may require a rigid distinction between the types of argument place of predicates. An argument place that is open to a singular argument could be reserved exclusively for such arguments. A similar restriction could be imposed on argument places open to plural arguments. Would this rigid distinction between singular and plural argument places reflect a feature of natural language? Different natural language predicates suggest different answers. Some predicates are flexible, combining felicitously with both singular and plural terms. Examples include 'own a house', 'lifted a boat', or, as in Frege's example, 'laid the foundations of spectral analysis'. (Of course, the conjugations of the verbs will have to be adjusted.) Other predicates appear to lack this flexibility, combining felicitously *only* with plural terms, as in 'cooperate with one another' and 'are two in number'. There is an interesting linguistic question as to the source of these felicity judgments: are they of syntactic, semantic, or pragmatic origin? We don't wish to take a stand on these matters. For our purposes, we can leave this question open, noting that the two kinds of argument place—apparently flexible and apparently inflexible—suggest different regimentation strategies, namely to admit flexible plural predicates, or not.[14]

Third, collective plural predicates are contrasted with distributive ones, such as 'are students', 'visited Rome', 'are prime'. Roughly speaking, these are predicates that apply to some things if and only if they apply to each of those things. How best to make this precise will depend on one's stand on the issue of flexible plural predicates mentioned just above. A flexible plural predicate P is distributive just in case the following equivalence holds:

$$P(xx) \leftrightarrow \forall x(x \prec xx \rightarrow P(x))$$

A slight modification is needed for inflexible plural predicates. Let P^s be the singular analogue of P. Then an inflexible plural predicate P is distributive just in case the following equivalence holds:

[14] The possibility of flexible plural predicates raises deep and interesting questions. In the philosophical and logical tradition, it is widely assumed that if an expression can be replaced by another expression *salva congruitate* in *one* context, then it can be so replaced in *all* contexts. This assumption of "strict typing" is true of the language of first-order logic, as well as of standard presentations of second-order logic. However, the assumption fails if some, but not all, plural predicates are flexible.

$$P(xx) \leftrightarrow \forall x(x \prec xx \rightarrow P^s(x))$$

Finally, if a plural predicate has no singular analogue (as is arguably the case for 'cooperate with one another' and 'are two in number'), then it is collective by default.[15]

Owing to these definitions, distributive plural predicates can be obtained by paraphrase from their corresponding singular forms. Such predicates can therefore be omitted from PFO+ without loss of expressibility—although admittedly with some violence to style.

Other useful notions can be obtained by paraphrase. One is the many-many relation of plural inclusion, symbolized as '$\preccurlyeq$' and defined thus:

$$xx \preccurlyeq yy \leftrightarrow_{\text{def}} \forall z(z \prec xx \rightarrow z \prec yy)$$

This relation is expressed by 'are among', as used in 'Annie and Bonnie are among the students'. Then, according to the definition, Annie and Bonnie are among the students just in case anything that is one of Annie and Bonnie is one of the students. Another notion is plural identity (symbolized as '$\approx$'), which can be defined as mutual plural inclusion.[16] In symbols:

$$xx \approx yy \leftrightarrow_{\text{def}} (xx \preccurlyeq yy \wedge yy \preccurlyeq xx)$$

That is, two pluralities are identical just in case they are coextensive.

To illustrate the use of PFO+, let us provide some examples of regimentation.

(2.22) Some students cooperated.

(2.23) $\exists xx\,(\forall y(y \prec xx \rightarrow S(y)) \wedge C(xx))$

(2.24) Bunsen and Kirchhoff laid the foundations of spectral analysis.

(2.25) $\exists xx\,(\forall y(y \prec xx \leftrightarrow (y = b \vee y = k)) \wedge L(xx))$

[15] What is the status of these equivalences? If PFO+ is to capture entailment relations in natural language, we must regard them as analytic (or near enough). This is because, for example, 'Annie and Bonnie visited Rome' entails 'Annie visited Rome'. Notice that our definition of distributivity takes the form of (analytic) equivalences. Some authors (e.g. McKay 2006, 6) tie distributivity solely to the left-to-right implication. For discussion and references, see Oliver and Smiley 2013, 114–15. For an overview of linguistic treatments of distributivity, see among others Winter and Scha 2015 and Champollion forthcoming.

[16] Of course, if flexible predicates are allowed, then plural identity can arguably be expressed by the ordinary identity predicate '='.

(2.26) Some critics admire only one another.

(2.27) $\exists xx\,(\forall x(x \prec xx \rightarrow C(x)) \wedge$
$$\forall x \forall y[(x \prec xx \wedge A(x, y)) \rightarrow (y \prec xx \wedge x \neq y)])$$

We now turn to the basic proof-theoretic aspects of plural logic.

2.4 The traditional theory of plural logic

The formal system PFO+ comes equipped with logical axioms and rules of inference aimed at capturing correct reasoning in the fragment of natural language that is being regimented. The axioms and rules associated with the logical vocabulary of ordinary first-order logic are the usual ones. For example, one could rely on introduction and elimination rules for each logical expression. The plural quantifiers are governed by axioms or rules analogous to those governing the first-order quantifiers.

Plural logic is often taken to include some further, very intuitive axioms. First, every plurality is non-empty:

(Non-empty) $\forall xx \exists y\, y \prec xx$

Then, there is an axiom scheme of indiscernibility stating that coextensive pluralities satisfy the same formulas:

(Indisc) $\forall xx \forall yy[xx \approx yy \rightarrow (\varphi(xx) \leftrightarrow \varphi(yy))]$

We need to make some remarks. First, the formula φ may contain parameters. So, strictly speaking, we have the universal closure of each instance of the displayed axiom scheme. Henceforth, we assume this reading for similar axiom schemes, including the one below, and for axioms with free variables in general. Second, as customary, we write $\varphi(xx)$ for the result of replacing all free occurrences of some designated plural variable vv with 'xx' whenever 'xx' is substitutable for vv in φ (see for example Enderton 2001, 113).) Third, (Indisc) is a plural analogue of Leibniz's law of the indiscernibility of identicals, and as such, the scheme needs to be restricted to formulas $\varphi(xx)$ that don't set up intensional contexts.

Finally, there is the unrestricted axiom scheme of *plural comprehension*, an intuitive principle that provides information about what pluralities there are. For any formula $\varphi(x)$ containing 'x' but not 'xx' free, we have an axiom stating that if $\varphi(x)$ is satisfied by at least one thing, then there are the things each of which satisfies $\varphi(x)$:

(P-Comp) $\exists x\varphi(x) \rightarrow \exists xx\forall x(x \prec xx \leftrightarrow \varphi(x))$

We refer to an axiomatization of plural logic based on the principles just described as *traditional plural logic*. This is to emphasize its prominence in the literature.

Traditional plural logic can, of course, be challenged. We will be particularly concerned with unrestricted plural comprehension. A challenge to this axiom scheme will be examined in Chapters 11 and 12. To talk about some things, we presumably need to circumscribe the things in question. Perhaps this circumscription isn't a trivial matter. That is, perhaps some conditions $\varphi(x)$ fail to circumscribe some things. For example, the trivial condition '$x = x$' might fail to do so because there is no properly circumscribed lot of "all objects whatsoever". We will eventually take this kind of challenge seriously and develop an alternative, and slightly weaker, "critical" plural logic. However, for the time being we will work with traditional plural logic, which includes the unrestricted plural comprehension scheme.

2.5 The philosophical significance of plural logic

The significance of plural logic is not only linguistic: it is not exhausted by its helpfulness in capturing natural language reasoning involving plural expressions. Plural logic is *philosophically* significant in that it has a claim to provide a suitable framework in which various philosophical projects can be successfully developed. This philosophical significance largely depends on two features that plural logic has been thought to possess: first, plural logic is in some sense "pure logic"; second, it provides greater expressive power than first-order logic. These two alleged features are at the core of the common picture of plural logic and explain why it has become an important component of the philosopher's toolkit. In this section, we flesh out this picture and describe how it sustains the main philosophical applications of plural logic.

Many aspects of this common picture of plural logic will be challenged throughout the book. Although this calls into question some popular applications of plural logic, we also develop some new applications; in particular, we show how plural logic can be used to shed light on set theory.

The first alleged feature of plural logic concerns its status as "pure logic". Surveying and assessing the debate about what counts as pure logic would take us too far afield. In the present context, we find it more fruitful to regard logicality, not as an all-or-nothing feature of a system, but as a cluster of conditions that are of independent philosophical interest. There are at least three such conditions that might underwrite the philosophical significance of plural logic: *topic-neutrality*, *formality*, and *epistemic primacy*. Let us discuss each in turn.

Topic-neutrality is based on a simple, intuitive idea: logical principles should be applicable to reasoning about any subject matter. By contrast, other principles are only applicable to particular domains. The laws of physics, for instance, concern the physical world and do not apply when reasoning about natural numbers or other abstract entities. Plural logic seems to satisfy this intuitive notion of topic-neutrality: the validity of the principles of plural logic does not appear confined to specific domains. As partial evidence for the topic-neutrality of plural logic, one may point out that, when available, pluralization as a morphological transformation does not depend in any systematic way on the kind of objects one speaks about. For example, both concrete and abstract nouns exhibit plural forms. The same goes for many other categorial distinctions.[17]

Another mark of logicality is formality. Logical principles are often thought to hold in virtue of their form, not their content. There are different ways of articulating the notion of formality, some of which are tightly connected to the notion of topic-neutrality just discussed (see MacFarlane 2000). We focus on two conditions that tend to be associated with formality. One is that formal principles are *ontologically innocent*: they do not commit us to the existence of any objects.[18] Another is that formal principles *do not discriminate between objects*: they cannot single out particular objects or classes thereof.

[17] On a closer look, we must distinguish between the weaker claim that *some* system of plural logic has topic-neutrality and the stronger claim that plural logic *as formulated above* has this neutrality. The latter may be challenged while retaining the former, as noted in footnote 6 and further explored in Chapter 12. We have in mind the view defended by Yablo (2006) and Linnebo (2010), according to which every plurality is extensionally definite, or circumscribed, in a way that the entire universe is not. This means that the plural comprehension scheme must be restricted when the domain of discourse is the entire universe (e.g. the formula '$x = x$' does not define a plurality).

[18] Of course, the choice of a non-free logic requires the existence of one object.

Is plural logic ontologically innocent? In particular, are plural quantifiers ontologically innocent? The usual answer to these questions is affirmative. Plural quantifiers do not incur ontological commitments beyond those incurred by the first-order quantifiers. Plural logic indeed originated as an ontologically innocent alternative to second-order logic. This view is sustained by a particular semantics for plural logic—due to Boolos (1985a)—which differs from the set-based semantics ordinarily employed for logics of first and second order. To see how, let us briefly sketch Boolos's semantics.

The key feature of this semantics is that it adopts plural resources in the metatheory and uses them to represent the semantic values of the plural terms of the object language. On this semantics—which many philosophers now regard as the canonical one—the difference between singular and plural terms is explained, not on the basis of *what* these terms signify, but on the basis of *how* they signify. A plural variable is not interpreted as a set (or set-like entity) of objects in the first-order domain. Instead, it is interpreted directly as *many objects* in this domain, without the mediation of a set (or set-like entity). In other words, plural variables do not range over a *special domain* but range in a *special, plural way* over the usual, first-order domain. Since the range of plural variables is the first-order domain, the truth of sentences involving plural quantifiers does not seem to make ontological demands that exceed those made by sentences involving first-order quantifiers. In this sense, plural logic is said to be ontologically innocent.[19]

As noted above, there is another condition associated with formality: formal principles must not discriminate between objects. The standard way of making this condition precise is to claim that logical principles are those that remain true no matter how the non-logical expressions of the language are reinterpreted. This presupposes a distinction between logical and non-logical expressions of the language, which is typically captured by defining logical notions in terms of isomorphism invariance and then characterizing as logical the expressions that are suitably related to logical notions.[20] Alfred Tarski (1986) observed that isomorphism invariance captures the standard

[19] Boolos's semantics has been widely used in philosophical logic. See, among others, Yi 1999, Yi 2002, Yi 2005, and Yi 2006; Hossack 2000; Oliver and Smiley 2001 and Oliver and Smiley 2016; Rayo 2002; McKay 2006. Authors who use this semantics tend to emphasize the ontological innocence of the resulting logic.

[20] See Tarski 1986, Sher 1991, and McGee 1996. Denoting a logical notion has been claimed to be necessary but not sufficient for an expression to be logical. An additional semantic connection would be required (as argued, for instance, by McCarthy 1981 and McGee 1996; but see also Sagi 2015 for a critical evaluation of these arguments).

logical notions expressible in higher-order logic.[21] Thus higher-order logic should count as formal according to this way of explicating the notion of formality. It is natural to think that analogous arguments ought to apply to plural logic, delivering the result that plural quantification and plural membership are logical notions, and that plural logic too should count as formal.

The final mark of logicality, we recall, is epistemic. The thought is that logical notions and principles permit a special kind of *epistemic primacy*.[22] Logical notions can be grasped without relying on non-logical notions. Likewise, logical truths, if knowable, can be known independently of non-logical truths. Do the principles of plural logic enjoy this kind of epistemic primacy? Since some of these principles are counterparts of principles of first-order logic (for example, the introduction and elimination rules for the quantifiers), it is plausible to assume that they enjoy the same epistemic status as their first-order counterparts. However, plural logic encompasses distinctive principles—chiefly plural comprehension—and the question is whether *they* are subject to epistemic primacy. For the moment, let us simply record the fact that many philosophers find plural comprehension to be obviously true. For example, Boolos writes that every instance of comprehension "expresses a *logical* truth if any sentence of English does" (Boolos 1985b, 342). Similarly, Keith Hossack finds plural comprehension to be a "harmless *a priori* truth" and, together with the other axioms of plural logic, regards it as a genuine logical truth (Hossack 2000, 422).

If logicality is the first key feature of the common picture of plural logic, the second is expressive power. Because of its metalogical properties, first-order logic has well-known expressive limitations. In particular, important mathematical theories formulated in first-order terms are subject to non-standard interpretations. For example, first-order arithmetic has uncountable models, while first-order analysis and set theory have countable ones. So first-order logic badly fails to express the intended models of such theories. By contrast, plural logic is usually ascribed metalogical properties that lead to greater expressive power. Indeed, it is often held that, when formulated with the help of plural quantification, arithmetic, analysis, and set theory avoid the non-standard interpretations just mentioned.[23] The resulting view, which we dispute in Chapter 8, is that plural logic does better than first-order logic in securing a gain in expressive power.

[21] See also Lindenbaum and Tarski 1935.

[22] See, for example, how Frege frames his logicist project in Frege 1879, Frege 1884, and Frege 1893/1903.

[23] See footnote 3 on p. 152 for references.

To sum up: on the common picture, plural logic has two key features, logicality and expressive power. As noted above, instead of thinking of logicality as an all-or-nothing matter, we find it more fruitful to regard it as a cluster of conditions. We isolated three such conditions: topic-neutrality, formality, and epistemic primacy. Under formality, we distinguished two further conditions: formal principles are ontologically innocent, and they cannot single out particular objects or classes of objects.

2.6 Applications of plural logic

The philosophical significance of plural logic lies in its promise to provide an essential tool for various philosophical projects. An obvious such project is to provide an account of plurals in thought and language. There are less obvious uses of plural logic as well. We now wish to describe some particularly important applications to the philosophy of mathematics, metaphysics, and semantics. As will become clear, these applications rely on various aspects of the common picture of plural logic discussed above. Some of these aspects will be challenged in the course of the book, especially the ontological innocence and expressive power of plural logic (see Chapter 8) and its epistemic primacy (see Chapter 12).

There is a well-known technical result that sheds lights on many of these applications. As first shown by Boolos (1984b), monadic second-order logic can be interpreted in PFO. (Monadic second-order logic is the fragment of second-order logic that allows quantification into predicate position only when the predicate is monadic.) The converse is true as well: PFO can be interpreted in monadic second-order logic.[24] From a syntactic point of view, the two theories are therefore equivalent. However, this mutual interpretability by no means guarantees that the two systems share certain philosophically important features and, hence, that they are equivalent in their potential for philosophical applications. Since second-order logic has faced a number of criticisms that are usually thought to be avoided by plural logic, one might hope to be able to replace at least some uses of monadic second-order logic with corresponding uses of plural logic.[25]

[24] See Chapter 6 for a detailed discussion of the result.

[25] Second-order logic has been criticized on various grounds, e.g. for involving an illegitimate form of quantification, for being ontological committal, and for being too entangled with mathematics to count as pure logic (see Linnebo 2011 for a survey of the standard objections to second-order logic).

As developed in the work of Frege and his followers, logicism is the thesis that a significant portion of mathematics is analytic in the sense of being derivable from general logical laws and definitions. Second-order logic provides the standard framework for the development of logicism. Thus the success of Fregean logicism depends crucially on the logicality of second-order logic. Which features of logicality matter here? When discussing the philosophical significance of logicism, Frege and more recent logicists have tended to emphasize topic-neutrality, ontological innocence, and epistemic primacy. Since these are features that plural logic is alleged to have, this logic promises be of immediate relevance to the logicist project. One concern, which we discuss in Section 12.5, is that plural logic doesn't actually enjoy epistemic primacy but on the contrary carries non-trivial set-theoretic content.

There is a more clear-cut worry, however. The resources needed for the standard implementation of the logicist project exceed those of monadic second-order logic, and therefore those of plural logic. For instance, the statement of one of the main stepping stones of logicism, Hume's Principle, requires quantification over dyadic relations. The principle asserts that, for any two monadic second-order entities F and G, the number associated with F is identical with the number associated with G if and only if there is dyadic relation witnessing the equinumerosity of F and G.

Plural logic may still have an important role to play in logicism. First, there are alternative implementations of logicism that rely on plural logic coupled with a thin understanding of relations (see Boccuni 2013). Second, one might be able to capture quantification over relations by supplementing monadic second-order logic with ordered pairs obtained by first-order abstraction principles (see Shapiro and Weir 2000, Tennant 2007), by embracing extensions of plural logic like the one devised in Hewitt 2012a, or by regarding equinumerosity (or some kindred notion) as primitive (Antonelli 2010). Moreover, even if plural logic cannot sustain the full logicist project, it could still serve a more modest form of logicism, such as Boolos's *sublogicism*. As Boolos describes it, sublogicism is "the claim that there are (many) interesting examples of mathematical truths that can be reduced (in the appropriate sense) to logic" (Boolos 1985b, 332). His case for sublogicism relies on plural logic. It essentially involves a plural interpretation of Frege's definition of the ancestral of a relation (see Boolos 1985b).

Another important philosophical application of plural logic, underpinned by its alleged ontological innocence, concerns various eliminative projects in metaphysics. For example, plural logic has been used to eliminate reference to certain kinds of complex objects.[26] Instead of quantifying over tables, say, one may quantify plurally over mereological atoms of some appropriate kind, namely those "arranged tablewise". A sentence involving tables, such as 'some table is in the room', can thus be rendered as a sentence involving pluralities of mereological atoms, namely 'some mereological atoms arranged tablewise are in the room'. Since the predicate 'arranged tablewise' is a collective predicate, this eliminative strategy can be carried out in PFO+. An interesting question raised by this strategy is how *plural* quantification over complex objects should be treated (Uzquiano 2004b). Since we have already "used up" ordinary plural quantification to paraphrase singular talk of complex objects, eliminating plural talk of such objects requires additional resources. We would need a form of quantification that stands to plural quantification as plural quantification stands to singular quantification. The availability of such expressive resources is discussed in Chapter 9.

Relatedly, plural logic has been used to eliminate reference to abstract objects. In particular, quantification over sets can sometimes be replaced by plural quantification over concrete objects. This nominalist strategy is of interest also to non-nominalists. In set theory, for example, quantification over proper classes might be eliminated in favor of plural quantification over sets (see Uzquiano 2003 and Burgess 2004).

The next application of plural logic we would like to highlight has to do with semantics. An example was already mentioned in the previous section. While discussing ontological innocence, we outlined the semantics for plural logic developed by Boolos. The key idea was to employ plural resources in the metalanguage and interpret each plural variable as standing for one or more objects rather than a set or some set-like entity. Boolos's semantic insight is applicable in other contexts as well. Plural resources can also be used to formulate a semantics for first- and second-order logic (see Rayo and Uzquiano 1999, Rayo and Williamson 2003). As we discuss at length in Chapter 11, an important aspect of this semantics is that it enables us to capture interpretations of the language whose domain of quantification encompasses absolutely everything there is. Let us briefly explain.

In the usual set-theoretic semantics, domains of quantification are represented by sets. However, since there is no universal set in standard set theory, there is no way of representing interpretations whose domain encompasses absolutely everything. Arguably, this is problematic. For certainly it *seems* that absolutely general quantification is possible; consider, for example:

[26] See van Inwagen 1990, Hossack 2000, Rosen and Dorr 2002, and Uzquiano 2004b.

(2.28) The empty set has no elements.

(2.29) Everything is physical.

These are *prima facie* cases in which the quantifiers 'no' and 'every' range over absolutely everything there is.

If we accept that quantification over absolutely everything is possible, set-theoretic semantics appears inadequate in this respect. This apparent inadequacy might be overcome by developing the semantics with the help of plural logic. Rather than describing a domain as a set-like entity whose members constitute the range of quantification, one may describe it as *some objects*, without assuming that there is a single entity to which the elements of the domain all belong as members. How does this help us with absolute generality? Once we let a domain be a plurality of objects, it seems, we can capture absolute generality by means of the universal plurality, that is, the plurality of absolutely everything there is. The existence of a universal plurality is guaranteed by the plural comprehension scheme available in traditional plural logic, for example, by using the formula '$x = x$'.

It is noteworthy that the ability to do justice to absolute generality depends on the ontological innocence of plural logic, at least in the narrow sense that it introduces no new commitments to sets or other set-like objects. If plural logic was committed in this sense, our use of it to capture absolute generality would likely be undermined. For in that case, there could be no universal plurality, contrary to traditional plural logic. This can be shown under minimal assumptions by an argument analogous to that of Russell's paradox. Suppose that plural talk is not ontologically innocent, in the sense that the existence of a plurality xx requires the existence of a corresponding set (or set-like object) $s(xx)$. The correspondence between xx and $s(xx)$ is understood in terms of coextensionality: anything is one of xx if and only if it is a member of $s(xx)$. An assumption we need is that the membership relation for these sets (or set-like objects) is subject to a principle of separation.[27] Now let uu and $s(uu)$ be, respectively, a universal plurality and its corresponding set (or set-like object). By separation, there is an object r whose members are all and only the things in $s(uu)$ that are not members of themselves:

[27] In this context, we can state the principle as follows. Given any set (or set-like object) s and any condition $\varphi(x)$, there is a set (or set-like object) r whose members are all and only the members of s which satisfy the condition. That is, there is a set (or set-like object) r such that, for any x, x is a member of r if and only if x is a member of s and $\varphi(x)$.

$$\forall x\big(x \text{ is a member of } r \leftrightarrow (x \text{ is a member of } s(uu) \wedge x \text{ is not a member of } x)\big)$$

Since uu is universal, so is $s(uu)$. Thus anything is a member of $s(uu)$ and, therefore, the members of r are all and only the things that are not members of themselves:

$$\forall x(x \text{ is a member of } r \leftrightarrow x \text{ is not a member of } x)$$

By instantiating the universal quantifier with r, we reach a Russell-style inconsistency:

$$r \text{ is a member of } r \leftrightarrow r \text{ is not a member of } r$$

The conclusion is that, if plural talk is not ontologically innocent in the mentioned sense, there cannot be a universal plurality. If there were such a plurality, there would be a corresponding set (or set-like object), leading to a version of Russell's paradox. But if there is no universal plurality, plural logic cannot serve to capture interpretations whose domain contains absolutely everything. Thus, an important application of plural logic in semantics would have to be renounced.

The last application we consider pertains to the philosophy of mathematics and relates to the expressive limitations of first-order languages mentioned in Section 2.5. Many attempts to overcome these expressive limitations resort to higher-order languages. In particular, second-order resources are often employed with an aim to provide a categorical axiomatization of the natural number structure, the real number structure, and certain initial segments of the hierarchy of sets. The ability to provide characterizations of this sort plays a major role in some philosophical accounts of mathematics, such as various forms of structuralism (see, for example, Hellman 1989 and Shapiro 1991).

However, the view that second-order logic is more expressive than first-order logic is not uncontroversial. It depends essentially on a particular semantics for second-order logic, which may be rejected. Since it is commonly assumed that plural logic does better than first-order logic in securing a gain in expressive power, plural logic has emerged as an appealing alternative to second-order logic in philosophy of mathematics. We critically assess this application of plural logic in Chapter 8.

2.7 Our methodology

This book is first and foremost a contribution to philosophical logic, although we occasionally aim to contribute to the philosophy of language and linguistics as well. Our primary interest is in exploring possible language forms and their philosophical significance. Indeed, what matters for philosophical purposes is often just *the availability* of certain language forms, rather than their *actual realization* in some natural language or other. For instance, the study of modalities exhibits a wide range of operators governing modal scope, some of which can be of philosophical use despite not having a correlate in natural language. Higher-order logic has also found a number of important applications even though it is controversial whether there is a genuine form of quantification into predicate position in natural language. Moreover, formal languages include predicates with arbitrarily high arity that do not correspond to any predicate of natural language. (There are presumably no primitive 17-adic predicates in English.)

This approach allows us to separate the study of possible language forms from the more straightforwardly empirical question of which of these language forms are in fact realized in natural language. We sometimes take a stand on the latter question. But there are also occasions when we set aside considerations of faithfulness to natural language in favor of an exploration of possible language forms, which is subject to fewer empirical constraints.

Even when we do consider natural language, our main focus is on regimentation rather than on a perfect representation of some underlying logical form, as this notion is understood in early analytic philosophy or in contemporary linguistic semantics.[28] The notion of regimentation we employ is Quinean in spirit.[29] To regiment a language is to paraphrase it into a fragment of ordinary or semi-ordinary language so as to lay bare structural features of relevance to the theoretical goals at hand, usually to address questions about logical consequence (see Chapter 3 for details). For example, our target in regimenting plural discourse into PFO or PFO+ is not to provide a faithful representation of *the* logical form underlying the discourse. The target is more modest: we aim to provide a representation of plural discourse that captures the logical features that are important in

[28] See, for instance, Pietroski 2016.
[29] See e.g. Quine 1960, 159.

the given context of investigation. This means that a regimentation might not capture all the logical relations observed in the regimented language. However, we do make the converse demand that all the logical relations obtaining in the regimenting language be reflected in the regimented one.[30] In any case, departures from what might be regarded as *the* logical form of a sentence will be justified on the basis of particular theoretical interests.

Our emphasis on the availability of possible language forms rather than their realization in natural language contributes to our goal of putting some philosophically interesting questions into sharper focus. In fact, the richness of natural language might be a hindrance to this goal. Consider the case of logical paradoxes. The significance of these paradoxes will of course depend on one's project. Paradoxes may not be of paramount importance in linguistic investigations concerned with a faithful representation of a given language, where the paradoxical aspects may simply be regarded as traits of that language. But paradoxes are clearly relevant to a project in philosophical logic that aims to explore possible language forms and identify those that can serve certain purposes in scientific theorizing. In that context, modeling a paradox is not enough: the paradox must somehow be resolved.

Our concern with paradoxes also explains the emphasis we place on absolute generality. Many of the arguments leading to paradox depend essentially on the assumption that quantification over absolutely everything is possible and that an adequate model theory must do justice to this possibility. The role played by absolute generality in the semantics of plurals is clarified in Chapter 7, where we discuss the connection between such generality and different approaches to model theory.

Additional questions that will benefit from an investigation of the kind we pursue concern ontological commitment, the legitimacy of various forms of quantification, the determinacy of plural quantifiers, and the relation between plurals and modalities. These are some of the main themes of this book.

[30] We will refer to this requirement as *logical adequacy* (see Section 3.1).

3
The Refutation of Singularism?

In mathematics, linguistics, and science more generally, pluralities are often eliminated in favor of sets or mereological sums. This affords unification and theoretical economy. In the philosophical literature, by contrast, it is easy to find arguments with sweeping conclusions to the effect that we need primitive plurals, and that this need cannot be filled by any of the singularist alternatives, such as adding *sets* of objects already recognized, adding *mereological sums* of such objects, or using *second-order logic* to quantify over all the ways for the objects already recognized to be. For example, Alex Oliver and Timothy Smiley write that "changing the subject"—which is their name for singularist attempts to eliminate plurals—"is simply not on" (2001, 306). Similar views have been defended by Byeong-uk Yi (1999, 2005, 2006), Tom McKay (2006), and others. In this chapter, we clarify and evaluate these arguments.

If successful, these arguments establish something important. Not only are primitive plurals available in English and many other natural languages, they are also scientifically legitimate and indeed indispensable. Since these are strong claims, however, we will play devil's advocate and examine whether primitive plurals might, after all, be dispensed with for scientific purposes.

3.1 Regimentation and singularism

It is useful to begin by asking: for what purposes are the singularist alternatives "not on"? Sweeping conclusions like the one just mentioned are usually made in the context of discussions about regimentation. Let us elaborate on our understanding of this notion, which we briefly discussed in Section 2.7. The process of regimentation takes as input sentences of a meaningful object language ($\mathcal{L}_O$) and yields a translation into a regimenting language ($\mathcal{L}_R$). This may be a natural language or a formal one. Even when $\mathcal{L}_R$ is a formal language, we may follow Quine in treating it as a "special

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part [. . .] of ordinary or semi-ordinary language" (Quine 1960, 145). From this perspective, both $\mathcal{L}_O$ and $\mathcal{L}_R$ are interpreted languages. We take $\mathcal{L}_O$ to be a fragment of natural language containing plurals.

To say that we need primitive plurals for regimentation does not immediately answer our initial question. *The adequacy of a regimentation is always relative to some purpose.* Whether a particular regimentation succeeds will thus depend on the purpose of the regimentation. Let us recall some of the main theoretical purposes that regimentation has served.

One of the most widespread uses of regimentation concerns "the application of logical theory" (Quine 1960, 145) and is illustrated by the process of translation into logical notation familiar from any logic course. Here the translation provides a perspicuous model of the object language that enables us to formulate a precise account of deductive reasoning and logical consequence. To provide such a model, it is not necessary to capture faithfully the meanings of the sentences translated. As emphasized by Quine (1960), a translation might convey more or less information than the sentence it translates. What matters is that, in virtue of the vocabulary being analyzed, the translation mostly reflects what follows from what in the object language.[1]

Regimentation can also serve the purpose of representing ontological commitments. The ontological commitments of statements of the object language are not always fully transparent. The translation might help clarify them. Following Donald Davidson, one might for instance regard certain kinds of predication as implicitly committed to events. As a result, one might be interested in a regimentation that, by quantifying explicitly over events, brings these commitments to light.

Our focus in this chapter is largely on "the application of logical theory". We will discuss some arguments purporting to show that singularist regimentations mischaracterize logical relations in the object language or mischaracterize the truth values of some sentences. There are various requirements one could put forward in this context. A minimal requirement is that the regimentation be *logically faithful* in the sense that, if an argument in $\mathcal{L}_O$ is invalid, then so should be its regimentation. Let τ be a translation. Then logical faithfulness requires that

[1] Note that this use of regimentation is consistent with different attitudes towards logical consequence. In particular, it is consistent with logical monism as well as logical pluralism. Of course, one's view about logical consequence will be reflected in one's approach to regimentation. So, unlike the pluralist, the monist will see regimentation as a tool to capture the "correct" notion of logical consequence for the object language.

$$\text{if } \tau(\varphi_1), \ldots, \tau(\varphi_n) \vDash \tau(\psi), \text{ then } \varphi_1, \ldots, \varphi_n \vDash \psi.$$

The converse requirement, which we call *logical adequacy*, is that if an argument in $\mathcal{L}_O$ is valid, then its regimentation be valid as well:

$$\text{if } \varphi_1, \ldots, \varphi_n \vDash \psi, \text{ then } \tau(\varphi_1), \ldots, \tau(\varphi_n) \vDash \tau(\psi).$$

The requirement of logical adequacy is less compelling than that of faithfulness, as can be seen by considering the case of sentential logic. There are arguments in natural language whose validity depends on their quantificational structure and thus cannot be captured by the usual regimentation in sentential logic. While this regimentation is logically faithful, it is not logically adequate. Logical adequacy is sometimes more plausible, however, when relativized to a theory T. That is, one may require that if an argument in $\mathcal{L}_O$ is valid, its regimentation be valid *when supplemented with the axioms of* T. Thus, we have:

$$\text{if } \varphi_1, \ldots, \varphi_n \vDash \psi, \text{ then } \tau(\varphi_1), \ldots, \tau(\varphi_n), T \vDash \tau(\psi).$$

Some of the analyses of plurals we will examine satisfy only this weaker adequacy condition.

There are parallel requirements concerning truth. Here it makes sense to require both faithfulness and adequacy, that is, to demand that a regimenting sentence be true if and only if the regimented one is true.

We turn now to some arguments to the effect that primitive plurals are needed for regimentation. The bone of contention is whether, for logical purposes, we can dispense with plurals in the regimentation of $\mathcal{L}_O$. This presupposes that we can determine whether the regimentation contains plurals. Since $\mathcal{L}_R$ is an interpreted language, however, we can presumably establish whether it contains plurals by relying on an antecedent understanding of the distinction between singular and plural expressions.

A singularist regimentation attempts to paraphrase away plural expressions. The alternative approach advocated by Boolos resists this elimination by taking plurals at face value. On this alternative, which we call *regimentation pluralism*, $\mathcal{L}_R$ does contain plural expressions. The languages $\mathcal{L}_{\mathrm{PFO}}$ and $\mathcal{L}_{\mathrm{PFO+}}$ are the main examples of regimentation pluralism.

Which approach is correct? As observed, the recent philosophical literature abounds with arguments against regimentation singularism. The principal aim of this chapter is to assess some of these arguments and gain a better understanding of the limits of regimentation singularism. While we think that regimentation pluralism has important applications, we also think that regimentation singularism is a more serious rival to regimentation pluralism than the mentioned literature suggests. So we want to give it a fair hearing. In fact, we identify some conditions under which regimentation singularism is perfectly benign. Since these conditions tend to be satisfied in the cases that interest linguists, their singularist proclivities are less problematic than many philosophers claim.

3.2 Substitution argument

One argument against regimentation singularism, put forward by Yi (2005, 471–2), turns on a substitution of plural and singular terms. We therefore dub it the *substitution argument*. This argument is meant to apply to any singularist regimentation, no matter how it paraphrases plural expressions.[2] For concreteness, we focus on a regimentation that uses sets.

Consider the plural term 'Russell and Whitehead' and its set-theoretic regimentation, the set term '{Russell, Whitehead}'. Letting 'Genie' abbreviate this set term, we now formulate the following sentences:

(3.1) Genie is one of Genie.

(3.2) Genie is one of Russell and Whitehead.

While (3.1) is arguably true (and logically so), (3.2) is false. But given the way in which 'Genie' was introduced, aren't 'Genie' and 'Russell and Whitehead' intersubstitutable *salva veritate*? If so, it follows that the two sentences have the same truth value. But this appears not to be the case.

Let us examine the argument more closely. Does it concern sentences of $\mathcal{L}_O$ or $\mathcal{L}_R$? Since $\mathcal{L}_R$ is supposed to be free of plurals, the argument must be concerned with sentences of $\mathcal{L}_O$.

Thus understood, the argument assumes that $\mathcal{L}_O$ contains plural resources and is able to express claims about sets (or whatever other objects are used

[2] Note that a semantic version of this argument is also present in Yi's discussion. The semantic version targets the view that a plural term *denotes* a set or a set-like entity. In this chapter, our focus is on regimentation and thus on the view that plural expressions can be *paraphrased* by means of singular constructions. We think that it is important to keep the two views separate. See Chapter 7 for a discussion of the semantics of plurals.

in $\mathcal{L}_R$ to paraphrase plurals). The problem—if there is one—stems from an *unintended interaction* between these plural resources and talk about the objects that are used to represent pluralities. Suppose $\mathcal{L}_O$ could not talk about the objects used to represent pluralities. Then 'Genie' would not be part of $\mathcal{L}_O$, and the argument would not get off the ground. This suggests that the argument can be blocked by denying the assumption that $\mathcal{L}_O$ can talk about the objects used to represent pluralities. We explore this option in the next section. In the remainder of this section, we will show that the argument can be resisted even when this assumption is granted.

The argument relies on the reasonable requirement that a proper regimentation of (3.1) and (3.2) do justice to the fact that the two sentences differ in truth value. But it is not hard to think of a simple translation that meets this demand. For example, we may translate (3.1) and (3.2) as respectively:

(3.3) Genie = Genie.

(3.4) Genie $\in$ {Russell, Whitehead}.

This regimentation captures the truth values of (3.1) and (3.2). It maps a (logically) true sentence to a (logically) true sentence, and it maps a false sentence to a false sentence.

It might be objected that we didn't translate 'is one of' uniformly. However, this non-uniformity seems justified by the peculiar grammatical status of (3.1). One might even complain that (3.1) is ungrammatical, since 'is one of' requires a plural term in its second argument place.

Yi proposes a variant of the argument intended to avoid this complication. Consider the following two sentences:

(3.5) Genie is one of Genie and Frege.

(3.6) Genie is one of Russell and Whitehead and Frege.

Again, the two sentences appear to differ in truth value: while (3.5) is (logically) true, (3.6) seems false. (Presumably, something is one of Russell and Whitehead and Frege just in case it is identical to one of the three named logicians.)

Even in this case it is not hard to think of a translation that captures the difference in truth value. For instance, we may translate (3.5) and (3.6) as respectively:

(3.7) Genie $\in$ {Genie, Frege}.

(3.8) Genie $\in$ {Russell, Whitehead, Frege}.

The translation of 'is one of' is now uniform. In addition, the translation preserves the truth value of the two sentences. The moral is that, even though regimentation singularism paraphrases a plural term like 'Russell and Whitehead' by means of a singular expression such as '{Russell, Whitehead}', it need not license in $\mathcal{L}_O$ the intersubstitution *salva veritate* of the two terms.

We conclude that the substitution argument does not undermine regimentation singularism. First, the argument relies on an assumption that may be resisted, namely that $\mathcal{L}_O$ can talk about the objects used in $\mathcal{L}_R$ to paraphrase plurals. Second, it overlooks the potential of some singularist regimentations to capture the intuitive truth values of the relevant sentences of $\mathcal{L}_O$.

We now turn to two further objections to regimentation singularism that share with the substitution argument the assumption that $\mathcal{L}_O$ can talk about the entities used to paraphrase plurals in $\mathcal{L}_R$.

3.3 Incorrect existential consequences

A colorful formulation of the next objection is contained in Boolos's famous passage quoted in Chapter 2:

> There are, of course, quite a lot of Cheerios in that bowl, well over two hundred of them. But is there, in addition to the Cheerios, also a set of them all? [...]
>
> It is haywire to think that when you have some Cheerios, you are eating a set [...]. [I]t doesn't follow just from the fact that there are some Cheerios in the bowl that, as some who theorize about the semantics of plurals would have it, there is also a set of them all. (Boolos 1984: 448–9)

In one reading of the passage, the objection is that a singularist regimentation validates incorrect inferences in $\mathcal{L}_O$ and, in particular, incorrect existential generalizations.

If the purpose of regimentation is the application of logical theory, we want the regimentation to be logically faithful. Consider the relevant inference (with '*cc*' naming the Cheerios in the bowl):

(3.9) $\dfrac{\text{George ate } cc.}{\text{George ate a set.}}$

As emphasized by Boolos, this inference is invalid. Compare now a set-theoretic translation of the argument:

(3.10) $\dfrac{\text{George ate* } \{cc\}.}{\text{George ate* a set.}}$

where 'ate*' is the translation of 'ate' and '{*cc*}' is the set-theoretic rendering of '*cc*' (say, '{*x*: *x* is a Cheerio in the bowl}').[3]

Unlike (3.9), (3.10) is valid. So we have a violation of logical faithfulness: an invalid inference has a valid translation.

The argument can be extended to other types of singularist regimentation. The general point is that regimentation singularism seems to permit illicit existential generalizations, allowing us to transition *as a matter of logic* from some objects to a single object that comprises or somehow represents these objects.

As in the case of the substitution argument, it is assumed that $\mathcal{L}_O$ can talk about the objects used in $\mathcal{L}_R$ to paraphrase plurals. Let us call these objects *proxies*. The alleged problem stems from an unintended interaction between the plurals and the talk of proxies. Thus, if $\mathcal{L}_O$ was precluded from talking about the proxies, the argument could not get off the ground. How might this be achieved? Since the object language is just given to us, it is not an option to ban certain expressions from $\mathcal{L}_O$ if they already occur in it. By contrast, the regimenting language $\mathcal{L}_R$ is not given but can freely be chosen to serve our needs. So we might well be able to choose our proxies so as to avoid problematic interactions with any resources found in $\mathcal{L}_O$. In many of the cases studied by linguists, such a choice is indeed possible.

Philosophical analysis, on the other hand, often aims for greater generality. Suppose that $\mathcal{L}_O$ already talks about sets. Then we might be able to avoid the problematic interaction by finding some other proxies—let us call them "supersets"—with which to regiment the plurals of $\mathcal{L}_O$. But even if an appropriate notion of superset can be found, we are not done. We may want to include talk of supersets in $\mathcal{L}_O$. Thus, for *any* kind of object that extensions

[3] Since 'ate' is used here as plural predicate, it is regimented by means of a singular counterpart 'ate*'. By contrast, the predicate 'set' is singular and thus remains unchanged in the regimentation.

of $\mathcal{L}_O$ might talk about, we must be able to regiment plural talk about such objects using proxies of a *new* kind.

Is this possible? The answer will depend on the generality to which our analysis aspires. Suppose we want a fully specified regimentation strategy that works for any given object language whatsoever. We thus specify a certain kind of proxy that will always be used to paraphrase plurals. When this general strategy is applied to an object language that talks about proxies of this kind, the problem under discussion arises. We conclude that the singularist's only hope is that her regimentation strategy need *not* be fully specified, thus allowing her to wait and see what expressive resources a given object language contains and *only then* choose her proxies—in a way that avoids the problematic interactions. This might well be doable. Thus, even if the most ambitious form of regimentation singularism succumbs to the objection from incorrect existential consequences, there are less extreme forms that avoid it. These forms will likely suffice for linguists' purposes. To achieve the kind of generality that philosophers seek, however, any viable form of regimentation singularism must refrain from a fixed choice of proxies.

3.4 The paradox of plurality

What is often regarded as the most serious objection to regimentation singularism is *the paradox of plurality*, first foreshadowed in Section 2.1. Suppose that we use sets to eliminate plurals and that $\mathcal{L}_O$ has the resources to talk about sets. Then the following sentence of $\mathcal{L}_O$ appears to be a truism:

(3.11) There are some objects such that any object is one of them if and only if that object is not an element of itself.

The contention is that set-theoretic singularism is bound to regiment (3.11) as follows:

(3.12) There is a set of which any object is an element if and only if that object is not an element of itself.

In symbols:

(3.13) $\exists x\big(\mathrm{set}(x) \land \forall y(y \in x \leftrightarrow y \notin y)\big)$

But this is an instance of the familiar, inconsistent Russell sentence. Thus, the true sentence (3.11) is regimented by means of the false sentence (3.12), which is unacceptable.[4]

Let us make explicit the generality of the argument.[5] As before, let proxies be the entities used in $\mathcal{L}_R$ to paraphrase plurals. Thus, a proxy need not be a set, but could equally well be a class, a mereological sum, or a group. Plural quantification is regimented as singular quantification over proxies. Let η be the translation into $\mathcal{L}_R$ of the one-many relation 'to be one of'. So '$x\eta y$' regiments the statement that x is one of the objects represented by the proxy y. Since η is a meaningful predicate, nothing precludes its introduction into $\mathcal{L}_O$—or so the argument goes. Thus we may suppose that the predicate is available in $\mathcal{L}_O$ as well.

We now face an awkward dilemma. Is η reflexive? Suppose not. Then there is an object that satisfies the open formula '$\neg(x\eta x)$'. Applying plural comprehension to this formula, we obtain:

(3.14) There are some objects such that any object is one of them if and only if that object does not bear η to itself.

The singularist regimentation of this sentence is:

(3.15) There is a proxy to which any object bears η if and only if that object does not bear η to itself.

And this, in turn, is formalized as:

(3.16) $\exists x[\text{proxy}(x) \land \forall y(y\eta x \leftrightarrow \neg(y\eta y))]$

But (3.16) is inconsistent! So again, a true sentence is regimented by means of a false one, which is unacceptable.

Alternatively, suppose that η is reflexive. Then there is no object that satisfies the open formula '$\neg(x\eta x)$'. This blocks the previous argument. Instead, another problem arises: the reflexivity of η entails that different pluralities must be represented by one and the same proxy and hence cannot

[4] The proposed regimentation also involves a violation of logical faithfulness. For (3.11) is not only true but intuitively valid, whereas (3.12) is not only false but logically so.

[5] The argument, inspired by Boolos (1984b, 440–4), is also discussed in Lewis 1991, 68; Schein 1993, Chapter 2, Section 3.3; Higginbotham 1998, 14–17; Oliver and Smiley 2001, 303–5; Rayo 2002, 439–40; and McKay 2006, 31–2.

be distinguished in the regimentation. To prove this entailment, suppose there are at least two objects, a and b. When η is reflexive, the singleton plurality of a must have itself as its proxy, and likewise for b. Consider now the plurality of a and b, and let c be its proxy. By the reflexivity of η, we have $c\eta c$. By the definition of a proxy, we also know that only a and b bear η to c. Hence c is identical with either a or b. But we observed that each of a and b is already used as a proxy for a distinct plurality, namely the singleton pluralities of a and b. Thus, as promised, different pluralities are represented by one and the same proxy.

Just like the previous two arguments, the paradox of plurality relies on the assumption that talk of proxies is available in $\mathcal{L}_O$. The lesson is that, if $\mathcal{L}_O$ can talk not only about pluralities but also about their proxies, then the regimentation validates unintended interactions of the sort just seen. To block the paradox, we would therefore have to prevent such problematic interactions.

One possibility, suggested by our discussion in the previous section, is to refrain from making a fixed choice of proxies to be used in the analysis of all object languages. Instead, the singularist can let her choice of proxies depend on the particular object language she is asked to regiment. All she needs to do is to choose new proxies, not talked about by the given object language. In this way, the problematic interactions are avoided.

In fact, there are other responses to the paradox of plurality that are compatible even with a fixed choice of proxies. One such response is that there is variation in the range of the quantifiers involved in the paradoxical reasoning.[6] In particular, one can avoid the paradox by assuming that in (3.16) the quantifier '$\exists x$' has a wider range than '$\forall y$'. To see why this assumption blocks the paradox, consider the reasoning leading from (3.16) to contradiction:

(3.16) $\exists x\big[\text{proxy}(x) \land \forall y(y\eta x \leftrightarrow \neg(y\eta y))\big]$

(3.17) $\forall y(y\eta r \leftrightarrow \neg(y\eta y))$

(3.18) $r\eta r \leftrightarrow \neg(r\eta r)$

In the step from (3.17) to (3.18), the witness to '$\exists x$' is used to instantiate '$\forall y$'. If the domain of '$\exists x$' extends that of '$\forall y$', the step becomes illicit.

[6] See, e.g., Parsons 1974a, 1974b; Glanzberg 2004, 2006.

This response brings to the fore the topic of absolute generality. Indeed, the response presupposes that the range of the quantifiers of the object language is not unrestricted and thus that a domain expansion is possible. The question of absolute generality is explored in Chapter 11, where we defend the permissibility of such generality. If we are right, then the response under discussion is unavailable. Still, since linguists are typically interested in ordinary discourse where absolute generality is not present, they can often bypass the argument.

Yet another response to the paradox of plurality is developed in Chapter 12, where we take a more critical stance towards plural comprehension. In particular, we block the paradox by developing a reason to reject instances of plural comprehension underlying the paradoxical reasoning, such as (3.14).

3.5 Plural Cantor: its significance

The paradox of plurality is closely related to a generalization of Cantor's theorem. Let us begin by reminding ourselves of the familiar set-theoretic version of Cantor's theorem, which can be formulated as follows.

Cantor's theorem (informal)

For any set A, the subsets of A are strictly more numerous than the elements of A.

The plural version of Cantor's theorem makes an analogous claim concerning pluralities.

Plural Cantor (informal)

For any plurality xx with two or more members, the subpluralities of xx are strictly more numerous than the members of xx.

There is only one tiny disanalogy: we need to assume that xx have two or more members, whereas no such assumption is required concerning A. The reason for this minor discrepancy is that pluralities, unlike sets, are required to be non-empty. If this requirement were lifted, the analogy between the set-theoretic and plural versions of the theorem would be perfect.

Of course, the cardinality comparisons involved in these two informal statements need to be explicated, and the resulting plural version of Cantor's theorem needs to be proved. But before we do so in the next section, we would like to explain the significance of Plural Cantor for our discussion.

The theorem can be seen as a diagnosis of the problem exploited by the paradox of plurality. Assume, as is done in traditional plural logic, that there is a "universal plurality" encompassing every object whatsoever. (This assumption will be challenged in Chapter 12.) Applied to this universal plurality, the theorem entails that there are more pluralities than objects. This implies that it is impossible to assign to each plurality a distinct object as its proxy. We now find ourselves in the "awkward dilemma" described in Section 3.4. Suppose we require that each plurality be assigned a unique proxy. Then, as we have just seen, we land in a contradiction. Alternatively, we may relax this requirement. But this means that some statements of $\mathcal{L}_O$ will receive an incorrect regimentation. For example, if distinct pluralities xx and yy are assigned the same proxy z, the true statement of $\mathcal{L}_O$ that these pluralities are not coextensive will be regimented as the contradictory statement that something does and does not bear η to z.

Can this dilemma be resisted? Once again, the question of absolute generality turns out to be central. Suppose that absolute generality is possible. Then, as we have observed, traditional plural logic yields an instance of Plural Cantor concerned with the universal plurality. We therefore obtain that there are more pluralities than objects. This means that each plural variable of the object language can have more possible values—namely each plurality—than there are objects or proxies. By contrast, suppose that absolute generality is not possible. Then the object language ranges over some plurality of objects aa which, when the domain is expanded, can be seen not to be universal. This makes it unproblematic that aa has more subpluralities than members. Each of these subpluralities can be represented by a distinct proxy—provided that most of these proxies are not among aa but are drawn from elsewhere. And there is no reason why such proxies should not be available when the object language has a restricted domain.

3.6 Plural Cantor: its statement and proof

We now turn to the task of explicating the cardinality comparison involved in Plural Cantor. As is turns out, there are various ways to do so, resulting in different versions of the theorem.

There are several ways to define what it means for one set Y to be "strictly more numerous than" another set X. One option is that there is no surjective function from X to Y; another is that there is no injective function from

Y to X.[7] In both cases, the notion of a function can be understood in the usual set-theoretic way.[8]

Consider now the cardinality comparison involved in Plural Cantor. What is it for the subpluralities of xx to be strictly more numerous than xx themselves? Let us try to imitate the answer given in the set-theoretic case. Suppose we add to our formalism variables of a new and primitive type for functions *from pluralities to objects*, that is, functions that take one or more objects as input and then output a single object as the value. We can then state that there is no injective function from subpluralities of xx to xx themselves by denying the existence of a function g from pluralities to single objects among xx such that:

$$\forall yy \forall zz(yy \preccurlyeq xx \wedge zz \preccurlyeq xx \rightarrow (g(yy) = g(zz) \rightarrow yy \approx zz)) \tag{3.19}$$

Alternatively, we might add variables of a new and primitive type for functions *from objects to pluralities*, that is, functions that take a single object as input and then output one or more objects as values. To state that there is no surjective function from xx to subpluralities of xx, we deny the existence of a function f from objects to pluralities such that:

$$\forall yy(yy \preccurlyeq xx \rightarrow \exists x(x \prec xx \wedge f(x) \approx yy)) \tag{3.20}$$

For each of these formulations, it is straightforward to prove the resulting formal version of Plural Cantor. The version using a function from objects to pluralities provides a good example. Assume, for contradiction, that there is a surjective function f of the relevant sort. We contend there is an $x \prec xx$ such that $x \not\prec f(x)$, as we shall prove shortly. Thus, plural comprehension allows us to define a subplurality $\delta\delta$ of xx such that:[9]

[7] Recall that a function f from X to Y is said to be *surjective* if and only if

$$(\forall y \in Y)(\exists x \in X)f(x) = y,$$

and *injective* if and only if $f(x) = f(x') \rightarrow x = x'$.

[8] More precisely, f is a function from X to Y if and only if (i) for every $x \in X$ there is a $y \in Y$ such that $\langle x, y\rangle \in f$, and (ii) if both $\langle x, y\rangle$ and $\langle x, y'\rangle$ are in f, then $y = y'$.

[9] Note that the instance of comprehension used is predicative; that is, the condition '$x \prec xx \wedge x \not\prec f(x)$' used to define the plurality does not itself quantify over pluralities. See Uzquiano 2015b, Section 3.1. Furthermore, note that this instance of plural comprehension is a case of what we will later (see Appendix 10.A and Section 12.5) call *plural separation*, namely, a comprehension axiom where a given plurality (in this case, xx) is cut down to a subplurality comprising the members of xx that satisfy some formula. The same applies to other instances of comprehension used in this and similar proofs. Thus, the proofs in question go through in the alternative system of critical plural logic that we defend in Chapter 12.

$$\forall x(x \prec \delta\delta \leftrightarrow x \prec xx \ \wedge \ x \nprec f(x)) \tag{3.21}$$

Since f is surjective, there is $\delta \prec xx$ such that $f(\delta) \approx \delta\delta$. By instantiating the quantifier of (3.21) with respect to δ, we easily derive

$$\delta \prec \delta\delta \leftrightarrow \delta \nprec f(\delta) \tag{3.22}$$

which is inconsistent because $f(\delta) \approx \delta\delta$.[10]

It remains only to prove our contention that there is an $x \prec xx$ such that $x \nprec f(x)$.[11] Assume not. As already assumed in the statement of the theorem, there are at least two distinct objects, a and b, among xx. By the former assumption, f maps each of these objects to the corresponding singleton plurality. By the assumed surjectivity of f, there is an object c that f maps to the plurality of a and b. So c must be distinct from each of a and b, since we have established that these two objects are mapped by f to other pluralities. Since the members of $f(c)$ are a and b, this entails that $c \nprec f(c)$, which contradicts our assumption that there is no $x \prec xx$ such that $x \nprec f(x)$. This concludes our proof.

Of course, these formulations and proofs assume that we have variables of a new and primitive type, either for functions from objects to pluralities or for functions in the reverse direction. Fortunately, our proof requires no special assumptions concerning these new functions, only that quantification over them obeys the usual logical principles.[12] Even so, it is important to realize that the new type of function is not required. In Appendix 3.A, we provide some alternative formulations of the relevant cardinality comparisons. Some of these avoid the new type of function, thus making the theorem available also in systems that are less expressive. Other formulations achieve greater generality by regarding functions as just a special kind of relation. Moreover, by considering all these formulations side by side, we obtain a more complete picture of the assumptions that this important theorem requires.

[10] The version of Plural Cantor using a function from pluralities to objects is proved similarly. Assume there is an injective g of the relevant sort. Since g is injective, there is an inverse function g^{-1}. We can now use plural comprehension to define a subplurality $\delta\delta$ of xx such that:

$$\forall x(x \prec \delta\delta \leftrightarrow x \prec xx \wedge x \nprec g^{-1}(x)) \tag{3.23}$$

To obtain a contradiction, let $\delta = g(\delta\delta)$ and instantiate the quantifier of (3.23) with respect to δ.

[11] This proof is nearly identical with the one on p. 40.

[12] It might be objected that the proof mentioned in footnote 10 assumes the existence of an inverse function g^{-1}. But this assumption is easily eliminated in favor of an alternative use of (impredicative) plural comprehension. In particular, plural comprehension yields the existence of a subplurality $\delta\delta$ of xx such that:

$$\forall x(x \prec \delta\delta \leftrightarrow x \prec xx \wedge \forall yy(g(yy) = x \rightarrow x \nprec yy))$$

Since g is injective, a contradiction follows.

Just as we have generalized the ordinary version of Cantor's theorem to Plural Cantor, so Plural Cantor admits of further generalizations. Suppose there are "superpluralities", that is, pluralities of pluralities. Then, using resources analogous to those used for the proof of Plural Cantor, one can show that, given any domain with two or more objects, the superpluralities based on that domain are strictly more numerous than the pluralities based on the same domain. This is done by proving that, relative to the given domain, there is no surjective function from pluralities to superpluralities (and no injective function in the reverse direction).

3.7 Conclusion

We have considered four arguments against regimentation singularism: the substitution argument, the argument from unintended existential consequences, the paradox of plurality, and the argument based on Plural Cantor. Although the arguments differ in important respects, we also found some common themes.

The first three arguments turn on problematic forms of interaction between plurals and talk of proxies. These arguments can therefore be blunted by giving up the requirement that a fixed sort of proxies be used in all regimentations. Suppose this requirement is lifted. Then, for any given object language, it may well be possible to choose new proxies, that is, proxies that are not among the objects that this language can talk about. If new proxies can always be found, the problematic interactions can be avoided.

A central question is therefore whether new proxies are always available. In fact, their availability is called into doubt by the fourth argument against singularism, which uses a generalization of Cantor's theorem to argue that there are more pluralities than objects and thus *a fortiori* too many pluralities for each to be assigned a unique object as its proxy.

We found, however, that even this fourth argument relies on some assumptions that can be challenged, namely the possibility of absolute generality and the validity of traditional plural logic. These assumptions are discussed at length in Chapters 11 and 12. If either assumption fails, this will provide an additional and more definitive response to the third argument, that is, the one based on the paradox of plurality.

Overall, we conclude that the prospects for regimentation singularism are not nearly as bleak as many philosophers make them out to be. As we have seen, there are promising responses to the anti-singularist arguments. It is noteworthy that these responses are particularly strong in many of the cases that concern linguists. For their purposes, it is often unproblematic to assume that the proxies are new vis-à-vis the objects that the object language talks about. Moreover, linguists often have independent reasons to foresake the ambition of absolute generality (see Peters and Westerståhl 2006, 47–9). These considerations explain why linguists' singularist tendencies are less problematic than many philosophers and logicians claim.

As mentioned at the beginning of this chapter, there are several ways to talk about many objects simultaneously. In addition to using the primitive plurals available in many natural languages, we can add *sets* of objects already recognized, add *mereological sums* of such objects, or use *second-order logic* to quantify over all the ways for the objects already recognized to be. We therefore asked whether primitive plurals are necessary or even scientifically legitimate. While we grant that there is a presumption in favor of taking expressive resources available in natural language to be scientifically legitimate, it would be good to do better. So this chapter has discussed some very general anti-singularist arguments that purport to establish the need for primitive plurals. We have shown that these arguments make limited progress.

We will now change tack and undertake a detailed comparison of plural logic with each of the other ways to talk about many objects simultaneously. This is our agenda for Part II of the book. We will find that, although the four alternatives have some important structural similarities, there are also some significant philosophical and formal differences between them. Based on these differences, we defend the thesis that none of them should be eliminated in favor of any other. This yields, in particular, a more robust argument for the scientific legitimacy of primitive plurals than this chapter has produced.

Appendix

3.A Alternative formulations of Plural Cantor

Suppose we want to avoid primitive functions from pluralities to objects or from objects to pluralities, both of which we invoked in Section 3.6. We now outline two alternatives: one that uses higher-order relations, and another that "codes" these relations in terms of pluralities of ordered pairs. This yields several formulations of Plural Cantor.

A plural comprehension axiom

$$\exists x\, \varphi(x) \to \exists xx \forall y (y \prec xx \leftrightarrow \varphi(y))$$

is said to be *impredicative* if $\varphi(y)$ contains plural quantifiers, and *predicative* if not. In what follows, we pay close attention to the question of whether impredicative plural comprehension is needed to prove the different formulations. This question is theoretically important. Even by its defenders, impredicative comprehension is often regarded as a strong commitment (see Bernays 1935). While we are prepared to make this commitment, at least in our reasoning about plurals (see Appendix 10.A), it is important to keep track of when the commitment is needed. It should also be noted that our discussion of Plural Cantor carries over, with minor modifications, to a second-order version of Cantor's theorem. This says, loosely speaking, that there are more values of second-order variables based on any domain than objects in the domain. As a corollary of our discussion, one thus easily obtains results about when impredicative *second-order* comprehension is required for the proof of a second-order version of Cantor's theorem.

Suppose we wish to use relations to state that it is impossible to "tag" each subplurality of xx with a unique member of xx. So we consider relations of the form $R(x, yy)$, that is, dyadic relations whose first and second argument places are open to objects and pluralities, respectively. We can now state that there is no relation that effects the described "tagging" by saying that there is no R of the mentioned form such that:

- (R is functional) $\quad yy \preccurlyeq xx \wedge yy' \preccurlyeq xx \wedge R(x, yy) \wedge R(x, yy') \to yy \approx yy'$
- (R is surjective) $\quad (\forall yy \preccurlyeq xx)(\exists x \prec xx) R(x, yy)$

This provides a useful relational statement of Plural Cantor.

It is interesting to reformulate that statement in terms of the converse of R, that is, the relation $\bar{R}$ defined by $\forall x \forall yy(\bar{R}(yy, x) \leftrightarrow R(x, yy))$. It is easy to verify that the two mentioned requirements on R are logically equivalent to the following requirements on $\bar{R}$, respectively:

- ($\bar{R}$ is injective) $\quad yy \preccurlyeq xx \land yy' \preccurlyeq xx \land \bar{R}(yy, x) \land \bar{R}(yy', x) \rightarrow yy \approx yy'$
- ($\bar{R}$ is total) $\quad (\forall yy \preccurlyeq xx)(\exists x \prec xx)\bar{R}(yy, x)$

Thus, the statement that there is no relation specifying a surjective function from members of xx to subpluralities of xx is equivalent to the statement that there is no relation that associates subpluralities of xx with members of xx in a way that is injective and total. This equivalence relies only on the extremely weak (and obviously predicative) assumption that every relation has a converse. We have thus achieved a pleasing unification of the surjectivity-based and the injectivity-based characterizations of the cardinality comparison: the two characterizations are logically equivalent *modulo* an extremely weak assumption.[13]

The claim that there is no relation specifying an injective and total *function* from subpluralities of xx to members of xx is strictly stronger than our pleasing unification. For the mentioned claim adds a third requirement on $\bar{R}$, namely that $\bar{R}$ be functional; that is:

$$\forall x \forall x' \forall yy(\bar{R}(yy, x) \land \bar{R}(yy, x') \rightarrow x = x')$$

Let us now prove our relational statement of Plural Cantor. Suppose, for contradiction, that there is a relation R satisfying the conditions laid out above. We want to use plural comprehension to define a subplurality $\delta\delta$ of xx such that:

$$\forall x(x \prec \delta\delta \leftrightarrow x \prec xx \land \exists yy(R(x, yy) \land x \not\prec yy)) \tag{3.24}$$

Of course, (3.24) is the consequent of a plural comprehension axiom whose antecedent is $\exists x(x \prec xx \land \exists yy(R(x, yy) \land x \not\prec yy))$. It is easy to prove this antecedent, on the assumption that xx comprise at least two objects, by imitating our proof of an analogous claim in Section 3.6. So let us return to our proof of the relational statement of Plural Cantor. By the assumed surjectivity

[13] By contrast, Uzquiano (2019) sees a deeper difference between these two characterizations.

of R, there is thus a $\delta \prec xx$ such that $R(\delta, \delta\delta)$. We now ask whether $\delta \prec \delta\delta$. By standard Russellian reasoning, it is straightforward to derive that this holds if and only if it does not.

It is important to notice that this proof relies on an impredicative plural comprehension axiom. For the plurality $\delta\delta$ is defined by quantifying over all sub-pluralities of xx, to which the defined plurality itself belongs. In fact, the reliance on impredicative comprehension can be shown to be essential.[14]

So far, we have made use of primitive relations involving pluralities. An alternative is to "code" such relations by means of pluralities of ordered pairs. This alternative is available in systems without quantification over primitive functions or relations, as is the case for most systems of plural logic found in the literature. The basic idea is to represent the fact that a is related to the plurality xx by pairing a with each object in xx. The resulting ordered pairs represent that a is related to the plurality of objects with which a has been paired. A visual example will help.

$$\begin{array}{c|ccc} c & \langle a, c\rangle & \langle b, c\rangle & \\ b & \langle a, b\rangle & & \langle c, b\rangle \\ a & & \langle b, a\rangle & \langle c, a\rangle \\ \hline & a & b & c \end{array}$$

Consider the six ordered pairs displayed. This plurality codes a relation of objects with pluralities. Specifically, an object x is related to the plurality of objects that figure as second coordinates in pairs with x as its first coordinate. This can be read off by attending to each column. Visually, each column represents the fact that the object along the horizontal axis is related to the plurality of objects that figure as second coordinates in this column. Thus, the diagram above represents that a is related to the plurality of b and c, that b is related to a and c, and that c is related to a and b.

[14] This follows from the fact that Frege's "Basic Law V" is consistent in second-order logic with only predicative comprehension axioms (see Heck 1996). We begin by rewriting Basic Law V with plural variables in place of second-order ones:

$$\{xx\} = \{yy\} \leftrightarrow xx \approx yy$$

Now define '$R(x, yy)$' as '$x = \{yy\}$'. Heck's model can now be tweaked to produce a model of plural logic with predicative comprehension and the statement that R is functional and surjective.

Equipped with this notion of coding, we obtain a precise way of expressing the plural version of Cantor's theorem using only plural resources.

Plural Cantor (formal)

For any plurality *xx* with two or more members, there is no plurality that codes a functional and surjective relation of members of *xx* with subpluralities of *xx*.

The proof of this version of the theorem is based on the same idea as before, although with a subtle but important difference. Suppose, for contradiction, that there is a plurality *rr* of ordered pairs that code a relation of the mentioned sort. We want to define a diagonal plurality $\delta\delta$ of each and every object $x \prec xx$ such that x is not a member of the plurality with which x is related by the relation coded by *rr*. This requires some unpacking. The plurality of objects with which x is related in the mentioned way are all the y such that $\langle x, y\rangle \prec rr$. Thus, the claim that x is not a member of this plurality is just the claim that $\langle x, x\rangle \nprec rr$. As before, it is easy to show that, if *xx* have two or more members, then there is at least one x that satisfies this condition. Thus, a plural comprehension axiom ensures the existence of our desired diagonal plurality $\delta\delta$. The advertised difference is that this comprehension axiom is fully predicative.[15] From this point on, the argument proceeds precisely as before. Since the coded relation is surjective, there is a δ that stands in this relation to $\delta\delta$. We now ask whether $\delta \prec \delta\delta$. Familiar Russellian reasoning enables us to prove that the answer is affirmative if and only it is negative.

The following table summarizes our findings concerning the need for impredicative plural comprehension:[16]

	primitive functions	higher-order relations	pluralities and pairs
no surjective function	predicative	impredicative	predicative
no injective total relation	—	impredicative	predicative
no injective function	impredicative (predicative if inverse functions are permitted)	impredicative	predicative

[15] We owe this surprising observation to Gabriel Uzquiano (see especially Uzquiano 2015b) and are grateful to him for discussion of its significance.

[16] In fact, every relevant instance of plural comprehension can be replaced by a corresponding instance of plural separation, as indicated in footnote 9 on p. 43.

This provides a richer and more interesting picture than would have been obtained had we focused solely on primitive functions. Our table raises the question of why predicative plural comprehension suffices to prove some formulations of the theorem, while others require impredicative comprehension. While this is not the place for a comprehensive assessment, we wish to make two remarks.

First, the resources needed to prove a formulation of Cantor's theorem are highly sensitive to the language in question. A striking example concerns the two formulations in terms of primitive functions. The "no surjective function" version uses a primitive function f from objects to pluralities. Assume f is surjective. Then, for any xx, there is x such that $f(x) \approx xx$. Generalizing, we establish the following equivalence:

$$\forall xx\, \varphi(xx) \leftrightarrow \forall x\, \varphi(f(x))$$

Using this equivalence, all plural quantification can be eliminated in favor of singular quantification. It is therefore unsurprising that predicative plural comprehension suffices for the proof. Since all plural quantification can be eliminated, it can obviously be avoided in the comprehension axioms. By contrast, no such elimination is available in the "no injective function" version, which uses a primitive function g from pluralities to objects.

Second, notice that all the plural versions of Cantor's theorem are negative existential claims to the effect that there isn't a function or relation that would establish that there are no more pluralities on a domain than objects in the domain. The strength of a negative existential claim obviously depends on the domain: the larger the pool of possible counterexamples, the stronger the negative existential. Compare the results recorded in the middle and right-hand columns of our table. The results in the middle column state that there isn't a counterexample in the large pool of all relations of the form $R(x, yy)$. By contrast, the results in the right-hand column state that there isn't a counterexample in what might prove to be a smaller pool of such relations *that can be coded by means of pluralities and ordered pairs alone*.

To investigate this possibility, let us compare the two pools of relations. Suppose that only predicative plural comprehension is accepted. Then there is no guarantee that every functional and surjective relation of objects to pluralities can be coded by means of a plurality of ordered pairs. To see this, consider a relation $R(x, yy)$ of the mentioned sort. If we had impredicative plural comprehension, we could establish that this relation is coded by means of the plurality of ordered pairs $\langle x, y\rangle$ defined by the impredicative condition $\exists yy(R(x, yy) \wedge y \prec yy)$. Without impredicative plural comprehension, however, this strategy for coding relations by means of pluralities of ordered pairs is unavailable and the two pools of relations will therefore differ in size. In fact, when only predicative comprehension is accepted, we cannot prove in general that all relations of the relevant type can be coded by means of a plurality of ordered pairs.[17]

Equipped with this observation, let us return to the difference between the middle and right-hand columns. We can now better understand the source of the difference. We found that, without impredicative plural comprehension, the middle column is concerned with a strictly larger pool of possible counterexamples than the right-hand column, namely the pool of all relations of the relevant type, not just those that can be coded by means of a plurality of ordered pairs. And it stands to reason that strictly stronger assumptions are needed to prove a negative existential claim when this claim is concerned with a strictly larger pool of possible counterexamples.

[17] The model construction described in footnote 14 on p. 49 provides an example of a relation that cannot be coded in this way: let $R(x, yy)$ be defined by $x = \{yy\}$.

II

COMPARISONS

4
Plurals and Set Theory

What is the relation between some things and their set? This is a hard question which has confounded many brilliant minds. We recall, for example, that Russell wrestled with the question:

> Is a class which has many terms to be regarded as itself one or many? Taking the class as equivalent simply to the numerical conjunction "*A* and *B* and *C* and etc.," it seems plain that it is many; yet it is quite necessary that we should be able to count classes as one each, and we do habitually speak of a class. Thus classes would seem to be one in one sense and many in another.
>
> (Russell 1903, Section 74)

We begin with a formal comparison between plural logic and set theory, which clarifies an important technical aspect of the question. After that, we address some philosophical issues concerning the relation between some things and their set. Our discussion yields an argument for primitive plurals, which we believe has more force than any of the arguments discussed in the previous chapter. More specifically, we argue that the expressive resources of plurals are needed to account for sets.

4.1 A simple two-sorted set theory

Assume we start with a singular first-order language whose quantifiers range over certain objects. Let us refer to these objects as *individuals*. We are interested in ways to talk simultaneously about many individuals.

The most familiar option, at least to anyone with some training in mathematics, is to use set theory. A set is a *single object* that has zero or more *elements*. Talking about a single set thus provides a way to talk about all of its elements simultaneously. For example, we can convey information about two individuals, say Russell and Whitehead, by talking about their

The Many and the One: A Philosophical Study of Plural Logic. Salvatore Florio and Øystein Linnebo,
Oxford University Press.
DOI: 10.1093/oso/9780198791522.003.0004

set {Russell, Whitehead}. The information that they are philosophers can be conveyed by saying that every element of the set is a philosopher. Similarly, we can convey information about the natural numbers by talking about their set. The information that they are infinitely many can be conveyed by saying that their set is infinite. Suppose, more generally, that we want to talk about some objects. According to the present strategy, we can achieve this by talking about an associated set.

It is not obvious, however, that such a set exists. After all, the lesson of the set-theoretic paradoxes is that not every formula defines a set. The most famous example is Russell's paradox of the set of all sets that are not elements of themselves. Consider the formula that serves as a condition for membership in this would-be set: $x \notin x$. Suppose this formula defines a set R. Now ask: is R an element of itself? The answer is affirmative if and only if R satisfies the membership condition. In other words: $R \in R$ if and only if $R \notin R$. But this is a contradiction!

Thankfully, the problem posed by the set-theoretic paradoxes can be put off, at least for a while. The paradoxes do not arise when we consider only sets of individuals drawn from a fixed first-order domain. And for present purposes, this is all we need. So let us consider a very simple set theory, which satisfies our present needs but does not give rise to paradoxes.

We distinguish between individuals and sets of individuals. To do so, it is convenient to use a *two-sorted* language. Such languages are easily explained because they are implicit in various mathematical practices. For example, in geometry we often use one set of variables to range over points (say, $p_1, p_2, \dots$) and another set of variables to range over lines (say, $l_1, l_2, \dots$). We adopt a similar approach in our simple set theory, letting lower-case variables range over individuals ($x, y, \dots$) and upper-case variables ($X, Y, \dots$) range over sets of individuals. We refer to these as *individual variables* and *set variables*, respectively. If desired, we can of course add constants of either sort. There are sortal restrictions on the formation rules. For instance, the language has a membership predicate '$\in$' whose first argument can only be an individual term and whose second argument can only be a set term. Thus, '$a \in X$' means that the individual a is an element of the set X. In addition to the ordinary identity predicate, which can be flanked by any two individual terms, our extended language contains a set identity predicate, which can be flanked only by set terms. For convenience, we use the standard identity sign for both identity predicates. Given the restrictions just mentioned, it is impermissible to make identity claims involving both an individual and a set term (such as '$a = X$').

This two-sorted language, which we call $\mathcal{L}_{\text{SST}}$, will be the language of our simple set theory, SST. Let $\mathcal{L}^{+}_{\text{SST}}$ be the extended language obtained by adding predicates that take set terms as arguments. This is an optional extra to which we will return.

We now formulate SST based on the axioms and rules of two-sorted classical logic. First, we adopt the axiom of extensionality for sets:

(S-Ext) $$\forall x(x \in X \leftrightarrow x \in Y) \rightarrow X = Y$$

Then, we adopt an axiom scheme of set comprehension:

(S-Comp) $$\exists X \forall x(x \in X \leftrightarrow \varphi(x))$$

where X does not occur free in $\varphi(x)$. The theory SST+ is obtained by adapting the rules and axioms of SST to the richer language $\mathcal{L}^{+}_{\text{SST}}$.

Notice how Russell's paradox is blocked by the use of separate sorts for individuals and their sets. In our two-sorted language, the membership condition for the offending set, namely $x \notin x$ is not even well formed.

4.2 Plural logic and the simple set theory compared

Let us compare how plural logic and the simple set theory talk about the many. Consider a domain of individuals to which both systems are applicable. (We will later address the important question of what, exactly, the conditions are under which each system is applicable.) Suppose we wish to talk about many individuals simultaneously. As we will now show, these two ways to talk about the many share a common structure.

The two languages share a common stock of variables x_i that take as their values one individual at a time. And each language has an additional stock of variables that are used to convey information about (loosely speaking) collections of individuals: plural variables xx_i, which take as their values many individuals simultaneously, or set variables X_i, which take as their values a single set of individuals. In addition, each language has a predicate for membership in a collection: $x_i \prec xx_j$ for "x_i is one of xx_j" or $x_i \in X_j$ for "x_i is an element of X_j".

This suggests that it should be straightforward to translate back and forth between the two languages. One can simply replace $\prec$ with $\in$ and xx_j with

X_j, and *vice versa*. In fact, things are *nearly* that simple. There are just two wrinkles to be ironed out:

- $\mathcal{L}_{\text{SST}}$ has an identity predicate that can be flanked by set terms, whereas $\mathcal{L}_{\text{PFO}}$ has no identity predicate that can be flanked by plural terms.
- SST postulates an empty set, whereas PFO has an axiom stating that every plurality is non-empty.

Fortunately, both problems are easily overcome. In Appendix 4.A, we show how to define a translation from each language to the other such that each sentence and its translation convey the same information, at least as far as the individuals are concerned, only that one sentence does so by utilizing plural resources, while the other uses set-theoretic resources.

As we explain in Appendices 4.A and 4.B, the translations satisfy the following important conditions:

(i) each translation is recursive, that is, there is an effective algorithm for carrying out the translation;
(ii) each translation commutes with the logical connectives (for example, the translation of a negation $\neg\varphi$ is the negation of the translation of φ);
(iii) every theorem of each of the two theories is translated as a theorem of the other theory (for example, every theorem of PFO is translated as a theorem of SST).

More generally, let τ be a translation from the language of one theory T_1 to that of another theory T_2 such that these three conditions are satisfied. Then, τ is said to be an *interpretation* of T_1 in T_2. Thus, what we show in the appendices is that each of our two theories PFO and SST can be interpreted in the other, and likewise with PFO+ and SST+.

It is important to be absolutely clear about what the mutual interpretability of two theories does and does not establish. Interpretability is a purely formal notion, which also allows us to recursively turn a model of one theory into a model of another. Thus, two mutually interpretable theories are equivalent for the purposes of formal logic. However, there is no guarantee that the equivalence will extend beyond those purposes.

To see this, suppose the two languages are meaningful. Then, there is no guarantee that the translation preserves the kinds of extra-logical properties that philosophers commonly discuss. For example, the translation need not preserve features of sentences such as:

- truth value;
- meaning (perhaps understood as the set of possible worlds at which a sentence is true);
- epistemic status (for example, a priori or a posteriori);
- ontological commitments.

It is often controversial whether a translation preserves these features. The translations presented in this chapter are no exception. Consider a nominalist who accepts a certain plural sentence but rejects its set-theoretic translation. This provides a perspective from which the translation does not preserve truth value and hence meaning.

4.3 Plural logic vs. set theory: classifying the options

What is the significance of the shared structure (or mutual interpretability) that we just observed? Is this *merely* a technical result? Or does the technical result have some broader philosophical significance?

When the structure of one theory can be recovered within that of another, this raises the question of whether one of the theories can be eliminated in favor of the other. In the present context, there are three options. First, one may eliminate pluralities in favor of sets. Second, one may proceed in the opposite direction and eliminate sets in favor of pluralities. Finally, one may refrain from any elimination and retain both pluralities and sets. All three options have their defenders.

First, some philosophers hold that the plural locutions found in English and many other natural languages should be eliminated in favor of talk about sets. We mentioned in Chapter 2 that Quine is an advocate of this view; see also Resnik 1988. For Quine at least, this is at root a claim about regimentation into our scientific language. It is indisputable that many natural languages contain plural locutions. But our best scientific theory of the world has no need for such locutions. This theory is to be formulated in a singular language—that is, a language lacking plural resources—whose quantifiers also range over sets. When regimenting natural language into this scientific language, the plural locutions of the former should be analyzed by means of the set talk of the latter. In short, for scientific purposes, we should eschew plural resources and instead rely on set-theoretic resources. These resources also suffice to interpret "the vulgar" (as Quine once put it), that is, to regiment the plural resources indisputably found in English and other natural languages.

Second, other philosophers insist that sets should be eliminated in favor of pluralities. That is, we can and should interpret ordinary set talk without relying on set-theoretic resources ourselves. A classic paper by Black (1971) can be read as advocating this view.[1] More recently, Oliver and Smiley have expressed considerable sympathy for the view, claiming to have at least shifted the burden of proof onto its opponents (2016, 316–17).

Lastly, one may hold that neither system should be eliminated in favor of the other, because both plural logic and set theory are legitimate and earn their keep in our best scientific theory. Following Cantor and Gödel, this is the view that we will defend. Suppose we are right that both systems should be retained. Then a host of questions arise concerning their relation. We will be particularly concerned with two such questions.

(a) Every non-empty set obviously corresponds to a plurality, namely the elements of the set. What about the other direction? Does every plurality correspond to a set? If not, under what conditions do some things form a set?[2]
(b) Suppose that some objects form a set. Can these objects be used to shed light on, or give an account of, the set that they form?

Before addressing these two questions, let us explain why we reject both the elimination of pluralities in favor of sets and that of sets in favor of pluralities.

4.4 Against the elimination of pluralities in favor of sets

One reason against the elimination of pluralities in favor of sets is the paradox of plurality, discussed in Section 3.4. The paradox arises in untyped approaches to sets, where sets are regarded as objects alongside others. Ordinary set theory is such an approach, unlike our system SST. The argument

[1] Rafal Urbaniak (2013) has argued that Leśniewski can be read in the same way. This reading is disputed by Oliver and Smiley who take Leśniewski to be "an orthodox singularist" about plurals (2016, 15).
[2] See Hewitt 2015 for a useful overview of this issue.

begins, we recall, by observing that the following sentence seems trivially true:

(4.1) There are some objects such that any object is one of them if and only if that object is not an element of itself.

Suppose that plural resources are to be eliminated in favor of set-theoretic ones. Then it is natural to regiment (4.1) as follows:

(4.2) There is a set of which any object is an element if and only if that object is not an element of itself.

In symbols:

$$\exists x\big(\text{set}(x) \land \forall y(y \in x \leftrightarrow y \notin y)\big) \tag{4.3}$$

This, of course, is an instance of the familiar Russell sentence, which is inconsistent.

While this is a powerful argument, we saw that several responses are possible. Quine might try to dismiss the plural talk about sets in (4.1) as just confused talk about sets in two different guises and as having no place in the ideal language of science. This is a logically coherent view for him to take. However, this blunt dismissal of "the vulgar" is ultimately hard to sustain. We find it difficult to deny that English speakers do understand plural talk about sets. A charitable interpretation of "the vulgar" should not deny this fact.

A more promising option is to deny the possibility of absolutely general quantification. If absolute generality is unattainable, then the door is open to claiming that (4.2) is true but that the witness to the existence claim lies beyond the range of the embedded universal quantifier ('$\forall y$' in the formalization), with the result that paradox is averted.[3]

Yet another option is to deny that (4.1) is true. Of course, (4.1) is just an instance of plural comprehension. But perhaps plural comprehension isn't always permissible! Any plurality must presumably be properly circumscribed. So when we are reasoning about a domain that cannot be circumscribed—such as the domain of absolutely everything—not every condition can be used to define a plurality.

[3] See discussion in Section 3.4.

We won't attempt to resolve the matter here. The last two responses to the paradox of plurality raise big questions that we discuss in the final two chapters. Instead, we wish to lay out another—and, we believe, more compelling—reason why pluralities should not be eliminated in favor of sets. The reason is simply that pluralities are needed to give an account of sets.[4] So if pluralities were eliminated in favor of sets, we could not use plural reasoning to give such an account. In sum, to retain an attractive account of sets in terms of pluralities, we cannot eliminate plurals.

What is the promised account of sets in terms of pluralities? It is useful to recall how Cantor, the father of modern set theory, sought to explain the concept of set.

> By a 'manifold' or 'set' I understand in general every many which can be thought of as one, i.e. every totality of determinate elements which can be bound together into a whole through a law [. . .].
>
> (Cantor 1883, 43; our translation)[5]

That is, a set is a "many thought of as one". Of course, it is far from clear how this is to be understood. (An explication will be proposed shortly.) But there can be no doubt that Cantor sought to understand a set in terms of the many objects that are its elements and that are somehow "thought of as one".

> By a 'set' we understand every collection into a whole M of determinate, well-distinguished objects m of our intuition or our thought (which will be called the 'elements' of M). We write this as: $M = \{m\}$.
>
> (Cantor 1895, 481; our translation)[6]

It is tempting to read Cantor's variable 'm' as a plural variable (see also Oliver and Smiley 2016, 4–5). So, in line with our notation, let us replace

[4] One might attempt to deny the need for such an account by adopting a more structuralist conception of set, where a set is characterized in terms of its structural relations to all other sets rather than in terms of some particularly intimate relation to its elements. See Parsons 2008, Chapter 4, for useful discussion. However, we insist that there is also a more ontological conception of set, especially in the case of hereditarily finite sets, which regards a set as "constituted" by its elements. In fact, such a conception is suggested by a liberal view of definitions, to be described shortly.

[5] The original reads: 'Unter einer Mannichfaltigkeit oder Menge verstehe ich nämlich allgemein jedes Viele, welches sich als Eines denken lässt, d.h. jeden Inbegriff bestimmter Elemente, welcher durch ein Gesetz zu einem Ganzen verbunden werden kann [. . .].'

[6] The original reads: 'Unter einer Menge verstehen wir jede Zusammenfassung M von bestimmten wohlunterschiedenen Objekten m unsrer Anschauung oder unseres Denkens (welche die Elemente von M genannt werden) zu einem Ganzen. In Zeichen drücken wir dies so aus: $M = \{m\}$.'

this variable with 'mm'. A set M is then said to be a collection into one of some well-distinguished objects mm, namely the elements of M. And we write $M = \{mm\}$.

What about the empty set? Here there is a threat of a mismatch. While standard set theory accepts an empty set, traditional plural logic does not accept an empty plurality. But we are confident that this threat can be addressed. One option is to break with traditional plural logic and accept an empty plurality, perhaps on the grounds that, although this isn't how plurals work in English and many other natural languages, there are coherent languages where plurals do behave in this way (see Burgess and Rosen 1997, 154–5). Another option is to break with standard set theory and abandon the empty set. However, we would prefer not to deviate from successful scientific practice, in this case set theory, unless there are compelling reasons to do so. Finally, an elegant option proposed (in a different context) by Oliver and Smiley (2016, 88) is to allow "co-partial functions", that is, functions that can have a value even where the argument is undefined. Suppose the 'set of' operation $xx \mapsto \{xx\}$ is such a function. Then, applied to an undefined argument, this function can have the empty set as its value.

What is it for many objects to be "thought of as one" or collected "into a whole"? Let us attempt to shed some light on this idea. Many philosophers and mathematicians believe that the elements of a set are somehow "prior to" the set itself and that the set is somehow "constituted" by its elements.[7] Assume xx form a set $\{xx\}$. Then the objects xx can be used to give an account of $\{xx\}$. That is, properties and relations involving the set are explained in terms of properties and relations involving the plurality of its elements. Why is a an element of $\{xx\}$? An answer immediately suggests itself: because a is one of xx. Why is $\{xx\}$ identical with $\{yy\}$? Again, the answer seems obvious: because xx are the very same objects as yy.[8]

Of course, in their current form, these remarks are highly programmatic. The promised account needs to be spelled out. We do this in Section 12.3 by defending a liberal view of definitions. Here is the rough idea behind the view: it suffices for a mathematical object to exist that an adequate definition of it can be provided. The adequacy in question is understood as follows. Suppose we begin with a "properly circumscribed" domain of

[7] See, e.g., Parsons 1977a and Fine 1991.

[8] This account contrasts with some earlier contributions to the metaphysics of sets, e.g. Lewis 1991 and Oliver and Smiley 2016 (Chapter 14). We believe our account coheres better with the remarks by Cantor (discussed above) and by Gödel (discussed in Section 4.6).

objects standing in certain relations.[9] We would like to define one or more additional objects. Suppose our definition determines the truth of any atomic statement concerned with the desired "new" objects by means of some statement concerned solely with the "old" objects with which we began. Then, according to the liberal view, the definition is permissible.

To illustrate the point, let us apply the view to the case of sets. Suppose we begin with some properly circumscribed domain of objects. For every plurality of objects xx from this domain, we postulate their set $\{xx\}$, with the understanding that atomic statements concerned with any new sets should be assessed in the following way.

(i) $\{xx\} = \{yy\}$ if and only if $xx \approx yy$.
(ii) $a \in \{xx\}$ if and only if $a \prec xx$.

Notice how this account determines the truth of any atomic statement concerned with the "new" sets solely in terms of the "old" objects with which we began, as required by the liberal view.[10]

We also observe that this account distinguishes a set from its singleton, as is customary in comtemporary set theory. By (i), we have $\{xx\} = \{\{xx\}\}$ just in case xx is coextensive with $\{xx\}$. We contend that this coextensionality claim is false. Suppose xx are two or more in number. Then cardinality considerations alone ensure its falsity. Alternatively, suppose xx consist of a single object a. Then the coextensionality claim is equivalent to $a = \{a\}$, which is false because a is an element of its own singleton but not, we may suppose, of itself.

To sum up, we argue that pluralities should be retained alongside sets, so that the former can be used to shed light on the latter. This account of sets draws essentially on our liberal view of definitions.

4.5 Against the elimination of sets in favor of pluralities

The view discussed in the preceding section retains sets but gives an account of them in terms of pluralities. One may wonder whether a more radical approach is possible. Why not simply eliminate sets in favor of pluralities?

[9] In Part IV of the book, the notion of being properly circumscribed will play an important role and will be analyzed under the label of *extensional definiteness*.

[10] Mereological sums provide another example of this liberal view of definitions; see Section 5.8.

Black's (1971) classic discussion suggests a view of this sort.[11] He observes that ordinary language often talks about sets: expressions such as 'my set of chessmen' or 'that set of books' feel fairly natural to English speakers. By reflecting on ordinary uses of the word 'set', he argues, we can come to see the intimate connection between talk about a set and about its elements. More specifically, we can come to realize that basic uses of the word 'set' are simply substitutes for plural expressions such as plural descriptions or lists of terms. In his example, the sentence 'a certain set of men is running for office' is what he calls an "indefinite surrogate" for the statement that, say, Tom, Dick, and Harry are running for office (Black 1971, 631).

Black recognizes that there is a gap between ordinary uses of the word 'set' and its uses in mathematics. For instance, ordinary speakers untrained in abstract mathematics often have misgivings about the empty set. If sets are collections of things, how can there be a collection of nothing whatsoever? Despite such misgivings, Black contends that we can rely on our ordinary understanding of plurals to make sense of "idealized" uses of the word 'set' as it occurs in mathematics.

There is an obvious difficulty for Black's contention. Talk of *sets of sets* is ubiquitous in mathematics and, as we will see shortly, such "nested" sets are essential to the now-dominant iterative conception of set. How can we account for these uses of the word 'set'? If talk about sets is shorthand for talk about pluralities, then sets of sets would seem to correspond to higher-level pluralities, that is, "pluralities of pluralities".[12]

It is controversial whether such higher-level pluralities make sense, but a putative example is given in following sentence.

(4.4) My children, your children, and her children competed against each other.

The subject of this sentence appears to be a "nested" plural, that is, a plural expression formed by combining three other plural expressions. Arguably, this nesting of the subject is semantically significant. The claim is not merely that all the children in question compete against each other but that they do so in teams, each team comprising the children of each parent. We return to the question of whether there are higher-level pluralities in Chapter 9.

[11] We should note that this is not the only way to read Black. It is not entirely clear whether he proposes an eliminative reduction or favors some form of non-eliminative reductionism. An eliminative proposal is developed by Hossack (2000), who appeals to plurals and plural properties to eliminate sets.

[12] For proposals along these lines, see Simons 2016 and Oliver and Smiley 2016, Section 15.1.

While the availability of higher-level pluralities is a necessary condition for the envisaged elimination of sets, it is not sufficient. As observed, the language of mathematics talks extensively about sets and appears to treat these as objects. If possible, it would be good to take this language at face value. The account of sets in terms of pluralities outlined in the previous section allows us to do just that. This provides a reason to retain sets even if higher-level pluralities are available. The reason is even stronger for those who accept other mathematical objects such as numbers. If numbers are accepted, why not also accept sets?

4.6 The iterative conception of set

Suppose we retain both pluralities and sets, giving up on any attempt to eliminate one in favor of the other. How, then, to account for nested sets? This means going beyond the simple set theory discussed in Section 4.1 to form a stronger set theory, where the threat of paradox re-emerges. The standard response to this threat is the so-called iterative conception of set. One of the first clear expressions of this conception is given in a famous passage by Gödel.[13]

> The concept of set, however, according to which a set is anything obtainable from the integers (or some other well-defined objects) by iterated application of the operation "set of", and not something obtained by dividing the totality of all existing things into two categories, has never led to any antinomy whatsoever; that is, the perfectly "naive" and uncritical working with this concept of set has so far proved completely self-consistent.
>
> (Gödel 1964, 180)

The passage calls for some explanation. First, Gödel distinguishes the iterative conception of set from a problematic conception based on the idea of "dividing the totality of all existing things into two categories." Consider a condition that any object may or may not satisfy. One might then attempt to use this condition to divide the totality of all objects into two sets: the set of objects that satisfy the condition and the set of those that don't. But this approach to sets is problematic: as we have seen, it gives rise to Russell's paradox.

[13] The passage contains some footnotes, which we elide.

By contrast, the iterative conception starts with the integers or "some other well-defined objects". We are then told to consider iterated applications of the operation "set of". An example will help. Suppose we start, at what we may call stage 0, with two objects, say a and b. The "set of" operation can be applied to any plurality of objects available at stage 0 to form their set. Thus, at stage 1, which results from the application of this operation to the objects available at stage 0, we have the following sets: $\varnothing$, $\{a\}$, $\{b\}$, and $\{a, b\}$. So, at stage 1, we have six objects, namely a and b together with four sets that were not available at stage 0. Now we can apply the "set of" operation again, this time to the objects available at stage 1. This yields sets such as $\{\varnothing, a\}$, $\{\{a\}, \{b\}\}$, and many others. Note that, by this procedure, the objects available at any given stage form a set at the next stage.

There is a more systematic way to describe what takes us from one stage α to the next stage $\alpha + 1$. For any set S, let its powerset, $\wp(S)$, be the set of all subsets of S, that is:

$$\wp(S) = \{x : x \subseteq S\}$$

Suppose the objects available at stage α are the elements of V_α. Then at stage $\alpha + 1$ we form all the subsets of V_α. So, at stage $\alpha + 1$, we have the elements of V_α as well as those of $\wp(V_\alpha)$. In symbols: $V_{\alpha+1} = V_\alpha \cup \wp(V_\alpha)$. Again, we have by this procedure that all the sets available at stage α, taken together, form a set at stage $\alpha + 1$.

In fact, we want to consider really long iterations of the "set of" operation. The first step is to define V_ω as the result of continuing in this way as many times as there are natural numbers. We do this by letting V_ω be the union of all of the collections V_n generated at a finite stage: $V_\omega = \bigcup_{n<\omega} V_n$. More generally, for any limit ordinal λ, we let V_λ be the union of all the collections of sets we have generated: $V_\lambda = \bigcup_{\gamma<\lambda} V_\gamma$.

The *cumulative hierarchy of sets*, V, is the union of all of the V_α. As Gödel observes (in a footnote to the passage quoted above), V isn't a set. There is no stage at which all sets are available to form a universal set. For any stage, there is a later stage containing even more sets. As a result, we ban the universal set and any other set that would lead to paradox.

Of course, this raises the question of the status of the cumulative hierarchy itself, including the question of whether "it" even exists as an object. We will encounter one appealing response to this question in Section 4.8: perhaps we can invoke plurals and simply regard the cumulative hierarchy as *all the sets* that are formed in the construction described above.

4.7 Zermelo-Fraenkel set theory

The iterative conception motivates much of today's standard set theory, Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC), which is adequate for nearly all of ordinary mathematics. This is a theory of pure sets, formulated in a one-sorted language with only one non-logical predicate, '$\in$' for membership. All other set-theoretic notions are defined in terms of this single predicate. The axioms are as follows.

Extensionality: Coextensive sets are identical. That is:

$$\forall u(u \in x \leftrightarrow u \in y) \rightarrow x = y$$

Empty set: There is an empty set. That is:

$$\exists x \forall y\ y \notin x$$

Pairing: Every two objects have a pair set. That is:

$$\forall x \forall y \exists z \forall u(u \in z \leftrightarrow u = x \vee u = y)$$

Union: For every set x, there is a set y whose elements are precisely those objects that are an element of some element of x. That is:

$$\forall x \exists y \forall u[u \in y \leftrightarrow \exists z(u \in z \wedge z \in x)]$$

Powerset: Every set has a powerset. That is:

$$\forall x \exists y \forall u(u \in y \leftrightarrow u \subseteq x)$$

Infinity: There is an infinite set, that is, a set with $\varnothing$ as an element and such that, whenever y is an element, so too is $y \cup \{y\}$. That is:

$$\exists x[\varnothing \in x \wedge \forall y(y \in x \rightarrow y \cup \{y\} \in x)]$$

Separation: For any set x and any condition φ, there is a set of precisely those elements of x that satisfy φ. That is:

$$\forall x \exists y \forall u(u \in y \leftrightarrow u \in x \wedge \varphi)^{14}$$

[14] This is an axiom *scheme*, which yields an axiom for each φ. The same goes for Replacement, stated below.

Foundation: Every non-empty set x has an element that is disjoint from x. That is:

$$\forall x(x \neq \varnothing \to \exists y(y \in x \wedge x \cap y = \varnothing))$$

Replacement: For every set x and functional condition ψ, there is a set of precisely those objects that are borne ψ by some element of x.[15] That is:

$$Func(\psi) \to \forall x \exists y \forall u[u \in y \leftrightarrow \exists z(z \in x \wedge \psi(z, u))]$$

This axiom is based on a simple and intuitive idea. Consider any set. For each of its elements, choose either to keep this element or to replace it with some other object. Then the resulting collection is also a set.

Choice: Every set x of non-empty disjoint sets has a choice set, that is, a set containing precisely one element of each element of x. An example due to Russell might be useful to understand the Axiom of Choice. Suppose you have infinitely many pairs of shoes. Then it is easy to define a set containing precisely one member of each pair, namely the set of left shoes. What if you have infinitely many pairs of socks where the two members of each pair are indistinguishable? Then we are unable to define a set containing precisely one member of each pair. The Axiom of Choice tells us that such a set exists, irrespective of our ability to define it.

As observed, ZFC is a theory of pure sets. It is easy, however, to modify the theory to make room for urelements, that is, objects that aren't sets. To do so, we first add to the language a predicate S for being a set. Using this predicate, we then formalize the axioms so as to match precisely their informal statements provided above. For example, the axiom of Extensionality is rewritten as:

$$\forall x \forall y[S(x) \wedge S(y) \ \to \ (\forall u(u \in x \leftrightarrow u \in y) \to x = y)]$$

The modified system is often known as ZFCU.

The iterative conception motivates many of the axioms of ZFC or ZFCU. The Powerset axiom provides a nice example. Suppose x is available at some stage s. Then the elements of x were available before s. Hence each subset of x is also available at s. Thus, the set of all these subsets is available at the stage immediately after s. We need not here take a stand on precisely which axioms

[15] The condition ψ is *functional* just in case, for every x there is a unique y such that $\psi(x, y)$.

of set theory are motivated by the iterative conception. Gödel appears to have taken the answer to be "all of them"; others disagree.[16]

4.8 Proper classes as pluralities

Let us use the word 'collection' in an informal way for anything that has a membership structure, such as a set, class, plurality, or indeed even a Fregean concept (where the relation between instance and concept is regarded as a membership structure). We often wish to talk about collections that are too large to form sets, such as the entire cumulative hierarchy of sets or all the ordinals. We will now explain the apparent need for such collections, why these are sometimes regarded as problematic, and finally a brilliant proposal due to Boolos, namely that plural logic provides a way to make sense of these collections.

Let us begin with the need for a novel type of collection, in addition to sets. There are several reasons for this need. Boolos mentions two. First, collections are needed to make sense of the cumulative hierarchy V, which is the domain of set theory. For example, we would like to say that V is the subject matter of set theory and that V is well founded.

Second, collections are needed to understand and justify two axiom schemes that are part of ZFC, namely Replacement and Separation.[17] Both of these take the form of an infinite family of axioms. Consider Separation. ZFC contains an axiom

(Sep) $$\forall z \exists y \forall x (x \in y \leftrightarrow x \in z \wedge \varphi)$$

for each of the infinitely many formulas φ of its language. Behind this infinite lot of axioms lies a single, unified idea that can be expressed by reference to collections.[18] For every collection C and every set x, there is a set y of all those elements of x that belong to C. Suppose we can quantify over collections. Then the infinitely many Separation axioms could be unified as the single axiom:

(C-Sep) $$\forall C \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge z \text{ belongs to } C)$$

[16] On this topic, see e.g. Boolos 1971 and Paseau 2006.
[17] Analogous considerations apply to the arithmetical principle of induction.
[18] See also Kreisel 1967.

In the literature, the desired collections are often known as *classes*, some of which can be shown to be "too big" to be sets. These are called *proper classes*. But what would these proper classes be? Just like sets, they are collections of many objects into one. But why, then, are proper classes not sets? As Boolos (1984b, 442) nicely observes, "[s]et theory is supposed to be a theory about *all* set-like objects".

Adding proper classes to a theory of sets is just like adding yet another layer of sets on top of the sets already recognized. In light of this, why shouldn't the proper classes count as just more sets? William Reinhardt puts the point well:

> [O]ur idea of set comes from the cumulative hierarchy, so if you are going to add a layer at the top it looks like you forgot to finish the hierarchy.[19]

Plural logic seems to provide precisely what we need. A proper class does not have to be *a single object* that somehow collects together many things into one. Instead of referring in a singular way to a proper class, construed as an object, why not simply refer plurally to its many members? In this way, we eliminate singular talk about proper classes in favor of plural talk about their members. For example, the cumulative hierarchy does not have to be an object. It suffices to talk plurally about *all the sets*.

Consider now the axiom scheme of Separation. This can be turned into a single axiom using a plural formulation. Given any objects pp and any set x, there is a set y of precisely those elements of x that are also among pp:

(P-Sep) $$\forall pp \forall x \exists y \forall z (z \in y \leftrightarrow z \in x \wedge z \prec pp)$$

Let us make two final observations. To represent all of the classes that we might be interested in, we would need an unrestricted form of plural comprehension, namely:

(P-Comp) $$\exists x \varphi(x) \rightarrow \exists xx \forall x (x \prec xx \leftrightarrow \varphi(x))$$

Moreover, we need plural logic to be ontologically innocent. If plural variables commit us to new objects, using plurals in the formulation of Separation or Replacement is not essentially different from using proper classes.

[19] Reinhardt 1974, 32. For a useful elaboration of the point, see Maddy 1983, 122.

4.9 Are two applications of plural logic compatible?

We have described two very attractive applications of plural logic: as a way of giving an account of sets, and as a way of obtaining proper classes "for free". Regrettably, it looks like the two applications are incompatible. The first application suggests that any plurality forms a set. Consider any objects xx. Presumably, these are what Gödel calls "well-defined objects". If so, it is permissible to apply the "set of" operation to xx, which yields the corresponding set $\{xx\}$. The second application, however, requires that there be pluralities corresponding to proper classes, which by definition are collections too big to form sets. For example, there must be a plurality of all sets whatsoever to serve as the proper class V. But, when the "set of" operation is applied to this plurality, we obtain a universal set, which is unacceptable.

Is there any way to retain both of the attractive applications of plural logic? To do so, we would have to restrict the domain of application of the "set of" operation so that the operation is *undefined* on the very large pluralities that correspond to proper classes, while it remains defined on smaller pluralities. The obvious concern is that this restriction would be *ad hoc*. The operation does apply to vast infinite pluralities, thus forming large sets in the cumulative hierarchy. But once we allow that these infinite pluralities form sets, why are other infinite pluralities suddenly too large to do so?

To respond to this challenge, we might seek inspiration from Gödel, who points to a restriction when he requires that the "set of" operation be applied to "well-defined objects". How should this restriction be understood? One option is to understand Gödel as requiring that the objects in question be *properly circumscribed*. Perhaps a collection corresponding to a proper class fails to satisfy this requirement. We explore this idea in Chapter 12 and find that there are indeed "collections" that fail to be properly circumscribed. However, we also argue that every plurality is (in the appropriate sense) properly circumscribed and can thus figure as an argument of the "set of" operation. Thus, if our argument succeeds, the two applications of plurals remain incompatible, and we must choose between them. We recommend retaining the first application of using plurals to give an account of sets, while looking elsewhere for an interpretation of talk about proper classes that aren't properly circumscribed and therefore cannot figure as arguments of the "set of" operation. A natural option is to look to second-order logic. We discuss this in Section 12.8.

Appendices

4.A Defining the translations

We wish to define a translation τ from the language of our simple set theory SST to that of plural logic. The central idea is obvious: let us replace talk about a set with talk about the objects that are elements of the set. Thus, instead of saying that x_i is an element of the set X_j, we say that x_i is one of xx_j. So we adopt:

$$\tau(x_i \in X_j) = x_i \prec xx_j$$

Identity statements involving set terms are translated as the corresponding plural coextensionality statements. For example, '$X_i = X_j$' is translated as:

$$\forall x_0(x_0 \prec xx_i \leftrightarrow x_0 \prec xx_j)$$

Atomic predications concerning individuals are left unchanged by the translation. Next, the translation commutes with the logical connectives. For example, the translation of a negated formula is the negation of the translation of the formula:

$$\tau(\neg\varphi) = \neg\tau(\varphi)$$

Finally, we need to translate existentially quantified formulas. (For simplicity, we may treat universal quantifiers as abbreviations in the usual way.) The individual existential quantifier poses no problem: here too we let the translation commute with the logical operator.

The set existential quantifier is slightly harder. Suppose we let the translation commute, setting '$\tau(\exists X_j\, \varphi)$' to be '$\exists xx_j\, \tau(\varphi)$'. This does not quite work. For we want to have an empty set but no empty plurality. Boolos (1984b, 444) proposes a trick to iron out this wrinkle. Let τ translate '$\exists X_j \varphi$' as

$$\exists xx_j \tau(\varphi) \vee \tau(\varphi') \tag{4.5}$$

where φ' is the result of substituting '$x_i \neq x_i$' everywhere for '$x_i \in X_j$'. The second disjunct simulates an expansion of the range of quantification, thus accommodating the possibility that a set is empty. (To see how this works, suppose X_j is empty. Then '$x_i \in X_j$' always has the same truth value as '$x_i \neq x_i$', namely *false*.) By induction on formal derivations, one can easily prove that each theorem of SST is mapped to some theorem of PFO.

It is easy to define a "reverse" translation that maps formulas of the language of plural logic to formulas of our two-sorted set-theoretic language. As expected, one can prove that this translation maps theorems of the former to theorems of the latter. So we can translate in both directions between PFO and SST while preserving theoremhood. Analogous results can be obtained for PFO+ and SST+.

4.B Defining the interpretation

The two translations we have just encountered illustrate an important general notion, which will provide a useful conceptual tool in subsequent discussions. So let us make explicit the relevant properties of the translations.

Suppose we are comparing two theories, T_1 and T_2, which are formulated in two multi-sorted languages $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively. (Note that all the formal languages we consider in this book can be viewed as languages of this kind.) And suppose we have specified a translation τ from $\mathcal{L}_1$ to $\mathcal{L}_2$ such that

(i) τ is recursive, that is, there is an effective algorithm that specifies how to translate any given formula of $\mathcal{L}_1$;
(ii) τ commutes with the logical connectives (for instance, $\tau(\neg\phi) = \neg\tau(\phi)$);
(iii) τ maps every theorem of T_1 to a theorem of T_2.

We wish to make some remarks about the translation of quantified formulas. First, the translation should permit a change in the type of variables. In particular, we sometimes want to map plural variables to set variables and vice versa. Second, a quantified formula is usually translated as a restricted quantification:

$$\exists v\, \phi \overset{\tau}{\longmapsto} \exists u(\theta(u) \wedge \tau(\phi))$$

But this requirement is unnecessary. In fact, to accommodate Boolos's trick, which simulates an expansion of the range of a quantifier, we must refrain from requiring that every quantified formula be translated in this way.

A translation that satisfies these three properties is said to provide *an interpretation of* T_1 *in* T_2. When there are such translations in both directions—as in the examples mentioned in the previous section—the two theories are said to be *mutually interpretable*.

As just defined, the notion of interpretability is entirely proof-theoretic: it is concerned with syntax, not semantics. However, by the soundness of the proof systems we use for our logic, the notion has a semantic upshot as well. Suppose τ is an interpretation of T_1 in T_2. Then any model of T_2 allows us to define, in a recursive manner, a model of T_1. The basic idea is simply to interpret each predicate of $\mathcal{L}_1$ in accordance with its τ-translation into $\mathcal{L}_2$ and to let the domain(s) of $\mathcal{L}_1$ be interpreted in accordance with how its quantifiers are translated into $\mathcal{L}_2$.

5
Plurals and Mereology

In the previous chapters, we discussed two ways of conveying information about many objects simultaneously. The first uses primitive plurals, while the second uses sets. We now examine a third alternative based on mereology.

Mereology is the theory of part-whole relations. Instances of such relations are easy to find. Consider a hydrogen atom that is part of a water molecule, which in turn is part of the contents of a bottle. Mereology aims to capture the general principles governing various relations of parthood. For instance, we may ask whether it follows that the hydrogen atom, in the mentioned example, is part of the contents of the bottle. The intuitive answer is affirmative. This points to a general principle that is assumed to hold of most relations of parthood, namely transitivity.

In this chapter, we present a basic development of mereology and compare it with plural logic. As we will see, the formal relation between these two systems is analogous to that between plural logic and the simple set theory of Section 4.1. This raises questions parallel to those encountered in the preceding chapter. Can we eliminate plurals in favor of mereology? Can we eliminate mereology in favor of plural logic? Or are there reasons to retain both systems?

5.1 Mereology

Let us begin by developing a basic formal framework for mereology. We start with the usual language of first-order logic and expand it with a new primitive predicate '$\leq$' for parthood. So we read '$x \leq y$' as 'x is part of (or equal to) y'. The resulting language is one-sorted: its only variables are the ordinary first-order ones.

The new primitive predicate allows us to define a number of important mereological relations. First, there is proper parthood, which we write

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as '$<$' and define by letting '$x < y$' abbreviate '$x \leq y \wedge y \neq x$'.[1] For example, England is a proper part of the United Kingdom. Next, let us say that x and y *overlap* when they have a common part: $\exists z(z \leq x \wedge z \leq y)$; we symbolize this as '$x \circ y$'. For example, Scandinavia and the European Union overlap, as both have Denmark as a part. Finally, let us say that x and y are *disjoint* when they do not overlap; we symbolize this as '$x \perp y$'. For example, the United Kingdom and Scandinavia are disjoint.

We turn now to the theory of mereology. For our purposes, the most relevant theory is so-called *Classical Extensional Mereology* (sometimes also known as *General Extensional Mereology*). We first adopt axioms stating that $\leq$ is a partial order (that is, $\leq$ is reflexive, transitive, and anti-symmetric).[2] Next, we adopt the axiom of Strong Supplementation, which states that when x is not part of y, there is a part z of x that does not overlap y:

$$x \not\leq y \rightarrow \exists z(z \leq x \wedge z \perp y)$$

For example, since the European Union is not part of Denmark, the former must have a part that is disjoint from the latter. Finally, we adopt an axiom scheme asserting the existence of arbitrary mereological sums (or "fusions", as they are also called). Suppose some object is φ. Then there is an object y that overlaps something z just in case z overlaps an object that is φ; y is said to be *the sum of all objects that are* φ. The existence of arbitrary sums is captured by the following axiom:

(M-Sum) $$\exists x\varphi(x) \rightarrow \exists y\forall z\big(y \circ z \leftrightarrow \exists w(z \circ w \wedge \varphi(w))\big)$$

We often denote sums by means of the familiar '+' symbol; for example, the sum of a and b is written '$a + b$'.[3]

5.2 Can mereology represent the plural?

It is fairly obvious why set theory is an attractive tool for conveying information about many objects simultaneously. Instead of talking about some objects, we can talk about the set whose elements are precisely these objects.

[1] Given antisymmetry (see footnote 2), an equivalent reading of '$x < y$' is '$x \leq y \wedge y \not\leq x$'.
[2] Recall that a relation R is said to be anti-symmetric if and only if $\forall x\forall y(Rxy \wedge Ryx \rightarrow x = y)$.
[3] Formally, y is said to be the sum of a and b if and only if $\forall z(y \circ z \leftrightarrow z \circ a \vee z \circ b)$.

Given this set, we can always retrieve the objects in question, namely as the elements of the set. Above, we described mereology as an alternative to set theory (and primitive plurals) for the purpose of conveying information about many objects simultaneously. It is less obvious how mereology can serve this purpose. When we consider the mereological sum of some objects, we *cannot* in general retrieve the objects with which we started.

For an example of this phenomenon, consider the following two pluralities: Russell and Whitehead, and the molecules of Russell and Whitehead. These are obviously entirely different pluralities. While the former things are two in number, the latter things are far more numerous. Yet the two pluralities appear to have one and the same mereological sum. Indeed, to overlap Russell or Whitehead comes to the same thing as overlapping one of the molecules of these two logicians.

This suggests that the mereological sum of some objects is insufficient to represent these objects. An example by Oliver and Smiley (2001) makes the problem vivid. Consider the following inference:

$$\text{(5.1)} \quad \frac{\text{Russell and Whitehead were logicians}}{\text{The molecules of Russell and Whitehead were logicians}}$$

Because of the distributive predicate 'were logicians', the conclusion is false and hence the argument is invalid. Suppose we want to represent some objects by means of their mereological sum. This representation of the argument seems to yield a different logical verdict. As Oliver and Smiley remark:

> 'Whitehead and Russell' and 'the molecules of Whitehead and Russell' represent different decompositions of the same sum, but giving them that sum as their common reference forces the conclusion that the molecules of Whitehead and Russell were logicians. (2001, 293)

So they conclude that "mereological sums or fusions are ineligible" for the task of representing many objects simultaneously. Similar examples have been put forward by others (see, for example, Rayo 2002, 444–5, and McKay 2006, 42).

In light of these considerations, it may be surprising that mereology is a far more popular tool among linguists interested in plurals than set theory. Suppose we start with some objects. Whatever the merits of set-theoretic representations in general, the set of these objects at least enables us to retrieve the objects in question. By contrast, there appears to be no guarantee that we can retrieve the objects with which we began from their mereological sum. As we just saw, one and the same sum can be obtained by taking the sums of two logicians and of their many molecules.

However, this problem isn't fatal for the project of using mereology to represent many objects simultaneously. To see why, consider three *atomic* particles, say a, b, and c (where by "atomic" we mean that they have no proper parts). To talk about the three atoms simultaneously, we may talk about their sum $a + b + c$. Given this sum, we can retrieve the three particles that jointly compose it: there is a unique way to break this sum down into its three atomic parts.[4] By talking about this single sum, we can therefore convey information about its three atomic parts. For example, the information that the three particles are collinear can be conveyed by saying that there is a line on which each atomic part of the sum lies.

Thus, provided that each of the objects in question is a mereological atom, mereology is a perfectly good tool for talking about all these objects simultaneously. But what if we wish to talk simultaneously about many objects that are not atomic but have proper parts? If we could somehow regard each object as an atom, the use of mereology to represent pluralities would be available more generally. Might this be possible?

A solution, developed and defended by Link (1983, 1998), goes as follows. Even if the objects with which we start have material parts, they can figure as atoms in a different sense: each is, in Link's phrase, an *individual atom*. That is, each object is an atom with respect to a different relation of parthood, namely *individual parthood*. While many complaints against mereological representations of plurals are appropriate for the ordinary notion of parthood, they do not apply to Link's notion.

Let us look at an example. In the material sense, the sum of Russell and Whitehead is the same as the sum of the molecules of Russell and Whitehead. This is not true, however, if the mereological notions are construed according to the relation of individual parthood. In that sense, the sum of Russell and Whitehead is the sum of Russell and Whitehead *conceived as atomic individuals*, that is, taken as atoms in the domain of quantification. This means that, in the individual sense, Russell and Whitehead are the only proper parts of the sum of Russell and Whitehead. It follows that, in the individual sense, the sum of Russell and Whitehead is *not* identical with

[4] Readers may find it an interesting exercise to prove this from the axioms of Classical Extensional Mereology.

the sum of their molecules. The former but not the latter has Russell and Whitehead as its only proper parts.

More generally, individual mereology starts with a domain of individuals that are treated as mereological atoms, ignoring other mereological relations in which those individuals may stand. A mereological structure is then defined on top of that domain. The relation of individual parthood satisfies the axioms of Classical Extensional Mereology. In addition, it satisfies the principle of Atomicity, which states that everything has an atom among its parts. Formally:

(M-Atomicity) $$\forall x \exists y (\mathrm{At}(y) \wedge y \leq x)$$

where '$\mathrm{At}(y)$' abbreviates '$\neg\exists z\, z < y$'. We call the resulting theory *Atomistic Classical Extensional Mereology*.

In fact, mereological sums have some advantages vis-à-vis sets, which have motivated their use in semantics. First, mereological sums are presumably just as concrete as their parts. While the set of Russell and Whitehead is frequently taken to be abstract, the sum of Russell and Whitehead is plausibly taken to be concrete. So, if we want our semantics to assign concrete entities to certain ordinary expressions, this recommends using sums rather than sets for that purpose.

Second, we might want to assign the same semantic value to 'Alice' and 'the objects that are identical with Alice'. Mereology allows this, since the sum of a single object is identical to this very object. By contrast, standard set theory does not allow this kind of identification, since a singleton set is distinct from its sole element. In fact, this problem has occasionally motivated the adoption of a non-standard set theory that allows exactly this kind of identification (Schwarzschild 1996, 1).

The appeal to individual mereology does raise an obvious question, however. Is it *permissible* to invoke mereological notions in the individual sense? The question can be split into two. First, is it even logically coherent to speak in this way? Second, assuming that it is coherent, is this just a manner of speaking or do the described mereological sums really exist?

We defend the claims of logical coherence and existence in Sections 5.3 and 5.8, respectively. Suppose we are right. Then we can assume that each of our initial objects is an individual atom. So we may consider sums of individual atoms. This ensures that pluralities of these initial objects are uniquely represented by the corresponding individual sum.

5.3 One-sorted plural logic

There is no concern about the logical coherence of individual mereology. As we will now show, plural logic can be developed in a way that realizes precisely this structure.

Let us explain. As presented in Chapter 1, plural logic is based on a two-sorted language, since it contains two sets of variables. A singular variable (x, y, ...) ranges over a single object, while a plural variable (xx, yy, ...) ranges over one or more objects. It is also possible to dispense with the singular variables and provide a *one-sorted* version of plural logic. Our recent foray into mereology makes this straightforward: a one-sorted plural logic can be obtained as a mere notational variant of mereology. Instead of the usual singular variables, we use plural variables. And instead of the parthood predicate $\leq$, we use the symbol '$\preccurlyeq$' as a new primitive, though we continue to read '$xx \preccurlyeq yy$' as "xx are among yy". Finally, there is an identity predicate that takes plural arguments. Indeed, in the one-sorted plural language, this is the *only* identity predicate.

What is it for some objects xx to comprise a single object (and in this sense be an individual)? We define 'Ixx' as '$\forall yy(yy \preccurlyeq xx \rightarrow xx \preccurlyeq yy)$'. That is, xx comprise just a single object if and only if xx are contained in each of its "subpluralities", which means that xx has no strictly smaller "subplurality".

How should we axiomatize plural logic in this one-sorted presentation? A straightforward but clumsy option is simply to translate the axiomatization already adopted (see Section 2.4) into the new one-sorted language. For some axioms, the result is not bad. For example, the axiom stating that every plurality is non-empty, $\forall xx \exists y\, y \prec xx$, translates as:

$$\forall xx \exists yy(Iyy \wedge yy \preccurlyeq xx)$$

But the translations of other axioms are needlessly long and unintuitive.[5]

A more elegant option is to exploit the close connection we have observed between one-sorted plural logic and the atomistic version of Classical Extensional Mereology. So let us simply adapt the axioms of the latter to the former. First, we lay down that $\preccurlyeq$ is a partial order and obeys the Strong

[5] Plural comprehension provides a good illustration. This axiom scheme translates as

$$\exists xx(Ixx \wedge \varphi(xx)) \rightarrow \exists xx(\forall yy(Iyy \rightarrow (yy \preccurlyeq xx \leftrightarrow \varphi(yy))))$$

This does not enjoy the immediate plausibility of its two-sorted analogue, namely (P-Comp).

Supplementation principle. Next, we require an analogue of atomicity, that is, that every plurality has a subplurality comprising just a single individual:

(P-Atomicity) $$\forall xx \exists yy (Iyy \wedge yy \preccurlyeq xx)$$

Finally, we require the existence of arbitrary sums. One way to implement this requirement is by adopting a principle to the effect that, for every instantiated condition $\varphi(xx)$, there is a unique smallest plurality zz that includes everything that satisfies the condition. That is, if $\varphi(xx)$ is instantiated, there are zz such that:

(i) zz include every xx that satisfy the condition $\varphi(xx)$:

$$\forall xx(\varphi(xx) \rightarrow xx \preccurlyeq zz)$$

(ii) zz is the smallest plurality verifying requirement (i):

$$\forall ww(\forall xx(\varphi(xx) \rightarrow xx \preccurlyeq ww) \rightarrow zz \preccurlyeq ww)$$

This principle can be given a more compact formalization as follows:

(P-Sum) $$\exists xx \varphi(xx) \rightarrow \exists zz \forall ww \big(\forall xx(\varphi(xx) \rightarrow xx \preccurlyeq ww) \leftrightarrow zz \preccurlyeq ww\big)$$

An alternative way to require the existence of arbitrary sums is by adopting a plural analogue of (M-Sum). As we prove in Appendix 5.B, the two alternatives are in fact equivalent, given background assumptions that are currently in place.

The possibility of a one-sorted approach to plural logic is theoretically important. This approach is just as serious as its more familiar two-sorted cousin about the fact that plural terms can stand for many objects simultaneously. But this insight is represented in two very different ways. On the one-sorted approach, the insight is captured by means of the 'among'-predicate '$\preccurlyeq$'. Its argument places belong to the same sort, a sort that is given a plural interpretation. By contrast, on the two-sorted approach, the insight also has a syntactic manifestation in the sortal distinction between terms representing individual objects and terms representing many objects simultaneously. But clearly, this syntactic manifestation of the distinction between one and many is not obligatory. As we show shortly, many linguists prefer to do without it.

It should be unsurprising, in light of our discussion, that we can translate between the languages of one- and two-sorted plural logic. One direction is straightforward. Apart from the special logical predicate '$\preccurlyeq$', the one-sorted language is a sublanguage of the two-sorted one. So the translation from the one-sorted to the two sorted-language leaves unchanged every formula not containing '$\preccurlyeq$'. It remains to specify how this predicate can be translated. A plural 'among'-statement, '$xx \preccurlyeq yy$', can be translated as a generalized individual 'among'-statement, namely

$$\forall z(z \prec xx \rightarrow z \prec yy)$$

So we have the translation in one direction.

Let us now describe a translation τ in the reverse direction. Consider '$x \prec yy$'. This individual 'among'-statement can be translated as a plural 'among'-statement conjoined with a statement to the effect that some things comprise just a single object:

$$xx \preccurlyeq yy \wedge I(xx)$$

Quantification over individual objects can be translated as plural quantification restricted to singleton pluralities. Thus, '$\exists x\, \varphi$' is translated by τ as:[6]

$$\exists xx(Ixx \ \wedge \ \tau(\varphi))$$

Again, it turns out that both translations provide an interpretation of one theory in the other. Formal definitions and proofs are given in the appendices.

It is also no surprise that we can translate between one-sorted plural logic and Atomistic Classical Extensional Mereology. After all, we have formulated the former theory by simply adapting the axioms of the latter. It follows that all of the three theories considered in this section—the mereological one and plural logic with one or two sorts—are mutually interpretable. Again, see the appendices for details.

5.4 Classifying some ways to talk about the many

This chapter and the previous one have described three different ways to talk about the many. In addition to the use of primitive plural resources, we can use sets or (individual) mereology. We have seen that there are close connections between these different systems. But let us be more systematic.

[6] We assume a convention is in place to ensure that xx does not occur in $\tau(\varphi)$.

The alternatives we have considered differ along two dimensions: they can be one- or two-sorted, that is, they have one or two distinct registers of variables and constants; and they may or may not allow an "empty entity". Our results are summarized by the following table:

	one-sorted	two-sorted
empty entity	—	SST+
no empty entity	individual mereology, one-sorted plural logic	PFO+

In fact, the top left-hand quadrant is populated as well. It is straightforward to modify individual mereology so as to allow an "empty sum", much as SST+ modifies PFO+ by allowing an empty set.

We can provide translations that interpret any one of the theories in any other. Translations that establish the mutual interpretability of the two entries in the right-hand column were sketched in Section 4.1. And translations that establish the mutual interpretability of the two entries in the bottom row were outlined in Section 5.3. Thus, by composing these translations, it follows that any system in the table can be interpreted in any other such system.

As observed in Section 4.2, the existence of these translations and the possibility of interpreting one system in another leave wide open various questions of great philosophical interest. The translations do not necessarily preserve meaning. In fact, the translations may not even preserve truth value on the intended interpretation of the languages in question. Consider a nominalist, who believes that everything is concrete and thus that there are no abstract objects such as sets. This theorist would take various set-theoretic statements to be false although their translations into the plural and mereological idiom are true. Moreover, even philosophers without nominalist scruples will reject as false certain set-theoretic statements whose plural analogues they regard as true. The statement that there is a universal plurality (discussed in Sections 2.6 and 3.5) provides an example. Its translation into ordinary single-sorted set theory is the statement that there is a universal set, which is false according to the standard contemporary conception of set. This apparent mismatch between plural logic and set theory will be a major theme in Chapters 11 and 12.

In the remainder of this chapter, we will consider the relation between pluralities and mereological sums. Does one explain, or even afford an elimination of, the other? Or should both notions be retained? As in the previous chapter, we end up favoring the more liberal option of retaining both notions.

5.5 Mereological singularism in linguistic semantics

Mereology is a popular tool among linguists interested in plurals. Indeed, the most influential analysis of plurals in linguistic semantics invokes individual mereology.[7] The popularity of the mereological analysis of plurals is supported by a number of theoretical considerations.[8]

To begin with, mereology provides a framework for the analysis of both plurals and mass terms. The key idea is that plurals are analyzed by means of individual mereology, while mass terms are analyzed by means of material mereology. By appealing to shared mereological structures, one can explain the common features of these two classes of expressions with a high degree of unification. Consider the property of *cumulative reference*. If some people are students and some other people are students, then all of those people are students. Similarly, if some stuff is water and some other stuff is water, then all of that stuff is water. On a mereological analysis, this general phenomenon is captured by assuming that certain properties P "transmit upwards" from the parts to the whole:

$$\forall x \forall y (P(x) \land P(y) \rightarrow P(x + y))$$

Moreover, mass nouns, like plurals, can give rise to collective and distributive readings. Compare:

(5.2) This jewelry is expensive.

(5.3) These pieces of jewelry are expensive.

Both sentences can mean that the jewelry as a whole is expensive. But they can also mean that each piece of jewelry is expensive. A mereological semantics permits a highly unified explanation, for example by assuming

[7] See, e.g., Link 1983, Link 1998, and Champollion 2017. For an overview, see Champollion and Krifka 2016. Alternative approaches based on mereology can be found in Gillon 1992, Moltmann 1997, Landman 2016, and Sutton and Filip 2016.

[8] This section and the next draw from Florio and Nicolas 2020.

that distibutive properties "transmit downwards" from the whole to its salient parts.

Furthermore, there are constructions that combine with plurals and mass nouns but exclude singular count nouns, for instance comparative constructions ('more pieces of jewelry' and 'more jewelry' are grammatical but 'more piece of jewelry' is not) and the proportional quantifier 'most' ('most pieces of jewelry' and 'most jewelry' are grammatical but 'most piece of jewelry' is not). In this case too, one can provide a highly unified analysis by assuming a shared mereological structure of plural and mass nouns.

Another appealing feature of mereology is that it can easily be integrated with the rest of linguistic semantics. Let us explain. In linguistic semantics, one usually interprets natural language by first assigning semantic values to the basic expressions of the language and then deriving the semantic value of more complex expressions compositionally. The stock of available semantic values belongs to a hierarchy generated in the following way. First, one postulates semantic values of some basic types, say objects and truth values. Then, one obtains more semantic values by means of set-theoretic operations applied to the semantic values of the basic types. Any set of objects, for example, is now available as a possible interpretation of a one-place predicate. More generally, the stock of available semantic values may include sets of truth values, and functions between any two sets already available. So the available semantic values inhabit a cumulative hierarchy of sets generated by the entities of the basic types. Mereological sums can be added to the pool of semantic values without fundamentally altering the rest of semantics. These new entities become available for the set-theoretic operations that yield other types of semantic values. The full power of set theory thus becomes available across the semantics. So there is no special difficulty in capturing the fact that plurals, mass terms, and singular count nouns combine in the same way with other grammatical expressions, such as adjectives and verbs, several determiners (for example, 'the', 'some', 'any', and 'no'), and partitive constructions.

Things look different if we try to add pluralities to the stock of semantic values. A plurality is not a special object and hence requires the introduction of a new semantic type. Consider the semantic value of a plural predicate. On the mereological approach, this might be a set of individual sums, where each such sum represents some objects to which the predicate applies. Since the individual sums are objects, they are eligible to figure as elements of a set. Suppose we used primitive plurals instead of individual sums. A plurality is not an object and is thus not eligible to figure as an element of a set, which precludes a set-theoretic representation of the semantic value of a plural predicate. This raises the broader question of how to integrate the new type of primitive plurals with the rest of semantics.

Take the case of 'some'. As shown by the following sentences—all instances of the scheme 'some φ ψ'—this determiner can combine with singular count nouns, plural count nouns, and mass nouns:

(5.4) Some wolf can be found on the North Pole.

(5.5) Some wolves can be found on the North Pole.

(5.6) Some ice can be found on the North Pole.

The mereological translations of these sentences have the same form:

$$\exists x(\varphi(x) \wedge \psi(x))$$

Each asserts that there is an object that satisfies both φ and ψ. Thus, on the mereological analysis, the determiner can be seen as making the same semantic contribution in all cases, requiring a common instance of φ and ψ.

By contrast, these sentences do not have the same representation in plural logic. While (5.4) and (5.6) have the form displayed just above, (5.5) has a different form, namely:

$$\exists xx(\varphi(xx) \wedge \psi(xx))$$

Therefore 'some' appears to have one type of meaning when it combines with a plural count noun and another type of meaning when it combines with a singular count noun or a mass noun.[9]

Thus, we see that linguists have multiple reasons to be attracted to mereological analyses of plurals. Do these analyses have any *philosophical* consequences? Do they reveal, say, how plural talk in natural language should really be understood and thus suggest that plural logic should be eliminated in favor of (individual) mereology? No doubt, the analyses open the possibility of this sort of elimination. But, by themselves, the linguistic reasons for such analyses don't support this philosophical conclusion. Link, however, can be read as suggesting this further, eliminative step:

[9] Another instance of this issue concerns the formulation of a generalized quantifier theory and is discussed in Studd 2015. See Yi 2016 for further discussion.

> While [Boolos] thinks that plural quantification is a self-understood notion I want to argue that this idiom is both in need and capable of a theoretical explanation, which I submit is mereology. (Link 1998, 331–2)

In Section 4.4, we argued that primitive plurals are needed to explain set theory. This argument has an important consequence concerning the possibility of eliminating plurals in favor of (individual) mereology. Plurals provide a more natural basis for the explanation of set theory than mereological sums. For it is more illuminating to explain a set in terms of its many elements than to explain it in terms of the *mereological sum* of these elements. Thus, our argument provides a reason to retain plurals and not eliminate them in favor of mereological sums.

5.6 Assessment of singularism in linguistic semantics

The use of individual sums in linguistic semantics requires that we think of sums as objects rather than pluralities. For sums can figure as elements of sets, while pluralities cannot. Therefore mereological talk in linguistic semantics is not one-sorted plural logic in disguise but a genuine form of singularism. As such, it faces the objections already considered in Chapter 3. Our assessment there was that the objections are not compelling, at least not in the absence of substantive assumptions.

In this section, we do two things. First, we discuss a new objection, which has particular force against mereological singularism in the context of linguistic semantics. Then, we revisit one of the substantive assumptions behind some arguments discussed in Chapter 3, namely the possibility of absolute generality. We examine the plausibility of this assumption in the particular context in which we now find ourselves.

The mereological analysis of plurals has raised a concern about the intelligibility of plural predication.[10] Consider the following collective predication:

(5.7) Annie and Bonnie cooperate.

This sentence is perfectly intelligible to competent speakers. According to the mereological analysis, its truth conditions are as follows:

(5.8) 'Annie and Bonnie cooperate' is true if and only if the individual sum denoted by 'Annie and Bonnie' satisfies the predicate 'cooperate'.

[10] See McKay 2006, 24.

However, it may be objected that the right-hand side of (5.8) is unintelligible. We do understand what it is for two people to satisfy the predicate 'cooperate' but—the objection goes—we do not understand what it is for *a sum* to satisfy that predicate.

In response, one may observe that we do understand what it is for a single entity like a group, a team, or a committee to satisfy the predicate 'cooperate'. For (5.9) is perfectly intelligible:

(5.9) This group/team/committee cooperates.

So one may insist that the sense in which a sum satisfies the predicate 'cooperate' is the same sense in which a group, a team, or a committee does.

An alternative response relies on an event-based analysis of predication that generalizes the influential proposal of Davidson 1967. If we broaden the notion of event to include states, we can regard all predicates as properties of events. We can then analyze a sentence like (5.7) in one of two ways.[11]

(5.10) 'Annie and Bonnie cooperate' is true if and only if there is an event of cooperating and the individual sum denoted by 'Annie and Bonnie' is the agent of that event.

(5.11) 'Annie and Bonnie cooperate' is true if and only if there is an event of cooperating, each atom of the individual sum denoted by 'Annie and Bonnie' is a co-agent of that event, and nothing else is a co-agent of that event.

The sole difference concerns the relation between the sum denoted by 'Annie and Bonnie' and the underlying event of cooperating. In the first analysis, the sum is the agent of the event. That is, the sum plays the thematic role of agent of the event. In the second, the sum simply provides the atoms that share the role of agent and, in this sense, function as co-agents of the event. No matter which proposal is adopted, the intelligibility problem should be less pressing: clauses (5.10) and (5.11) appear to be intelligible. The mereological notions involved are given to us through axioms, and we can certainly rely on our ordinary understanding of events for a basic grasp of the event-theoretic notions employed in the semantics. But event semantics is a well-established and successful framework, routinely used by many linguists and

[11] See Landman 2000, Chapter 3, Section 3.2–3.3. For historical details and references, see Oliver and Smiley 2016, 44–5.

philosophers. We see no reason to doubt the coherence of their research and the intelligibility of the event-based analysis of predication.

A theme that emerged in Chapter 3 is that a singularist analysis, such as the mereological one, might not be available in the presence of absolute generality, provided that traditional plural logic is assumed. Let the domain of quantification of our plural object language comprise absolutely everything. We observed in Section 5.2 that mereology can represent the plural only if the objects in the range of the first-order quantifiers are mereological atoms in the individual sense. Since the first-order domain contains absolutely everything, it follows that every object whatsoever is an individual atom. To apply the mereological analysis, however, we would need further objects, namely sums of atoms. Because absolutely every object is now regarded as an atom, no such sums are available. We have, as it were, run out of objects to serve as sums.

How strong is this objection? We will ultimately respond to it by restricting traditional plural logic. But a more immediately appealing response is to deny the possibility of absolute generality. If there is no such thing as absolute generality, then the objection under discussion gets no foothold.

Even if there is some sense to be made of absolute generality—as we argue in Chapter 11—a closely related response nonetheless remains available, namely to observe that, for the vast majority of their purposes, linguists can set aside the problem of absolute generality. They are anyway assuming that the domain is given as a set, for instance when they do generalized quantifier theory. And as we have seen, there is no set of absolutely all objects. Thus, the objection poses no *additional* problem for linguists. Given their purposes, linguists are entitled to proceed precisely as they do.

5.7 The elimination of mereology in favor of plural logic

The thesis that mereology should be eliminated in favor of plural logic has found a number of supporters in metaphysics.[12] A systematic development is given by Keith Hossack (2000), who advocates an atomistic metaphysics. According to this view, there really are no complexes such as masses, composite objects, or sets; only metaphysical atoms exist. The view relies essentially on plural logic.

[12] An influential use of this idea is found in van Inwagen 1990; see also Rosen and Dorr 2002.

Hossack points out that none of the usual axioms of mereology, including the ones stated above, seems to hold in general. For example, we can find uses of the word 'part' for which transitivity fails. A page is part of a book, which is part of a library, although the page is not part of the library. And it is highly controversial whether the axiom (M-Sum), which asserts the existence of arbitrary sums, is correct.

According to Hossack, "[a]bout the only interpretation on which the mereological axioms are indisputable logical truths is a plural one" (Hossack 2000, 423). The formal translation from mereology to one-sorted plural logic can be seen as vindicating this point. Indeed, he gestures at the result and concludes that:

> it seems plausible that we can use the *are-some-of* relation to give an analysis of our ordinary talk of parts and wholes that is superior to the account given by extensional mereology. (Hossack 2000, 424)

Finally, he shows how the analysis can be carried out for various complexes.

Simplifying a bit, the proposed strategy is illustrated by the following examples of elimination concerning masses and complex objects.

(5.12) There is some water.

(5.13) Some atoms are φ.

(5.14) There is a chair.

(5.15) Some atoms are ψ.

Here φ and ψ are collective predicates true of atoms that constitute water and atoms that constitute a chair, respectively. In the literature, the latter is usually rendered as "are arranged chairwise".

There are three main issues with this approach. First, what guarantee do we have that all composite objects decompose into atoms? Aristotle famously held that matter is indefinitely divisible. Any bit of matter contains an even smaller bit of matter. Whether or not he was right about that, it certainly seems possible that there could be atomless gunk, that is, some stuff without atomic parts.[13] Thus, the proposed analysis depends on a risky and controversial metaphysical assumption.

[13] More formally, x consists of atomless gunk if and only if any part y of x has a proper part z.

Second, how should we analyze plural talk about composite objects?[14] Consider the following collective predication about a plurality of composite objects:

(5.16) The chairs are arranged in a circle.

The problem is that talk about a single chair already uses plurals, in the form of plural talk about some atoms arranged "chairwise". So we have already "used up" the plural resources of the language in which we give our analysis. Pluralities of composite objects would therefore require superplurals. (We discuss the legitimacy of superplurals in Chapter 9.)

Finally, there appears to be a mismatch between the modal profiles of a plurality and that of a composite object. Plural membership is modally rigid. If a is one of bb, then necessarily so (at least on the assumption that all of the objects in question continue to exist). And likewise for not being one of some things. In short, a plurality doesn't vary with respect to which members it has in different circumstances or possible worlds. (This view is defended in Chapter 10.) By contrast, there are composite objects for which parthood appears non-rigid. Consider one of your cells. It seems possible for you to exist even though this cell is no longer to be part of you. And a good thing too, since the life expectancy of most cells is far shorter than that of the organism to which they belong![15]

5.8 Keeping both plural logic and mereology

Where does this leave us? We argued in Chapter 4 that both pluralities and sets should be retained. Should mereological sums too be retained alongside pluralities and sets?

Our previous discussion suggests an "algebraic conception" of mereology. The axioms of mereology describe a certain kind of abstract structure, which can be realized in many different—indeed non-isomorphic—ways.[16]

[14] For a discussion of this objection, see Uzquiano 2004a.

[15] Even if our claims about a mismatch of modal profiles is right, this isn't the end of the story—as so often in philosophy. The mismatch is analogous to that of the modal properties of the statue and the clay in the famous problem of material constitution. In both cases, a proponent of the relevant reduction can attempt to address the mismatch by invoking a counterpart relation (see Lewis 1971 and Gibbard 1975).

[16] See Fine 2010, Section II, for a similar view based on a "pluralist" conception of parthood.

We have seen various examples of such realizations: the material interpretation, where x is part of y just in case the matter of x is contained in that of y; and the plural interpretation in the one-sorted formulation of plural logic. This suggests that mereology, unlike set theory, does not have a single canonical interpretation. Mereology is the abstract theory of part-whole structures, which are realized in many different ways. In this respect, mereology is rather like the theory of partial orders. It makes no sense to ask what is the true partial order of reality. A plethora of different partial orders are realized throughout reality. Likewise, we submit, it makes no sense to ask what is the true part-whole structure of reality. There are many such structures.

The question, then, is: what part-whole (or mereological) structures are there? We have already mentioned two examples: material parthood and the among-relation defined on pluralities. For our purposes, the most important aspect of the question concerns individual mereology, which as we have seen plays a key role in many linguistic approaches to plurals. Are there individual sums?

This metaphysical question has no easy answer. A comprehensive discussion would take us too far afield. Instead, we will briefly present two reasons to accept the existence of individual sums. First, individual sums are very useful in semantics in order to account for various natural language phenomena. This provides a broadly naturalistic reason to accept them, namely that individual sums figure in respectable scientific practice.

Second, as explained in Section 4.4, we are attracted to a liberal view of definitions. According to this view, it suffices for a mathematical object to exist that an adequate definition of it can be provided, where the adequacy in question is understood as follows. Suppose we start with a domain of objects standing in certain relations and would like to define one or more additional objects. Suppose our definition determines the truth of any atomic statement concerned with the desired "new" objects by means of some statement concerned solely with the "old" objects with which we began. Then, according to our liberal view, the definition is permissible.

Let us apply this approach to our question about the existence of individual sums. Suppose we start with some domain of objects. For every plurality of objects xx from this domain, we postulate their individual sum $\Sigma(xx)$, which contains each member of xx as an individual atom. Atomic predications concerned with these objects are to be assessed as follows.

(a) If xx consist of just a single object y, then the sum $\Sigma(xx)$ is identical with y.
(b) $\Sigma(xx) \leq \Sigma(yy)$ if and only if $xx \preccurlyeq yy$.

Clearly, (b) entails:

(c) $\Sigma(xx) = \Sigma(yy)$ if and only if $xx \approx yy$.

This yields an account of the desired individual sums and their relations in terms of pluralities of the objects with which we started and *their* relations.

The account of individual sums clashes with certain metaphysicians' attempts to eliminate mereology in favor of pluralities. The result of adopting the liberal view of definitions yields mereological sums as objects, much as the application of Gödel's "set of" operation to pluralities yields sets as objects. These objects live alongside the pluralities from which they are formed. By contrast, the *eliminative* project surveyed in Section 5.7 rejects the existence of all mereological non-atoms. Overall, our view is that we should retain two kinds of derived objects—sets and individual mereological sums—both of which can be accounted for in terms of pluralities. Because of this account, our view is a form of *non-eliminative* reductionism.

We have advocated retaining pluralities, sets, and mereological sums. How do these three kinds of object interact? This question raises a number of interesting and difficult issues. We shall content ourselves with commenting on one particularly important point. How does the individual sum $\Sigma(xx)$ differ from the set $\{xx\}$? Part of the answer has to do with clause (a): while the sum of a singleton plurality is identical with the sole member of this plurality, the set formed by a singleton plurality is distinct from its sole member. Another part of the answer emerges when sum formation is iterated. Clause (b) must then be replaced by a more general criterion of identity. Let '$z \circ xx$' abbreviate $\exists x(z \circ x \land x \prec xx)$. Then this more general criterion can be formulated as:

$$(\mathrm{b}^+) \qquad \Sigma(xx) \leq \Sigma(yy) \;\leftrightarrow\; \forall z(z \circ xx \rightarrow z \circ yy)$$

which, unlike (b), is valid even when xx and yy are not all individual atoms. Clearly, (b^+) entails:

$$(\mathrm{c}^+) \qquad \Sigma(xx) = \Sigma(yy) \;\leftrightarrow\; \forall z(z \circ xx \leftrightarrow z \circ yy)$$

These clauses show that sum formation is "flat" in a way that set formation is not.[17] That is, taking the sum of some objects and some other objects is the same as taking the sum of the former objects and *the sum of* the latter objects. The analogous set-theoretic claim is false: taking the set of some objects and some other objects is not the same as taking the set of the former objects and *the set of* the latter objects. To be precise, let us formalize these observations. Let t_1 and t_2 be two terms, either singular or plural, and let tt be the plural term referring to all of the objects referred to by either t_1 or t_2. Using $\Sigma(t_1, t_2)$ as a shorthand for $\Sigma(tt)$, we then have that $\Sigma(xx, \Sigma(yy)) = \Sigma(xx, yy)$, while the analogous set-theoretic claim, $\{xx, \{yy\}\} = \{xx, yy\}$, is false. Thus, as advertised, sums and sets behave in importantly different ways.

[17] Kit Fine develops a similar but more general view. His "sums" (2010, 574) correspond to our individual sums.

Appendices

5.A Partial orders and principles of decomposition

The appendices to this chapter have two main aims. First, we want to provide a useful introduction to mereology. We begin with the axioms of the atomistic version of Classical Extensional Mereology, mentioned in Section 5.1. We present the axioms in natural groups, where each group captures one fairly unified idea. Second, based on the resulting understanding of mereology, we prove the mutual interpretability of our official two-sorted plural logic PFO+ and the appealing one-sorted alternative based on the described mereological theory.

We begin by rehearsing some definitions. Let '$\leq$' be an atomic predicate representing 'is part of (or equal to)'. Then we make the following definitions.

Definition 5.1 (Basic notions)

(a) $x < y$ (x is a *proper part* of y) iff $x \leq y \land x \neq y$.
(b) $x \circ y$ (x *overlaps* y) iff $\exists z(z \leq x \land z \leq y)$.
(c) $x \perp y$ (x *is disjoint from* y) iff $\neg x \circ y$.
(d) $\mathrm{At}(x)$ (x is *an atom*) iff $\neg\exists y(y < x)$.

The first group of axioms, which is already familiar, consists of those of a partial order.

Definition 5.2 (Partial order) $\leq$ is a *partial order* iff:

(PO1) $$x \leq x$$

(PO2) $$x \leq y \land y \leq x \rightarrow x = y$$

(PO3) $$x \leq y \land y \leq z \rightarrow x \leq z$$

Let *PO* be the first-order theory whose axioms are (PO1)–(PO3).[18]

The second group of axioms are principles of decomposition. They sanction that the mereological relations that obtain between two objects are a matter of these objects' parts. First, there are the *supplementation axioms*:

[18] In the statement of these axioms, we rely on our convention from Section 2.4 of omitting initial universal quantifiers, as is often done in mathematical prose.

(WS)	$x < y \rightarrow \exists z(z \leq y \wedge z \perp x)$	(Weak Supplementation)
(SS)	$x \nleq y \rightarrow \exists z(z \leq x \wedge z \perp y)$	(Strong Supplementation)

Next, there is the principle of complementation:

(C) $x \nleq y \rightarrow \exists z \forall w(w \leq z \leftrightarrow w \leq x \wedge w \perp y)$ (Complementation)

If $x \nleq y$, the object z said to exist by (C) is easily seen to be unique; this object is often written '$x \setminus y$' (pronounced "x minus y").[19]

Our first result orders the principles of decomposition by their logical strength. Let us say that φ is *strictly stronger than* ψ *relative to a theory* T iff $T, \varphi \vdash \psi$ but $T, \psi \nvdash \varphi$. Then:

Lemma 5.1 Relative to the theory PO of partial orders, we have: (C) is strictly stronger than (SS), which is strictly stronger than (WS), which is strictly stronger than just PO.

Proof. The implications are straightforward. First, $x \setminus y$ can serve as the object z said to exist by (SS). Second, we use the fact that the definition of $x < y$ assures $y \nleq x$. The three non-implications are established by means of counterexamples. See Varzi 2019, Section 3, for details. ⊣

Strong Supplementation is particularly important because it ensures that parthood admits of a very useful characterization in terms of overlap, namely:

(∗) $$\forall z(z \circ x \rightarrow z \circ y) \leftrightarrow x \leq y$$

Let us call $\forall z(z \circ x \rightarrow z \circ y)$ *the overlap criterion* for the parthood claim $x \leq y$. Thus, (∗) asserts the validity of the overlap criterion for parthood. Our next result reveals the tight connection between Strong Supplementation and the validity of the overlap criterion.

Lemma 5.2 (SS) is equivalent to (∗) relative to the theory PO.

Proof. First, observe that some simple first-order logic allows us to rewrite (SS) as:

(SS′) $$\forall z(z \leq x \rightarrow z \circ y) \rightarrow x \leq y$$

[19] To prove uniqueness, assume there were two such objects, z_1 and z_2. Then we would have $z_1 \leq z_2$ and $z_2 \leq z_1$, whence $z_1 = z_2$ after all.

Next, PO proves the equivalence of $\forall z(z \leq x \rightarrow z \circ y)$ and $\forall z(z \circ x \rightarrow z \circ y)$. It follows that (SS) is equivalent, relative to PO, to the left-to-right direction of $(*)$. Our claim therefore follows because the other direction of $(*)$ is a theorem of PO. $\dashv$

Strong Supplementation has another attractive consequence as well, which is recorded in the following lemma.

Lemma 5.3 The following statements are equivalent relative to PO plus (SS):

(i) $x = y$
(ii) $\forall z(z \leq x \leftrightarrow z \leq y)$
(iii) $\forall z(z \circ x \leftrightarrow z \circ y)$
(iv) $\forall z(z \perp x \leftrightarrow z \perp y)$

Proof. Relative to PO, (i) implies (ii), which in turn implies (iii). We now use Lemma 5.2 to establish that (iii) implies (i) relative to PO + (SS). Thus, the first three conditions are equivalent. Finally, we observe that (iii) is equivalent to (iv) because $z \perp x \leftrightarrow z \perp y$ can be rewritten as $\neg z \circ x \leftrightarrow \neg z \circ y$. $\dashv$

5.B Some notions of sum

We now describe two conceptually different notions of sum that are available in the context of any partial order $\leq$.

The first notion is that of a *least upper bound*. Let us say that z is an *upper bound of x and y* iff $x \leq z$ and $y \leq z$. A *least upper bound of x and y* is an upper bound z of x and y such that, for any other upper bound w, we have $z \leq w$. The statement that z is a least upper bound of x and y can be formalized as:

$$\forall w(z \leq w \leftrightarrow x \leq w \land y \leq w) \tag{5.17}$$

Clearly, when a least upper bound of two objects exists, it is unique.[20]

A second notion of sum is defined in terms of the notion of overlap, namely that z is *a fusion of x and y* iff:

$$\forall w(w \circ z \leftrightarrow w \circ x \lor w \circ y) \tag{5.18}$$

[20] Suppose both z_1 and z_2 were least upper bounds of x and y. Then we would have $z_1 \leq z_2$ and $z_2 \leq z_1$, which entails $z_1 = z_2$.

That is, a fusion of x and y is an object z such that, to overlap z is equivalent to overlapping either x or y. Assume Strong Supplementation. Then, if there is a fusion of x and y, this fusion is unique. To see this, suppose that z_1 and z_2 are fusions of x and y. By our definition of a fusion, an object w overlaps z_1 iff w overlaps z_2 (namely, iff w overlaps either x or y). By the overlap criterion of identity—which by Lemma 5.2 is available on the assumption of PO and Strong Supplementation—it follows that $z_1 = z_2$.

What is the relation between the two notions of sum? The next result provides the answer.

Lemma 5.4

(a) Assume Strong Supplementation. Then any fusion of x and y is also a least upper bound of x and y.

(b) Assume Complementation. Then any least upper bound of x and y is also a fusion of x and y.

Proof. (a) Assume z is a fusion of x and y, that is, (5.18). The overlap criterion for parthood immediately yields $x, y \leq z$. It remains to show that z is the least upper bound of x and y. So assume $x, y \leq z'$. This assumption, combined with (5.18), yields $\forall w(w \circ z \to w \circ z')$, whence by the overlap criterion again, we have $z \leq z'$, as desired.

(b) Assume z is a least upper bound of x and y, that is, (5.17). Because $x, y \leq z$, we have that $w \circ x \vee w \circ y$ implies $w \circ z$. It remains to establish the converse implication. Assume, for contradiction, that $w \circ z$ but $w \perp x \wedge w \perp y$. This means that $z \nleq w$, whence by Complementation, we can let u be $z \setminus w$, that is:

(5.19) $$\forall v(v \leq u \leftrightarrow v \leq z \wedge v \perp w)$$

Instantiating '$\forall v$' with respect to x and y, it follows that $x, y \leq u$. But $w \circ z$, so $u < z$. This result contradicts our assumption that z is *the least* upper bound of x and y. That establishes the implication we set out to prove. ⊣

We now formulate axioms stating that *any two objects* have a sum in each of our two senses:

(LUB) $$\exists z(\forall w(z \leq w \leftrightarrow x \leq w \wedge y \leq w))$$

(Fus) $$\exists z(\forall w(w \circ z \leftrightarrow w \circ x \vee w \circ y))$$

What about *larger collections of objects*: do these too have sums? Let us start with sums understood as least upper bounds. A partial order $\leq$ is said to be *complete* iff, for any instantiated condition $\varphi(x)$, there is a least upper bound of all the objects that satisfy the condition; that is, iff:

$$(\text{LUB}^+) \qquad \exists x\, \varphi(x) \rightarrow \exists z \forall w(z \leq w \leftrightarrow \forall x(\varphi(x) \rightarrow x \leq w))$$

Next, the partial order is said to *permit unrestricted fusion* iff the following axiom scheme holds:

$$(\text{Fus}^+) \qquad \exists x\, \varphi(x) \rightarrow \exists z \forall w(w \circ z \leftrightarrow \exists x(\varphi(x) \wedge w \circ x))$$

This is meant to capture the idea that any non-empty collection of objects has a fusion.

As before, and under the same assumptions as before, we can prove that a least upper bound (or fusion), if there is one, is unique. Also as before, and under the same assumptions as before, we can prove that these unrestricted notions of least upper bound and fusion are equivalent.

Equipped with this understanding of unrestricted sums, we are now ready to define Classical Extensional Mereology.

Definition 5.3 The theory of *Classical Extensional Mereology* consists of:

(i) the theory PO of partial orders
(ii) the axiom of Strong Supplementation
(iii) the axiom scheme (Fus^+)

Various alternative (but equivalent) axiomatizations exist as well. Here is one important example: we may replace (ii) and (iii) with the axiom of Complementation and the axiom scheme (LUB^+), respectively.[21]

[21] To see that this alternative yields an equivalent axiomatization, we first invoke the mentioned generalization of Lemma 5.4. It then remains only to show that Classical Extensional Mereology, as defined above, entails Complementation. (Hint: Show that $x \setminus y$ can be defined as the fusion of w such that $w \leq x \wedge w \perp y$.) See Varzi 2019, Section 4.4, for a useful overview of further alternatives, including more minimalistic ones. The articulation of the axioms of Classical Extensional Mereology into the three groups (i) through (iii) is nevertheless historically important and, we think, conceptually more illuminating than the more minimalistic alternatives.

5.C Atomicity

Definition 5.4 A partial order $\leq$ is said to be *atomistic* iff every object has an atomic part:

(At) $$\forall x(\exists u(u \leq x \wedge \mathrm{At}(u))$$

Let *Atomistic Classical Extensional Mereology* be the result of adding (At) to Classical Extensional Mereology.

Lemma 5.5 Let $\leq$ be an atomistic partial order with Strong Supplementation.

(a) Then parthood can be tested on atoms, in the following sense:

$$x \leq y \leftrightarrow \forall u(\mathrm{At}(u) \rightarrow (u \leq x \rightarrow u \leq y))$$

(b) Assume $\leq$ is also complete; that is, (LUB$^+$) holds. Then every object is is identical to the fusion of its atoms.

Proof sketch. The proof of (a) is routine and is therefore omitted. For (b), consider any object x, and let y the fusion of the atoms in x. By Atomicity, anything that overlaps x is easily shown also to overlap y. Moreover, anything that overlaps y overlaps an atom in x and thus also x itself. By the overlap criterion of identity, it follows that $x = y$. ⊣

Our main reason for being interested in atomistic mereology is that the among-relation $\preccurlyeq$ is atomistic. Recall that this relation is defined in the two-sorted system, but primitive in the one-sorted system. Consider the plural comprehension scheme:

(P-Comp) $$\exists x\, \varphi(x) \rightarrow \exists xx \forall x(x \prec xx \leftrightarrow \varphi(x))$$

The analogue of this principle in the atomistic mereology of $\preccurlyeq$ is the principle stating that, provided φ is instantiated by an atom, there is a sum whose atomic parts are all and only the atomic φs:

(At-Sum) $$\exists x(\mathrm{At}(x) \wedge \varphi(x)) \rightarrow \exists z \forall w(\mathrm{At}(w) \rightarrow (w \leq z \leftrightarrow \varphi(w)))$$

Let us now compare this atomistic principle with our unrestricted fusion principle (Fus^+).

Theorem 5.1 (Fus^+) is strictly stronger than (At-Sum) relative to the theory of partial orders with Strong Supplementation. However, if we add the assumption that the partial order is atomistic, the two principles become equivalent.

Proof. Assume $\exists x(\text{At}(x) \wedge \varphi(x))$. Consider the fusion of all atomic φs. This fusion is easily shown to be a witness for the existential claim in the consequent of (At-Sum). The converse implication is easily seen to fail when $\leq$ is non-atomic. Finally, assume that $\leq$ is atomistic. Suppose there is a φ. We want to show that there is a fusion of all φs. Let z be the sum of all atomic parts of φs, which is ensured by (At-Sum) to exist. That is, we instantiate (At-Sum) with the formula '$\exists y(\varphi(y) \wedge x \leq y)$'. Then, since $\leq$ is atomistic, to overlap z is equivalent to overlapping some atomic part of some φ. And to overlap some atomic part of some φ is (again, since $\leq$ is atomistic) equivalent to overlapping some φ. By transitivity, to overlap z is equivalent to overlapping some φ. Thus, z is our desired fusion of all φs. $\dashv$

5.D One- and two-sorted plural logic compared

Let us compare the one- and two-sorted formulations of plural logic. The former, we recall from Section 5.3, states that the among relation $\preccurlyeq$ satisfies the axioms of Atomistic Classical Extensional Mereology. More precisely, this theory is just like Atomistic Classical Extensional Mereology, as formulated above, except that it uses the predicate '$\preccurlyeq$' rather than '$\leq$' and that all of its variables are doubled (for example, 'xx' instead of 'x'). The latter is the familiar system PFO+. We also provided translations from each language into the other. Let us now prove the promised result that these translations are interpretations of each formulation of plural logic in the other.

Consider first the result of translating the system PFO+ into one-sorted plural logic. The axioms and inference rules of PFO+ are easily seen to be mapped to theorems and derived rules of one-sorted plural logic. First, the axioms and rules of sentential logic and the quantifiers rules are straightforward. Next, consider the indiscernibility principle:

(Indisc) $$xx \approx yy \rightarrow (\varphi(xx) \leftrightarrow \varphi(yy))$$

The antecedent is translated as the statement that xx and yy have the same atomic parts. By Lemma 5.5(a) and the assumption that $\preccurlyeq$ is a partial order, we derive the identity statement $xx = yy$, whence the translation of the consequent of (Indisc) follows by Leibniz's Law.

Then, there is the axiom which says that every plurality is non-empty:

$$\text{(Non-Empty)} \qquad \forall xx \exists y (y \prec xx)$$

This is mapped to the axiom that states that the order $\preccurlyeq$ is atomistic.

Finally, each instance of the plural comprehension scheme (P-Comp) is mapped to an instance of the atomistic principle (At-Sum), which is a theorem of our one-sorted plural logic by the first half of Theorem 5.1.

We turn now to the reverse direction, namely the interpretation of one-sorted plural logic in PFO+. As mentioned in Section 5.3, the primitive among-relation $\preccurlyeq$ of one-sorted plural logic is translated as its defined counterpart in PFO+:

$$xx \preccurlyeq yy \leftrightarrow_{\text{def}} \forall u (u \prec xx \rightarrow u \prec yy) \tag{5.20}$$

So we must verify that this defined relation satisfies the required properties.

Theorem 5.2 The defined relation $\preccurlyeq$ in PFO+ satisfies the axioms of Atomistic Classical Extensional Mereology.

Proof. It is immediate from its definition that the relation $\preccurlyeq$ is a partial ordering. Using (P-Comp) we can easily derive Strong Supplementation and, via (Non-Empty), Atomicity. The completeness axiom (Fus^+) also follows from an appropriate use of (P-Comp). That is, an application of (P-Comp) to the formula '$\exists xx(\varphi(xx) \wedge u \prec xx)$' delivers the fusion of all pluralities which satisfy φ. (This is, essentially, the second half of Theorem 5.1.) $\dashv$

Putting everything together, we obtain our main result.

Theorem 5.3 The two-sorted system PFO+ and the one-sorted plural logic are mutually interpretable.

6
Plurals and Second-Order Logic

We have encountered three ways to talk about the many, using primitive plurals, set theory, or mereology. Let us now examine a fourth and final way, namely using second-order logic. We begin with a brief introduction to second-order logic. We will then examine whether this system can be eliminated in favor of plural logic or vice versa.

6.1 Second-order logic

Consider the statement that Socrates thinks, which we formalize as:

(6.1) $T(s)$

Classical *first-order logic* allows us to generalize into the noun position occupied by 'Socrates' to conclude that there is an object x that thinks:

(6.2) $\exists x\, T(x)$

By allowing additional forms of generalization, we can obtain more expressive logics. *Second-order logic* (SOL) studies another form of generalization: it allows us to generalize into the predicate position occupied by T in (6.2) to conclude:

(6.3) $\exists F\, F(s)$

Following Frege, we describe the values of monadic second-order variables as *concepts*. So we gloss (6.3) as follows: there is a concept, F, such that Socrates falls under F.[1]

[1] Different glosses are found in literature, e.g. that a concept "applies to" an individual, that an individual is "in the extension of" a concept, or that an individual "instantiates" a concept. We will make use of these glosses when stylistically convenient.

The Many and the One: A Philosophical Study of Plural Logic. Salvatore Florio and Øystein Linnebo, Oxford University Press.
DOI: 10.1093/oso/9780198791522.003.0006

Variables taking predicate position—called *second-order variables*—belong to special sorts and are written as upper-case letters. There is one sort for each type of predicate. First, we have a sort for variables taking the position of monadic predicates. Variables of this sort are marked by the superscript '1' (X^1, Y^1, ...). Then, we have another sort for variables taking the position of binary predicates. Variables of this sort are marked by the superscript '2' (X^2, Y^2, ...). And so on. When no confusion arises, we omit the superscripts.

Second-order logic is thus a multi-sorted system, with a sort for individual variables and multiple sorts for second-order variables. As mentioned in Section 2.6, monadic second-order logic (MSOL) is the subsystem of second-order logic that adds to first-order logic only monadic variables. We can expand MSOL with predicates taking monadic variables as argument. We refer to the resulting system as MSOL+.

The key observation in this context is that a monadic second-order term allows us to talk about many things simultaneously. For a concept can be used to represent all the things that fall under it. For example, the concept F represents precisely the φs if and only if $\forall x(Fx \leftrightarrow \varphi(x))$.

Monadic second-order logic must nevertheless be carefully distinguished from plural logic. While the former allows us to generalize into predicate position, the latter allows us to generalize plurally into noun position. Plural logic thus allows us to infer from (6.1) that there are one or more objects xx that think:

(6.4) $\exists xx \forall y\,(y \prec xx \rightarrow T(y))$

As is apparent, plural and monadic second-order logic permit different kinds of generalization.

This difference distinguishes MSOL from the three other approaches to representing many objects simultaneously that we considered above. While all four approaches enable us to talk about "collections" of some given objects, MSOL is unique among these approaches in representing the "collections" by means of the semantic values of predicates. So, on the second-order approach, there will be interactions between ordinary first-order predication and the representation of "collections" that are not found in any of the other approaches.

Let us describe the second-order logic that we will adopt. The rules associated with the singular vocabulary—logical connectives and quantifiers—

are the usual ones, for example introduction and elimination rules for each logical expression. Second-order quantifiers (that is, the quantifiers binding second-order variables) have introduction and elimination rules analogous to those of the singular quantifiers. In addition, there is the second-order comprehension scheme:

(SO-Comp) $$\exists F \forall x(Fx \leftrightarrow \varphi(x))$$

where F does not occur free in φ, as well as the polyadic analogues of this scheme.[2]

Is there a natural language counterpart of second-order quantification? In other words, does quantification into predicate position occur in natural language? Some authors have defended an affirmative answer. For example, Higginbotham (1998, 3) points to the following sentence.[3]

(6.5) John is everything we wanted him to be.

A natural regimentation of (6.5) involves bound variables in predicate positions. If this analysis is correct, MSOL is not only an available language but it is actually in use.[4]

Even if there isn't always a good natural language counterpart, we need not give up on second-order languages. For the lack of correspondence might be due to an expressive limitation of natural language. Still, if we are to use a second-order language in theorizing, we need to learn it. Is it possible for us to do that? Williamson suggests that we use "the direct method":

> We may have to learn second-order languages by the direct method, not by translating them into a language with which we are already familiar. After all, that may well be how we come to understand other symbols in contemporary logic, such as $\supset$ and $\Diamond$: we can approximate them by 'if' and 'possibly', but for familiar reasons they may fall short of perfect synonymy [...]. At some point we learn to understand the symbols directly; why not

[2] For ease of readability, we will often omit parentheses around argument positions that immediately follow second-order variables.

[3] See also Rayo and Yablo 2001.

[4] A fan of plurals might attempt a further generalization as well, namely to pluralize not only first-order variables but also higher-order ones. For example, one might try to regiment "You are several things that I am not", using a plural version of second-order logic, as $\exists FF\ \forall G(G \prec FF \rightarrow G(\text{you}) \wedge \neg G(\text{I}))$. See Fine 1977 for a system with a very general form of pluralization.

use the same method for $\forall F$? We must learn to use second-order languages as our home language. (Williamson 2003, 459)

If successful, this method establishes the legitimacy of second-order resources.

6.2 Plural logic and second-order logic compared

What is the relation between second-order and plural logic? Let us compare the languages of MSOL+ and PFO+. We can suppose that the singular first-order fragments of these languages coincide. Each language adds to this shared base a new stock of variables, as well as new predicates whose argument places admit such variables. Although the new variables of each language function in very different ways, there is (as we will now explain) a natural correspondence between their values.

The new variables of MSOL+ are supposed to range over monadic concepts, which (as Frege pointed out) we may think of as functions from individuals to truth values.[5] For example, one of these variables can replace the predicative expression '…is a boy' and have as its value the function β defined as follows:

$$\beta(x) = \begin{cases} \text{the true,} & \text{if } x \text{ is a boy;} \\ \text{the false,} & \text{if not.} \end{cases}$$

By contrast, each of the new variables of PFO+ is allowed to have one or more individuals as its values. For example, one of these variables can replace the plural noun phrase 'the boys' and have as its many values all and only the boys in the domain, say *bb*.

We can now explain the promised natural correspondence between the values of the new kinds of variable. The correspondence is nicely illustrated by the function β and the plurality *bb*. Suppose we start with β. Then *bb* can be defined as all and only the objects that β maps to the true. Suppose instead we start with the plurality *bb*. Then β can be defined as the function that maps these objects, and only these, to the true. As mathematicians like to put it, β is *the characteristic function* of *bb*. This allows us to define either β or *bb* in terms of the other.

[5] Although our target is a syntactic translation between MSOL+ and PFO+, it is convenient to refer to semantic concepts such as *values* and *ranges* of variables. This is done for ease of exposition. We turn to semantic matters in Chapter 7.

Although this single example captures the essence of the comparison we want to make between MSOL+ and PFO+, it is useful to state things in proper generality. This requires some notation for talking about the various types of expression. As customary, let e be the type of ordinary singular terms and t be the type of truth values. Moreover, for any two types θ_1 and θ_2, we let $\langle\theta_1, \theta_2\rangle$ be the type of functions from entities of type θ_1 to entities of type θ_2. Thus, $\langle e, t\rangle$ is the type of one-place predicates, which (as we have seen) stand for functions from individuals to truth values. But we make a single important addition to this customary setup: we add another basic type, *ee*, as the type of plural terms.

Using this notation, we can rehearse the above explanation—only now stated in proper generality. While MSOL+ adds variables of type $\langle e, t\rangle$, PFO+ adds variables of type *ee*. But there is a natural correspondence between the two types *ee* and $\langle e, t\rangle$. Given any objects *aa*, there is an associated function α that sends an object x to the true just in case $x \prec aa$. We may think of α as the semantic value of the predicative expression '. . . is one of *aa*'. Conversely, given any function α of type $\langle e, t\rangle$, there is—at least according to traditional plural logic—a plurality *aa* of all and only those objects that α sends to the true. Finally, each of MSOL+ and PFO+ adds predicates applying to any number of arguments of type e and of the one additional type available in that language, namely $\langle e, t\rangle$ or *ee*.

With these explanations on board, it is easy to describe translations between the two languages. The basic idea is simply to map each variable xx_i to X_i and vice versa, and to map a predicate with an argument place of type *ee* to a predicate with a corresponding argument place of type $\langle e, t\rangle$ and vice versa. For example, 'cooperate(xx)' is mapped to 'cooperate(X)'.

There is only one small bump in the road. While a monadic concept may apply to no individuals at all, a plurality is ordinarily taken to consist of at least one individual. This bump is easily handled by incorporating into the translations a trick due to Boolos, which we described in Appendix 4.A. The trick is nicely illustrated by considering the behavior of the resulting translations on existentially quantified statements. Let σ be the translation from the plural to the second-order language. Then a plural existential generalization is translated as the corresponding conceptual existential generalization restricted to non-empty concepts, that is:

$$\exists xx\, \varphi \xmapsto{\sigma} \exists X(\exists y Xy \wedge \sigma(\varphi))$$

Let τ be the translation in the opposite direction. Then a conceptual existential generalization is translated as a disjunction:

$$\exists X\,\varphi \quad \overset{\tau}{\longmapsto} \quad \exists xx\,\tau(\varphi) \vee \tau(\varphi')$$

where φ', defined as the result of substituting '$x_i \neq x_i$' everywhere for '$X_j x_i$', is a way of simulating existential generalizations involving empty concepts.

Equipped with these translations, let us now consider the axioms of the two theories. It can be shown that the comprehension axioms of the two theories match via the translations. A second-order comprehension axiom

$$\exists X \forall x(Xx \leftrightarrow \varphi(x))$$

translates as a formula equivalent to the corresponding plural comprehension axiom:

$$\exists x\varphi(x) \rightarrow \exists xx \forall x(x \prec xx \leftrightarrow \varphi(x))$$

And clearly, a plural comprehension axiom translates as a formula equivalent to a second-order comprehension axiom.

It remains only to consider some additional axioms of plural logic. The axiom stating that every plurality is non-empty translates as the trivial claim that every non-empty concept is non-empty. More interestingly, a plural indiscernibility axiom is translated as a corresponding indiscernibility principle in the second-order language:

$$\forall x(Xx \leftrightarrow Yx) \rightarrow (\varphi(X) \leftrightarrow \varphi(Y))$$

Indiscernibility principles of this kind are implausible on a conceptual interpretation of the second-order language, and for that reason are not part of MSOL+, as defined above. Two coextensive concepts might be discerned by modal properties. Assume, for example, that *being a creature with a heart* and *being a creature with a kidney* are coextensive. Even so, these two concepts can be discerned by a modal property such as *possibly being instantiated by something that lacks a heart*. We return shortly to the philosophical significance of this observation.

6.3 The elimination of pluralities in favor of concepts

The formal results just presented open up the possibility of two eliminative strategies. We might try to eliminate pluralities in favor of concepts, or we might try to effect the opposite elimination. Michael Dummett advocates the former option:

> [A] plural noun-phrase, even when preceded by the definite article, cannot be functioning analogously to a singular term. [. . .] [I]t is only as referring to a concept that a plural phrase can be understood [. . .]. To say that it refers to a concept is to say that, under a correct analysis, the phrase is seen to figure predicatively. (Dummett 1991, 93)

As an illustration of his eliminative strategy, he proposes to analyze (6.6) as (6.7) and (6.8) as (6.9).[6]

(6.6) All whales are mammals.

(6.7) If anything is a whale, it is a mammal.

(6.8) The Kaiser's carriage is drawn by four horses.

(6.9) There are four objects each of which is a horse that draws the Kaiser's carriage.

Here, plurals nouns ('whales', 'mammals', 'horses') are replaced by corresponding singular predicates ('is a whale', 'is a mammal', 'is a horse'); moreover, the plural 'four objects' is eliminated in the usual way in favor of first-order quantifiers and identity statements.

To see how Dummett's proposal could be generalized, it is useful to start from a basic example of collective predication:

(6.10) Russell and Whitehead wrote *Principia Mathematica*.

How should this use of plurals be eliminated? One option is to use Dummett's analysis of (6.8) as a model and to regiment (6.10) as (6.11):

(6.11) Russell is a co-writer of *Principia Mathematica*, Whitehead is as well, and no one else is.

However, the formal translation given in Section 6.2 suggests a more systematic approach. First, we regiment (6.10) in PFO+. Then, we apply the formal translation to eliminate pluralities in favor of concepts. The steps are as follows.

[6] For discussion, see for example Rumfitt 2005 and Oliver and Smiley 2016, Chapter 4.

(6.10) Russell and Whitehead wrote *Principia Mathematica*.

(6.12) $\exists xx(\forall y(y \prec xx \leftrightarrow (y = r \lor y = w)) \land \text{wrote}(xx, p))$

(6.13) $\exists X(\forall y(Xy \leftrightarrow (y = r \lor y = w)) \land \text{wrote}(X, p))$

We refer to this eliminative approach as the *predicative analysis*.

This analysis has received much criticism.[7] Before going into details, a general observation is in order. A precondition for eliminating plurals in favor of second-order resources is that plural logic can be interpreted in second-order logic.[8] The objections we will now consider purport to show that the former theory cannot even be interpreted in the latter.

One objection concerns *flexible predicates*, which can combine felicitously with both singular and plural terms. We observed in Section 2.3 that 'own a house' and 'lifted a boat' appear to be flexible. However, the translation defined in Section 6.2 implicitly assumes that there are no flexible predicates. Thus, if such predicates are added to the language, the interpretability result from that section is no longer available.[9]

Oliver and Smiley (2016, 59) discuss the problem, calling attention to the following sentence:

(6.14) Wittgenstein wrote the *Tractatus*, not Russell and Whitehead.

The predicate 'wrote' appears to be flexible, and a natural formalization of this sentence in plural logic would employ a single predicate applying to both Wittgenstein and the duo Russell and Whitehead:

(6.15) $\text{wrote}(w_1, t) \land \exists xx(\forall y(y \prec xx \leftrightarrow (y = r \lor y = w_2)) \land \neg\text{wrote}(xx, t))$

Applying the predicative analysis to (6.15) yields:

(6.16) $\text{wrote}(w_1, t) \land \exists X(\forall y(Xy \leftrightarrow (y = r \lor y = w_2)) \land \neg\text{wrote}(X, t))$

[7] See especially Yi 1999, Yi 2005, and Oliver and Smiley 2016, Chapter 4.

[8] At least, all theoretically useful parts of our plural logic must be so interpretable. If some aspects of this theory were found to be of no scientific use, they might perhaps be abandoned and therefore ignored for the purposes of the elimination.

[9] The need for flexible predicates disappears if one adopts a one-sorted plural logic. In this system, individuals become singleton pluralities. By raising the type of individuals to that of pluralities, one restores uniformity among the argument places of predicates.

However, this sentence is not even well-formed in MSOL+. According to the standard formulation of second-order logic, predicates are *strictly typed*: each of their argument places can be occupied by expressions belonging to a unique sort. So the first argument of 'wrote' cannot be an individual variable in one conjunct and a second-order variable in the other.

In response, the proponent of the predicative analysis could relax the requirement that predicates be strictly typed and allow certain predicates to apply to objects and concepts alike. In the context of higher-order logic, this flexibility is known as *cumulativity*. It is not assumed in standard presentations of second-order logic but there is no formal obstacle to adopting it. A cumulative version of higher-order logic is perfectly consistent. Indeed, Gödel once referred to strict typing as a "superfluous restriction" of higher-order logic (Gödel 1933, 46). In sum, if we allow flexible predicates in plural logic, we can allow flexible predicates in second-order logic too. If second-order logic is modified in this way, we can at least *simulate* predicates of the former kind using predicates of the latter kind.

In linguistics, a variant of the predicative analysis has been proposed by Higginbotham and Schein (1989). It centers around an event-based account of predication and resembles closely the combination of events and mereology discussed in Section 5.6. There we mentioned two ways of analyzing plural predications in terms of events. According to the first, a mereological sum can serve as the agent of the event described by the predicate. According to the second, a mereological sum can serve to represent the atoms that function as co-agents of the event, that is, participate in the event as agents. It is easy to see that the role played by mereological sums in each account could be played by concepts. A concept can serve as the agent of an event or, perhaps more plausibly, it can serve to represent the individuals (namely its instances) that function as co-agents of the event.

Higginbotham and Schein (1989) develop the second approach and analyze (6.17) as (6.18):

(6.17) Some apostles lifted the piano.

(6.18) $$\exists X\big(\exists yXy \wedge \forall y(Xy \rightarrow \text{apostle}(y)) \wedge \exists e(\text{lift-the-piano}(e) \wedge \forall y(Xy \leftrightarrow y \text{ is a co-agent of } e))\big)$$

This approach avoids the objection from flexible predicates. To see why, let us return to the example that illustrated the need for such predicates:

(6.14) Wittgenstein wrote the *Tractatus*, not Russell and Whitehead.

On Higginbotham and Schein's analysis, the predicate 'wrote' applies to events, not to objects or concepts. This means that the predicate's arguments are uniform. Applying their analysis to (6.14) yields someting along the following lines:

$$
\begin{aligned}
(6.19)\quad & \exists e(\text{writing-the-}\mathit{Tractatus}(e) \land w_1 \text{ is an agent of } e) \land \\
& \neg\exists e \exists X\big(\text{writing-the-}\mathit{Tractatus}(e) \land \forall y(Xy \leftrightarrow (y = r \lor y = w_2)) \land \\
& \qquad \forall y(Xy \leftrightarrow y \text{ is a co-agent of } e)\big)
\end{aligned}
$$

So the objection from flexible predicates does not get off the ground.

A second objection to the predicative analysis concerns *extensionality*. The thought is that, because pluralities and concepts differ with respect to extensionality, the former cannot be eliminated in favor of the latter. A clear manifestation of this problem has already emerged. At the end of Section 6.2, we remarked that the main difficulty for interpreting PFO+ in MSOL+ has to do with the indiscernibility principle, which states that coextensive pluralities satisfy the same formulas. The principle strikes most logicians and philosophers as highly plausible and is included in our axiomatization of plural logic. Its second-order translation states that coextensive concepts satisfy the same formulas:

$$\forall x(Xx \leftrightarrow Yx) \rightarrow (\varphi(X) \leftrightarrow \varphi(Y))$$

However, this second-order indiscernibility principle is not plausible (see the example on p. 109), let alone a good candidate for a logical truth. Still, the principle is required if our translation is to yield an interpretation of PFO+ in MSOL+.

Let us illustrate this objection with an example discussed by Yi (1999, 2005). Consider the following inference, which is intuitively valid:

(6.20)

Russell and Whitehead cooperate.
Russell and Whitehead are philosophers who wrote *Principia Mathematica*.

Some philosophers who wrote *Principia Mathematica* cooperate.

The inference can be formalized in PFO+ as:

$$
\text{(6.21)} \quad \frac{\begin{array}{l} \exists xx(\forall y(y \prec xx \leftrightarrow (y = r \vee y = w)) \wedge \text{cooperate}(xx)) \\ \exists xx(\forall y(y \prec xx \leftrightarrow (y = r \vee y = w)) \wedge \\ \qquad\qquad \forall y(y \prec xx \rightarrow \text{philosopher}(y)) \wedge \text{wrote}(xx, p)) \end{array}}{\exists xx(\forall y(y \prec xx \rightarrow \text{philosopher}(y)) \wedge \text{wrote}(xx, p) \wedge \text{cooperate}(xx))}
$$

We obtain a formalization of the inference in MSOL+ by applying to (6.21) the translation procedure described in Section 6.2. The result is as follows:

$$
\text{(6.22)} \quad \frac{\begin{array}{l} \exists X(\forall y(Xy \leftrightarrow (y = r \vee y = w)) \wedge \text{cooperate}(X)) \\ \exists X(\forall y(Xy \leftrightarrow (y = r \vee y = w)) \wedge \\ \qquad\qquad \forall y(Xy \rightarrow \text{philosopher}(y)) \wedge \text{wrote}(X, p)) \end{array}}{\exists X(\forall y(Xy \rightarrow \text{philosopher}(y)) \wedge \text{wrote}(X, p) \wedge \text{cooperate}(X))}
$$

As is easy to verify, the conclusion of each formal argument can be derived from its premises with the help of instances of the appropriate indiscernibility principle. Without them, the validity of the initial inference would be left unexplained.

Since (6.20) is logically valid and does not appear to be enthymematic, the indiscernibility principles must be assumed to be logical. In particular, it must be assumed that *as a matter of logic* a predicate like ‘cooperate’ is extensional in the sense that it does not distinguish between coextensive concepts. Yi concludes:

> it is one thing to hold the extensional conception, quite another to hold, more implausibly, that the truth of the conception rests on logic alone. [...] One cannot meet the objections [...] under the assumption that the property indicated by “*COOPERATE*” is one that Russell calls *extensional* (that is, a second-order property instantiated by any first-order property coextensive with one that instantiates it). This does not help unless the assumption holds by logic [...]. (Yi 2005, 475; see also Yi 1999, 173)

Is this objection fatal to the eliminative project under consideration? We think not. Let a *first-level concept* be a concept of objects, and a *second-level concept* be a concept of first-level concepts.[10] Then the key observation is that

[10] Thus, a first-level concept can be the value of a second-order variable, and more generally, a concept of level n can be the value of a variable of order $n + 1$.

the following fact is provable in a basic extension of MSOL+: every second-level concept has a counterpart that doesn't discern between coextensive first-level concepts. That is, for every second-level concept **P**, there is another second-level concept **P*** applying to all and only the first-level concepts coextensive with those to which **P** applies:

$$\mathbf{P}^*(X) \leftrightarrow \exists Y(\mathbf{P}(Y) \wedge \forall x(Xx \leftrightarrow Yx))$$

Let us call **P*** *the undiscerning counterpart of* **P**. The proponent of the eliminative strategy can use these undiscerning counterparts to capture the extensional behavior of plural predicates.[11] This move does not require that the second-order indiscernibility principle be logical and is indeed consistent with some failures of the principle. In sum, we can admit undiscerning second-level concepts alongside "discerning" ones. We just need to ensure that all plural predicates are translated by means of undiscerning second-level concepts. This shows that plural logic can be *simulated* using concepts.

In fact, there is another reason to think that the objection from extensionality is not fatal. We have seen that the problem posed by flexible predicates can be avoided if we combine plurals and events along the lines indicated by Higginbotham and Schein. Remarkably, their framework also manages to avoid the objection from extensionality. Consider again the potentially problematic inference (6.20):

(6.20)

Russell and Whitehead cooperate.
Russell and Whitehead are philosophers who wrote *Principia Mathematica*.

Some philosophers who wrote *Principia Mathematica* cooperate.

Its validity can easily be explained in Higginbotham and Schein's framework. Roughly put, the premises are understood as stating that Russell and Whitehead are co-agents of events of three kinds: cooperating, being a philosopher, and writing *Principia Mathematica*. It follows from second-order comprehension that there is a concept X whose instances are co-agents of events of those three kinds. But, on Higginbotham and Schein's analysis, this is precisely what the conclusion states.

[11] See Florio 2014a, 12.

A third and final objection is a modal analogue of the objection from extensionality. Membership in a plurality is typically taken to be modally rigid. That is, if a is one of bb, then necessarily so, at least on the assumption that the objects in question exist; and likewise with non-membership. (See Chapter 10 for a defense of this view.) By contrast, falling under a concept is almost universally taken to be modally non-rigid. Although Socrates in fact falls under the concept *philosopher*, he might not have done so. Thus, when modalities are added, we obtain an extension of PFO+ that may no longer be interpreted in the corresponding extension of MSOL+.

One might try to counter this objection by imitating the response given to the extensionality problem. We saw that it is possible to model the behavior of plural predicates using undiscerning second-level concepts, even though many such concepts are discerning. Something similar might work here. It might be possible to model the rigid behavior of pluralities using rigid concepts, even though many such concepts are non-rigid. The crucial question, though, is what assurance we have that the requisite rigid concepts exist. In the presence of plural logic, a compelling argument for their existence is available. For every plurality aa, we can use second-order comprehension to define a corresponding concept:

$$\exists F \Box \forall x(Fx \leftrightarrow x \prec aa)$$

The rigidity of the plurality ensures the rigidity of the concept defined in terms of it. In the absence of plural logic, however, it is unclear that the existence of rigid concepts can be motivated without invoking highly controversial forms of modalized second-order comprehension (see Williamson 2013, Chapter 6). We conclude that the most compelling reason against the elimination of pluralities in favor of concepts has to do with their modal behavior. The modal rigidity of pluralities isn't easily secured in second-order logic without relying on plurals.

6.4 The elimination of concepts in favor of pluralities

Let us now consider the attempt to eliminate in the opposite direction, that is, to eliminate concepts in favor of pluralities. As we will see, this project is more challenging and therefore more likely to fail.[12]

[12] See Williamson 2003, Section IX, with whom we are in broad agreement.

Some difficulties have to do with the fact that there are some natural ways to generalize MSOL+ that have no obvious analogues in the case of plurals. One example is third-order quantification, that is, quantification over second-level concepts. This raises the question of whether there is any plural analogue of quantification of third and higher orders. A monadic second-level concept would correspond to a plurality of pluralities. We are thus led to the question of superplurals, which has surfaced in our discussion from time to time. We defer a proper discussion of the matter to Chapter 9.

Another example concerns polyadic concepts. There are not only monadic concepts but also polyadic ones. If it is permissible to quantify over monadic concepts, it should be equally permissible to quantify over polyadic ones. By contrast, there is no obvious polyadic analogue of plural quantification.[13] So it is unclear whether polyadic second-order logic can be interpreted in some form of plural logic.

One might respond that the desired polyadic analogues can be defined provided that ordered pairs are available. Suppose that for all individuals a and b, there is an ordered pair $\langle a, b\rangle$ such that:

$$\langle a, b\rangle = \langle a', b'\rangle \leftrightarrow a = a' \wedge b = b'$$

Using ordered pairs, it is easy to define a plural analogue of any relation. For example, a plurality of ordered pairs can be used to represent the extension of a dyadic relation, provided that the relation is non-empty. This plural analogue of a relation might perhaps be criticized for being too indirect and unnatural. Just as a monadic concept of ordered pairs can represent a dyadic relation but isn't *really* one, a plurality of ordered pairs too can represent a dyadic relation but isn't *really* one.

Although we find talk about what relations "really are" somewhat nebulous, a clear objection can be extracted from the fog. Consider again the mismatch between the modal profile of pluralities and that of concepts. We observed in the previous section that it is possible to model the behavior of pluralities using rigid concepts, even though concepts are in general non-rigid. But the reverse direction is problematic. With only modally rigid material at our disposal we are unable to model any non-rigid phenomenon. For example, while a plurality of ordered pairs can model *the extension* of a dyadic relation, it cannot in general represent all of its *intensional* features.

[13] See Hewitt 2012a for an attempt to develop a non-obvious analogue.

Yet another reason against the elimination of concepts in favor of pluralities emerges in the final chapter, where we argue that the comprehension scheme found in traditional plural logic must be restricted. In particular, we deny that there is a plurality of absolutely every object, which renders our system unable to represent the universal concept.

6.5 Conclusion

We have compared plural and second-order logic. As part of this undertaking, we have investigated to what extent one of these systems can be interpreted in the other.

We found that second-order logic in its entirety cannot be interpreted in plural logic. It is unclear how to handle polyadic concepts or concepts of higher levels, and, most seriously, there is no way to handle the intensionality of second-order logic, using only modally rigid pluralities. Since interpretability is a precondition for elimination, we conclude that second-order logic cannot be eliminated in favor of plural logic.

What about the other direction? We found that plural logic can be interpreted in terms of second-order logic, at least in the absence of modality. This raises the question of whether plural logic should be eliminated in favor of a subsystem of second-order logic. We regard the proposed elimination as problematic, for several reasons. The identification is *prima facie* implausible because of the deep and pervasive differences in how plurals and predication are represented in English and other natural languages. Being a member of a plurality and falling under a concept are, it seems, simply different things.

Our detailed analysis identifies a further, more robust, reason against the proposed elimination. As already stressed, a precondition for this elimination is the interpretability of plural logic in second-order logic. We have isolated various assumptions that are needed for this interpretability result to obtain. First, we found that plural predicates are extensional, while predicates of concepts are not. We showed how to circumvent this problem by invoking undiscerning second-level concepts. Second, we observed that plural membership and predication have different modal profiles. While the former is a matter of necessity (at least conditional on the continued existence of all the objects in question), the latter is not. This difference was neutralized, in our interpretability result, only by the assumptions that plural terms stand for rigid concepts. But the most natural and compelling argument for the existence of such concepts seems to rely on plurals.

The conclusion of this chapter echoes those of the preceding ones. There are several ways to represent many objects simultaneously, at least in ordinary circumstances where our domain is a set. In addition to taking plurals at face value, we may use sets, mereological sums in the individual sense, or monadic concepts. Although these systems share a common formal structure, at least in ordinary circumstances, the notions that they represent are different and must be kept apart.

What if ordinary circumstances do not obtain? In Chapter 12, we provide an account of domains that do not form a set and argue that, in such domains, another deep difference between plurals and predicates emerges: while the former are subject to a form of limitation of size, the latter are not. If correct, this account provides yet another reason not to attempt to eliminate second-order logic in favor of plural logic.

Let us briefly summarize all of Part II. We have examined four different ways to talk about many objects simultaneously. By identifying various philosophical and formal differences, we have argued that none of these ways should be eliminated in favor of any other. Thus, the detailed, pairwise comparisons of Part II yield an argument for primitive plurals that Chapter 3 failed to produce.

III

PLURALS AND SEMANTICS

7
The Semantics of Plurals

Plural logic has emerged as an appealing framework for the regimentation of natural language plurals and for the development of various philosophical projects. As we will now see, however, the choice of a regimenting language leaves wide open the semantic question of how such a language should be interpreted. Since plural logic is characterized by a precise axiomatic theory, one may wonder why we should be interested in its semantics.

The semantic question is important, for at least two reasons. First, a semantics is needed for a complete account of plural logic. It is by means of a semantics that we define a relation of logical consequence, which can be used to identify valid as well as invalid arguments. The axioms of plural logic help us reason correctly but, by themselves, do not tell us which arguments are invalid. This limitation can be overcome by studying not only which meanings the expressions of the language have but also which meanings such expressions *might* have. The part of semantics concerned with possible meanings is model theory. In model theory, logical consequence is defined as truth preservation under every interpretation (model) of the language, where an interpretation is simply an assignment of possible meanings. Starting from the notion of interpretation, we thus obtain a fully general way of characterizing whether or not a conclusion is a logical consequence of some premises. Second, the alleged features of plural logic that underlie the philosophical applications discussed in Section 2.6—ontological innocence, expressive power, and absolute generality—are really semantic features and hence can only be assessed in light of a worked-out semantics.

In this chapter, we are primarily concerned with traditional plural logic. In particular, the various semantic accounts we consider validate the unrestricted axiom scheme of plural comprehension. These accounts have to be adjusted if, as we suggest in Chapter 12, an alternative logic is adopted.

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7.1 Regimentation vs. semantics

The regimentation of plurals was discussed in Chapter 3, where we focused on the following two views.

> REGIMENTATION SINGULARISM
>
> A singular language suffices to regiment a fragment of natural language containing plurals, where a language is said to be "singular" if it has no plural resources, unlike, say, $\mathcal{L}_{\mathrm{PFO+}}$.
>
> REGIMENTATION PLURALISM
>
> Plural terms, variables, and predicates are required to regiment the relevant fragment of natural language.

We asked whether singularism can provide a satisfactory regimentation of plurals and found that, though often benign, this approach has some shortcomings as a general strategy. Especially if we assume that absolute generality is possible and that traditional plural logic is valid, there are good reasons to favor regimentation pluralism. In light of this, we examined the relation between plural logic and other systems.

While regimentation is relevant to semantics, it does not determine how semantic interpretations should be specified, at least not in any obvious way. So it is important to distinguish two questions that are often conflated:

(Q1) How should a given fragment of natural language be regimented?
(Q2) Once a regimenting language has been chosen, how should we specify the semantic interpretations of that language?

While the first question is entirely about the object language, the second is also about the metalanguage.

The importance of the distinction between regimentation and semantics becomes clear when we look at other cases of semantic analysis. Consider, for example, modalities. The first question concerns the proper regimentation of modal notions. Should they be regimented as predicates of sentences? Or should they rather be regimented as operators? As is well known, considerations related to paradox strongly recommend the second approach. Once a particular regimentation has been chosen, a second question arises as to how semantic interpretations for the regimented language should be specified. The most popular option, embodied in standard possible world semantics, is simply to characterize models as set-theoretic constructions.

Modalities are therefore absent from the semantics. An alternative approach would take modal notions as primitive in the metalanguage and use them to define the interpretations of the regimented language. But, as the set-theoretic approach shows, the metalanguage need not embrace the notions being analyzed. This means that the semantics cannot be directly read off the regimented language. The transition from regimentation to semantics is a delicate one.

The case of plurals is no different: here too we have options. In perfect analogy with the case of modalities, we could provide a set-theoretic specification of the interpretations of the language, or we could take plural talk as primitive in the metalanguage and use it to formulate the semantics. For our purposes, it is useful to characterize two broad methodological approaches to the semantic question.

> SEMANTIC SINGULARISM
>
> Once the regimenting language has been chosen, semantic interpretations can be specified within a theory formulated in a singular metalanguage.
>
> SEMANTIC PLURALISM
>
> Once the regimenting language has been chosen, semantic interpretations must be specified within a theory formulated in a plural metalanguage.

The second approach was Boolos's great innovation and marks a clean break with broadly Quinean approaches, which might acknowledge the availability of plural resources but would not permit their use in rigorous theorizing.

Combining the two semantic approaches with the approaches to regimentation discussed earlier, we end up with four alternatives, shown in the table below together with some of their supporters.

	regimantation singularism	regimentation pluralism
semantic singularism	Quine	(some linguists)
semantic pluralism	—	McKay, Oliver & Smiley, Simons, Rayo, Yi

The bottom left-hand quadrant is empty because regimentation singularism makes semantic singularism almost inevitable. If plurals are just singular expressions in disguise, why should we appeal to plurals in the semantics? But as emphasized above, regimentation pluralism does not make semantic pluralism inevitable. If plurals are *not* just singular expressions in disguise, however, why avoid plurals in the semantics? Possible reasons have to do not only with simplicity and ideological economy, but also with expressibility problems that arise when one embraces type-theoretic hierarchies, as we will see in Chapter 11.

7.2 Set-based model theory

Model theory provides the standard setting for the characterization of logical properties. Given an object language $\mathcal{L}$, one first defines the notions of *interpretation* (or *model*) of $\mathcal{L}$ and *truth in an interpretation* (or *satisfaction*). Then one uses these notions to characterize the key relation of logical consequence for sentences of $\mathcal{L}$. A sentence ψ is said to be a logical consequence of a set of sentences Σ just in case ψ is true in every interpretation in which every member φ of Σ is true. When this holds, we write:

$$\Sigma \models \psi$$

Other logical properties (such as logical truth and consistency) can easily be defined in terms of consequence.

The possible interpretations of $\mathcal{L}$ are obtained by varying two features: the domain of quantification and the interpretation of the non-logical terminology of $\mathcal{L}$ (that is, its constants and predicates). Thus, an interpretation is fixed by specifying these two features. The second feature is specified by means of an *interpretation function* (or *interpretation*, for short).

The ordinary implementation of model theory is based on sets. Working within set theory, an interpretation of $\mathcal{L}$ is taken to be a pair $\langle d, f \rangle$, where d is a set representing the domain and f is an interpretation function from the non-logical terminology of $\mathcal{L}$ to set-theoretic constructions generated by d. When $\mathcal{L}$ is the language of first-order logic, the situation is familiar. For example, f assigns an element of the domain to singular constant of $\mathcal{L}$, and it assigns a subset of the domain to each monadic predicate.[1]

The next step is to define what it is for a sentence to be true in an interpretation. Again, the situation is familiar when $\mathcal{L}$ is the language of first-order logic. We obtain the definition of truth in an interpretation from the more general relation of truth in an interpretation relative to a

[1] Our semantics treats all terms as denoting, as is usually done. The semantics could, if desired, be generalized to allow non-denoting terms. For discussion of non-denoting terms in the context of plurals, see Oliver and Smiley 2016, Chapter 5.

variable assignment. Let $i = \langle d, f \rangle$ be an interpretation, and let s be a *variable assignment* relative to d, namely a function assigning an element of d to each variable of $\mathcal{L}$. We use the notation $[\![E]\!]_{i,s}$ as follows:

$$[\![E]\!]_{i,s} = \begin{cases} f(E), & \text{if } E \text{ is a constant or a predicate;} \\ s(E), & \text{if } E \text{ is a variable.} \end{cases}$$

(We may omit one or both subscripts when the intended notation is clear from context.) So $[\![E]\!]_{i,s}$ stands for the semantic value of E (that is, its "denotation") according to i or s.

To define when a formula φ is true in i relative to s, written $i \vDash \varphi\ [s]$, we proceed by induction on the complexity of φ via satisfaction clauses. If φ is an atomic formula, say St, then:

(Sat-A) $\quad i \vDash St\ [s]$ if and only if $[\![t]\!]_{i,s} \in [\![S]\!]_{i,s}$

Since we treat the identity predicate as logical, it is always interpreted homophonically (namely by means of the analogous predicate in the meta-language). That is:

(Sat-=) $\quad i \vDash t_1 = t_2\ [s]$ if and only if $[\![t_1]\!]_{i,s} = [\![t_2]\!]_{i,s}$

If φ is a negation $(\neg\psi)$ or a conjunction $(\psi_1 \wedge \psi_2)$, then we have the obvious clauses:

(Sat-¬) $\quad i \vDash \neg\psi\ [s]$ if and only if it is not the case that $i \vDash \psi\ [s]$

(Sat-∧) $\quad i \vDash \psi_1 \wedge \psi_2\ [s]$ if and only if $i \vDash \psi_1\ [s]$ and $i \vDash \psi_2\ [s]$

If φ is an existential generalization $(\exists v\, \psi)$, then

(Sat-∃) $\quad i \vDash \exists v\, \psi\ [s]$ if and only if $i \vDash \psi\ [s(v/x)]$ for some $x \in d$

where $s(v/x)$ is an assignment just like s, with the possible exception that $s(v/x)$ assigns x to v.

We are now ready to define our target notion, namely truth in an interpretation. The definitions just given ensure that if φ is a sentence (that is, it has no free variable), we can ignore variable assignments. More precisely, φ is true in i relative to a variable assignment if and only if φ is true in i relative to any other variable assignment. Thus, we can define φ to be true in i, written $i \vDash \varphi$, if φ is true in i for some (equivalently, every) variable assignment.

So far, we have only recapitulated the standard, set-based definition of truth in an interpretation for the language of first-order logic. But there is a straightforward extension of this definition to the richer language $\mathcal{L}_{\mathrm{PFO+}}$. An interpretation of this language is a pair $\langle d, f \rangle$, just as before, only that f now also assigns set-theoretic semantic values to plural constants and predicates. For example, f assigns a non-empty subset of d to every plural constant, and it assigns a (possibly empty) set of non-empty subsets of d to every monadic plural predicate. Likewise, a variable assignment s relative to d is extended by assigning a non-empty subset of d to each plural variable.

The extended definition of truth in an interpretation relative to an assignment is achieved by adding the following satisfaction clauses to the previous ones. If φ is an atomic plural predication, say Ptt, then

(Sat-PA) $\quad i \vDash Ptt\ [s]$ if and only if $[\![tt]\!]_{i,s} \in [\![P]\!]_{i,s}$

For the special case of plural membership, we have:

(Sat-$\prec$) $\quad i \vDash t \prec tt\ [s]$ if and only if $[\![t]\!]_{i,s} \in [\![tt]\!]_{i,s}$

This means that the interpretation of plural membership does not vary: it always corresponds to set-theoretic membership. Finally, if φ is a plural existential $(\exists vv\, \psi)$, then

(Sat-P$\exists$) $\quad i \vDash \exists vv\, \psi\ [s]$ if and only if $i \vDash \psi\ [s(vv/x)]$ for some non-empty $x \subseteq d$

where $s(vv/x)$ is an assignment just like s, with the possible exception that $s(vv/x)$ assigns x to vv.

It is worth highlighting an important implication of the last clause: plural quantifiers are taken to range over the full powerset of the first-order domain d, minus the empty set.[2] This semantic treatment of plural quantifiers corresponds to a *standard* interpretation of second-order logic, that is, an

[2] Since the range of plural quantifiers is always determined by the first-order domain, there is no need to specify it separately.

interpretation in which the second-order quantifiers range over all subsets of the first-order domain. In the next chapter, we develop an analogue of Henkin semantics, which permits plural quantifiers to have a narrower range.

To explain the standard semantics with which we are currently concerned, it might help to consider a particular interpretation of $\mathcal{L}_{\mathrm{PFO}}$. This example will also be useful later in our discussion. To keep things simple, let us assume that $\mathcal{L}_{\mathrm{PFO}}$ contains only the following items:

A. *singular terms*: two constants t and r, plus the usual variables $(v, v_1, v_2, \ldots)$;
B. *plural terms*: two constants tt and rr, plus the usual variables $(vv, vv_1, vv_2, \ldots)$;
C. *singular predicates*: a monadic predicate S;
D. parentheses and the usual logical symbols ($\neg, \wedge, \exists, \prec$, etc.).

The interpretation we want to consider is $i = \langle d, f \rangle$, with $d = \{a, b, c\}$ and f defined by the following identities:

$$\begin{array}{ll} f(t) = a & f(r) = b \\ f(tt) = \{a, b\} & f(rr) = \{b, c\} \\ f(S) = \{a, b\} & \end{array}$$

Then, for example, the next two sentences are true in i:

(7.1) $r \prec tt$

(7.2) $\exists v(v \prec rr \wedge \neg Sv)$

That is easy to verify using the clauses given above. For (7.1), we reason like this:

$$\begin{array}{ll} i \vDash r \prec tt & \text{if and only if} \\ i \vDash r \prec tt\,[s], \text{ for some } s & \text{if and only if} \\ [\![r]\!]_{i,s} \in [\![tt]\!]_{i,s}, \text{ for some } s & \text{if and only if} \\ f(r) \in f(tt) & \text{if and only if} \\ b \in \{a, b\} & \end{array}$$

For (7.2), an analogous series of steps yields:

$$i \vDash \exists v(v \prec rr \wedge \neg Sv) \quad \text{if and only if}$$
$$x \in \{b, c\} \text{ and } x \notin \{a, b\}, \text{ for some } x \in d$$

Similarly, we could verify that these two sentences are false in i:

(7.3) $t \prec rr$

(7.4) $\neg\exists v(v \prec rr \wedge Sv)$

It might also help to consider a particular interpretation of $\mathcal{L}_{\text{PFO+}}$. This example too will be useful later in our discussion. For simplicity, we assume that $\mathcal{L}_{\text{PFO+}}$ augments $\mathcal{L}_{\text{PFO}}$ with just one (monadic) plural predicate P. Our interpretation of PFO+ is an extension of $i = \langle d, f \rangle$, the interpretation of $\mathcal{L}_{\text{PFO}}$ presented just above. We only need to specify a semantic value for P. Let f^+ be an extension of f such that $f^+(P) = \{\{a, b\}\}$. Then $i^+ = \langle d, f^+ \rangle$ is an interpretation of $\mathcal{L}_{\text{PFO+}}$. It follows by construction that any sentence of $\mathcal{L}_{\text{PFO}}$ that is true in i remains true in i^+. Here are some sentences available in $\mathcal{L}_{\text{PFO+}}$ but not in $\mathcal{L}_{\text{PFO}}$:

(7.5) Ptt

(7.6) $\exists vv \neg Pvv$

(7.7) $\exists v \exists vv((Sv \wedge v \prec vv) \wedge Pvv)$

Using the appropriate clauses, it would be easy to verify that, in i^+, the first two sentences are true while the last one is false.

On the semantics just given, PFO and PFO+ have metalogical properties that distinguish them from first-order logic. Neither system is complete or compact, and both lack the Löwenheim-Skolem property. Indeed, both systems are able to provide categorical characterizations of arithmetic and analysis, and a quasi-categorical characterization of set theory. In this sense, the expressive power of both systems goes beyond that of first-order logic.

7.3 Plurality-based model theory

We have seen that the familiar set-based model theory is easily extended to PFO and PFO+. However, there is nothing inherent in the idea of a model theory that requires it to be set-theoretic. So why not adopt plural resources in the metalanguage and exploit these richer resources to represent the semantic values of plural expressions? This alternative approach, which we call *plurality-based model theory*, was initiated by Boolos (1985a). On Boolos's new semantic paradigm, the semantic value of a plural variable is not a set (or any kind of set-like object) whose members are drawn from the ordinary, first-order domain. Rather, a plural variable has many values from this ordinary domain and thus ranges plurally over it. This semantic approach to plurals has become very popular among philosophers.[3]

To develop a plurality-based model theory, we proceed much as before. Working within plural logic, we first define a notion of interpretation. Then, we use this notion to characterize, via satisfaction clauses, that of truth in an interpretation. And as before, we rely on variable assignments as an intermediate step.

Let us spell out these steps. In set-based model theory, we defined domains and interpretations functions as special kinds of set-theoretic objects. This allowed us to define an interpretation as a pair $\langle d, f \rangle$. These definitions were possible because the semantic values of terms and predicates were themselves objects. But now we want the semantic value of a plural term to be one or more objects. So an interpretation function can no longer be a function in the usual set-theoretic sense. We need a different strategy.

The model-theoretic characterization of logical consequence requires that we can quantify over interpretations: an argument is valid just in case it is truth-preserving under *every interpretation* of the language. Since our metalanguage has just two sorts of quantifiers—singular and plural—interpretations must be either objects or pluralities. Given the semantic shift sanctioned by Boolos's approach, it is natural to consider the idea that interpretations themselves might be pluralities rather than objects. As it turns out, this idea leads to a nice formulation of the plurality-based model theory.

If we postulate a pairing operation on objects, there is a relatively simple way to proceed. Recall that an interpretation is fixed by specifying a domain of quantification and the interpretation of the non-logical terminology of the language. We can represent a domain of quantification by pairing a conventional symbol, say the symbol '$\exists$', with each element of the domain. If we want the domain to consist of the objects a and b, for example, we

[3] See, e.g., Oliver and Smiley 2005; Yi 2005; Yi 2006; McKay 2006, Chapter 3; Rayo 2006; and Oliver and Smiley 2016, Sections 11.5, 12.5, and 13.2.

will represent that by means of the pairs $\langle `\exists', a\rangle$ and $\langle `\exists', b\rangle$. (For simplicity, we omit the quotation marks in this type of ordered pairs and write: $\langle \exists, a\rangle$ and $\langle \exists, b\rangle$.) Similarly, we can represent an interpretation function by pairing the relevant expressions with their semantic value or values. For example, if we want to assign the plurality a and b to the term *tt*, we will do that by means of the pairs $\langle tt, a\rangle$ and $\langle tt, b\rangle$. An interpretation will just be a plurality *ii* of pairs representing the relevant semantic information. Among *ii* there will be pairs representing information about the domain and pairs representing an interpretation function. Quantifying over interpretations amounts to quantifying over the appropriate pluralities of pairs.

Let us illustrate the new definition of interpretation by showing how to convert the set-based interpretation $i = \langle d, f\rangle$ from the previous section into a plurality-based interpretation. This way of coding a plurality-based interpretation goes back to Boolos 1985a. First, the domain $d = \{a, b, c\}$ is represented by these three pairs:

$$\langle \exists, a\rangle \quad \langle \exists, b\rangle \quad \langle \exists, c\rangle$$

Call these pairs *dd*. Next, there is the interpretation function *f*, which was defined by the following identities:

$$\begin{array}{ll} f(t) = a & f(r) = b \\ f(tt) = \{a, b\} & f(rr) = \{b, c\} \\ f(S) = \{a, b\} & \end{array}$$

We can represent *f* by means of eight pairs:

$$\begin{array}{ll} \langle t, a\rangle & \langle r, b\rangle \\ \langle tt, a\rangle\, \langle tt, b\rangle & \langle rr, b\rangle\, \langle rr, c\rangle \\ \langle S, a\rangle\, \langle S, b\rangle & \end{array}$$

Call these pairs *ff*. The plurality-based interpretation corresponding to i is the plurality *ii* combining *dd* and *ff*. Thus *ii* consists of the eleven pairs shown above.

Our next goal is to provide the satisfaction clauses defining the relation of truth in a plurality-based interpretation relative to a variable assignment. In this context, a variable assignment is a plurality of pairs *ss* representing the assignment of an object to each singular variable and of one or more objects to each plural variable. For instance, an assignment ss containing precisely the pairs $\langle v, a\rangle$, $\langle vv, b\rangle$ $\langle vv, c\rangle$ is one that assigns a to v and the plurality b and c to vv. The notation for semantic values will follow our earlier convention. That is, we let the symbol $[\![E]\!]_{ii,ss}$ indicate the interpretation of E according to ii if E is a term or a predicate, and we let it indicate the assignment to E according to ss if E is a variable. In both cases, the result can be one or more things. For instance, if we consider the interpretation ii and the assignment ss just introduced, we have that $[\![tt]\!]_{ii,ss}$ indicates a and b, whereas $[\![vv]\!]_{ii,ss}$ indicates b and c.

We are finally ready to state the clauses that define when a formula φ is true in ii relative to ss, written $ii \vDash \varphi\ [ss]$. If φ is an atomic formula, say St, then:

(Sat-A*) $\quad ii \vDash St\ [ss]$ if and only if $[\![t]\!]_{ii,ss} \prec [\![S]\!]_{ii,ss}$

A small wrinkle needs to be ironed out. A predicate may obviously have an empty extension, but there is no empty plurality. This mismatch is easily handled, for example by always adding an arbitrary triple to the interpretation of any predicate. This convention will henceforth be implicit in model theories where predicates are given a plural interpretation.

For plural membership, we have:

(Sat-≺*) $\quad ii \vDash t \prec tt\ [ss]$ if and only if $[\![t]\!]_{ii,ss} \prec [\![tt]\!]_{ii,ss}$

Notice that plural membership is always interpreted homophonically, in accordance with our decision to treat it as logical. If φ is a negation $(\neg\psi)$ or a conjunction $(\psi_1 \wedge \psi_2)$, then we have:

(Sat-¬*) $\quad ii \vDash \neg\psi\ [ss]$ if and only if it is not the case that $ii \vDash \psi\ [ss]$

(Sat-∧*) $\quad ii \vDash \psi_1 \wedge \psi_2\ [ss]$ if and only if $ii \vDash \psi_1\ [ss]$ and $ii \vDash \psi_2\ [ss]$

Let dd be the domain of ii. If φ is a singular existential $(\exists v\, \psi)$, then

(Sat-∃*) $\quad ii \vDash \exists v\, \psi\ [ss]$ if and only if $ii \vDash \psi\ [ss(v/x)]$ for some $x \prec dd$

where $ss(v/x)$ is an assignment just like ss, with the possible exception that $ss(v/x)$ assigns x to v. If φ is a plural existential $(\exists vv\, \psi)$, then

(Sat-P∃*) $\quad ii \vDash \exists vv\, \psi\ [ss]$ if and only if $ii \vDash \psi\ [ss(vv/xx)]$ for some $xx \preccurlyeq dd$

where $ss(vv/xx)$ is an assignment just like ss, with the possible exception that $ss(vv/x)$ assigns xx to vv.

Plural quantification receives, again, a *standard* interpretation. Plural quantifiers are taken to range over *every* subplurality of the first-order domain. Interestingly, it is possible to formulate an alternative, Henkin-style semantics even within a plurality-based model theory. We develop this idea in the next chapter, where we also explore its significant philosophical implications.

We have obtained a definition of the relation of truth in an interpretation relative to a variable assignment for formulas of $\mathcal{L}_{\mathrm{PFO}}$. However, our definition is carried out in a richer metalanguage, namely $\mathcal{L}_{\mathrm{PFO+}}$. This is because the relation being defined ("φ is true in ii relative to ss") has a singular argument for formulas and two plural arguments, one for interpretations and one for assignments. So, in this setting, '... is true in ... relative to ...' is a plural predicate, which takes us beyond PFO into PFO+. This is not an accident but a manifestation of Tarski's theorem on the undefinability of truth. We examine this phenomenon more closely in Chapter 11. A consequence of immediate concern is that we should expect the model theory for PFO+ to require an even richer metalanguage (see Section 7.5).

As in the case of set-based model theory, the satisfaction of a sentence is independent of the choice of variable assignment. So we can define truth in an interpretation as follows. For any sentence φ, φ is true in ii, written $ii \vDash \varphi$, if φ is true in ii for some (equivalently, every) variable assignment.

Going back to the interpretation ii we used as an example, it is easy to verify that ii makes true the same sentences that were made true by its set-based counterpart i. Recall that

(7.1) $r \prec tt$

was shown to be true in i (p. 129). We can verify that this sentence is true in ii by means the following reasoning:

$ii \vDash r \prec tt$	if and only if
$ii \vDash r \prec tt\,[ss]$, for some ss	if and only if
$[\![r]\!]_{ii,ss} \prec [\![tt]\!]_{ii,ss}$, for some ss	if and only if
$b \prec a$ and b	

For another example, apply the above clauses to (7.2), which is easily seen to yield:

$ii \vDash \exists v(v \prec rr \wedge \neg Sv)$ if and only if

$x \prec b$ and c, and $x \not\prec a$ and b, for some $x \prec dd$

where *dd* is the domain of *ii*. These truth conditions now have a pleasing homophonic feel.

We mentioned above some of the key metalogical properties of PFO on the set-based model theory. PFO turns out to have the same metalogical properties on the plurality-based model theory just outlined. In particular, PFO is neither complete nor compact, and it lacks the Löwenheim-Skolem property. So PFO continues to have more expressive power than first-order logic.

While the two model theories are on a par with regard to such metalogical properties, it has nevertheless been argued that there are reasons to prefer the plurality-based model theory over its set-based analogue. Let us turn to this issue.

7.4 Criticisms of the set-based model theory

There is an apparent element of artificiality in the set-based model theory. Plural terms are taken to denote sets. So plural quantification is interpreted as quantification over sets. By contrast, the plurality-based model theory does justice to the intuitive idea that a plural term does not stand for a set of things, but it stands for the things themselves. Intuitively, the term 'Paris and Rome' does not stand for the set of the two cities; it stands for the cities themselves. The plurality-based approach captures this intuitive idea. It assumes that a plural term refers *plurally* to some things, without the mediation of a set that stands proxy for them.

The issue becomes especially pressing when the things intuitively denoted by a plural term are too many to form a set. Consider a plural constant intended to refer to all the sets. Assuming traditional plural logic, we can construct a plurality-based interpretation in which this term refers plurally, as intended, to all sets.[4] There is no corresponding interpretation in the set-based model theory. We must interpret a plural term by means of a single set.

[4] If the correct plural logic is the "critical" one we propose in the final chapter, then this interpretation is unavailable. We discuss the semantic significance of this approach in Section 12.8.

But there is no set of all sets in standard set theory. So the semantic value of our constant cannot encompass all sets.

Another manifestation of this issue concerns the domain of quantification. By requiring that the domain of quantification be a set, the set-based model theory rules out any interpretation whose domain is too big to form a set. In particular, there is no set-based interpretation whose domain includes all sets. This means that there is no interpretation corresponding to the intended model of set theory. The set-based model theory is thus unable to capture all the intuitive interpretations of the language.

The plurality-based model theory avoids these limitations, again assuming traditional plural logic. Since the domain of quantification is given by a plurality, it is possible to represent a domain encompassing all sets. Consider all and only the pairs such that their first coordinate is the symbol '$\exists$' and the second coordinate is a set. Plural comprehension and the existence of a pairing operation on objects jointly entail that there is a plurality of exactly those pairs. This plurality represents a domain encompassing all sets. Likewise, there is a plurality that represents a domain encompassing every object whatsoever. Therefore, the plurality-based model theory can be said to capture not only the intended interpretation of set theory but also absolute generality.

As stated, these considerations pertain to *intuitive* limitations of the set-based model theory, namely its inability to represent intuitive semantic values or intended interpretations. But are such considerations relevant to logic? We can move beyond the intuitive level by focusing on a key fact that has been implicit in our discussion. While every set-based interpretation can be converted into a plurality-based one, it is a consequence of the plural version of Cantor's theorem that the reverse claim isn't true.[5] This fact is relevant to logic. For logical consequence is defined by quantifying over every interpretation and hence depends on which interpretations are admitted. Let us elaborate on this claim.

Imagine two parties A and B wishing to characterize the relation of logical consequence for sentences of a given language $\mathcal{L}$. Suppose B has a richer conception of interpretation than A. That is, every interpretation

[5] The theorem states that the subpluralities of *xx* are strictly more numerous than the members of *xx*, provided that *xx* has two or more members (see Section 3.5). Using traditional plural logic, we let *xx* be the universal plurality—that is, the plurality of absolutely all objects. It follows that the pluralities are more numerous than the objects. Since any plurality can be the domain of a plurality-based interpretation, it follows in turn that the plurality-based interpretations are more numerous than the set-based interpretations, which are objects.

countenanced by A is also countenanced by B, but not the other way around. So, letting $\vDash_A$ and $\vDash_B$ be the two parties' consequence relations, we have:

$$\Delta \vDash_B \varphi \Rightarrow \Delta \vDash_A \varphi$$

for any set of sentences Δ and any sentence φ. But there is no guarantee that the opposite implication holds and hence no guarantee that the two relations of consequence are coextensive. The proponent of the set-based model theory and the proponent of the plurality based model theory are in the same situation as A and B.

When the language is first-order, Georg Kreisel's famous "squeezing argument" can be used to establish that the two model theories yield an equivalent relation of consequence (Kreisel 1967). Let $\vDash_P$ and $\vDash_S$ be the relation of consequence sanctioned, respectively, by the plurality-based model theory and by the set-based model theory. But let us restrict attention to first-order sentences. The argument goes as follows. In the preceding paragraph, we established that:

$$\Delta \vDash_P \varphi \Rightarrow \Delta \vDash_S \varphi$$

By the completeness theorem for first-order logic, we have

$$\Delta \vDash_S \varphi \Rightarrow \Delta \vdash \varphi$$

where $\vdash$ is the usual provability relation for first-order logic. Finally, we observe that the plurality-based account of consequence is sound with respect to this relation:

$$\Delta \vdash \varphi \Rightarrow \Delta \vDash_P \varphi$$

This closes the circle of implications. It follows that:

$$\Delta \vDash_S \varphi \Leftrightarrow \Delta \vDash_P \varphi$$

In sum, despite the fact that the the proponent of the plurality-based model theory has a richer conception of interpretation than its set-based competitor, their definitions of logical consequence yield exactly the same verdict for arguments involving first-order sentences.

An essential premise of Kreisel's argument is that the set-based relation of consequence satisfies the completeness theorem. But this premise might not hold when we move beyond first-order logic. In fact, it fails for PFO, which is not complete according to the plurality-based model theory presented above. So Kreisel's argument is not available for PFO. We can, however, get the same effect by appealing to set-theoretic reflection principles. For a simple example, consider the principle which asserts that any sentence of PFO that is true in the universe of sets is also true in some set-based model:

$$\text{(PR)} \qquad \varphi \rightarrow \exists\alpha\,(\varphi^{V_\alpha})$$

where φ^{V_α} is the result of restricting the quantifiers of φ to the set V_α, whose elements are all the sets of rank less than α (see Section 4.6). The principle (PR) turns out to be equivalent to the claim that any sentence that has a plurality-based model also has a set-based model (Shapiro 1987). This ensures the extensional equivalence of two definitions of *logical truth*: one in terms of truth in every set-based model, the other in terms of truth in every plurality-based model. An analogous result is available for the notion of *logical consequence*, although the required reflection principle is stronger than the one just mentioned (again, see Shapiro 1987).

These results may assuage the worries with which the section started. The inability of a model theory to represent some intuitive semantic values or intended interpretations need not have an effect on the logic. In particular, the apparent artificiality of the set-based model theory need not manifest itself at the level of logical consequence. For example, the model theory does not validate incorrect existential consequences of the kind discussed in Section 3.3. In other words, although plural terms have sets as semantic values, a sentences like *Ptt* does not logically entail that sets exist. A parallel case is that of predication in the usual set-based model theory for first-order logic. Predicates have sets as semantic values. Yet a predication like *St* does not logically entail that sets exists. Boolos (1984b, 448–9) insisted that "it doesn't follow just from the fact that there are some Cheerios in the bowl that, as some who theorize about the semantics of plurals would have it, there is also a set of them all." A set-based model theory does not sanction that it follows *logically* from the fact that there are some Cheerios in the bowl that there is also a set of them.

As far as logic is concerned, and assuming the appropriate reflection principle, we have found no reason to think that adopting a set-based model theory for PFO is any more problematic than adopting a set-based model theory for first-order logic. As we will see shortly, however, other considerations may help us decide which is the more appropriate type of model theory.

7.5 The semantics of plural predication

The interpretation of plural predicates raises a number of interesting questions. We presented a plurality-based model theory for PFO in Section 7.3. Let us now examine how this model-theoretic approach can be extended to PFO+. There are two main ways to proceed, depending on whether we want to give plural predicates an extensional or an intensional interpretation.

As formulated above, the plurality-based model theory for PFO incorporates an extensional treatment of singular predication. For the semantic value of a monadic singular predicate S is the plurality of objects to which S applies, that is, its extension. This choice of semantic value aligns with the choice of semantic value in the set-based model theory, where S is assigned the set of objects in the domain to which S applies.

One might instead take an *intensional* approach to predication and interpret predicates not as pluralities, but as properties. Suppose—if only temporarily—that properties are objects, and let us interpret S by means of the property σ. Then we could simply include the pair $\langle \exists, \sigma \rangle$ in our interpretation function. Recall the plurality-based interpretation function *ff* for $\mathcal{L}_{\text{PFO}}$ described in Section 7.3 (p. 132):

$$
\begin{array}{ll}
\langle t, a \rangle & \langle r, b \rangle \\
\langle tt, a \rangle \; \langle tt, b \rangle & \langle rr, c \rangle \; \langle rr, d \rangle \\
\langle S, a \rangle \; \langle S, b \rangle &
\end{array}
$$

On an intensional approach, *ff* would be replaced by the following plurality of pairs:

$$
\begin{array}{ll}
\langle t, a \rangle & \langle r, b \rangle \\
\langle tt, a \rangle \; \langle tt, b \rangle & \langle rr, c \rangle \; \langle rr, d \rangle \\
\langle S, \sigma \rangle &
\end{array}
$$

The clause for singular predication would be revised accordingly:

$$ii \vDash St\,[ss] \text{ if and only if } [\![t]\!]_{ii,ss} \text{ has } [\![S]\!]_{ii,ss}$$

That is, the truth of a singular predication St amounts to the fact that the semantic value of the term t (an object) has the semantic value of S (a property).

We thus have two approaches to singular predication, one extensional and one intensional. How do these approaches extend to *plural* predication?

Let us start with the extensional approach. Suppose the semantic value of a singular predicate S is a plurality of objects. Then the most natural choice of semantic value for a plural predicate P would be a "plurality of pluralities", or a "superplurality" as we have called it. The intuitive reason is this. The predicate S applies to objects and hence its semantic value is the plurality of the objects to which it applies. Since P applies to pluralities, its semantic value must be the plurality of pluralities to which it applies.

But what is a "plurality of pluralities"? Throughout the book, we use 'plurality' as a shorthand for a plural construction. Thus, to talk about "a plurality of dogs" is just a shorthand for talking about one or more dogs. It is controversial whether it makes sense to talk about "a plurality of pluralities" and hence whether the expressive resources needed to formulate an extensional version of plurality-based model theory for PFO+ are legitimate. We address the question of superplurals in Chapter 9. For the time being, we would like to make two remarks. First, it is relatively straightforward to develop a formal system of superplural quantification suitable to develop our model theory (Rayo 2006). Moreover, natural language offers at least some help. We can think of a superplurality as some things articulated into distinct subpluralities, such as: Russell and Whitehead, and Hilbert and Bernays, or: these things, those things, and these other things.[6]

Assuming the legitimacy of the expressive resources needed, we must find a way to incorporate the proposed interpretation of plural predicates into the model theory. We would like to proceed much as in the case of singular predicates, where a singular predicate S was interpreted by means of a plurality of ordered pairs, $\langle S, a\rangle$, $\langle S, b\rangle$, et cetera. Each of these ordered pairs, say $\langle S, x\rangle$, represents that S applies to x. Now consider a *plural* predicate P. We would like to interpret P by means of a bunch of ordered pairs, which we may think of as $\langle P, aa\rangle$, $\langle P, bb\rangle$, et cetera. Each of these ordered pairs, say $\langle P, xx\rangle$, represents that P applies to xx. The problem, however, is that no sense has yet been assigned to expressions such as '$\langle P, aa\rangle$'. After all, an ordered pair is an ordered pair *of objects*. Thus, the second coordinate of an ordered pair must be an object; it cannot be a plurality of two or more objects.

Fortunately, there is a natural way to assign sense to the mentioned expressions.[7] Suppose we want to talk about the ordered pair of P and aa.

[6] For more examples from natural language, see Section 9.4.
[7] For details and a more general treatment, see Appendix 11.A.

Consider all ordered pairs of the form $\langle P, a \rangle$, where $a \prec aa$. The plurality *pp* of such ordered pairs can be used to represent the desired but problematic pair $\langle P, aa \rangle$. To see that this representation works, observe first that the representing plurality *pp* is well defined, and second, that the representation uniquely determines *P* and *aa* and thus does all the work that the problematic ordered pair was meant to do. To wit: given the mentioned plurality *pp* of ordered pairs, *P* can be retrieved as the unique object that figures as the first coordinate of all the ordered pairs *pp*, and *aa* can be retrieved as the plurality of objects each of which figures as the second coordinate of one of these ordered pairs. In light of this, it is unproblematic to use the familiar notation '$\langle P, aa \rangle$' as a suggestive shorthand for the mentioned plurality *pp*.

With this convention in place, we can proceed to state the interpretation of a plural predicate *P* as a bunch of ordered pairs, $\langle P, aa \rangle$, $\langle P, bb \rangle$, et cetera—keeping in mind that this bunch will be a superplurality, as it is a bunch of pluralities. We can then say, informally, that an atomic plural predication, such as *Ptt*, is true if and only if *tt* stands for a plurality that appears as second coordinate in one of the pairs $\langle P, aa \rangle$, $\langle P, bb \rangle$, et cetera.

To provide a formal clause capturing these truth conditions, we need to define a notion of interpretation capable of representing the target interpretation of plural predicates as well as other non-logical expressions. As it turns out, this can be done by letting an interpretation be a superplurality *iii*.[8] Its components will be a domain *dd* and an interpretation function *fff*, which now consists of a superplurality. A variable assignment remains a plurality *ss*. Let us use 'are among' to indicate the membership relation between a plurality *xx* and a superplurality *xxx*. So, loosely speaking, *xx* are among *xxx* just in case *xx* are one of the pluralities comprising *xxx*. We are finally in a position to state the satisfaction clause for an atomic plural predication:

$$iii \models Ptt\,[ss] \text{ if and only if } [\![tt]\!]_{iii,ss} \text{ are among } [\![P]\!]_{iii,ss}$$

where $[\![tt]\!]_{iii,ss}$ is a plurality and $[\![P]\!]_{iii,ss}$ a superplurality.

Let us now turn to the intensional approach to predication, which takes properties rather than pluralities (or superpluralities) as semantic values of predicates. To interpret plural predicates, we need *plural properties*, that is, properties that (if instantiated) are instantiated by many things *jointly*. By

[8] Again, see Appendix 11.A for technical details.

contrast, *singular properties* (if instantiated) are instantiated by many things *separately*. For example, the property of cooperating is plural while that of being human is singular.

Suppose that plural properties are objects.[9] Then, given any plural property π, we could obtain an interpretation function ff^{+} for $\mathcal{L}_{\mathrm{PFO+}}$ by simply adding the pair $\langle P, \pi \rangle$ to the interpretation function ff considered above. For an atomic predication, we would therefore have:

$$ii \vDash Ptt\,[ss] \text{ if and only if } [\![tt]\!]_{ii,ss} \text{ have } [\![P]\!]_{ii,ss}$$

which is perfectly analogous to the satisfaction clause for singular predication on the intensional approach.

However, there is significant pressure to reject the supposition that plural properties are objects. For this supposition is subject to a variant of a Russell-style argument put forth by Williamson (2003), and it clashes with a plural version of Cantor's theorem.[10] Let us briefly comment on the last claim. As discussed in Section 3.5, the instance of Plural Cantor concerned with the universal plurality entails that there are more pluralities than objects. If properties are objects, it follows that there cannot be a plural property corresponding to every plurality. But this may seem implausible. If there are some things, why shouldn't there also be a plural property had by them and only them?

The natural reaction is to postulate a type distinction between objects and properties.[11] Thus, we may think of properties as higher-level entities, which, following Frege, we call concepts. This view leads us to combine the resources of plural logic with those of second-order logic. The resulting system is an extension of PFO+ where quantification into predicate position is allowed for both singular and plural predicates. For example, these generalizations are legitimate:

(7.8) $\exists F \neg Ftt$

(7.9) $\exists F \forall vv\, Fvv$

(7.10) $\forall vv \exists F\big(Fvv \wedge \forall uu(Fuu \rightarrow \forall v(v \prec uu \leftrightarrow v \prec vv)\big)$

[9] In the context of plural logic and its semantics, this assumption is endorsed by Hossack 2000 and McKay 2006.

[10] See Florio 2014c for a detailed exposition of these arguments.

[11] This approach is endorsed, for example, by Oliver and Smiley and by Yi (see footnote 3 on p. 131).

In fact, they are also provable with the help of the appropriate instances of this scheme of comprehension for plural concepts:

(PSO-Comp) $\exists F \forall xx(Fxx \leftrightarrow \varphi(xx))$

where F does not occur free in φ. We assume this scheme as well as its polyadic analogues.

Postulating a type distinction between objects and properties avoids the two problems mentioned above. The Russell-style argument is blocked for essentially the same reason Russell's paradox was blocked in the simple set theory discussed in Chapter 4. That is, owing to the sortal distinctions between individuals, pluralities, and properties (now concepts), the key condition driving the paradoxical argument cannot even be formulated (see Williamson 2003, Section IX). Moreover, if properties are no longer objects, it is consistent to hold *both* that there are more pluralities than objects *and* that there is a plural property corresponding to every plurality. So there is no clash with the plural version Cantor's theorem.

How can we accommodate the view that predicates stand for concepts in the plurality-based model theory? As in the case of superplurals, we cannot supply the interpretation of predicates by simply adding some ordered pairs to the interpretation function. Concepts are not objects. So we cannot represent the fact that the semantic value of S is the singular concept X, and the semantic value of the predicate P is the plural concept Y, by means of $\langle S, X\rangle$ and $\langle P, Y\rangle$. For these are not proper pairs.

In fact, there is a way to represent not only single such "pairs" but also many of them simultaneously. We resort to concepts of an even higher level than X and Y. To represent a bunch of pairs of the form $\langle S, X\rangle$, we use a second-level concept $\mathbf{R}$ with two argument places, one open to objects and the other open to first-level concepts, such that $\mathbf{R}(S, X)$ just in case $\langle S, X\rangle$ is one of the target bunch of pairs.[12] By quantifying over the appropriate sort of higher-level concepts, we can then define a notion of interpretation capturing the informal idea that an atomic plural predication Ptt is true if and only if Fxx, where xx is the plurality for which tt stands and F is the plural concept interpreting P.

[12] This approach, in its current version, assumes that concepts are individuated extensionally. That assumption can be dispensed with, if desired. One option is to adopt suitable modalized comprehension principles of the form $\exists F \Box \forall x(Fx \leftrightarrow \varphi(x))$.

7.6 The problem of choice

We have surveyed a number of ways in which the model theory for plural logic may be developed. On the one hand, we could opt for a set-based model theory. We argued in Section 7.4 that this model theory is no more problematic than the usual set-based model theory for first-order logic, at least from a purely logical point of view and assuming the relevant reflection principle. On the other hand, we could opt for an extensional or intensional version of a plurality-based model theory. The formulation of this style of model theory has led us to introduce richer expressive resources. The model theory for PFO was carried out in PFO+, as it construed interpretations as pluralities and thus relied on a plural predicate to characterize the notion of truth in an interpretation. The model theory for PFO+ was carried out in a metalanguage including either superplural quantification or quantification over concepts. As noted, the ascent to more expressive metalanguages is not an accident but a robust Tarskian phenomenon, which we explore systematically in Chapter 11.

The existence of multiple model theories presents us with a problem: how are we to choose between the available options? This *problem of choice*, as we will call it, is connected with large and difficult philosophical questions. So we will not attempt to reach a final verdict. We do, however, aim to paint as complete a picture as possible of the considerations that are relevant to solving the problem.

It is useful to classify the available options on the basis of whether or not they use certain resources. Does a given option use plural resources? And does it use conceptual resources? The table below summarizes the alternatives and indicates some of the authors who have adopted them.

	no plural resources	plural resources
no conceptual resources	Link and other linguists, Quine, Resnik	Boolos, Hossack, McKay, Rayo
conceptual resources	Higginbotham & Schein, Florio	Oliver & Smiley, Yi, Rayo & Yablo, Williamson, Linnebo

Three of the four alternatives are exemplified by model theories discussed above: the set-based model theory uses neither plural not conceptual resources; the extensional version of the plurality-based model theory uses plural but not conceptual resources; the intensional version of plurality-based model theory uses both plural and conceptual resources.

However, we have not presented any option that uses only conceptual resources. Let us briefly mention two such options.[13]

The first imitates the eliminative strategy of Higginbotham and Schein discussed in Chapter 6. They analyze predication in terms of events and eliminate pluralities in favor of concepts. The same resources (events and concepts) could be put to use in developing a model theory for PFO+. The second way to construct a model theory that uses only conceptual resources imitates another eliminative strategy discussed in Chapter 6. We have seen that there is a translation of PFO+ into MSOL+. Similar results can be obtained for extensions of these theories. In particular, the systems needed to formulate the extensional or intensional version of plurality-based model theories for PFO+ can be interpreted in a fragment of higher-order logic containing a few layers of concepts. These conceptual resources can be used to develop a model theory for PFO+. What's more, it can be shown that the resulting model theory delivers the same relation of consequence as the plurality-based model theory (see Florio 2014a).

Recall that the set-based model theory can deliver the same relation of consequence as the plurality-based model theory for PFO, assuming the appropriate reflection principle (Section 7.4). This continues to hold when we add plural predicates to the object language. It follows that all the options considered are on a par, as far as logic is concerned. If we are to solve the problem of choice, we must look beyond the logic.

In light of this, someone with a purely instrumental view of the semantics may deny that there is a problem of choice after all. If model theory is a mathematical tool whose sole purpose is to characterize a relation of consequence for a given language, then any option is just as good as any other option—at least in the case at hand. And if an option must be selected, one might well select the simplest and most economical one. However, if we want model theory to be not just a mathematical study of logical relations but a study of possible meanings of natural and formal languages, then there might be further constraints capable of discriminating between the options.

[13] The existence of all these alternatives may be unsurprising: owing to the interpretability results presented in Part II, we know that it is possible in principle to imitate any of the relevant systems in any other of those systems.

7.7 Absolute generality as a constraint

One such constraint is absolute generality. Since this issue will receive a detailed discussion in Chapter 11, we limit ourselves to some brief remarks. Absolutely general quantification seems possible; for example, it seems possible to assert that the empty set has no elements, none whatsoever. What would be the domain of an absolutely general quantifier? It cannot be a set, since standard set theory recognizes no universal set. Perhaps it can be a plurality (though see Chapter 12). For example, plural comprehension implies that there are some objects uu such that:

$$\forall x(x \prec uu \leftrightarrow x = x)$$

This plurality includes absolutely everything there is and hence every set. Using uu as a domain, the plurality-based model theory is able to represent an absolutely general interpretation of the quantifiers. Thus, absolute generality—if there is such a thing—can serve as a constraint that narrows down the options, ruling out the set-based model theory.

Could this conclusion be resisted by using some non-standard set theory that does accept a universal set?[14] We might for example use New Foundations (see Forster 2019) or some "logical" notion of set (or property) of the kind developed in Fine 2005c and Linnebo 2006. Any such maneuver would only shift the bump in the carpet. Recall the plural version of Cantor's theorem. In the context of traditional plural logic, this implies that there are more pluralities than objects. Since any plurality can be used to interpret a one-place predicate—namely by letting the predicate be true of precisely these objects—the theorem means that there are more interpretations of the predicate than can be represented by objects—no matter what kind of singular representation one chooses, including any non-standard conception of set. If we want our model theory to capture every possible interpretation of a predicate, then the plural version of Cantor's theorem serves to rule out any form of semantic singularism cashed out in first-order terms.

While absolute generality promises to be a powerful weapon against any such form of semantic singularism, it is perfectly consistent with a model theory formulated within higher-order logic. If we accept higher-order logic,

[14] Class theories such as von Neumann-Bernays-Gödel and Morse-Kelley take a major step in this direction by accepting a class of all sets. But there can be no class of all objects, since a class is prohibited from being an element of a set or class.

there is good reason to accept a universal concept. This corresponds to the following instance of second-order comprehension:

$$\exists F \forall x(Fx \leftrightarrow x = x)$$

The universal concepts can then serve to represent a domain of quantification containing absolutely all objects. As indicated in Section 7.6, it can be shown more generally that every plurality-based model has an isomorphic model described with purely conceptual resources (Florio 2014a).

So absolute generality does not single out the plurality-based model theory as the only viable option. Put in terms of our diagram on p. 144, absolute generality takes out the upper left-hand quadrant, but is neutral with respect to the three remaining options.

7.8 Parity constraints

There are additional constraints one might impose on the model theory. One might require that certain theoretical and empirical desiderata be satisfied. For example, one might want the metatheory to be ontologically parsimonious or the representation of possible meanings to be psychologically plausible. Moreover, one might want to be able to integrate the model theory for PFO+ with the model theory for a language including a broader class of expressions (such as generalized quantifiers, adverbs, and modalities). As we have seen in Chapter 5, a similar thought is an important motivation behind the analysis of plurals in terms of individual mereology favored by some linguists.

In this section, we want to examine some constraints that demand some form of parity between the language being analyzed and the language used to analyze it. We therefore call them *parity contraints*. We consider four such constraints.

First, we remarked above that the truth conditions of the plurality-based model theory have a pleasing homophonic feel. This contrasts with the set-based model theory whose truth conditions appear more artificial: they equate truth in an interpretation to facts about sets. Could we rule out semantic singularism, and any model theory based on purely conceptual resources, by assuming *homophonicity* as a constraint? We need to be careful. Since we are dealing with model theory, not truth theory, it is not really an option to require homophonicity across the board. This would conflict with the aim of letting non-logical expressions be reinterpreted from model to model.

Note that this also applies to more restricted requirements of homophonicity, such as the following *deflationist constraint on predication*:

(7.11) $\forall y$('F' is true of $y \leftrightarrow Fy$)

It states that a predicate can be said to be true of whatever it can be truly predicated of. Friederike Moltmann cites this constraint in support of plural reference and notes that it reflects "what is generally considered an important condition on a semantic theory, namely that the object language be included in the metalanguage" (2016, 108). A plural predicate can be truly predicated of pluralities and hence these must be the semantic values of the terms with which the predicate combines. Is this a reasonable constraint? Again, we need to be careful. Since we are dealing with model theory, our ability to enforce the constraint is limited. While the constraint may be plausible for truth theory, it is not an option for model theory.

Our question, then, is whether one might capture the spirit of homophonicity and of the deflationist constraint on predication through a requirement that is applicable to model theory. A second constraint does just that—by requiring that the semantic value of an expression of the object language be given by an expression of the same type in the metalanguage. Let us call this the principle of *type preservation*. This constraint is less demanding than the previous two, requiring only that the type of an expression be preserved by its semantic value. On the plausible assumption that the relevant types are determined by the logico-linguistic categories of PFO+, type preservation solves the problem of choice in favor of the intensional version of the plurality-based semantics. Type preservation rules out the extensional version of plurality-based model theory because this model theory interprets predicates as superpluralities rather than concepts. Moreover, it rules out the remaining model theories because they fail to preserve the type of plural terms, interpreting them as objects or concepts rather than pluralities.

However, the power of type preservation comes with far-reaching, revisionary consequences. It can often be illuminating to analyze expressions of one type using resources from other types. An example is the analysis of tense in terms of explicit reference to, and quantification over, moments of time. Another example is the usual set-based model theory for ordinary first-order languages, which interprets predicates by means of sets rather than concepts and therefore violates the principle of type preservation.

Indeed, much of the set-based model theory employed in linguistics and mathematics would have to be rewritten. Especially in the case of linguistics, it is unclear whether this can be done successfully while holding on to the principle of type preservation. We have no guarantee that the array of types needed to regiment natural languages can be systematically incorporated in a unified and adequate model theory. A case in point is that of modals: these expressions are usually interpreted using possible worlds, not primitive modalities in the metalanguage.

A final parity constraint concerns the *modal profile* of plural terms. These are generally thought to be rigid.[15] That is, the following principles are supposed to hold. Let E be an existence predicate (paraphrasable as $\exists zz\, zz \approx \ldots$), and let $\Box$ stand for metaphysical necessity. Then:

(RGD$^+$) $$\Box\forall x\, \forall yy\, (x \prec yy \rightarrow \Box(Eyy \rightarrow x \prec yy))$$

(RGD$^-$) $$\Box\forall x\, \forall yy\, (x \nprec yy \rightarrow \Box(x \nprec yy))$$

Informally, the principles state that if this object is one of those objects, then necessarily, whenever those objects exist, this object is one of them. Similarly, if this object is not one of those objects, then necessarily this object is not one of them. But unlike plural terms, predication is not rigid, as illustrated by the next example.

(7.12) John is tall but might not have been.

Now consider this constraint concerning modal profile: semantic values should have the same modal profile as the expressions of which they are semantic values. It would follow that predicates cannot have superpluralities as semantic values, as the latter but not the former are rigid. Similarly, plural terms could not have concepts as semantic values, as the former but not the latter are rigid. Thus, the constraint helps with the problem of choice by eliminating the off-diagonal options in our diagram (p. 144).

Once again, the constraint can be challenged. For example, Kripke semantics for modal logic is widely regarded as illuminating, despite using semantic values with the "wrong" modal profile. Although sets possess their members necessarily, in Kripke semantics they are successfully used as semantic values of predicates. The key is to allow these semantic values to vary from world to world.

[15] See Chapter 10 for details and a defense.

7.9 Conclusion

We have discussed various constraints that may help us choose among the available model theories for plural logic. First, we have absolute generality, which rules out the options in the upper left-hand quadrant of our diagram. Then, we have two parity constraints applicable to model theory: type preservation—which selects an option in the bottom right-hand quadrant—and modal profile—which rules out the off-diagonal options. Putting everything together, only the bottom right-hand quadrant remains.

However, neither of these parity constraints was found to be absolutely compelling. So there seems to be no simple solution to the problem of choice. What is required to make progress, it seems to us, is greater clarity on what model theory is supposed to do. On a minimal conception of the role of model theory, such as the one espoused by an instrumentalist about semantics, the existence of several, equally good options is perfectly acceptable. This is less so if model theory is supposed to capture certain features of the "true nature" of our expressive resources. In that case, one might insist on a model theory that is not only extensionally but also intensionally correct. This is especially important when we lack independent means of determining the correct extension. If so, we can establish an extensionally correct theory by relying on an intensionally correct one. These considerations lend further support to the option in the bottom right-hand quadrant. This option will play a role in Chapter 11, where we again discuss semantic matters in the presence of absolute generality.

8
On the Innocence and Determinacy of Plural Quantification

8.1 Introduction

Plural logic has undoubtedly become an important component of the philosopher's toolkit.* Many of its applications depend on two alleged virtues: ontological innocence and expressive power. In this chapter, we want to assess whether plural logic has these virtues and thus whether those applications are ultimately justified.

It is commonly assumed that plural logic is ontologically innocent in the sense that plural quantifiers do not incur ontological commitments beyond those incurred by the ordinary first-order quantifiers. This alleged virtue of plural logic is supported by the plurality-based model theory pioneered by Boolos (1985a) and further developed by Agustín Rayo and Gabriel Uzquiano (1999). (For an overview and discussion of this form of model theory, see Chapter 7.) On this model theory, the value of a plural variable is not a set (or any kind of set-like object) whose members are drawn from the ordinary, first-order domain. Rather, a plural variable has *many values* from this ordinary domain and thus *ranges plurally* over this domain. Of course, in ascribing to a plural variable many values, the plurality-based model theory makes essential use of the plural resources of the metalanguage. In a nutshell, on the traditional set-based model theory, a plural variable ranges in an ordinary way over a special domain reserved for variables of its type, whereas on the new kind of plurality-based model theory, a plural variable ranges in a special, plural way over the ordinary domain.[1]

The second alleged virtue of plural logic is expressive power. To see this point, consider first the case of second-order logic with its two kinds of

* Most of this chapter derives from Florio and Linnebo 2016.

[1] Defenses of the innocence of plural logic in the sense just defined are put forth, among others, by Boolos 1984b, and Boolos 1985a, Yi 1999, Yi 2002, Yi 2005, Yi 2006; Hossack 2000, Oliver and Smiley 2001 and 2016; Rayo 2002; McKay 2006.

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Oxford University Press.
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traditional set-based model theory. In standard semantics, the second-order quantifiers range over the full powerset of the first-order domain, whereas in Henkin semantics the second-order quantifiers may range over a subset of this powerset. This gives rise to an interesting debate about semantic determinacy. That is, does our linguistic practice single out, relative to a given domain, the interpretation given by the standard semantics as the correct one?[2] An important aspect of this question is that it is only on the standard semantics that second-order logic can truly be said to offer more expressive power than first-order logic. For second-order logic on the Henkin semantics may be regarded as a version of first-order logic, namely a first-order system with two sorts of quantifiers. As such, it has all the main metalogical features of first-order logic: it is complete, compact, and has the Löwenheim-Skolem property. But, for the same reason, it fails with respect to the main accomplishments of second-order logic with the standard semantics. Chiefly, it does not discriminate between importantly different classes of structures, such as countable and uncountable ones, and it fails to ensure the categoricity of arithmetic and analysis, and the quasi-categoricity of set theory.

In this respect, plural logic on the plurality-based model theory, as well as higher-order logic on a parallel higher-order model theory, is thought to provide a significant improvement over second-order logic on the set-based model theory. Indeed, one finds many claims to the effect that plural logic, on the plurality-based model theory, is immune to the threat of non-standard (Henkin) interpretations that confronts higher-order logics on their more traditional, set-based model theory. Nearly all writers who have embraced plural logic on the plurality-based model theory ascribe to this system metalogical properties which presuppose that the semantics is standard rather than Henkin, but without flagging this as a substantive presupposition as one would do as a matter of routine in the case of systems with a set-based model theory.[3] The failure to make this presupposition explicit strongly suggests that the only plurality-based interpretation is the standard one. So it is naturally interpreted as a commitment to the standard semantics rather than the Henkin alternative. Why else claim that plural logic—not plural logic *with standard semantics*—lacks a complete axiomatization and

[2] See Shapiro 1991, Chapter 8. A notable consequence of the view that second-order quantification is determinate is the thesis famously held by Kreisel and others that the Continuum Hypothesis is either true or false (for discussion, see Weston 1976).

[3] See, for instance, Rayo and Uzquiano 1999, 315–18; Yi 1999, 180–1; Hossack 2000, 439–41; Yi 2006, 256–7; McKay 2006, 139–43; Rayo 2007, 215; Oliver and Smiley 2016, 246–9.

compactness, yielding the categoricity of arithmetic and analysis, and the quasi-categoricity of set theory?

In any case, a striking feature of the literature on this novel kind of model theory for plural logic is the near-absence of debate about the semantic determinacy of plural quantification thus interpreted.[4] Indeed, on the plurality-based approach, the only interpretation of the plural quantifiers that has been articulated is the standard one. No analogue of Henkin semantics has been developed. The following diagram sums up the kinds of semantics currently available:

	standard semantics	Henkin semantics
set-based semantics	A. Tarski	L. Henkin
plurality-based semantics	G. Boolos	—

The apparent absence of a plurality-based Henkin semantics has no doubt influenced the ensuing debate. It has encouraged the thought that plural logic on the plurality-based model theory is immune from non-standard interpretation, and thus the thought that plural logic does better than higher-order logic on the set-based model theory in securing a gain in expressive power.

As appealing as this common picture of plural logic may be, we believe that it is far too optimistic. Our aim in this chapter is to develop an alternative picture, one in which both alleged virtues of plural logic—ontological innocence and expressive power—are much less significant than they are made out to be. We argue that set-based and plurality-based model theory are on a par with respect to worries about indeterminacy. So no progress is made by switching from the former to the latter. We do not take a stand on which side of the debate prevails; though in the absence of a compelling argument, we urge caution about the determinacy claims. Moreover, we articulate a generalized notion of ontological commitment according to which plural logic is not, after all, innocent. This provides, for the first time, a precise development of some ideas by Parsons 1990 (section 6), Hazen 1993, Shapiro 1993, and Linnebo 2003. Our focus is on plural logic, though much of what we say would apply, *mutatis mutandis*, to second- and higher-order logics that quantify into predicate position.

[4] The same is true for a higher-order model theory for higher-order logic, though Rayo and Yablo 2001 provides a rare exception.

Our pursuit of the mentioned aims uses as its main tool a semantics for plural logic that fills the gap in the above diagram. Accordingly, the first part of the chapter is devoted to the development and defense of a plurality-based Henkin semantics. (Technical details can be found in this chapter's appendices.) In the second part of the chapter, we reconsider the alleged virtues of plural logic in light of the new semantics. The resulting picture is one in which the role of plural logic as a philosophical tool is substantially diminished.

8.2 A plurality-based Henkin semantics

As announced, our first step is to construct a plurality-based Henkin semantics for plural logic and thus populate the empty quadrant in the above diagram. Although from a technical standpoint this is largely a straightforward adaptation of the familiar set-based Henkin semantics, arguing for its philosophical legitimacy is not straightforward. Once the resources needed to develop a plurality-based Henkin semantics are identified, they must be shown to be in good standing vis-à-vis the resources used to develop the plurality-based standard semantics.

Our object language will be $\mathcal{L}_{\mathrm{PFO}}$. As with the set-based model theory, the plurality-based Henkin models consist of a domain for the first-order quantifiers, a representation of the range of the plural quantifiers, and an interpretation function that specifies the semantic values of the non-logical terminology of the language. The crucial difference is that, in our case, the first-order domain, the range of the plural quantifiers, and the interpretation functions will not be set-theoretic objects.

A domain dd for the first-order quantifiers will consist of *some things*—any things in the domain of the metatheory. Next, to represent the range of the plural quantifiers, we need a "collection" D of pluralities. We will think of D as a plural concept, but an alternative interpretation is available: D may be taken to be a superplurality. We remain neutral between these interpretations.

The pluralities 'in' D will be exactly those that instantiate D. We require that the two domains be connected in the following way: for every xx such that $D(xx)$ (that is, xx instantiate D), xx are among dd. In symbols:

$$\forall xx(D(xx) \rightarrow xx \preccurlyeq dd)$$

Finally, we continue to assume the model-theoretic framework presented in Section 7.3. In particular, we assume that the metatheory is equipped with a pairing operation so that an interpretation function can be defined as some ordered pairs *ii* specifying the semantic value or values of each non-logical item in the vocabulary of the object language. (A more precise formulation of the semantics is provided in Appendix 8.A.) As is well known, the standard deductive system for second-order logic is sound and complete with respect to set-based Henkin semantics. As one would expect, this result carries over to the case of plurality-based Henkin semantics for plural logic. A completeness proof is given in Appendix 8.B.

Two aspects of our semantics deserve to be highlighted. First, as in the plurality-based standard semantics, plural quantifiers in our plurality-based Henkin semantics do not range over any special kind of set-like objects. Rather, they range plurally over things in the domain of the first-order quantifiers. Second, the formulation of the semantics requires expressive resources that go beyond those of plural logic. The variable *D*, used to represent the non-standard interpretations for the plural quantifiers, introduces a form of third-order quantification. As interpreted above, *D* stands for a plural concept. The alternative is to give *D* a superplural interpretation. (See Chapter 9 for a discussion of superplurals.) Either interpretation of *D* might raise worries about the legitimacy of the additional expressive resources required by our semantics. So let us address this issue next.

8.3 The legitimacy of ascending one order

As shown in Chapter 7, a standard version of the plurality-based model theory for PFO does not require expressive resources beyond those of PFO+. So, when describing standard interpretations of $\mathcal{L}_{\text{PFO}}$, there is no need to invoke a variable *D*. This is only needed if we wish to "select" a non-standard range for the plural quantifiers. In the plurality-based standard semantics, a sentence of the form $\exists vv\ \varphi$ is true in a model of the language just in case *some things* among those in the first-order domain satisfy the formula φ. The formulation of this clause relies only on plural quantification. In our Henkin semantics, we want to impose the additional requirement that the things satisfying the formula also be among the pluralities represented by *D*.

The expressive economy of the plurality-based standard semantics may be thought to constitute an important advantage of that semantics over our Henkin alternative, especially when coupled with some skepticism about the legitimacy of expressive resources going beyond PFO+. However, we believe that this advantage of the plurality-based standard semantics over our Henkin alternative is not significant. For, as we will now argue, the additional expressive resources required by our semantics are available, and they are needed anyway for independent semantic reasons.

As observed in Section 7.5, it is relatively straightforward to develop a formal system of third-order quantification suitable to develop the plurality-based Henkin semantics (see Rayo 2006). Thus the expressive resources under discussion are available at least in the sense of belonging to the inventory of possible semantic mechanisms. Moreover, there is evidence from natural language that such resources are available also in the stronger sense of being actually in use. On the one hand, familiar arguments for the presence in natural language of quantification into predicate position extend from singular to plural predicates. In Section 6.1, we observed that examples such as 'John is everything we wanted him to be' are naturally regimented using bound variables in predicate position (Higginbotham 1998, 251, but see also Rayo and Yablo 2001). The same conclusion vis-à-vis plural predicates is suggested by analogous examples involving plural predication, such as 'John and Mary are everything we wanted them to be'. This vindicates the interpretation of *D* in terms of plural concepts. On the other hand, it has been argued that natural languages such as English contain superplural expressions (see Oliver and Smiley 2004, Oliver and Smiley 2005, and Oliver and Smiley 2016, Section 8.4; Linnebo and Nicolas 2008), which provides at least *prima facie* support for the superplural interpretation of *D*. (Again, see Chapter 9 for details.)

An important reason why the expressive resources required by our semantics are needed anyway has to do with absolute generality. This emerged in Chapter 7, where a plurality-based standard semantics for PFO was carried out in PFO+ and the same kind of semantics for PFO+ was carried out in a richer metalanguage including either superplural quantification or quantification over concepts. Let us recapitulate the main idea.[5]

An attractive feature of the plurality-based standard semantics is that it allows us to capture models whose first-order domain of quantification contains absolutely everything. By means of the plural resources available in the metalanguage, one can define models in which the first-order quantifiers range over *all objects*. But, if quantification over absolutely everything is

[5] As we noted in Chapter 7, the appeal to resources going beyond PFO+ can also be motivated by parity constraints other than absolute generality.

possible, developing a model theory for plural logic requires the introduction of a new non-logical predicate. Specifically, it requires the introduction of a plural predicate functioning as a satisfaction predicate (see Rayo and Uzquiano 1999). However, once the original language of plural logic has been expanded to include plural predicates, ascending one order higher becomes unavoidable. For it is now known that a model theory for the language expanded to include plural predicates will require a language that is one order higher than plural logic (see Chapter 11). So, if one wants to do justice to the possibility of quantifying over absolutely everything, semantic considerations push the expressive resources up one order.

Whether one interprets this higher-order quantification as quantification over plural concepts or as superplural quantification, semantic reflection will eventually lead the proponent of the plurality-based standard semantics to embrace the expressive resources needed to formulate the plurality-based Henkin semantics. Since the additional resources needed to formulate our Henkin semantics are available and needed anyway for independent semantic reasons, we conclude that the expressive economy of plurality-based standard semantics does not constitute a significant advantage over our plurality-based Henkin semantics.

8.4 Does ontological innocence ensure determinacy?

The previous two sections establish that there exist plurality-based yet non-standard interpretations of a plural language. This is significant. For it is commonplace to maintain that plural logic on the plurality-based model theory is determinate. The view goes back at least to Boolos's famous argument that plural logic is non-firstorderizable. The argument is based on plural logic's alleged ability to distinguish standard from non-standard models of arithmetic (Boolos 1984a, Boolos 1984b, and Boolos 1985a). But of course, if our plurality-based non-standard interpretations are admitted, then plural logic is no better equipped to make such distinctions than, say, a first-order set theory. This contrasts with the widespread view that, when formulated with the help of plural quantification, arithmetic and analysis are categorical, and set theory is quasi-categorical; and relatedly, that plural logic is not axiomatizable (see footnote 3). To be perfectly clear: we are not claiming that all proponents of this view deny or fail to recognize the existence of plurality-based non-standard interpretations. Our claim is that their remarks are potentially misleading because they suggest that the only plurality-based interpretation is the standard one.

It might be responded that, while we have shown that plurality-based non-standard interpretations exist, they can safely be set aside as unintended or illegitimate. Doing so would restore the determinacy of plural logic, which the views just referenced all presuppose. The key question, it seems to us, is whether this response is any better than the analogous response for traditional set-based interpretations. That is, does plural logic on a plurality-based model theory have a *better* claim to determinacy than plural logic on a set-based model theory? Let *Plural Robustness* be the view that the plurality-based model theory is superior in this regard. A defense of Plural Robustness would have to show that the plurality-based standard interpretations are in better standing vis-à-vis their (plurality-based) Henkin rivals than the set-based standard interpretations are vis-à-vis their (set-based) Henkin rivals. Our aim in this section is to articulate and reject a natural defense of Plural Robustness. In the next section, we argue that the two forms of standard semantics are equally well (or poorly) placed against their respective Henkin rivals and that Plural Robustness should therefore be rejected.

Plural Robustness has considerable initial plausibility. An explicit defense is due to Hossack, who nicely lays out the argument as follows:

> The singularist [a proponent of a set-based model theory] cannot solve the problem of indeterminacy, but the pluralist [a proponent of a plurality-based model theory] can. [...] Plural set theory has no non-standard models, so the indeterminacy problem does not arise for pluralism. [...] [P]lural variables range plurally over the very same particulars that the singular variables range over individually. *Therefore the pluralist does not confront an independent problem of identifying what the plural variables range over.* [...] Plural sentences therefore provide the missing additional constraint we were seeking on admissible interpretations. This is why the pluralist [a proponent of a plurality-based model theory] is able to solve the indeterminacy problem, though the singularist cannot do so.
>
> (Hossack 2000, 440–1, our emphasis)

As we understand it, the argument has as its point of departure the other virtue that plural logic is widely believed to enjoy, namely ontological innocence. According to this view—which we call *Plural Innocence*—plural quantification does not incur ontological commitments to entities beyond those in the first-order domain. In particular, plural quantification is not reducible to singular quantification over sets or mereological sums, nor does it involve reference to such entities. Rather, plural variables range plurally over objects in the ordinary, singular domain. And the use of such variables incurs ontological commitments only to objects in this ordinary domain, not to any sets or sums of such objects.

Of course, Plural Innocence is not uncontroversial (see Resnik 1988, Parsons 1990, Hazen 1993, and Linnebo 2003); we too take issue with it below. But if the thesis is false, so is an essential premise of the argument we wish to reject, and we are done. In the remainder of this section we therefore proceed on the assumption that the thesis is true.

It would be very natural to think that Plural Innocence supports Plural Robustness. Since the plural quantifiers do not range over any kind of "plural objects", such as the subsets of the first-order domain, we do not—as Hossack observes—"confront an independent problem of identifying what the plural variables range over." Plural quantifiers just range plurally over the very same domain that the singular quantifiers range over. This contrasts with the set-based model theory for second-order logic, where the standard interpretation requires one to single out a range for the second-order quantifiers that contains *all* the subsets of the first-order domain. The possibility of failing to single out such a range gives rise to the possibility of non-standard interpretations in the set-based model theory. Since Plural Innocence ensures that no new range of entities needs to be singled out for the plural quantifiers, this thesis renders plural logic on the plurality-based model theory immune to non-standard interpretations, or at least more immune than plural logic on the set-based model theory.

However, we contend that our plurality-based Henkin semantics is just as innocent as the plurality-based standard semantics. On both semantics, plural variables range plurally over objects in the ordinary, first-order domain. The only difference is that, on our semantics, the range of the plural variables can be so restricted as to make room for general interpretations in addition to the standard one.

In fact, this notion of ontological innocence can be understood in a less and in a more demanding way. The less demanding way requires the ontological innocence of the plural quantifiers. Then our claim that plural quantification is innocent on the plurality-based Henkin semantics is incontrovertible. Since the semantics is plurality-based, the plural quantifiers do not range over special kinds of objects. They range plurally over the objects in the first-order domain. This is the sense of ontological innocence operative in the argument from Plural Innocence to Plural Robustness spelled out above.

One might also want innocence in a more demanding form that includes the resources employed by the model theory itself. (For instance, the plurality-based model theory uses a pairing operation which is not ontologically innocent.) Our semantics may possess a high degree of innocence even in this more demanding sense. For there are arguments, akin to the one developed by Boolos himself, for the ontological innocence of the third-order quantification that binds the variable D. This is fairly straightforward in the case of the "superplural" interpretation of D. As for the official interpretation of D as a plural concept, one may argue for its innocence along the lines of Rayo and Yablo 2001 (see also Wright 2007). Moreover, in the more demanding sense of innocence the two semantics appear to be on equal footing. As argued above, an appeal to higher-order resources is unavoidable when the defender of the plurality-based standard semantics attempts to articulate a model theory for a language containing plural predicates (as she will have to do when formulating the model theory for her own metalanguage). So, when seen from this perspective, the semantic machinery of the plurality-based standard semantics is no more innocent than that of its Henkin competitor.

We conclude that, no matter which understanding of Plural Innocence is assumed, the plurality-based Henkin semantics has as good a claim to innocence as the standard semantics. This shows that Plural Innocence does not support Plural Robustness. For there is an innocent semantic option, namely the plurality-based Henkin semantics, for which Plural Robustness fails. This poses a challenge for defenders of Plural Robustness. If their claim is not supported by Plural Innocence, then what, if anything, does support it?

8.5 The semantic determinacy of plural quantification

The question of semantic determinacy, we recall, is whether the unique correct interpretation of our quantificational practice is the one associated with the standard interpretations. We contend that plural logic with the traditional set-based model theory and plural logic with plurality-based model theory are on a par with regard to semantic determinacy.

Two remarks about this *parity thesis*—as we shall call it—are in order. First, our contention is that the determinacy claims concerning plurality-based model theory stand or fall with the corresponding determinacy claims concerning set-based model theory. We remain agnostic about whether they stand together or fall together; though as mentioned, in the absence of compelling arguments, we urge caution about the determinacy claim. Second, the parity thesis includes, but goes beyond, the claim that Plural Robustness is false. If Plural Robustness is false, then no *additional* assurance of determinacy is gained by switching from a set-based to a plurality-based model theory. Our parity thesis consists of this claim and its converse.

We submit that the parity thesis has a great deal of plausibility whenever the domain of quantification is set-sized, as is the case of higher-order quantification over the natural numbers or the reals. Assume that the domain is a set d, and let dd be its elements. (We indicate this relationship by writing $d = \{dd\}$.) In the case of the set-based model theory, we need to single out a special object—the standard interpretation—from a large pool of other objects—the Henkin interpretations. In the case of the plurality-based model theory, we need to single out a special way of ranging over the domain dd—the standard way—from a large pool of other ways of ranging over dd—the Henkin ways. *Why should it be any easier—or harder—to single out an object from a pool of objects than to single out a way from an isomorphic pool of ways?* Since the two tasks are isomorphic, whatever can be said in one case carries over to the other.

While these considerations capture the gist of our argument, some work remains to be done to establish the parity thesis in full generality, that is, independently of the assumption that the domains of the plurality-based model theory are set-sized.[6] Consider first the possibility that plural logic is determinate on the plurality-based model theory and *in*determinate on the set-based model theory. If plural logic is determinate on the plurality-based model theory, this means that the only plurality-based Henkin interpretation is the standard one. *A fortiori*, no non-standard plurality-based Henkin interpretation can be countenanced in which the elements dd of the domain form a set d. But this is incompatible with the idea that non-standard *set-based* Henkin interpretations are legitimate, since the legitimacy of an interpretation would then depend entirely on the way in which the interpretation is described. Non-standard Henkin interpretations with set-sized domains would be legitimate when described set-theoretically but illegitimate when described with the help of higher-order resources. So we must conclude that plural logic on the set-based model theory is determinate too, and thus Plural Robustness is false.

[6] On the critical plural logic we develop in Chapter 12, every plurality defines a set and the mentioned assumption is always satisfied. Thus, if we are right, the considerations of this paragraph and the next become redundant.

We now consider the converse. Might plural logic be determinate on the set-based model theory but not on the plurality-based model theory? We believe the answer is negative. The determinacy of plural logic on the set-based model theory rules out non-standard interpretations whenever the domain is set-sized. So, if plural logic admits non-standard interpretations on the plurality-based model theory, such interpretations could only arise when the domain is too large to form a set. As a result, the type of interpretation legitimate for the plural quantifiers would vary depending on the size of the domain. That is, the interpretation of the plural quantifiers would be standard whenever the domain forms a set but may be non-standard when the domain is too big to form a set. Why should that be so? Since plural quantifiers are treated as logical, this asymmetry would be implausible. Thus, it appears that if plural logic is determinate on the set-based model theory, it must also be determinate on the plurality-based model theory.

8.6 The metaphysical determinacy of plural quantification

We now briefly examine a different determinacy question pertaining to plural and other forms of higher-order quantification. This question is metaphysical and challenges a presupposition of the semantic determinacy question discussed above. Consider a domain $d = \{dd\}$. Is there a determinate maximal set of subsets of d or a determinate maximal concept of being a subplurality of *dd*? Where the semantic question asks whether our practice uniquely singles out as correct a maximal interpretation of the plural and higher-order quantifiers, the metaphysical question asks whether the sort of thing we are attempting to uniquely single out even exists.

Many philosophers and mathematicians have defended a negative answer in cases where the domain is infinite. Their skepticism is fueled in part by our inability to answer some fairly immediate questions about the set of subsets (or its analogue in the case of plurals). A well-known example is Cantor's Continuum Hypothesis, which provably resists an answer by ZFC and has so far resisted an answer from widely accepted further axioms.

The metaphysical question is interesting in part because it might provide a reason to prefer the plurality-based model theory over the set-based model theory. For metaphysical determinacy might hold in the case of pluralities but fail in the case of sets. However, we don't think that this is so. More generally, we believe there is a determinate totality of subpluralities of the things *dd* that serve as our domain if and only if there is a determinate totality of subsets of $d = \{dd\}$. To see this, consider the conditions that would define these two totalities, namely '$xx \preccurlyeq dd$' and '$x \subseteq d$'. We contend that the pluralities satisfying the former condition are in one-to-one correspondence with the sets satisfying the latter. Provided that this contention is right, it is hard to see how one of the conditions could define a determinate totality while the other fails to do so.[7]

It remains only to defend our contention. For every set satisfying the condition '$x \subseteq d$', the plurality of its elements satisfies the condition '$xx \preccurlyeq dd$'. Next, we observe that every plurality satisfying the latter condition forms a set, by the axiom of Separation and the fact that $d = \{dd\}$. Moreover, this set satisfies the former condition. Thus, we can go back and forth between sets and pluralities satisfying the two conditions. Indeed, we obtain the promised one-to-one correspondence by observing that a set is sent to the plurality of its elements, which are sent back to the original set.

What about the case where *dd* do not form a set? Our considerations leave open whether in this case there is a determinate totality of subplutalities of *dd*. But any trouble here would only serve to limit the advantage of the plurality-based model theory over its set-based rival.

8.7 A generalized notion of ontological commitment

Let us finally consider the debate about the ontological commitments of plural logic. According to Boolos and his followers, plural languages are ontologically innocent. For instance, when you say that you had a bowl of Cheerios for breakfast, you are talking exclusively about the Cheerios, not about a set of them, their sum, or any kind of "plural entity". Call this the *narrow notion* of ontological commitment. It will be made precise below. We have seen how to develop a model theory for a plural object language in a plural metalanguage in which the semantic values of a plural variable is one or more objects from the ordinary first-order domain. This model theory upholds the view that the use of plural quantifiers incurs no new commitments to sets, sums, or any other kind of plural entities (Boolos 1985a).

The opposite side responds by disputing the *prima facie* case for the ontological innocence of plural quantification. For instance, commenting on

[7] Here we rely on an analogue of Replacement for our intuitive notion of determinate totality. In Chapter 12 we develop a notion of extensional definiteness that obeys the mentioned principle.

Boolos's example 'there are some sets which are all and only the non-self-membered sets', Parsons writes:

> in a context of this kind a quantifier like 'there are some sets' is saying that there is a plurality of some kind. Cantor's notion of 'multiplicity' and Russell's of 'class as many' were more explicit versions of this intuitive notion, both attempting to allow that pluralities might fail to constitute sets. (Parsons 1990, 326)

(See also Hazen 1993, Shapiro 1993 and Linnebo 2003, as well as Resnik 1988 for a more "singularizing" version of the view.) The model theory developed in a plural metalanguage cuts both ways. Both parties to the debate can agree that if the use of the plural quantifiers in the metalanguage is innocent, then so is their use in the object language. One party will assert the antecedent, while the other will deny the consequent. Thus there are two internally coherent views on the matter, and we appear to have reached a standoff.

The best way to make progress, we believe, is by considering two competing construals of the notion of ontological commitment. If one understands this notion in the *narrow sense* (as concerned exclusively with the existence of objects) and takes an object to be the value of a singular first-order variable, then the plurality-based model theory does indeed show that plural logic is ontologically innocent. For this model theory does not use singular first-order variables to ascribe values to the plural variables of the object language; rather, this ascription is made by means of plural variables of the metalanguage.

There is, however, a *broad notion* of ontological commitment. According to this notion, ontological commitment is tied to the presence of existential quantifiers of *any logical category* in a sentence's truth conditions. If this notion is operative, then even the plurality-based model theory shows that plural locutions incur additional ontological commitments. The resulting view is an analogue of that espoused by Frege when he held that quantification into predicate position incurs its own distinctive kind of commitment, not to objects but to concepts.

Before a meaningful debate can take place about which notion of commitment is more interesting and appropriate, both notions need to be clearly articulated. We will now show that our plurality-based Henkin semantics is precisely the tool we need in order to articulate the more inclusive notion.

Let us begin with the narrow notion, which ties ontological commitment to the values of singular first-order variables. Here is one of Quine's more helpful statements of his view.

> The ontology to which an (interpreted) theory is committed comprises all and only the objects over which the bound variables of the theory have to be construed as ranging in order that the statements affirmed in the theory be true. (Quine 1951, 11)

This suggests the following precise definition. A theory T is committed to κ objects that are φ if and only if every model of T contains at least κ objects satisfying the formula φ.

In light of our work in earlier sections, it is straightforward to extend this criterion of commitment to plural variables. In both cases, the formulation of the criterion relies on the use of quantifiers that are assumed to be antecedently understood in the metatheory. A theory T is committed to κ pluralities that are φ if and only if every *plurality-based Henkin model* of T has a range D of the plural quantifiers containing at least κ pluralities satisfying the formula φ. (Of course, the proper way to talk about many pluralities is by means of plural concepts or super-pluralities, as discussed above.)[8] It is important to note that the appeal to plurality-based *Henkin* models is essential. If we had instead appealed to Boolos-style plurality-based *standard* models, then the ontological commitment of any theory involving plural quantifiers would be trivially determined by the ontological commitments of the first-order quantifiers of the theory. For any theory would incur commitments to all and only the pluralities based on the objects to which the theory is committed. By contrast, the definition of commitment to pluralities that we have proposed has the desirable feature that a theory's commitment to pluralities can add information over and above its commitment to objects.

The value of this information is most easily appreciated when it is denied that there is a single maximal interpretation of the plural quantifiers, that is, when the metaphysical determinacy of these quantifiers is denied. When this is denied, there can be no hope of determining the theory's commitments to pluralities directly on the basis of its commitments to objects. Instead, one must assess the commitments to pluralities independently, using the

[8] In a perfectly analogous way, we can define a notion of ontological commitment incurred by quantification into predicate position.

generalized Quinean criterion set out above. To illustrate this point and, more generally, the value of our notion of commitment to pluralities, let us consider a puzzle due to Hazen (1993, 135). Consider the scheme of plural comprehension:

$$\exists x\, \varphi(x) \;\rightarrow\; \exists xx\, \forall x\, (x \prec xx \leftrightarrow \varphi(x))$$

Which instances of the scheme should we accept? The traditionalist (whose position is enshrined in the standard semantics for plural logic) accepts *all instances*—with the obvious and uncontroversial proviso that $\varphi(x)$ not contain xx free. This traditional view faces various challenges. According to predicativists, for example, we should only accept plural comprehension axioms that are predicative in the sense that $\varphi(x)$ does not contain any bound plural variable. And according to the critical plural logic we develop and defend in Chapter 12, plural comprehension needs to be restricted so as to avoid commitment to a universal plurality or other pluralities that are not properly circumscribed. As Hazen observes, there is a clear and intuitive sense in which these non-traditional views are committed to fewer pluralities than the traditionalist. Thus, if a notion of commitment is to be worth its salt, it must capture this sense. And this is exactly what our broad notion of ontological commitment enables us to do. Using this notion, we can maintain that the traditionalist, unlike the predicativist, takes on commitments to impredicatively defined pluralities. By contrast, had we assumed the plurality-based standard semantics, this conclusion would not have been available.

Our notion of commitment to pluralities is also useful in cases where the metaphysical determinacy of plural quantification is granted. When this is granted, there is a notion of commitment to pluralities—namely the one associated with the maximal interpretation of the plural quantifiers—according to which these commitments supervene on the commitments to objects. Once the commitments to objects of a theory have been determined, so have the commitments to pluralities associated with the maximal interpretation. It must therefore be conceded that there is no *further* question concerning the theory's commitments to pluralities. However, the supervenience of one parameter on certain others does not mean that there is no genuine and theoretically interesting question as to the value of this parameter! In our case, even if metaphysical determinacy ensures that the commitments to pluralities of a theory are uniquely determined by its commitments to objects, we still want to know how many, and what kind of, pluralities the theory is committed to. Even if one believes in the metaphysical determinacy of plural quantification, one may have views about how strong, or mathematically rich, one's notion of subplurality is (e.g. Shapiro 1993 and Parsons 2013). The notion of commitment to pluralities that we have articulated allows such views to be expressed.

An example might be helpful. Assume that the commitments to objects of a theory involve an omega-sequence, which we may think of as the natural numbers. If metaphysical determinacy holds, then there is a sense in which the commitments to pluralities are determined by the commitments to objects. Even so, we can ask *which* pluralities the theory is committed to. Different answers are possible. For instance, a theorist who believes the axiom of constructibility, $V = L$, may answer that the only subpluralities of the "natural numbers" to which the theory is committed are the ones that are constructible (in the sense that they correspond to sets in the constructible hierarchy L). Another theorist—who rejects the axiom of constructibility—may disagree and insist that the commitments to pluralities go beyond the constructible ones.

It may be objected to the broad notion of commitment that the commitments associated with plural and higher-order quantifiers is not a form of *ontological* commitment but perhaps, following Quine, of *ideological* commitment. We see little point in quarreling over terminology. A more interesting question is whether ideological commitments in this sense give rise to fewer philosophical problems, or whether they are philosophically less substantive, than ontological commitments narrowly understood. It is far from obvious why this should be so. Indeed, it seems to us that questions involving the broad notion of commitment can be just as interesting and problematic as those involving the narrow ones. How are we to understand the values of different sorts of variables—in extensional or intensional terms? Which such values are there and which comprehension axioms should we therefore accept? How do we trace a value from one context (e.g. time or possible world) to another?

In light of these considerations, we are inclined to agree with Parsons when he writes that, on the narrow notion,

> ontological commitment may just not have the significance that both nominalists and many of their opponents attribute to it, or that Boolos seems to attribute to it in the case of proper classes. That might be a victory for the Innocence Thesis, but it would be a Pyrrhic victory.
>
> (Parsons 2013, 173)

Thus, if Parsons is right, then either Plural Innocence is false, or else it is true but not nearly as interesting as one might have thought.

Our primary goal in this section has been not so much to adjudicate this debate as to prepare the ground for a precise and well-informed debate. We have done so by using our plurality-based Henkin semantics to provide a clear articulation of a generalized notion of commitment. Still, on the picture emerging from our discussion, the role of plural logic as a philosophical tool appears substantially diminished. As we have shown, plural logic is not immune from the threat of non-standard interpretations, and the promised gain in expressive power has not been established. Although we do not take a stand on which side of the debate prevails, we have, in the absence of a compelling argument, urged caution about the determinacy claims.

Further, there is a precise and interesting sense in which plural logic may be said to be committing. Whether this commitment is ontological or ideological, it is a full-fledged form of commitment nonetheless.

8.8 Applications reconsidered

The conclusion we have just reached is in stark contrast to the common picture of plural logic canvassed in Section 2.5. According to that picture, plural logic is "pure logic" and hence also ontologically innocent, and it provides greater expressive power than first-order logic. In Section 2.6, we explained how this common picture has sustained some important applications of plural logic, thereby contributing to the view that plural logic has great philosophical significance. We focused on four such applications, which concern logicism, nominalism, semantics, and categoricity arguments in philosophy of mathematics. Let us briefly reconsider these applications in light of the preceding discussion.

By itself, our rejection of Plural Innocence and Plural Robustness does not force any logical revision. Our arguments can be accepted while retaining the traditional version of plural logic that we have used so far. This means that our arguments do not affect technical applications of plural logic, including to logicism. Logicists can employ plural logic in developing their views, provided that such views are compatible with the failure of Plural Innocence.

The case of nominalism is different. The use of plural logic in some nominalistic projects relies essentially on the alleged ontological innocence of plural quantification. Eliminating certain kinds of complex objects in favor of pluralities will be less significant if one accepts that plural quantification incurs commitments that go beyond those of first-order quantification. Nominalists can trade some commitments to objects for new commitments to pluralities. But they will still face some substantive metaphysical and epistemological questions about the nature and extent of the new commitments.

In semantics, the main application of plural logic was to develop a plurality-based model theory. This application is unaffected by our conclusions concerning the innocence and determinacy of plural quantification. Indeed, our argument for the existence of non-standard interpretations of plural logic used precisely the framework of plurality-based model theory. What about absolute generality? Since the Henkin semantics subsumes all the standard interpretations, the new semantics is just as congenial to absolute generality as the standard one.[9] (As mentioned, however, the use of plural logic to represent absolute generality faces an entirely different challenge; see Chapters 11 and 12.)

Finally, plural logic has been held up as an appealing alternative to second-order logic in order to overcome the expressive limitation of first-order logic and hence make available categorical characterizations of important mathematical structures. Given our rejection of Plural Robustness, this application of plural logic becomes highly problematic. Because Plural Robustness fails, plural logic is *not* immune to the threat of non-standard interpretations, and the desired gain in expressive power remains in doubt.

In sum, we have found that plural logic lacks some key features that pure logic has been thought to have, in particular ontologically innocence; nor is the logic immune to worries about indeterminacy.[10] This calls into question some popular applications of the system. As we have stressed, however, plural logic has other important applications, particularly in accounting for sets, which do not require those features. Plural logic is thus of great interest and theoretical value, just not in the way that many of its earlier proponents have argued.

[9] In Section 2.6, we claimed that if plural talk is not ontologically innocent, then the use of plural logic to capture absolute generality would appear to be undermined. The claim was made in the context of what we now call the narrow notion of ontological commitment and was explicitly linked to the existence of set-like objects (see p. 27).

[10] A more comprehensive summary of our view on the extent to which plural logic counts as pure logic can be found in the final section of the book.

Appendices

8.A Henkin semantics

Let us provide a more precise formulation of the plurality-based Henkin semantics for PFO. This semantics is a variant of the standard semantics illustrated in Section 7.3. The difference is that some key definitions are relativized to a plural concept D functioning, in effect, as a domain for the plural quantifiers.

We want to characterize a Henkin interpretation. We start with a plurality dd serving as the first-order domain. Then we relativize to D the previous definition of an interpretation function ff (Section 7.3) by adding this requirement: for every plural constant tt, there is at least one x such that $\langle tt, x\rangle \prec ff$, and for all xx such that

$$\forall y(y \prec xx \leftrightarrow \langle tt, y\rangle \prec ff)$$

it holds that $D(xx)$. The requirement captures the idea that, in any interpretation function, a plural constant tt denotes some things that instantiate D, specifically those appearing as second coordinates of pairs whose first coordinate is tt.

An interpretation of the object language is obtained by combining the domains dd and D with an interpretation function ff relative to dd and D. Given how these three components have been characterized, an interpretation is not an object or the value of a single higher-order variable. But such components can be 'merged' so as to be represented by a single variable I, whose value is a plural concept (or, alternatively, a superplurality) that codes the three components. Quantifying over interpretations then amounts to quantifying over plural concepts (or superpluralities). For convenience, however, we speak of an interpretation as a triple and represent it as $\langle dd, D, ff\rangle$.[11]

[11] Here is one way of doing the coding. In keeping with the notation introduced in Section 7.5, we let '$\langle y, xx\rangle$' stand for the ordered pairs obtained by pairing y with each x in xx. Then, given dd, D, and ff, there is I such that for all yy, $I(yy)$ if and only if one of the following holds:

(1) $yy \approx \langle a, dd\rangle$;
(2) there are zz such $D(zz)$ and $yy \approx zz$;
(3) $yy \approx \langle b, ff\rangle$;

where a and b are any two distinct objects. The plural concept (or superplurality) so characterized can be used as a surrogate for the triple $\langle dd, D, ff\rangle$.

We also relativize to D the previous definition of a variable assignment *ss*: we require that for every plural variable *vv*, there is at least one x such that $\langle vv, x\rangle \prec ss$, and for all *xx* such that

$$\forall y(y \prec xx \leftrightarrow \langle cc, y\rangle \prec ss)$$

it holds that $D(xx)$. This means that a plural variable *vv* is assigned some things that instantiate D, specifically those appearing in the assignment as second coordinates of pairs whose first coordinate is *vv*.

Before defining the notion of truth in an interpretation, let us introduce some additional notation, following our convention in Section 7.2. For any model $\langle dd, D, ff\rangle$, variable assinglment *ss*, and non-logical expression E, let $[\![E]\!]_{\langle dd,D,ff\rangle,ss}$—but, in fact, we will write $[\![E]\!]_{ff,ss}$ leaving the domains implicit—indicate the semantic value or values of the expression E relative to the model $\langle dd, D, ff\rangle$ and the variable assignment *ss*.

We are ready to give the inductive characterization of truth in an interpretation via satisfaction clauses. In the Henkin semantics, a formula φ is true in an interpretation $\langle dd, D, ff\rangle$ relative to a variable assignment *ss* based on D, written $\langle dd, D, ff\rangle \vDash_H \varphi\ [ss]$, just in case:

(i) if φ is $t_1 = t_2$, then $[\![t_1]\!]_{ff,ss} = [\![t_2]\!]_{ff,ss}$;
(ii) if φ is $S^n(t_1, \ldots, t_n)$, then $\langle [\![t_1]\!]_{ff,ss}, \ldots, [\![t_n]\!]_{ff,ss}\rangle \prec [\![S^n]\!]_{ff,ss}$;
(iii) if φ is $\exists v\, \psi$, then $\langle dd, D, ff\rangle \vDash_H \psi\ [ss(v/x)]$ for some $x \prec dd$;
(iv) if φ is $\exists vv\, \psi$, then $\langle dd, D, ff\rangle \vDash_H \psi\ [ss(vv/xx)]$ for some $xx \preccurlyeq dd$ such that $D(xx)$;
(v) the clauses for the logical connectives are the obvious ones.

As usual, the satisfaction clauses ensure that if φ is a sentence, we can ignore variable assignments.

We say that an interpretation $\langle dd, D, ff\rangle$ is *faithful* if it satisfies every instance of the plural comprehension scheme:

$$\exists v\, \varphi(v) \;\rightarrow\; \exists vv \forall v(v \prec vv \leftrightarrow \;\varphi(v))$$

Logical consequence is defined with respect to faithful models only. (Of course, when we are interested in systems with restricted plural comprehension, we modify the definition so as to consider all models that satisfy the relevant comprehension scheme.) A sentence φ is a consequence of a set of sentences Δ in the Henkin semantics (written '$\Delta \vDash_H \varphi$') if, for every faithful interpretation $\langle dd, D, ff \rangle$ satisfying every member of Δ, $\langle dd, D, ff \rangle \vDash_H \varphi$.

8.B Completeness of the Henkin semantics

Let us now prove that traditional plural logic defined in Section 2.4 is sound and complete with respect to the plurality-based Henkin semantics formulated above. We use the symbol $\vdash$ to denote the relation of provability is this system. We want to show that, for any sentence φ and set of sentences Δ, $\Delta \vDash_H \varphi$ (if and) only if $\Delta \vdash \varphi$. The shortest and most elegant way of proving this is through a squeezing argument.

First, it is a routine exercise to verify that traditional plural logic is sound with respect to the plurality-based Henkin semantics, which means that

(8.1) if $\Delta \vdash \varphi$, then $\Delta \vDash_H \varphi$.

Now consider the familiar set-based Henkin semantics for second-order logic. (See, for instance, Shapiro 1991, Section 4.3.) It is relatively straightforward to adapt this semantics to PFO. An interpretation is given by a triple $\langle d_1, d_2, f \rangle$, where d_1 is a non-empty set, d_2 (the range of the plural quantifiers) is a set of non-empty subsets of d_1, and f is interpretation function from the non-logical vocabulary of the language to elements of d_1 (for singular terms), elements of d_2 (for plural terms), and possibly empty sets of n-tuples from d_1 (for singular n-ary predicates). Plural membership ('is one of') is systematically interpreted as set-theoretic membership. Let us use the symbol $\vDash_h$ for the resulting relation of logical consequence when confined to *faithful* interpretations, namely those satisfying every instance of plural comprehension. So $\Delta \vDash_h \varphi$ means that φ is a logical consequence of Δ in the set-based Henkin semantics. In other words, for every faithful interpretation $\langle d_1, d_2, f \rangle$, if $\langle d_1, d_2, f \rangle \vDash_h \psi$ for every member ψ of Δ, then $\langle d_1, d_2, f \rangle \vDash_h \varphi$.

It is evident that every set-theoretic model just described corresponds to a plurality-based Henkin model. Take any model $\langle d_1, d_2, f \rangle$. Then its corresponding plurality-based model $\langle dd, D, ff \rangle$ is one in which dd are the elements of d_1, D is the concept of being a plurality that forms a set in d_2, and *ff* is an interpretation function that matches *f*.[12] This correspondence establishes the following:

(8.2) If $\Delta \vDash_H \varphi$, then $\Delta \vDash_h \varphi$.

Finally, we can easily adapt the standard proof that second-order logic is complete with respect to the set-based Henkin semantics (Henkin 1950) to show that traditional plural logic is complete with respect to the set-based Henkin semantics outlined in the paragraph just above.

This gives us that

(8.3) if $\Delta \vDash_h \varphi$, then $\Delta \vdash \varphi$.

Putting together the last three numbered claims, we obtain the result we wanted to prove:

(8.4) $\Delta \vDash_H \varphi$ (if and) only if $\Delta \vdash \varphi$.

So traditional plural logic is complete with respect to the plurality-based Henkin semantics. Therefore, it is also compact and axiomatizable.

[12] Specifically, if $f(t) = x$, then $\langle t, x\rangle \prec ff$. If $f(aa) = \{xx\}$, then $\forall y(\langle aa, y\rangle \prec ff \leftrightarrow y \prec xx)$. And, for any n-tuple $\langle x_1, \ldots, x_n\rangle$, $\langle S^n, \langle x_1, \ldots, x_n\rangle\rangle \prec ii$ if and only if $\langle x_1, \ldots, x_n\rangle \in f(S^n)$.

9
Superplurals

9.1 Introduction

Superplurals have come up time and again in the preceding chapters. We have observed on several occasions that the expressive resources afforded by superplurals would have useful applications. Superplurals would permit the formulation of a higher-order version of Cantor's theorem; they would help with the elimination of mereology and second-order logic in favor of plural logic; and they could be extensively used in semantics, for example by serving as values of plural predicates, coding models of PFO+, and restricting the range of plural quantifiers in Henkin interpretations.

However, the legitimacy of superplural resources cannot be taken for granted. This is why our appeals to superplurals have always been conditional so far. It is time to examine the matter more closely.

Can the step from the singular to the plural be iterated? Is there such a thing as the superplural that stands to the plural the way the plural stands to the singular? Some examples from natural language suggest an affirmative answer. We have already mentioned expressions that denote things articulated into distinct subpluralities, namely 'Russell and Whitehead, and Hilbert and Bernays' and 'these things, those things, and these other things'. These and other examples will be discussed shortly. First we need to clarify and sharpen the questions about superplurals that we seek to answer.

9.2 What superplural reference would be

The morphological operation of pluralization cannot be iterated—at least not in English or any other natural language with which we are familiar. For example, while 'cat' and 'cats' are permissible, 'catses' is not. So, as far as common nouns are concerned, the singular and the plural exhaust the options in English. This means that the quest for English superplurals cannot be based on morphology alone. Rather, it must be based on some

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semantic feature. The most natural candidate is reference. Focusing on terms, we may thus attempt to single out superplural terms on the basis of what kind of reference they effect.

Terms may be semantically classified according to the number of things they are capable of referring to. Singular terms can refer to at most *one* thing, while plural terms are capable of referring to *a plurality* of things, that is, *many* things at once. What is semantically distinctive of superplural terms?

In earlier chapters, we often described the reference of a superplural term as a "plurality of pluralities". But we noted that this expression does not have a clear meaning, given our stipulation that 'plurality' be used as a shorthand for a plural construction. In this sense, a plurality of things just is many things. So a plurality of pluralities would have to be, in Russell's terms, *many many's* (1903, 516). However, one may reasonably doubt that this gloss is meaningful. As a result, one may worry that the very the notion of a superplural term is semantically unintelligible (see, for example, Ben-Yami 2013).

On a set-based formulation of the semantics, there is a perfectly acceptable way to characterize superplural reference. Let our domain of discourse be a set d. Then a set-based interpretation assigns an individual in d to a singular term and a non-empty set of individuals in d to a plural term. So a superplural term would be one to which the interpretation assigns a *higher-level set*, namely a *set of sets* of individuals in d (subject to the conditions that all the relevant sets be non-empty). This characterization makes precise both the idea of iteration and the analogy used above, namely that the superplural stands to the plural the way the plural stands to the singular.

Linguistic semantics bears witness to the fact that the notion of superplural reference thus characterized is of substantial theoretical interest. For there is a rich and subtle debate over whether certain expressions of natural language should be interpreted by means of higher-level sets.[1] The debate concerns, in particular, the interpretation of plural noun phrases obtained by conjoining other plural noun phrases. Let α be a noun phrase of the form

$$\beta_1 \text{ and} \ldots \text{and } \beta_n$$

where $\beta_1, \ldots, \beta_n$ are themselves plural noun phrases. The question is whether α should be interpreted as the union of the semantic values of $\beta_1, \ldots, \beta_n$ or as the higher-level set whose elements are those semantic values. So we must

[1] See, for example, Gillon 1987, Lasersohn 1989, Gillon 1990, Lasersohn 1995, Schwarzschild 1996, and Winter 2001.

choose between a rule of semantic composition that makes α a plural term, namely

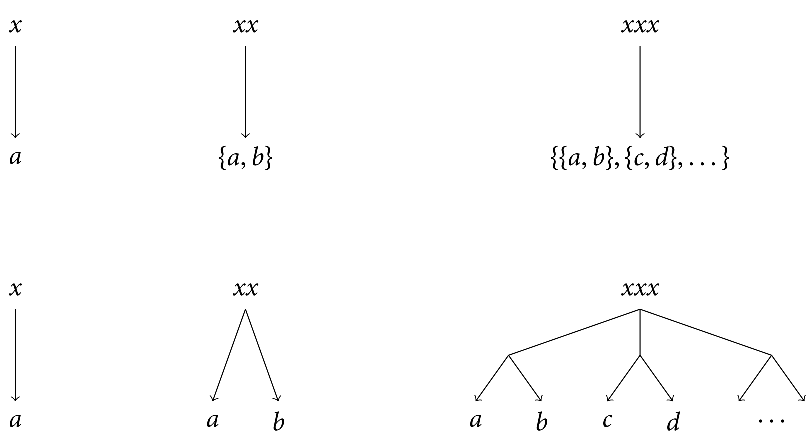
$$[\![\alpha]\!] = [\![\beta_1]\!] \cup \ldots \cup [\![\beta_n]\!]$$

and an alternative rule that makes α a superplural term, namely

$$[\![\alpha]\!] = \{[\![\beta_1]\!], \ldots, [\![\beta_n]\!]\}$$

We will return to this debate below. For the moment, we simply want to emphasize that the notion of superplural reference has a clear and theoretically fruitful characterization within a set-based semantics. In this framework, there is no intelligibility problem.

A new problem arises, however, when we adopt a plurality-based semantics. If a plural term refers to many things, how are we to understand the reference of a superplural term? This question differs from the one discussed by linguists working within a set-theoretic framework (though the data they consider and many of their analyses remain relevant). The desired superplural reference would be a form of *articulated reference*: a superplural term refers not just to many objects but to *many objects articulated in a certain way*. The set-based semantics provides one way to make this articulation explicit. But the following visual analogy suggests another way to represent the articulation, consistent with a (generalized) plurality-based semantics.



Our aim, then, is to "articulate" some objects, just like a higher-level set "articulates" these objects through its members. If higher-level sets are required for a successful set-based semantics, something with the same functional role is required to carry out the semantics in a plurality-based setting.

If natural language contains no devices with this functional role, we may have to develop the needed devices ourselves. From a methodological point of view, the situation is not unusual. Sometimes theorizing requires the introduction of new expressive resources. Think of the huge expressive gains in the history of science afforded by the introduction of new mathematical resources. The justification for these resources derives from their theoretical fruits, not from their antecedent availability in natural language. The proponent of a plurality-based semantics may find herself in a similar situation. She might need to develop superplural resources to account for semantic phenomena that in the set-based semantics require higher-level sets. These new resources too are justified in part by their theoretical fruits.

An additional way to address the intelligibility problem is by adapting a lesson from plurals. It is customary to assume that denotation is a functional relation between a term and its semantic value. So, for any plural term *tt*, there are some things such that *tt* refers to them and to no other things. This requires the use of plurals in the specification of the semantic value of *tt*. But if we relax the functionality requirement and construe reference as a one-many relation, the use of plurals can be avoided. To characterize the reference of *tt*, we can say that *tt* refers to at least one thing, leaving open the possibility that it refer to some other thing. To express that *tt* achieves plural reference to *a* and *b*, we then say that *t* refers to *a* and *t* refers to *b*. The semantic characterization of a plural term is now given using only singular vocabulary. By making the right semantic adjustments, this notion of reference is entirely adequate, as shown by the model theory presented in Sections 7.3 and 7.5. Of course, as we learned there, plurals are not eliminated from the semantics. They are still needed, for example to state the satisfaction clause for plural predication. But it is possible to reduce the complexity of the definition of plural reference so that it only relies on singular resources.

An analogous strategy can be employed for superplural terms. We can say that a superplural term *ttt* refers to at least one plurality, leaving open the possibility that it refer to some other plurality. To express that *ttt* achieves superplural reference to *aa* and *bb*, we then say that *ttt* refers to *aa* and *ttt* refers to *bb*. This definition does not presuppose superplural notions. Relying only on our understanding of plurals, we now have access to a semantic characterization of superplural reference. Of course, this does not mean that we can fully carry out the semantics without invoking superplural notions. But the strategy suffices at least to make intelligible what superplural reference would be in the context of a plurality-based semantics. So we submit that the notion of superplural reference is in good standing.

9.3 Grades of superplural involvement

Questions about superplural reference are part of a broader range of interesting questions concerning different ways in which superplurals may be said to be "available". Affirmative answers to such questions correspond to different "grades of superplural involvement", as we shall now explain.

The first question has to do with the availability of a certain logical system.

(Q1) Can we formulate a superplural logic?

This question receives a definitive answer in Rayo 2006. It is relatively straightforward to formulate an extension of plural logic that, in addition to singular and plural vocabulary, includes a third category of superplural expressions. These new expressions (terms, predicates, and quantifiers) are governed by logical rules similar to those governing singular and plural expressions. In the resulting system we can express, for instance, the plausible axiom that every superplurality is non-empty:

(9.1) $\forall xxx \, \exists yy \;\; yy \prec xxx$

Once a superplural logic has been formulated, an obvious semantic question arises:

(Q2) Can we provide a semantics, especially a model theory, for a superplural language?

A set-based model theory for such a language can easily be formulated as an extension of the set-based model theory for PFO+. If we are willing to countenance metalanguages with superplural resources, a plurality-based model theory is available as well (see Rayo 2006, Linnebo and Rayo 2012, and Section 11.A below).

The next two questions ask whether superplurals are available in a more demanding sense.

(Q3) Is superplural reference a legitimate expressive resource for beings like us?

This is a question in philosophical logic, unlike the fourth question, which is empirical and requires input from linguists.

(Q4) Is superplural reference realized in some (human) natural language?

Our primary target is the grade of superplural involvement associated with (Q3). This question involves the notion of "legitimate expressive resource". How should this notion be understood? Some indications are provided in Lewis 1991, Hazen 1997, Linnebo 2003, and Rayo 2006. As Hazen and Lewis point out, we can show the legitimacy of certain expressive resources by showing that they can be taught and employed. Indeed, as discussed in Section 6.1, this process of training can include learning by what Williamson calls "the direct method".

We will also be concerned with the last—and highest—grade of superplural involvement. While this is of obvious relevance to natural language semantics, our primary interest in the fourth question has to do with the fact that actuality entails possibility, or legitimacy, as we put it above. If some natural language truly realizes superplural reference, then such reference is certainly possible, which answers our primary question, (Q3).

9.4 Possible examples from natural language

Before reviewing a number of purported examples of superplurals in English, let us briefly consider two phenomena from other languages.[2] The first involves number words in Icelandic. Such words have plural forms which count not individual objects, but pluralities of objects that form natural groups. For instance,

[2] These examples are mentioned in Linnebo 2017. Grimau forthcoming offers an interesting discussion of a wider range of examples.

einn skór	means	one shoe
einir skó	means	one pair of shoes
tveir skór	means	two shoes
tvennir skó	means	two pairs of shoes

So we can talk about a "pluralities of pluralities" of shoes without the mediation of any singularizing device. This feature is lost in the translation, since 'two pairs of shoes' involves a singularization (via 'pair of') and thus amounts to regular plural talk of objects (the pairs of shoes).

For another example, consider these quantificational expressions in Lithuanian:

vienas batas	means	one shoe
vieneri batai	means	one group of shoes
du batai	means	two shoes
dveji batai	means	two groups of shoes
abu batai	means	both shoes
abieji batai	means	both groups of shoes
keleri batai	means	several groups of shoes
keli batai	means	both shoes

Here too we can talk about "pluralities of pluralities" of shoes, again without the mediation of any singularizing device.

Turning to English, there are many purported examples of superplural terms. Most of these examples involve *lists of plural terms*. Let us first reflect on the case of lists of singular terms. Some predicates applying to such lists appear to take a variable number of arguments, that is, they appear to be *multigrade*:

(9.2) (a) Annie and Bonnie met/cooperated/helped each other.
(b) Annie, Bonnie, and Connie met/cooperated/helped each other.
(c) Annie, Bonnie, Connie, and Danny met/cooperated/helped each other.

These predicates are plausibly analyzed as *plural* predicates with a single argument filled by a plural term. So, on this analysis, the predicates have fixed adicity after all, and each list of singular terms is in fact a plural term. By analogy, one might expect that predicates applying to lists of plural terms also have fixed adicity and should be analyzed as *superplural* predicates. If so, each list of plural terms would be a superplural term.

Here are some promising example involving lists.

(9.3) The cards below seven and the cards from seven up have been separated. (Landman 1989a, 574)

(9.4) The Beatles and the Rolling Stones gave a joint concert. (Inspired by Rosen and Dorr 2002, footnote 21, 172–3)

(9.5) The joint authors of multivolume classics on logic are Whitehead and Russell, and Hilbert and Bernays. (Oliver and Smiley 2005, 1065)

(9.6) These people, those people, and these other people played against each other. (Linnebo and Nicolas 2008, 193)

(9.7) The square things, the blue things, and the wooden things overlap. (Linnebo and Nicolas 2008, 193)

It is worth noting that (9.5) also gives us a purported example of a superplural term other than a list, at least if one assumes that the sentence expresses a form of identity. On this assumption, the plural description has the same semantic status as the list.

Oliver and Smiley maintain that this is not an isolated case: there is a class of plural descriptions whose members are in fact uniformly superplural. These are descriptions of the form 'the *F*' that purport "to denote all the joint satisfiers of *F*" rather than "the unique joint satisfiers of *F*" (Oliver and Smiley 2016, 132). In addition to the description in (9.5), they mention various other descriptions as belonging to this class, including 'the twin primes' and 'the creators of a great comic opera'. Let us therefore add another of their examples to our stock:

(9.8) 3 and 5 are among the twin primes. (Oliver and Smiley 2016, 139)

On the superplural reading they propose, 'are among' is a higher-order counterpart of the relation of plural membership. It holds between a plural term ('3 and 5') and superplural one ('the twin primes') when, so to speak, the plurality denoted by the former is one of the pluralities denoted by the latter.

9.5 The possible examples scrutinized

To present a convincing case for a superplural interpretation of the examples just considered, we must rule out alternative analyses. The harder it is to find an alternative analysis, the stronger the case for superplurals. We will therefore examine a range of attempts to find an alternative analysis, without necessarily endorsing these attempts.

For some of the examples, alternative analyses are clearly available (see Linnebo and Nicolas 2008 for discussion). For instance, (9.3) can be analyzed as follows:

(9.9) The cards below seven have been separated from the cards from seven up.

This sentence contains no *prima facie* superplurals. It features just ordinary plural descriptions. However, the possibility of this analysis depends on the fact that the original list is composed of *two* terms. So the analysis does not directly apply when a list has more terms:

(9.10) The cards below four, the cards between four and six, and the cards from seven up have been separated.

To analyze (9.10), we can integrate the idea behind (9.9) with a *conjunctive analysis*:

(9.11) The cards below four have been separated from the cards between six and four, the cards between four and six have been separated from the cards from seven up, and the cards from seven up have been separated from the cards below four.

The success of this conjunctive analysis requires the logical equivalence of (9.10) and (9.11). As is easy to see, this equivalence obtains, although one might question whether meaning is fully preserved.

However, the conjunctive analysis does not provide a general method for eliminating purported examples of superplural terms. Its limitations become apparent when we examine another example given above:

(9.4) The Beatles and the Rolling Stones gave a joint concert.

As before, there is a simple analysis that avoids the list:

(9.12) The Beatles gave a concert with the Rolling Stones.

Still, the following variant of (9.4) is not amenable to a conjunctive analysis:

(9.13) The Beatles, the Rolling Stones, and Led Zeppelin gave a joint concert.

For it is clear that (9.13) is not equivalent to this conjunction:

(9.14) The Beatles gave a concert with the Rolling Stones, the Rolling Stones gave a concert with Led Zeppelin, and Led Zeppelin gave a concert with the Beatles.

While (9.13) implies that there is a single concert featuring all three bands, (9.14) does not.

Although it resists a conjunctive analysis, (9.13) does not yet provide a compelling example of superplurals. That is because it can plausibly be analyzed in a different way. Indeed, it has been suggested that terms like 'the Beatles' and 'the Rolling Stones', though syntactically plural, denote *groups* and are therefore semantically singular (see, e.g., Landman 1989a, Landman 1989b, and Landman 2000, Lectures Four, Five and Six). It is possible that the Beatles and the Rolling Stones gave a joint concert, though Ringo Starr was ill and was replaced by someone else. Assuming that pluralities have their members necessarily (see Chapter 10), this possibility shows that 'the Beatles' and 'the Rolling Stones' do not refer plurally and thus lends support to the group-based analysis. On this analysis, lists such as 'the Beatles and the Rolling Stones' amount to ordinary plural terms.

Is the conjunctive analysis applicable to the sentences suggested by Oliver and Smiley? It is not applicable directly to (9.5). For, clearly, these two sentences are not equivalent:

(9.5) The joint authors of multivolume classics on logic are Whitehead and Russell, and Hilbert and Bernays.

(9.15) Whitehead and Russell are joint authors of multivolume classics on logic, and Hilbert and Bernays joint authors of multivolume classics on logic.

To bridge the gap, one may apply the conjunctive analysis indirectly through a Russellian elimination of the plural description:

(9.16) Whitehead and Russell are joint authors of multivolume classics on logic, Hilbert and Bernays are joint authors of multivolume classics on logic, and any joint authors of multivolume classics on logic are either Whitehead and Russell or Hilbert and Bernays.

So the force of (9.5) as an example of superplurals depends on the extent to which this elimination is objectionable.[3]

The sentence concerning twin primes is somewhat difficult to assess. In mathematics, it is common to treat 'twin primes' distributively, taking a twin prime to be a prime that differs by two from another prime. This makes 'the twin primes' an ordinary plural description. So a conjunctive paraphrase would be perfectly in order:

(9.8) 3 and 5 are among the twin primes.

(9.17) 3 is a twin prime and 5 is a twin prime.

It is also common to use 'twin prime' as synonymous with 'twin prime pair'. Then the semantic value of 'the twin primes' would be the collection of *pairs* of the form $\langle p, p+2\rangle$, where p and $p+2$ are prime. While this understanding is closer to the superplural interpretation, it will not persuade the skeptic who takes a pair to be an object and hence sees a collection of pairs as an ordinary plurality. This kind of skeptic may regard a true utterance of (9.8) as tantamount to:

(9.18) The pair $\langle 3, 5\rangle$ is among the twin primes.

Linnebo and Nicolas (2008) argue that examples (9.6) and (9.7) strengthen the case for superplurals in English in that they are not amenable to either a conjunctive analysis or a group-based interpretation.

(9.6) These people, those people, and these other people played against each other.

(9.7) The square things, the blue things, and the wooden things overlap.

[3] For objections, see Oliver and Smiley 2016, 135–6.

The group-based analysis is not available because, in each example, all the terms in the lists are paradigmatic instances of semantically plural terms. What about a conjunctive analysis? In its intended reading, (9.6) conveys that there is a three-way competition between these people, those people, and these other people. So a conjunctive analysis is not applicable (compare with 'The Beatles, the Rolling Stones, and Led Zeppelin gave a joint concert'). Something similar may be said of (9.7). As noted by Linnebo and Nicolas (2008, 198), however, this sentence can be given an indirect conjunctive paraphrase:

(9.19) There is a thing such that it is one of the square things, it is one of the blue things, and it is one of the wooden things.

So the force of (9.6) as example of superplurals depends on whether this paraphrase is objectionable.[4]

Let us summarize our discussion so far. Some alleged examples of superplurals in English have limited force, since they can be given a conjunctive analysis or a group-based one. Other examples cannot easily be analyzed in those ways and are therefore more persuasive.

How persuasive? Again, the answer turns on the availability of alternative analyses. In the next section, we discuss an analysis according to which lists cannot give rise to superplural terms because they are not terms at all. If correct, this analysis undermines any example that essentially relies on lists. In Section 9.7, we consider an analysis purporting to undermine *all* alleged examples of superplural terms. It holds that lists, as well as plural descriptions of any kind, are ordinary plurals.

9.6 The multigrade analysis

If some lists are to count as superplural terms, they must be terms in the first place. The assumption that lists are terms is *prima facie* plausible and, as remarked in Section 9.4, it enables us to interpret seemingly multigrade predicates applying to lists of *singular* terms as plural predicates. By parity of reasoning, we can regard at least some seemingly multigrade predicates applying to lists of *plural* terms as superplural predicates.

[4] Linnebo and Nicolas suggest that the paraphrase could be resisted on grounds that 'overlap' is, plausibly, lexically primitive.

The assumption that lists are terms may, however, be disputed. One may instead interpret lists as strings of separate terms, taking predicates to be genuinely multigrade. There are several advocates of this multigrade analysis of predication, and Oliver and Smiley have concluded that the view that lists are string is just as plausible as the view that lists are terms (see Oliver and Smiley 2016, Chapter 10, for discussion and references).

To see how the multigrade analysis works, let us start with some basic examples.

(9.20) (a) Annie and Bonnie met.
(b) Annie, Bonnie, and Connie met.
(c) Annie, Bonnie, Connie, and Danny met.

If we treat the predicate 'met' as having variable adicity, these sentences can be regimented as follows.

(9.21) (a) $M(a, b)$
(b) $M(a, b, c)$
(c) $M(a, b, c, d)$

Note that a general implementation of the multigrade analysis requires a distinction between argument places and argument positions.[5] Consider these sentences:

(9.22) (a) Annie and Bonnie rescued Connie.
(b) Annie rescued Bonnie and Connie.

To avoid ambiguities, the regimentation must clearly separate the agents from the patients of the relation. So we must first have *argument places*, roughly corresponding to relevant thematic roles, such as agent and patient. Then, within each argument place, we must have *argument positions*. Using semicolons to separate argument places and commas to separate argument positions, we can regiment (9.22a) and (9.22b) as (9.23a) and (9.23b), respectively.

(9.23) (a) $R(a, b; c)$
(b) $R(a; b, c)$

[5] For an elaboration of the distinction, see Oliver and Smiley 2016, 172–4.

The key point is that, on the multigrade analysis, 'Annie', 'Bonnie', and 'Connie' do not combine to form new terms. Rather, they occupy different argument positions, sometimes sharing the same argument place.

If we adopt this analysis, lists are strings of separate terms. As a result, we forsake all examples of superplurals based on lists. For instance, (9.6) has the form of (9.24), a formula containing only ordinary plural variables.

(9.6) These people, those people, and these other people played against each other.

(9.24) $\varphi(xx, yy, zz)$

However, the multigrade analysis faces some difficulties. To begin with, there is the question of how to develop the approach systematically so that it can account for inferential relations involving sentences with lists. As observed in Florio and Nicolas 2015 (454–5), one would have to rely heavily on meaning postulates in order to account for the validity of inferences such as the following:

(9.25) (a) Annie and Bonnie are students and best friends.
(b) Therefore, some students are best friends.

(9.26) (a) Bonnie and Connie are students.
(b) Annie, Bonnie, and Connie met.
(c) Therefore, Annie and some students met.

(9.27) (a) Annie, Bonnie, and Connie visited Paris.
(b) Therefore, Annie and Bonnie visited Paris.

(9.28) (a) Annie and Bonnie met.
(b) Therefore, Bonnie and Annie met.

According to the multigrade analysis, these inferences have the following regimentations:

(9.29) (a) $S(a, b) \wedge B(a, b)$
(b) $\therefore \exists xx(S(xx) \wedge B(xx))$

(9.30) (a) $S(b, c)$
(b) $M(a, c, b)$
(c) $\therefore \exists xx(S(xx) \wedge M(a, xx))$

(9.31) (a) $V(a, b, c; p)$
(b) $\therefore V(a, b; p)$

(9.32) (a) $M(a, b)$
(b) $\therefore M(b, a)$

To capture the validity of these inferences, we must introduce special rules governing the relevant classes of expressions. While there is no obvious obstacle, one may regard the need for special rules as a disadvantage over alternative accounts that can explain the validity of the inferences from simpler and more basic semantic principles.

Another difficulty has to do with the implementation of the analysis in a plurality-based setting. If the multigrade analysis is to serve as a genuine strategy for avoiding superplurals, it cannot rely on them in its implementation. Yet it is not clear that a plurality-based model theory for multigrade expressions can avoid superplurals. For example, to describe the interpretation of argument positions, one must refer to arbitrary long sequences of pluralities, which naturally calls for superplurals.

9.7 Covers

If successful, the multigrade analysis would undermine alleged examples of superplurals based on lists. However, there is an alternative analysis that is more far-reaching in its potential to weaken the case for superplurals. According to this analysis, lists as well as plural descriptions of any kind are *ordinary plurals.*

This idea might seem a non-starter. Aren't there obvious counterexamples to any ordinary plural analysis of alleged superplural terms? As we have seen above, there are clear cases in which the syntactic articulation of a plural term matters to the truth conditions. Other cases are not hard to produce. Consider these two sentences in a context where the participants in a competition are either students or teachers at a local school:

(9.33) The participants played against each other.

(9.34) The students and the teachers played against each other.

Given the context, these two sentences can differ in truth value. This suggests that the semantic values of their respective subjects are not the same, contrary to the thesis that lists and plural descriptions are always ordinary plurals. If 'the students and the teachers' was an ordinary plural term, its semantic value would be same as that of 'the participants'. So the two sentences would have to be equivalent, which they are not. By assuming, instead, that 'the students and the teachers' is not an ordinary plural term but a superplural one, we easily explain why the sentences are not equivalent.

Recall the dispute over the interpretation of a complex plural noun phrase α of the form

$$\beta_1 \text{ and} \ldots \text{and } \beta_n$$

where $\beta_1, \ldots, \beta_n$ are themselves plural noun phrases (Section 9.2). We can choose between a rule of composition that renders α a plural term and one that renders it a superplural term. That is, we have two options:

$$[\![\alpha]\!] = [\![\beta_1]\!] \cup \ldots \cup [\![\beta_n]\!]$$

and

$$[\![\alpha]\!] = \{[\![\beta_1]\!], \ldots, [\![\beta_n]\!]\}$$

Examples such as (9.33) and (9.34) put pressure on the plural interpretation. However, linguists such as Brendan Gillon (1987, 1990) and Roger Schwarzschild (1996) have argued that this interpretation can in fact be defended.

According to Gillon and Schwarzschild, articulated plural noun phrases ('the students and the teachers') have the same type of denotation as unarticulated ones ('the participants'). The role of the articulation in the truth conditions is explained by appeal to *covers*.

To see how the analysis works, let us examine a sentence with multiple readings:

(9.35) Annie and her dogs weigh 50 kg.

There is, of course, a collective reading and a distributive one: Annie and her dogs may be said to weigh 50 kg collectively or individually. But another reading is easily available. This is an *intermediate* reading on which Annie is said to weigh 50 kg individually, and her dogs are said to weigh 50 kg collectively.

The three readings correspond to three *partitions* of the set containing Annie and each of her dogs. For concreteness, assume that the dogs are two,

d_1 and d_2, and hence the set can be represented as $D = \{\text{Annie}, d_1, d_2\}$. So the three readings correspond to partitions of D with one, three, and two cells, respectively:

$$P_1 = \{\{\text{Annie}, d_1, d_2\}\}$$
$$P_2 = \{\{\text{Annie}\}, \{d_1\}, \{d_2\}\}$$
$$P_3 = \{\{\text{Annie}\}, \{d_1, d_2\}\}$$

By appealing to these partitions, we can capture the multiple readings of (9.35) while retaining the assumption that

$$[\![\text{Annie and her dogs}]\!] = [\![\text{Annie}]\!] \cup [\![\text{her dogs}]\!]$$

This can be done by stating the truth conditions as follows:

> 'Annie and her dogs weigh 50 kg' is true in some context Q if and only if there is a partition P of $[\![\text{Annie and her dogs}]\!]$ that is salient in Q, and every element of P is in $[\![\text{weigh 50 kg}]\!]$.

Thus, the three readings described above (and potentially more) can be seen to arise from contextually salient partitions. For instance, if the salient partition is P_3, the sentence is true if and only if $\{\text{Annie}\} \in [\![\text{weigh 50 kg}]\!]$ and $\{d_1, d_2\} \in [\![\text{weigh 50 kg}]\!]$. That is, the sentence is true if and only if Annie weighs 50 kg individually and her dogs weigh 50 kg collectively.[6]

However, if this analysis is to succeed, we need something more general than partitions. This is shown by the next example. Suppose that Annie, Bonnie, and Connie variously cooperated to build sand castles; specifically, Annie cooperated with Bonnie, Bonnie with Connie, and Connie with Annie. None of them built any other sand castle. Then it seems that we can truly say:

(9.36) Annie, Bonnie, and Connie cooperated.

A moment's reflection reveals that no partition of the denotation of 'Annie, Bonnie, and Connie' will capture the intended truth conditions. Indeed, the relevant articulation is represented by a set whose cells overlap:

[6] For convenience, here we follow Schwarzschild in assuming that a singleton set may be identified with its sole element. Since the sentence 'Annie weighs 50 kg individually' is true if and only if Annie $\in [\![\text{weigh 50 kg}]\!]$, it follows that the same sentence is true if and only if $\{\text{Annie}\} \in [\![\text{weigh 50 kg}]\!]$.

$$\{\{a, b\}, \{b, c\}, \{c, a\}\}$$

The notion of *cover* provides what is needed. A cover is just like a partition except that the cells are allowed to overlap. In the set-theoretic framework, the formal definition is as follows. A set C is a cover of a set D if and only if:

(i) C is a set of non-empty subsets of D
(in symbols: $\forall x\,(x \in C \rightarrow x \neq \varnothing \land x \subseteq D)$;

(ii) every element in D is in at least one set in C
(in symbols: $\forall x\,(x \in D \rightarrow \exists y\,(y \in C \land x \in y))$.

Using covers, we can generalize the analysis of 'Annie and her dogs weigh 50 kg' presented above (see Schwarzschild 1996, 67):

> $[_{\mathrm{S}}\ \mathrm{NP}_{\mathrm{plural}}\ \mathrm{VP}]$ is true is some context Q if and only if there is a cover C of the denotation of the NP that is salient in Q, and every element in C is in the extension of the VP.[7]

Let us call this the *cover analysis*.

It is now straightforward to account for the multiple readings of the sentences encountered in this section. The different truth conditions arise from contextual variations that select different covers. If a plural subject is syntactically structured, this structure might obviously influence the context and render salient a cover with the corresponding articulation. So the syntactic structure of a plural subject can affect the truth conditions even if one rejects the assumption that the subject's semantic value is articulated.

The cover analysis is fully general and applies even when the plural subject is a definite description or a demonstrative noun phrase:

(9.37) The children cooperated.

(9.38) These children cooperated.

If the children in question are Annie, Bonnie, and Connie, these sentences are predicted to exhibit the same range of readings as:

[7] Gillon (1987) requires the additional condition that the cover be *minimal*, namely that no cell be properly contained in any other cell. For our purposes, we can safely leave out this complication. In so doing, however, we do not mean to rule out its correctness.

(9.36) Annie, Bonnie, and Connie cooperated.

This is because the noun phrases in the three sentences co-refer and are therefore associated with the same range of possible covers. Nevertheless, the difference in syntactic structure means that the sentences may make different covers salient.

We now wish to consider an objection to the cover analysis. Let us return to one of the sentences that we used to motivate superplurals in Section 9.4:

(9.7) The square things, the blue things, and the wooden things overlap.

Suppose that the denotations of the three definite descriptions are as follows:

$$\begin{aligned} &[\![\text{the square things}]\!] = \{a, b\} \\ &[\![\text{the blue things}]\!] = \{a, c\} \\ &[\![\text{the wooden things}]\!] = \{a, d\} \end{aligned}$$

Then, according to the cover analysis, we have:

$$[\![\text{the square things, the blue things, and the wooden things}]\!] = \{a, b, c, d\}$$

where, in the absence of further contextual cues, the salient cover is given by the syntactic structure of the plural subject:

$$C = \{\{a, b\}, \{a, c\}, \{a, d\}\}$$

It follows from the analysis that (9.7) is true if and only if every element in C is in the extension of 'overlap'. This means that the truth of (9.7) requires:

$$[\![\text{overlap}]\!] = \{\ldots, \{a, b\}, \{a, c\}, \{a, d\}, \ldots\}$$

But that seems incorrect. For this extension would also sanction the truth of:

(9.39) The square things overlap.

which is clearly false in the given context. Intuitively, the correct extension of 'overlap' is as follows:

$$[\![\text{overlap}]\!] = \{\{\{a, b\}, \{a, b\}\}, \ldots, \{\{a, b\}, \{a, c\}, \{a, d\}\}, \ldots\}$$

That is, X is an element of the extension of 'overlap' just in case X is a set of non-empty sets that have some element in common. To make use of this extension, however, we would have to revise the cover analysis so that a sentence is true if and only *the cover itself* is in the extension of the verb phrase. But this revision would not fit with the analysis of our previous examples where the extension of the verb phrase must contain *the elements* of the cover, not the cover itself.

Let us take a step back. The cover analysis is supported by evidence concerning intermediate readings. Still, at least as formulated above, it cannot handle some of the alleged examples of superplurals, as shown in the previous paragraph. Perhaps some alternative version of the analysis, coupled with a closer linguistic examination of the examples, can overcome this apparent difficulty. Even so, implementing the proposal in the plurality-based setting would have significant consequences for the debate over superplurals.

Indeed, superplurals seem required to formulate the cover analysis in the plurality-based setting. Consider the following examples:

(9.40) The ordinals and the cardinals overlap.

(9.41) The sets and the classes overlap.

If we assign to the descriptions their intended interpretations (the pluralities of ordinals, cardinals, and so on), we would have to rely on superplural terms to describe the salient covers. So appealing to covers in this setting concedes that some sentences express "superplural thoughts". This suffices for the third grade of superplural involvement described in Section 9.3. It does not matter whether natural language is linked to superplural thoughts in a somewhat roundabout way.

The third grade of superplural involvement is secured even on a more instrumental conception of the semantic machinery associated with covers. The need to invoke superplurals to develop a semantics of natural language would still show that these resources are legitimate for theorizing and hence legitimate in the sense of the third grade.

9.8 Mixed-level predications

We now turn to an important challenge for the superplural analysis.[8] The challenge is based on a simple idea. Consider a purported example of a sentence with a superplural subject:

[8] See, e.g., Landman 1989a, 598–601; Schwarzschild 1996, 45–8; Ben-Yami 2013, 87.

(9.42) My children, your children, and her children played against each other.

Now extend the sentence as follows:

(9.43) My children, your children, and her children played against each other and then ate ice cream.

Notice that the added predicate, unlike the original one, does not call for an argument with a superplural articulation. Thus, the two predicates seem to belong to different levels, yet they are applied to the very same subject. How, then, can this subject involve superplural reference?[9]

One might try to handle such mixed-level predications by claiming that, when the semantic value of a noun phrase involves more articulation than is required by a predicate, the redundant articulation is simply "thrown away", resulting in a less articulated semantic value that *is* appropriate for the predicate in question. To this end, one may postulate a type-shifting operation that maps a superplurality to the underlying plurality; for example:

$$a_1, a_2; b_1, b_2; c_1, c_2, c_3 \quad \mapsto \quad a_1, a_2, b_1, b_2, c_1, c_2, c_3$$

However, type-shifting should not be postulated beyond necessity. And in fact, it may be unnecessary to invoke this kind of mechanism in the present case. There is no need to shift the semantic value of a noun phrase by "throwing away" its redundant articulation. A predicate that does not *require* this extra articulation might nevertheless be *tolerant* of it—in the sense that the predicate simply ignores the articulation.[10] To ensure that the semantic values are tolerant of extra articulation in this way, we require that they be *upwards closed*: if X is an element of the semantic value, then so is any Y that is based on the same individuals but has a strictly richer articulation than X. As an example, consider the distributive plural predicate 'ate ice cream'. For convenience, we represent its semantic value set-theoretically. Suppose we have:

$$\{a, b, c, d\} \subseteq [\![\text{ate ice cream}]\!]$$

[9] These examples also pose a challenge for the multigrade analysis. For this analysis has no obvious explanation of why (9.42) follows from (9.43). See the discussion in Section 9.6.

[10] This strategy is critically discussed in Schwarzschild 1996, Chapter 4. Schwarzschild favors a cover-based approach, as noted in Section 9.7.

Assuming that the semantic value of the predicate is upwards closed, it will contain all other sets based on the same individuals but 'articulated' in more complex ways. For example, we have:

$$\{\{a, b\}, \{c, d\}\} \subseteq [\![\text{ate ice cream}]\!],$$

which corresponds to a superplural articulation of the four individuals. We give this idea a precise mathematical definition in Appendix 9.A.

To sum up, provided that the semantic values of predicates are required to be upwards closed, the challenge under discussion can be answered. Recall our guiding example, (9.43). Suppose the predication of 'played against each other' is analyzed as having a superplural subject, namely my children, your children, and her children. The upwards closure of $[\![\text{ate ice cream}]\!]$ ensures that the associated predication is true of the mentioned superplurality just in case each of the children ate ice cream, precisely as desired. More generally, a highly articulated semantic value of some noun phrase is never a problem so long as predicates that do not require this degree of articulation simply ignore it.

9.9 Mixed-level terms, order, and repetition

The superplural analysis assumes that the semantic value of some plural noun phrases is articulated, namely structured into multiple sets or pluralities. Additional evidence for semantic articulation may come from the linguistic phenomena illustrated by the following sentences:

(9.44) Annie, Annie's sisters, and Bonnie competed.

(9.45) 'a', 'b', 'c', and 'd' are consecutive letters.

(9.46) 1, 1, and 4 have mean 2.

The first example involves a mixed-level list, where singular terms are combined with a plural description. The other examples show sensitivity to order and to repetition. In (9.45), changing the order of the terms affects the truth conditions. So does removing the repetition in (9.46).

Those who accept the superplural analysis can simply take these examples to show the need for yet other forms of articulation. In particular, they can account for the data by countenancing these additional forms of articulated reference:

(a) reference to *mixed-level pluralities*, including combinations of single objects and pluralities;
(b) reference to *ordered pluralities*, respecting order;
(c) reference to *multipluralities*, allowing for repetition.

Suggestions of this kind have been put forward by Hewitt (2012a), Ben-Yami (2013), and Hossack (2020).[11] Hewitt and Hossack claim that plural terms can refer to things in an order (serial reference). According to Ben-Yami, the reference of plural terms can be articulated in various ways (articulated reference): when the articulation represents information about order or repetition, this information can play a role in the truth conditions.

This appeal to additional forms of articulated reference can be challenged, however. Building on earlier work by Kay (1989), Oliver and Smiley (2004; 2016, Chapter 10), and Chaves (2012), Florio and Nicolas (2015) argue that order and repetition should be explained in a different way. They point out that assuming additional forms of articulated reference has limited scope and is unnecessary to give a unified account of the broad range of cases in which order and repetition are semantically relevant. On the account they propose, order and repetition enter the truth conditions through salient indexings introduced by context and by the meaning of special expressions (for example, 'consecutive', 'in that order', and 'respectively').

To see how their proposal works, consider this sentence:

(9.47) Annie, Bonnie, and Connie arrived in the order they were called. (Florio and Nicolas 2015, 449)

Here the order of mention is irrelevant. There are two relevant orders: first, the one in which Annie, Bonnie, and Connie arrived; then, the one in which they were called. The sentence conveys that these two orders are, in some sense, the same. This can be explicated as follows: the sentence is true if and only if the indexing of the plurality of Annie, Bonnie, and Connie according to their time of arrival is isomorphic to the indexing of the same plurality according to the order of calling. There is no need to assume that the semantic

[11] Fine (2010) develops a more comprehensive approach, which can serve as a framework for these suggestions. At the heart of the approach is a general composition operation that can be specialized to obtain any form of articulated reference discussed here. In particular, this operation can be set to respect or ignore each of the following features: order, repetition, and articulation into higher-level pluralities. For a precise statement of these choices, see Fine 2010, 573.

contribution of 'Annie, Bonnie, and Connie' goes beyond that of supplying the mere plurality of objects on which the salient indexings operate.[12] So the proposal can be developed without postulating that the semantic value of a plural term is articulated.

Might the multigrade analysis provide an alternative way to account for the examples that seem to motivate the postulation of additional forms of articulated reference? This analysis, we recall, promises an alternative to superplurals. It is naturally extended to treat mixed-level terms, order, and repetition. Using multigrade predicates, one can provide the following regimentation of the examples discussed at the beginning of this section:

(9.44) Annie, Annie's sisters, and Bonnie competed.

(9.48) $C(a, tt, b)$

(9.45) 'a', 'b', 'c', and 'd' are consecutive letters.

(9.49) $C(a, b, c, d)$

(9.46) 1, 1, and 4 have mean 2.

(9.50) $M(1, 1, 4; 2)$

Thus, apparent mixed-level terms are simply cases in which the positions of an argument place are occupied by terms of different levels. Moreover, the semantic relevance of order and repetition is explained by an obvious fact about predication: changing the order of the arguments, as well as adding or removing an argument, does not in general preserve truth.

This extended use of the multigrade analysis has serious limitations, however. The problem is that order and repetition can matter even in the presence of a single, non-conjunctive argument:[13]

(9.51) There are some consecutive letters.

(9.52) These letters are consecutive.

(9.53) Some numbers have mean 2.

(9.54) These numbers have mean 2.

[12] For details, see Florio and Nicolas 2015, Section 5.

[13] See also Ben-Yami 2013, 96–7, and Florio and Nicolas 2015, 455.

The multigrade account of order and repetition just sketched does not apply to these cases, and it is unclear how it could explain them without borrowing from the other approaches discussed earlier in this section. In particular, articulated pluralities might still have to be invoked as covers or as possible values of plural variables. But if so, it becomes harder to justify the introduction of the multigrade apparatus.

It remains an open problem, then, how best to deal with the examples of mixed-level terms, order, and repetition. The case for additional forms of articulated reference is not as strong as that for superplural reference.

9.10 Conclusion

We have examined four questions concerning the availability of superplurals, corresponding to increasingly higher "grades of superplural involvement". The first two grades are unproblematic: there is no obstacle to formulating a superplural logic and a model-theoretic semantics for a superplural language. It is not immediately obvious, however, whether superplural reference is a legitimate expressive resource for beings like us. Nor is it immediately obvious whether superplural reference is realized in natural language. So the third and fourth grades are less straightforward and require careful investigation.

Our investigation showed that a strong case can be made for the highest grade. Some sentences of natural language can plausibly be analyzed in superplural terms, and this analysis fares well compared to alternatives such as the multigrade analysis and the cover analysis. Let us review the pros and cons of each analysis.

The superplural analysis is supported by *prima facie* evidence from articulated noun phrases. However, it may require the assumption that the semantic values of predicates be upwards closed, which some might find problematic. Moreover, by avoiding superplural reference, the alternative analyses can claim better ideological economy.

The multigrade analysis offers a simple way to handle some cases of order and repetition, but it faces difficulties in accounting for all cases in which order and repetitions are semantically relevant. Moreover, it must rely heavily on meaning postulates to capture data about logical consequence.

The cover analysis is supported by evidence from intermediate readings. But it seems to falter on promising examples of superplurals, where the cover itself, rather than its members, has to be included in the extension of the predicate.

What about the third grade of superplural involvement? If there is a good case for the highest grade, there is also a good case for the third grade. That is, if some natural language realizes superplural reference, it is hard to deny that this form of reference is a legitimate expressive resource for beings like us. In fact, the case for the third grade is probably even stronger than that for the highest grade. In particular, since we have accepted the plurality-based model theory as a new and valuable alternative to the traditional set-based model theory, it becomes difficult to avoid superplurals. Furthermore, in that style of model theory, superplurals are likely to be involved even in the cover analysis and in the multigrade one. The theoretical need to adopt superplural resources thus shows that these resources are legitimate for theorizing, which would also establish our third grade of superplural involvement.

Appendix

9.A The notion of upwards closure

In this appendix, we make formally precise some of the ideas outlined in Section 9.8. Our first task is to define an analogue of the sets of finite rank based on a domain of individuals D, though modified so as to consider only sets of cardinality greater than or equal to 2. This modification is natural when modeling superplural phenomena. Sets are used to represent pluralities. But since a singleton plurality is identical with its single member, there is no need for a singleton set to represent this singleton plurality, since this plurality is already represented by whatever represents its single member. We therefore proceed as follows. First, we define our modified powerset operation by letting $\wp^{\geq 2}(X)$ be the set of subsets of X of cardinality greater than or equal to 2. Next, we define the analogue of the sets of finite rank based on D:

- $V_0^{\geq 2}(D) = D$
- $V_{i+1}^{\geq 2}(D) = \wp^{\geq 2}(V_i^{\geq 2}(D))$
- $V_\omega^{\geq 2}(D) = \bigcup_{i<\omega} V_i^{\geq 2}(D)$

Finally, let $D^\star = V_\omega^{\geq 2}(D)$.

We are now ready to define the desired operation of upwards closure. Consider some $X \in D^\star$. What is it for some $Y \in D^\star$ to be the result of imposing a richer articulation on X? Let us proceed in steps. First, let us say that Y is a *simple articulation of* X just in case Y is the result of replacing zero or more members of X with subsets of X, ensuring that all of the members of X are "used" in the construction, that is, that all members of X are either in Y or in some member of Y. For example, let X be $\{a, b, c\}$. Then two simple articulations of X are $\{a, \{b, c\}\}$ and $\{a, \{a, b, c\}\}$. Next, let us say that Y is an *articulation of* X just in case Y can be obtained by a finite chain of simple articulations starting with X.

Notice that this definition works for the representation of higher-order pluralities as well. For example, let X be $\{\{a, a'\}, \{b, b'\}, \{c, c'\}\}$. Then $\{\{a, a'\}, \{\{b, b'\}, \{c, c'\}\}\}$ is a simple articulation of X.

These definitions can be made precise as follows.

Definition 9.1 Let $X \in D^\star$. Then Y is a *simple articulation of* X if and only if there are $U, V \in D^\star$ such that

(i) $Y = (X \setminus U) \cup V$
(ii) $x \in V \to x \subseteq X$
(iii) $U \subseteq \bigcup V$

Next, Y is an *articulation of* X if and only if there is a finite chain $X_0, X_1, \ldots, X_n$ such that

(i) $X = X_0$
(ii) $Y = X_n$
(iii) X_{i+1} is a simple articulation of X_i for each $i < n$.

Finally, let $A \in D^\star$ be the semantic value of some predicate. Its *upwards closure* is defined as:

$$\mathrm{UC}(A) = \{B \in D^\star : B \text{ is an articulation of } A\}$$

Suppose we require that semantic values of predicates be upwards closed. Then there is no problem about predications of the sort discussed in Section 9.8. Consider our main example:

(9.43) My children, your children, and her children played against each other and then ate ice cream.

Provided that the semantic value of 'ate ice cream' is upwards closed, it will apply to the articulated semantic value of 'my children, your children, and her children' just in case it applies to the individuals which are thus articulated.

In fact, the machinery just developed enables us to define what it is for a predicate to be plural (= plural of order 1), superplural (= plural of order 2), or plural of order n.

Definition 9.2 Assume the semantic value of a predicate P is Y. Then P is *plural of order n* if and only if there is a $X \subseteq V_n^{\geq 2}(D)$ such that $Y = \mathrm{UC}(X)$.

That is, a predicate P on a domain D is plural of order n just in case its semantic value Y can be generated as the upwards closure of a set X whose members are of rank n.

IV

THE LOGIC AND METAPHYSICS OF PLURALS

10
Plurals and Modals

10.1 Introduction

Just as the interaction between first-order quantification and modalities raises a number of interesting and difficult questions, so does the interaction between plural quantification and modalities.* In this chapter, we discuss the central aspects of the problem of how plurals and modalities should be combined. We have two main goals. The first is to provide a useful map of the current literature. The second is to argue for the metaphysical claim that pluralities are modally rigid. What does this claim mean?

Consider some things, and choose any one of them. Is the chosen thing *necessarily* one of the things from which it was chosen? It is usually assumed that the answer is positive—so long as the things in question still exist.[1] If some things did not include our chosen thing, then these things would simply not be the things with which we started, that is, the things from which we made a choice. If the things from which we chose exist at all, then *necessarily*, whenever they exist, they include the chosen thing. Likewise, if some other thing is *not* one of the things from which we chose, then this too is a matter of (conditional) necessity. With the help of an existence predicate E, these two modal constraints on plural membership can be formalized as follows:[2]

$$(\textsc{Rgd}^{+}) \qquad \Box\forall x\forall yy(x \prec yy \rightarrow \Box(Eyy \rightarrow x \prec yy))$$

$$(\textsc{Rgd}^{-}) \qquad \Box\forall x\forall yy(x \not\prec yy \rightarrow \Box(x \not\prec yy))$$

The claim that pluralities are *rigid* is the conjunction of these constraints, which we abbreviate as (Rgd).

* This chapter is a revised and expanded version of Linnebo 2016.

[1] See Williamson 2003, 456–7; Rumfitt 2005, section VIII; Williamson 2010, 699–700; Uzquiano 2011; Williamson 2013, 246–8. The view is challenged in Hewitt 2012a.

[2] One might have thought that more existential presuppositions were needed. We will later see that that isn't so.

The Many and the One: A Philosophical Study of Plural Logic. Salvatore Florio and Øystein Linnebo, Oxford University Press.
DOI: 10.1093/oso/9780198791522.003.0010

The rigidity principles can be regarded as a form of extensionality, stating that pluralities are tracked extensionally across possible worlds. The ordinary principle of extensionality states that if every one of *these things* is one of *those things* and vice versa, then *these things* and *those things* are the very same things. (We will return to the question of how best to formalize this principle.) This is widely assumed to provide a criterion of identity for pluralities; and like criteria of identity in general, the principle is widely thought to hold of necessity. Even so, the principle provides no information about how pluralities are tracked across possible worlds. The rigidity principles fill this gap. They tell us that necessarily any given things have their members by necessity. A plurality is therefore not allowed to vary in its membership across possible worlds. Any variation in membership would result in our talking about some other things, not the things in question.

To appreciate how the rigidity claims for pluralities have real bite, it is useful to contrast pluralities with *groups*, such as teams, clubs, committees, and the like.[3] These entities do not have their members necessarily. Consider the Department of Philosophy of the University of Athens. It might have had other members than it in fact has: Sophus might have been hired instead of Sophia. Someone who *is* a member of this department might not have been so; and someone who *is not* a member might nevertheless have been one. Thus, a group such as a philosophy department does not have its members necessarily. The same is true of other typical groups.

If pluralities are rigid but groups are not, what explains the difference? One might try to appeal to the distinction between many and one. A plurality is many, while a group is one. But this distinction cannot explain the relevant modal difference between pluralities and groups. For a set is one and yet has its members necessarily. So rigidity is compatible with being one.

A far more promising response arises from the following basic thought: a plurality is nothing over and above its members and is thus fully specified when we have circumscribed its members. Tracking a plurality across possible worlds is therefore trivial: it is simply a matter of tracking its members. Unlike a plurality, a group *is* something over and above its members: it is not fully specified when we have circumscribed its members. For example, we additionally need to specify its membership criterion. A group such as a department of philosophy will be associated with a membership criterion that is sanctioned by the statutes of the university. So tracking a group across

[3] For useful discussions of groups, see, e.g., Landman 1989a, Landman 1989b, and Uzquiano 2004b.

possible worlds is not trivial; it goes beyond tracking each of its members.[4] By contrast, since a plurality is nothing over and above its members, there is no material available that might underwrite a non-trivial tracking across possible worlds. All we have to go on are the members. So the only way to track a plurality is the trivial one, which ensures plural rigidity.[5]

Sets—understood according to the iterative conception—resemble pluralities in this respect, with the additional and complicating factor that their members are "bound together" into a single object (see Section 4.4).

In what follows, we attempt to clarify and develop the basic thought that a plurality is fully specified when we have circumscribed its members. The result will be a disentangling and clarification of several aspects of the basic thought. We will find that plural rigidity figures at the heart of a network of ideas having to do with what we will call *the extensional definiteness* of pluralities.

10.2 Why plural rigidity matters

The question whether pluralities are rigid has emerged as the central question about the interaction between plural quantification and modalities. The reason for this has to do with the important ramifications of the question in philosophical logic, metaphysics, and the philosophy of mathematics.

One example is the debate about the relation between plural logic and second-order logic discussed in Chapter 6. Can plural logic be replaced by monadic second-order logic or even reduced to it? Or is some reduction in the opposite direction possible? If pluralities are rigid, then the two forms of logic have different modal profiles. For the modal behavior of predication is clearly non-rigid, as the following sentences illustrate.

(10.1) Timothy Williamson is a philosopher, but he might not have been one.

(10.2) Hillary Clinton is not a philosopher, but she might have been one.

[4] Uzquiano 2018 provides a systematic development of the idea that, since tracking them is trivial, pluralities can be seen as a limiting case of generally non-rigid groups.

[5] Roberts (forthcoming) provides a systematic investigation of this basic thought, resulting in a defense not only of (RGD) (which is our main concern in this chapter) but also some further modal principles.

As we have seen, however, the difference between the modal profile of predication and that of plural membership makes at least one kind of reduction problematic (see Section 6.4).

A related example concerns the semantics of predication. In Section 7.5, we showed that if we have ordered pairs at our disposal, it is technically possible to use plurals to give a semantic analysis of predication. Specifically, we can take the semantic value of a predicate to be the plurality of tuples of which the predicate is true. In addition to the lack of homophonicity even on the intended interpretation and the need for *ad hoc* tricks to handle predicates that are true of nothing, there is a violation of the constraint that semantic values should have the same modal profile as the expressions of which they are semantic values. So considerations pertaining to modal rigidity can help us decide among competing semantics.

Next, the rigidity of pluralities plays a central role in one of Williamson's main arguments for *necessitism*, the metaphysical view that, necessarily, everything necessarily exists.[6] The denial of this view is *contingentism*. When we go on to consider arguments for the rigidity of plurals, it will be important to keep in mind whether the argument is intended to be given in a necessitist setting (which is always easier) or in a contingentist setting (which requires greater care).

Finally, the question of the rigidity of pluralities plays an essential role in an approach to mathematics and to the phenomenon of indefinite extensibility developed in recent work by one of us (Linnebo 2010 and Linnebo 2013).[7]

10.3 Challenges to plural rigidity

We aim to survey a number of arguments for the claim that pluralities are rigid. Before doing that, however, we should address some alleged counterexamples to plural rigidity.

The first one involves plural descriptions. Assume that Sophia is one of *the philosophers*. Does it follow that she is *necessarily* one of the philosophers? (For simplicity, we leave implicit the assumption that the entities in

[6] See Williamson 2010, 2013.

[7] While this approach to mathematics and indefinite extensibility draws inspiration from Parsons 1983b and to some extent also Putnam 1967 and Hellman 1989, these earlier views do not rely in the same way on the rigidity of pluralities.

question still exist.) If so, we would have to accept the implausible claim that, necessarily, Sophia is a philosopher. Thus, we must deny that Sophia is necessarily one of the philosophers. Does the case of Sophia provide us with a counterexample to plural rigidity?

Long ago Kripke taught us how to respond. Let *pp* be the things such that anything is one of them if and only it is in fact a philosopher. What is necessarily the case is that Sophia is one of *pp*. But it is not necessary that *pp* are all and only the philosophers. Sophia might have become a psychologist, not a philosopher. Then she would not have been included in the ranks of the philosophers, although she would still have been one of *pp*. It is important not to misunderstand the rigidity claim.

Other apparent counterexamples involve pronouns rather than plural descriptions. A nice example is the following ad we once saw for a gym:

(10.3) Join, and become one of us!

The plural pronoun 'us' is naturally taken to stand for a plurality. But when so interpreted, the message presupposes that it is possible to become a member of a plurality of which one is not already a member.[8] If there were such a possibility, we would have a failure of rigidity.

How are we to respond to these apparent counterexamples to the rigidity of pluralities? Interesting though they are, these examples are inconclusive. Consider, for instance, a bohemian parent who upon seeing some particularly smug business school students tells her daughter:

(10.4) I'm glad you're not one of them.

It is natural to understand the parent as expressing joy that her offspring is not (in some salient respect) *like* the students in question rather than pleasure with a fact about plural non-membership. Thus, (10.4) poses no more of a challenge to the rigidity of pluralities than the following sentence poses to the necessity of identity:

(10.5) I'm glad you're not him.

In particular, the apparent counterexamples can be explained away if we allow that a plural pronoun can sometimes function as a covert description

[8] A similar example is attributed to Dorothy Edgington in Rumfitt 2005. Further examples are found in Hewitt 2012b.

or refer to a group.[9] Either way, the behavior of the pronoun would be consistent with plural rigidity: as observed above, plural descriptions can function non-rigidly, and groups need not be rigid. Of course, more work would be needed to dispel an apparent counterexample such as (10.3) and establish an alternative explanation consistent with plural rigidity. What our discussion does show, however, is that it is advantageous to base our assessment of plural rigidity on more systematic and theoretical considerations.

As mentioned, we will see that plural rigidity figures at the heart of a network of ideas having to do with the extensionality of pluralities. Since the ideas in this network are true of a *core use* of our plural resources in ordinary language and thought, we commend them as an explication of these resources. We do, however, accept the existence of non-rigid groups. So we have no trouble admitting that there may be uses of plural resources (including plural variables) to stand for groups. Nor do we have any trouble admitting that there are plural expressions (e.g. some plural descriptions) that fail to satisfy rigidity.

10.4 An argument for the rigidity of sets

It will be useful to begin our investigation of the rigidity of pluralities by reminding ourselves of an argument for the necessity of identity and distinctness made famous by Saul Kripke (1980, Lecture III) and often attributed to Ruth Barcan Marcus (1947).[10] As we will see, this argument has striking consequences for the metaphysics of sets. Throughout this chapter, we assume the modal system T as our background modal logic. When stronger modal axioms are used, this will be noted explicitly.

The argument turns on Leibniz's law:

(Leibniz) $$\Box\forall x\forall y\big(x = y \rightarrow (\varphi(x) \leftrightarrow \varphi(y))\big)$$

where, as usual, the relevant argument place of φ occurs in a transparent context, namely outside the scope of quotations and propositional attitudes.

[9] For arguments in support of the first strategy, see e.g. Heim 1990 and Neale 1990, Chapter 5. The possibility of the kind of reference assumed in the second strategy is shown by examples such as 'Yesterday, the committee/club/team met. They decided to issue a press release.' Sentences such as these are natural and contain a plural pronoun, 'they', that appears to be anaphoric on a group.

[10] See Burgess 2014 for a recent discussion of the origin of the argument.

Given the assumption $\Box(x = x)$, Leibniz's law entails $\Box\forall x\forall y(x = y \rightarrow \Box x = y)$. Moreover, given the Brouwerian axiom (or just B for short)

$$\varphi \rightarrow \Box\Diamond\varphi$$

we can also derive the necessity of distinctness: $\Box\forall x\forall y(x \neq y \rightarrow \Box\, x \neq y)$.[11] (Proof sketches of these arguments are provided in Appendix 10.B.)

A contingentist may object to the assumption of $\Box(x = x)$. After all, in a negative free logic, '$x = x$' can be used as an existence predicate, in which case what is assumed is the necessary existence of x. The problem is easily circumvented. The contingentist will have no problem with the assumption that x satisfies the following open formula whose sole argument is represented by '. . .':

$$\Box(x = x \rightarrow x = \ldots)$$

Applying (Leibniz), this enables us to derive formulations of the necessity of identity and distinctness that are acceptable to the contingentist:[12]

(□ =) $\qquad \Box\forall x\forall y(x = y \rightarrow \Box(x = x \rightarrow x = y))$

(□ ≠) $\qquad \Box\forall x\forall y(x \neq y \rightarrow \Box\, x \neq y)$

The derivation of the latter from the former relies, as before, on B.

As Kripke realized, Leibniz's law has important metaphysical consequences. The case of sets provides a nice illustration. Consider the set-theoretic principle of extensionality:

(Set-Ext) $\qquad \forall x\forall y(\forall u(u \in x \leftrightarrow u \in y) \leftrightarrow x = y)$

Leibniz's law reveals a respect in which this is quite a strong principle. Let x and y be coextensive sets. By (Set-Ext), x and y are identical. Observe now that x satisfies the open formula

$$\Box\forall u(u \in \ldots \leftrightarrow u \in x)$$

[11] In fact, as Williamson (1996) has pointed out, (□ ≠) can also be derived without use of the Brouwerian axiom by invoking suitable principles of actuality.

[12] Since the necessitation of (Leibniz) ensures $\Box(x = y \rightarrow x = x)$, the existential presupposition '$x = x$', present in (□ =), would be redundant in (□ ≠). For were $x \neq y$ to fail, the mentioned presupposition would anyway be satisfied. (Thanks to Tim Williamson for this observation.)

So by Leibniz's law, y too satisfies this formula. We conclude that two coextensive sets are subject to necessary *covariation*:

(Set-Cov) $$\forall x \forall y(\forall u(u \in x \leftrightarrow u \in y) \rightarrow \Box \forall u(u \in x \leftrightarrow u \in y))$$

Thus, the set-theoretic principle of extensionality logically entails, via (Leibniz), that two coextensive sets are necessarily coextensive.[13] In fact, since (Set-Ext) holds of necessity, (Set-Cov) does too.

This consequence of Leibniz's law is important. It brings out a fundamental difference between sets and other kinds of collections, such as groups, which are tracked across possible worlds in some intensional way. Consider the covariation claim concerning groups: "If two groups in fact have the same members, they are identical and thus necessarily have the same members." This claim is wildly implausible. Membership in a group is contingent and thus subject to "drift", in the sense that the group may have different members at different possible worlds. But once membership drift is permitted, there is no guarantee that two groups that in fact coincide will necessarily coincide. In the case of sets, by contrast, the principle of extensionality and Leibniz's law entail that there can be no such drifting apart.

The observed difference between sets and groups derives from the fact that sets, unlike groups, are subject to the principle of extensionality. Having the same members suffices for two sets to be identical, but not for two groups. Let us dig deeper. *Why* does having the same members suffice for two sets to be identical, but not for two groups? The only explanation, we contend, is that sets, unlike groups, are constituted by their members. A set is fully characterized by specifying its members. So when two sets have the same members, they are identical. By contrast, a group has additional features that go beyond its members, which means that having the same members need not suffice for identity.

It is these additional features of groups that allow them to be tracked from possible world to possible world in non-trivial ways, permitting, for example, a philosophy department to have different members at different possible worlds. Sets are fundamentally different. Since a set is constituted by its members, there is nothing other than its members to go on when tracking it from world to world. Sets are therefore tracked rigidly. Just as in the case

[13] Here and in the remainder of this section, we leave implicit the proviso that the sets still exist.

of pluralities, the claim that sets are rigid, abbreviated as (Set-Rgd), can be formalized as the conjunction of two statements:[14]

(Set-Rgd$^+$) $$\Box\forall x\forall y(x \in y \to \Box(Ey \to x \in y))$$

(Set-Rgd$^-$) $$\Box\forall x\forall y(x \notin y \to \Box(x \notin y))$$

Let us take stock. Although our argument for (Set-Rgd) does not rest on purely logical premises, it is hard to resist. The principle of extensionality holds for sets, unlike groups, because sets, unlike groups, are constituted by their members. Thus, when tracking a set across possible worlds, there is nothing other than its members to go on. This ensures that the tracking is rigid. Not so for groups, which have additional features that go beyond their members and thus permit non-trivial tracking.

These considerations give rise to a dilemma that applies not only to sets but to any other notion of collection (including groups). *Either* we have to give up the principle of extensionality (as in the case of groups), *or else* we have to accept rigidity as well (as in the case of sets).[15] There is no stable middle ground. The only explanation for the principle of extensionality also supports rigidity.[16]

We will now consider two objections to our argument for the dilemma. The first objection is based on a mereological analogue of the argument for the rigidity of sets. Let $\leq$ indicate the relation of parthood. Assume that x and y share all their parts; that is, $\forall z(z \leq x \leftrightarrow z \leq y)$. Provided that parthood is reflexive and anti-symmetric, it follows that x and y are identical. Furthermore, since necessarily x shares all of its parts with itself, it follows by (Leibniz) that necessarily x and y share all of their parts. Yet these conclusions seem compatible with parthood being non-rigid! This calls into question our claim that analogous conclusions concerning sets support the rigidity of set membership.[17]

[14] Even for a contingentist, no further existential presuppositions are required, for reasons analogous to those that apply in the case of plurals (see Section 10.6).

[15] For sets, the former option is unattractive. As Boolos (1971, 229–30) reminds us, if ever there was an example of an analytic truth, then the extensionality of sets is one.

[16] These considerations pose a challenge to Fregean and neo-Fregean approaches to collections (or extensions, or *Wertverläufe*). On the one hand, such approaches adopt the principle of extensionality as a criterion of identity. On the other hand, they view a collection as somehow "derived from" its defining (Fregean) concept, which is plausibly regarded as non-rigid. So they are potentially on a collision course with the rigidity thesis. See Parsons 1977b for a discussion of Frege's concept of extension.

[17] Thanks to Jeremy Goodman for articulating this objection.

Our response is to deny that the two cases are analogous. The crux of our argument is the claim that any reason to accept the set-theoretic principle of extensionality is also a reason to accept the rigidity of set membership. By contrast, there is a reason to accept that sameness of parts suffices for identity that is *not* also a reason to accept the rigidity of parthood. Here is the intuitive idea. To make sense of contingent parthood, it is useful to think of objects as involving both matter and form.[18] For instance, a molecule that is part of you might not have been so because tracking you across possible worlds involves more than merely tracking your matter. On this hylomorphic conception, it is natural to take parthood to be sensitive to both matter and form. Mutual parthood would then ensures identity not only of matter but also of form—and hence also ensures the identity of the objects in question. But this explanation is perfectly compatible with objects involving form, not only matter, and thus being tracked non-trivially from world to world. In short, the principle that sameness of parts ensures identity admits of an explanation that does not support the rigidity of parthood.[19]

The second objection takes its departure from the well-known fact that Leibniz's law needs to be restricted. Assume that Nikita is the shortest spy. Of course, necessarily the shortest spy is the shortest spy. But it does not follow that necessarily Nikita is the shortest spy. It is often proposed that Leibniz's law be restricted to *rigid designators*—defined as terms that refer to the same object at every world at which they refer at all—thus excluding terms like 'the shortest spy'. Ordinarily, this restriction works well. But when reasoning about sets or other kinds of collection, the restriction threatens to undermine our dilemma between denying the principle of extensionality and accepting the rigidity principles.

To understand this threat, we need to distinguish between two completely different notions of rigidity. Until the previous paragraph, we have been concerned exclusively with a metaphysical notion of rigidity. Sets and other kinds of collection are said to be rigid if their membership is a matter of necessity, in the precise sense laid down by the kind of rigidity claims stated above. But as we have just seen, there is also the semantic notion of a rigid designator.

[18] Abstract objects would be a limiting case where the material contribution is nil.

[19] A better analogue of the set-theoretic principle of extensionality is the principle that sameness of *material* parts ensures identity. Now the analogy with our argument is restored. Any reason to accept the mentioned mereological principle is also a reason to accept the rigidity of material parthood. Of course, anyone attracted to non-rigid parts should respond to this observation by denying that sameness of material parts ensures identity.

The problem is that it can be hard to disentangle the two kinds of rigidity. Assume that a term t refers at w_1 to a collection comprising a and b, where a and b are all and only the Fs at w_1. Assume that t refers at w_2 to the singleton collection of a, where a is the one and only F at w_2. Is t a rigid designator? The question cannot be answered until we have been told how to track the relevant kind of collection from world to world. If the collections are tracked extensionally, we are considering different collections, with the result that t is not a rigid designator. But if the collections are tracked intensionally in terms of their membership criterion, we may well be considering one and the same object, namely the collection of Fs, in which case t is a rigid designator after all.

The threat to our dilemma is now apparent. To show that our use of Leibniz's law is permissible, we must first show that the terms in question are rigid designators. This involves showing that they refer to the same set across possible worlds. But this presupposes that we already know how to track sets across possible worlds! As we have seen, this is a matter of answering the question of metaphysical rigidity. Our argument therefore appears powerless to answer the question of metaphysical rigidity. The permissibility of its appeal to Leibniz's law presupposes that the question has already received an affirmative answer.

Fortunately, the threat can be avoided by reformulating the restriction on Leibniz's law. Say that a term is *purely referential* if its semantic contribution to linguistic contexts in which it occurs is exhausted by its referent, or, as Quine put it, if the term "is used purely to specify its object, for the rest of the sentence to say something about" (1960, 177). Instead of restricting Leibniz's law to rigid designators, we can restrict the law to purely referential terms. After all, the semantic contribution of such terms is exhausted by supplying their referents. Assume that $t_1 = t_2$ is a true identity involving two purely referential terms, and that the relevant argument place of a formula φ occurs in an transparent context. Then of course $\varphi(t_1) \leftrightarrow \varphi(t_2)$ is true as well, as this merely says of the common referent of t_1 and t_2 that it is φ if and only if it is φ. By restricting Leibniz's law to purely referential terms rather than to rigid designators, our problem dissipates. The only terms involved in our argument are variables. And a variable is purely referential because its semantic contribution is nothing but its value. Thus, our argument for metaphysical rigidity goes through.

There is a more general lesson here as well. The problem of disentangling metaphysical rigidity from semantic rigidity points to an unfortunate feature of the notion of a rigid designator: it runs together two kinds of considerations that are best kept apart. First, there is the semantic question of whether a term is purely referential. Then, there is the metaphysical question of how its referent is to be tracked from one possible world to another. It is true that every purely referential term is a rigid designator. But our discussion shows that we get a cleaner separation of the metaphysical and semantic questions by focusing on the notion of pure reference rather than rigid designation.[20] Thus, in what follows, the default notion of rigidity will once again be the metaphysical one.

Let us sum up. As observed, Leibniz's law entails the necessity of identity. We have examined whether an analogous argument can be given for the rigidity of sets. Given (Set-Ext), we found that Leibniz's law entails (Set-Cov), which states that two coextensive sets are necessarily coextensive. This falls short of the rigidity of sets, though it is a step in that direction. To establish the desired rigidity claim, we argued as follows. Any reason to accept (Set-Ext), we argued, is also a reason to accept the rigidity of sets. For (Set-Ext) holds because sets are constituted by their members, and this insight about the nature of sets also ensures that there is nothing other than the members in terms of which a set can be tracked.

10.5 An argument for plural rigidity

We will now extend the argument from the previous section to the case of pluralities. Previously, we started with Leibniz's law. Now, we propose to start with the principle that any coextensive pluralities are indiscernible. As before, we use $xx \approx yy$ to abbreviate the claim that xx and yy are coextensive (see Section 2.3). Thus, our proposed starting point is the principle:[21]

(Indisc) $$\Box\forall xx\forall yy\big(xx \approx yy \rightarrow (\varphi(xx) \leftrightarrow \varphi(yy))\big)$$

[20] See Stalnaker 1997 and essays 1–3 of Fine 2005b for some closely related considerations.

[21] This starting point allows us to remain neutral on the question of whether there is a notion of identity between pluralities. Clearly, (Indisc) requires no such notion. If such a notion is nevertheless available—denote it with the ordinary identity sign—then (Indisc) is merely the result of contracting into a single principle the law of extensionality for pluralities

(Ext) $$\Box\forall xx\forall yy(\forall u(u \prec xx \leftrightarrow u \prec yy) \leftrightarrow xx = yy)$$

and the plural analogue of Leibniz's law:

(Leibniz*) $$\Box\forall xx\forall yy(xx = yy \rightarrow (\varphi(xx) \leftrightarrow \varphi(yy)))$$

Two concerns arise. First, as we have seen, the ordinary singular version of Leibniz's law needs to be restricted. Analogous considerations apply in the plural case. Fortunately, it is easy to see that (Indisc) is suitably restricted. Since plural variables are purely referential just as much as singular ones are (only in a plural way), (Indisc) is entirely legitimate. In particular, it presupposes no prior answer to the question of the rigidity of pluralities and can thus safely be employed in an argument for this rigidity thesis.

Second, is (Indisc) acceptable from a contingentist point of view? To assess this issue, we need to be more explicit about what semantics we adopt. It is natural to use an extension of the plurality-based semantics on which '$x \prec xx$' is true at a world w relative to an assignment ss if and only if the objects assigned to 'xx' by ss exist at w and the object assigned to 'x' is one of them. This semantics makes it natural to adopt a negative free logic.[22] The inference rules for the quantifiers must then be formulated so as to make existential assumptions explicit; for instance, from $\forall x\, \varphi(x)$ we can infer $Et \rightarrow \varphi(t)$, and likewise for the plural universal quantifier. (We will shortly have more to say about the plural existence predicate.) Given these choices, it is easy to verify that (Indisc) remains a valid principle even in a contingentist setting.

We are ready to develop our argument for the rigidity of pluralities. The next step is to derive from (Indisc), an analogue of the necessity of identity. As in the set-theoretic case, we call this analogue *covariation*:

$$\text{(Cov)} \qquad \Box\forall xx\forall yy(xx \approx yy \rightarrow \Box(xx \approx yy))$$

It asserts that, as matter of necessity, two coextensive pluralities are necessarily coextensive. Given the Brouwerian axiom B, we can derive the necessity of non-coextensiveness as well.

We now come to the heart of the argument. Recall the case of sets, where Leibniz's law and (Set-Ext) entail (Set-Cov). While (Set-Cov) is formally compatible with the non-rigidity of sets, it is far more plausible with rigidity. In particular, any reason to accept (Set-Ext) is also a reason to accept the rigidity of sets. Precisely the same goes for pluralities. That is, (Indisc) entails (Cov). While (Cov) is formally compatible with the non-rigidity of pluralities, it is far more plausible with rigidity.[23] In particular, any reason to

[22] Notice that this enables us to drop the existential assumptions Ex and Eyy from (Rgd^{-}) on p. 205.

[23] If a notion of plural identity is available (see footnote 21), the case of pluralities is perfectly parallel to that of sets. For the plural analogue of Leibniz's law and (Ext) entail (Cov).

accept (Cov) is also a reason to accept the rigidity of plural membership. For (Cov) holds because a plurality is nothing over and above its members, and this insight about the nature of pluralities also ensures that there is nothing other than the members in terms of which a plurality can be tracked.

We therefore face a dilemma, as in the case of sets. Either we have to give up (Indisc), which implies (Cov), or else we have to accept plural rigidity. In other words, either we need to give up the ordinary principle of extensionality encapsulated in (Indisc), or else we have to accept the full transworld extensionality associated with plural rigidity. Just as in the case of sets, the former horn is deeply unattractive, as it comes close to just changing the subject. So we conclude that plural rigidity holds.

It is worth noting that (Cov) is logically weaker than (Indisc). The covariation principle gives us precisely what its name suggests, namely that two overlapping pluralities necessarily covary. By contrast, (Indisc) states that all properties of pluralities supervene on membership. To see that the latter principle goes beyond the former, consider a department whose statutes decree that all and only tenured faculty are to be members of the Hiring Committee and of the Graduate Admissions Committee.[24] Then the two committees necessarily covary in membership. Nevertheless, the two committees have different powers, namely to hire new faculty and to admit graduate students, respectively.

To be even more specific about the relation between (Indisc) and (Cov), one can observe that the former "factorizes" into the latter and the claim that the properties of a plurality *supervene* on what we may call its *modal membership profile*:

$$\text{(Sup)} \qquad \Box\forall xx\forall yy\big(\Box(xx \approx yy) \to (\varphi(xx) \leftrightarrow \varphi(yy))\big)$$

We see this as follows. Clearly, (Cov) and (Sup) entail (Indisc), which in turn entails each of the former two principles. Moreover, (Cov) and (Sup) are logically independent and encapsulate different philosophical ideas, namely covariation in membership and supervenience of properties on modal membership profile, respectively. A more comprehensive factorization of a cluster of ideas associated with the extensionality of pluralities will be offered in Section 10.10.

[24] We assume that the statutes are partially constitutive of the committees, in the sense that, were one to change the statutes, the original committees would cease to exist and be replaced by new ones. If necessary, this persistence condition for the committees can be written into the statutes.

We wish to end this section by briefly commenting on another argument for plural rigidity based on the covariation principle (Cov). This argument, due to Williamson (2013, 245–51), begins by unpacking the two biconditionals in (Cov) to obtain:

$$\Box\forall xx\forall yy(xx \preccurlyeq yy \land yy \preccurlyeq xx \;\rightarrow\; \Box xx \preccurlyeq yy \land \Box yy \preccurlyeq xx)$$

Williamson then makes the following, interesting observation: $xx \preccurlyeq yy$ gives no support to $yy \preccurlyeq xx$, and $yy \preccurlyeq xx$ gives no support to $\Box xx \preccurlyeq yy$. Thus, he contends, $yy \preccurlyeq xx$ should imply $\Box yy \preccurlyeq xx$, and $xx \preccurlyeq yy$ should imply $\Box xx \preccurlyeq yy$. Williamson therefore concludes that (Cov) "stands or falls" with the following, inferentially stronger principle:

$$\Box\forall xx\forall yy(xx \preccurlyeq yy \;\rightarrow\; \Box xx \preccurlyeq yy)$$

It is hard not to agree.

We are now only a small step away from (Rgd$^+$). All it takes to make this step is a principle asserting the existence of singleton pluralities, namely:

(Single$\prec$) $$\Box\forall x\exists xx\Box\forall y(y \prec xx \leftrightarrow x = y)$$

As Williamson shows, this natural principle, combined with (10.5), entails (Rgd$^+$).

We find this argument rather convincing. However, we believe the argument developed above is more explanatory. This argument, we recall, is an explication of the basic thought that a plurality is nothing over and above its members. Since a plurality is nothing over and above its members, the only basis for tracking it across different possible worlds is in terms of these members. We thus obtain an explanation of why pluralities are tracked rigidly, with the result that the rigidity principles are true.

10.6 Towards formal arguments for plural rigidity

We have developed an argument for plural rigidity. But, as it stands, the argument is not formally valid. Starting from (Indisc), our best formal result so far is (Cov), which states that coextensive pluralities are necessarily coextensive. Rigidity, our target, states that a plurality has the same members at any world at which it exists. We now investigate some ways to formally bridge this gap.

Since we are now aiming for formal rigor, the time has come to be entirely precise about the existential assumptions involved in our arguments. This requires a plural existence predicate that we can use to say of some things xx that they exist. As we have seen, the existence of a single object x can be expressed simply as $x = x$ (sometimes written Ex). But what about the plural existence predicate?

One may try to define plural existence distributively in terms of singular existence; that is, to define Exx as $\forall x(x \prec xx \rightarrow x = x)$. But this is unsuccessful. For a contingentist, the initial quantifier ranges only over objects that exist at the relevant world, which renders the quantified claim trivially true for any plurality xx whatsoever. Another natural but unsuccessful idea is to define Exx as $xx \approx xx$ in an attempt to imitate the definition of its singular analogue Ex as $x = x$. This too is easily seen to trivialize, for exactly the same reason as the previous attempt.

One safe option is simply to adopt a primitive collective plural existence predicate Exx, which we stipulate to be satisfied by some things at a world just in case all these things exist at the world. Another option is available as well, given the axiom that every plurality is non-empty: $\forall xx \exists y(y \prec xx)$. We can then define Exx as $\exists y(y \prec xx)$. To confirm that this works, suppose that xx don't exist at some world w. The semantics we are assuming (p. 217) ensures that Exx is false at w. Suppose instead that xx do exist at w. Then the axiom ensures that Exx is true at w. We adopt this option, rather than the first, as it is more economical.

Recall that we have assumed a negative free logic as our background for contingentist reasoning. As we already noted, this requires some restrictions on the axioms for the quantifiers.[25]

Next, we adopt the following "being constraint":

(BC) $$\Box \forall x \forall yy \, \Box(x \prec yy \rightarrow Ex)$$

That is, necessarily, if x is one of yy, then x exists. Clearly, this is valid on our semantics. Notice also that, given our definitions of the singular and plural existence predicates, (BC) entails:

(10.6) $$\Box(x \prec yy \rightarrow Ex \wedge Eyy)$$

[25] See Hughes and Cresswell 1996, Chapter 16, for a system of free logic in the context of modal logic.

Finally, we adopt an axiom stating that any plurality is ontologically dependent on each of its members:

$$\text{(Dep)} \qquad \Box\forall x\forall yy(x \prec yy \to \Box(Eyy \to Ex))$$

We contend that this axiom is plausible and observe that it is compatible with x being one of yy at some worlds but not at others.[26] The axiom is useful because it enables us to formulate (RGD$^+$) as we have done, rather than adopt the following, more cautious formulation with an additional existential presupposition Ex:

$$\Box\forall x\forall yy(x \prec yy \to \Box(Ex \wedge Eyy \to x \prec yy))$$

Moreover, by means of (Dep) and (BC), we are able to formulate (RGD$^-$) as we have done, rather than adopt the following, more guarded formulation:[27]

$$\Box\forall x\forall yy(x \not\prec yy \to \Box(Ex \wedge Eyy \to x \not\prec yy))$$

In the necessitist setting, of course, no existential presuppositions are needed.

With these preliminary questions clarified, we will now consider some formal arguments for plural rigidity. Each argument will first be developed from a necessitist point of view, as this is simpler. We will then use our plural existence predicate to reformulate the argument so as to work in a contingentist setting.

10.7 The argument from uniform adjunction

The first formal argument relies on an operation $+$ of adjoining one object to a plurality. It is reasonable to assume that, necessarily, to be one of these things and that thing is to be one of these things or to be identical with that thing. We call this principle uniform adjunction:

[26] As observed by Roberts (forthcoming), (Dep) does not entail the principle that a plurality is ontologically dependent on each of its *subpluralities*:

$$\Box\forall xx\forall yy(xx \preccurlyeq yy \to \Box(Eyy \to Exx))$$

While this principle is not needed in the present context, Roberts points out that it is useful elsewhere.

[27] To verify this claim, observe first that $\Box\psi$ and $\Box(\varphi \to \psi)$ are equivalent schemes of modal propositional logic when $\Box(\neg\psi \to \varphi)$. Now let φ and ψ be $Ex \wedge Eyy$ and $x \not\prec yy$, respectively. Then $\Box(\neg\psi \to \varphi)$ is just (10.6), which was established above.

$$(\text{UNIADJ}) \qquad \Box\forall xx\forall y\Box\forall x(x \prec xx + y \;\leftrightarrow\; x \prec xx \vee x = y)$$

where '$xx + y$' figures as a complex plural term. We now argue as follows. Assume $y \prec xx$. Then, by (UNIADJ), we have $xx \approx xx + y$. So by applying (COV) to xx and $xx + y$, we obtain:

$$(10.7) \qquad \Box(xx \approx xx + y)$$

Next, we observe that (UNIADJ) also entails

$$(10.8) \qquad \Box(y \prec xx + y)$$

From (10.7) and (10.8), some simple modal logic ensures our desired conclusion that $\Box(y \prec xx)$.

Gabriel Uzquiano has raised a legitimate concern about the argument.[28] Recall that (10.7) is a consequence of Leibniz's law and must be restricted to purely referential terms. Is it permissible to assume that '$xx + y$' is purely referential? This can be disputed. Fortunately, we can sidestep the problematic assumption by reformulating (UNIADJ). The above argument proceeds from the assumption that uniform adjunctions exist. We can express this assumption as the closure of the following plural comprehension principle:

$$(\text{UNIADJ}^*) \qquad \Box\forall xx\forall y\exists zz\Box\forall u(u \prec zz \;\leftrightarrow\; u \prec xx \vee u = y)$$

This principle is very weak. Indeed, it is something that even an opponent of plural rigidity should assent to, as the principle retains its plausibility even when the plural variables are allowed to range over groups.[29]

We now give our improved and official version of the argument from uniform adjunction. As before, assume $y \prec xx$. By (UNIADJ*), let zz be the uniform adjunction of y to xx. From this point onward the argument proceeds exactly as before, only with zz in the role previously played by $xx+y$. (The argument is spelled out in detail in Appendix 10.B.) Notice that this argument makes no appeal to (INDISC) other than its single instance, (COV). In this respect, the argument from uniform adjunction is like the argument from Section 10.5. This establishes (RGD$^+$).

[28] For a mereological analogue of this concern, see Uzquiano 2014, 42.

[29] Here, as elsewhere, it is interesting to inquire whether an analogous argument can be given in mereology, to the effect that parthood is rigid. We believe the answer is negative, but this isn't the place for a proper investigation.

What does it take to obtain (RGD$^-$)? It turns out that, in the system S5, a necessitated version of (RGD$^+$) entails (RGD$^-$). We prove this useful fact in Appendix 10.B and invoke it repeatedly in what follows.

Finally, let us adapt the argument to a contingentist setting. Then, uniform adjunction requires the following, more guarded formulation:

(UNIADJ-C) $\Box\forall xx\forall y\Box\big(Exx \wedge Ey \rightarrow \forall x(x \prec xx + y \leftrightarrow x \prec xx \vee x = y)\big)$

As before, this principle can be formulated in a way that sidesteps worries one may have about the complex term '$xx + y$', namely by means of the following comprehension principle:

(UNIADJ*-C) $\Box\forall xx\forall y\exists zz\Box\big(Exx \wedge Ey \rightarrow \forall u(u \prec yy \leftrightarrow u \prec xx \vee u = y)\big)$

Thankfully, it can be verified that (RGD$^+$) follows and that, using axiom B, so does (RGD$^-$). In sum, we find the argument from uniform adjunction convincing, both in a necessitist and in a contingentist setting. None of the arguments we will proceed to consider does any better, or so we will argue.

10.8 The argument from partial rigidification

Another formal argument is proposed in Williamson 2010 (699–700). The argument requires that, for any objects *xx*, there be some objects *yy* that are a partial rigidification of *xx* in the sense that $xx \approx yy$ but it is impossible for *yy* to lose any of their members. To be precise, we assume the following plural comprehension axiom:

(PARTRIG) $\Box\forall xx\exists yy(xx \approx yy \wedge \forall x(x \prec yy \rightarrow \Box x \prec yy))$

We can now argue as follows. Assume $y \prec xx$. Let *yy* be the partial rigidification of *xx*. Thus, we have $\Box(y \prec yy)$. By (COV), we also have $\Box(xx \approx yy)$. The latter two claims entail $\Box(y \prec xx)$, as desired. Using (RGD$^+$), we can, as before, obtain (RGD$^-$).

Let us now try to develop the argument from a contingentist point of view. As usual, the comprehension axiom needs to be formulated with greater care:

(PARTRIG-C) $\Box\forall xx\exists yy(xx \approx yy \wedge \forall x(x \prec yy \rightarrow \Box(Eyy \rightarrow x \prec yy)))$

Assume $y \prec xx$, and let yy be the partial rigidification of xx. Applying the same strategy as in the case of necessitism, we derive $\Box(Eyy \to z \prec yy)$. By (Cov), we also have $\Box(xx \approx yy)$. This establishes $\Box(Eyy \to z \prec xx)$. The final step to our desired target, namely $\Box(Exx \to z \prec xx)$, follows by using (Cov) to show that the existence of xx necessitates the existence of yy. (The argument is spelled out in detail in Appendix 10.B.)

How does Williamson's argument compare with the argument from uniform adjunction? Let us begin by addressing the question in the necessitist setting. Both arguments rely on a single instance of (Indisc), namely (Cov). The arguments differ only with respect to the plural comprehension axioms that they invoke. We are thus left with the task of comparing the two comprehension axioms, namely (UniAdj*) and (PartRig). A careful formal investigation reveals that, against the background of (Cov) and S5, the two axioms are equivalent with each other and also with the rigidity claim. An analogous claim holds in the contingentist setting.[30] Thus, at least in the context of S5, the choice between the argument from partial rigidification and uniform adjunction is merely a matter of taste and which heuristics one prefers.[31]

10.9 The argument from uniform traversability

The last formal argument for plural rigidity that we will consider is inspired by an observation made by Ian Rumfitt (2005, 117–18). Like above, we first give a simple version of the argument that is acceptable from a necessitist point of view, and then consider how the argument can be adapted to suit the contingentist.

A finite plurality can be *traversed*, in the sense that its members can be exhaustively listed. Assume for instance that aa is the plurality whose members are a, b, and c, and that these members have names $\bar{a}$, $\bar{b}$ and $\bar{c}$, respectively. Then aa can be traversed:

[30] Does this three-way equivalence mean that the arguments for rigidity are begging the question? To think so would be to conflate deductive validity with begging the question. It is particularly important to notice that any one member of the pairs of assumptions sufficient to prove rigidity (i.e. (Cov) plus a comprehension axiom) is compatible with the failure of rigidity.

[31] Linnebo 2016 claims that, in the contingentist setting, (PartRig-c) is less plausible than (UniAdj*-c). While this may be true when the axioms are considered in isolation, our present point is that the axioms are equivalent *modulo* the mentioned assumptions. In the context of weaker modal logics, the argument from uniform adjunction can be shown to require weaker modal assumptions than the argument from partial rigidification.

$$\forall x(x \prec aa \leftrightarrow x = \bar{a} \vee x = \bar{b} \vee x = \bar{c})$$

In fact, this traversability is uniform, in the sense that it holds by necessity:[32]

$$\Box\forall x(x \prec aa \leftrightarrow x = \bar{a} \vee x = \bar{b} \vee x = \bar{c})$$

What about infinite pluralities? A straightforward generalization is available if we allow infinitary disjunctions and assume that every object a has a name $\bar{a}$:[33]

(UniTrav) $$\Box\forall x(x \prec aa \leftrightarrow \bigvee_{a \prec aa} x = \bar{a})$$

We now argue as follows. Assume $y \prec aa$. Then we can find a such that $y = \bar{a}$. By the necessity of identity, we have $\Box(y = \bar{a})$. This entails the necessitation of $\bigvee_{a \prec aa} y = \bar{a}$. Some simple modal logic ensures our target $\Box(y \prec aa)$. See Appendix 10.B for details.

Let us now consider matters from a contingentist point of view. Equation (UniTrav) must be reformulated so as to make all existential presuppositions explicit. Given any objects aa, we can name all of its members and use this to state that, provided aa still exist, to be one of aa is just to be identical with one of the aforementioned members. In symbols:

(UniTrav-c) $$\Box(Eaa \rightarrow \forall x(x \prec aa \leftrightarrow \bigvee_{a \prec aa} x = \bar{a}))$$

As far as we can see, this modified principle is just as plausible, given contingentism, as the original principle is, given necessitism. It is therefore satisfying to be able to verify that the original argument for rigidity goes through much as before.

We find the argument from uniform traversability less explanatory than the previous two formal arguments for plural rigidity. One problem is the lack of "conceptual distance" between the premise and the conclusion. Uniform traversability is little more than an infinitary restatement of our

[32] In fact, as Jeremy Goodman observed, if a singleton plurality is uniformly traversed by its sole member, then Uniform Adjunction allows us to *prove* that any finite plurality is uniformly traversed by its members.

[33] Of course, the choice of names depends on the particular plurality aa. This means that 'aa', in the subscript to the disjunction sign, can only be understood as a plural constant, not a variable.

target claim that a plurality is fixed in its membership as we shift attention from one possible world to another (see Hewitt 2012b, 860–2, for a similar objection).[34] Moreover, this premise is needlessly strong—concerning both *what* it says and *how* it says it.

The clearest way to appreciate the strength of uniform traversability is by observing that it entails all the premises of the previous two arguments. One of these premises is (Cov). This is entailed by uniform traversability, as can be seen by a simple transitivity argument available in S5. Indeed, uniform traversability entails plural rigidity, which in turn entails (Cov), as observed at the end of Section 10.5. Moreover, it can be verified that uniform traversability entails the comprehension axioms employed in the arguments from uniform adjunction and partial rigidification.

Next, the infinitary resources employed by the argument from uniform traversability are very strong. To see this, it is useful to separate these resources from the modal claim that they are used to express. We can do so by considering what we may call *traversability*, which is (UniTrav) without the initial necessity operator. In fact, even this non-modal version of traversability has some strong consequences. As we explain in Appendix 10.A, this principle legitimizes what Paul Bernays (1935) calls "quasi-combinatorial" reasoning, that is, reasoning about infinite totalities as if they were finite.

10.10 Pluralities as extensionally definite

We have surveyed various formal and informal arguments for plural rigidity. It may be useful to summarize the most important principle that we have discussed and their inferential relations to one another.

For a necessitist, the picture looks as follows:

$$\begin{tikzcd}
(\textsc{Indisc}) \arrow[d] \arrow[dr, dotted] \arrow[ddr] & & (\textsc{UniTrav}) \arrow[dl] \arrow[d] \\
(\textsc{Sup}) & \textsc{Rigidity} \arrow[d, bend left] & \textsc{Traversability} \\
& (\textsc{Cov}) \arrow[u, dotted, bend left] &
\end{tikzcd}$$

[34] By contrast, as observed in footnote 30, the previous two argument both relied on *two* assumptions—(Cov) and a comprehension principle—each one of which is compatible with the failure of plural rigidity.

Solid arrows represent one-way implications. Dotted arrows represent non-deductive support, but can be transformed into implications by adding suitable comprehension axioms, as discussed in Sections 10.7 and 10.8. Formal theses are in parentheses, as usual. RIGIDITY abbreviates the conjunction of the rigidity claims (RGD$^+$) and (RGD$^-$), and, in a contingentist setting, also the dependence claim (DEP).

A contingentist can use the same diagram, with two exceptions. First, (UNITRAV) must be replaced with (UNITRAV-C), whose left diagonal implication then only yields the two rigidity claims, not (DEP).[35] Second, RIGIDITY entails only a restricted version of (COV) (see p. 235).

Where does this leave us? We raised the question of whether plural rigidity, just like the rigidity of identity, can be established from purely logical premises. We have given at least a conditional answer. If (INDISC) and the "suitable comprehension axioms" identified in Sections 10.7 and 10.8 count as purely logical, then so does plurality rigidity. But is the antecedent true? As it stands, we find the notion of pure logic insufficiently clear to give a definitive answer.

We find it more productive to recall the basic thought that has animated much of our discussion in this chapter, namely that a plurality is nothing over and above some circumscribed lot of objects. *Every plurality thus exhibits extensionality in its purest form. All we have are some objects, properly circumscribed.* Pluralities are, as we will put it, *extensionally definite*. This basic thought motivates some of our central principles, especially (INDISC) and (UNITRAV), which explicate different aspects of the extensional definiteness of pluralities. The picture that emerges is thus that plural rigidity figures at the heart of a tightly interwoven network of principles that all have to do with the extensional definiteness of pluralities. These principles mutually support each other. In particular, it would be difficult and unmotivated to excise plural rigidity from this network. Plural rigidity is an integral part of our analysis of pluralities as extensionally definite.

It is particularly interesting to examine the "second floor" of the above diagram. For the principles that figure on this floor provide a factorization of all the aspects of the extensional definiteness of pluralities that are represented in the diagram. To explain what we mean, let us begin by observing that each principle on this floor represents a simple and natural idea.

[35] If desired, one can tweak (UNITRAV-C) so as to ensure that (DEP) too follows, namely by adding the following as a third (and perfectly sensible) conjunct: $(Eaa \leftrightarrow \bigwedge_{a \prec aa} E\bar{a})$.

(a) The properties of any plurality supervene on its modal membership profile, as expressed by (Sup).
(b) A plurality has a rigid membership profile: it has the very same members at any possible world at which it exists.
(c) A plurality is traversable, thus ensuring the permissibility of quasi-combinatorial reasoning applied to the plurality.

Next, we observe that the three factors entail each of the aspects of the extensional definiteness of pluralities. It suffices to verify that the items on the top floor of the diagram are entailed by those on the second. As observed on p. 218, (Indisc) can be factorized into (Sup) and (Cov). And it can be verified that (UniTrav) (or its contingentist cousin) factorizes into Rigidity and Traversability.

What remains is to verify that the three factors are logically independent of one another. To see that property supervenience, (Sup), does not follow from the other two aspects of extensional definiteness, consider again the case of committees. Imagine an oligarchic department where three senior academics *a*, *b*, and *c* have written into the department statutes that they, and they alone, are to be on the Hiring Committee and the Graduate Admissions Committee. Both committees have a rigid membership profile and are clearly traversable. Yet the two committees are not subject to property supervenience as different powers of decision are vested in them.

Next, to show that a rigid membership profile does not follow from the other two aspects, consider the case of properties, understood as objects that are individuated by the necessary coextensionality of their defining concept or condition, and tracked across possible worlds in terms of this concept or condition. Thus understood, properties exemplify the second aspect of extensional definiteness: all the characteristics of any given property are shared by any necessarily coextensive property. However, a property can be subject to contingent membership (or, perhaps better, contingent application), including when its instances are traversable. And as we have seen, the traversability of a domain ensures the traversability of any property on this domain.

Finally, we observe that traversability is not a formal consequence of the other two aspects of extensional definiteness. The principles that explicate these other two aspects do not ensure the availability of the infinitary resources needed for traversability. As Bernays observed, traversability is based on an extrapolation from the finite into the infinite. How far are we willing to extrapolate? The first two aspects of the extensional definiteness of pluralities do not, by themselves, provide any answer to this question.

10.11 The status of plural comprehension

We wish to end the chapter with some remarks about the status of the plural comprehension axioms. Many philosophers regard such axioms as utterly trivial and insubstantial.[36] Provided that a condition is well defined and has at least one instance, *of course* the condition can be used to define a plurality of all and only its instances.

This view seems to us misguided. Indeed, we suspect the view is the result of an excessive focus on ontology at the expense of other important concerns. Because plural logic is thought to incur no ontological commitments over and above those already incurred by the singular quantifiers, the plural comprehension axioms are assumed to be ontologically innocent. And because of the intense focus on ontology, one therefore concludes that these axioms are trivial and insubstantial. One of the main upshots of this chapter is that, irrespective of the question of ontological commitment, plurals are governed by strong extensionality principles whose satisfaction is a non-trivial matter. Since the plural comprehension axioms make claims about possible assignments to the plural variables, which would accordingly be governed by the non-trivial extensionality principles, these axioms too should be regarded as non-trivial (see also Williamson 2016).

To elaborate, let us consider the three factors of the extensional definiteness of pluralities. First, it is not hard to see that *traversability* is a non-trivial assumption. To say that plural comprehension is permissible on a condition φ is to say that we may reason quasi-combinatorially about all the φs. A number of disputes in the foundations of mathematics testify to the non-triviality of this assumption.[37]

Second, *property supervenience* too is a non-trivial matter. Consider the following:

(10.9) The Hiring Committee met yesterday. They decided to make an offer to Sophia.

[36] In Section 2.5, we mentioned the claims that plural comprehension axioms as "genuine logical truths" found in Boolos 1985b (342) and Hossack 2000 (422).

[37] See Feferman 2005 for a survey of debates concerning the legitimacy of impredicative reasoning in mathematics.

Is it permissible to apply the rule of plural existential generalization to 'they'? The answer must be 'no'. Generalizing in this way would ascribe the property of making a job offer to the members of the committee *considered as a mere plurality*, where in reality the property can only be ascribed to *the committee as such*. It is only the committee, not the plurality of its members, that has the power to make job offers. Indeed, the property ascribed in the second sentence of (10.9) fails to supervene on the modal membership profile. As our earlier examples show, two committees can share the same modal membership profile while differing in the powers that are vested in them.[38]

The final factor of the extensionality of pluralities is their *rigid membership profile*. This rules out, for example, the existence of a plurality of all actual and merely possible objects, that is, a plurality aa such that $\Box\forall x(x \prec aa \leftrightarrow x = x)$. For such a plurality would vary in membership from world to world. (See Linnebo 2010.)

Summing up, we have argued that pluralities are rigid and that this is in fact just one of several extensionality principles that govern pluralities. These principles explicate our basic thought that pluralities are extensionally definite. Although the principles can be split into three independent factors (of which plural rigidity is one), they go naturally together as a package. Since the extensionality principles are non-trivial, so are the plural comprehension axioms, which assert the existence of pluralities governed by these principles. This non-triviality plays an essential role in our development of a critical plural logic in Chapter 12, which restricts the plural comprehension scheme.

[38] Our example from p. 218 of the Hiring Committee and the Graduate Admissions Committee will do.

Appendices

10.A Traversability and quasi-combinatorial reasoning

We claimed in Section 10.9 that the non-modal traversability principle (UniTrav) licences what Bernays (1935) calls "quasi-combinatorial" reasoning, that is, reasoning with infinite totalities as if they were finite. Let us now spell out and defend this claim.

First, we claim that (UniTrav) ensures the permissibility of impredicative plural separation axioms of the following form:

$$\exists x(\varphi(x) \wedge x \prec xx) \rightarrow \exists yy \forall u\big(u \prec yy \leftrightarrow \varphi(u) \wedge u \prec xx\big) \tag{10.10}$$

That is, given any xx that include a φ, there are some objects that are all of the φs among xx. We show this as follows. We begin by finding a bunch of names $\bar{a}$ that provide a traversal of xx. We would like another bunch of names $\bar{b}$ that provide a traversal of just those members of xx that satisfy φ. This is easily achieved by going through the former bunch, deleting every item that names a non-φ. The resulting sub-traversal yields a quantifier free—and thus fully predicative—definition of the desired sub-plurality of xx. The upshot is that traversability functions like an axiom of reducibility, in Russell and Whitehead's famous sense, that is, as an axiom stating that every higher-order entity has a predicative definition. The reducibility afforded by (UniTrav) becomes particularly far-reaching if there is an all-encompassing or universal plurality, as is standardly assumed. We would then obtain a justification for the full impredicative comprehension scheme.

Second, when we work in the context of an intuitionistic theory, traversability ensures that quantification restricted to any plurality behaves classically. Assume that a formula $\psi(x)$—which may have further free variables—is decidable on any given argument:

$$\forall x(\psi(x) \vee \neg\psi(x))$$

In effect, this means that the property defined by $\psi(x)$ behaves classically on any given argument. Then tranversability ensures that quantification restricted to xx behaves classically as well, in the precise sense that we have the following decidability property:

$$(\forall x \prec xx)\psi(x) \vee (\exists x \prec xx)\neg\psi(x)$$

To see this, observe that by traversability this restricted quantification reduces to a conjunction of its instances, each of which has been assumed to behave classically.

10.B Proofs

We now provide proof sketches of various arguments referred to in Chapter 10. We work within modal extensions of first-order logic (FOL) and of PFO. Sentential and quantificational reasoning will apply to expressions in the extended language. When dealing with arguments in a contingentist setting, we rely on a standard negative free logic. This means that the rules for singular and plural quantifiers are restricted so as to ensure that quantifiers range over existing objects or pluralities. For instance, from $\forall x\varphi(x)$ we can infer $Ey \rightarrow \varphi(y)$. Similarly, we can infer $\exists x\varphi(x)$ from $Ey \wedge \varphi(y)$, but not from $\varphi(y)$ alone. Moreover, there are rules guaranteeing that atomic predications are false if at least one of the terms involved is empty.

The necessity of identity and distinctness. Both arguments rely on Lebniz's law. We develop them in a necessitist setting.

(1)	$x = x$	FOL
(2)	$\Box(x = x)$	1, Necessitation
(3)	$\Box\forall x\Box(x = x)$	2, FOL, Necessitation
(4)	$\Box\forall x\forall y\big(x = y \rightarrow (\Box(x = x) \leftrightarrow \Box(x = y))\big)$	Leibniz
(5)	$\Box\forall x\forall y(x = y \rightarrow \Box(x = y))$	3, 4, K, FOL

The argument for the necessity of distinctness appeals to the Brouwerian axiom:

$$\varphi \rightarrow \Box\Diamond\varphi$$

The argument goes as follows:

(0)	$\Box\forall x\forall y(x = y \rightarrow \Box(x = y))$	as shown in the previous proof
(1)	$x = y \rightarrow \Box(x = y)$	0, T, FOL
(2)	$\neg\Box(x = y) \rightarrow x \neq y$	1, FOL
(3)	$\Box(\neg\Box(x = y) \rightarrow x \neq y)$	2, Necessitation
(4)	$\Box\neg\Box(x = y) \rightarrow \Box(x \neq y)$	3, K
(5)	$\Box\Diamond(x \neq y) \rightarrow \Box(x \neq y)$	4, Definition of $\Diamond$
(6)	$x \neq y \rightarrow \Box(x \neq y)$	5, B
(7)	$\Box\forall x\forall y(x \neq y \rightarrow \Box(x \neq y))$	6, FOL, Necessitation

The proofs of the necessity of identity and distinctness in the contingentist setting are simple adaptations of the proofs just given. ⊣

Necessary set covariation. We observed in Section 10.4 that the set-theoretic principle of extensionality (Set-Ext) implies that two coextensive sets are necessarily coextensive:

(Set-Cov) $\forall x \forall y(\forall u(u \in x \leftrightarrow u \in y) \rightarrow \Box \forall u(u \in x \leftrightarrow u \in y))$

The proof is similar to that of the necessity of identity:

(1)	$\forall x \forall y(\forall u(u \in x \leftrightarrow u \in y) \leftrightarrow x = y)$	(Set-Ext)
(2)	$\forall x \forall y\big(x = y \rightarrow (\Box \forall u(u \in x \leftrightarrow u \in x) \leftrightarrow \Box \forall u(u \in x \leftrightarrow u \in y))\big)$	Leibniz
(3)	$\forall u(u \in x \leftrightarrow u \in x)$	FOL
(4)	$\Box \forall u(u \in x \leftrightarrow u \in x)$	3, Necessitation
(5)	$\forall x \forall y(\forall u(u \in x \leftrightarrow u \in y) \rightarrow \Box \forall u(u \in x \leftrightarrow u \in y))$	1, 2, 4, FOL

Note that this reasoning is available to necessitists and contingentists alike. The necessitation of (Set-Cov) can be easily obtained from the necessitation of (Set-Ext). ⊣

The remaining proofs concern modal properties of pluralities. Thus we work in a modal extension of PFO. Because of the complexity of the reasoning involved, a fully formal presentation of some arguments will not be particularly illuminating. In those cases, we prefer to reason informally about the system rather than formally within the system. All the argumentative strategies employed are meant to be essentially available, *mutatis mutandis*, to both necessitists and contingentists.

Necessary plural covariation. First, we show that (INDISC) entails that two coextensive pluralities are necessarily coextensive:

(Cov) $\Box \forall xx \forall yy(xx \approx yy \rightarrow \Box(xx \approx yy))$

(1)	$\Box \forall xx \forall yy(xx \approx yy \rightarrow (\Box(xx \approx xx) \leftrightarrow \Box(xx \approx yy))$	(INDISC)
(2)	$xx \approx xx$	PFO
(3)	$\Box(xx \approx xx)$	2, Necessitation
(4)	$\Box \forall xx \Box(xx \approx xx)$	3, PFO, Necessitation
(5)	$\Box \forall xx \forall yy(xx \approx yy \rightarrow \Box(xx \approx yy))$	1, 3, PFO

There is a perfect analogy with the case of identity. In the presence of axiom B, (Indisc) also entails that two distinct pluralities are necessarily distinct:

$$\text{(Cov}^{-}\text{)} \qquad \Box\forall xx\forall yy(xx \not\approx yy \rightarrow \Box(xx \not\approx yy))$$

⊣

The Barcan formula for bounded quantifiers. In the context of S5, a restricted version of the Barcan formula is derivable:

$$\text{(BFR)} \qquad \forall xx\big(\forall x(x \prec xx \rightarrow \Box\varphi(x)) \rightarrow \Box\forall x(x \prec xx \rightarrow \varphi(x)\big)$$

Here is the contingentist argument for it (the necessitist argument can be easily read off from it). Let xx be an existing plurality. Assume $\forall x(x \prec xx \rightarrow \Box\, \varphi(x))$. We conjoin Exx and apply B to the resulting conjunction to obtain a claim that will be used shortly:

$$(^*) \qquad \Box\Diamond\big(Exx \wedge \forall x(x \prec xx \rightarrow \Box\varphi(x))\big)$$

Assume for reductio that the consequent of (BFR) is false, that is, $\Diamond\exists x(x \prec xx \wedge \neg\, \varphi(x))$. By the necessitation of (Rgd^{+}) and S4, we can add a conjunct to this formula so as to obtain:

$$\Diamond\exists x(x \prec xx \wedge \neg\varphi(x) \wedge \Box(Exx \rightarrow x \prec xx))$$

We now use (*) to add a conjunct, namely the result of removing the outermost $\Box$ from (*). This yields:

$$\Diamond\exists x\big(x \prec xx \wedge \neg\varphi(x) \wedge \Box(Exx \rightarrow x \prec xx) \wedge \Diamond(Exx \wedge \forall x(x \prec xx \rightarrow \Box\varphi(x))\big)$$

A bit of modal logic on the last two conjuncts of this formula yields:

$$\Diamond\exists x\big(x \prec xx \wedge \neg\varphi(x) \wedge \Diamond\Box\varphi(x)\big)$$

Reasoning in S5, we can derive the possible existence of some x which is both φ and $\neg\varphi$. We turn this possible contradiction into an actual contradiction. So we deny that there could be something among xx that is $\neg\varphi$ and conclude $\Box\forall x(x \prec xx \rightarrow \varphi(x))$, which completes our proof. Since the proof relies on no extra-logical assumption, (BFR) may be necessitated. ⊣

Plural rigidity entails a restricted version of (Cov). The proof appeals to (BFR). Let xx and yy be two pluralities and suppose that rigidity holds. Assume $xx \approx yy$. We want to show $\Box(xx \approx yy)$. By the assumption, if $x \prec xx$, then $x \prec yy$. It follows from (Rgd^+) that $\Box(Eyy \rightarrow x \prec yy)$; in the case of necessitism, the antecedent can be dropped. Thus, in the case of necessitism, we have:

$$\forall x(x \prec xx \rightarrow \Box(x \prec yy))$$

By (BFR), we obtain:

$$\Box\forall x(x \prec xx \rightarrow x \prec yy)$$

By symmetrical reasoning, we obtain:

$$\Box\forall x(x \prec yy \rightarrow x \prec xx)$$

The last two displayed formulas entail our target claim: $\Box(xx \approx yy)$. In the case of contingentism, parallel reasoning can be carried out, though contingent on the continued existence of the pluralities in question, thus yielding a restricted version of the target claim: $\Box(Exx \wedge Eyy \rightarrow xx \approx yy)$. ⊣

(Rgd^-) from (Rgd^+). In the presence of axiom B, a necessitated version of (Rgd^+) entails (Rgd^-). Assume

($\Box\text{Rgd}^+$) $$\Box\Box\forall x\forall yy(x \prec yy \rightarrow \Box(x \prec yy))$$

Suppose for reductio that (Rgd^-) is false, that is:

$$\Diamond\exists x\exists yy(x \not\prec yy \ \wedge \ \Diamond(x \prec yy))$$

The two displayed formulas entail:

$$\Diamond\exists x\exists yy(x \not\prec yy \ \wedge \ \Diamond(x \prec yy \ \wedge \ \Box(x \prec yy)))$$

By B, we can add a third conjunct:

$$\Diamond\exists x\exists yy(x \not\prec yy \ \wedge \ \Diamond(x \prec yy \ \wedge \ \Box(x \prec yy)) \ \wedge \ \Box\Diamond(x \not\prec yy))$$

A bit of modal reasoning yields:

$$\Diamond\exists x\exists yy\Diamond\Diamond(x \prec yy \ \wedge \ x \not\prec yy)$$

That is,

$$\neg\Box\forall x\forall yy\Box\Box\neg(x \prec yy \;\wedge\; x \nprec yy)$$

But this is inconsistent in K. From this reductio, we conclude that (Rgd$^-$) holds. Note that, since axiom 4 enables us to necessitate (Rgd$^+$), we have also shown that (Rgd$^-$) follows from (Rgd$^+$) in S5. ⊣

A central concern in Chapter 10 was to provide formal arguments in support of plural rigidity. We focused on three principles that can yield such arguments: uniform adjunction, partial rigidification, and uniform traversability.

(UniAdj*) $\Box\forall xx\forall z\exists yy\Box\forall u(u \prec yy \leftrightarrow u \prec xx \vee u = z)$

(PartRig) $\Box\forall xx\exists yy(xx \approx yy \wedge \forall x(x \prec yy \rightarrow \Box x \prec yy))$

(UniTrav) $\Box\forall x(x \prec xx \leftrightarrow \bigvee_{a \prec xx} x = \bar{a})$

We now prove the various claims made in the main text.

(UniAdj*) entails (Rgd). The proof relies on axiom B. We first derive (Rgd$^+$). Assume that $z \prec xx$. It follows from (UniAdj*) and axiom T that are yy such that:

$$\forall u(u \prec yy \leftrightarrow u \prec xx \vee u = z) \;\wedge\; \Box\forall u(u \prec yy \leftrightarrow u \prec xx \vee u = z)$$

An obvious consequence of this fact is that $\Box(z \prec yy)$. Since $z \prec xx$, so we also have that $xx \approx yy$. By (Cov), $\Box(xx \approx yy)$. But $\Box(z \prec yy)$ and $\Box(xx \approx yy)$ entail that $\Box(z \prec xx)$. Thus:

$$z \prec xx \rightarrow \Box(z \prec xx)$$

The variables are arbitrary and the reasoning relies only on a necessary non-logical premise, so we conclude:

$$\Box\forall z\forall xx(z \prec xx \rightarrow \Box(z \prec xx))$$

The other component of the rigidity claim, (Rgd$^-$), can be proved similarly by appealing to (Cov$^-$), which was proved to follow from (Indisc) and B. Alternatively, we can obtain (Rgd$^-$) from (Rgd$^+$) in S5, as shown above. ⊣

(PartRig) entails (Rgd). The axioms of S5 are used. Assume that $z \prec xx$. By (PartRig) and T, there are yy such that:

$$xx \approx yy \wedge \forall x(x \prec yy \rightarrow \Box x \prec yy)$$

So $\Box(z \prec yy)$. By (Cov), $\Box(xx \approx yy)$. Therefore, $\Box(z \prec xx)$. So we have shown that:

$$\forall z \forall xx(z \prec xx \rightarrow \Box(z \prec xx))$$

The modal status of (PartRig) guarantees that this holds of necessity. We can then derive (Rgd^-) from (Rgd^+) in S5. ⊣

So far we have two arguments for plural rigidity. The first relies on (UniAdj^*) and makes use of axiom B. The second relies on (PartRig) and can be carried out in S5. We claimed that, against the background of (Cov) and S5, the following three statements are equivalent: (UniAdj^*), (PartRig), and (Rgd). An analogous claim holds in the contingentist setting. We have already proved that each of the former statements entails the third. So it suffices to establish the two converse entailments.

(Rgd) entails (UniAdj^*). Let xx and z be arbitrary. By plural comprehension, there are yy such that:

$$\forall u(u \prec yy \leftrightarrow u \prec xx \vee u = z) \tag{†}$$

We want to show that this generalization holds by necessity. Let us first prove that the left-to-right direction of (†) holds by necessity. Suppose that $u \prec yy$. Then either $u \prec xx$ or $u = z$. If $u \prec xx$, then (Rgd) implies $\Box(u \prec xx)$ and thus $\Box(u \prec xx \vee u = z)$. If $u = z$, then $\Box(u = z)$ and thus $\Box(u \prec xx \vee u = z)$. So, in either case, $\Box(u \prec xx \vee u = z)$. Since u is arbitrary, we have established that:

$$\forall u(u \prec yy \rightarrow \Box(u \prec xx \vee u = z))$$

By (BFR), we obtain:

$$\Box \forall u(u \prec yy \rightarrow u \prec xx \vee u = z) \tag{*}$$

Now we prove the necessity of the other direction. We proceed by *reductio* and suppose the opposite, which can be written as:

$$\Diamond\exists u((u \prec xx \vee u = z) \wedge u \nprec yy)$$

A bit of logical manipulation yields the following disjunction:

$$\Diamond\exists u(u \prec xx \wedge u \nprec yy) \ \vee \ \Diamond\exists u(u = z \wedge u \nprec yy)$$

But each disjunct leads to contradiction. Consider the first disjunct. By the converse of (BFR), that is:

$$\Diamond\exists u(u \prec xx \wedge \varphi(u)) \ \rightarrow \ \exists u(u \prec xx \wedge \Diamond\varphi(u))$$

we obtain:

$$\exists u(u \prec xx \ \wedge \ \Diamond(u \nprec yy))$$

But given how yy were introduced, if $u \prec xx$, then $u \prec yy$. By (Rgd), $\Box(u \prec yy)$, which contradicts $\Diamond(u \nprec yy)$. Now consider the second disjunct. It entails that $\Diamond(z \nprec yy)$. But this contradicts $\Box(z \prec yy)$, which follows from $z \prec yy$ by (Rgd). We conclude from the reductio that:

(**) $$\Box\forall u(u \prec xx \vee u = z \rightarrow u \prec yy)$$

Our target claim, (UniAdj*), is an immediate consequence of the conjunction of (*) and (**). ⊣

(Rgd) entails (PartRig). This is straightforward and requires no special modal assumption.

(1)	$\Box\forall xx\forall x(x \prec xx \rightarrow \Box(x \prec xx))$	(Rgd)
(2)	$\Box\forall xx\big(xx \approx xx \ \wedge \ \forall x(x \prec xx \rightarrow \Box(x \prec xx))\big)$	1, PFO, K
(3)	$\Box\forall xx\exists yy\big(xx \approx yy \ \wedge \ \forall x(x \prec yy \rightarrow \Box(x \prec yy))\big)$	2, PFO, K ⊣

The last formal argument for plural rigidity considered in Chapter 10 relies on the principle of uniform traversability:

(UniTrav) $$\Box\forall x(x \prec xx \leftrightarrow \bigvee_{a \prec xx} x = \bar{a})$$

The formulation of this principle (and of the resulting argument) requires an infinitary extension of PFO.

(UNITRAV) entails each instance of (RGD). Let xx be any plurality. By (UNITRAV), we can find a traversal:

$$(*) \qquad \Box\forall x(x \prec xx \leftrightarrow \bigvee_{a \prec xx} x = \bar{a})$$

By T and Universal Instantiation, we obtain $z \prec xx \leftrightarrow \bigvee_{a \prec xx} z = \bar{a}$. Now, the necessity of identity entails $\bigvee_{a \prec xx} z = \bar{a} \rightarrow \bigvee_{a \prec xx} \Box(z = \bar{a})$. And basic modal logic ensures $\bigvee_{a \prec xx} \Box(z = \bar{a}) \rightarrow \Box \bigvee_{a \prec xx} z = \bar{a}$. Combining the three preceding formulas, we obtain:

$$z \prec xx \rightarrow \Box \bigvee_{a \prec xx} z = \bar{a}$$

From this and (*) we derive:

$$z \prec xx \rightarrow \Box(z \prec xx)$$

Since z is arbitrary, we can universally generalize to establish our desired conclusion:[39]

$$\forall x(x \prec xx \rightarrow \Box(x \prec xx))$$

The modal profile of (*) ensures that this conclusion holds of necessity. ⊣

[39] Can we proceed to universally generalize on 'xx' as well? In fact, this move is unavailable because (UNITRAV) is a axiom *scheme*, which for each xx states that there is a traversal, but which provides no uniform way of specifying such a traversal.

11
Absolute Generality and Singularization

11.1 Absolute generality

Is it possible to assert something of absolutely everything there is? It certainly *seems* so; consider for instance the following assertions:

(11.1) Everything is physical.

(11.2) The empty set has no elements.

The truth of these assertions, it seems, rules out the existence of absolutely any ghost or element of the empty set. Any ghost or element of the empty set, no matter how remote or unfamiliar, is incompatible with what has been asserted. Let *absolute generality* be the view that it is possible to quantify over absolutely everything there is.

Plausible though it appears, absolute generality faces some challenges. We begin by laying out what we take to be the most interesting and powerful one. We do this in some detail, as it will be important to understand exactly which options we have for responding. We then argue, following Williamson (2003), that the rejection of absolute generality faces serious expressibility problems. Next, we examine Williamson's defense of absolute generality, which gives up the thesis that there are universal devices of singularization. We show how this proposal leads to an ascent to languages of ever higher orders and argue that the resulting outlook suffers from expressibility problems that are very similar to those that Williamson sought to avoid.[1] Motivated by this, we explore an alternative approach to the challenge, which allows a form of absolute generality but denies that the associated domain is extensionally definite (that is, properly circumscribed), and on this basis denies that the domain is an all-encompassing plurality of objects.

Absolute generality has emerged as one of the central themes in the book, figuring as an essential premise in several arguments in the preceding

[1] Here we draw on arguments from Linnebo 2006 and Linnebo and Rayo 2012.

The Many and the One: A Philosophical Study of Plural Logic. Salvatore Florio and Øystein Linnebo,
Oxford University Press.
DOI: 10.1093/oso/9780198791522.003.0011

chapters. For instance, absolute generality was crucial to the more promising attempts to refute regimentation singularism. Moreover, it was invoked in the comparisons between alternative ways to talk about the many; for example, it emerged as an obstacle to the elimination of pluralities in favor of sets and to the mereological analysis of plurals. Furthermore, absolute generality was proposed as a constraint in the choice of a model theory for plural logic, a constraint that would rule out any form of semantic singularism cashed out in first-order terms. In order to reach a verdict on all these arguments, we need to resolve the question of absolute generality.

11.2 A challenge to absolute generality

Let us now formulate a challenge to absolute generality. The challenge is based on the plural version of Cantor's theorem that we presented in Section 3.5.

> Plural Cantor
>
> For any plurality *xx* with two or more members, the subpluralities of *xx* are strictly more numerous than the members of *xx*.

In particular, consider the *universal plurality*, that is, the plurality of every object there is. On the uncontroversial assumption that there are two or more objects, the corresponding instance of Plural Cantor can be formulated as follows:

> Plural Profusion
>
> There are more pluralities than objects.

This means that there can be no injective mapping of pluralities into objects.

The problem is that this technical result clashes with a wide range of views in metaphysics, philosophy of mathematics, and semantics. To get the problem in focus, consider first all the stars in the universe, *ss*. There appear to be many examples of injective mappings from subpluralities of *ss* into objects. Consider the operator 'the set of'. According to its typical usage, this operator defines an injective mapping from subpluralities of *ss* to objects. If *xx* and *yy* are distinct, so are their object-level correlates, that is, the set of *xx* and the set of *yy*. (For a detailed discussion see Linnebo 2010.) Metaphysics provides other important examples. An interesting case is that of propositions or facts (see McGee and Rayo 2000). It is widely assumed that if *xx* and *yy* are distinct, then the propositions or facts expressed by '*xx* exist' and '*yy* exist' are also distinct. Thus, the functional expressions 'the proposition that . . . exist' and 'the fact that . . . exist' define injective mappings from subpluralities of *ss* to objects. However, if these functional expressions were applicable not just to subpluralities of *ss* but to all pluralities, we would run into trouble. Suppose that, for any things *xx* whatsoever, there is the set of *xx* (or the proposition that *xx* exist, or the corresponding fact). Then we would have an injective mapping *from pluralities to objects*—in violation of Plural Profusion.

Let a *singularization* be an injective mapping from the subpluralities of some objects *xx* into objects.[2] The plural version of Cantor's theorem constrains what singularizations are possible. For the theorem says that, provided that *xx* have two or more members, there can be no injective mapping of the subpluralities of *xx into these very objects*. That is, if there is a singularization of all the subpluralities of *xx*, then the values of the subpluralities under this singularization cannot all be among *xx* but must "overflow" this plurality. Of course, in some of the examples mentioned above, this is precisely what one would expect. No one expected a set of stars to be itself a star, and likewise for facts, properties, and propositions concerned with stars. The problem arises when this overflowing is impossible, as in the case of the universal plurality. This implies that there can be no singularization of its subpluralities. Any such singularization would have to overflow the universal plurality, which by its assumed universality is impossible.

This is a puzzling result. In our first example, it seemed completely incidental that we considered the subpluralities of all the stars, as opposed to the subpluralities of some other lot of objects. It is not as if stars are more amenable to figuring as elements of sets than any other objects, or, for that matter, to be involved in propositions, facts, or properties. One might therefore have thought that these are *universal* singularizations, in the sense that they are available for any plurality whatsoever.[3] Plural Profusion seems to show, in one fell swoop, that none of the mentioned singularizations, nor any other, can be universal: there simply aren't enough objects to enable a singularization of absolutely all pluralities.

[2] As before, this talk of mappings can either be taken as primitive or be understood as shorthand for claims that officially talk merely about pluralities of ordered pairs (see Appendix 3.A). For ease of communication, we will mostly indulge in talk about mappings, which could always be translated into talk about pluralities of pairs.

[3] This is also a consequence of the liberal view of definitions canvassed in Section 4.4 and developed in more detail below.

A common reaction in the literature has been to take this result at face value as a surprising limit on what singularizations there can be.[4] This reaction is not without problems, however. Singularization seems to play an important role in natural language and in a wide range of theoretical contexts, from mathematics to semantics. What are we to say about all these apparent singularizations? Since it is not an option to reject singularizations altogether, the most promising response is to find a way to restrict their availability. We can allow these singularizations to be undefined on certain pluralities or lift the requirement that the associated mappings always be injective. However, this "compromising" response faces a threat of arbitrariness. Restricting the scope of a device of singularization raises the question of whether the restriction is adequately motivated.

We have ended up in an awkward position. Plural Cantor seems to show that there can be no universal singularization, and as we have just seen, this threatens to introduce some arbitrary and unmotivated restriction on what singularizations there can be. Let us therefore reexamine the argument. Might there be some way to reconcile Plural Cantor with the availability of universal singularization? Given Plural Cantor, we know that any singularization of the subpluralities of some things would have to overflow these things. So a reconciliation would have to find a way to permit this kind of overflow without exception. There have been two attempts to permit this.

The better known strategy is *generality relativism*, which denies that absolute generality is possible. This view entails that no plurality is universal in an absolute sense. The most we can ever have is a relative kind of universality, which can always be surpassed. Although a plurality xx may be universal with respect to our current interpretation of the quantifiers—that is, $\forall x(x \prec xx)$—it is possible to find an extended interpretation with respect to which xx is no longer universal—that is, $\exists^+ x \neg(x \prec xx)$ (where the $\exists^+$ indicates that the quantifier is taken in this extended sense). This yields an operation that extends any given interpretation I to a strictly more inclusive interpretation I^+. On the resulting view, any plurality—including one that is universal with respect to our current interpretation of the quantifiers—can be surpassed once we adopt a more inclusive interpretation of the quantifiers. Clearly, this view makes the world safe for singularization. Since no

[4] See, e.g., McGee and Rayo 2000, Rayo 2002, and Uzquiano 2015a. Note also that this attitude towards singularization is implicit in much of the philosophical literature on plural logic. An analogous point is true with respect to the parallel case of nominalization and higher-order logic.

plurality is universal in an absolute sense, there is no obstacle to unrestricted singularization. Provided that singularization leads to an expansion of the interpretation of the quantifiers, we can safely accept its effect of always surpassing any plurality with which we begin.

11.3 A trilemma

There is another strategy for blocking the argument against universal singularizations, namely to restrict the axiom scheme of plural comprehension. This strategy has no truck with generality relativism and accepts that an absolute interpretation of the quantifiers is possible. Nor does the strategy have any quarrel with Plural Cantor: it is perfectly true, for any plurality *xx* with two or more members, that the subpluralities of *xx* are strictly more numerous than the members of *xx*. Rather, unlike the generality relativist, who seeks to retain traditional plural logic, the strategy in question challenges our naive assumptions concerning what pluralities there are. After all, in the argument above, trouble arose only when we assumed that there is a universal plurality, which enabled us to derive the problematic instance of Plural Cantor, namely Plural Profusion. (Of course, we would get the same effect from any other plurality that is too large to allow of singularization, as its correspondingly small complement cannot accommodate the overflow that would result.)

Needless to say, the big challenge for this strategy is to explain why there are no pluralities that are so large that they cannot be singularized. The existence of such pluralities is underwritten by the unrestricted plural comprehension scheme of traditional plural logic. Any rejection of the currently accepted version of plural logic will of course have to be well motivated. Attempts to provide such a motivation have in fact been made, targeting the unrestricted plural comprehension scheme in particular. A promising idea derives from the thought that domains of quantifications might be *extensionally indefinite*, or not properly circumscribed, while every plurality is extensionally definite. The idea is nicely summarized in the following passage by Stephen Yablo:

> The condition $\phi(u)$ that (I say) fails to define a plurality can be a perfectly determinate one; for any object x, it is a determinate question whether x satisfies $\phi(u)$ or not. How then can it fail to be a determinate matter what are *all* the things that satisfy $\phi(u)$? I see only one answer to this. Determinacy of the ϕ's follows from

(i) determinacy of $\phi(u)$ in connection with particular candidates,
(ii) determinacy of the pool of candidates.

If the difficulty is not with (i), it must be with (ii). (Yablo 2006: 151–2; some notation and terminology has been modified)

Perhaps the price of absolute generality is that the range of our quantifiers becomes extensionally indefinite (or "indeterminate", as Yablo might put it). Since pluralities are extensionally definite, however, this would give us a reason to restrict plural comprehension so as to reject the universal plurality. Of course, the main challenge for this approach will be to articulate the notion of extensional definiteness and show that it has the right properties. This task was begun in the previous chapter and will be completed in next.

Let us take stock. Assume that absolute generality is possible and there is a plurality that is universal in this absolute sense. Then Plural Profusion entails that there cannot be a universal singularization. For if there were, such a singularization would yield an injective mapping from subpluralities of the universal plurality to objects, contradicting Plural Profusion. Moreover, since this argument can be given with the quantifiers interpreted absolutely, which is assumed to be possible, it is not an option to object that the argument equivocates by expanding the interpretation of the quantifiers somewhere along the way. So we have a trilemma. We must accept one of the following three horns.

First horn

Universal singularizations are impossible.

Second horn

It is impossible to quantify over absolutely everything.

Third horn

There is no plurality that is universal or all-encompassing.

The trilemma confronts us with a difficult theoretical choice.[5] As we have seen, there are several examples of devices of singularization in natural

[5] Might a fourth option be possible, namely to challenge Plural Cantor? Since this is a mathematical theorem, it can be no more controversial than the assumptions on which it rests. The only point of attack seems to be the impredicative plural comprehension axiom that is involved in some versions of the theorem. We don't find this challenge at all promising, for two reasons. First, some versions of the theorem require only predicative comprehension, as we saw in Appendix 3.A. Second, even for the versions that require impredicative comprehension the requisite impredicative plural comprehension was defended in Appendix 10.A.

language and some of them appear to be fully general. Next, there is a wealth of examples of assertions that seem to be about absolutely everything. Finally, the existence of a universal plurality is underpinned by the principle of plural comprehension enshrined in the traditional version of plural logic and thus appears to stand on solid ground. How should the trilemma be resolved? In what follows, we shall assess each of its three horns.

11.4 Relativism and inexpressibility

It turns out that the rejection of absolute generality is fraught with difficulties. Let us mention three problems. One is simply that absolute generality very much appears to be possible, for instance when we truly assert that the empty set has absolutely no elements. It would take a very good reason to go against such a robust appearance. A second problem is that absolute generality is needed in order to express various general views that we find interesting, such as the physicalist claim that absolutely everything is physical. To disallow the expression of these views would be to disallow a lot of potentially fruitful theorizing. As Williamson remarks, "[i]f the unexamined life is not worth living, the credentials of a life without absolutely general thought are shaky" (2003, 452). We take this to be a very serious complaint.

The most intriguing argument against generality relativism, however, is that the view cannot coherently be expressed. David Lewis states the point with characteristic verve:

> Maybe the singularist replies that some mystical censor stops us from quantifying over absolutely everything without restriction. Lo, he violates his own stricture in the very act of proclaiming it! (Lewis 1991, 68)

Some unpacking may help. Consider the claim that my current language does not quantify over absolutely everything. This entails that there is something over which my quantifiers do not range. But this is incoherent, as I am now using a quantifier to assert the existence of something not in the range of my quantifiers.

A relativist might attempt to do better by expressing her view as a claim about interpretations, namely that for every interpretation of our quantifiers, there is a more expansive interpretation. We use subscripts to indicate the interpretation given to the quantifiers. Let $I \subset J$ abbreviate $\exists_J x \forall_I y(x \neq y)$. One may then attempt to express relativism as follows:

$$\forall I\, \exists J(I \subset J) \tag{11.3}$$

Promising though it may be, the attempt fails, as shown by the following dilemma. Assume first that the quantifiers '$\forall I$' and '$\exists J$' in (11.3) range over absolutely all interpretations. Then (11.3) expresses what it is meant to express. But in so expressing it, one is violating the view expressed. For just as there are arguments that it is impossible to quantify over all ordinal numbers or all sets (see Florio 2014b, Section 3.1), there are analogous arguments concerning quantification over interpretations. Alternatively, assume that the quantifiers in (11.3) do not range over absolutely all interpretations.[6] Thus understood, (11.3) is compatible with the view it is meant to express. The problem is now that (11.3) fails to express the view properly. All that is expressed is that every interpretation *in some limited range of interpretations* can be extended. But this is compatible with there being a maximal interpretation outside of this limited range.

The standard response by generality relativists, advocated for instance in Glanzberg 2004 and Parsons 2006, is to invoke *schematic generality*. This idea traces back to Russell's use of free variables to achieve a form of generality that goes beyond that afforded by the quantifiers.i

> For our purposes [the distinction between 'all' and 'any'] has a different utility, which is very great. In the case of such variables as propositions or properties, 'any value' is legitimate, though 'all values' is not. Thus we may say: '*p* is true or false, where *p* is any proposition', though we can not say 'all propositions are true or false'. (Russell 1908, 229–30)

In effect, we use free variables to achieve a version of absolutely general universal quantification. Consider an operation which, when applied to any interpretation I yields an extended interpretation I^+; an example is the operation described in Section 11.2. Using free variables, relativism can now be expressed schematically as follows:

$$I \subset I^+ \tag{11.4}$$

The use of schematic generality is severely limited, however. Schematic statements cannot be negated and cannot be freely combined in other truth-functional ways. Consider for instance the negation of (11.4) and

[6] See, for instance, the relativist argument developed (as a foil) in Williamson 2003, Section IV.

read this schematically. The formula would express a universally generalized negation, not the desired negated universal generalization. More generally, schematic generality enables us to express absolutely general Π_1-sentences, but not Σ_1-sentences or beyond.[7] We find this expressive limitation hard to accept. Anything that can be expressed can also be denied.

Can we do better by exploiting alternative expressive resources? An interesting option is to formalize reasoning about expansions of quantifier interpretations by means of modalities.[8] So (11.3) receives a modal reading:

(11.5) Necessarily, for any interpretation *I*, there could be an extended one *J*

or in symbols:

(11.6) $\Box\forall I\,\Diamond\exists J\,(I \subset J)$

How should the modal operators be interpreted? The ordinary metaphysical interpretation is problematic. For the existence of the relevant objects, such as pure sets, is often assumed to be metaphysically necessary, which rules out any variation of the domain of such objects across metaphysical possibilities. Some writers favor an interpretational understanding of the modality, where the modal operators enable us to theorize about the result of certain changes to the interpretation of the language.[9]

Suppose this understanding of the modality can be made out. What would have been achieved? A desire for greater expressive adequacy led to the adoption of resources that allow us to retrieve, or at least to simulate, full absolute generality. For the strings '$\Box\forall$' and '$\Diamond\exists$' can now be used as devices of generalization: not just over everything in the range of the quantifiers as *currently* interpreted, but over everything in their range on *any possible* interpretation. Indeed, the "mirroring theorem" of Linnebo 2010 shows that, under plausible assumptions, the "modalized quantifiers" '$\Box\forall$' and '$\Diamond\exists$' behave precisely like ordinary quantifiers as far as logic is concerned. So these can be seen as ways to recover a form of absolute generality from within a theoretical standpoint that shares many of the motivations of relativism.

Where does this leave us? We set out to develop a form of relativism. We ended up defending a form of absolute generality—albeit with an uncon-

[7] We can, however, use operators or function symbols to make *some* existence claims even in the scope of schematic generalization. An example is (11.4) where '+' represents an operation we can apply to any given interpretation *I* so as to form an extended interpretation I^+.

[8] See Fine 2006, Linnebo 2010, and Studd 2013.

[9] See Fine 2006; Linnebo 2018, Sections 3.5–3.6; Studd 2019, Sections 4.4 and 6.1.

ventional understanding of absolute generality. This unconventional feature emerges particularly clearly in connection with the following modalized analogue of the ordinary plural comprehension scheme:

$$\Diamond\exists xx\Box\forall y(y \prec xx \leftrightarrow \varphi(y))$$

Recall from Chapter 10 that every plurality is rigid: it has the same members at every world at which it exists. This entails that the above scheme is invalid. For example, let $\varphi(y)$ be the condition '$y = y$'. Since the domain can vary from possible world to possible world, so can the extension of this condition. By contrast, a plurality cannot vary in membership. It is therefore impossible for there to be a plurality that is necessarily coextensive with this condition. The upshot is that, when the strings '$\Box\forall$' and '$\Diamond\exists$' are used to recover absolute generality, the plural comprehension scheme, couched in terms of this form of generality, needs to be restricted. In short, an attempt to defend the first horn of our trilemma has morphed into a view that is more usefully regarded as a defense of the third horn. Specifically, we have used pluralities, which are tracked rigidly across possible worlds, to explicate the notion of an extensionally definite collection, or Yablo's corresponding notion of a "determinate pool of objects".

For the purposes of this book, we prefer a more direct approach. Instead of adding modal operators to shore up a version of the first horn, we would like to develop the third horn directly—which we do in the next chapter.

11.5 Traditional absolutism and ascent

In the remainder of this chapter, we will explore the second horn of our trilemma. This horn concedes that there cannot be any universal singularizations, while retaining absolute generality over an extensionally definite domain, represented by a universal plurality. Throughout this discussion, traditional plural logic will therefore be assumed. We call the resulting view *traditional generality absolutism*.

One of the main challenges confronting this view is to develop a model theory for a language whose quantifiers are interpreted absolutely. The usual set-based model theory is obviously unavailable, since the domain now consists of absolutely everything there is and there is no universal set according to standard set theory (see Section 7.7). How, then, should advocates of absolute generality represent the domain of their absolute quantifiers and the semantic values of the predicates defined on this all-inclusive domain?

An answer that has recently gained a lot of support is that the model theory for a first-order language with absolute generality can and must be given in a plural or higher-order metalanguage.[10] In this metalanguage, we let domains be pluralities or concepts. The domain of a language with absolute generality will then correspond to the universal plurality or a universal concept. This way to talk and reason about domains requires no singularization whatsoever. Although we talk informally about "the domain", using a singular definite description, we officially have in mind the many objects, or the many instances of a concept, over which the quantifiers range. A similar strategy allows us to ascribe semantic values to predicates. Although we informally talk about "the semantic value" of a predicate, officially there are many objects or a concept representing the predicate's semantic contribution. In short, in order to develop the model theory for a first-order language with absolute generality, we must ascend to a language with plural or second-order resources.

Our discussion in Section 7.5 showed another instance of this phenomenon. A model theory for PFO+ with absolutely unrestricted quantification can only be given in a language with another layer of quantification, such as superplural quantification or quantification over plural concepts. In fact, this ascent phenomenon can be shown to continue further, as we will now explain in detail. We will focus on the plural hierarchy, although it is not difficult to adapt our discussion to the corresponding conceptual hierarchy. For the two hierarchies have a common type-theoretic structure. So to emphasize the parallel between them, we will often speak of a *type-theoretic* hierarchy. Recall our terminology when the types receive a plural interpretation: a language of order 1 is just a regular first-order language, while order 2 adds plural quantification, order 3 adds superplural quantification, and so on. Thus, a language of order $n + 1$ quantifies over what we call pluralities of level n.

A more detailed argument for the ascent can now be set out as follows.

Premise 1

Traditional plural logic is valid.

Premise 2

Absolute generality is possible at every order of the hierarchy; that is, for every order, it is possible to quantify over absolutely all entities at that order.

[10] See, e.g., Rayo and Uzquiano 1999, Rayo and Williamson 2003, and Williamson 2003.

To formulate the third and final premise, let a *generalized semantics* be a theory of all possible interpretations that a language might take, without any artificial restrictions on the domains, interpretations, and variable assignments. A generalized semantics is thus an instance of model theory, in our liberal sense of that term. The premise can now be stated as follows.

PREMISE 3 (SEMANTIC OPTIMISM)

Given any legitimate language, it should be possible to develop a generalized semantics for that language.

Finally, we have the following theorem.

ASCENT THEOREM (BASIC FORM)

Assume traditional plural logic and the possibility of absolute generality. Then a generalized semantics for a first-order language cannot be given in another first-order language but can be given in a language with plural quantification.

The result of the three premises and the theorem is that we are pushed from a first-order to a plural language.

The question now arises: what about the semantics of a language with absolutely general plural quantification? It turns out that the considerations that require the initial ascent from a language of order 1 to a language of order 2 require further ascents as well. For we have the following:

ASCENT THEOREM (ARBITRARY FINITE FORM)

Assume traditional plural logic and the possibility of absolute generality at every finite order $n \geq 1$. Then a generalized semantics for a language of order n cannot be given in another language of order n but can be given in a language of order $n + 1$.

So at every finite order, the desire for a generalized semantics pushes us one step up. This results in an ascent up through all the finite orders.[11]

[11] More fine-grained results are possible as well. The Ascent Theorem applies to languages that are sometimes called *full* (or *plenary*) in the sense that they contain predicates whose arguments can be variables of order n, where n is the order of the language. If a language of order n is not full, the formulation of a generalized semantics for it requires only that we ascend to a *full* language of order n. This is why, for example, the plurality-based model theory for PFO was carried out in PFO+. For a summary of these results and references to the literature, see Florio 2014b, Section 4.1.

In fact, as shown by Linnebo and Rayo (2012), an additional principle, broadly in the spirit of those behind the theorems, extends the hierarchy of higher-order languages into the transfinite. The additional principle states, roughly, that for any collection of languages in the hiearchy, we can form a "union" language that encompasses all the languages in the collection.[12] Linnebo and Rayo (2012) prove a generalization of the ascent theorem into the transfinite.

> Ascent theorem (transfinite form)
>
> Assume traditional plural logic and the possibility of absolute generality at every order. Then we cannot develop a generalized semantics for a language of order α in another language of order α. But for every successor ordinal α, we can develop a generalized semantics for a language of order α in a language of order $\alpha + 1$.[13]

In fact, there is reason to think that Tarski knew all of this (and more):

> [T]he setting up of a correct definition of truth for languages of infinite order would in principle be possible provided we had at our disposal in the metalanguage expressions of higher order than all variables of the language investigated. (Tarski 1935, 72)

The end result is that defenders of the new orthodoxy, seeking to secure the possibility of absolute generality while holding on to semantic optimism, are pushed by the ascent theorems higher and higher up through the orders of the hierarchy.

The only way to stop this ascent would be to give up on semantic optimism, which insists on a generalized semantics. Why insist on this? As remarked, a generalized semantics is just a higher-order version of ordinary set-based model theory. It is therefore natural to think that a generalized semantics is required in order to give an appropriate definition of logical

[12] To avoid inconsistency, we must restrict the collections to which the principle applies. If languages are indexed by ordinals, a plausible restriction is to any bounded collection of languages: for a limit ordinal λ, if one is prepared to countenance languages of order β for every $\beta < \lambda$, then one should also countenance a language of order λ. Without this restriction, we would be able to prove an inconsistency. See Florio and Shapiro 2014 and Linnebo and Rayo 2014 for further discussion.

[13] What about a language whose order is an infinite limit ordinal λ? Since this language is contained in the language of order $\lambda + 1$, the theorem ensures that its generalized semantics can be developed in a language of order $\lambda + 2$.

consequence, just like Tarski's notion of logical consequence requires model theory.

One might try to resist this natural thought by recalling an observation made in Section 7.4. By appealing to an appropriate set-theoretic reflection principle, we can ensure that the standard definition of logical consequence as truth preservation in every set-based model is extensionally equivalent to the definition of logical consequence as truth preservation in every plurality-based model. Does this show that a generalized semantics isn't required after all? We don't think so. As indicated in Section 7.9, one response is that we would like a theory of logical consequence that is not only extensionally but also intensionally correct. For instance, by the downwards Löwenheim-Skolem theorem, we know that it would be extensionally correct to define logical consequence by quantifying solely over finite and countably infinite models. Although extensionally correct, this definition of logical consequence would be inferior to Tarski's, because it badly fails to capture the intended intension. An analogous argument can be made for considering not only set-based models but all models, as is done in generalized semantics. Moreover, the ability to capture the intended intension, which here requires quantification over all models, is essential when we lack independent means of determining the correct extension. In that case, it is only against the backdrop of an intensionally correct theory that we can check whether any other theory is in fact extensionally adequate.

Another, more direct response is that generalized semantics is legitimate and interesting in its own right, irrespective of its contribution to theorizing about logical consequence. Our language has one interpretation. But there are myriad other interpretations that it might have had. It is a legitimate undertaking to study all these interpretations and how the truth of sentences is affected by the choice of interpretation.

11.6 Ascent and inexpressibility

Let us therefore accept that traditional generality absolutism pushes us up through the orders of the hierarchy and proceed to inquire about the significance of this ascent phenomenon. It turns out that the ascent gives rise to three complaints that mirror those we leveled against relativism in Section 11.4.

First, type-unrestricted generality appears possible. For example, it appears meaningful to ask whether the law of extensionality holds at every order of the type-theoretic hierarchy. It would take a very good reason to go against such a robust appearance. Yet no such generalization is available on the type-theoretic view. While quantification of any specific order is available, there is no such thing as quantification across all orders at once.

Second, type-unrestricted generality is needed to engage in many interesting and potentially valuable forms of theorizing. We already mentioned the question of extensionality, which figures in some theoretically important claims, for example, that extensionality holds at every order of the plural hierarchy but not of the conceptual hierarchy. Likewise, it is an important insight, which deserves to be properly expressed, that a version of Cantor's theorem holds at every order of the type-theoretic hierarchy. Coupled with widely held assumptions about plural comprehension, this means that there are more pluralities than objects, more superpluralities than pluralities, and so on up through all the possible levels. For a final example, consider the claim that the principle of compositionality holds at every order, that is, that at every order, the semantic value of a complex expression is determined as a function of the semantic values of the expression's simpler constituents.[14] None of these questions can be properly expressed and discussed in the type-theoretic setting. We thus seem to be confronted with examples of expressive limitations that curtail certain forms of systematic and valuable theorizing.

In fact, the view that type theory suffers from expressive limitations has a long history. Wittgenstein alludes to it in the *Tractatus* (Proposition 4.1241) and formulates it explicitly in his pre-Tractarian period:

> Types can never be distinguished form each other by saying (as is often done) that one has these *but* the other has those properties, for this presupposes that there is a *meaning* in asserting all these properties of both types. (Wittgenstein 1979, 106)

For essentially these reasons, he concludes two pages later that "a THEORY of types is impossible". Very similar considerations are echoed by Gödel twenty years later:

> The theory of simple types [...] has the consequence that the objects are divided into mutually exclusive ranges of significance, [...] and that therefore each concept is significant only for arguments belonging to one of these ranges, i.e., for an infinitely small portion of all objects. What makes

[14] See Linnebo 2006 for some further examples.

> the above principle particularly suspect, however, is that its very assumption makes its formulation as a meaningful proposition impossible [...]. Another consequence is that the fact that an object x is (or is not) of a given type also cannot be expressed by a meaningful proposition. (1944, 466)

It might be objected that our examples of expressive limitations are biased.[15] From our point of view, there are indeed important generalizations that the type theorist cannot express. But from the type theorist's point of view, the alleged examples of inexpressible insights can be dismissed as ungrammatical gibberish. This is a perceptive and interesting complaint, which leaves us in a difficult dialectical situation. From one point of view, there is evidence against the opposing view. From the opposing point of view, this alleged evidence isn't even meaningful!

How can we get beyond this apparent impasse? It is true that the type theorists can stubbornly reject the attempted examples of expressive limitations without any fear of thereby contradicting themselves. But we claim it would be bad methodology to do so. Greater expressive power appears possible; there are consistent ways to develop this greater expressibility; and the greater expressibility promises to be theoretically useful.[16] In such cases, we contend, it is good methodology to press ahead, despite the protestations of the coherent naysayers—though obviously with the epistemic caution that behooves every exploration of an unconventional hypothesis.

Third, an objection to the semantic ascent through the type-theoretic hierarchy is that we are precluded from properly stating the type theorists' view that there is a hierarchy strictly divided into levels and without a top level. This is analogous to the case of relativism. Recall the relativist's predicament: to state that every quantifier interpretation can be extended, we need to avail ourselves of absolute quantification over interpretations. Likewise, to state that quantification of every order can be extended by quantification of some even higher order, we need to generalize across all the orders simultaneously. The hierarchy has no maximal level, yet we are precluded from properly expressing that.

Might the expressive limitations be overcome by appealing to schematic generality, as discussed in Section 11.4? In this case, the schematic generality would reside in the type indices. Where τ is a type, a claim $\varphi(x^\tau)$ would be

[15] A version of this objection is discussed in Krämer 2014.
[16] For the second point, see Section 11.7.

understood as conveying that the claim holds for any type τ.[17] However, as noted in our discussion of generality relativism, the logical complexity of the generalizations that can be captured by schematic generality is extremely limited.

In sum, generality relativism and traditional generality absolutism have far more in common than has been acknowledged in the existing literature: both suffer from expressibility problems. We have discussed three such problems: the apparent meaningfulness of certain absolute generalizations, their potential theoretical utility, and the inability to properly express one's own view without access to such absolute generalizations. Our discussion motivates taking a closer look at the third alternative, namely absolute generality over a domain that is extensionally *in*definite, or not properly circumscribed. This is the task of the next and final chapter.

Before turning to this task, however, we would like examine a strategy that might allow the traditional absolutist to restore full expressibility. We will find that, while promising, this strategy ends up transforming traditional absolutism into a view that has much in common with the third alternative.

11.7 Lifting the veil of type distinctions

As we have seen, semantic considerations push the traditional absolutist higher and higher up through the type-theoretic hierarchy. But this ascent phenomenon leads to expressibility problems. We now explore another perspective on the debate. Consider the entire plural hierarchy to which the traditional absolutist ends up committed. At the bottom, there is an extensionally definite domain of individuals, which make up level 0. Then there is level 1, which adds pluralities; level 2, which adds superpluralities; and so on. Let the traditional absolutist make her choice about how high to go. Our only assumption is that there is no maximal level of the hierarchy. That is, for every level α in the plural hierarchy, there is also level $\alpha + 1$. These levels are reflected in the type distinctions of our language: variables of each type take their values exclusively from the corresponding level.

What happens if we abandon these type distinctions and bring all the different sorts together? Doing so would be a radical change in perspective. We would, as it were, lift the veil of type distinctions and thus gain a new perspective on reality. We will now defend the coherence of this new

[17] Russell famously exploits this kind of "typical ambiguity"; see Russell 1908, 251.

perspective. We will also show that, from the new perspective, traditional plural logic is no longer valid.[18]

As a warm-up case, imagine a community of extreme Cartesian dualists whose language involves a strict type distinction between mental and physical vocabulary. Members of this community regard the application of mental predicates to physical terms as meaningless rather than false. Likewise, they regard the application of physical predicates to mental terms as meaningless rather than false. This prevents them from being able to generalize at once over both the mental and the physical domain. For example, these dualists cannot express claims such as:

(11.7) Everything is either mental or physical.

(11.8) Nothing is both mental and physical.

The situation is analogous to the one described in the previous section. Like the type theorist we encountered there, the dualists can dismiss these alleged examples of inexpressible claims as ungrammatical gibberish. How can we convince them to abandon their type distinction and adopt a perspective that permits the expression of the above claims? From our point of view, the dualists' type distinction is dogmatic and parochial. But this charge is supported by evidence which, from their point of view, isn't even meaningful! We face an impasse, which again can only be overcome by showing to the dualists the methodological flaws of their dogmatism. Greater expressive power appears possible, there are consistent ways to develop this greater expressibility, and the greater expressibility promises to be theoretically useful. As before, we contend that it is good methodology in such cases to explore the expressively richer perspective. So let us describe a suitable language in which that can be done.

The language of the community of extreme Cartesian dualists, we recall, has distinct sorts for mental and physical vocabulary. We translate this language into a one-sorted language in which all syntactic restrictions based on the two sorts have been removed. We add two new predicates 'M' and 'P' for being mental and being physical, respectively. Using these predicates together with the "all-purpose" variables of the one-sorted language,

[18] Simons (2016) and Oliver and Smiley (2016, Chapter 15) share our aim of developing a logic of higher-level pluralities in an untyped language and discusses some axioms that might be appropriate for this logic. Unlike them, however, we pursue this aim indirectly by first formulating a typed logic of higher plurals and then translating it into an untyped system.

we can track the dualists' sortal distinction and interpret it as a form of quantificational restriction. When they assert, relative to the mental sort, that everything is F, we interpret them as asserting that every M is F. Likewise, when they assert, relative to the physical sort, that something is G, we interpret them as asserting that some P is G. By means of this translation, we regain full expressibility. For example, we can now state the claims that the dualists were unable to express:

(11.9) $\forall x(Mx \vee Px)$

(11.10) $\neg\exists x(Mx \wedge Px)$

Let us return to the typed language that is our real concern, namely the language of the plural hierarchy—call it $\mathcal{L}_1$. Proceeding as in our warm-up case, let us bring its many sorts together by translating this language into a standard one-sorted language $\mathcal{L}_2$. We want to capture the sortal distinctions of $\mathcal{L}_1$ in the one-sorted setting of $\mathcal{L}_2$. To this end, we let $\mathcal{L}_2$ contain a new two-place predicate 'L' for the level of a plurality. Intuitively, '$L(x, 0)$' means that x is an individual; '$L(x, 1)$', that x is a plurality; '$L(x, 2)$', that x is a superplurality; and so on. (We are assuming the language can quantify over enough ordinal numbers to index all the levels of the plural hierarchy.)

Let us describe a translation τ from $\mathcal{L}_1$ to $\mathcal{L}_2$. Every atomic formula containing no plural vocabulary is translated as itself. An atomic formula containing plural vocabulary is translated by replacing each plural expression with a singular counterpart. We avoid clashes of terminology by ensuring that the translation of singular and plural vocabulary does not overlap. Moreover, we reserve the special symbol 'η' for a *membership* relation that translates plural membership. Here are some examples:

$$\begin{array}{rcl} Fa & \overset{\tau}{\longmapsto} & Fa \\ Gxx_i & \overset{\tau}{\longmapsto} & Gx_i \\ R(xx_i, y_j) & \overset{\tau}{\longmapsto} & R(x_i, y_j) \\ y_j \prec xx_i & \overset{\tau}{\longmapsto} & y_j \ \eta \ x_i \end{array}$$

The translation commutes with the logical connectives:

$$\begin{array}{rcl} \neg\varphi & \overset{\tau}{\longmapsto} & \neg\tau(\varphi) \\ \varphi \wedge \psi & \overset{\tau}{\longmapsto} & \tau(\varphi) \wedge \tau(\psi) \end{array}$$

Finally, we come to the action of τ on the quantifiers. Singular quantifiers retain their sort, since this is the only sort available in $\mathcal{L}_2$, but we restrict them by means of the formula '$L(x, 0)$'. Plural quantifiers are translated as singular quantifiers restricted by means of the formula '$L(x, 1)$'; superplural quantifiers, by means of '$L(x, 2)$'; and so on. So we have:

$$\begin{array}{rcl} \forall y_j\ \varphi & \stackrel{\tau}{\longmapsto} & \forall y_j(L(y_j, 0) \rightarrow \tau(\varphi)) \\ \forall xx_i\ \varphi & \stackrel{\tau}{\longmapsto} & \forall x_i(L(x_i, 1) \rightarrow \tau(\varphi)) \\ \forall xxx_i\ \varphi & \stackrel{\tau}{\longmapsto} & \forall x_i(L(x_i, 2) \rightarrow \tau(\varphi)) \end{array}$$

In a nutshell, a speaker of $\mathcal{L}_1$ is interpreted as a speaker of an untyped language whose sortal distinction is merely the syntactic expression of a quantificational restriction.

The kind of translation we are proposing is not entirely unprecedented. (See also Quine 1956.) Suppose $\mathcal{L}_1$ is the language of PFO+, which has only two types: singular and plural. Then $\mathcal{L}_2$ is essentially the kind of one-sorted version of plural logic that we discussed in Section 5.3.[19] Returning to the general case, we may think of $\mathcal{L}_2$ as a generalization of this one-sorted plural language. The single sort of variables of $\mathcal{L}_2$ permits different forms of reference: singular, plural, superplurals, and so on. That is, we have a single sort of "all-purpose" variables whose possible assignments include those of all the different types of variable that are available in $\mathcal{L}_1$.

Is this one-sorted language $\mathcal{L}_2$, and the translation into it, permissible? Let us begin by examining whether we run a risk of inconsistency by translating in this way. We need to examine how theories formulated in $\mathcal{L}_1$ relate to their translations. Let T be an $\mathcal{L}_1$-theory and let T^* be an $\mathcal{L}_2$-theory whose axioms are the translations of the axioms of T. Then, we have the following key fact.[20]

Fact 11.1 *T is consistent if and only if T^* is consistent.*

Thus, formal consistency is not a concern when we lift the veil of type distinctions.

We therefore turn to more philosophical issues concerning the language $\mathcal{L}_2$. Should its single sort of terms be taken to refer to sets? If so, the traditional absolutist might protest that the translation involves a change of

[19] The only difference is the unimportant one that $\mathcal{L}_2$ has a predicate 'η' for membership, whereas the other language has a predicate '$\preccurlyeq$' for being among. As we have seen, however, these two predicates are interdefinable.

[20] See Enderton 2001, 300, Theorem 44A.

subject. The plural variables of $\mathcal{L}_1$ are used to refer plurally, not to effect singular reference to sets; so to be adequate, a translation needs to respect that fact. But in fact, it is neither obligatory nor particularly natural to take the single sort of terms of $\mathcal{L}_2$ to effect singular reference to sets. Many theorists take sets to lack spatiotemporal location, and almost all take them to lack causal powers. But pluralities of every level can have location, time, and causal powers; for example, some children may be located in the garden, break a window, and so on.

If not to sets, to what *do* the variables of $\mathcal{L}_2$ refer? As mentioned, our proposal is that each of these variables is capable of a variety of different forms of reference: singular, plural, superplural, and so on. The assignment to each such variable will be made in some metalanguage by means of another variable with the same capabilities concerning its forms of reference. This view isn't objectionable to traditional absolutists in the way it would be objectionable to interpret the terms of $\mathcal{L}_2$ as referring to sets. True, $\mathcal{L}_2$ lifts the veil of the syntactic type distinctions found in $\mathcal{L}_1$. But after lifting the veil, each term retains precisely the form of reference it had before.[21]

So far, we have acquitted $\mathcal{L}_2$ of the charges of risking inconsistency and of changing the subject by translating terms that refer plurally as terms that refer to sets. What positive reason might we have to accept $\mathcal{L}_2$? Our answer is that in $\mathcal{L}_2$ we can express everything we wanted to, but couldn't, express in $\mathcal{L}_1$. Here are some examples. First, we can raise the question of cumulativity: can a plurality of level $n + 1$ have members only of level n or also of any level lower than $n + 1$? For example, does 'my children, your children, and Bob' refer to such a mixed-level plurality? Second, what is the relation between a singleton plurality and its single member? Should these be identified or not? For example, do 'the objects identical to Bob' and 'Bob' co-refer? Third, do extensionality principles hold at every level of the plural hierarchy? For example, should we accept an indiscernibility principle (Sections 2.4 and 10.5) governing each level? Based on these considerations, we contend that traditional absolutists have good reason to accept the translation of their plural logic, generalized to pluralities of all levels, into the one-sorted language of higher pluralities, $\mathcal{L}_2$.

We now face a crucial question: can the all-purpose variables of the one-sorted language $\mathcal{L}_2$ be "pluralized"? In other words, can we introduce

[21] The view that all-purpose variables can effect generalized forms of plural references is embraced by some theorists who develop higher-level plural logic directly rather than indirectly by lifting the veil. See footnote 18.

variables that relate to the all-purpose variables the way ordinary plural variables relate to ordinary singular variables? Consider the question from the point of view of our opponent, the traditional absolutist. We are supposing, recall, that her plural language $\mathcal{L}_1$ contains *all* the forms of pluralization that are available. Moreover, pluralization is a relationship that holds between an expression of order $\alpha + 1$ and expressions of order α (or, in the case of cumulativity, orders $\leq \alpha$). Every form of pluralization corresponds to some level of the plural hierarchy associated with $\mathcal{L}_1$. Transposed to the one-sorted setting of $\mathcal{L}_2$, this means that every pluralization of its single sort of variable would have to have values at some level α (or, in the case of cumulativity, at levels $\leq \alpha$). If we are to pluralize the all-purpose variables of the language $\mathcal{L}_2$, it follows that each of the resulting pluralities would be bounded by some level.[22] In particular, there can be no universal plurality with respect to the single sort of variable of $\mathcal{L}_2$—precisely as in the alternative version of plural logic that we will defend in the next chapter.

In short, we have shown how full expressibility can be restored to the traditional absolutist's language while retaining plural reference, superplural reference, and so on. Moreover, we have argued that the traditional absolutist should accept this move. Full expressibility is restored by means of a one-sorted language $\mathcal{L}_2$ that lifts the veil of type distinctions. Crucially, we have found that traditional plural logic is not valid in this new one-sorted setting. When we attempt to pluralize the all-purpose variables of $\mathcal{L}_2$, the interpretation must be confined to some level, and hence some instance of plural comprehension fails—this is the case, for example, for any instance yielding the universal plurality. So, even by her own lights, the traditional absolutist has a reason to countenance an alternative plural logic where plural quantification is bounded by some level. The most plausible development of traditional absolutism thus ends up transforming it into a view that has much in common with our third alternative of developing a critical version of plural logic. In the final chapter, we take a more direct approach to this third alternative.

[22] Oliver and Smiley (2016, Chapter 15) reach the same conclusion via somewhat different reasoning. For us, the boundedness requirement has its root in the typed system and is revealed when the veil is lifted; for them, it is proposed as a natural response to a version of Russell's paradox that would afflict the untyped plural logic if (axioms equivalent to) unrestricted plural comprehension were accepted.

Appendix

11.A The Ascent Theorem

Recall that traditional plural logic assumes unrestricted plural comprehension at every order of the type-theoretic hierarchy. (This gives us, in particular, a universal plurality.) Then, as we saw above, we have the following theorem.

Ascent theorem (arbitrary finite form)

Assume traditional plural logic and the possibility of absolute generality at every finite order $n \geq 1$. Then a generalized semantics for a language of order n cannot be given in another language of order n but can be given in a language of order $n + 1$.

Let us begin by reminding ourselves of the proof of the *basic form* of the theorem, which states that a generalized semantics for a first-order language cannot be given in another first-order language but can be given in a language with plural quantification, such as $\mathcal{L}_{\text{PFO}+}$. First, there is the positive part of the theorem: this was shown in Section 7.3, where we provided a generalized semantics for PFO, and hence for its first-order fragment, in PFO+.

Then, there is the negative part of the theorem. This result relies heavily on the following thesis:

Plural Profusion

There are more pluralities than objects.

As we saw in Section 11.2, this thesis follows from Plural Cantor together with the assumption that there is a universal plurality and two or more objects. The negative part of the Ascent Theorem now has a straightforward proof.

Proof. Under the assumption of absolute generality, an ordinary singular predicate can be interpreted by means of any plurality. But by Plural Profusion, there are more pluralities than objects. It follows that interpretations of a first-order language cannot be objects but must be represented by means of higher-order resources. ⊣

We now turn to the proof of the Ascent Theorem in its *arbitrary finite form*. This proof is somewhat involved but can be broken down into three components. First, we need a way of coding ordered pairs of pluralities of arbitrary finite level. Second, we extend the plurality-based model theory to higher level. This is a version of the recursive characterization of truth in a model familiar from Tarski (1935). Finally, taking a cue from Frege (1879) and Dedekind (1888), we show how we can convert a recursive definition to an explicit one by ascending one level in the hierarchy.

Coding of *n*-tuples of higher-level pluralities

We have shown in Section 7.5 that interpretations and variable assignments can be taken to consist of pluralities of ordered pairs carrying the appropriate semantic information. For instance, an interpretation includes a domain, which is represented by a plurality of pairs of the form $\langle \exists, x \rangle$.

Having defined interpretations and variable assignments, we can talk about the semantic value of an expression E according to an interpretation ii or a variable assignment ss, indicated by $[\![E]\!]_{ii,ss}$. So, for a plural constant tt, we have:

$$\forall x\big(x \prec [\![tt]\!]_{ii,ss} \leftrightarrow \langle tt, x \rangle \prec ii\big)$$

Now we want to generalize these definitions to higher orders. We need some notation for expressions of each finite order. For convenience, we use single lowercase variables for terms and upper case variables for predicates. As usual, the superscript indicates the order of a term. The hierarchy has a plural interpretation but could also be given a conceptual interpretation. We count objects as pluralities of level 0. The following examples illustrate the relation between this notation and the one we have used throughout the book:

$$\begin{array}{ccc} x^0 \prec x^1 & \mapsto & x \prec xx \\ P(x^2) & \mapsto & P(xxx) \end{array}$$

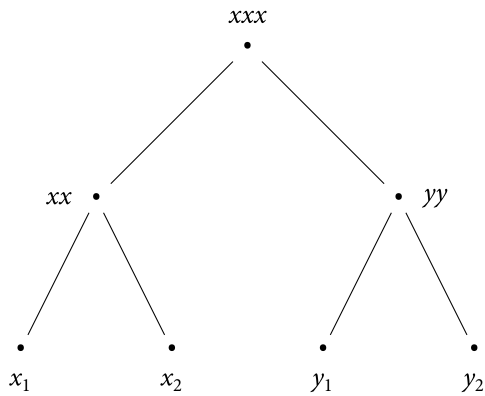
We leave the predicates' arity unmarked. We also suppress the superscript of the symbol '$\prec$' for membership between any two successive levels of the hierarchy. So we write '$x^0 \prec x^1$', '$x^1 \prec x^2$', and so on.

It is essential to have a device for handling higher-level analogues of n-tuples, that is, n-tuples of pluralities of arbitrary (and possibly different) orders. If this can be done, the characterization of an interpretation for a higher-order language will be routine. Thankfully, we have the following theorem:

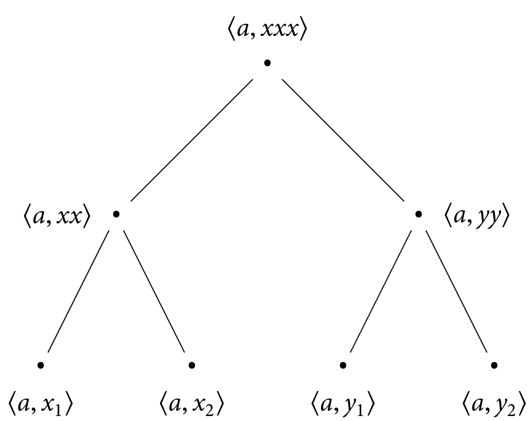
Theorem (n-tuples) Assume that for any two objects there is another object that serves as their ordered pair. Given any pluralities $x_1^{k_1}, \ldots, x_n^{k_n}$ whose levels are indicated by the superscripts, we can then code for the ordered n-tuple of these pluralities by means of a single plurality x^k, where k is the maximum level among the k_i.

We will designate this plurality x^k as $\langle x_1^{k_1}, \ldots, x_n^{k_n} \rangle$.

As an initial exercise, consider the case of superplurals. Assume we want to pair an object a with xxx to form $\langle a, xxx \rangle$. As we have seen, xxx is usefully represented in terms of its articulation, for example:



Now, to code the desired ordered pair, all we need to do is add a as a first coordinate to every object that occurs at the base level while retaining the superplural articulation:



This idea, which we have illustrated visually, can now be developed formally and with appropriate generality.

Proof idea.[23] The key step is to define the ordered pair of x^0 and an n-th level plurality x^n. We proceed by induction on n. Assume we have defined $\langle x^0, x^n \rangle$. Then we let $\langle x^0, x^{n+1} \rangle$ be the unique plurality y^{n+1} described by the following equivalence:

$$\forall u^n \big(u^n \prec y^{n+1} \leftrightarrow \exists x^n (u^n = \langle x^0, x^n \rangle \wedge x^n \prec x^{n+1}) \big) \tag{11.11}$$

This key step enables us to attach a "tag" x^0 to any higher plurality x^n. And this tagging, in turn, enables us to represent the n-tuple $\langle x_1^{k_1}, \ldots, x_n^{k_n} \rangle$, which can be seen as follows. First, we attach the unique tag i to each $x_i^{k_i}$. Then, we wish to form the union of all of the tagged higher pluralities. Assume for the moment that $k_i = k$ for each i. Then the desired union can be defined as the higher plurality y^k whose members are any z^{k-1} that figures as a member of one of the tagged higher pluralities $x_i^{k_i}$.

We claim that this union y^k represents the desired n-tuple. To establish this claim, we must show how each entry can be retrieved from the union. Suppose we want to retrieve the i-th entry. First, we delete each object at the base level of the union whose tag is distinct from i, while retaining the articulation of the remaining base-level objects. Second, we delete all occurrences of the tag i, again retaining the articulation. This yields $x_i^{k_i}$.

Let us now lift the simplifying assumption that $k_i = k$ for each i and let one of the k_i be less than k. We wish to handle this by raising the level of $x_i^{k_i}$ up to k. We achieve this raising by considering the singleton plurality of $x_i^{k_i}$, and its singleton plurality in turn, and so on until we obtain a higher plurality of level k. We record the number of singleton operations applied by means of a supplementary tag. We now proceed as before but use the resulting plurality instead of $x_i^{k_i}$. When the time comes for retrieving the i-th entry from the union of all the k-th level pluralities, we apply the two steps described in the preceding paragraph and then finish by undoing the j topmost singleton operations, where j is the number recorded by means of the supplementary tag. This yields $x_i^{k_i}$. $\dashv$

[23] See Linnebo and Rayo 2012, Appendix B, for a detailed proof of their closely related Theorem 1.

Tarski on satisfaction and model theory

We now want to characterize the notion of truth in an interpretation (satisfaction) for a language of order n. As done in Chapter 7, we proceed by first defining the notion of interpretation (as a combination of a domain and an interpretation function) and the notion of variable assignment (and a variant thereof). Then we obtain the definition of truth in an interpretation from the more general relation of truth in an interpretation with respect to a variable assignment, which we characterize recursively. Thus we have generalized the model theory encountered above (Sections 7.3 and 7.5).

Definition (truth in an interpretation) Assume that we have defined an interpretation $i^{n+1} = \langle d^{n+1}, f^{n+1} \rangle$ and a variable assignment s^n. Then we define truth in i^{n+1} with respect to s^n by means of the following clauses.

1. If φ is a formula of the form $P(t_1, \ldots, t_m)$ where P is an m-place predicate and the t_i are of appropriate order (that is, are of order at most n), then:
   $i^{n+1} \vDash \varphi\ [s^n]$ if and only if $\langle [\![t_1]\!]_{i^{n+1},s^n}, \ldots, [\![t_m]\!]_{i^{n+1},s^n} \rangle \prec [\![P]\!]_{i^{n+1},s^n}$
2. If φ is a formula of the form $t^0 = u^0$, then:
   $i^{n+1} \vDash \varphi\ [s^n]$ if and only if $[\![t^0]\!]_{i^{n+1},s^n} = [\![u^0]\!]_{i^{n+1},s^n}$
3. If φ is a formula of the form $t_1 \prec t_2$, where t_1 and t_2 are of the appropriate order (that is, the order of t_1 is strictly below that of t_2), then:
   $i^{n+1} \vDash \varphi\ [s^n]$ if and only if $[\![t_1]\!]_{i^{n+1},s^n} \prec [\![t_2]\!]_{i^{n+1},s^n}$
4. If φ is a formula of the form $\neg\psi$, then:
   $i^{n+1} \vDash \varphi\ [s^n]$ if and only if it is not the case that $i^{n+1} \vDash \psi\ [s^n]$
5. If φ is a formula of the form $\psi_1 \wedge \psi_2$, then:
   $i^{n+1} \vDash \varphi\ [s^n]$ if and only if $i^{n+1} \vDash \psi_1\ [s^n]$ and $i^{n+1} \vDash \psi_2\ [s^n]$
6. If φ is a formula of the form $\exists v^m\ \psi$, where $m \leq n$, then:
   $i^{n+1} \vDash \varphi[s^n]$ if and only if there is x^m in d^{m+1} such that $i^{n+1} \vDash \psi[s^n(v^m/x^m)]$
   where d^{m+1} is the domain of pluralities of level m encoded in d^{n+1} and $s^n(v^m/x^m)$ is a variant of s^n, namely an assignment just like s^n, with the possible exception that $s^n(v^m/x^m)$ assigns x^m to v^m.

Frege and Dedekind on recursive definitions

Frege (1879) and Dedekind (1888) discovered that recursive definitions can be turned into explicit ones by generalization over "collections" of the entities related by the recursive definition. Consider the case of addition in arithmetic. Using the prime symbol for the successor operation, $\textsc{Add}(x, y, z)$ ("z is a sum of x and y") can be defined recursively as follows:

(i) $\textsc{Add}(x, 0, x)$
(ii) $\textsc{Add}(x, y, z) \rightarrow \textsc{Add}(x, y + 1, z + 1)$

Then $\textsc{Add}(x, y, z)$ can be defined explicitly as follows:

$$\textsc{Add}(x, y, z) \leftrightarrow_{\text{def}} \forall R\big(\forall u R(u, 0, u) \wedge \forall u, v, w(R(u, v, w) \\ \rightarrow R(u, v', w')) \rightarrow R(x, y, z)\big)$$

Tarski (1935) realized that his own recursive definition of satisfaction could be turned into an explicit one in this way. The same obviously goes for his later definition of truth in a model. (See Appendix B.1 of Linnebo and Rayo 2012 for details.) This completes our proof sketch for the positive part of the Ascent Theorem.

As for the negative part, we already described how to prove this in the basic case, utilizing Plural Profusion. Assuming unrestricted comprehension for higher pluralities as well, it is easy to establish higher-level analogues of Plural Profusion, namely that there are more pluralities of level $n + 1$ than pluralities of level n. Equipped with this result, the observation we used to prove the basic case is easily extended to prove the arbitrary finite case.

12
Critical Plural Logic

12.1 Introduction

An inconsistent triad figured centrally in the previous chapter. We cannot simultaneously accept universal singularizations, unrestricted plural comprehension, and absolute generality. So at least one of these *prima facie* attractive assumptions has to be abandoned. Which one?

We began by rejecting generality relativism, which abandons absolute generality. This left us with a choice between rejecting universal singularizations and rejecting unrestricted plural comprehension. We proceeded to take a closer look at the former option, which has received far more attention than the latter. Our discussion revealed some serious problems with this popular option. One problem is that there remains pressure to accept universal singularizations, in particular when we examine how plural logic can be used to illuminate set theory. Another problem is that this version of absolute generality faces difficulties akin to those of generality relativism. Overall, the previous chapter thus motivates taking a closer look at the last remaining option, namely to restrict the plural comprehension scheme.

At the outset, this option seems unpromising. How could there *not* be some things that are all and only the φs? Provided there is at least one φ, the mentioned claim seems obviously true, as observed for example by Boolos and Hossack (see Section 2.5). Clearly, an explanation would be needed of why the plural comprehension scheme must be restricted. Our proposed explanation is simple, at least in essence. True, all we need to do to define a plurality is circumscribe the objects in question; in particular, there is no postulation of a set or any other "plural entity" over and above these objects. *But the objects in question do need to be circumscribed.* And as we shall see, on some metaphysical views, reality as a whole resists proper circumscription. If a view of this sort is right, there can be no universal plurality, as this

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would require circumscribing something uncircumscribable. It follows that the plural comprehension scheme must be restricted.[1]

The aim of this chapter is to explore and develop these considerations. This leads to a third way, we argue, between generality relativism and the traditional form of generality absolutism.

12.2 The extendability argument

Let us reconsider the kind of extendability argument that is typically used to motivate generality relativism. In a nutshell, the argument takes the following form.[2]

> Assume that we quantify over absolutely everything. For every condition $\varphi(x)$, we can define a set $\{x : \varphi(x)\}$ of all objects satisfying the condition:
>
> $$(*) \qquad \forall x(x \in \{x : \varphi(x)\} \leftrightarrow \varphi(x))$$
>
> Consider the condition '$x \notin x$' and the resulting set $R = \{x : x \notin x\}$. If R is in the range of the quantifier '$\forall x$' in the associated instance of $(*)$, a contradiction follows by familiar Russellian reasoning. Therefore, R is outside the range of this quantifier, and we weren't quantifying over absolutely everything after all.

Clearly, the crux of the argument is the use of an arbitrary condition to define a set subject to the requirement $(*)$.

Arguments of this form can be frustrating, however. Why is it permissible to define the mentioned sets? Generality relativists take an extremely liberal view of what constitutes a permissible mathematical definition. They claim that this liberalism supports the mentioned crux and thus also their relativist conclusion. From the point of view of a generality absolutist, however, this extreme liberalism is unacceptable—indeed provably so, as can be seen by considering the following simple truth of first-order logic:

[1] Some alternative strategies for defending this thesis can be found in Spencer 2012 and Hossack 2014. Essentially the same view is also defended in Linnebo 2010, although the quantifiers used here correspond to his "modalized quantifiers" '$\Box\forall$' and '$\Diamond\exists$'.

[2] See Dummett 1991 (especially Chapter 24), Parsons 1974b, Glanzberg 2004, and Fine 2006.

$$\neg\exists y \forall x (Rxy \leftrightarrow \neg Rxx)$$

Our inability to define the offending Russellian set is just an instance of this logical truth, obtained by replacing 'R' with the relativists' desired notion of membership in a set.

Thus, in its standard form, the extendability argument fails to resolve the debate about the possibility of absolute generality. The argument turns on the permissibility of certain definitions, which absolutists have good reason to reject. We believe progress can be made by means of a more nuanced formulation of the argument. To explain what we have in mind, it is useful to start with an analogy.

Suppose you detest web pages that link to themselves.[3] So you wish to create a web page that links to all web pages that are innocent of this bad habit. In other words, you wish to create a web page that links to all and only the web pages that do not link to themselves. Can your wish be fulfilled? The answer depends on how your wish is analyzed. Should the scope of the crucial plural description—'the web pages that do not link to themselves'—be narrow or wide? Depending on the scope of the description, your wish can be analyzed in either of the following two ways:

(N) You wish to design a web page y such that, for every web page x, y links to x if and only if x does not link to itself.

(W) There are some web pages xx such that, for every web page x, x is one of xx just in case x does not link to itself, and you wish to design a web page y that links to all and only xx.

On the narrow scope reading (N), your wish is flatly incoherent. The desired web page would have to link to itself just in case it does not link to itself. On this reading, your wish is no better than the wish to bring about the existence of a Russellian barber:

(B) You wish there to be a barber y such that, for all x, y shaves x if and only if x does not shave himself.

On the wide scope reading (W), by contrast, there is no conceptual or mathematical obstacle to the fulfillment of your wish. First, you identify all

[3] See Linnebo 2016, Section 7, and Linnebo 2018, Section 3.3. This example has been independently used by Brian Rabern in teaching and on social media.

the web pages xx that refrain from the bad habit of self-linking. Then, you create a new web page that links to all and only xx.

What explains this stark difference between the two readings? The heart of the matter is how one specifies the target collection, that is, the web pages of which you wish to create a comprehensive inventory. (As before, we use the word 'collection' in an informal way for anything that has a membership structure, such as a set, class, plurality, or indeed even a Fregean concept—where the relation between instance and concept is regarded as a membership structure.) On (N), the target is specified intensionally by means of the condition 'x does not link to itself'. This intensional specification means that the target shifts with the circumstances. First, you find that there is no web page of the sort you wish for. So you attempt to fulfill your wish by changing the circumstances, that is, by creating a web page of the desired sort. But since the target is specified intensionally, this new web page must itself be taken into account when assessing whether your wish has been satisfied—which of course it has not, as logic alone informs us.

By contrast, on the wide scope reading (W), the target is specified extensionally by means of the plurality xx. This extensional specification ensures that the target stays fixed when you change the circumstances. (Here we invoke the modal rigidity of pluralities, which was defended in Chapter 10.) You can thus fulfill your wish by creating a new web page that links to all and only xx. Although xx are described, in the original circumstances, by means of a condition that is prone to paradox, there is no requirement that xx should remain so described in alternative circumstances. Like any other plurality, xx are tracked rigidly across alternative circumstances, not in terms of any description that these objects happen to satisfy.

With this analogy in mind, let us return to the question of what is a reasonable liberalism about mathematical definitions. Suppose you care about sets, not web pages. You wish to define a set by specifying its elements. As our web page analogy reveals, it is essential to distinguish between two different ways in which the elements of the would-be set might be specified. You might specify the elements *intensionally*, by means of a condition $\varphi(x)$:

(I) You wish to define a set y such that, for every object x, x is an element of y if and only if $\varphi(x)$.

Alternatively, you might specify the elements of the would-be set *extensionally*, by means of a plurality xx:

(E) You wish to define a set y such that the elements of y are precisely xx.

Can either wish be fulfilled?

This is a question about what it takes for a mathematical definition to be permissible. We claim that the proposed definition is often problematic when the target is specified intensionally, but always permissible when the target is specified extensionally. Our defense of these claims will be informed by our web page analogy.

Let us begin with the negative claim that (I) is often problematic. The reason is simple. We can hardly be more liberal about mathematical definitions than we are about objects that we literally (and easily) construct, such as web pages. This means we need to be extremely cautious about which definitions of sets we deem permissible when the target is specified intensionally. To illustrate how such definitions can be problematic, observe that one instance of the intensionally specified wish (I) is an analogue of the problematic narrow-scope wish (N) concerning web pages:

(N′) You wish to define a set y such that, for every object x, x is an element of y if and only if x is not an element of itself.

Just as (N) is flatly incoherent, so, we contend, is (N′). This takes care of the negative claim, showing also that the standard extendability argument is too quick. In the next section, we defend the positive claim that (E) is always permissible.

12.3 Our liberal view of definitions

Suppose that the target set is specified extensionally by means of a plurality xx. Then this specification ensures that the target won't shift with the circumstances. We therefore have no difficulty making sense of circumstances in which xx define a set, much as we have no difficulty making sense of circumstances in which some given web pages yy are precisely the ones to which some new web page links.

We can be far more specific, though. Consider a dispute between a proponent and an opponent of the proposed definition. Suppose both parties accept a domain dd. The proponent now wishes to define one or more sets of the form $\{xx\}$, where xx are drawn from dd. She does not insist that the sets to be defined be among dd; in this sense, the sets may be "new". To shore up the proposed definition, she provides the following account of what it takes for a "new" set to be identical with another set or have a certain element:[4]

(i) $\{xx\} = \{yy\}$ if and only if $xx \approx yy$
(ii) $y \in \{xx\}$ if and only if $y \prec xx$

These clauses achieve something remarkable. They provide answers to all atomic questions about the "new" sets of the form $\{xx\}$ in terms that are concerned solely with the "old" objects in *dd*, objects that were available before the definition. That is, all atomic questions about the "new" objects receive answers in terms of the "old" objects that both parties to the dispute accept.

In fact, this is merely an instance of the more general liberal view of definitions encountered in Sections 4.4 and 5.8. According to this view, it suffices for a mathematical object to exist that an adequate definition of it can be provided, where the adequacy is understood as follows. Consider a domain *dd* of objects standing in certain relations. We would like to define one or more additional objects. Suppose our definition provides truth conditions for every atomic predication concerned with the desired "new" objects in the form of some statement concerned solely with the "old" objects with which we began. Thus, every atomic question about the "new" objects receives an answer in terms that are solely about the "old" objects. Then, according to our liberal view, the definition is permissible. This idea can be applied not only to sets but also to mereological sums (as we saw in Section 5.8), cardinal numbers, and so on. For example, the cardinal numbers of two pluralities are identical if and only if the pluralities are equinumerous.[5]

Our liberal view of definitions is an explication of a theme that one often encounters in mathematicians' own reflections on their practice. A striking example is the following passage by Cantor.

> Mathematics is in its development entirely free and only bound in the self-evident respect that its concepts must both be consistent with each other and also stand in exact relationships, ordered by definitions, to those concepts which have previously been introduced and are already at hand and established. [...] [T]he *essence* of *mathematics* lies precisely in its *freedom*. (Cantor 1883, 896)

[4] In fact, the right-to-left direction of (i) follows from the plural indiscernibility principle (Indisc) introduced in Section 2.4.

[5] See discussion in Linnebo 2018, Section 3.3.

A similar sentiment is expressed by other mathematicians, such as David Hilbert and Henri Poincaré. Hilbert writes:

> As long as I have been thinking, writing and lecturing on these things, I have been saying [...]: if the arbitrarily given axioms do not contradict each other with all their consequences, then they are true and the things defined by them exist. This is for me the criterion of truth and existence.
>
> (Letter to Frege of 29 December 1899, in Frege 1980, 39–40)

According to Poincaré, "[m]athematics is independent of the existence of material objects; in mathematics the word 'exist' can have only one meaning; it means free from contradiction" (1905, 1026).

Let us apply our liberal view of definitions to the case of sets. It is instructive to compare with the situation where the desired set is specified intensionally, by means of a membership condition. Again, we start with some objects *dd* accepted by both parties. A more extreme proponent of liberal definitions may wish to define sets of the form $\{x : \varphi(x)\}$, where any parameters in the membership condition $\varphi(x)$ are drawn from *dd*. As before, she does not insist that these sets be among *dd*; they may be "new". The opponent will rightly challenge her to provide an account of what it takes for "new" sets to be identical or to have certain elements. Given the intensional specification of the desired sets, her answers will be as follows:

(i′) $\{x : \varphi(x)\} = \{x : \psi(x)\}$ if and only if $\forall x(\varphi(x) \leftrightarrow \psi(x))$
(ii′) $y \in \{x : \varphi(x)\}$ if and only if $\varphi(y)$

These answers are potentially problematic in a way that their extensional analogues, (i) and (ii), are not. An interesting example is the attempt to define a set $a = \{x : x \in x\}$. If this definition is to succeed, there must be an answer to the question of whether a is an element of itself. But the only answer we receive from clause (ii′) is that $a \in a$ if and only if $a \in a$. Of course, this is useless.[6] More tellingly, the answer is not stated in terms of the objects accepted by both parties to the dispute. An atomic question about the "new" object a receives an answer that essentially involves *this very object*; there is no reduction to the "old" objects among *dd*.

[6] It could be worse. When we ask whether the Russell set $b = \{x : x \notin x\}$ is an element of itself, we receive an inconsistent answer.

Notice that it is of no avail for the extreme liberal to allow a to lie outside of dd, that is, in our parlance, to be "new". The set a is specified intensionally, by means of the membership condition '$x \in x$', and we cannot "outrun" this specification. Even in a domain that strictly extends dd, a is, by definition, the set of all and only the objects that satisfy the condition '$x \in x$'. By contrast, when a set is specified extensionally by means of a plurality xx, it does help to consider a domain that strictly extends dd. Even if xx are, say, all the sets among dd that are not elements of themselves, xx need not satisfy this plural description in an extended domain. For xx are tracked rigidly into the extended domain, not by means of the description. This makes the world safe for the desired set $\{xx\}$, provided that the set is located outside of dd. Notice also the striking parallelism with the case of web page design. Suppose you want a web page to link to all and only the members of some collection of web pages, for example, the collection of web pages that do not link to themselves. If the target collection is specified intensionally, it is of no avail to create a *new* web page: you cannot "outrun" this problematic specification. By contrast, if the collection is specified extensionally, there is no obstacle to the creation of the desired web page.

The picture that emerges is that there is a fundamental difference between the proposed definitions of sets depending on whether the target is specified extensionally or intensionally. In the former case, every atomic question about the "new" objects receives an answer expressed solely in terms of the "old" objects, whereas in the latter case, this kind of reduction is often unavailable. The proposed definitions are therefore often unacceptable when the target is specified intensionally. In the case of an extensional specification, on the other hand, a proponent of liberal definitions is in a much stronger position. She has laid out certain definitions, which are mathematically fruitful and have the desirable property that all atomic questions about the "new" objects receive answers in terms that are acceptable to her opponent. Granted, she cannot force her opponent to accept the proposed definitions: he does not contradict himself when he rejects them. But she can justifiably accuse her opponent of dogmatism that stifles scientific progress. He dogmatically clings to certain beliefs that stand in the way of fruitful mathematics. By insisting that dd are all-encompassing—and thus that there can be no "new" objects outside of dd—he privileges certain metaphysical or logical dogmas over good mathematics.

12.4 Why plural comprehension has to be restricted

In the previous chapter, we defended the permissibility of absolute generality. And we have just argued that any given objects can be used to define a set. Thus, we have defended two of the three assumptions that we know to form an inconsistent triad. We therefore have no alternative but to reject the third assumption, that is, to restrict the plural comprehension scheme.

Here is an intuitive and more direct version of our argument for that conclusion. To define a plurality, we need to circumscribe some objects. But when we circumscribe some objects, we can use these objects to define yet another object, namely their set, in a way that would not be possible were the objects in question not circumscribed. And since yet another object can be defined, it follows that the circumscribed objects cannot have included *all* objects. Thus, reality as a whole cannot be circumscribed: there is no universal plurality. Consequently, the plural comprehension scheme needs to be restricted.

It might be objected that traditional plural logic is so compelling that the correct response to our findings is not to reject it but to reconsider our defenses of absolute generality and the view that every plurality can be used to define a set. This response deserves a hearing. So let us explain why we believe it is appropriate to reject traditional plural logic. First, we have, as already mentioned, offered positive arguments for the two other assumptions that make up the inconsistent triad.

Second, we have identified major difficulties with each of the two other responses to the inconsistent triad. Both relativism and traditional absolutism suffer from a serious expressibility deficit—the former because of its relativism, the latter because it is pushed up through the type-theoretic hierarchy. Our alternative solution avoids these expressibility problems. Without a universal plurality, Plural Cantor no longer entails Plural Profusion. And without Plural Profusion, we can no longer prove the Ascent Theorem, which appeared to show that the need for a generalized semantics forces us higher and higher up in the type-theoretic hierarchy. In fact, even if one accepts a type-theoretic hierarchy despite not being *forced* up it, there is a way to restore full expressibility, as we explained in Section 11.7.

Ultimately, though, we believe the debate must be decided by theoretical considerations. Which of the three horns of the trilemma is theoretically most satisfying? We hope this book as a whole will show that the widely ignored third horn has major theoretical attractions, which very likely exceed those of its two rivals.

Let us compare our third alternative—*critical (generality) absolutism*, as we shall call it—with its two rivals. We begin by reminding ourselves of how the three types of view respond to the inconsistent triad.

type of view	universal singularization	absolute generality	unrestricted plural comprehension
traditional absolutism	✗	✓	✓
relativism	✓	✗	✓
critical absolutism	✓	✓	✗

Just like the traditional absolutist, our view accepts that our quantifiers can achieve a form of absolute generality. We differ from the traditional absolutist only in our insistence that absolute generality is generality over an extensionally *in*definite domain, which consequently does not sustain unrestricted plural comprehension.

More nuanced comparisons are possible as well. Although we deny that reality as a whole can be circumscribed, there are also restricted domains of quantification that are extensionally definite and can thus be specified as a plurality. Let dd be one such domain. Consider a condition $\varphi(x)$ that has an instance among dd. We can then use a plural separation principle to define some objects xx that are all and only the objects among dd that satisfy $\varphi(x)$.[7] This shows that the unrestricted plural comprehension scheme is valid whenever the domain of quantification is extensionally definite. Instead, an extensionally definite domain does not permit a universal singularization within this domain. For recall that our argument for the permissibility of singularizing pluralities as sets forces us out of any given extensionally definite domain. So universal singularization fails when all our quantifiers are relativized to such a domain. The following table provides a summary of these observations:

type of domain	universal singularization	unrestricted plural comprehension
extensionally definite	✗	✓
extensionally indefinite	✓	✗

[7] See Appendix 10.A for a justification.

This table enables some illuminating comparisons of our view with its two main rivals. First, when a domain is extensionally definite, the correct view on plural comprehension and universal singularization is precisely that of traditional absolutism. This traditional view is entirely correct—when relativized to any given extensionally definite domain. Its only error is the assumption that reality as a whole can be circumscribed, that is, that there is a universal plurality. *Modulo* this single—though important—error, we are in agreement with the traditional absolutist.

Second, consider relativism. Of course, we disagree with the relativists on the important question of the possibility of absolute generality. But there is something more interesting to be said. There are, as noted, many extensionally definite domains, namely those that can be specified as a plurality. Suppose we restrict our attention to generality over such domains. Thus restricted, the relativist's claim is right: given any extensionally definite domain, there is indeed an even larger such domain. But the relativists fail to appreciate the important distinction between domains that are extensionally definite and those that are not. They appear tacitly to assume that all domains are extensionally definite. It is only restricted to such domains that their extendability claim has force, as argued in Sections 12.2 and 12.3. Moreover, by ignoring the possibility of extensionally *in*definite domains, they inflict upon themselves a gratuitous expressibility deficit, which we avoid by accepting generality over an extensionally indefinite domain of absolutely everything.[8]

12.5 The principles of critical plural logic

By advocating a restriction of the plural comprehension scheme, we depart from the traditional formulation of plural logic. To emphasize this departure, let us call our approach *critical plural logic*.[9]

How, exactly, does our critical plural logic differ from the traditional version? We accept standard classical first-order logic. Furthermore, we allow the plural quantifiers to be governed by axioms and rules analogous to

[8] To be fair, many relativists achieve a form of absolute generality through schematic generality, as discussed in Section 11.4. But this form of generality is either subject to its own expressibility deficit (by permitting only Π_1-generalizations) or will be transformed into a version of our preferred form of absolute, but extensionally indefinite, generality.

[9] This label is inspired by Charles Parsons's 'Infinity and a Critical View of Logic' (2015). Some examples of this approach to logic are discussed in a forthcoming special issue of *Inquiry* edited by Mirja Hartimo, Frode Kjosavik, and Øystein Linnebo.

those governing the first-order quantifiers.[10] We also retain the axiom stating that pluralities are non-empty and the axiom scheme stating that coextensive pluralities are indiscernible (see Section 2.4). Our quarrel with traditional plural logic concerns only the question of what pluralities there are, or, in other words, the question of which plural comprehension axioms to accept. It is therefore incumbent on us to clarify what pluralities we take there to be. It is insufficient merely to observe that the plural comprehension scheme needs to be restricted in some way or other to avoid a universal plurality. We need some "successor principles" to the unrestricted plural comprehension scheme to tell us what pluralities there in fact are.

How should these successor principles be chosen and motivated? When discussing this question, it is useful to recall the intuitive version of our argument for restricting the plural comprehension scheme. To define a plurality, we need to circumscribe some objects. But when we circumscribe some objects, we can use these objects to define yet another object, namely their set. It follows that the circumscribed objects cannot have included *all* objects and thus, in particular, that reality as a whole cannot be circumscribed.

Clearly, this argument hinges on the idea that every plurality is circumscribed, or, as we also put it, *extensionally definite*. Can this notion of extensional definiteness guide our search for successor principles and help us justify, or at least motivate, the resulting principles? Here we face a fork in the road, depending on whether or not we attempt to provide an analysis of extensional definiteness in more basic terms, and on this basis, try to provide the requisite guidance and justification.

There have been several attempts to provide such an analysis. Linnebo 2013 proposes a modal analysis inspired by Cantor's famous distinction between "consistent" and "inconsistent" multiplicities. Here is how Cantor explains the distinction in a famous letter to Dedekind of 1899:

> [I]t is necessary . . . to distinguish two kinds of multiplicities (by this I always mean definite multiplicities). For a multiplicity can be such that the assumption that all of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I call *absolutely infinite* or *inconsistent multiplicities* . . . If on the other hand the totality of the elements of a multiplicity can be thought of without contradiction as 'being together', so that they can be gathered together into 'one thing', I call it a *consistent multiplicity* or a 'set'. (In Ewald 1996, 931–2)

[10] But, of course, we should insist that the formulation of logical rules be neutral with respect to which comprehension axioms are validated.

Using the resources of modal logic, it is relatively straightforward to formalize Cantor's notion of a multiplicity being "one finished thing", namely, that all possible members of the multiplicity can exist or "be together". Or, changing the idiom slightly, there is no possibility of the multiplicity gaining yet more members at more populous possible worlds.[11] Based on this analysis, Linnebo 2013 proves various principles of extensional definiteness, which in the present context amount to principles concerning the existence of pluralities.

Another analysis of extensional definiteness is inspired by Michael Dummett's suggestion that a domain is definite just in case quantification over this domain obeys the laws of classical logic, not just intuitionistic.[12] Intriguingly, it turns out that a fairly natural development of this Dummettian suggestion validates almost the same principles of extensional definiteness as the modal analysis.[13] Yet other analyses may be possible as well. We invite the readers to explore.

Here we wish to pursue the other fork in the road, namely to leave the notion of extensional definiteness unanalyzed and instead to use our intuitive conception of the notion, coupled with abductive considerations, to motivate principles of extensional definiteness. This strategy has both advantages and disadvantages: it is more general, as it avoids specific theoretical commitments; but it also provides less leverage and thus less of an independent check on the proposed principles of definiteness. In any case, we believe this is an option worth exploring. We thus ask what it is for a collection to be circumscribed or extensionally definite.

First, since every single object can be circumscribed, there are singleton pluralities:

$$\forall x \exists yy \forall z (z \prec yy \leftrightarrow z = x)$$

Second, because the result of adding one object to a circumscribed plurality is also circumscribed, we accept a principle of adjunction. Given any plurality xx and any object y, we can adjoin y to xx to form the plurality $xx + y$ defined by:

$$\forall u (u \prec xx + y \leftrightarrow u \prec xx \vee u = y)$$

[11] See pp. 248–9 for an explication of this idea in a modal language.

[12] A closely related idea is found in Solomon Feferman's widely circulated and discussed manuscript, "The Continuum Hypothesis is neither a definite mathematical problem nor a definite logical problem" (Feferman unpublished).

[13] See Linnebo 2018.

Moreover, we already argued that a plural separation principle is well motivated (see Appendix 10.A). Suppose you have circumscribed a collection and have formulated a sharp distinction between two ways that members of the collection can be. Then the subcollection whose members are all and only the objects that lie on one side of this distinction is in turn circumscribed. More formally, given any plurality xx and any condition $\varphi(x)$ that has an instance among xx, there is a plurality yy of those members of xx that satisfy the condition:

$$\exists x(\varphi(x) \land x \prec xx) \rightarrow \exists yy \forall u(u \prec yy \leftrightarrow u \prec xx \land \varphi(u))$$

Next, there are some plausible union principles. Let us begin with a simple case. Since two circumscribed collections can be conjoined to make a single such collection, a principle of pairwise union is plausible. Given any plurality xx and any objects yy, there is a union plurality zz defined by:

$$\forall xx \forall yy \exists zz \forall u(u \prec zz \;\leftrightarrow\; u \prec xx \lor u \prec yy)$$

A generalized union principle can also be motivated. Consider some circumscribed collections, each with its own unique tag. Suppose that the collection of tags is also circumscribed. Then the "union collection" comprising all the items that figure in at least one of the tagged collections is circumscribed. This motivates a generalized union principle to the effect that the union of an extensionally definite collection of extensionally definite collections is itself extensionally definite. We can formulate this as the following schema. Suppose there are xx such that:

$$\forall x(x \prec xx \rightarrow \exists yy \forall z(z \prec yy \leftrightarrow \psi(x,z)))$$

Then there is zz such that:

$$\forall y(y \prec zz \leftrightarrow \exists x\,(x \prec xx \;\land\; \psi(x,y)))$$

Although the generalized union principle does not, on its own, entail the pairwise one, this entailment does go through in the presence of the singleton and adjunction principles.[14] It therefore suffices to adopt the generalized union principle.

[14] *Proof sketch*. Consider two pluralities xx and yy. Assume there are two distinct objects, say a and b, to tag these pluralities. (If there is only a single object, the pairwise union of xx and yy is a singleton plurality.) Now apply the generalized union principle to the formula '$(x = a \land y \prec xx) \lor (x = b \land y \prec yy)$', observing that a and b form a plurality. This yields the pairwise union of xx and yy.

The principles accepted so far do not entail the existence of any infinite pluralities; indeed, they have a model where every plurality is finite. Is it possible for an infinite collection to be circumscribed and thus to correspond to a plurality? This question calls to mind the ancient debate about the existence of completed infinities. Aristotle famously argued that only finite collections can be circumscribed, and that a collection can be infinite only in the potential sense that there is no finite bound on how many members the collection might have. This remained the dominant view until Cantor boldly defended the actual infinite and the existence of completed infinite collections. The natural numbers provide an example. Aristotle denied, whereas Cantor affirmed, the existence of a completed collection of all natural numbers.

We are interested in an analogous question concerning pluralities. Let '$P(x, y)$' mean that x immediately precedes y. Following first-order arithmetic, we accept that every natural number immediately precedes another:[15]

$$\forall x \exists y\, P(x, y) \tag{12.1}$$

We would like to know whether there is a circumscribed collection, or plurality, of all natural numbers. More precisely, we would like to know whether there are some objects xx containing 0 and closed under P, in the following sense:

$$\exists xx(0 \prec xx \land \forall x \forall y(x \prec xx \land P(x, y) \rightarrow y \prec xx)) \tag{12.2}$$

Although asserting the existence of such a plurality is a substantial step, it has also been a tremendous theoretical success, as mathematics since Cantor has clearly demonstrated. On abductive grounds, we therefore recommend accepting (12.2), conditional on (12.1), as a plural analogue of the set-theoretic axiom of Infinity.

It will be objected that this conditional principle is concerned specifically with the natural numbers and thus lacks the topic neutrality of a logical law. The objection is entirely reasonable and points to the need for a more general principle that justifies transitions such as the one from (12.1) to (12.2). There is nothing special about 0 and the functional relation P. So, for *any* plurality xx and functional relation, there should be a plurality yy

[15] Aristotle would only accept a weaker, modal analogue of this principle, namely $\Box\forall x \Diamond \exists y\, P(x, y)$, where the modal operators represent metaphysical modalities.

containing xx and closed under that function. We therefore claim that the desired generalization is the schematic principle that every plurality can be closed under function application:

$$\forall x \exists! y\, \psi(x,y) \rightarrow \forall xx \exists yy(xx \preccurlyeq yy \wedge \forall x \forall y(x \prec yy \wedge \psi(x,y) \rightarrow y \prec yy)) \tag{12.3}$$

We adopt this as the official plural principle of infinity. In practice, however, it doesn't much matter whether we accept this more general schematic principle or merely (12.2), conditional on (12.1). For in the presence of first-order arithmetic, ordered pairs, and the other principles concerning pluralities, these two principles of infinity are provably equivalent.[16]

A plural analogue of the axiom of Replacement is plausible as well. Consider a plurality of objects. Now you may replace any member of this plurality with any other object, or, if you prefer, leave the original object unchanged. Then the resulting collection is also circumscribed and thus defines a plurality of objects. We formalize this as follows.

$$\forall xx[\forall x(x \prec xx \rightarrow \exists! y\, \psi(x,y)) \rightarrow \exists yy \forall y(y \prec yy \leftrightarrow \exists x(x \prec xx \wedge \psi(x,y)))]$$

It is pleasing to observe that this plural version of Replacement follows from the generalized union principle and the singleton principle. And, as in the case of sets, the plural principle of replacement entails that of separation.[17]

To sum up, we started with some core assumptions shared with traditional plural logic: first-order logic, axioms and rules governing the plural quantifiers, and the principles (Non-empty) and (Indisc). Next, our intuitive

[16] *Proof sketch.* The only hard direction is to show that the specific conditional entails the general one. Consider any xx, and assume that ψ is functional. For every member $a \prec xx$, we contend that there is a plurality zz_a containing a and closed under ψ. Given this contention, the generalized union principle enables us to define the desired plurality yy as the union of all the pluralities zz_a. To prove the contention, we observe that, using ordered pairs and plural quantification, we can produce a formula $\theta(n,y)$ which expresses that n is a natural number and that y is the nth successor of a in the series generated by ψ. We do this by letting $\theta(n,y)$ state that $\langle n,y\rangle$ is a member of every plurality containing $\langle 0,a\rangle$ and closed under the operation $\langle m,u\rangle \mapsto \langle m+1,v\rangle$, where v is the unique object such that $\psi(u,v)$. Now we apply the generalized union principle to the plurality of all natural numbers and the formula θ to obtain the desired plurality zz_a.

[17] *Proof sketch.* Consider xx and a condition $\varphi(x)$. Assume $\varphi(a)$ for some member a of xx. Now apply the principle of replacement to the condition $\psi(x,y)$ defined as $(\neg\varphi(x) \wedge y = a) \vee (\varphi(x) \wedge y = x)$. This yields the subplurality of those members of xx that satisfy $\varphi(x)$.

conception of extensional definiteness motivates the following three principles concerning pluralities:

- singleton
- adjunction
- generalized union

An additional principle receives a more theoretical justification:

- infinity

These four principles, in addition to the core assumptions just mentioned, constitute the system we call *critical plural logic*.

As observed, the first three of these principles entail some other plausible principles:

- separation
- pairwise union
- replacement

Moreover, it is straightforward to verify that each principle of critical plural logic can be derived from traditional plural logic. In essence, each of the pluralities we licence is a subplurality of the universal plurality licenced by traditional plural logic. Critical plural logic is therefore strictly weaker than the traditional system. This relative weakness is for a good cause, as will emerge clearly in Section 12.7, where we explore the connection between critical plural logic and set theory. This connection is far simpler and, we believe, more natural than in the case of traditional plural logic.

12.6 Extensions of critical plural logic

When stronger expressive resources are accepted, various extensions of critical plural logic can be formulated and justified. The addition of superplural resources provides an obvious example. This addition enables us to express analogues of the principles of critical plural logic. Here we will focus on two novel and more interesting principles.

First, we can formulate a principle of extensional definiteness that corresponds to the familiar set-theoretic axiom of Powerset. We can do this entirely without mention of sets by using superplurals. For any plurality *xx*, there is a superplurality *yyy* of all subpluralities of *xx*:

$$\forall xx \exists yyy \forall zz(zz \prec yyy \ \leftrightarrow \ zz \preccurlyeq xx)$$

The justification for this "powerplurality" principle is less straightforward than in the case of the earlier principles. It relies on what Bernays (1935) calls "quasi-combinatorial" reasoning: a combinatorial principle that is compelling for finite domains is extrapolated to infinite domains. The powerplurality principle is certainly reasonable when the plurality *xx* is finite: we can then list all of its subpluralities. The general principle is a big and admittedly daring extrapolation of the finitary principle into the infinite. Its justification is thus partially abductive: the big and daring extrapolation has proved to be a theoretical success. Just like its set-theoretic analogue, the principle fits into a coherent and fruitful body of theory, as will be explained shortly. The principle also provides important information about which superpluralities there are.

Second, superplurals make it possible to formulate plural choice principles. For example, given a superplurality *xxx* of non-overlapping pluralities, there is a "choice plurality" whose members include one member of each plurality of *xxx*. That is, for each such *xxx* we have:

$$\exists yy \forall zz(zz \prec xxx \ \rightarrow \exists ! y(y \prec zz \ \wedge \ y \prec yy))$$

As in the case of the powerplurality principle, plural choice principles are extrapolations from the finite into the infinite, and their is justification is partially abductive.[18]

In sum, the addition of superplural resources enables us to formulate and justify an extended critical plural logic. Two distinctive principles are:

- powerplurality
- choice

[18] See Pollard 1988 for a defense of the Axiom of Choice on the basis of a plural choice principle. If ordered pairs are available, there is less of a need for superplurals to express choice principles. For example, we can assert that for any relation coded by means of a plurality of ordered pairs, there is a functional subrelation with the same domain, again coded by means of a plurality of ordered pairs.

Of course, yet stronger principles can be countenanced as ever greater expressive resources are considered.

12.7 Critical plural logic and set theory

The various plural principles we have discussed provide valuable information about sets. To see this, recall the correspondence we have advocated between pluralities and sets:

(i) $\{xx\} = \{yy\}$ if and only if $xx \approx yy$
(ii) $y \in \{xx\}$ if and only if $y \prec xx$

Using this correspondence, the plural principles entail analogous set-theoretic axioms.

However, there are two reasons to worry that the plural principles will not lead to Zermelo-Fraenkel set theory. First, since we do not ordinarily admit an empty plurality, there is a threat of losing the empty set. Some ways to address this threat were discussed in Section 4.4. One solution is to allow an empty plurality. Another is to allow the "set of" operation $xx \mapsto \{xx\}$ to be what Oliver and Smiley (2016, 88) call a "co-partial" function, which can thus take the value $\varnothing$ on an undefined argument. Either way, we can prove the existence of an empty set.

Second, since plural logic is applied to all sorts of objects, the mentioned correspondence introduces impure sets, that is, sets of non-sets. The relevant comparison is therefore not ZFC, but ZFCU—the modified system which accommodates urelements (see Section 4.7). Recall that this system is obtained by making explicit the quantification over sets in the axioms of ZFC. Whenever a quantifier of an axiom of ZFC is intended to range over sets even when urelements are introduced, we explicitly restrict this quantifier to sets by means of a predicate 'S' intended to be true of all and only sets.

Our aim, then, is to use critical plural logic and the correspondence principles (i) and (ii) to justify axioms of ZFCU. We define '$S(x)$' as '$\exists xx(x = \{xx\})$'. This enables us, it turns out, to derive the axioms of Empty Set, Pairing, Separation, Union, Infinity, and Replacement. (The proofs are relatively straightforward.) Moreover, the axiom of Extensionality follows immediately from the correspondence between pluralities and sets, and

Foundation can be seen as explicating how sets are successively formed from pluralities of elements, and as justified on that basis.[19]

To derive the axioms of Powerset and Choice, we need to go beyond critical plural logic. Choice follows naturally from the superplural choice principle discussed in the previous section. Deriving Powerset is less straightforward. Given any set a, we want to prove the existence of its powerset. To do so, we need to show that there is a plurality comprising all of a's subsets. How might this be done? One option, inspired by the iterative conception of set, is to postulate the existence of such a plurality, on the grounds that when a was formed, all its elements were available, thus giving us the ability also to form all of a's subsets. We prefer to utilize the powerplurality principle of the previous section, reasoning as follows. Let aa be the elements of a, and consider their superplurality bbb. For every subset x of a, if $x = \{xx\}$ for some xx, then $xx \prec bbb$. That is, bbb circumscribe all the subpluralities of aa. But if some pluralities are jointly circumscribed, so are the unique sets formed from precisely these pluralities. This gives us the desired plurality of subsets of a. (This reasoning assumes that the extended, superplural logic contains a replacement principle that allows us to replace each plurality of a superplurality with a unique object and thus arrive at a plurality.)

Our discussion shows that critical plural logic, and the plausible superplural extensions thereof, have great explanatory power, especially in connection with the correspondence principles (i) and (ii). Still, one might worry that things are too good to be true. Do we even know that our assumptions—the mentioned plural logics and the correspondence principles—are jointly consistent? This worry can be put to rest by proving that these assumptions are consistent relative to ZFC. For critical plural logic and the correspondence principles, we do this by translating plural quantifiers as first-order quantifiers restricted to non-empty sets. An analogous relative consistency result can be given for the described extension of critical plural logic. In that case, superplural quantifiers are translated as first-order quantifiers restricted to non-empty sets of non-empty sets.

[19] Relative to the other axioms of ZFC, Foundation is equivalent to the following induction scheme:

> Suppose that every urelement is φ and that, for every xx each of which is φ, $\{xx\}$ too is φ. Then everything is φ.

This induction scheme explicates the idea that every set is generated by means of the "set of" operation.

Let us end with some more general observations. First, on the view we have defended, plural logic lacks one of the features commonly ascribed to pure logic, namely epistemic primacy vis-à-vis all other sciences (see Section 2.5). To see this, we need only recall the extent to which our defense of critical plural logic relies on abductive considerations, in particular, on considerations about what constitutes a permissible mathematical definition. Moreover, some of the principles of critical plural logic—infinity, powerplurality, and choice—specifically received an abductive justification.

Second, our view forges a close connection between the principles of critical plural logic and the axioms of set theory, which suggests that critical plural logic and its extensions have non-trivial mathematical content. Let us explain. We have provided a factorization of set theory into two components: the correspondence principles, which link pluralities and their corresponding sets, and critical plural logic, which provide information about what pluralities there are and how these behave. Clearly, the strong mathematical content of set theory derives from these two components. It is the correspondence principles that introduce sets as mathematical objects by characterizing what Gödel called the the "set of" operation (see Section 4.6). What sets there are, however, will depend on what pluralities are "fed into" this operation and is determined in large part by the plural logic that is brought to bear. We can study this dependence by keeping the correspondence principles fixed, while varying the plural logic to which they are applied. As just observed, our extended critical plural gives rise to full Zermelo-Fraenkel set theory. If we remove the plural principle of infinity, the result is a comparatively weak theory of hereditarily finite set. Alternatively, suppose we retain that plural principle of infinity but impose a predicativity requirement on the generalized union principle (and thus also plural replacement and separation).[20] Then a broadly predicative set theory ensues. In short, when we keep the correspondence principles fixed but vary the plural logic, we obtain set theories with wildly different mathematical content. This observation strongly suggests that some of the mathematical content of the resulting set theory derives from the plural logic to which the correspondence principles are applied, not solely from these principles. If this is correct, it follows that a theory can have substantial mathematical content without any commitment to mathematical objects.

[20] Specifically, we require that the formula $\psi(x, y)$ be predicative, in the sense that it contain no bound plural variables.

To come to terms with the possibility of mathematical content even in the absence of mathematical objects, it is useful to recall Bernays's notion of quasi-combinatorial reasoning, whereby principles that are compelling in finite domains are extrapolated to infinite ones. Bernays and others regard such reasoning as distinctively mathematical and a major watershed in the foundations of mathematics, marking the onset of serious infinitary reasoning. Since critical plural logic and its extensions embody, and are motivated by, such reasoning, Bernays would regard both the notion of a plurality and the principles of critical plural logic as distinctively mathematical in character. This is particularly clear for the plural principles of infinity, powerplurality, and choice, whose justification explicitly relied on quasi-combinatorial reasoning.

Is the mathematical content of plural logic compatible with our view that pluralities can be used to explain sets? We believe it is. The explanation in question is a broadly metaphysical one: we make sense of a set $\{xx\}$ as "formed" from its elements xx. There is no conflict between this explanation and the view that plural logic has non-trivial mathematical content. Indeed, on this view, the indisputable mathematical content of set theory is in part inherited from that of plural logic.[21]

Finally, the view that logic can have mathematical content has important consequences concerning how we choose a "correct" logic. Some starkly different views are found in the literature. At one extreme we find Frege, who claims that logic codifies "the basic laws" of all rational thought, and the laws of logic must therefore be presupposed by all other sciences. He writes:

> I take it to be a sure sign of error should logic have to rely on metaphysics and psychology, sciences which themselves require logical principles. (Frege 1893/1903, xix)

This "logic first" view has been very influential. Following Frege, logic is often regarded as epistemologically and methodologically fundamental. All disciplines, including mathematics, are answerable to logic rather than vice versa.

At the opposite extreme we find Quine, whose radical holism leads him to assimilate logic and mathematics to the theoretical parts of empirical science. Logic and mathematics, he claims, are not essentially different

[21] Thanks to Hans Robin Solberg for raising this concern.

from theoretical physics: although they go beyond what can be observed by means of our unaided senses, they are justified by their contribution to the prediction and explanation of states of affairs that *can* be observed.

These extremes are not the only views, however. In particular, one need not be a radical holist to reject the Fregean logic-first view. What are sometimes called "critical views of logic" represent a less dramatic departure from Frege.[22] These views hold that the logical principles governing some subject matter may depend on features of this subject matter or of our discourse about it. The views thus stop short of Quine's radical holism and emphasize instead a more local entanglement of logic with some particular discipline, such as mathematics, semantics, or some part of metaphysics. As a result of this entanglement, logic is answerable to one's views in this other discipline.

The revision of plural logic that we have defended provides a good example of such a critical view of logic. Avoiding any commitment to Quinean holism, we have argued that the principles of plural logic are entangled with our theory of correct mathematical definitions. Specifically, we have defended a liberal theory of mathematical definitions, and on the basis of this theory, we have argued that plural comprehension needs to be restricted more than has traditionally been assumed.

12.8 Generalized semantics without a universal plurality?

We wish to address an open question that, despite not being directly about plurals, is nevertheless relevant to the view we have defended in this chapter and the previous one.

The question concerns how semantics should be done if we adopt critical plural logic. When a language quantifies over an extensionally definite domain, the answer is straightforward: we may as well employ the usual set-based model theory, since all of the relevant constructions, relative to this domain, can be done in set theory. But what about languages that quantify over an extensionally *in*definite domain, such as the important domain of absolutely everything? For such languages, plural logic is no better off than set theory for the purposes of developing a generalized semantics; after all, we have argued that there is no plurality corresponding to the domain of absolutely everything. What to do?

[22] See footnote 9 on p. 278.

Our recommendation is to use Fregean concepts to do the job previously done by pluralities. Instead of giving a plurality-based semantics, we should give a second-order semantics based on Fregean concepts, as explained in Section 7.5. Of course, this means that we must accept enough Fregean concepts to serve the needs of semantics; in particular, we need a universal concept to serve as our absolutely unrestricted domain of quantification.

While promising, this strategy raises some hard follow-up questions, two of which are particularly pressing. First, we argued that every plurality defines a set, which forced us to restrict plural comprehension (see Sections 12.3 and 12.4). Can an analogous argument be given that every Fregean concept defines some sort of object, thus forcing us to restrict second-order comprehension? If so, this might imperil our strategy of using higher-order logic to develop a generalized semantics. Second, we argued that traditional plural logic, when combined with absolute generality and a desire for a generalized semantics, forces us to ascent to higher and higher levels of the plural hierarchy (see Section 11.5). This gives rise to an expressibility deficit akin to that which afflicts generality relativism. Does our use of second-order logic, combined with the same assumptions, force an analogous ascent in the conceptual hierarchy, resulting in an analogous expressibility deficit?

Let us take the questions in order. Concerning the first, recall that our argument that every plurality defines a set relies essentially on the extensional definiteness of pluralities. Without the assumption of extensional definiteness, our liberal view of definitions would not licence the relevant definitions. Suppose we wish to use some sort of collection to define a set whose elements are precisely the members of this collection. We showed that when the collection is extensionally definite—as any plurality is—the assumptions of our liberal view are satisfied and the attempted definition succeeds. But when the collection fails to be extensionally definite—as is often the case with Fregean concepts—the assumptions are not satisfied and the definition can be dismissed as illegitimate. Thus, our argument does not extend from pluralities to Fregean concepts. As far as the views defended in this book are concerned, it is wide open which Fregean concepts, if any, define corresponding objects and what the resulting restriction on second-order comprehension, if any, would have to be.

The second question, concerning an ascent into the conceptual hierarchy and a resulting expressibility deficit, is harder. As discussed, we wish to retain absolute generality and to develop a generalized semantics. Suppose that we also accepted traditional second-order logic, with its unrestricted second-order comprehension scheme. Then all the assumptions of the argument developed in Section 11.5 would be satisfied, and we would thus be forced to ascend higher and higher in the conceptual hierarchy. Wouldn't this be unacceptable? Let us consider three increasingly ambitious responses.

The least ambitious response is to bite the bullet and admit the forced ascent into the conceptual hierarchy as well as the ensuing expressibility deficit. The resulting view would be on a par with generality relativism, which suffers from an analogous expressibility deficit. But the resulting view would at least have one advantage vis-à-vis traditional generality absolutism, namely that it retains full expressibility with respect to the plural hierarchy, accommodating the universal singularization provided by the transition from some things to their set. Even so, we do not find this option very appealing: it is much too close to the generality relativism that we sought to avoid.

A more ambitious response, if forced to ascend into the conceptual hierarchy, is to lift the veil of type distinctions along the lines explained in Section 11.7, so as to return to an untyped language that ensures full expressibility. We regard this strategy as promising but are mindful of the fact that it will be significantly harder to implement it in the case of the conceptual hierarchy than in the case of the plural one. The intuitive barrier to lifting the veil appears greater for the conceptual hierarchy than for the plural one. Perhaps this is because we are familiar with cumulativity in the latter case but not the former.

Moreover, in the conceptual case, we face the question of how to handle predication in the one-sorted language used to lift the veil. When the veil is lifted, all the predicates of the original typed language are subsumed under a single category. (Of course, predicates would still be distinguished according to their adicity.) But the one-sorted language will include predicates of its own, such as a generalized identity predicate and the application predicate 'η', both of which figured crucially in the translation into the one-sorted language described in Section 11.7. Presumably, these predicates too must have semantic values. But if so, we face a treacherous dilemma. Can these semantic values figure as values of the single sort of variables? If they can, then paradox will threaten, for example, by considering the interpretation of '*x* does not apply to itself'. If they cannot, we will have failed to restore full expressibility.[23]

The most ambitious response is to reject traditional second-order logic in favor of a more critical one that restricts the second-order comprehension scheme. This would mean that the assumptions on which the ascent phe-

[23] See Hale and Linnebo forthcoming for discussion of whether, and if so how, this dilemma can be resolved.

nomenon relies are no longer granted, with the result that the pressure to ascend ceases. What kind of restriction might be appropriate? The desired critical higher-order logic would have to balance two potentially opposing needs. It would have to be weak enough to block the ascent, while simultaneously being strong enough to serve the needs of generalized semantics. Although much remains to be investigated, there are indications that this balancing act might be doable.[24]

12.9 What we have learnt

This book has centered around the three overarching questions outlined in Chapter 1. We would like to end our discussion by revisiting those questions and recapitulating the answers we have developed throughout the book.

First there was:

> The legitimacy of primitive plurals
>
> Should the plural resources of English and other natural languages be taken at face value or be eliminated in favor of the singular?

As emphasized already in Chapter 1, different considerations pull in different directions. On the one hand, the success of set theory suggests that we may be able to dispense with the many: the "ones" provided by set theory suffice for our theoretical needs. On the other hand, considerations from natural language and the paradoxes appear to show that it is impossible consistently to eliminate every "many" in favor of corresponding "ones".

While natural language does indeed provide some evidence in favor of primitive plurals, we have argued that this evidence is not entirely conclusive (Chapter 2). We have even stronger reservations about the arguments from paradox (Chapter 3). The stronger of these arguments depend on both absolute generality and traditional plural logic. While we accept absolute generality, we have argued that this form of generality makes traditional plural logic problematic. However, this does not mean that we give up on primitive plurals. There is a significant though underappreciated reason in their favor: plural resources are of great value for the explanation of sets (Chapter 4). In short, while some of the well-known arguments for primitive

[24] Promising approaches are developed in Fine 2005a, Linnebo 2006, Schindler 2019, and work in progress by Sam Roberts.

plurals are less compelling than many philosophers think, a less well-known argument is quite powerful.

Once primitive plurals have been recognized, our second overarching question arises, namely:

> How primitive plurals relate to the singular
>
> What is the relation between the plural and the singular? We have been particularly interested in the circumstances under which many objects correspond to a single, complex "one" and whether any such correspondence can shed light on the complex "ones".

Attempts to answer these questions are constrained by a key fact:

> Plural Cantor
>
> For any plurality *xx* with two or more members, the subpluralities of *xx* are strictly more numerous than the members of *xx*.

Plural Cantor leads to a tricky trilemma (Chapter 11): we cannot simultaneously accept the possibility of absolute generality, the existence of universal singularizations, and traditional plural logic. Part IV explored the theoretical consequences of choosing among its three horns, which amounts to accepting one of the following three "package deals":

(i) generality relativism;
(ii) generality absolutism with traditional plural logic but no universal singularizations;
(iii) generality absolutism with critical plural logic and universal singularizations.

We have defended the third package based on three sets of considerations. To begin with, the first two options are afflicted with expressibility deficits. Moreover, if we restore full expressibility by lifting the veil of type distinctions, there is a compelling argument for abandoning traditional plural logic, as is done by the third option. Finally, this option is also supported by a plausible account of permissible definitions, according to which a mathematical object exists whenever an adequate definition of it can be provided.

Our third and final overarching question addresses the significance of plural logic as a tool:

The significance of primitive plurals

What are the philosophical and (more broadly) scientific consequences of taking plurals at face value?

The fact that primitive plural resources are available in thought and language is itself highly significant. Philosophers often make strong further claims on behalf of primitive plurals, in particular, that they:

(i) help us eschew problematic ontological commitments, thus greatly aiding metaphysics and the philosophy of mathematics;
(ii) ensure the determinacy of higher-order quantification;
(iii) require us to redo semantics in a way that uses primitive plurals not only in the object language but also in the metalanguage.

We have argued that these three claims are severely exaggerated. Plural quantification is not ontological innocent, at least not in the most interesting sense of this expression (Chapter 8). While it may not carry ontological commitments to *objects*, it does carry substantive commitments that can be precisely measured by means of a Henkin semantics. The same semantics reveals that plural logic, even on its plurality-based model theory, is not immune from non-standard interpretations and fails to secure a gain in expressive power.

Finally, employing primitive plurals in the semantics is not always required. When the domain is extensionally definite—which domains of natural language typically are—the domain can be represented either as a plurality or as its corresponding set. So wherever we can use plurals, we can also use sets (or, for that matter, individual sums). Linguists can therefore largely be acquitted of the charge of error. The only exception concerns certain uses of language of particular interest to philosophers, namely those where the domain is all-encompassing and thus, on our view, extensionally *in*definite. In such cases, we must look beyond the extensional resources of plural logic, set theory, and individual mereology.

Let us finally return to the question of whether plural logic is really “pure logic”. Three alleged features of pure logicality were identified in Section 2.5: topic-neutrality, formality, and epistemic primacy. We have no quarrel with the idea that plural logic is topic neutral, that is, that it is applicable to reasoning about any domain. However, we have argued that some domains are extensionally *in*definite and for that reason require critical rather than traditional plural logic. We focused on two components of formality. We grant that plural logic does not discriminate between objects, but we deny that it is ontologically innocent, at least in the most interesting sense of that term. Finally, does plural logic permit a special kind of epistemic primacy? A negative answer was defended in Section 12.5, where we argued that both the concepts of plural logic and the justification of its principles are entangled with set theory.

All in all, we hope to have shown that plural logical is a tool of great value and theoretical interest, both in its own right and for the hard questions it raises concerning the relation between the many and the one—in semantics, metaphysics, and the philosophy of mathematics. The value of this tool does not require that it qualify as "pure logic" in any robust sense.

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Index

absolute generality 3, 13, 26–8, 30, 41–2, 45–6, 61, 88, 90, 123–4, 136, 146–7, 150, 156–7, 169, 240–56, 262, 268–70, 276–8, 291, 293–4
  critical generality absolutism 277, 294
  generality relativism 3, 243–4, 246–7, 256, 268–9, 291–2, 294
  traditional generality absolutism 249, 253, 256, 269, 292, 294
  universal plurality *see* universal; plurality
Antonelli, Aldo 25
Aristotle 91, 282
arithmetic 23, 70n17, 130, 152–3, 157, 267, 282–3
ascent (semantic) 144, 240, 249–53, 255–6, 276, 291–3
  theorem 262–7

Barcan formula (bounded) 234
Barcan Marcus, Ruth 210
being constraint 220
Ben-Yami, Hanoch 175, 193n8, 196, 197n13
Bernays, Paul 47, 226, 228, 231, 285, 289
Black, Max 14, 60, 65
Boccuni, Francesca 25
Boolos, George 9, 11, 14–15, 22–6, 33, 36–7, 39n5, 70–1, 73–4, 88, 108, 125, 131–2, 138, 144, 151, 153, 157, 160, 163–5, 167, 213n15, 229n36, 268
Burgess, John 15n12, 26, 63, 210n10
Burnet, John 4

Cantor, Georg 2, 60, 62, 63n8, 164, 273, 279–80, 282
Cantor's theorem 3, 41, 45, 51
  higher-order 174, 254
  Plural; *see* Plural Cantor
  second-order 47
cardinality comparisons 41–4, 48
Cartesian dualists (example) 257–8; *see also* type distinctions; lifting the veil of
categoricity 28, 130, 152–3, 157, 168–9
Champollion, Lucas 13n8, 13n9, 18n15, 85n7
Chaves, Rui Pedro 196
circumscribed, properly 12n6, 20, 21n17, 61, 63–4, 72, 166, 206–7, 227, 240, 244, 256, 268, 276–83, 287; *see also* extensional definiteness
classes
  as many and as one 14, 55, 164
  proper 26, 70–2, 167
  theories of 146n14
collections (of a general sort) 9, 12–13, 57, 69–70, 72, 105, 154, 214–15, 267, 271, 275, 280–3, 291
  specified extensionally or intensionally 271–2
compactness 130, 135, 152–3, 173
completeness (logical) 130, 135, 137–8, 152–3
  of plural logic on the Henkin semantics 155, 172–3
completeness (mereological) 100–1, 103
compositionality, principles of 86, 176, 189, 196n11, 254
comprehension axioms
  predicative 43n9, 47–8, 49n14, 50–2, 166, 231, 245n5, 288
  impredicative 44n12, 47, 49–52, 166, 229n37, 231, 245n5
  modalized plural 249
  modalized second-order 116, 143n12
  plural second-order 143
  plural 19–20, 21n17, 23, 27, 39, 41, 43–4, 47–52, 61, 71, 81n5, 101, 103, 109, 117– 18, 123, 136, 146, 166–7, 171–2, 222–4, 226–7, 229–31, 237, 244–6, 254, 261–2, 267–9, 276–9, 290–2
  second-order 47, 106, 109, 115–16, 147
  set 57
concepts 104, 107–9, 111–19, 142–5, 147–9, 164, 250, 291
  higher-level 143

concepts (*cont.*)
non-extensionality of 113–15; *see also* predicates; undiscerning counterpart
non-rigidity of *see* non-rigidity; of predicates
plural 143, 154–7, 160, 165, 170–1, 250
rigid 116–18
contingentism *see* modality; contingentism
Continuum Hypothesis 152n2, 162, 280n12
conjunctive analysis 10, 182–5
correspondence (between sets and pluralities) 2, 27, 64, 163, 286–8
covers 188–93, 198
definition of 191
Cresswell, Maxwell John 220n25
criteria of identity 94, 99, 101, 206, 213n16

Davidson, Donald 13, 32, 89
Dedekind, Richard 263, 267, 279
definitions, liberal view of *see* liberal view of definitions
deflationist constraint on predication 148
determinacy (of plural quantification)
metaphysical 162, 165–7
semantic 4, 30, 151–62, 169, 295
direct method (of learning languages) 106–7, 179
domains (of quantification) 21n17, 22, 26–8, 40, 45, 47, 51, 56–7, 61, 63–4, 70, 79, 90, 93, 119, 126, 128–9, 131–6, 141, 146–7, 151–2, 154–6, 158–9, 161–3, 170, 240, 244, 249, 250–1, 256, 272, 275, 277–8, 280, 285, 290–1, 295 *see also* quantifiers
expansion 41–2, 73–4, 244–6, 248
extensionally definite and indefinite *see* extensional definiteness and indefiniteness
Dorr, Cian 25n26, 90n12, 181
Dummett, Michael 109–10, 269n2, 280

Edgington, Dorothy 209n8
eliminative reductions 1–3, 10–11, 13, 31, 33, 51, 59–60, 119, 145, 174, 293
in metaphysics 25–6, 94
of concepts in favor of pluralities 116–18
of mereological sums in favor of pluralities 90–2
of pluralities in favor of concepts 109–16
of pluralities in favor of mereological sums 87–8
of pluralities in favor of sets 60–4
of proper classes in favor of pluralities 70–2
of sets in favor of pluralities 64–6
empty plurality 63, 73, 133, 286
Enderton, Herbert B. 19, 259n20
entanglement (of logic with other disciplines) 3, 24n25, 290, 296
epistemic primacy 21, 23–5, 288, 295
events 13, 32, 89–90, 112–13, 115, 145
existence predicate 149, 205, 211
plural 220–1
expressive limitations 23, 28, 106, 169, 248, 254–5; *see also* inexpressibility
expressive power 20, 23–4, 26, 28, 123, 130, 135, 151–3, 168–9, 255, 257, 295
extendability argument 269–72
extensionality
generalized 253–4, 260
of pluralities 113, 115, 206, 210, 216, 218, 227, 229–30
of sets 57, 68–9, 211–14, 233, 286
extensional definiteness and indefiniteness 12n6, 21n17, 64n9, 163n7, 207, 226–30, 240, 244–5, 249–50, 256, 277–81, 284, 290–1, 295

Feferman, Solomon 229n37, 280n12
Filip, Hana 85n7
Fine, Kit 63n7, 92n16, 95n17, 106n4, 146, 196n11, 216n20, 248n8, 248n9, 269n2, 293n24
first-order logic 9, 15–16, 17n14, 19–23, 28, 76, 97, 104–5, 126, 128, 130, 135, 137–8, 144, 152, 168–9, 232, 278
Florio, Salvatore 9n*, 85n8, 115n11, 142n10, 144–5, 147, 151n*, 187, 196, 197n12, 197n13, 247, 251n11, 252n12
FOL *see* first-order logic
formality 21–4, 295–6
Forster, Thomas 146
Frege, Gottlob 10–11, 13–15, 17, 23n22, 25, 104, 107, 142, 164, 213n16, 263, 267, 274, 289–90
his Basic Law V 49n14
Fregean concept *see* concepts

functions
characteristic 107–8
co-partial 63, 286
primitive 43–5, 50–1
fusions *see* mereology; sums

Geach-Kaplan sentence 11
generality absolutism and generality relativism *see under* absolute generality
generalized quantifier theory 87n9, 90
generalized semantics *see* semantics; generalized
Gibbard, Allan 92n15
Gillon, Brendan 85n7, 175n1, 189, 191n7
Glanzberg, Michael 40n6, 247, 269n2
Goodman, Jeremy 213n17, 225n32
Gödel, Kurt 60, 63n8, 66–7, 70–2, 94, 112, 254, 288
Grimau, Berta 179n2
groups 2, 11, 39, 89, 179, 183–5, 206–7, 210, 212–13, 222

Hale, Bob 292n23
Hartimo, Mirja 278n9
Hazen, Allen P. 153, 159, 164, 166, 179
Heck, Richard Kimberly 49n14
Heim, Irene 210n9
Hellman, Geoffrey 28, 208n7
Henkin, Leon 153, 173
Henkin semantics 129, 152–3
plurality-based 134, 154–7, 159–60, 164–9, 170–3, 295
Hewitt, Simon 25, 60n2, 117n13, 196, 205n1, 209n8, 226
Higginbotham, James 12, 13n9, 39n5, 106, 112–13, 115, 144–5, 156
higher-order logic 23, 29, 112, 145–6, 152, 153n4, 153, 243n4, 291, 293 *see also* second-order logic
higher-order pluralities *see* superpluralities
homophonicity 127, 133, 135, 147–8, 208
Hossack, Keith 22n19, 23, 26n26, 65n11, 90–1, 142n9, 144, 151n1, 152n3, 158–9, 196, 229n36, 268, 269n1
Hume's principle 25
Hughes, George Edward 220n25

ideological commitment 167–8; *see also* ontological commitment
inconsistent triad 245, 268, 276–7
indefinite extensibility 208
indiscernibility
plural 19, 102–3, 109, 113–14, 216–19, 222, 224, 226–8, 233–4, 236, 260, 273n4, 279
second-order 109, 113–15
inexpressibility
and generality relativism 246–7
and traditional generality absolutism 249–56 *see also* ascent (semantic)
infinitary disjunctions 225–6, 238–9
infinite pluralities 72, 225, 282
infinite totalities *see* quasi-combinatorial reasoning
intended interpretation 84, 136, 138, 158, 193, 208; *see also* model theory; intended model
intensionality 117–18, 139–42, 144–5, 148, 150, 167, 212, 215, 253, 271–2, 274–5
interpretability (of theories) 58–9; *see also* translations
mutual *see* mutual interpretability
van Inwagen, Peter 26n26, 90n12
isomorphism invariance 22; *see also* formality

Kaplan, David 11n4
Kay, Paul 196
Kjosavik, Frode 278n9
Klement, Kevin C. 14n10
Krämer, Stephen 255n15
Kreisel, Georg 70n18, 137–8, 152n2
Krifka, Manfred 13n8, 85n7
Kripke, Saul 209–11
Kripke semantics 149; *see also* modality; possible world semantics

Landman, Fred 10n2, 13n9, 85n7, 89n11, 181, 183, 193n8, 206n3
Lasersohn, Peter 175n1
Leibniz's law 19, 103, 210–17, 222, 232–3
Leśniewski, Stanisław 60n1
Lewis, David 39n5, 63n8, 92n15, 179, 246
liberal view of definitions 62n4, 63–4, 93–4, 242n3, 269, 271–5, 290–1
limitation of size 119
Lindenbaum, Adolf 23n21
Link, Godehard 10n2, 12, 13n8, 79, 85n7, 87–8, 144

Linnebo, Øystein 9n*, 9n1, 12n6, 15n12, 21n17, 24n25, 144, 146, 151n*, 153, 156, 159, 164, 178–9, 181–2, 184–5, 205n*, 208, 224n31, 230, 240n1, 241, 248, 252, 254n14, 265n23, 267, 269n1, 270n3, 273n5, 278n9, 279–80, 292n23, 293n24
logic
  choice of 3, 289
  critical plural *see* plural logic, critical
  critical view of 278n9, 290
  first-order *see* first-order logic
  higher-order *see* higher-order logic
  intuitionistic 231, 280
  modal *see* modality; modal logic
  negative free 211, 217, 220, 232
  sentential 33, 102, 232
  second-order *see* second-order logic
  traditional plural *see* plural logic, traditional
"logic first" orientation 3, 289–90
logical adequacy *see* regimentation; logical adequacy
logical consequence 29, 32, 123, 126, 131, 136–8, 171–2, 198, 253
logical faithfulness *see* regimentation; logical faithfulness
logical form 29–30
logicality 21–5, 295
logicism 25, 168
  sub- 25
Löwenheim-Skolem 130, 135, 152, 253

MacFarlane, John 21
Maddy, Penelope 71n19
mass terms 85–6
McCarthy, Timothy 22n20
McGee, Vann 22n20, 241, 243n4,
McKay, Thomas J. 9n1, 16n12, 18n15, 22n19, 31, 39n5, 78, 88n10, 125, 131n3, 142n9, 144, 151n1, 152n3
membership
  modal profile 116, 205–6, 218, 228, 230
  plural 16, 23, 92, 118, 128, 133, 172, 181, 208, 258
  set-theoretic 27, 56-7, 68, 128, 172, 213–14
mereology
  atom 12–13, 26, 79–83, 89–94, 96, 101–3
  atomistic classical extensional 80, 83, 101–3
  atomless gunk 91
  classical extensional 77, 79, 80–1, 100
  hylomorphic conception of parthood 214
  individual parthood 12–13, 79–80
  material parthood 12, 79, 85, 93, 214n19
  principles of decomposition 96–7
  sums 12, 31, 46, 64n10, 77–80, 82, 84, 86, 88, 90–5, 98–103, 112, 119, 159, 163, 273, 295
metalanguage 4, 26, 124–5, 131, 134, 144, 148–9, 151, 156, 160, 163–4, 250, 252, 260, 295
modality
  contingentism 208, 211, 213n14, 217, 220–1, 223–5, 227–8, 232–5, 237
  interpretational 248
  metaphysical 15, 248, 282n15
  mirroring theorem 248
  modalized quantifiers 248, 269n1
  modal logic 149, 210, 220n25, 224n31, 280
    B (Brouwerian axiom) 211, 217, 232
    S4 (axiom) 234
    S5 (system) 223–4, 226, 234, 236–7
    T (system) 210
  modal rigidity; *see* rigidity
  necessitism 208, 221, 223–6, 232–5
  possible world semantics 124–5, 149
model theory 30, 123, 249–51, 266
  Henkin *see* Henkin semantics
  higher-order 152, 153n4
  intended model 23, 136
  non-standard model 23, 152–3, 155, 157–9, 161–2, 168–9, 295
  plurality-based 130–5, 139–43, 151–5, 157–64, 169, 188, 199, 263, 295
  problem of choice 144–50, 241
  set-based 126–30, 135–8, 152, 159–62, 178, 252–3, 290
Moltmann, Friederike 10n2, 13n8, 85n7, 148
MSOL and MSOL+ *see* second-order logic; monadic
multi-sorted languages 55–7, 74, 84, 102–5, 257–61
mutual interpretability 24, 58–9, 74, 83–4

n-tuples (higher-order) 263–4
natural language 1, 9, 15–17, 18n15, 19–20, 29–33, 46, 59–60, 63, 106, 118, 123–4, 243, 293, 295; *see also* regimentation
  higher-order plural resources in 140, 156, 174–5, 177, 179, 193, 198–9
  semantics of 4, 86–7, 149
necessary covariation
  of pluralities 217–18, 233–4
  of sets 212, 233
necessitism *see* modality; necessitism
Neale, Stephen 210n9
necessity of identity and distinctness 210–11, 232
New Foundations 146
Nicolas, David 10n2, 85n8, 156, 181–2, 184–5, 187, 196, 197n12, 197n13
nominalism 26, 59, 84, 167–9
non-eliminative reductionism 2, 65n11, 94
non-rigidity *see also* rigidity
  of groups 209–10
  of parthood 92, 213–14
  of predication 116–18, 149, 207
non-standard interpretations *see* model theory; non-standard model

Oliver, Alex 9n1, 10n3, 14n11, 16n12, 16n13, 18n15, 22n19, 31, 39n5, 60, 62–3, 65n12, 78, 89n11, 110n6, 111, 125, 126n1, 131n3, 142n11, 144, 151n1, 152n3, 156, 181, 183, 184n3, 186, 196, 257n18, 261n22, 286
omega-sequence 167
ontological dependence 221
ontological innocence 21–2, 24–8, 71, 123, 151, 153, 157–60, 163–4, 168–9, 229, 295–6
ontological commitment 4, 22, 24n25, 30, 32, 59, 151, 158–9, 165, 229, 295
  generalized notion of 153, 163–67
  narrow notion of 163–5, 167, 169n9
ordered pairs 25, 47, 49–52, 117, 140–1, 208, 242n2, 263–5, 283, 285n18

pairing operation 131, 136, 155, 160
paradox of plurality 12, 38–42, 45, 60, 62
Parmenides 4–5
Parsons, Charles 40n6, 62n4, 63n7, 153, 159, 164, 167–8, 208n7, 213n16, 247, 269n2, 278n9
parthood *see under* mereology
partial order 77, 81, 93, 96–8, 100–3
  definition of 96
partial rigidification 223–4, 226, 236
Paseau, Alexander 70n16
Peters, Stanley 46
PFO and PFO+ *see* plural logic, traditional; theory of
Pietroski, Paul 29n28
pluralism
  regimentation 33–4, 124–5
  semantic 125
Plural Cantor 41–3, 45, 50–1, 142, 241, 243–5, 262, 276, 294
  primitive functions 42–5
  relational 47–9
  with pluralities of ordered pairs 49–50
plural descriptions 65, 181–2, 184–5, 188–9, 195, 208–10, 270
plural identity 18, 217n23
Plural Innocence 158–60, 168
plural logic, critical 3, 20, 43n9, 161n6, 166, 230, 286–90
  consistency of 287
  extensions of 284–6
  infinity 282
  principles of 278–84
  powerplurality 285
  replacement 283
  separation 43n9, 231, 277, 281
plural logic, traditional 3, 5, 20, 27, 42, 45, 63, 90, 108, 118, 123–4, 135–6, 146, 172–3, 244, 249– 52, 257, 261–2, 276, 279, 283–4, 291, 293–4, 296
  language of 15–19
  one-sorted 81–3
  plurality-based Henkin semantics for *see* Henkin semantics; plurality-based
  plurality-based semantics for 130–5, 139–43
  set-based semantics for 126–30
  theory of 19–20
    indiscernibility *see* indiscernibility; plural
    non-emptiness of plurals 19, 41, 58, 81, 103, 109, 178, 220
    plural comprehension *see* comprehension axioms; plural
plural predicates
  collective vs. distributive 17–18

plural predicates (*cont.*)
flexible vs. inflexible 17
of order *n* 201 *see also* superpluralities
Plural Profusion 241–2, 244–5, 262, 267, 276
Plural Robustness 158–61, 168–9
plural terms
list of 180–1, 185
Poincaré, Henri 274
Pollard, Stephen 285n18
possible language forms 29–30
predicates
argument places and positions 186–7
cumulativity of 112, 260–1, 292
mixed-level predication 193–5
multigrade analysis 180, 185–8, 194, 197–9
undiscerning counterparts 115–16, 118
predicative analysis (of plurals) 111–13
primitive plurals 1–2, 4, 15, 31–3, 46, 55, 76, 78, 83, 86–8, 104, 119, 293–5
properties (as objects) 139, 141–3
pure logic 20–1, 24n25, 168–9, 227, 288, 295 *see also* logicality
Putnam, Hilary 208n7

quantifiers
absolutely general *see* absolute generality
bounded 234, 261
classical 231–2
determinacy of *see* determinacy
domain of *see* domains (of quantification)
modalized *see* modality; modalized quantifiers
plural 4, 16, 19, 22–3, 26, 30, 39, 47, 51, 83, 88, 117, 128–9, 134–5, 151, 153–5, 157– 9, 162–70, 174, 205, 207, 232, 250–1, 259, 261–2, 278, 283, 287, 295
polyadic 117–18
range of 3, 22, 27, 40–1, 61, 73–4, 128n2, 154, 159, 172, 174, 245–6, 248, 269
restricted to a plurality 231, 278
second-order 4, 106, 129, 152, 157, 161–2, 167, 295
singular 26, 39, 51, 106, 159, 229, 259
superplural 140, 144, 156–7, 250, 256, 259, 287
third-order 117, 155–6, 160
unrestricted 250 *see also* absolute generality
quasi-combinatorial reasoning 226, 229, 231, 285, 289
Quine, Willard V. 11–12, 14–15, 29, 31–2, 59–61, 125, 144, 165–7, 215, 259, 289–90

Rabern, Brian 270n3
radical holism 289–90
Rayo, Agustín 9n1, 15n12, 22n19, 26, 39n5, 78, 106n3, 125, 131n3, 140, 144, 151, 152n3, 153n4, 156–7, 160, 178–9, 240n1, 241, 243n4, 250n10, 252, 265n23, 267
reduction *see* eliminative reductions *and* non-eliminative reductionism
reference 25–6, 70, 259–61
articulated 176, 195–8
cumulative 85
plural 148, 177, 260n21, 261
purely referential term 215–17, 222
superplural 174–9, 194, 198–9, 261
reflection principles *see under* set theory
regimentation 11–12, 14, 17–18, 29, 31–40, 42, 45, 59, 106, 123–5, 186, 187, 197, 241
logical adequacy 29–30, 33
logical faithfulness 32–3, 37, 39
object language and regimentation language 32
pluralism *see* pluralism; regimentation
proxies 37–40, 42, 45–6
singularism *see* singularism; regimentation
Reinhardt, William 71
Resnik, Michael 59, 144, 159, 164
rigidity 116, 206–19, 221–2, 224–8, 230, 271; *see also* non-rigidity
formal arguments 219–226, 231–9
metaphysical notion of 214
RGD^+ and RGD^- 149, 205, 219, 221–3, 227, 234–8
rigid designators and semantic notion of 214–16
of sets 210–16
Roberts, Sam 207n5, 221n26, 293n24
Rosen, Gideon 15n12, 26n26, 63, 90n12, 181

Rumfitt, Ian 16n13, 110n6, 205n1, 209n8, 224
Russell, Bertrand 2, 4–5, 14–15, 55, 69, 114, 164, 175, 184, 231, 247, 256n17
Russell's paradox 2, 27–8, 39, 49–50, 56–7, 61, 66, 143, 261n22, 269–70, 274n6

Sagi, Gil 22n20
satisfaction 126–8, 131–2, 134, 157, 171, 266–7
Scha, Remko 18n15
Schein, Barry 10n2, 13n9, 39n5, 112–13, 115, 144–5
Schindler, Thomas 292n24
schematic generality 247–8, 255–6, 278n8
Schwarzschild, Roger 10n2, 80, 175n1, 189, 190n6, 191, 193n8, 194n10
second-order logic 17n14, 22, 24–6, 28, 31, 46, 49n14, 72, 104–8, 111–12, 117–18, 128, 142, 151–2, 155, 159, 172–4, 207, 291–2
  monadic 24–5, 104–5, 107–8, 117, 119, 126, 207
semantics
  generalized 251–3, 262, 276, 290–3
  instrumental view of 145, 150, 193
  linguistic 9–10, 29, 85–9, 175
  parity constraints 147–150, 156n5
  plurality-based *see* model theory; plurality-based
  set-based *see* model theory; set-based
semantic optimism 251–2
semantic values 22, 86, 105, 128, 131, 133, 136, 138, 141, 148–9, 154, 163, 208, 249–50, 254, 292
  upwards closed 194–5, 198, 200–1
sets
  the cumulative hierarchy of 67–8, 70–2, 86
  empty set 58, 63, 65, 68, 73, 84, 146, 240, 246, 286
  iterative conception of 65–70, 207, 287
  logical notion of 146
  nested 65–6
  plurality-based account of 2, 62–66, 71–2, 169, 273
  pure 68–9, 248
  "set of" operation 63, 66–7, 72, 94, 288
  with urelements 69, 286
set theory 1, 11, 23, 26
  axioms 68–9
  derivation of (in critical plural logic) 286–7
  non-standard 80, 146
  reflection principles 138, 144–5, 253
  simple 55–7
  ZFC 68–70, 162, 286–7
  ZFCU 69, 286
Shapiro, Stewart 25, 28, 138, 152n2, 153, 164, 167, 172, 252n12
Sher, Gila 22n20
Simons, Peter 14, 65n12, 125, 256n18
singleton pluralities 40, 44, 83, 94, 111n9, 200, 219, 225n32, 260, 265, 280, 281n14
singularism
  incorrect existential generalization 36–8, 138
  in linguistics 2, 4, 13–14, 46, 85–90, 295
  mereological 85–88
  regimentation 33–4, 36–8, 45, 124–5, 241
  semantic 125, 146–7, 241
singularization 2–4, 164, 180, 242
  universal 240, 242–6, 249, 268, 277–8, 292, 294
Smiley, Timothy 9n1, 10n3, 14n11, 16n12, 16n13, 18n15, 22n19, 31, 39n5, 60, 62–3, 65n12, 78, 89n11, 110n6, 111, 125, 126n1, 131n3, 142n11, 144, 151n1, 152n3, 156, 181, 183, 184n3, 186, 196, 256n18, 261n22, 286
SOL *see* second-order logic
Solberg, Hans Robin 289n21
sorts 56; *see also* multi-sorted languages
Spencer, Joshua 269n1
squeezing argument 137, 172–3
SST *see* set theory; simple
Stalnaker, Robert 216n20
strictly typed predicates 112
structuralism 28, 62n4
Studd, James P. 87n9, 248n8, 248n9
substitution argument 34–7, 45
superpluralities 45, 92, 140–1, 144, 148–9, 154–5, 160, 170, 174–201, 250, 254, 256, 258– 61, 264, 284–7
  availability in natural language 66, 156, 177, 179
  grades of superplural involvement 178–9
  reference to *see* reference; superplural
  intelligibility problem 176–7

supervenience (of properties) 218, 228–9
Sutton, Peter 85n7

Tarski, Alfred 22, 23n21, 134, 144, 153, 252–3, 263, 266–7
Tennant, Neil 25
topic-neutrality 21, 24–5, 282, 295
translations 31–2, 35–7, 58, 84
  between one-sorted and two-sorted plural logic 81–3, 102–3
  between a multi-sorted and a one-sorted metalanguage 258–60
  between mereology and plural logic 81–3, 102–3
  between second-order and plural logic 107–9
  between set theory and plural logic 73–4
  as interpretations 58–9, 74–5; *see also* mutual interpretability
truth
  explicit and recursive definition of 263, 266–7
  in an interpretation 126–8, 131–2, 134, 138, 144, 147, 171, 263, 266–7
  undefinability of 134
types
  hierarchy of 86, 250–6, 258, 260–3, 276, 291
  notation 108
  preservation of 148–50
  type-shifting operation 194
  type-unrestricted generality 253–4
type distinctions 142–3
  lifting the veil of 256–61, 292

uniform adjunction 221–4, 225n32, 226, 236
uniform traversability 224–6, 228–9, 231–2, 236, 238
universal
  concept 118, 147, 250, 291
  plurality 27–8, 42, 84, 136n5, 142, 166, 231, 241–2, 244–6, 249–50, 261–2, 268, 276, 278–9, 284, 290–3
  set 26, 67, 72, 84, 146, 249
Urbaniak, Rafal 60n1
Uzquiano, Gabriel 26, 43n9, 48n13, 50n15, 92n14, 151, 152n3, 157, 205n1, 206n3, 207n4, 222, 243n4, 250n10

V (class) 67–8, 70, 72; *see also* sets; the cumulative hierarchy of
$V = L$ (axiom) 167
Varzi, Achille 97, 100n21

web pages (example) 270–2, 275; *see also* liberal view of definitions
Weir, Alan 25
Westerståhl, Dag 46
Weston, Thomas 152n2
Williamson, Timothy 3, 26, 106–7, 116, 142–4, 179, 205n1, 208, 211n11, 211n12, 219, 223–4, 229, 240, 246, 247n6, 250n10
Winter, Yoad 18n15, 175n1
Wittgenstein, Ludwig 254
Wright, Crispin 160

Yablo, Stephen 21n17, 106n3, 144, 153n4, 156, 160, 244–5, 249
Yi, Byeong-Uk 9n1, 16n12, 22n19, 31, 34–5, 87n9, 111n7, 113–14, 125, 131n3, 142n11, 144, 151n1, 152n3