Meeting of the Aristotelian Society, held at Senate House, University of London, on 20 February 2023 at 6:00 p.m.

VI—On Type Distinctions and Expressivity

Salvatore Florio

Quine maintained that philosophical and scientific theorizing should be conducted in an untyped language, which has just one style of variables and quantifiers. By contrast, typed languages, such as those advocated by Frege and Russell, include multiple styles of variables and matching kinds of quantification. Which form should our theories take? In this article, I argue that expressivity does not favour typed languages over untyped ones.

I

Introduction. Formal languages have come to play a key role in the pursuit of philosophical and scientific rigour. In choosing a formal language, however, we face an embarrassment of riches, given the wide variety of available options. One of the fundamental differences concerns the basic structure of the language and, in particular, whether or not the language includes type distinctions.

Philosophical analysis often relies on the familiar language of first-order logic, which has a simple type-theoretic structure. It features 'a single style of quantifiable variable' (Quine 1956, p. 267). In other words, the language is *untyped*. The semantic picture associated with this language is correspondingly simple: there is one domain of entities, which can be characterized using the predicates available in the language.

Several alternative frameworks are based on *typed* languages, which exhibit richer structures of types. A prominent example is second-order logic, which adds to first-order logic a second style of variables and a matching kind of quantification. The semantic picture associated with this system is less straightforward but, on a popular view, the language describes two incommensurable domains: objects and properties of objects (or concepts). Each domain is then characterized using predicates appropriate to that domain. One can

Proceedings of the Aristotelian Society, Vol. XX*, Part* XX
https://doi.org/10.1093/arisoc/aoad007

introduce more styles of variables and quantification, as is in full higher-order logic, which includes infinitely many styles beyond the second.

Does a single style of variables suffice for theoretical inquiry? According to the philosophical tradition famously represented by Quine, the answer is positive: an untyped system is enough for our philosophical and scientific needs. The tradition associated with Frege and Russell supports a negative answer: we require typed languages. These languages have received renewed philosophical attention in the past two decades, and recent philosophical work in metaphysics (see Skiba 2021; Fritz and Jones forthcoming b) as well as philosophy of mathematics (for example, Linnebo and Rayo 2012; Ladyman and Presnell 2018; Corfield 2020) has explored new applications of typed systems, including sophisticated frameworks such as type theories with transfinite types and homotopy type theory.

As also suggested by long-standing philosophical disputes, the contrast between these two positions seems to be significant. Moreover, it is of foundational importance, since it concerns the basic form of our theories. What considerations support the adoption of a framework with one style of variables rather than many? It has been argued, for example, that the richer type-theoretic structure of second-order logic is superfluous, illegitimate, or even unintelligible (for example, Quine 1956; Quine 1986, pp. 66–8). But one can also find many arguments defending the theoretical importance, or even the indispensability, of the richer structure (see Williamson 2003, 2013; Dorr 2016; Jones 2018; Trueman 2021). For instance, Russell held that avoiding paradoxes involving self-reference *necessitates* 'the division of objects into types' (1908, p. 237).

Our focus here is on the important but puzzling role that the notion of *expressivity* plays in this debate. On the one hand, any untyped language can be extended to a typed one, and can therefore be seen as part of a broader, typed language. As a result, typed languages may seem to afford *greater expressive power* than untyped ones and, on this basis, be preferable. On the other hand, type-theoretic hierarchies have been thought to have expressive limitations that can be overcome in an untyped language. Wittgenstein and Gödel, among others, expressed views of this kind (Wittgenstein 1979, pp. 106 ff.; Gödel 1944, p. 466). Moreover, there are arguments that untyped

Proceedings of the Aristotelian Society, Vol. XX, *Part* XX
https://doi.org/10.1093/arisoc/aoad007

languages are *too expressive*, since they permit the formalization of nonsense or force us to make unnecessary theoretical choices, which can be avoided in typed languages.

In this article, I will develop some of these argumentative strategies, clarifying how expressivity could be understood. I will defend two main claims. The first is that, in this context, appealing to greater expressive power in support of typed languages is not effective. The second is that untyped languages are not too expressive in the sense described above. So if there are compelling reasons to adopt a typed language, they are not based on expressivity. Overall, my discussion will provide a partial vindication of untyped languages.

II

Many-Sorted Logic as a Framework for the Debate. To set the stage for the debate and compare alternative approaches, I believe it is illuminating to use the framework of many-sorted logic. I will start by giving a brief, informal presentation of the framework, before commenting on its benefits.

The signature of a many-sorted logic includes a set of objects representing the *sorts* admitted by the logic. The set could be finite or infinite. Each sort has an associated style of variables and quantifiers. For example, one may countenance infinitely many sorts and use the natural numbers as superscripts to represent them. Then, for each natural number n, there are variables of sort n (x^n, y^n, z^n, …) and corresponding quantifiers ($\forall x^n$, $\forall y^n$, $\forall z^n$, …). The non-logical terminology comprises constants of each sort (for example, a^0, a^1, a^2, …), as well as predicates with appropriate sortal restrictions.[1] More specifically, each argument place is restricted to one sort. For example, one may have a binary relation R taking terms (variables or constants) of sort 1 in its first argument place and terms of sort 0 in its second argument place. This means that $R(x^1, a^0)$ is well formed, whereas $R(x^0, a^1)$ is not. To mark sortal restrictions, predicates are usually decorated with a sequence representing the sorts of their arguments. So in our example, the relation R would be presented as $R^{\langle 1,0\rangle}$. (Decorations will be omitted when possible.)

[1] For simplicity, I leave out function symbols. This will not affect my main points, though function symbols are relevant to the theme of §vi. See footnote 5 below.

Proceedings of the Aristotelian Society, Vol. XX*, Part* XX
https://doi.org/10.1093/arisoc/aoad007

An important choice point has to do with the predicates that are admitted in the language. One could be maximally liberal and, for any finite combination of sorts, allow predicates whose arguments correspond to that combination. That is, for any sequences of sorts $\langle i_1, \ldots, i_n \rangle$ one would admit predicates like $R^{\langle i_1, \ldots, i_n \rangle}$. Alternatively, one could be more restrictive and ban predicates whose arguments correspond to certain combinations of sorts. For example, one may allow predicates like $R^{\langle 1,0 \rangle}$ but ban predicates like $R^{\langle 1,2 \rangle}$. In this case, one allows relations between entities of sort 1 and entities of sort 0, but not relations between entities of sort 1 and entities of sort 2. As we shall see later, some debates about the correct form of type theory can be interpreted here as debates about which predicates to admit.

A striking difference between the framework of higher-order logic and that of many-sorted logic concerns predication. In higher-order logic, predication is normally treated as a logical notion and is not represented by any symbol in the language. By contrast, predication is introduced as a non-logical symbol in many-sorted logic. Suppose we want express that a property applies to an object. In second-order logic, one would simply concatenate the appropriate terms in a formula like $x^1(y^0)$ or $X(y)$. In many-sorted logic, one would rely on a special relation symbol, such as $P^{\langle 1,0 \rangle}$, and express predication using a formula like $P^{\langle 1,0 \rangle}(x^1, y^0)$. This is in keeping with the fact that higher-order logic, but not many-sorted logic, permits quantification into predicate position.

It is straightforward to recapture the standard systems of first- and second-order logic within many-sorted logic. First-order logic is the limit case in which there is only one sort. Second-order logic corresponds to the case in which there are two sorts, let's say 0 and 1, and predicates are constrained as follows. All predicates of first-order logic are allowed. In addition, there are relational symbols expressing predication relations holding between an entity of the second sort (sort 1) and entities of the first sort (sort 0). The second-order axiom scheme of comprehension asserts that for any appropriate formula φ, there is a second-order entity that holds of exactly the first-order entities satisfying φ. In symbols:

$$\exists X \forall y_1 \ldots \forall y_n \left(X\left(y_1, \ldots, y_n\right) \leftrightarrow \varphi\left(y_1, \ldots, y_n\right)\right)$$

where X does not occur free in φ. In many-sorted logic, an analogous schema expresses the principle that for any appropriate formula φ,

Proceedings of the Aristotelian Society, Vol. XX, *Part* XX
https://doi.org/10.1093/arisoc/aoad007

there is an entity of the second sort that stands in the predication relation exactly to the entities of the first sort that satisfy φ. That is,

$$\exists x^1 \forall y_1^0 \ldots \forall y_n^0 \left(P^{<1,0,\ldots,0>} \left(x^1, y_1^0, \ldots, y_n^0 \right) \leftrightarrow \ \varphi \left(y_1^0, \ldots, y_n^0 \right) \right)$$

where x^1 does not occur free in φ. It is straightforward to extend this translation procedure to recapture standard logics of higher orders.

From a semantic point of view, it is common to interpret a many-sorted system by means of a set-theoretic model (or structure) consisting of many non-empty domains, one for each sort, and relations respecting the sortal restrictions of the predicates in the language. For instance, the relational symbol $R^{\langle 1,0 \rangle}$ is interpreted as a set of pairs $\langle u, t \rangle$, where u is in the domain of sort 1 and t is in the domain of sort 0. The interpretation of the symbols for predication is not fixed, but can vary across models. This marks an important difference with respect to the standard semantics for second-order logic, where predication has a fixed interpretation. This semantic approach to many-sorted logic leads to a system that shares crucial metalogical properties with first-order logic, for example, completeness, compactness, and the Löwenheim-Skolem property. In this sense, the system can be said to be *first-order* (see, for example, Shapiro 1991, p. 14).[2]

Many-sorted logic can be a valuable tool when comparing different typed languages. It reduces some debates about the correct form of type theory to debates about which predicates to admit. This yields a simpler and arguably more tractable rendering of the original debates. For instance, the debate between critics and supporters of cumulative type theory (Degen and Johannsen 2000; Linnebo and Rayo 2012; Williamson 2013, pp. 237–8; Krämer 2017; Florio and Jones 2021; Button and Trueman 2022) reduces to the question whether predication symbols of the form $P^{\langle n,m \rangle}$ may be admitted whenever $n > m$. Moreover, the treatment of predication in many-sorted logic seems better suited to represent some metaphysical ideas. For instance, it is more hospitable to views according to

[2] Second-order logic with standard semantics is said to have greater expressive power than first-order logic. This is because, on the relevant semantics, second-order logic can describe certain classes of structures that resist first-order characterization (see Shapiro 1991, §4.2). However, the language of many-sorted logic can, through a suitable semantics, be endowed with similarly great expressive power, and so can the language of first-order logic. This understanding of expressive power is therefore not central to the present discussion.

Proceedings of the Aristotelian Society, Vol. XX, *Part* XX
https://doi.org/10.1093/arisoc/aoad007

which predication is just one among many fundamental ontological relations, such as exemplification, participation and realization (for example, Smith 2005). It also allows a very clean separation between ontology (given by quantification) and ideology (given by the predicates admitted in the language). The separation is less clear in the standard formulation of higher-order logic, which relies on quantification into predicate position. The question arises how to understand the commitments incurred by this form of quantification and their relation to the commitments of quantifiers in nominal position. Additional considerations in favour of many-sorted logic can be found in Manzano 1996, which provides a systematic defence of this framework emphasizing its unifying power.

Many-sorted logic can also be a valuable tool when comparing typed languages and untyped ones. Translating sentences of a many-sorted language into sentences of a one-sorted language is, as we shall see, straightforward. Furthermore, a number of potentially useful results concerning theoretical equivalence become immediately available. In philosophy of science, there is a well-known problem of characterizing various senses in which two theories can be said to be equivalent. The problem becomes more complex when we compare theories with very different type-theoretic structures. This is because the standard definitions of a number of relevant notions (for example, mutual interpretability and definitional equivalence) apply directly only to theories formulated in languages with similar types. Many-sorted logic offers a fruitful setting to tackle this problem. Recent work by Thomas William Barrett and Hans Halvorson has clarified some of the ways in which systems with different sortal structures, such as a one-sorted system and a system with many sorts, can be said to be definitionally equivalent (Barrett and Halvorson 2016; Halvorson 2019, chs. 4-5).[3]

So I propose to adopt many-sorted logic as a framework for our discussion. This means formulating typed theories as many-sorted theories. Our overarching question ('Should we adopt a typed language?') becomes a question about sorts ('Should we admit more than one sort?'). The main contrast is thus between one-sorted

[3] Even definitional equivalence, a very strong notion of theoretical equivalence, fails to guarantee sameness of meaning or truth-values between corresponding sentences of two theories. So there are interesting questions about the relation between theoretical equivalence and the notions of expressivity articulated in this article. For reasons of space, I will not be able to pursue this topic here.

Proceedings of the Aristotelian Society, Vol. XX, *Part* XX
https://doi.org/10.1093/arisoc/aoad007

systems and systems with more than one sort. As noted, a one-sorted system is a limit case of a many-sorted system. However, it will be convenient henceforth to abuse terminology and call a system, language or logic *many-sorted* if and only if it has more than one sort.

I am now ready to develop and assess two arguments against untyped languages, as I set out to do in §1. Recall the gist of the arguments: typed languages have more expressive power than untyped languages; untyped languages are too expressive in problematic ways.

Let us start with the first argument. Do many-sorted systems really have *more expressive power* than those with one sort only? There are at least two ways of developing the argument, depending on how expressive power is understood. One understanding is semantic, the other syntactic. On the semantic understanding, the argument is that many-sorted logic has more expressive power than one-sorted logic because more can be represented in many-sorted logic than in one-sorted logic. On the syntactic understanding, the argument is that many-sorted logic has more expressive power because more can be said and proved in many-sorted logic than in one-sorted logic.

III

Can Many-Sorted Logic Represent More than One-Sorted Logic? For now, let us assume the common model-theoretic semantics sketched in the previous section. This semantics relies on standard first-order set theory, an untyped theory, to describe the range of models that can be represented by a many-sorted language. A model of the language consists of many non-empty sets representing domains of quantification, one for each sort, and set-theoretic relations denoted by predicates of the many-sorted object language. The domains need not be disjoint. Denotations respect the sortal constraints of the object language. Let the domains be U_i with $i \in I$, the set of sorts. Then a constant a^i of sort i denotes an element of U_i, and a predicate $R^{\langle i_1, \ldots, i_n \rangle}$ denotes a subset of $U_{i_1} \times \ldots \times U_{i_n}$. A many-sorted theory selects a class of models, those making all sentences of the theory true.

Models of many-sorted languages ('many-sorted models' for short) are seemingly more complex than models of one-sorted languages ('one-sorted models' for short). In effect, a many-sorted model combines multiple one-sorted models into one, with additional

Proceedings of the Aristotelian Society, Vol. XX, *Part* XX
https://doi.org/10.1093/arisoc/aoad007

information about relations among entities of different sorts. This means that any one-sorted model is included in some many-sorted model. It may thus appear that the range of models that can be represented by many-sorted languages extends the range of models that can be represented by one-sorted languages.

However, this appearance is superficial. It is well known that every many-sorted model can be easily converted into an equivalent one-sorted model (see, for example, Enderton 2002, pp. 296–9). The simple idea behind this *model conversion* is to capture sorts using special predicates. Start with a many-sorted language. Treat its signature as if it were one-sorted. To the resulting one-sorted language, add a new predicate S_i for each sort i. To obtain a one-sorted model from a given many-sorted model, proceed as follows. Set the domain to be the union of domains U_i of the many-sorted model. Let each predicate S_i denote the set U_i. The rest of the non-logical vocabulary retains the denotation it has in the many-sorted model.

Let us explore how to capture formally the equivalence between a many-sorted model M and the one-sorted model M^* obtained from the model conversion just described (see Enderton 2002, pp. 297–8). The aim is to make more precise one way in which languages can be compared for expressivity. There are two steps. The first is just our model conversion. The second is the characterization of a syntactic translation • from the many-sorted language to the one-sorted language. The translation maps quantification of sort i to quantification restricted to S_i. The rest of the language is again treated as if it were one-sorted. More precisely, the action of • is given by the following recursive clauses, where $[\varphi]^\bullet$ is the result of applying • to φ, t^i and u^i are terms (constant or variable) of sort i, and $R_{\langle i_1,\ldots,i_n\rangle}$ as well as $t_{i_1},\ldots,t_{i_n}$ are one-sorted expressions in which the sortal decoration functions as a mere subscript to avoid clashes of terminology in the translation.

$$\left[R^{\langle i_1,\ldots i_n\rangle}\left(t^{i_1},\ldots t^{i_n}\right)\right]^\bullet = R_{\langle i_1,\ldots,i_n\rangle}\left(t_{i_1},\ldots,t_{i_n}\right)$$

$$\left[t^i = u^i\right]^\bullet = t_i = u_i$$

$$[\neg\varphi]^\bullet = \neg[\varphi]^\bullet$$

$$[\varphi \wedge \psi]^\bullet = [\varphi]^\bullet \wedge [\psi]^\bullet$$

Proceedings of the Aristotelian Society, Vol. XX, *Part* XX
https://doi.org/10.1093/arisoc/aoad007

$$\left[\forall x^i \varphi\right]^\bullet = \forall x_i \left(S_i\left(x_i\right) \rightarrow [\varphi]^\bullet\right)$$

The first two clauses clarify the sense in which terms and atomic formulas, including identities, are treated as if they were one-sorted. The translation commutes with negation and the logical connectives. Finally, quantification over entities of sort i is replaced by quantification restricted by the special predicate S_i.

Combining the two steps—model conversion and translation—we have that M satisfies a sentence σ if and only if M^* satisfies its translation $[\sigma]^\bullet$:

$$M \vDash \sigma \text{ if and only if } M^* \vDash [\sigma]^\bullet$$

The differences between M and M^* are unimportant to what we can express. On the semantics assumed so far, the two models are about the same entities, and the denotations of the corresponding primitive expressions are the same. It can be shown that any sentence σ and its translation $[\sigma]^\bullet$ have equivalent truth conditions when interpreted in M and M^*, respectively. This is obvious for atomic formulas: the denotations of terms and atomic predicates are the same. Now consider a quantified sentence, such as $\forall x^{i_1} \forall x^{i_2} R^{\langle i_1, i_2\rangle}\left(x^{i_1}, x^{i_2}\right)$, which can be read as 'everything of sort i_1 is related by R to everything of sort i_2'. This sentence and its translation

$$\forall x_{i_1}\left(S_{i_1}\left(x_{i_1}\right) \rightarrow \left(\forall x_{i_2}\left(S_{i_2}\left(x_{i_2}\right) \rightarrow R_{\langle i_1, i_2\rangle}\left(x_{i_1}, x_{i_2}\right)\right)\right)\right)$$

assert the same: every element of U_{i_1} stands in a certain relation to every element of U_{i_2}, the relation denoted by both $R^{\langle i_1, i_2\rangle}$ and $R_{\langle i_1, i_2\rangle}$. A simple induction can establish the equivalence in full generality.

Assuming that the extensions of the new predicates S_i are non-empty, the equivalence extends to logical consequence. Take any set Σ of sentences in the many-sorted language, and define $\Sigma^\bullet$ to be the set of their translations under $\bullet$. Let Δ be $\{\exists x_i S_i(x_i) \colon i \in I\}$, the set of sentences asserting that each S_i has a witness. Then $\Sigma \vDash \sigma$ in the many-sorted system if and only if $\Sigma^\bullet \cup \Delta \vDash [\sigma]^\bullet$ in the one-sorted system (see Theorem 43c of Enderton 2002, p. 298). This result will be used below in §v.

The preceding paragraphs establish that any model of a many-sorted language corresponds to a model of a one-sorted language, and any sentence in a many-sorted language has a one-sorted

Proceedings of the Aristotelian Society, Vol. XX*, Part* XX
https://doi.org/10.1093/arisoc/aoad007

counterpart with equivalent truth conditions. In this sense, many-sorted languages are not more expressive than one-sorted languages. The discussion assumed an *untyped metatheory*, since the semantics was formulated in standard first-order set theory. However, one may reject this assumption and insist on a *typed metatheory*. On this approach, a typed language is assumed as primitive in the metatheory, and the semantics is formulated using a typed system. Let us consider what happens if this option is chosen.

Adopting a typed metatheory opens up the possibility of selecting semantic values from the many types available. By contrast, an untyped metatheory can draw the possible semantic values only from the single type it comprises. The option of using semantic values from multiple types has become rather popular. Its main implementation is known as *higher-order semantics* (see Boolos 1984, 1985; Rayo and Uzquiano 1999; Rayo 2002; Rayo and Williamson 2003; Yi 2005, 2006; McKay 2006; Oliver and Smiley 2016). In the context of higher-order logic or plural logic, expressions of the object language are interpreted by means of corresponding expressions in the metalanguage. For example, the semantic value of a second-order variable is not a set or any other object. Rather, it is a second-level entity, which may be thought of as a *sui generis* property of objects described by primitive second-order vocabulary of the metalanguage. Similarly, a plural variable does not denote an object. It denotes some objects, which are specified by means of plural resources in the metalanguage.

From the perspective of a typed metatheory, the models of higher-order logic or plural logic appear richer than those of first-order logic, which only represents objects and their relations. The higher-order resources of the object language provide additional information: they represent higher-level entities (for instance, properties) and their relations. Differences between models of typed languages and models of untyped languages are therefore significant, again from the perspective of a typed metatheory. In higher-order semantics, sentences of typed languages are generally not about the same entities as sentences of untyped languages. Some sentences of typed languages do not have one-sorted counterparts with the same truth conditions. Typed languages thus appear more expressive.

https://doi.org/10.1093/arisoc/aoad007

Recast in the present framework, higher-order semantics interprets expressions of the many-sorted object language by means of entities of the corresponding sorts in the metatheory. For example, the denotation of a constant a^i is an entity of sort i, whereas quantification over a given sort is interpreted by the corresponding quantification in the metalanguage. The main point remains: from this semantic perspective, many-sorted languages appear more expressive than one-sorted languages. The truth conditions of some sentences of many-sorted languages involve entities of multiple sorts simultaneously, and thus go beyond the truth conditions of sentences of one-sorted languages.

Let us summarize the preceding discussion. We have reached a stand-off between two consistent perspectives. If we assume an untyped metatheory like standard first-order set theory, many-sorted languages do not appear more expressive than one-sorted languages. From the perspective of a typed metatheory, however, the situation looks different: many-sorted languages appear more expressive than one-sorted languages. It follows that appealing to expressivity in support of many-sorted languages has little dialectical weight on its own. These languages appear expressively superior only if they are presupposed in the metatheory.

IV

Absolute Generality. The semantic considerations of the previous section delivered a conditional conclusion: we can come to see a many-sorted language as more expressive if and only if we adopt a typed metatheory. Progress can be made by providing positive reasons to adopt a typed metatheory. A prominent argument due to Timothy Williamson (2003, pp. 425–7) aims to do precisely that. According to this argument, making reflective sense of absolute generality requires type distinctions. The argument can be given a general formulation, though we assume for definiteness that the relevant types are objects and properties, as in Williamson's original discussion. For present purposes, the argument may be formulated as follows.

Let P be a predicate applying to objects. In model-theoretic semantics, we are interested in all possible interpretations of the language. So the interpretation of P varies across models. In some model, the predicate applies to certain objects in the domain. In other models,

Proceedings of the Aristotelian Society, Vol. XX*, Part* XX
https://doi.org/10.1093/arisoc/aoad007

it applies to different objects in the same domain or in another domain. Here is an intuitive principle about the range of possible interpretations: for any formula φ of the metalanguage, there is an interpretation according to which P applies exactly to the objects in the domain that are φ. Call it the *liberal principle of interpretations*. To see how the principle works, suppose that the domain comprises the natural numbers and φ is the predicate 'is prime'. Then the principle yields an interpretation according to which P applies exactly to the natural numbers that are prime. The principle is intuitive in part because it helps express the plenitude of interpretations of the object language, a model-theoretic ideal.

Let us state the principle a bit more precisely. For any domain D and for any formula φ, there is an interpretation J according to which P applies to an object in D if and only if the object satisfies φ. If absolute generality is possible, there is an unrestricted domain of quantification. This domain contains absolutely every object and, when used in connection with the principle, it has the effect of rendering the restriction to domain D superfluous. This means that a specialized version of the principle holds: for any formula φ, there is an interpretation J according to which P applies to an object if and only if the object satisfies φ.

Assuming that interpretations are objects, a contradiction is obtained by taking φ to be 'is not an interpretation x according to which P applies to x'. Using an unrestricted domain, the liberal principle of interpretations entails that there is an interpretation J such that, for every object x, P applies to x according to J if and only if x is not an interpretation according to which P applies to x. It is possible to instantiate x with J, since J is assumed to be an object and is thus an appropriate instance of a universal quantifier ranging over objects. It is then straightforward to derive an inconsistency through classical reasoning: P applies to J according to J if and only if P does not apply to J according to J.

Williamson's preferred response relies on a type distinction. He concludes (2003, pp. 452–4) that interpretations are not objects but incommensurable entities described by primitive second-order vocabulary: interpretations are properties of objects. As noted above, the argument can be given a general formulation. One simply starts with the predicate $P^{\langle i \rangle}$, where i an arbitrary sort. The analogous conclusion would be that interpretations are not of sort i but incommensurable entities described by expressions of a different sort.

https://doi.org/10.1093/arisoc/aoad007

So theorizing about interpretations would require a many-sorted language. Thus we would have a positive reason to adopt a typed metatheory.

The argument presents us with a choice. We must reject one of three assumptions: the liberal principle of interpretations, the possibility of absolute generality, and the claim that interpretations are objects. Sacrificing the third, as Williamson does, is certainly appealing. However, I believe that greater insight is gained if we analyse the liberal principle of interpretation.

The plenitude of model-theoretic interpretations is naturally expressed as the idea that, for any combination of *suitable semantic values* for the expressions of the object language, there is an interpretation that assigns that combination of semantic values to those expressions. In particular, a predicate can be interpreted by means of any suitable semantic value. Suppose the suitable semantic values for predicates are properties. Then plenitude requires:

(i) for any property, there is an interpretation assigning that property to P as a semantic value.

On that interpretation, P applies to an object in the domain if and only if the object has the property assigned to P.

We can now factor the liberal principle of interpretations into two components. The first is (i). The second informs us about which properties there are:

(ii) for any formula φ, there is a property that an object has if and only if the object satisfies φ.

This more fined-grained analysis of the liberal principle of interpretations affords a different outlook on Williamson's argument. While (i) is very plausible, (ii) is a form of naive comprehension and is, in light of the paradoxes, very controversial.

If one wishes to do justice to (ii), the introduction of type distinctions is a natural outcome. It is a safe and well-understood way to avoid inconsistency. As remarked earlier, Russell went as far as claiming that the introduction of type distinctions is *necessitated* by the paradoxes. Paradoxical notions (for example, 'is predicated of itself', 'is not an interpretation x according to which P applies to x') can no longer be expressed because they violate type distinctions. However, contrary to Russell's claim, there is nothing inevitable about this outcome.

https://doi.org/10.1093/arisoc/aoad007

There are non-classical attempts to capture naive comprehension (for example, Field 2004). But even if we adhere to classical logic, it is possible to develop a framework that, arguably, does justice to naive comprehension (Schindler 2019). Moreover, one might also reject (ii) altogether, and this is what a number of type-free theories of properties do. Consider the analogy with set theory, where retaining naive comprehension is not a decisive factor and where, in fact, the rejection of naive comprehension prevails. That is because naive comprehension is ruled out by a given conception of set, or because it is outweighed by broader theoretical considerations. The same holds for a theory of properties. Some conceptions of property (for example, Linnebo 2006 and Roberts MS) do not sit well with naive comprehension, and broader theoretical considerations might militate against it (Jubien 1989; Menzel forthcoming).

To accommodate the possibility of absolute generality, one needs a *universal* entity representing an unrestricted domain of quantification. This would likely be a property, since standard set theory rules out the existence of a universal set. The introduction of type distinctions makes it easy to obtain a universal property. By contrast, obtaining such a property in an untyped metatheory is not straightforward. So accommodating absolute generality adds some complications if we are operating within an untyped metatheory.

Let us take stock. Williamson's argument can be used to motivate a typed metatheory. However, I have argued that this move assumes a certain stance on the principle of naive comprehension for the relevant entities. While viable, this stance is not inevitable. For example, if the relevant entities are properties, the assumption is that they satisfy a form of naive comprehension and that this is best captured by postulating a type distinction between objects and properties. Neither part of the assumption is forced on us. There might be a satisfactory one-sorted theory of properties that rejects the principle of naive comprehension, as is the case for set theory. Moreover, a one-sorted theory of properties might be able to do justice to the principle. This is very much an open debate, with interesting new approaches being developed. No one can plausibly have the last word yet.

V

Can Many-Sorted Logic Say and Prove More than One-Sorted Logic? In §III, we considered the thesis that many-sorted logic has

https://doi.org/10.1093/arisoc/aoad007

more expressive power than one-sorted logic. The notion of expressive power was understood semantically. Here the thesis is assessed on a syntactic understanding of the notion, namely, that many-sorted logic can say and prove more than one-sorted logic.

So understood, the thesis seems trivially true. Any one-sorted system can be properly extended by a many-sorted system. We simply add new sorts and new predicates connecting them. This already yields more validities, owing to the logical axioms of the system.

However, the mere ability to say and prove more is not necessarily significant. An obvious case is the ordinary extension of a system by means of explicitly defined predicates. The additional vocabulary and the new validities associated to it do not offer any significant expressive gain.

Recall the result, mentioned earlier, that the many-sorted relation of consequence has a one-sorted equivalent under the translation $\bullet$. That is, for any Σ and σ, $\Sigma \vDash \sigma$ in many-sorted logic if and only if $\Sigma^{\bullet} \cup \Delta \vDash [\sigma]^{\bullet}$ in one-sorted logic, where Δ conveys that each S_i has a witness. Both systems are sound and complete with respect to the model-theoretic semantics introduced in §iii. It follows that $\Sigma \vdash \sigma$ if and only if $\Sigma^{\bullet} \cup \Delta \vdash [\sigma]^{\bullet}$. This means that deducibility in a many-sorted system corresponds to a certain kind of deducibility in a one-sorted system. This fact is significant.

A central motivation for the programme of higher-order metaphysics, which advocates the use of typed resources, has to do with the ability to regiment metaphysical reasoning (see Fritz and Jones forthcoming a). Examples include metaphysical claims and arguments involving quantification over propositions, properties or modalities. Typed languages are shown to provide a fruitful framework for a systematic formalization of this kind of metaphysical reasoning (for example, Dorr 2016).

The deductive correspondence between reasoning in many-sorted logic and reasoning in one-sorted logic suggests that untyped languages are expressively powerful in the sense relevant here. Any many-sorted system that faithfully represents reasoning in a metaphysical domain can be matched inside a one-sorted system. Thus the regimenting potential of typed languages does not exceed that of untyped languages. It should be emphasized that, by itself, this conclusion does not undermine the value of higher-order metaphysics. It might well be that typed languages provide a particularly good,

https://doi.org/10.1093/arisoc/aoad007

perhaps even heuristically optimal, environment for metaphysical theorizing. Still, any deductive insights obtained in that environment can be recovered in an untyped theory.

What applies to higher-order metaphysics also applies, *mutatis mutandis*, to any other area of inquiry that can be faithfully regimented in a many-sorted system. The deductive correspondence in question is a fully general metatheoretic result.

VI

Nonsense and Unnecessary Theoretical Choices. It was remarked in the introduction that the notion of expressivity plays a puzzling role in this debate. Typed languages seem more expressive, and yet a long-standing criticism of typed languages is that they have expressive limitations. According to critics, there are natural generalizations that cannot be expressed because they require quantification over types, which is not available in standard typed languages. In particular, there are 'deep and interesting semantic insights that cannot properly be expressed' (Linnebo 2006, p. 154), including the semantic principle of compositionality. (For discussion, see Linnebo 2006, §6.4, and Krämer 2013.) It seems that these expressive limitations can be overcome in untyped languages, where types are objects and thus belong to the domain of quantification. In this respect, untyped languages appear more expressive than typed ones. In this section, I discuss some objections alleging that the expressivity of untyped language goes too far, permitting the formalization of nonsense or forcing us to make unnecessary theoretical choices.[4]

Switching from a many-sorted system to a one-sorted system opens up new theoretical questions, as emphasized by Quine (1956, p. 268–9). Questions that might not be grammatically formulated in the many-sorted system become expressible in the one-sorted system. The type-theoretic solution to the paradoxes advocated by Russell relies essentially on making problematic expressions, such as self-application, ungrammatical. In a many-sorted system rendering the theory of types, the relation of predication would not apply to two terms of the same sort. Once we switch to a one-sorted system, self-predication becomes expressible again, and hence we can inquire

[4] Related considerations can be found in Jones 2018 and Bacon forthcoming.

Proceedings of the Aristotelian Society, Vol. XX, *Part* XX
https://doi.org/10.1093/arisoc/aoad007

into whether there is anything that can be predicated of itself. This phenomenon has been used to argue against untyped systems.

Consider a natural language like English. There is no significant morphological or syntactic distinction between, for instance, predicates that apply to concrete objects and those that apply to abstract objects. A speaker of the language can freely apply predicates of each kind to any class of entities. The results might be typical examples of category mistakes: 'every concept walks' or 'every natural number is green'. A many-sorted language has the resources to avoid these results. One could reserve different sorts for concepts, natural numbers, and concrete entities. Then appropriate sortal restrictions ensure that no variables can simultaneously occupy the argument positions of the following pairs of predicates: 'is a concept' and 'walks'; 'is a natural number' and 'is green'. These category mistakes can no longer be grammatically formulated. If we adopt a one-sorted language, this is obviously not an option. There is only one sort, and thus no straightforward way to impose restrictions on how predicates can be linked by variables. Category mistakes can be expressed, just as in natural language.

How can these observations be turned into a philosophical argument against untyped languages? Let us consider two ways.

First, one might hold that, by allowing category mistakes, the one-sorted language permits the expression of nonsense, and that this is undesirable. There are several problems with this line of thought. The main problem is that there are good reasons to reject the view that category mistakes are nonsensical. This view is admittedly natural and has enjoyed a lot of support. However, a range of considerations strongly suggest that category mistakes are in fact meaningful, as argued by Ofra Magidor (2013, ch. 3). To begin with, it is generally accepted that category mistakes are grammatical. The one-sorted framework respects this fact, unlike the use of sorts to ban category mistakes as ungrammatical. Moreover, the assumption that category mistakes are grammatical but meaningless clashes with linguistic evidence and theoretical considerations. Let us summarize the key points (for a detailed discussion, see again Magidor 2013, ch. 3): (a) category mistakes are context-sensitive in a way that meaninglessness is not; (b) category mistakes can be felicitously embedded in attitude reports, but nonsense cannot; (c) a compositional theory of meaning will likely imply that many paradigmatic cases of category mistakes have meaning; (d) uses of category mistakes in figurative

Proceedings of the Aristotelian Society, Vol. XX, *Part* XX
https://doi.org/10.1093/arisoc/aoad007

speech are best explained if we assume that category mistakes are meaningful. While it may be debated how compelling each point is, they collectively make a strong case for the meaningfulness of category mistakes. It follows that formalizations permitting the expressions of category mistakes might be theoretically desirable rather than undesirable. Of course, category mistakes tend to be practically pointless: there is usually very little value in attempting to settle them. But protecting us from futility has never been the goal of a formal system.

A second way to argue against untyped systems focuses on the theoretical choices we face if we adopt a single style of variables, choices thought to be undesirable. Barrett and Halvorson write: 'eliminating sort distinctions forces us to make unnecessary conventional choices about how to extend predicates beyond their original range of application' (2017, p. 3578). Category mistakes give rise to such choices. If we can say 'every concept walks' or 'every natural number is green', we face the question whether we should take these sentences to be true or false. By contrast, the many-sorted framework can sidestep these conventional choices: 'it does not force us to apply predicates in cases where we have no good reason to say that they do (or do not) hold of the items in question' (Barrett and Halvorson 2017, p. 3578). As observed, in our examples we can simply avoid the problem by using different sorts for concepts, natural numbers, and concrete entities, with predicates restricted appropriately.

How significant is this problem? Let us consider two perspectives from which the question may be approached. From an axiomatic perspective, we are not always forced to make the relevant conventional choices. We may be happy for our theories to remain incomplete in these ways, given that the sentences left open are uninteresting. When conventional choices are required, either because we aim for completeness or because settling certain questions has theoretical value, there is no technical obstacle. Barrett and Halvorson observe that care is needed to ensure that the theory is extended consistently: on pain of contradiction, we cannot stipulate that all the sentences in question are false (2017, p. 3577). But we know that if care is taken, we can succeed. By Lindenbaum's lemma, any consistent first-order theory has a complete consistent extension.

Now consider the problem from a standard model-theoretic perspective. Any model of a consistent theory takes a stand with respect to the truth-value of every sentence in the language. So in building

https://doi.org/10.1093/arisoc/aoad007

a model (for example, to prove the consistency of a theory), conventional choices might have to be made. In particular, we need to fix the extension of every predicate in the object language, deciding for each object in the domain whether it satisfies the predicate. Still, these choices occur in the metatheory and are only relative to a model. They need not settle how the one-sorted theory is actually extended. That is, choices concerning the extensions of predicates in a *model* need not fix how the predicates' ranges of application are extended in the *theory*. In any case, if one is ultimately committed to avoiding such choices, one can still stick with a one-sorted language by adopting a model-theoretic approach that delivers a partial logic.

Note that the issue under discussion is not confined to category mistakes, but is more general. It concerns genuine theoretical possibilities that are not expressible in a many-sorted setting but emerge in a one-sorted setting.[5] While some of these cases seem to require unnecessary conventional choices, not all do. Some cases concern theoretical questions we wish to express and even settle in a certain way, because of a powerful intuitive pull towards one particular answer. This suggests that the one-sorted framework can afford greater expressivity of a kind that matters, as stressed in Florio and Linnebo 2021 (§§11.6 and 11.7). Quine (1956) offers some examples. One concerns the possibility of identifying classes with co-extensive sets after translating a two-sorted class theory into a one-sorted theory (p. 269). Another concerns the possibility of identifying empty classes from different types after translating the theory of types into a one-sorted set theory (p. 274).[6] Let me provide a different example from plural logic.

The standard formulation of plural logic is two-sorted. There is one sort for objects and another sort for plural talk about objects. The distinction is often marked by the use of single variables (x, y, z, …) for the first sort and double variables (xx, yy, zz, …) for the

[5] Functions provide a clear example. Consider a function taking an input of sort i_1 and returning an output of sort i_2. In a many-sorted setting, the domain of the function is strictly confined to sort i_1: the syntax of the language forbids applications to inputs of other sorts. This restriction is lifted when we switch to a one-sorted setting, where new theoretical possibilities emerge. One must choose whether to treat the function as partial or total. If the function is treated as total, additional choices concern the function's behaviour over new inputs.

[6] In that case, Quine points out, it is also possible to identify an individual with its singleton set. Though natural and useful in some contexts (for example, Schwarzschild 1996, p. 1; Fine 2010, pp. 574–5), this identification does not generally have a strong intuitive pull.

https://doi.org/10.1093/arisoc/aoad007

second sort. So we read '$\exists x Px$' as 'something is P' and '$\exists xx Qxx$' as 'some things are (collectively) Q'. The system includes a predicate for plural membership, corresponding to 'is one of' and relating variables of the first sort to variables of the second sort. Consider any object a. The axioms of plural logic entail that there are some things such that a and only a is one of them. Call these things aa. Roughly put, we have a 'plurality', aa, with one and only one member, namely a. An intuitive question is whether a and aa are identical. The answer we would intuitively like to give is that they are indeed identical. After all, there is nothing more to aa than a. The question cannot be addressed in the two-sorted setting, because there is no cross-type identity relation. By contrast, using special predicates (for example, 'is an object' and 'is a plurality'), the desired identification can be expressed in a one-sorted system for plural logic, where it is perfectly consistent to assume that a is identical to the plurality whose only member is a. It is interesting to note here that the one-sorted approach to plural logic has gained popularity in recent literature (see Oliver and Smiley 2016, ch. 15; Florio and Linnebo 2021, §11.7).

VII

Conclusion. Which form should our theories take? I have argued that expressivity does not clearly favour typed languages over untyped ones. First, appealing to greater expressive power in support of typed languages is not effective. On a semantic understanding of expressive power, we reach at best a conditional conclusion: we can come to see a typed language as more expressive if and only if such a language has already been adopted as a metatheoretic framework. On a syntactic understanding, the potential of typed languages to capture deductive reasoning does not exceed that of untyped languages. Second, untyped languages are not too expressive in problematic ways, despite permitting the formalization of category mistakes and opening up new theoretical questions.

Of course, typed languages might still be preferable because of other theoretical or practical considerations (Manzano 1996, §1.6; Halvorson 2019, §5.4; Fritz and Jones forthcoming b). Here I have offered only a partial vindication of untyped languages. As far as expressivity is concerned, one may continue to follow Quine's advice

https://doi.org/10.1093/arisoc/aoad007

to 'pool types and get on with homogeneous variables' (Quine and Carnap 1990, p. 353).[7]

Department of Philosophy
University of Birmingham
Edgbaston
Birmingham B15 2TT
UK

Department of Philosophy, Classics, History of Art and Ideas
University of Oslo
Postboks 1020
Blindern 0315, Oslo
NORWAY

References

Bacon, Andrew forthcoming: 'A Case for Higher-Order Metaphysics'. To appear in Fritz and Jones forthcoming b.

Barrett, Thomas William, and Hans Halvorson 2016: 'Morita Equivalence'. *Review of Symbolic Logic*, 9(3), pp. 556–82.

——2017: 'Quine's Conjecture on Many-Sorted Logic'. *Synthese*, 194(9), pp. 3563–82.

Benacerraf, Paul, and Hilary Putnam (eds.) 1983: *Philosophy of Mathematics: Selected Readings*. 2nd edn. Cambridge: Cambridge University Press.

Boolos, George 1984: 'To Be Is to Be a Value of a Variable (or to Be Some Values of Some Variables)'. *Journal of Philosophy*, 81(8), pp. 430–49.

——1985: 'Nominalist Platonism'. *Philosophical Review*, 94(3), pp. 327–44.

Button, Tim, and Robert Trueman 2022: 'Against Cumulative Type Theory'. *Review of Symbolic Logic*, 15(4), pp. 907–49.

Corfield, David 2020: *Modal Homotopy Type Theory: The Prospect of a New Logic for Philosophy*. Oxford: Oxford University Press.

Degen, Wolfgang, and Jan Johannsen 2000: 'Cumulative Higher-Order Logic as a Foundation for Set Theory'. *Mathematical Logic Quarterly*, 46(2), pp. 147–70.

Dorr, Cian 2016: 'To Be F Is To Be G'. *Philosophical Perspectives*, 30, pp. 39–134.

[7] For helpful comments and discussion, I would like to thank Laura Crosilla, Nick Jones, Øystein Linnebo, Guy Longworth, M. M. McCabe, Carlo Nicolai, Chris Partridge, Sam Roberts, Stewart Shapiro, Eric Snyder, and Matthew West. I am also grateful to audiences at the University of Oslo and at the meeting of the Aristotelian Society where the article was presented. This work was supported by the Arts and Humanities Research Council (grant number AH/V015516/1).

https://doi.org/10.1093/arisoc/aoad007

Enderton, Herbert B. 2002: *A Mathematical Introduction to Logic*. 2nd edn. San Diego, CA: Academic Press.

Field, Hartry 2004: 'The Consistency of the Naive Theory of Properties'. *Philosophical Quarterly*, 54(214), pp. 78–104.

Fine, Kit 2010: 'Towards a Theory of Part'. *Journal of Philosophy*, 107(11), pp. 559–89.

Florio, Salvatore, and Nicholas K. Jones 2021: 'Unrestricted Quantification and the Structure of Type Theory'. *Philosophy and Phenomenological Research*, 102(1), pp. 44–64.

Florio, Salvatore, and Øystein Linnebo 2021: *The Many and the One: A Philosophical Study of Plural Logic*. Oxford: Oxford University Press.

Fritz, Peter, and Nicholas K. Jones (eds.) forthcoming a: 'Higher-Order Metaphysics: An Introduction'. To appear in Fritz and Jones forthcoming b.

—— forthcoming b: *Higher-Order Metaphysics*. Oxford: Oxford University Press.

Gödel, Kurt [1944] 1983: 'Russell's Mathematical Logic'. In Paul A. Schilpp (ed.), *The Philosophy of Bertrand Russell*, pp. 125–53. Evanston, IL: Northwestern University Press. Reprinted in Benacerraf and Putnam 1983, pp. 447–69. Page references are to the reprint.

Halvorson, Hans 2019: *The Logic in the Philosophy of Science*. Cambridge: Cambridge University Press.

Jones, Nicholas K. 2018: 'Nominalist Realism'. *Noûs*, 52(4), pp. 808–35.

Jubien, Michael 1989: 'On Properties and Property Theory'. In Gennaro Chierchia (ed.), *Properties, Types and Meaning, Vol. I: Foundational Issues*, pp. 159–75. Dordrecht: Kluwer.

Krämer, Stephan 2013: 'Semantic Values in Higher-Order Semantics'. *Philosophical Studies*, 168(3), pp. 709–24.

——2017: 'Everything, and Then Some'. *Mind*, 126(502), pp. 499–528.

Ladyman, James, and Stuart Presnell 2018: 'Does Homotopy Type Theory Provide a Foundation for Mathematics?' *British Journal for the Philosophy of Science*, 69(2), pp. 377–420.

Linnebo, Øystein 2006: 'Sets, Properties, and Unrestricted Quantification'. In Agustín Rayo and Gabriel Uzquiano (ed.), *Absolute Generality*, pp. 149–78. Oxford: Oxford University Press.

Linnebo, Øystein and Agustín Rayo 2012: 'Hierarchies Ontological and Ideological'. *Mind*, 121(482), pp. 269–308.

McKay, Thomas J. 2006: *Plural Predication*. Oxford: Oxford University Press.

Magidor, Ofra 2013: *Category Mistakes*. Oxford: Oxford University Press.

Manzano, María 1996: *Extensions of First-Order Logic*. Cambridge: Cambridge University Press.

Menzel, Christopher forthcoming: 'Pure Logic and Higher-Order Metaphysics'. To appear in Fritz and Jones forthcoming b.

https://doi.org/10.1093/arisoc/aoad007

Oliver, Alex, and Timothy Smiley 2016: *Plural Logic: Second Edition, Revised and Enlarged*. Oxford: Oxford University Press.
Quine, W. V 1956: 'Unification of Universes in Set Theory'. *Journal of Symbolic Logic*, 21(3), pp. 267–79.
——1986: *Philosophy of Logic*, 2nd edn. Cambridge, MA: Harvard University Press.
Quine, W. V and Rudolf Carnap 1990: *Dear Carnap, Dear Van: The Quine-Carnap Correspondence and Related Work*. Edited and with an introduction by Richard Creath. Berkeley and Los Angeles, CA: University of California Press.
Russell, Bertrand 1908: 'Mathematical Logic as Based on the Theory of Types'. *American Journal of Mathematics*, 30(3), pp. 222–62.
Shapiro, Stewart 1991: *Foundations without Foundationalism: A Case for Second-Order Logic*. Oxford: Oxford University Press.
Skiba, Lucas 2021: 'Higher-Order Metaphysics'. *Philosophy Compass*, 16(10), pp. 1–11.
Rayo, Agustín 2002: 'Word and Objects.' *Noûs*, 36, pp. 436–64.
Rayo, Agustín and Gabriel Uzquiano 1999: 'Toward a Theory of Second-Order Consequence'. *Notre Dame Journal of Formal Logic*, 40(3), pp. 315–25.
Rayo, Agustín and Timothy Williamson 2003: 'A Completeness Theorem for Unrestricted First-Order Languages'. In J C Beall (ed.), *Liars and Heaps*, pp. 331–56. Oxford: Oxford University Press.
Roberts, Sam MS: 'The Iterative Conception of Properties'. Unpublished manuscript.
Schindler, Thomas 2019: 'Classes, Why and How'. *Philosophical Studies*, 176(2), pp. 407–35.
Schwarzschild, Roger 1996: *Pluralities*. Dordrecht: Kluwer.
Smith, Barry 2005: 'Against Fantology'. In Johann C. Marek and Maria E. Reicher (eds.), *Experience and Analysis*, pp. 153–70. Vienna: HPT and ÖBV.
Trueman, Robert 2021: *Properties and Propositions: The Metaphysics of Higher-Order Logic*. Cambridge: Cambridge University Press.
Yi, Byeong-Uk 2005: 'The Logic and Meaning of Plurals, Part I'. *Journal of Philosophical Logic*, 34, pp. 459–506.
——2006: 'The Logic and Meaning of Plurals, Part II'. *Journal of Philosophical Logic*, 35, pp. 239–88.
Williamson, Timothy 2003: 'Everything'. *Philosophical Perspectives*, 17, pp. 415–65.
——2013: *Modal Logic as Metaphysics*. Oxford: Oxford University Press.
Wittgenstein, Ludwig 1979: *Notebooks 1914–1916*. 2nd edn. Oxford: Basil Blackwell.

Proceedings of the Aristotelian Society, Vol. XX*, Part* XX
https://doi.org/10.1093/arisoc/aoad007