

# A Structuralist Proposal for the Foundations of the Natural Numbers

Desmond A Ford 2019  
mondford@slingshot.co.nz

## ABSTRACT

This paper introduces a novel object that has less structure than, and is ontologically prior to the natural numbers. As such it is a candidate model of the foundation that lies beneath the natural numbers. The implications for the construction of mathematical objects built upon that foundation are discussed.

Note: This paper is the last in a series of three linked papers, the other two being:

- [1] The Gedanken Ball-and-Stick Construction Problem: What is the Most Simple Structure that it is Possible to Construct?
- [2] Icosahedral Quasicrystalline Structure Modelled as a Dynamically Updating System.

## 1. Introduction

As a starting point we take the conventional position that it is yet an open question as to what lies beneath the natural numbers – sometimes referred to as a “crisis” in the foundations of mathematics (e.g., [3]). The prevailing view may be that of the logicist, that it is logic that lies at the foundations. Here, however, we take a structuralist approach in which it is considered that beneath the natural numbers is an underlying structure, specifically one that is exemplifiable in a concrete model. To quote Shapiro, “the role of concrete and quasi-concrete systems is the motivation of structures and the justification that structures with certain properties exist” [4]. The objective of this paper is to show that there is a fundamental mathematical object, exemplifiable in a concrete model, that has less structure than, and is ontologically prior to the natural numbers.

The following Sections introduce a candidate model for that foundational structure. First, an argument is made for the necessity of such a structure, and then the bottom-up construction of that structure is initiated, referring to associated paper [2] where a detailed description of the analogous concrete model can be found.

The proposal of a new mathematical object as the foundation that underlies the natural numbers has implications, naturally, for the construction of mathematical objects built upon those foundations. Some of those implications are discussed in this paper.

## 2. Identifying a Problem

Some basics accepted first are that the natural numbers can be referred to by symbols, where each symbol can be embodied in a concrete or quasi-concrete object, perhaps quite primitively by a simple mark such as a vertical stroke. From there the sequence of natural numbers can be generated recursively from the bottom up in successive steps. Below are examples of the first three steps quasi-concretely embodied as: (a) a primitive mark, (b) set theoretic notation, and (c) in graph format.

### Step 0:

- (a) | The first primitive mark.
- (b) {} The empty set
- (c) • The origin vertex  $O$

Step 0 has introduced three separate examples of quasi-concrete objects from which a structure that ties to the sequence of natural numbers can be constructed. In this setting we will call these first objects the *fundamental objects*. The operation that is required at this stage is addition that can act recursively on the fundamental objects, which gives the notion of the successor object.

### Step 1:

- (a) ||
- (b) {{ }}
- (c)  $\overset{O}{\bullet} \text{---} \overset{A}{\bullet}$  Vertices are labelled origin vertex  $O$  and the successor vertex  $A$ .

### Step 2:

- (a) |||
- (b) {{{ }}{}}
- (c)  $\overset{O}{\bullet} \text{---} \overset{A}{\bullet} \text{---} \overset{B}{\bullet}$

The operations outlined above require (i) the notion of the fundamental object, and (ii) the notion of the successor function that acts recursively to reproduce and concatenate the fundamental object to construct the sequence of objects.

With respect to the above examples, it is axiomatic that the fundamental object at Step 0 should be identical to the fundamental object at any subsequent step such that the structure formed is translation invariant. We can say further that it is a basic principle of the construction of the structure that underlies the natural numbers that it should be composed of identical fundamental objects. At Step 2, however, it is already apparent that there is a problem with this principle, most clearly evidenced in the graph format, example (c).

The relations that any object may have with immediate predecessor or successor objects is analogised in the graph by edges that span between pairwise adjacent vertices. The edges incident to each vertex can be quantified in terms of the valence or the degree. At Step 2, *O* is degree 1, *A* is degree 2, *B* is degree 1. We can refer to this as the “problem of the non-identical fundamental object”.

With respect to the inductive process of the bottom-up construction of the structure, to write the instruction that prescribes the fundamental objects fully will now require additional prescriptive information input to specify unique cases with respect to their state of relations. And this will be the case at each successive step.

Of course, the recursive process of adding new fundamental objects to the structure can continue indefinitely (and if extended to the integers, then indefinitely in both directions) so that the problem, as it has been identified, may be successively with each step pushed further out toward the extremities at infinity. In fact, the requirement of translation invariance can be said to imply that the structure is necessarily infinite, however, in the context of the first principles, bottom-up construction that is employed here, to invoke that as a solution to the problem would be a circular argument.

Clearly there is a broader philosophical discussion to be had involving infinite structures and processes that is relevant to the above problem – recognizing, for example, that Cantor introduced the philosophical notion of the infinite object as a complete object. The approach here, however, is to tentatively accept that the non-translation invariant structure that is composed of objects that include non-identical, then non-fundamental objects, as it has been identified above, is a problem, and to approach a solution to the problem from a structuralist perspective.

To clarify; this is, of course, not referring to a problem with the natural numbers as counting numbers. Rather, the problem that we are pointing to here is that the linearly recursive structures that have (implicitly or explicitly) been taken to lie at the foundations of the natural numbers are (at least with respect to the example pointed to above) not fundamental – they have added structure.

So when we ask what lies beneath the natural numbers the above argument is saying that if what is required is some ultimately fundamental precursor structure, then that is not exemplified in the graph shown in example (c). On that basis, the objective now is to investigate whether, for the indefinitely extensible structure that underlies the natural numbers, there is an optimal structure such that for any specific

subset of that structure that we may sample, that subset will be composed of identical fundamental objects such that the structure is translation invariant.

It is proposed that evaluation of the strength or weakness of the above argument that points to there being a problem with the conventional notion of the linear ordering of the structure that underlies the natural numbers should include an evaluation of the model that is developed here in response to that problem.

### 3. Initiating a Solution

Referring back to Section 2 and looking at the graph (c) from Step 2, the vertices of that graph are now reconfigured below (Fig. 1) so that they form a cyclic graph in which all vertices are now degree 2.

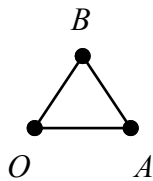


Figure 1. Cyclic graph

In the cyclic graph, Figure 1, the fundamental objects (represented by vertices) no longer have distinguishing features (i.e., non-identical degree) and can now, as a collection, be prescribed with minimal information input. There are no longer unique cases with distinct features that require individual specification.

Obviously, although the revised configuration in the cyclic graph has resolved the problem of non-identical degree count, the fundamental objects represented by vertices labelled  $O$ ,  $A$ ,  $B$  could be mapped to the natural numbers, 0, 1, 2, indicating successor/predecessor relationship between 0 and 2, which obviously does not model the sequential ordering of the natural numbers.

#### 3.1 Hierarchical layers of structure

Section 2 made the clarification that the problem of the non-identical fundamental object is not referring to a problem with the natural numbers per se, but with the structure that underlies the natural numbers.

We are differentiating between, first, a foundational structure that is the ontologically prior structure, that we will refer to as *level 1 structure*, and second, a structure that is merely a system that is an instantiation of that foundational structure, that we will refer to as *level 2 structure*. The examples (a), (b) and (c) shown at Step 2 are *level 2 structures* that have the additional structure that constrains them to a linear ordering, and example (b) also has the set theoretic notion of structure that presupposes the concept of set.

The proposal now is to develop the level 1 structure that is composed of fundamental objects in fundamental relations. This is a pre-ordered structure without the added level 2 structure of a linear ordering (or the concept of set), and it is also a requirement that it should not have the additional structure implied by non-identical fundamental objects.

The proposed structure is conceived of as the level 1 fundamental underlying structure over top of which there is a level 2 less-fundamental ordered structure that supports the sequence of successor relations that make up the natural numbers. As has been noted above, the cyclic graph shown in Figure 1 is obviously not sufficiently extensive to model the natural number sequence.

We also note that (as is the case with the construction of both set theory and the natural numbers) the proposed fundamental structure should be conceived of as a bottom-up construction employing the successor function that acts recursively on the fundamental object to produce a sequence of successor objects. Terminating the operation of the successor function at some specific fundamental object would require the additional structure of some instruction to effect that termination, and would, as a consequence, single out that terminal object as special. That structure would then no longer be considered to be fundamental. Therefore, in the absence of that instruction it follows that the fundamental level 1 structure is indefinitely extensible.

There are now some criteria established for the level 1 structure: 1) It is to be constructed from the bottom-up as the recursive sequence of fundamental objects. 2) The structure is indefinitely extensible but any subset that we sample must be translation invariant, meaning that the constituent fundamental objects are necessarily in all respects identical. Every sample sequence is made up of fundamental objects where there is no special boundary object.

Having made the distinction between the layers of structure, it appears that in a casual view of the natural numbers there is commonly, implicitly or explicitly, a conflation of level 1 and level 2 structures.

### **3.2 Quantifying structure: an informational approach**

Having the notion of hierarchical layers of structure as outlined above entails being able to evaluate the fundamentalness of the structure of an object (where that *object* can be a *structure*) which associates with determining how much structure that object has. In group theory, for example, the amount of structure is quantified in terms of the amount of symmetry that object has, typically expressed as the size of the object's automorphism group. The amount of structure an object has, or how fundamental that object is, is inversely proportional to the size of its automorphism group.

Group theoretic methods are appropriate to the task of comparing and evaluating structures that are extant, but the approach here (as in set theory and the natural numbers) is to begin with nothing and to construct from the bottom up. In this case it is necessary to ensure that at each stage the action taken best contributes to a resulting structure that is maximally fundamental, where that can be equated with being a maximally information entropic structure, meaning that for this situation an

informational approach is useful. (In fact, as we adopt this approach we see at later stages, i.e., subsection 6.2, that the group theoretic method of evaluating the amount of structure an object has presumes background structure that the bottom-up informational approach does not permit.)

At each step in the construction, the amount of structure the object has can be quantified in terms of the amount of prescriptive information that is required to produce it. A structure is maximally fundamental if it is the structure that expresses maximum information entropy, which will be the case if at each stage as it is constructed from the bottom up, the associated prescriptive information, or the instruction required to produce it, is minimal.

Given all of that, the objective now is to construct the maximally fundamental structure as a continuation of the cyclic graph (Fig. 1) under a regime where prescriptive information input is blocked from entering the process. At this introductory stage information can be dealt with informally as semantic content; i.e., instructional information (inputs) and factual information (outputs) can be recognized as ordinary language statements. Because the intent is that all potential information inputs are to be recognized and blocked from entering the system, there is not the requirement of formal information-theoretic methods of quantifying information precisely (as e.g. bits or byts), simply because (to the extent that the regime is successful) there should be no information entering the system to quantify.

With respect to employing the informational approach, we can have confidence that in constructing the proposed structure as a continuation the graph (Fig. 1), the objects of the structure (in this case the quasi-concrete vertices) are the fundamental, zero-information objects; and we can have confidence that it is in principle possible to ensure that, over and above the bottom up recursive process of accreting vertices to the graph (that parallels the formally understood recursive construction processes at the basis of set theory and the natural numbers), any potential extraneous assembly instruction (additional prescriptive information input) can be recognized and blocked – thus producing the maximally information entropic fundamental structure.

#### **4. The Candidate Structure: Introducing QPD Space**

In the Sections leading up to this point the objective of this paper has developed into that of producing (as a continuation of the graph construction, Figure 1) an optimal structure that satisfies the following criteria:

- 1) It is maximally information entropic.
- 2) It is constructed of fundamental objects, where the construction may be indefinitely extended.
- 3) Any finite substructure that is sampled is composed of fundamental objects such that there is no special boundary object. The proposed structure is

possibly infinite, and there is the condition that every finite substructure does not have a boundary object.

Note: Only the first criterion stating that the structure should be maximally information entropic is axiomatic (i.e., involves instruction input). It follows that subsequent criteria must have their logical origins there, and do not require information input.

In response to the need for a new structure that satisfies the above criteria we refer to a candidate model (as mentioned in the introduction) that has been developed in associated papers [1] and [2]. In those papers the structural analysis of the optimal fundamental object that satisfies the criteria set out above is initiated by constructing an analogous concrete model (ball-and-stick model). The construction of that model begins to coincide with general space-filling problems, and more specifically, the problem of constructing the ideal icosahedral quasicrystalline lattice. The structure that is produced in [2] is referred to as the *quasicrystalline pentakis dodecahedral space*, or QPD space.

#### **4.1 From the cyclic graph, Figure 1, to the candidate model, QPD space**

The objective in this subsection is to introduce QPD space as a continuation of the cyclic graph, Figure 1. The minimal prescriptive information basis means that there is no presupposed background space and no presupposed notion of the dimensionality.

In Section 2 the original problem of non-identical degree was evident at Step 2, graph (c). This was resolved in Section 3 by reconfiguring the vertices to form the cyclic graph, Figure 1, thus taking the graph from a one-dimensional structure to a two-dimensional structure. This resetting of the spatial dimensionality was not an arbitrary top-down decision to work in some preferred dimensional setting; rather, it was brought about by the intrinsically originating imperatives – specifically, the need to configure the structure such that it best produces uniform degree count across the constituent vertices.

Continuing the bottom-up construction of the graph by accreting successor vertices on the established basis (i.e., so that the resulting structure exhibits maximally uniform degree count across all vertices) will subsequently force the construction to produce a three-dimensional structure – i.e., from the two-dimensional graph at Figure 1 that is the equilateral triangle  $OAB$ , the accretion of the successor vertex  $C$  produces the three-dimensional tetrahedron,  $OABC$ . But in [2] it is observed that further continuation of the growth process (the accretion of successor vertex  $D$  or any number of additional vertices) does not similarly force the configuration of vertices from three-dimensional structure to a structure of  $>3$  spatial dimensions.

#### **4.2 The analogous concrete model, the ball-and-stick construction**

According to the above, the continuation of the graph construction is necessarily producing an innately three-dimensional structure. We now make the transition from

this paper's graph constructions to the three-dimensional analogue of that, as presented in [1, 2] with a modelling system based on a conventional ball-and-stick model. The original concept of the successor function that is exemplified by the accretion of vertices to a graph is now modelled as the assembly of the three-dimensional structure embodied in the ball-and-stick model. Note: In this paper "Ball" is now synonymous with (and used interchangeably with) "vertex" and the "stick", also referred to as a "rod", is synonymous with "edge".

In this paper's development of the graph up to this stage (and in [1]), the emphasis has been on uniformity of degree count, whereas in [2] the focus goes to producing a more general uniformity and the globally symmetric structure that results from the minimal prescriptive information basis. In both cases the effect is that the uniformity/symmetry requirement biases the construction toward producing cluster formation. In [2] subsections 2.1 and 2.2, this appears in the ball-and-stick model as balls clustered about the central origin ball  $O$ , resulting in the geometrically frustrated icosahedral quasicrystalline structure [5].

### 4.3 Recap

The problem that has origins in the objective that was set, i.e., to identify the structure composed of identical fundamental objects; becomes the problem of constructing the graph that has identical degree count across all of the vertices; which biases structure growth toward producing cluster formation; that becomes the problem of resolving geometrical frustration and the lack of translational periodicity in the three-dimensional icosahedral quasicrystalline structure; where in [1, 2] that structure has been exemplified with a ball-and-stick construction, and where the full development produces QPD space.

### 4.4 Key features of QPD space listed:

**1. QPD space as a dynamically updating system:** Where conventional models resolve the aperiodic and geometrically frustrated structure of the clustered growth model in an ideal structure theorised to live in hyperdimensional space, QPD space constructs, as an alternative approach, the ideal model for the icosahedral quasicrystal as a dynamically updating system; but where that includes explicit recognition of the role of the static model that only approximately captures the ideal structure's perfect symmetry and periodicity.

**2. Quantum geometric growth and the penetration twinned IQC building blocks:** Trialling a stepwise successive growth model showed that it was not possible to produce the required identical degree count across all of the vertices at all stages of construction. The correct growth mode was then identified to be centrally symmetric quantum growth that produces at each growth step a complete shell of the cluster configuration. The completion of the first shell produced the IQC, which is the



fundamental building block from which the entire QPD space is constructed; recognising the feature that those building blocks coalesce in penetration twinned morphology (once again, in order to satisfy the uniform degree count criterion).

**3. Fractal growth:** The growth model described above produces the first shell, the IQC, the second shell, the QPD, but continued growth does not produce further shells. Rather, there is a seamless phase transition to the second fractal layer that reproduces the construction of the prototypical QPD structure. Although the IQC is recognised as the fundamental building block, the QPD is identically the first atom and the final global structure, iterated through indefinitely extended fractal layers.

**4. Spatial migration, ubiquitous helicity and probabilistic fundamental objects:**

Note 1 above indicates that QPD space is a dynamically updating system that is necessarily interpreted (imperfectly) in a static model. Further to that, a static snapshot that gives the microstate of the system shows (at all resolution) two adjacent vertices (located within a delineated range of position) with a discrete interval between. It is conjectured that the dynamical system produces, over multiple updates, a general migration, or evolution, of the overall structure such that the static interval can be thought of as being filled with those vertices that, it can be inferred, would be produced over the multiple updates.

For the static model that captures a snapshot of the dynamically updating system, those geometric objects that infill the interval (i.e., virtual vertices that are in a state of creation or annihilation) can only be represented probabilistically, and are only locatable within a range of position. However, for each interval segment the geometry that delineates that three-dimensional range of position (a tubular volume) within which a vertex may be created is straightforward to produce, as are some basic concatenation rules that are implicit to the structure; so that those tubular interval segments link together in a specific way to form helically configured pathways (referred to as range-tube pathways) through QPD space ([2] Sections 14, 15, 16). The range-tube pathway is a new geometric object that tracks in the static model the three-dimensional tubular volume that gives the range of position within which the flow of vertex creation and annihilation occurs in the dynamical system.

The entire static model that captures the structure of QPD space can be constructed, propagating out from origin vertex  $O$  such that it produces the icosahedral quasicrystalline lattice structure that is mapped with helically configured pathways that give the range of position within which the fundamental objects of the structure can be probabilistically located.

## 4.5 Summary

QPD space, as outlined in the subsection above (and as presented more comprehensively in [1, 2]), is introduced in this paper as the candidate model of the optimal, minimal, level 1 structure that is ontologically prior to the natural numbers.

Three main criteria were set out at the beginning of this section. We will look at these again below in light of the description we now have of QPD space:

- 1) *The optimal structure should be maximally information entropic:* It is fundamental to the development of QPD space that its construction is based on a program that involves blocking the prescriptive information input. This is also discussed in the following section.
- 2) *The optimal structure is constructed of fundamental objects, where the construction may be indefinitely extended:* Because fractal character emerges in the construction of QPD space, this means that it can extend indefinitely, yet we can also describe it definitively.
- 3) *Any finite substructure that is sampled must be composed of fundamental objects such that there is no special boundary object:* First, the bottom-up construction and the paucity of prescriptive information means that there is no pre-existing body of information that provides a prior, background meta-structure external to the QPD structure; therefore the only valid observer viewpoint is that of an *intrinsic* observer located at a vertex. Secondly, the intrinsic observer's location must coincide with a "created" vertex, then is correlated with that dynamically updating structure such that it is hypothesised in [2], Section 18, that regardless of how the observer viewpoint transitions within the structure, that viewpoint remains always at the centre. Thus, for any legitimate observer, the global structure, QPD space, is without boundary; and given the fractal character, features of the global structure are iterated in every sampled substructure.

## 5. Preliminary Notes on a Complimentary Formal System

The philosophical discussion around the extent to which relations of objects in the real world coincide with relations of objects within an abstract structure constructed from first principles is obviously a key part of the structuralist enquiry. Without attempting to rehearse any of that here, we can say that a preliminary look at QPD space in relation to a comparable formal system suggests that (at least in the case of very fundamental models) the two – the formal and the concrete – do appear to coincide.

In this paper the model has been first constructed as a graph. This is not a schematic diagram of abstract relations, but rather, it is those abstract relations directly analogised in real world spatial relations. This is given an even more literal representation in [1, 2] where it appears as a conventional ball-and-stick model that is constructed in a non-predefined space that nevertheless matches with our conventional intuition of three-dimensional space and in which we see very literal geometric imperatives (such as objects that either physically fit together in that space or they don't). Reading these imperatives as visual inferences is predicated upon our

earlier stated (Section 1) structuralistic understanding of the role of concrete and quasi-concrete systems with respect to analogising the properties of structures.

It is also acknowledged, however, that obtaining visual inference by constructing a concrete model in a non-predefined space involves some trade-off in favour of intuition at the expense of rigor. It is on the basis of visual inference that we attribute certain characteristics to the relational structures – e.g., the way in which structure growth is considered to have auto-terminated, or auto-completed, at specific configurations (usually a shell) based on the observation that all of the geometrically available vertex positions are occupied by a vertex.

There is also the informational aspect of this paper’s approach that defines the structure produced in the bottom-up construction process to be fundamental in proportion as the prescriptive information is successfully blocked from entering the process. This implies a concern to show rigorously that no information has entered the system surreptitiously through visual inferences encoded within the three-dimensional concrete model, but is not accounted for.

An obvious way to evaluate the informational integrity of the construction of the graph/concrete model is to create a parallel formal system, absent any spatial referent. Preliminary work undertaken separately from this paper indicates that there is a formal system (independent of the graph/concrete model) that reproduces the relations between objects that we have otherwise obtained from construction of the graph/concrete model and visual inference based on that. Initial work has shown that constructing the fundamental (i.e., maximally information entropic) structure combinatorially as strictly a list of objects and relations between those objects, requiring only some minimal axioms, reproduces the configuration of relations that we see in the concrete model of the IQC constructed in [2].

For example, the fundamental objects, labelled  $O$ ,  $A$ ,  $B$ , that in Section 3 of this paper were represented in graph format, can instead be formatted as a list. New objects can then be recursively accreted to grow the list, and the status of each object’s relations is assessed at each step; and the status of the overall structure formed by the aggregate of objects in the list can also be evaluated at each step. Preferred structures are those in which there is uniformity with respect to the relations of all objects in the list (i.e., the counterpart to uniform “degree” or “valence” in the graph/concrete model).

The cardinality of the list naturally increases at each step where a new object is introduced, and the associated relations also increase, so that the number of trials that have to be run before the optimal structure (i.e. that which produces the most homogenous distribution of relations) is identified, increases exponentially. It becomes apparent that in sentential format development of the structure requires running an increasingly large number of trials before arriving at the correct configuration.

The ameliorating aspect is that the simplicity of the list system means that the trials are suited to being run by a machine. However, although a machine can readily facilitate those operations, the results still require translation into a suitable interface that allows human comprehension; and that is best achieved by moving from the

sentential list-based format to a visual map-based format, i.e., essentially returning to an image of the three-dimensional graph/concrete model.

For that same overall structure of the relations between objects that is essentially unintelligible in sentential format, in the format of the graph/concrete model the key relations are readily apparent from a cursory visual analysis (i.e. a quick glance) – all of which attests to the (only recently quantified) large bandwidth (8.75 megabits per second) that the human brain can apply to the task of visual information processing.

It is anticipated that in following work it will be possible to develop a list-based formal system that will bring additional rigor to the modelling. However, the list-based format is not a suitable model with which to carry out this initial investigation. In fact, it is not fully possible for human cognition to apprehend the full patterns of relations between objects if working with the raw data in sentential format alone. Although the list-based model can apply rigorous analysis to an *existing* model of QPD space, and may in fact be important in the future development, it would not have *discovered* QPD space.

## **6. What QPD Space Isn't**

### **6.1 Euclidean 3-space**

In the model of QPD space as a dynamically updating system it is hypothesised that the mechanics of the asynchronous update process will generate a migration of the overall structure so that vertices are created (in some iteration) at every position throughout the three-dimensional space, thus producing an ideal macrostate in which there is the uniform distribution of vertices in a homogeneous and isotropic ambient space (see [2], Section 3).

Superficially it may appear that for that dynamical model of QPD space described above there should be a static representation that is isomorphic to Euclidean 3-space. That is not the case however. Beneath the infinitely dense space that is theorised to be produced over infinitely many updates of the dynamically evolving system, QPD space retains a sinuous and porous, or “lumpy” fractal structure such that for any specific microstate the space includes (not as defects, but as an aspect of the fractal structure) unmappable regions into which the structure and the geometry do not penetrate (see also subsection 14.3). The QPD concept of points filling the space has a specific dynamical interpretation that is not compatible with the conventional model of Euclidean 3-space where every point attaches to a real number on (static) coordinates such that those points fill the entire volume.

In a conventional model a perfectly dense space-filling taken to infinity destroys structure completely – the space of objects in relations becomes one object with no relations, or the graph becomes the abstract black hole of an infinitely large single vertex with no edges. On that understanding, the two notions, structure and infinite

space-filling, are antithetical.

In QPD space, however, we saw in [1, 2] that the bottom-up construction produced the hierarchically nested fractal layers of structure in which at all scale edges are separated by voids (otherwise, obviously, the edges would not be defined). This means that there is definitely structure present in the static snapshot of QPD space (the microstate), but it is also the case that this structure is smoothed out in the dynamical model (the macrostate). It is hypothesised that the entire structure migrates through all space over successive iterations of dynamical update, thus producing vertex creation uniformly densely with respect to the complete evolution of the global structure.

A second factor is that the idealised observer is necessarily intrinsic to the structure because at the fundamental level the structure is all that there is – there is no background because QPD space *is* the fundamental background. So, in a conventional understanding we see through holes in objects, where those holes are filled with the ambient medium they are set within, down to the fundamental case where it is the background space that is defined by the hole – beyond which we have the case of QPD space, where that *is the background*, and the hole in the background is a void.

In the fundamental structure for which there *is no more-fundamental background* the only valid viewpoint is necessarily within the structure itself, or the intrinsic viewpoint, which detects structure only where there is structure. The voids are not detected. In the analogous spatial construction exemplified by the graph (or the ball-and-stick model) the vertices and edges are the full extent of everything that there is.

The topological method is able to discern holes by tracking the ways in which pathways through the structure may intersect or not intersect, which can be imagined as a bug crawling on the surface. The bug can thus determine topologically whether it is, for example, on a torus or a sphere. But without leaving the surface, the bug has no direct impression of the hole. Similarly, for the “correct” idealised intrinsic observer no “voids” are detected directly. There are not gaps in the space, but rather, in the geometry of the static microstate there are portions of the overall space where there “is no space”.

As the fundamental underlying structure of space the dynamical model of QPD space is completely dense and perfectly filled, yet it is also captured in the associated static model, the snapshot, and that view reveals a sinuous and porous fractal structure. QPD space is not conceptually or structurally compatible with Euclidean 3-space.

## 6.2 Group theory and simplicial triangulation

Both the group-theoretic approach and QPD’s informational approach broadly agree that the amount of structure an object has is inversely proportional to the size of its automorphism group, or the amount of symmetry that it has. However, there are key differences between conventional symmetry calculation and the way that it is calculated in QPD space.

Group theory deals with symmetry as a static phenomenon, an object either has symmetry or it doesn't, whereas the QPD structure has perfect symmetry (in the mean) in the dynamically updating model where it is the subject matter, while simultaneously the geometric constructions in QPD space have very little symmetry in their representations in the static model. These irregular static representations (e.g. the IQC) can be thought of as having “quasi-symmetry”, given that they are actually referring to a dynamical, perfectly symmetrical ideal model.

Under a group-theoretic approach, considering for example the automorphisms of K3, these include reflections about a central axis for which there is no location on the vertices of K3, but for which the axis is located at some central position on the background. In QPD space, however, the vertices of the structure are the entirety of the structure. Outside of that, there is no background to support locations for axes of symmetry. Consequently, in QPD space, for the three vertices that make an equilateral triangle, symmetry calculations can only be applied with respect to an axis that coincides with vertices – in which case there are no reflectional or rotational symmetries at all for substructures of the IQC.

This aligns with the observation central to [1, 2] that the complete IQC is the quantum geometric building block for which, other than the origin vertex  $O$ , there are no valid intermediary subunits such as triangles or tetrahedra – which observation effectively militates against employing the simplicial triangulation approach to modelling the fundamental underlying structure.

## **7. A Fundamental Dichotomy Between the Ontology and Epistemology**

In considering QPD space as the model of the fundamental structure that lies beneath the natural numbers it is relevant to take account of a generally under-examined and irresolvable dichotomy that exists between:

- (a) Models that fit best with the subject matter (the ontology).
- (b) Models that fit best with the way in which humans apprehend and process that subject matter (the epistemology).

We're referring in (b) to models that fit best with the human brain, which can be thought of as a black box. Then (a) is the stimulus, or the subject matter (i.e., the universe), which is everything outside of the black box – noting that the stimulus is a system, entirely, that is dynamically updating. The output (b) is formatted as determinate statements, or static models. It appears that the black box is structured to receive dynamically updating inputs from the subject matter and process that as finite data from which it produces static/determinate outputs, such as measurements or statements of fact.

QPD space, as a dynamically updating system, is an (a)-type model that has the most direct isomorphism with the overwhelmingly dynamical universe that is the

subject matter. However, the (a)-type dynamical system is also the model that least-well matches the way that human cognition (the black box) is formatted to represent that subject matter.

The body of human knowledge is fundamentally composed of determinate statements of fact that correspond to a (b)-type static model (that does not exist in nature). In particular, static models that project to the 2D image plane match well with the way that the human brain takes the finite light data that triggers sensors on the planar surface of the retina and converts that into sequentially updating discrete frames that make up the mental image.

Here, we are saying that the material universe, as the subject matter, can be modelled as a perfect dynamical system, but where that can only be represented in static models that are imperfect; considering nevertheless that the static model is best suited to the internal modelling processes that are innate to human cognition. The general thesis of this paper is that the mathematical formalization that will develop from a revised concept of the foundations of the natural numbers is optimally a dynamically updating system; but noting here that this has to be returned to a static model. However, that static model is not any of the currently available traditional models, especially given that those do not generally recognise in their structures the inherent ontology/epistemology dichotomy.

## **8. A First Look at QPD Space as the Structure that Underlies the Natural Numbers**

According to the previous Section we have to consider that there are two distinct layers to the modelling of QPD space. There is, first, the dynamical model that has the most direct isomorphism with the subject matter, where that subject matter includes real world objects that the natural numbers are counting. Secondly, there is the static model that has the most direct isomorphism with the way that human cognition can process that subject matter. These two cases will be looked at separately below, but first a more general recap:

Section 2 introduced the graph to analogise the underlying structure from which the natural numbers can be constructed. The graph analogises the uniform character of the binary relations between fundamental objects that make up the structure by setting the pairwise adjacent vertices at uniform separations apart. The minimal prescriptive information basis has meant that there is no prior information available to provide a background structure comprising any system of points as an embedding space that might infill that separation. For example, the graph is not embedded in Euclidean space, so there is not the notion of the straight-line spanning between vertices. Nor is the theory conceptually embedded in the broader structure of classical mathematics that would give the traditional notion of the real interval as a subset of the real line.

Rather, we begin with a zero information “separation” between vertices, where that separation can be embodied in the concrete model by any arbitrarily shaped

object, requiring only that uniformity across those objects (via the primitive notion of congruence) can be established (thereby giving, obviously, the uniformity of spatial relations, and where that is analogising the uniformity of abstract relations). It is strictly due to obvious practical considerations (that are in themselves interesting but not investigated here) that the graph is tentatively drawn with *straight* lines representing the edges; and the model in [2] is tentatively constructed with *straight* rods connecting the balls. Also, we now use the terms “separation” and “interval” interchangeably, however, as noted above this is not referring to classical mathematics’ real interval.

## 8.1 The Dynamical model

With respect to the ball-and-stick concrete model that exemplifies QPD space, it is definitely possible to conceive of and visualise the dynamically updating system as an animated mechanical construction in which the ball/rod connections click together (indicating a created vertex) or pull apart (annihilated vertex) in sequentially triggered updates throughout the three-dimensional array of balls and rods.

This dynamical update action has the effect that the model that defines the entire space migrates. This, in turn, means that there is a geometry that defines the separation between pairwise adjacent vertices in terms of “virtual vertices” that mark out potential positions at which vertex creation may occur in some update of the system.

Where the interval was first provisionally represented with a straight-line, the geometry of the dynamical system now describes a tubular volume delineated by compound curves; furthermore, there is the associated mechanics that describes the way in which the curved interval segments concatenate to form helically configured strands of tubular pathway ([2] Sections 14,15,16).

More generally, the overall picture of QPD space is of a dynamically updating system that fills a global space made up of vertices where nearest neighbours are separated by discrete/uniform intervals. The frustration that is encountered in the space-filling geometry of the static structure determines that vertices necessarily update between causally correlated states of creation or annihilation, but where this produces the complete symmetry at the macroscopic level so that the global structure is, in the mean, translation symmetric and perfectly unfrustrated.

To recap: QPD space has an isotropic and homogeneous structure – it is a space in which uniformly dispersed fundamental objects (vertices) dynamically update between states of creation and annihilation. Every pair of adjacent positions defines a discrete-uniform interval, so that every extension through QPD space is made up of commensurable multiples of those curvilinear intervals. The concatenation of those intervals produces helically configured tubular pathways that extend through the fractally-layered structure.

In Euclidean space we may say that the shortest distance between two points, e.g.,  $B$  and  $C$ , is a straight line (although that is, of course, not strictly a “definition”).



The counterpart to that in QPD space would say that the distance between  $B$  and  $C$  is the pathway that requires minimal prescriptive information, producing minimal structure, i.e., the fundamental pathway. Clearly this is not the Euclidean straight line, but rather it is the QPD geodesic that has a specific helical configuration for which the development of the model in [2] describes the geometry. Not only is every QPD interval curved so that the concatenated intervals form the helically configured pathway, but the dynamical character determines that for the intrinsic idealised observer traversing that pathway, leaving a starting point vertex, the distance to some other remote vertex involves a dynamical update (that is correlated with the observer) so that the position of that end-vertex is updated as an aspect of that measurement event.

We've noted that the statement, "the shortest distance between two points is a straight line," is not a formal definition. Contrastingly, from QPD space a formal definition of the shortest distance between two points can be constructed when the notion "shortest" is taken to mean "the most fundamental pathway", defined in terms of the prescriptive information content required to construct it. Those pathways are shown in [2] Section 17, to have the characteristic that they appear always configured as strands of coiled helices.

The update action that we've described in QPD space (as an abstract mathematical space) is analogised in the measurement effects that are an integral part of physical distance measurement – most obviously at the small and large scale extremes of quantum and cosmological measurement. The shortest path in spacetime, for example, is defined by the geodesic, or the path that light takes between two points, where that is curved over the large scale and helical at photon scale, and where travel to a distant point, such as e.g. the particle horizon, involves travelling through expanding space with the effect that the endpoint position has updated during the measurement event.

Or, in Zeno's second paradox, 'The Achilles', it seems that every time Achilles covers the distance to where the tortoise is, the tortoise has moved a small distance further. Zeno's apparent paradox is a result of the arbitrary decision regarding where the race is stopped so that the positions of the runners can be examined (i.e., the static model); but noting, relevantly, that the dichotomy reflects the inherent and irresolvable imprecision involved in obtaining determinate results that require a static model that is abstracted from the dynamical reality. Noting also that this is modelled in the finite update system of the QPD space geometry without involving any special notions of a continuum of infinite processes or time. All vertices are in a finite created or annihilated state in the dynamical macrostate, but the static snapshot that produces the microstate can only express the positions and the created vs. annihilated states probabilistically.

It is emphasised in [2] that the domino effect means that QPD space is bound to update in discrete step changes, without an external cause, and without the requirement of any background clock to which updates must conform. Just as QPD space was not embedded in a continuous coordinate space, but rather it produces the notion of an intrinsic discrete space, neither is it necessary for QPD space to be

embedded within a time structure (discrete or continuous), but rather, both the coordinate space and the concept of the phenomenological passage of time can appear as an emergent property of the dynamically updating model of QPD space, where it is modelling the fundamental underlying structure.

The construction that is outlined in [1] and detailed in [2] gives an introductory yet reasonably complete initial model of the geometry of QPD space as the dynamically updating, helically configured, fractally layered space through which the idealised intrinsic observer transitions while always remaining at the centre (with the analogy that is available in the unreachable horizon for the observer who remains always at the centre of the observable universe).

Conventionally, the natural numbers (as a subset of the integers) are of course considered to correspond to specially marked points evenly spaced along a straight line. For QPD space, being the ontologically prior fundamental space, there is no straight-line, rather, there are the most-simple (i.e. maximally information entropic) pathways through the ambient space that are helically configured and are marked off with uniformly spaced positions. However, given that the positions locate vertices that may be in a created or annihilated state we have to give up on precise values and allow for some uncertainty.

It appears that QPD space is a system that operates according to principles that, broadly, have a quantum character suggesting compatibility with principles and operating properties inherent to quantum computation. The advantage anticipated is that when it becomes possible to perform calculations on this basis, it will naturally involve quantum properties that parallel those of the ambient universe that the calculations are trying to simulate. The problem that is not explained at this stage, however, is how we interact with the dynamical system to carry out those calculations.

## **8.2 The static model**

The observation in Section 7 is that, regardless of the fact that a dynamical system most accurately models the dynamical universe, human cognition will nevertheless involuntarily revert to processing it in terms of a static model viewed from the extrinsic perspective – and some concessions are involved in this.

In discussing the dynamical model of QPD space in the previous subsection we have said that we have to give up on precise values and allow for some uncertainty. That, however, is not strictly correct with respect to the dynamical model. It is not exactly clear at this stage whether dynamical QPD space is fundamentally deterministic or nondeterministic, however, what is being referred to above is that any attempt to obtain determinate values from the dynamical model involves deriving the corresponding model for a static instance, or microstate, and where dropping from the dynamical to the static requires a trade-off in the precision and certainty with which it is possible to construct the relevant geometric model, which is reflected in the values we obtain from it.

Every extension between the pairwise adjacent fundamental objects of

dynamical QPD space is the unit extension, and every update is not a progression through continuous time, but rather it consists of the vertices updating (in response to discrete causal imperatives) between finite states of creation or annihilation at positions located in the network of unit intervals. Nevertheless, as was exemplified in the frustrated geometry of the three-dimensional ball-and-stick model [2], it is not possible for all vertices to be located definitively for any static microstate of the entire system (or even for a small sample of the system, e.g., the IQC).

This imprecision is analogised very literally in [2] where the ball-and-stick model is adapted to give it a flexibility so that there is a range of position possible in each connection. This allows the model to be constructed so that for any static microstate it is possible to describe the geometry that gives the range of position within which any vertex may be located. The bounds of that range produce a surface that encloses a tubular volume in the three-dimensional model, referred to as the range-tube pathway. These form the helically configured pathways that can be accurately defined for the entire indefinitely extensible model of QPD space (due to the fractal character, once defined for the initial construction, obviously it is thus defined for the entire structure).

Although the range-tube pathways are well defined, when it comes to obtaining the mean value of the range-tube, the extrema are located at the centreline of the tube and at the outer surface; so that the mean also can only be expressed as a three-dimensional tubular section that falls between those extrema. Consequently there is no precise value that the range-tube can be collapsed to and the values remain indeterminate (see [2] Section 13). The results obtained can be increasingly refined by zooming in to smaller-scale structure, however, the geometric object that defines the range of values has, at all scale, a tubular form. So the refining process (analogous of a converging infinite series) never arrives at a precise value. *In QPD space there is no mathematical singularity.*

The concept of the QPD space interval is referred to in [2] as the hybrid discrete/pseudo-continuous interval. Any static snapshot that gives the microstate of QPD space can zoom in on two adjacent vertices (that are locatable within a range of position) showing a discrete interval between. Furthermore, the dynamical update of the system produces a general migration of the overall structure so that it can be inferred that, in the macrostate, vertex creation will occur with non-zero probability somewhere within the tubular volume of the interval.

So, in the static QPD space there are the two end-vertices that are probabilistic geometric objects that are each located (in either created or annihilated states) within a defined range of position such that they delineate the static interval (with a fuzzy precision). And there are the objects that infill the volume of the range-tube that spans the interval – these are the *inferred*, or *virtual* vertices for which the range-tube interval delimits the spatial volume within which they may be created in one of the (possibly infinite) successor updates of the network. On this basis the end vertices that *define* the interval and the virtual vertices that *infill* the interval do not have equivalent ontological status. Furthermore, the finite actions that sequentially update the creation and annihilation of vertices generate in the dynamical model an *illusory motion* within

the tubular volume enclosed within the helically configured range-tube pathways, giving the illusion of a continuous directional flow throughout the entire icosahedrally ordered lattice structure of QPD space; and where that can be mapped out in the static model. This is the model that produces what is referred to as the hybrid discrete/pseudo-continuous interval.

No correlation is immediately available between Euclidean space and QPD space through comparing arbitrary measurements. However, work that will be appearing in a forthcoming paper is currently investigating a correlation by comparing the radius/circumference ratio constants for the unit sphere embedded in the respective spaces. Although this work is not available yet, it is nevertheless possible at this stage to discuss the principle employed. Essentially, the helically configured (or, waveform in 2D) range-tube pathway that traces out the circumference of the QPD sphere (see [2], Section 12, Figs 21 and 22) is collapsed to give the Euclidean circumference equivalent for which the transformation to a straight-line measurement is, of course, conventionally obtained through application of the ratio constant  $\pi$ .

This work is interesting in that it begins with the unit sphere embedded in QPD space – where that is the perfectly homogeneous, dynamically updating space in which every extension is the uniform, curved, unit interval extension such that all points in the space are connected by commensurable multiples of the unit interval. For the unit sphere in that dynamically updating space, the geometric object that gives the radius is the curved unit interval that is commensurable with the waveform circumference path made up of multiples of that curved unit interval; where the ratio between those is the rational radius/circumference ratio 1:6.

This forthcoming work shows the transformation from the unit sphere in QPD space, where it has the rational radius/circumference ratio, to the equivalent in Euclidean space, where of course  $\pi$  gives the ratio. Although it was not an aim of this work, QPD space is essentially providing the basis for a theory that explains the ratio constant  $\pi$ . Archimedes' Exhaustion Method is considered the first theoretical method of obtaining  $\pi$ , and since that time mathematical formulae for converging series give  $\pi$  to ever increasing decimal places, however, they do not provide any fundamental underlying theory that explains why the particular value of the ratio is what it is.

In considering the relationship between dynamical curved QPD space and static Euclidean space there is now the basis for an argument to explain, for example, the reason why  $\pi$  does not terminate at a determinate value (i.e., the tubular volume from which the range of values for the interval must be collapsed) and to explain why it has the approximate value that it has – essentially the value for  $\pi$  is the approximation that is accepted in the transformation from the perfect dynamical ideal model, QPD space, wherein the radius/circumference ratio value is 1:6, to the imperfect static Euclidean geometry where the ratio constant is the transcendental number  $\pi$ .

To recap: From the dynamically updating curved space (that is homogeneous in the mean) wherein the unit sphere has a rational radius/circumference ratio, and wherein, more generally, all distances between points are commensurable, it is possible to derive the ratio constants of static Euclidean space. That is, we will be able

to show the transformation from the perfectly homogeneous QPD space in which all ratios are commensurable, to Euclidean space in which there is, in some cases, only the incommensurable ratios that give rise to the irrational numbers that classical mathematics deals with in the hierarchy of mathematical objects by constructing over top of the rational numbers the theory of the real numbers.

More importantly, we can then also contemplate the structuralistic argument that there is a rational (albeit dynamical) structure that supports an alternative to the classical concept of all irrational and transcendental numbers. *We are saying that QPD space offers the prospect of a theory whereby all of that which is produced in classical mathematics has an alternative construction entirely with rational numbers that are mapped to the space as a dynamically updating system.*

Where classical mathematics relies on a large infrastructure of hyperabstract, largely philosophical arguments, QPD space presents an alternative model, initiating an entirely structural theory, constructible from first principles, where the explanation for incommensurable spaces lies in the concessions necessarily made in the transformation from the perfect ideal model available in the dynamically updating system, to the static model that is accessible to human cognition.

## 9. The straight-line Problem

The development of QPD space specifically did not require the straight-line as a fundamental object. Only the primitive notion of congruence (applied to any geometric object employed to instantiate the interval) was required in order to analogise the concept of uniform relations. The straight-line is not axiomatic in QPD space, whereas it is, of course, one of the five axioms of Euclidean geometry. Euclid gave no satisfactory definition for it and after well over two thousand years there still is none.

It can be proved that there exists a straight-line, but there is no definite procedure to construct one unless you first have one. It can readily be expressed in a formula if one first grants the continuum and system of coordinates, however, a definition on that basis obviously involves the circularity that subverts attempts at a definition for the straight-line generally. In a formal approach fundamental entities such as lines, planes, and points can be abandoned altogether, or more generally the straight-line just enters geometry axiomatically as a primitive object. From a structuralist perspective, however, those approaches are less than satisfying.

To summarise the conventional position:

- The straight-line is the primitive object with which it is possible to construct curvature, whether that is with coordinate systems or tangent lines.
- The geometric approach defines the curvature of a straight-line to be identically zero.
- The straight-line is considered to be the singular fundamental object that can act as a standard, while there are infinite other possible curvatures. Coordinate-

free spaces and spaces more generally that diverge from Euclidean geometry are still conceived of in terms of *how* they diverge from the canonical Euclidean notion of space.

In QPD space, however, all of the above is reversed:

- The straight-line is not a required primitive object, but rather, it can be constructed from within the fundamental QPD curved-space.
- The curve (or the circle) on the other hand, since Euclid's Elements, *does* have a satisfactory definition and it can be constructed from first principles. Furthermore, in common experience there is not an infinite range or variation in curvature at all, but there is one fundamental curve subject to scale (and composites thereof). Colloquially, holding up a billiard ball so that it is superimposed over a full moon in the night sky, the curved surfaces of the two objects obviously map to each other identically, demonstrating a difference in scale, not some other quality of curvature such as sharpness or shallowness etc. (Noting that, obviously, a "sharp" acute angle cannot in the same way be scaled up to a "shallow" obtuse angle.)

The extended model of QPD space is completely constructed of fractal layers of self-similar curved intervals that concatenate to form helical range-tube pathways. There is one curve subject to scale, none more curved, none less curved. Every extension in the fractal geometry of QPD space is produced by concatenating iterations of a singular fundamental curve. (Anything over and above that is constructed from that foundation and is less fundamental, i.e., has additional structure requiring an infusion of instructional information.)

On this basis it is the curved extension that is fundamental and for which a satisfactory first principles definition can be given. The Euclidean straight-line required for all vector spaces and coordinate spaces can, on the other hand, be constructed within that more general, underlying curved space.

In this we see that to some extent the conceptual basis of QPD space is the mirror image of that which underpins calculus and the differentiable coordinate spaces. In calculus a continuous curve is thought to be composed of infinitesimal straight lines. This project, however, argues that the straight-line is not fundamental, but can be constructed within the fundamental underlying structure that is composed of a multiplicity of curved intervals, but where those have a tubular structure such that they represent a range of values.

The suggestion here is that without the benefit of being fully derivable from first principles, calculus has been giving adequate results at classical scale through what is essentially an inverted theoretical underpinning. It is, however, where general relativity and quantum mechanics coincide at singularities that calculus breaks down and a new concept is required. Alternatively, at this same point QPD space returns results that are inherently probabilistic and are located within a smeared range-band that will not produce the singularities and resulting infinities that are problematic in the equations of, for example, black holes and the big bang.

## 10. Summary: An Overview of QPD Space to this Stage

The objective that this paper has set itself is to show that there is a fundamental object that has less structure than, and is then ontologically prior to the natural numbers – and where this object is exemplifiable in a concrete model. This approach requires that structure-objects can be arranged in hierarchies of fundamentalness, for which an informational basis is employed – structures are fundamental in proportion as the amount of prescriptive information that is required to instruct their construction is minimal.

We began in Section 2 with a conventional bottom-up recursive construction, however, it was observed that the specifically linearly ordered structure that was forming (as exemplified in the three formats, but most evident in graph format) was producing structure that was at each stage non-translation invariant; that is, there was the problem of the non-identical object that could not then be the fundamental object that is required for the composition of the proposed fundamental underlying structure.

From there the program became that of constructing the optimally fundamental structure from the bottom up under a regime that blocks additional instructional information input. Then, in addition to the problem that was initially identified (the lack of translation invariance that is evident in the linearly ordered structure) the informational approach also highlighted the associated problem that in the construction of the maximally information entropic fundamental structure there should be no instruction prejudicing the successor function to produce a specifically linear sequence of fundamental objects. (This vindicates the earlier decision to sidestep philosophical discussion that is grounded in arguments that are specific to linearly ordered structures.)

A casual observation may arrive at the view that the successor function applied so that it produces a linearly ordered structure is in fact rightly producing the default structure. The correct view, however, is that all degrees of freedom are available to the maximally information entropic construction so that it is more the case that information input would be required if one wishes to restrict any otherwise uninstructed construction process so that it can produce only a linear ordering. Essentially, this is saying that not only do we have to find a structure for which translational invariance applies, but it must apply generally, not just restricted to a specific subset of successor relations. Or, in the analogous concrete model, translational invariance must apply across the entire array of fundamental objects in the structure with respect to all possible spatial axes.

At this stage of the project there is a shift in the methodology: once the concept that the construction process is restricted to minimal information input is introduced, we can no longer proceed on the basis of presupposing information-heavy objectives such as resolving specific problems or any other aims or objectives that imply a predetermined optimal target structure. The program becomes strictly that of modelling the successor function applied to the most fundamental objects on the basis

of minimal instructional information input, i.e., constructing from the bottom up the maximally information entropic structure.

In practical terms, one of this paper's main conclusions is that any constructor who is equipped with a modelling system that (i) can represent the most fundamental object and the most fundamental relations between iterations of that fundamental object (i.e. objects and relations that are produced with minimal prescriptive information input), and (ii) can apply a most simple successor function to grow a structure from those components, including as part of that the capability that the overall system should be able to recognise prescriptive information and block it from entering the construction process; then the structure that will be produced will necessarily be that described in this paper as QPD space; and where, in [1, 2], the modelling system that explains that structure has been the modified ball-and-stick model, conceived of as a dynamically updating system.

From there we claim that (now having the model of QPD space) if the above setup can successfully demonstrate that there is no more-fundamental object that meets the definition of "structure" (i.e., no structure-object that can be constructed with less prescriptive information), then the QPD model analogises the structure with zero structure, or the structure that is in the state of maximal information entropy. The claim is not that QPD space presents an interesting structure among other interesting structures, but that QPD space presents the concrete model that analogises *the* fundamental underlying structure that sits at the foundation of all structure, including that of the natural numbers.

We know from information theory that a structure produced from minimal information input can in some cases have a large output. And from work with self-assembling systems we know that a maximally information entropic construction process can produce an "ordered" structure if there are more ways for it to produce order, than to do otherwise [5]. We see that the structure produced here, QPD space, is clearly not a mundane object, it is not an amorphous mash, but rather, it is an inherently interesting fractal structure.

A static snapshot of the dynamically updating QPD space shows a three-dimensional icosahedrally ordered lattice structure that, from a central origin vertex  $O$ , radiates outward along twelve rays that are aligned with the six icosahedral symmetry axes. Under magnification those axis rays can be enlarged to show the underlying substrative fractal layer in which they resolve to the prototypical helically configured rods formed of stacked, penetration twinned IQC building blocks. At every junction in the lattice where twelve such rods intersect at a (self-locating) point, the QPD is the polyhedron that naturally forms, noting also that in the fractally-layered structure the global QPD is identically the vertex.

The QPD is the first atom that is the fundamental object and identically it is the completed global structure, persisting through the hierarchically nested fractal layers of structure. This is the unique structure that is necessarily produced in any attempted bottom up construction that is restricted to minimal or zero instructional input, as the associated paper [2] describes in some detail.



In the construction of QPD space we claim to have achieved the initial objective, which was to show that for the indefinitely extensible structure that underlies the natural numbers there is an optimal structure such that for any specific subset of that structure that we may sample, that subset will be composed of identical fundamental objects such that the structure is translation invariant.

The fractal character of QPD space means that when we refer to the “finite substructure that is sampled”, the minimum such substructure is the fundamental atom, which is identically the global QPD structure that has been shown in this paper (subsection 4.5) to be without edge or boundary. Furthermore, QPD space can demonstrate the action that occurs at the structure periphery with an analogous concrete model. It is theorized that any directional shift of the observer position within QPD space is correlated with the system of dynamical update such that, in the churn of vertex creation/annihilation, vertices are created ahead (zipped together), causing vertices to be annihilated behind (unzipped) so that the observer viewpoint appears to drag the QPD structure along with it, thus the idealized intrinsic observer viewpoint never approaches any boundary, but rather, it remains always at the central origin vertex  $O$ .

All of this is, in principle, capable of demonstration in the concrete model. This structuralistic interpretation presents as a concrete alternative to the topological models in which the observer transitions through faces, where that action is shrouded in a cloud of unknowing. Or, equally, as an alternative to classical mathematics’ approach to resolving the translation invariance problem that relies on purely philosophical notions such as that of completed infinite objects (where that relies on the problematic axiom of infinity).

## 11. Dynamically Updating Space vs. Higher-dimensional Space

In the development of QPD space, perceiving the structure as a dynamically updating system viewed by an intrinsic observer has been referred to as the “correct” viewpoint. But it was also acknowledged in Section 7 that we must consider that there are two models of the structure – the ontologically preferred model and the epistemologically preferred model. For the *correct, perfect*, dynamically updating model we must also have the associated static model that is *imperfect* but nevertheless provides the interface that is compatible with human cognition.

This is the concession that we have traditionally accepted, largely without examination. Practically, in the QPD setting this “concession” that we are forced to accept manifests as the lack of global periodicity and the geometrical frustration that shows up as fissures that run through the *static* lattice of the icosahedral quasicrystalline structure. Essentially, the QPD space interpretation says that there is the perfect *dynamically updating* ideal model that is nevertheless represented in the imperfect, *static* quasicrystalline structure.

In Section 2 the structure that underlies the natural numbers was initially represented as points on a straight-line (a one-dimensional space) which was relaxed to the more general concept of an homogeneous but otherwise unspecified-dimensional space of objects in successor relations, without any arbitrary restriction to the degrees of freedom. The development of the model that evolved on that basis, as exemplified in the concrete model, produced icosahedral quasicrystalline structure. Equally, we can say more generally that the problem of constructing the maximally information entropic structure coincides with the crystallographic problem of constructing icosahedral quasicrystalline structure. Because that structure is aperiodic in three dimensions it is not straightforward to find the correct mathematical formalism to express the (static) ideal model.

The lack of periodicity and the problem of geometrical frustration that appears in, and requires a resolution in, the static QPD model can be seen more widely to confound all attempts to construct fundamental homogeneous structure in static three-dimensional space; e.g., the problem of filling space with regular identical tetrahedra to construct a globally symmetric structure. Traditionally, the approach is to consider that these problems are resolved in ideal models that live in hyperspace, and where that involves projections of lattice points from that hyperspace to physical space.

Theories of higher-dimensional spaces have, of course, a long history in the mathematics and physics literature – beginning with the pioneering work of Theodore Kaluza and Oskar Klein, and through to, for example, the exotic spaces of supergravity theory and then superstring theory. In higher dimensions the approach to actually doing the math involves working by analogy from two and three-dimensional space, but from there any intuitive connection is removed and we are left to reason about things analytically. It is not possible to carry over the intuitive visualisation of the familiar three dimensions into four or more dimensions, and it remains the case that there is no accessible concrete model of any  $>3$  dimensional manifold. To the extent that hyperspaces “exist”, they are effectively transcendental or extrasensory spaces that we are not physiologically or psychologically suited to observing or conceptualising.

In some cases, accepting the concept of higher-dimensional space means that we are required also to accept results that are deeply counter-intuitive, such as those that arise in sphere packing in higher dimensions [6]. There is the thought experiment that takes the problem of sphere containment within a cube to higher dimensions, where this requires that we accept that in dimensions  $>9$  the sphere that is confined within the cube has a diameter that is larger than the sides of the cube within which it is confined.

Counter-intuitive concepts can be separated by type into those that, once introduced and the prejudicial barriers are broken down, become accessible (e.g., aspects of general relativity, or, for that matter, this paper’s concept that the dynamically updating three-dimensional space is a more-simple space than a one or two-dimensional static space). Then there are others, such as higher-dimensional spaces generally, or the sphere-packing example mentioned, for which no amount of familiarity (or “expansion of consciousness” as some popularisations would have it)

makes the concept more accessible to intuition and visualisation, or to exemplification in the construction of a concrete model. Presumably some threshold is reached beyond which the theorist will not conclude that it is some quirky counter-intuitive reality that we have to accept, but rather, it is just wrong.

It is recognised, first, that there is a proven utility to the higher-dimensional approaches. Even the deeply paradoxical (wrong?) higher-dimensional dense sphere packing models have utility in practical applications (e.g., in error-correcting codes used by various forms of data transmission to improve sending signals through noisy channels). The proposal put forward here, however, is that alternative models based on dynamically updating systems should be explored. In the setting of this paper, with respect to the problem of the ideal model for the icosahedral quasicrystal, we argue that the dynamical approach produces a superior model; and on that basis we propose that it is worth investigating whether this will apply more generally?

If it proves that there is in fact the option to choose between either a dynamical approach or a higher-dimensional approach, then it should be factored into the discussion that *every* example of structure that we can point to in the physical universe is definitely at least residually dynamical (the third law of thermodynamics), whereas there are *no* examples in the physical universe that we can point to that give a concrete model that embodies the mathematical concept of higher-dimensional structure.

There is very little empirical data with which to determine how many spatial dimensions there are, however, recent data obtained from measuring the strength of gravitational waves confirms that those waves propagate in precisely 3+1 spacetime dimensions, implying that physical space is precisely three-dimensional [7].

Broadly, on empirical grounds at least, it seems reasonable to investigate the dynamical update option.

Taking an overview of the conceptual landscape we can now characterise these competing approaches – higher dimensional or dynamically updating – as a potential fork in the pathway of development. It is possible that a whole genus of problems previously attacked by invoking unphysical higher dimensional spaces may be open to the alternative dynamical space interpretation. If this turns out to be the case, then relations that otherwise had expression only as analytic facts will be able to be represented visually (in computer generated graphics) exposing the problems to enhanced mathematical intuition and the benefits that attend that.

## **12. A Second Look at QPD Space as the Structure that Underlies the Natural Numbers**

Previously, Section 8 took an initial look at QPD space as the structure that provides a substrate within which the natural numbers can be placed in a one-to-one correspondence. We can now take that further and speculate on computer-generated models in which the dynamical system is an asynchronous cellular automaton.

On a conventional understanding, at its most basic the construction of a cellular automaton (CA) involves gridding a plane into a lattice of cells with a set of rules that determine how those cells interact. The rules can be thought of as a field theory where the lattice is the field, and the cells are the elementary component of the field. Now, conceiving of the CA constructed in QPD space, rather than the grid on the plane, the lattice is the three-dimensional icosahedral quasicrystalline structure that propagates radially out from a central origin; the elementary components are vertices that update between created and annihilated states in simple causal interactions; and the field theory in this case is the most fundamental set of rules which are no rules, or as Wheeler phrased it, “a principle of organization which is no organization at all...” [8].

Rather than top-down rules, the QPD construction mode (i.e., self-assembly) and the rules that govern the dynamical update system are entropy driven. In this model we do not first conceive of the update system and then arbitrarily design a (static) grid to set it on, but rather, the maximally information entropic approach results in a structure that self-assembles to form the icosahedral quasicrystalline lattice, and where this structure tells us that it must dynamically update in order to satisfy the homogeneity required of a fundamental space.

We can speculate further and consider that within this dynamically updating structure the intrinsic idealized observer is modelled as an embedded calculating avatar that is managed by higher-level algorithms so that it traces pathways through the structure that essentially are computations. This is a departure from equation-based modelling (that fits best with a static picture of systems that are biased toward attaining equilibrium) and signals a move toward alternative methods of modelling computations that are based on algorithms.

The way in which QPD space operates as a system of finite updates of the position-states of discrete fundamental objects suggests an overall structure that will be compatible with computation in finite state machines. But an inherent indeterminacy that applies to the extrinsic observer’s ability to ascribe states (created or annihilated) to the fundamental elements, along with other quantum-theoretic characteristics, point to QPD space being compatible with the principles of quantum computing.

Broadly, QPD space as a dynamically updating system is an isotropic and homogeneous space composed of uniformly dispersed fundamental objects in which all ratios of distances are commensurable. Throughout this space the fundamental objects mark off the intervals in helically configured pathways that provide the structure that can support the recursive sequence of the natural numbers, thus providing the dynamical counterpart to the traditional static number line.

However, as has been pointed out, no matter how successful the development of QPD space as an intrinsically observed dynamically updating system may become, a transformation to the static model remains an essential component of the interface with human cognition. Subsection 8.2 gave the first look at QPD as the static substrate beneath the natural numbers. It described the interval that has typical end-vertices  $B$  and  $C$  that are located within a range of position, and there is the range-tube

that is the three-dimensional model of the interior of the interval that is populated with virtual vertices. This model of the segment of range-tube pathway that spans between end-vertices is the QPD counterpart to classical mathematics' real interval.

For any arbitrary scale that we set the unit interval as the distance between pairwise adjacent vertices, there is the successor function that propagates that interval through all space; and this space is completely mapped by the icosahedrally ordered quasicrystalline lattice that supports the helically configured range-tube pathways.

The range-tube, while novel in concept, has a straightforward geometric construction that has been described in some detail in [2]. The delimiting surface of the range-tube is defined in the geometry; however, as discussed in subsection 8.2, when it comes to obtaining the mean value of the range-tube, the extrema are located at the centreline of the tube and the outer surface. Therefore the mean also can only be expressed as a three-dimensional tubular section that falls between those extrema. Consequently, the refining process of constructing smaller scale range-tubes never arrives at a specific value with infinite precision.

It is conjectured that the universe has finite resolution, and in QPD space the abstract mathematics has finite resolution also. Just as the amount of energy that is required to obtain physical measurement smaller than a Planck length effectively causes a black hole with Planck length diameter, an informational approach to the foundations of mathematics indicates that to graduate the interval to infinite precision (or to attempt infinite space-filling) generates an informational black hole that destroys structure. QPD space as a mathematical object has finite resolution such that every number signifies, not an infinitely precise value, but a defined range of value.

To summarise: The QPD space model as the foundation for the natural numbers redefines them as new mathematical objects that signify a range of values. That range is definable as a function of the scale of the QPD lattice. This is not an arbitrary philosophical response to the question as to what comprises the foundations, but rather, QPD space emerges as the output of the bottom up, maximally information entropic, first principles construction. In this model QPD space is the perfect ideal space in which the fundamental objects' positions are precisely located in the dynamically updating system. However, in the transformation to the static microstate those objects have a virtual character so that their existence is probabilistic, and the position of those mathematical objects can only be given to within the defined range of precision. Within the theory of QPD space there is the straightforward geometrical construction that gives the bounds of the uncertainty of position.

### **13. QPD Space and the Traditional Number line**

The extent to which the number line has historically been considered to be a foundational mathematical object as opposed to merely a heuristic device is not always clear. It is known that both Dedekind and Cantor postulated that there is a one-to-one correspondence between all numbers and points on a line. Dedekind had

wanted to jettison the crutch of the geometric number line in pursuit of arithmetizing the completeness of the real numbers, however, he was unable to do that. In fact the Dedekind cut, which is of course a main plank in some approaches to the explicatory development of the concept of real numbers, treats the number line as an almost corporeal object. It is reasonable to presume that the “number line” being referred to is always the static linear ordering on a one-dimensional line. Given that  $\mathbf{R}^2$  and  $\mathbf{R}^3$  are a model of Hilbert's axioms, then the subset, the real line, corresponds to a line in Euclidean geometry that is presumed to be composed of infinitely many points.

The next question to consider is to what extent the traditional concept of the number line as a static linear ordering is presumed to be an especially fundamental structure? Rather than any explicit argument for most-fundamental status, the traditional number-line probably owes more to ancient origins in marks etched on tally sticks and the deep intuition that we can place things in order of size, or in order of “before” and “after”, where that order can be analogised by the spatial form of linear order.

Informally there is a sense that the conventional number line is the *most simple, straightforward and intuitive* model that should therefore underpin the concept of objects in successor and predecessor relations. However, with regard to being *most simple*, the argument that this paper is putting forward is that the first principles development of QPD space shows that points arranged along a static, one-dimensional straight line demonstrably do not constitute the *most simple* substructure as the base upon which to construct a sequential ordering. And of course it hasn't in practice been at all *straightforward* or *intuitive* to find within the points that make up that line a place for all numbers – famously,  $\sqrt{2}$  for instance.

Whether or not Dedekind found that place where  $\sqrt{2}$  is represented with a point on the real line by carrying out his cut method (let alone a place for less basic examples that tend not to be cited) is not important here. What is relevant is that there is definitely no method to produce that place on the real line that has a complete development from first principles, or is straightforward or intuitive – to the stage where the system of rational points, although it is everywhere dense, does not cover all of the number line (or even a small part of it). What started as an intuitively compelling system of linking of the arithmetical property of a successive ordering to its static, linear, geometric counterpart, is now, in the real line, no longer intuitive or compelling at all.

The central argument that has evolved from this paper's development of QPD space is that beneath classical mathematics there is at the basis a naïve structure (the Euclidean notion of the straight-line) that underpins the conceptual foundations of the real number system at the expense of then requiring difficult and highly abstract interpretations.

As Herman Weyl expressed it:

“The introduction of numbers as coordinates by reference to the particular division scheme of the open one-dimensional continuum is an act of violence [...].”

And since that comment, those concepts that are required to support the classical notion of the real number continuum, being largely of a philosophical nature, now fit even less well with the increasingly quantum theoretic perspective of the physical sciences, or with modern developments in the theory of computation and computer science – none of which was, of course, yet on the horizon at the time that classical mathematics was first being formulated in the 19<sup>th</sup> Century.

The Weyl comment quoted above was included in an article by J. A. Wheeler, where he goes on to comment further:

“The continuum of natural numbers, Weyl taught us, is an illusion. It is an idealization. It is a dream. With numbers of ever increasing mathematical sophistication we can approach that infinity ever more closely; but we commit a folly if we think we can ever get there. That, in poor man’s language, is the inescapable lesson of Gödel’s theorem and modern mathematical logic.” And, further on he asks: “Then how can physics in good conscience go on using in its description of existence a number system that does not even exist?”

However, both Weyl and Wheeler were mathematical pragmatists, as is evident in the completion of the first Weyl quote from above:

“ [...] an act of violence whose only practical vindication is the special calculatory manageability of the ordinary number continuum with its four basic operations.”

And Wheeler comments further in his article:

“Yet for daily work the concept of the continuum has been and will continue to be as indispensable for physics as it is for mathematics. In either field of endeavor, in any given enterprise, we can adopt the continuum and give up absolute logical rigor, or adopt rigor and give up the continuum, but we can’t pursue both approaches at the same time in the same application.”

QPD space, however, offers the prospect that we can in fact have both logical rigor *and* the continuum. Or, more correctly, we can have logical rigor and the *pseudo*-continuum. When Wheeler alludes figuratively to the continuum as an “illusion”, QPD space rather literally agrees with that. Subsection 8.2 gave a description of the illusory pseudo-continuous interval that is populated with virtual vertices that do not admit to conventional operations whereby they can be traversed (Zeno), or cut (Dedekind) or denumerated (Cantor) – but they yet have their origins in discrete concrete objects that, subject to finite atemporal actions, sequentially update as an innate, entropy driven feature of QPD space; and where that space is constructed from the bottom up on a minimally axiomatic first principles basis.

Accepting the label of “illusory”, we do not, however, also accept Dr Wheeler’s capitulation that the classical concept of the continuum is indispensable for physics, as it is for mathematics. There are, of course, examples in the equations of physics, specifically singularities, where the infinitely precise values that are analogized by the infinitely dense continuum are not “indispensable”, but rather, they are anathema. QPD space, alternatively, offers the hybrid discrete/pseudo-continuous interval that can function in some respects like the classical interval, but where,

critically, a point in that interval signifies, not an infinitely precise value, but rather, a smeared range of values that are embodied in the concrete model as a tubular volume (i.e., there is no mathematical singularity).

It is, after all, only in the realm of pure mathematics that it is supposed that there can be values of infinite precision constructed of infinite decimals (i.e., the real numbers). Applied mathematicians and scientists when reading gauges or taking measurements naturally work with approximate decimals, rounding off to the precision required. Definitively, general relativity combined with the Heisenberg uncertainty principle tell us that it is impossible to measure anything to a precision smaller than the Planck length (or Planck time) without creating a black hole in that region.

In summary, it is proposed that QPD space offers a bridging link between classical systems that work well for most applications, and a new model for the fundamentals of mathematics that will be foundationally secure and has the potential to work well in those applications where classical mathematics otherwise produces singularities that explode to infinities in the equations.

## **14. Motivations and Outlook**

To generalise we can say that mathematics is a bottom-up approach that seeks to build up a self-consistent formalism from an axiomatically minimal base. Philosophically, the degree and type of realism that should be assumed is, of course, a central debate. Most would agree, however, that mathematics must make sense as a purely informational system of abstract relations.

In contrast to mathematics' bottom-up approach physics has historically been, for the main part, a top-down process of stripping away structure to arrive at more-simple, more-fundamental structure. Ultimately, the most elementary components may be stripped of all properties other than a binary notion of created state or annihilated state (existence or non-existence) that is indistinguishable from pure information.

If the above characterisations are both broadly correct, then (without requiring the full debate over realism vs. anti-realism) mathematics and physics can be said to coincide at some fundamental informational substrate. On the face of it, it is difficult to see how the set-theoretic classical mathematics of the 19<sup>th</sup> century fits within this scenario. In contrast to that, QPD space does display characteristics that are generally sympathetic with the concept of such a fundamental substrate. The following subsections outline areas that give a preliminary indication of the fitness of QPD theory to interact with the current aims of modern physics.



## 14.1 Field theoretic models

The underlying structure of the universe is the end-model in cosmological “fate of the universe” scenarios where a continued evolution sees the universe progress toward a cold, low-density state, arriving at thermodynamic equilibrium beyond which mass fades out and all material manifestation of information is devolved from the system. The process thus described arrives at an underlying stratum that (if the Standard Model is correct) is best described by quantum field theory, below which it is speculated there is the massless, timeless, purely informational field theory that, broadly, fits with the model of QPD space as an abstract field theoretic space. Clearly this proposal is very speculative; however, in QPD there is a conceptual basis from which to investigate a correlation between physical structure and abstract mathematical structure – whereas there is no expectation of any such basis within the underlying structure of classical mathematics.

## 14.2 A ubiquitous helicity

Helicity is, of course, a signature characteristic of fundamental structure across all scales throughout the physical universe; from submicroscopic particle helicity, to the collagen triple helix at molecular scale, and of course DNA structure, and extending out to helical galactic trajectories at cosmological scale. Essentially, in some reference frame every object of the universe is carving out a helical trajectory. Contextualised in that way, it is significant to note that both the physical universe and QPD space have in common the feature of an underlying structural chassis that has a fundamentally helical configuration.

No presupposed notion of helicity was written into the QPD assembly instructions, in fact it was not expected, nor was it evident in the early stages. But, as the characteristic packing sequences formed along the symmetry axes of the icosahedral quasicrystal, the helical configuration became very evident, increasingly throughout all scales of the extended structure.

## 14.3 Fractal structure and the Swiss cheese universe

As was the case with helicity, fractal structure was not written into the QPD construction, and again it was not initially expected. In fact, with respect to the related problem of producing the ideal model of the icosahedral quasicrystal, most current approaches (such as, e.g., the cut-and-project method based on the E8 lattice [9]) consider a successful model to be one in which structure propagates to produce global periodicity through as many successive concentric shells as possible. QPD space, however, produces (as a function of the entropically driven construction) structure formation that completes two concentric shells, at which point growth auto-terminates at that phase of construction and seamlessly transitions to the production of the successor fractal layer that is a copy of the previous structure.

When we say that fractal structure was not anticipated, of course it should have been. The maximally information entropic construction process can expect no infusion of new information that might instruct novel construction. This means that the only way in which the structure can continue to produce indefinitely extensible growth is by recycling existing structure. Section 4 introduced the fractal character of QPD space wherein the first atom and the global structure are identically the QPD.

In physical cosmology a considerable number of theories propose that the distribution of matter in the Universe, or the structure of the universe itself, is a fractal across a wide range of scales. As one example, the quantum gravity theory, Causal dynamical triangulation [10], proposes that there is a fractal structure for spacetime at near Planck scale. It is also the case, however, that a fractal universe is at odds with the Cosmological Principle. The standard model of cosmology assumes that the large-scale structure of the universe is homogeneous and isotropic at all points. And in support of that, astronomical observations agree that the density of matter in the universe is relatively smooth.

The QPD space dynamical model appears capable of satisfying both of the competing claims of homogeneity and fractal structure. A static snapshot of the microstate shows that, fundamentally, QPD space has a fractal structure, but it is conjectured that in the dynamical macrostate this evens out in the mean to a homogeneous structure.

In subsections 4.3, and 8.1 we've described the QPD lattice in terms of the icosahedral quasicrystalline structure that propagates radially out from a central origin. The IQC building blocks are typically stacked such that they form rods along the symmetry axes of the icosahedral quasicrystal. Throughout the magnification that accords with each successive fractal layer we see filaments of rod surrounded by pockets of void. These are pockets of space where, for one static snapshot of the microstate, there is no space, indicating that QPD space has a sinuous and porous structure.

In all common types of structure, if there is a hole in the structure, the background fills the void (e.g., fills the hole in the donut [11]). However, when the structure of interest *is the background*, then the hole in the background has an ontology that is distinct from all other holes (as first discussed in subsection 6.1). The QPD model does not allow that voids can be detected directly. Translated to the physical model, we cannot “see” structure where there is no structure (i.e. the electromagnetic field) to carry the photon. But voids may, of course, be detected in other ways. For example, in cosmology there are measurement anomalies (conventionally attributed to dark energy) that recent theories suggest may be explained by a “Swiss Cheese” cosmological model [12]. That model proposes an underlying cosmological structure that is sympathetic with the proposed QPD space. (Noting that early references to a locally inhomogeneous cosmological model, or “Swiss Cheese Universe,” date back to Einstein and Strauss [13].)

There is the structure that is observed *in* the universe, where, in the large-scale structure we see dense filaments separated by voids that are filled by the background; and there is the structure *of* the universe, which *is* the background structure, and for

which we are proposing that the underlying geometry is that of QPD space in which a large portion of the volume is entirely missing.

In the context of this paper, both QPD space and Euclidean 3-space claim to represent a model of the physical universe in which all matter exists. In Euclidean 3-space every point attaches to a real number on static coordinates such that for the entire volume no points are missing, whereas QPD space is (in the microstate) honeycombed, with a high percentage of the volume occupied by voids. The respective volume and expected total density as calculated in these two different models obviously will not agree. This is setting up a line of investigation in which, rather than introducing new exotic constituents, dark matter and dark energy can instead be investigated as aspects of QPD measurement concepts applied to the underlying structure of space.

The concept that the fundamental underlying structure (of everything) has a fractal character, then the universe has a fractal character, has interesting philosophical implications. It provides an argument for applying inductive inference across the boundary of empirical phenomena within the observable universe, and into that which lies beyond observational limits. It would further imply a basis for the application of inductive inference across the boundaries of the categories of philosophy itself.

Again we see an example where the development of a mathematical formalism based on QPD space offers the prospect of interacting with current physical (and philosophical) theories in a way that we cannot similarly expect from classical mathematics.

#### **14.4 Pilot wave theory**

Pilot wave theory is considered to be a fringe theory of quantum mechanics, however, it is at the same time the most intuitive and physical of the complete and self-consistent theories. It has the advantage that there is not the need for quantum objects to transition in a mystical way between non-real waves and real particles. Rather, pilot wave theory offers an interpretation of quantum mechanics in which the wave function describes a real wave that guides the path of a real point-like particle.

This description assumes the traditional notion that there is some fundamental description of the background space, *on top of which* there is the concept of the *real wave* that guides the particle. This results in the theory being criticised for being unparsimonious or contrived. The bottom-up, maximally information entropic construction of QPD space, however, is showing us that the default, most-fundamental underlying space *just is waveform*.

The work in this paper is emphasising that the Euclidean/Newtonian trajectory is not the default most-simple or fundamental pathway through space. Rather, in the geometry of QPD space every extension is a helically configured geodesic (i.e. projects to a 2D wave) and position states of geometric objects are correlated (i.e., very basically, the asynchronous updates attributed to the domino effect, first

described in subsection 8.1, can be expected to causally ripple through the entire lattice structure). The QPD atemporal/causal domino effect that perpetuates the dynamical updates implies an abstract global hidden variable effect. It remains, at this stage, that the dynamical update process may be deterministic or non-deterministic, but in either case there is a limit to the precision with which the extrinsic observer can obtain values from QPD space. There is a limit to the certainty with which any required determinate statement can be ascertained.

Whether referring to the fabric of spacetime, vector spaces, fields, or any of the varieties of non-Euclidean spaces, in all cases there is at the basis the traditional notion that the static Euclidean straight-line is the primitive object in light of which it is possible to conceive of curvature; it is the singular fundamental object that can act as the default standard that gives the contextual basis within which all other coordinate systems are perceived (see also Section 9).

It is, however, against this traditional view that we argue that it is the dynamically updating and helical (i.e., waveform) QPD space that is the default fundamental underlying structure. The default, most-fundamental underlying space *just is waveform and probabilistic*. It is proposed that future work should investigate QPD space as a new (and now parsimonious) framework within which to revisit pilot wave theory as an alternative to current quantum mechanical interpretations such as e.g. the Copenhagen or Many Worlds.

## 14.5 String theory

It was hypothesised at the beginning of this section that ultimately mathematics and physics should converge at some primordial underlying structure. If physics' reductive process has (according to some) arrived at string theory as the picture of the fundamental irreducible structure of the fabric of reality, then the picture presented there matches, in some respects, with the QPD picture of the abstract structure that has no structure, or the maximally information entropic structure.

In the case of string theory, the graviton becomes a loop of string, not a point particle. The strings trace out 2D sheets so that even very high-energy interactions occur at a smeared position, which means that there is not the problem of the mathematics exploding into black-hole-creating infinities. In Feynman diagrams where two particles would normally meet at a point-like singularity, the diagram now instead shows the path as a smooth tube-like surface with the effect that there are no zero-dimensional singularities.

In the case of QPD space (and stressing, obviously, that QPD is constructed without prejudice that would be geared toward producing string theory type objects) the bottom-up minimal information input construction produces a fundamental mathematical object that matches, at least in initial respects, with string theory's picture of the graviton. In both cases a point-like object becomes a two-dimensional sheet that traces paths through the space, creating a tubular thickening of the volume within which the position of the object can be specified.

String theory holds the promise of being a theory of quantum gravity wherein the vibrating quantum strings are the basis of a mechanism to reproduce the theory of both general relativity and quantum theory, and as such it is considered by many to be generally pointing in the right direction. However, it is of course also criticised for several problems; mainly that it has failed to produce any confirmed predictions. But also, the theory's "strings" are hypothesised to be actual one-dimensional vibrating real physical strands that exist in space, without there being any hope of experimental evidence for these objects.

The question is whether it remains necessary to arbitrarily postulate new physical entities if QPD space offers a revised mathematical structure that, first, has the prospect of being foundationally superior to traditional classical mathematics, and also presents a mechanism that reproduces critical aspects of string theory, thus promising to resolve some of those same problems in the physics?

To summarise: In QPD space the model of fundamental elements in fundamental relations has a discrete geometric structure, but where a specific dynamical characteristic produces the pseudo-continuous interval, and where a key feature is that in this system there are now new mathematical objects to which attach values that lie within a range; and where that range maps to an analogous structure that has a three-dimensional, tube-like representation such that the process of arriving at a final value is inherently smeared and probabilistic.

## **14.6 The shape of the universe**

If we are hypothesising that there is a fundamental mathematical substrate that coincides with the primordial structure of the universe, then we should reasonably expect to see further signs of common origins additional to the features that we have so far tentatively identified.

One area of interest will be to look at QPD space in the context of the global geometry of the universe. Given that in key respects there is significant correlation between QPD space and Poincaré dodecahedral space, it is interesting to note that although current observational data are at this stage inconclusive regarding the shape of the universe, Poincaré dodecahedral space is one of only several models currently considered [14].

It is reasonable to anticipate that the development of QPD space will support further investigation of a dodecahedral shaped universe.

## **14.7 The fate of the universe**

Section 12 introduced the concept of the QPD dynamical system as an asynchronous cellular automaton that forms the substrate that supports the arithmetical property of a successive ordering of the natural numbers. And given that a main thesis developing in this section is that a model of the primordial structure of the universe coincides with the model of the fundamental mathematical substrate as a cellular automaton, it

is then perhaps not such a huge step further to speculate that the physical world itself may be, at its bottom, a discrete, digital automaton. This begins to link in with concepts such as, e.g., digital physics, pan-computationalism, the informational universe, the CA hypothesis and Wheeler's original "it from bit" proposal.

Related to, but separate from those approaches, it is proposed that future work based on this initial paper will develop the hypothesis that the fundamental structure that underlies all structure is the QPD CA in which the fundamental objects are essentially pure information – and we can hypothesise further that this structure is not perfectly deterministic. Given that the QPD CA is composed of an indefinitely extensible array of interlocked dynamically updating component fundamental objects, then as those interactions trigger created and annihilated states a small error would compound, leading to uncertainty and an evolving, chaotic system. We can then speculate on a CA model in which these interactions produce oscillations that lead to elementary particles, atoms, emergent laws of physics, physical structure and ultimately the universe.

If we adopt a position along the lines of Edward Fredkin's Finite Nature hypothesis [15], then on this type of model we can suppose that the QPD CA universe is a finite, isolated global system wherein information is conserved. This definition of the QPD CA entails that it is a finite construction of fundamental objects that have a relational structure that can be analogized in the geometry of a finite number of unique configurations that the dynamically updating system will evolve through.

For the QPD CA as a finite, isolated system that has fixed global information content, schematically we can draw a box around the global system such that the fixed information content is contained within that box. The evolution of the QPD CA involves expansion phases wherein the configurations are updating from those that have less structure/information, to those that have greater structure/information until peak structure/information (the minimum information entropy state) is reached, at which stage the system is expected to seamlessly transition to configurations with less structure, implying an information/structure contraction phase. This is, then, a cyclic model, and to the extent that we accept the hypothesis that the evolution of the informational system is tied to physical structure and ultimately the universe, then the QPD CA is modelling a cyclic universe (e.g., [16]) that repeatedly returns to a primordial ground state that is purely informational.

To explain the local gain and loss of information within the fixed global information content (that is isolated within the conceptualised box enclosure) we propose that the overall pool of information divides into *latent* information that in the expansion cycle transitions to *manifest* information as a function of the system transitioning from those permutations with less structure (smaller information content) to those with more structure (larger information content).

Following the peak structure/information stage the QPD CA cyclic universe model is expected to unwind through a contraction phase, passing through a series of successor configurations that (on average) each have less structure/information than the predecessor, until the system ultimately arrives at the underlying stratum that is modelled as the QPD CA ground state, i.e., the maximally information entropic

structure. This has its parallel in the evolution of the physical universe toward a cold low-density state, arriving at thermodynamic equilibrium beyond which mass fades out and all material manifestation of information is devolved from the system.

This is, however, not describing the transition to a state of the system associated with a loss or destruction of information, but, rather, the system has transitioned to the structure that has minimal *manifest* information content, but which has maximal *latent* information content so that the overall information content of the system is constant. In this model the QPD CA ground state represents the compressed information reservoir (all information in the system has transitioned to latent information) from which can be recovered all less-fundamental structure and the associated manifest information content.

Following that line of argument, the QPD CA is a candidate model for the fundamental primordial structure that in some sense contains the entire latent information content of the universe and can, in the cyclic universe model, spontaneously evolve to produce the entire manifest information content that reproduces the phenomenology of spacetime and matter.

## 14.8 Ethos

This Section has outlined several areas where the development of QPD space is expected to have relevance. Considering that broad overview, there appears to be a confluence where the QPD concept of the fundamental underlying structure has the potential to form a unifying basis for several streams of investigation across areas of physics, cosmology and mathematics.

A common theme that is apparent across all of these strands is that the QPD model offers an intuitive approach that is readily accessible to analogy in visualisable concrete models. And primarily this appears in areas for which, contrastingly, the currently preferred models tend to be deeply counterintuitive; to the extent that in some cases it has come to be considered naïve to expect that there should, even in principle, be such visualisable concrete models for the sophisticated abstract concepts at hand. A key example is where, as an alternative to classical mathematics' hyperabstract Cantorian concept of an infinity of different sized infinite sets that make up the real interval, the QPD structure introduces the first principles, geometrically constructible, discrete/pseudo-continuous interval.

This paper's prosaic approach stands in contrast to the general tone of the intellectual environment that has formed the mathematical landscape over the previous hundred years. Bertrand Russell, for example, refers to a "truth" that is available through (Weierstrassian) abstract mathematics as opposed to the "vulgar prejudices of common sense" [17]. We note the influence of the Bourbaki group's focus on group theory and the dogmatic commitment to abstraction, to the point that geometry became redefined as the study of groups of transformations and anything that did not fit within that paradigm was ignored.

It wouldn't have helped that since antiquity the tools for rendering geometry had remained the compass, ruler, and a flat drawing surface. Essentially, for most of the development of mathematics' foundations there has been no adequate method of visually interpreting the increasingly sophisticated concepts. Now, however, with the relatively recent development of the modern computer that has of course all changed. QPD space, for example, as a mathematical object that has been constructed in response to those original foundational questions can potentially be modelled as a three-dimensional, dynamically updating, asynchronous cellular automaton that forms the substrate that supports the natural numbers, and where calculations are performed by an intrinsically embedded mathematical avatar. Visualisation is no longer merely a poor tool to help elucidate existing mathematical objects, now it is the means to construct a new generation of mathematical objects.

## 15. Conclusions

The objective of this paper has been to show that there is a fundamental mathematical object, exemplifiable in a concrete model, that has less structure than, and is ontologically prior to the natural numbers. We consider that this objective has been met by putting forward the candidate model, QPD space, as outlined in this paper and as summarised in Section 10.

Essentially, the basis upon which it is argued that the structure produced (QPD space) is more fundamental than the linear ordering of the natural numbers is by constructing the structure that is more fundamental (has less structure) than *all* other structure. And the way that is achieved is to construct from the bottom up, starting from a base that involves the minimum prescriptive information input that is sufficient to establish the notion of "structure", and from there ensuring (informally at this stage) that no further prescriptive information is allowed to enter the construction process.

On this basis the bottom-up construction produces the QPD space that is populated with fundamental geometric objects that are necessarily in a dynamical flux of creation and annihilation. As an alternative to the linear ordering on the conventional static *number line*, QPD space is conceived of as a dynamically updating *number lattice*, or *number field*.

If at first this does not match with our intuitive concept of the most fundamental abstract structure upon which to base the successive ordering of the natural numbers, then we can make a comparison with the way in which it was also initially counter-intuitive to find that, as quantum field theory tells us, empty physical space, or the ground state space, is actually roiling with activity. So we now, similarly, find that the QPD model is telling us that empty *abstract* space (this maximally information entropic space, empty of all prescriptive information input – the informational ground state) is populated with fundamental geometric objects that are necessarily in a dynamical flux of creation and annihilation.



This proposal is not a philosophical response to the question as to where the foundations of mathematics lie, but rather, QPD space is the substrate that emerges from a first principles bottom-up construction, and within which is found a linear sequence of fundamental objects with which the natural numbers can be placed in a one-to-one correspondence.

However, in the transformation from QPD space as the dynamically updating system, to the representation in the static microstate, the natural numbers now correspond to, and signify, mathematical objects that do not have a determinate existence and do not have a precise value, but rather, they are probabilistic objects with locations that are expressed in values that fall within a definable range.

An attempt to encapsulate the theory should include the fact that the dynamical-to-static transformation that is required for the interface with human cognition involves a reinterpretation that sees the perfect, rational, discrete-dynamical system transformed into the imperfect, static, incommensurable, pseudo-continuous model.

The theory of QPD space incorporates the discrete and the pseudo-continuous, and the deterministic and the probabilistic, in such a way as to bring those under the umbrella of a unifying theory of the foundations that (unlike classical mathematics) is compatible with what computer science and quantum theory are currently telling us. This offers the prospect of a foundation upon which to formulate the new mathematics that is required to facilitate, for example, the unification aims of the quantum gravity program.

The task of unifying quantum gravity's two precursor theories, general relativity and quantum mechanics (and more broadly, all of the fundamental forces), into a single mathematical framework is subverted if that framework is not in the first instance foundationally correct. To quote J. Butterfield and C. J. Isham [18]:

Finally, we note that, from time to time, a few hardy souls have suggested that a full theory of quantum gravity may require changing the foundations of mathematics itself. A typical argument is that standard mathematics is based on set theory, and certain aspects of the latter (for example, the notion of the continuum) are grounded ultimately on our spatial perceptions. However, our perceptions probe only the world of classical physics – and hence we feed into the mathematical structures currently used in *all* domains of physics, ideas that are essentially classical in nature. The ensuing category error can be remedied only by thinking quantum theoretically from the very outset – in other words, we must look for ‘quantum analogies’ of the categories of standard mathematics.

As is suggested in the quote above, the view that the foundations of mathematics should be substantially revised is not yet widely accepted or very palatable. However, indications are that in order for physics to produce the unifying theory that is currently

lacking, that revision is necessary. In this context QPD space is put forward as a candidate structure upon which to base a revised model of the foundations of the natural numbers.

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