Causation as Constraints in Causal Set Theory

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Abstract

Many approaches to quantum gravity —the theory that should account for quantum and gravitational phenomena under the same theoretical umbrella— seem to point at some form of spacetime emergence, i.e., the fact that spacetime is not a fundamental entity of our physical world. This tenet has sparked many philosophical discussions: from the so-called empirical incoherence problem to different accounts of emergence and mechanisms thereof. In this contribution, I focus on the partial order relation of causal set theory and argue that causation can be characterized as an a-temporal constraint over the kinematic space defined by the theory. The relation constrains the growth of a new element/event with respect to the other elements/events of a given set. Therefrom, the flow of time emerges from the collection of the possible growths of the given set, where each possibility is characterized by a classical probability assigned by the dynamics of the theory.

1 Introduction

Perhaps, one of the most challenging enterprises in contemporary theoretical physics is the development of a unified theory of gravitation and quantum mechanics: a theory of quantum gravity. The theory is meant to solve some of the problems that still affect two of the most well-verified theories of modern science: the general theory of relativity and quantum field theory. Both theories deliver, in their own domains of applicability, some astounding experimental predictions, and yet they both fail to provide a comprehensive account of some fundamental questions about the beginning of the universe, black holes, quantum effects of gravity, and many others.

To make things worse, general relativity (GR) and quantum field theory (QFT) seem to be fundamentally incompatible with one another, to the point that even a brief (and incomplete) overview is enough to show the conceptual (and mathematical) difficulties of the unification task. General relativity is a deterministic theory about spacetime, gravity, and matter. As Wheeler's motto famously says: "matter tells spacetime how to curve, and spacetime tells matter how to move". Spacetime is treated as a dynamical and continuous entity that admits no preferred spatial or temporal directions, and the interactions of the many physical systems that live in such spacetime are local. Quantum field theory, on the other hand, studies the dynamical fields that 'live' on a non-dynamical background. The theory is characterized by the uncertainty principle, and thus by the lack of complete localizability. Finally, quantum mechanics is fundamentally probabilistic and all interactions are quantized.

Many physicists have tried to combine these apparently incompatible domains, leading to the proliferation of different approaches and theories of quantum gravity. Despite the many differences of these approaches, one common feature seems to be that spacetime ought to be treated as an entity that emerges from the interaction of some fundamental entities. This has sparked a prolific discussion among physicists and philosophers about issues such as: the definition of emergence, how to recover the manifest spacetime from non-spatiotemporal entities, how to experimentally probe a theory of quantum gravity, and many others.

With the present contribution, I focus on characterizing the fundamental relation of causal set theory as a constraint over the possible configurations admitted by the theory. More specifically, Section 2 offers a brief presentation of the main axioms of the theory and its kinematics. I shall emphasize the relationship between those causal sets that represent possible classical configurations of our universe, and the causal tree that represents the multiplicity of possible growths.

Section 2 discusses the role of the time parameter in causal set theory as bookkeeping device for the position of a given event relative to the other events of the causal set. Then, I maintain that causal relations in CST are more fundamental than their spatiotemporal counterparts. This latter claim opens up the possibility for interpreting the partial order relation as a constraint principle.

Section 4 begins with a brief overview of some common accounts of causation in philosophy of science. Then, starting from a pluralistic account, I will characterize the relation of causation in CST (the partial order relation) as a constraint over the possible growths of all possible configurations of the casual sets. Section 5 provides some concluding remarks.

2 Causal Set Theory

With this section I offer a review of some of the fundamental tenets and results of CST, but deeper reviews and discussions can be found in (among others): (Sorkin 2005), (Surya 2019), (Wüthrich 2019), (Henson 2009), (Wallden 2013), (Dowker 2005).

The central idea of Causal Set Theory (CST) is that spacetime is fundamentally discrete and its structure is that of locally finite partially ordered sets that represent possible kinematic configurations of the universe. The theory follows the sum-over-histories approach for which, starting from a space of possible histories, one assigns a measure to the individual trajectories to calculate a final amplitude as sum-over individual histories. The space of possible histories in the causal set theory consists of discrete structures to which Lorentzian manifolds are only an approximation, that is, the causal sets. There are typically two formulations of the theory in the literature and they are distinguished by the properties of their causal relation. The *reflexive* formulation, which is also the least common one, makes use of a causal relation that is reflexive, antisymmetric and transitive. The more common version of causal set theory makes use of an irreflexive relation which does not allow for instances of self-causation. Since the latter version is more common, I shall use it for the rest of this contribution. The theory is defined by the following six axioms (Dribus 2017, p. 151):

1. Binary Axiom: Classical spacetime may be modeled via a set $\mathcal{C}$, called a causal set, whose elements represent spacetime events, together with a binary relation $\prec_{CS}$ on $\mathcal{C}$, called the causal set relation, whose elements represent causal relationships between individual pairs of spacetime events.

2. Measure Axiom: $\mathcal{C}$ is equipped with a discrete measure μ_{CS}, called the causal set measure, which assigns to each subset of $\mathcal{C}$ a volume equal to its number of elements in fundamental units, up to Poisson-type fluctuations

3. Countability: $\mathcal{C}$ is countable.

4. Transitivity: Given three elements x, y, and z in $\mathcal{C}$, if $x \prec_{CS} y \prec_{CS} z$, then also $x \prec_{CS} z$.

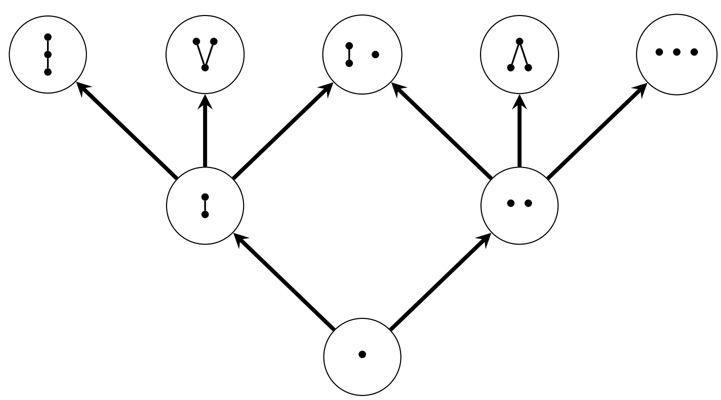
5. Interval Finiteness (or) Local Finiteness: For every pair of elements x and z in $\mathcal{C}$, the open interval

$$\langle\langle x, z\rangle\rangle \coloneqq \{y \in \mathcal{C} | x \prec CSz\} \tag{1}$$

has finite cardinality.

6. Irreflexivity: Elements of $\mathcal{C}$ are not self-related with respect to $\prec_{CS}$' i.e., $x \not\prec_{CS} x$

An example of how causal set theory models a universe composed of three elements is represented below, where each node represents a given causal set with its corresponding events and relations:



From the simplified three-elements model represented above, one individuates two distinct 'levels' in causal set theory: one the one hand we have the individual nodes that represent possible configurations of classical universes with one, two or three elements:

$$S_{cs} : \left\{ \bullet, \text{I}, \bullet\bullet, \bullet\bullet\bullet, \text{I}\ \bullet, \Lambda, \text{V}, \vdots \right\} \tag{2}$$

On the other hand, the tree-structure represents the multiplicity of possible growths from one universe-configuration to another. It corresponds to the quantum-level of the theory that represents a kinematic scheme S_{cs} of sequential possible growths.[1] As already pointed out in (Dribus 2017), the two levels of causal set theory seem to question the popular idea that classical mechanics is less fundamental than quantum mechanics, while here it seems that the classical nodes are more fundamental than the quantum kinematic scheme. Here, the analogy with the sum-over-histories can help us clarify the conundrum. Indeed, in the sum-over-histories approach (alternatively, path integrals), the individual trajectories can be considered as 'classical' for they are individuated by a classical action functional. The quantum properties emerge as soon as we consider the total ensemble of possible trajectories, in that each trajectory is assigned a probability amplitude which sums with all the other possible paths.

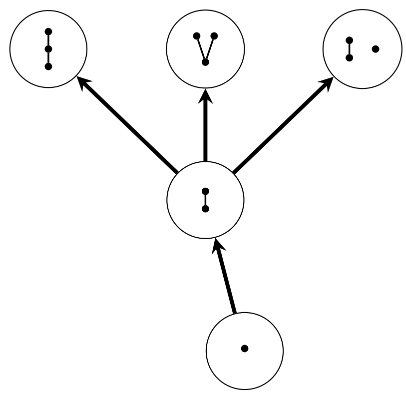
Notably, terms such as 'reduction' and 'fundamentality' become somewhat tricky here. Even without diving deep into the literature, consider the following two readings: on the one hand it is intuitive to consider quantum mechanics as more fundamental than classical mechanics —for example because the former operates at scales smaller than the latter. On the other hand, quantum mechanics dictates that all possibilities count —for example, all trajectories in the path integral— and it is only at the classical limit that such a multitude of possibilities reduces to one. This is especially evident if we look at how, at the limit $\hbar \to 0$, the many possible trajectories of quantum mechanics constructively interfere around the classical trajectory, thereby reducing the path integral to the least action principle.[2] Therefore, we shall rely on a temporary workaround to the problem of characterizing terms such as 'reduction' and 'fundamentality': we shall more timidly say that quantum mechanics and classical mechanics represent different structures and that while the former interests a specific physically possible scenario, the latter dictates that all possible physical scenarios count.

I will say more on the dynamics of the theory below, but for now it is enough to say that it provides a rule for the probabilities of the co-relative histories (or) transitions from one node to another. The most common approach is named Classical Sequential Growth (Rideout and Sorkin 1999) and it assigns classical probabilities to the possible transitions from one causal set to another in the kinematic scheme. For example, in the simplified tree

[1]We shall specify the concept of growth later in this paper.

[2](Forgione 2020) describes this mechanism of constructive and destructive interference of possible trajectories in the path integrals formulation of quantum mechanics.

below, the transition from empty set $(\emptyset)_{d_0}$ to $(\bullet)_{d_1}$ is $p(d_1 \to d_2) = 1$ and the probability of the realization of causal set $(\vdots)_{d_t}$ is $P(d_t) = 1/3$. Notably, The probabilities assigned by CSG are still classical, which testifies the incompleteness of causal set theory as a full-fledged theory of quantum gravity.



2.1 Why Causal Set?

Thus far, we have seen some of the general features of the theory, but I have not really mentioned why the use of Hesse diagrams and causal sets can be useful to recover relativistic spacetime. The central theorem of the causal set programme is the Hawking-Malament theorem: (Hawking, King, and McCarthy 1976) and (Malament 1977) —also refered to as the Metric Recovery Theorem (MRT) by (Dribus 2017) and (Dribus 2013). Broadly speaking, the theorem proves that: "the causal structure of relativistic spacetime determines the corresponding metric structure up to smooth conformal isometry" (Dribus 2017, p. 91). Here, *relativistic causal structure* means that the future timelike and past timelike directions are not dependent on the choice of frame of reference —and thus causal influences propagate always from causes to effects.[3] Physically, the theorem states that relativistic spacetime can be

[3] Alternatively, a relativistic spacetime manifold that satisfies some appropriate causal conditions can be characterized by a causal relation $\leq_{GR}$ that is a strict partial order. While (Malament 1977) identifies five such conditions, it is enough for our purposes to list the *past and future distinguishing condition* for which a "relativistic spacetime manifold X is future distinguishing if and only if for every x in X, and every open set U containing x, there exists a 'smaller' open set U' in U, containing x, such that no future-directed smooth timelike curve through x that leaves U' ever returns to it" (Dribus 2017, p. 101), (Malament 1977, p. 1400).

reconstructed starting from the description of all the causal relations between the events populating spacetime. In this sense, Wheeler's motto: *space tells matter how to move, and matter tells space how to curve* emphasizes the prescriptive character of geometry and matter, which is captured by the theory of general relativity. But, instead of prescribing how space bends and matter moves, one can provide a description of all the causal relations between the events populating spacetime. The metric recovery theorem states that prescriptions and descriptions convey the same information. However, this latter statement is imprecise and the theorem adds the condition *up to smooth conformal isometry*: a tricky requirement that requires some unpacking.

First, we shall begin with the definition of *conformal equivalence*, (Dribus 2017, p. 87): "two pseudo-Riemannian metrics g and g' on a smooth manifold X are called conformally equivalent if there exist a smooth positive function $\omega : X \to \mathbb{R}$ called the conformal factor, such that $g'(v, w) = \omega(x)^2 g_x(v, w)$ for every point $x \in X$ and every pair of tangent vectors $v, w \in T_x X$". In layman's terms, two structures are conformally equivalent as long as there is a function between them that (locally) preserves angles, while the measuring rods (lengths) might vary. A conformal isometry is an even stricter condition, since it imposes the existence of a diffeomorphism between two metrics such that the metric $f * g'$ on X is conformally equivalent to g on X. Therefore, the problem of reconstructing relativistic geometry from its causal structure is the lack of the conformal factor (that is: scale data) on the relativistic manifold. This is intrinsic to GR, since the diffeomorphism invariance, for example, makes it impossible to obtain those scale data from some other ambient scale (e.g.: an embedding manifold).

The workaround to the missing conformal factor is the axiom of local finiteness for which all ordered intervals in the causal set have finite cardinality, thereby corresponding to a finite cut-off which is interpreted as a measure of volume. Therefrom the slogan coined by Sorkin: *Order plus number equals geometry.* While the term *order* stands for the binary relation on the events of a causal set, the term *number* stands for the natural scale — that is, the local finiteness axiom associated with a measure of volume μ_{CS} which consists of counting the elements of a given causal set C.[4] Finally, the term *geometry* stands for relativistic spacetime. Bombelli et al. (1987,

[4]The use of the counting measure was already suggested by (Myrheim 1978, p. 1): "If spacetime is assumed to be discrete, then the counting measure is the natural measure, and the causal counting is the only structure needed. Coordinates and metric may be derived as secondary, statistical concepts."

p. 522) suggested that the observables of causal set theory correspond to topological and metric observables and that the latter should be expressed in terms of a deeper notion of order:

> In this view volume is number and macroscopic causality reflects a deeper notion of order in terms of which all the 'geometrical' structures of space-time must find their ultimate expression.

Continuum based geometry is then recovered *via* smoothing at large scales the structure of causal set C and binary relation $\prec_{CS}$. The operation is warranted by the notion of *embedding*, which consists of an injective map $\phi : C \rightarrow (M, g)$ from a causal set to a pseudo-Riemannian manifold such that given $x, y \in C$:

$$x \prec_{CS} y \longleftrightarrow \phi(x) \prec_M \phi(y) \tag{3}$$

But, not all causal sets can be embedded into a spacetime (M, g) and, even if they could, this does not guarantee that a given spacetime is approximated by a given causal set. One shall require that *number* approximates *spacetime volume*, a stricter condition called *faithful embedding* (Surya 2019, p. 14): "every finite spacetime volume V is represented by a finite number of elements $n \approx \rho_C V$ in the causal set", where $\rho_C = V_C^{-1}$. In addition, to recover relativistic spacetime one needs to ensure the covariance of the distribution of the elements of the causal set, and this is obtained by inducing a random sprinkling of elements (Poisson process) on pseudo-Riemannian manifolds, (Surya 2019, p. 16):[5]

> We say that a causal set C is approximated by a spacetime (M, g) if C can be obtained from (M, g) via a high probability Poisson sprinkling. Conversely, for every $C \in \mathcal{C}(M, \rho_C)$ there is a natural embedding map
>
> $$\phi : C \rightarrow M$$

where $\mathcal{C}$ is an ensemble of causal sets obtained from the imposition of a partial order on the elements of the Poisson sprinkling. It remains to determine how a manifold-like causal structure can uniquely determine large-scale manifolds. The uniqueness of the continuum approximation is warranted by a conjecture, the *Hauptvermutung* (fundmanetal conjecture of causal set theory):

[5] This amounts to constructing a causal set from a pre-existing manifold, rather than obtaining the manifold bottom-up from the individual elements of the set.

> ***The Hauptvermutung of CST:*** C can be faitfully embedded at density ρ_C into two distinct spacetimes, (M, g) and (M', g') iff they are approximately isometric (Surya 2019, p. 19).

The conjecture is not proven yet, but some work in this direction has been put forward by, among others: (Bombelli 2000), (Noldus 2004), (Bombelli, Noldus, and Tafoya 2012).

Finally, a few words on the most common dynamical approach to causal set theory: the causal sequential growth (CSG), see: (Rideout and Sorkin 1999). According to the CSG exploratory model, a causal set is built via evolutionary steps i.e., by adding one element to the set at each transition. Each transition (co-relative history) consists of a morphism between two causal sets, thereby making spacetime a dynamical entity that evolves following a co-relative history. As it was mentioned earlier, the kinematic scheme of the theory describes the potential evolutionary steps of a given causal set, while the dynamics (CSG) supplies with (classical) probabilities that weight each possible transition.

3 Emergence of Time

The presentation of the theory, thus far, has left the notions of causation and time evolution somewhat blurry. Indeed, the individual causal sets do not track a proper time evolution in that they are part of the kinematic scheme. The causal relation of CST does not correspond to a strictly forward in time relation, but this should not be much of a surprise since the relativistic nature of spacetime admits spacelike separated events.[6] For example, while the sequence $(\bullet) \rightarrow \left(\begin{smallmatrix}\bullet\\ |\\ \bullet\end{smallmatrix}\right)$ might suggest a passage of time from one causal set to the other, the co-relative history $(\bullet) \rightarrow (\bullet\,\bullet)$ has the same probabilities to occur. The partial order relation of causal set theory is hardly a causal relation in a traditional sense, and that is because causal sets are not enough to relate each cause to the corresponding effect. The point was raised by Wüthrich (2019) and it is justified by the fact that not all timelike or null

[6] In addition, the classical sequential growth dynamics imposes a locality condition called *Bell Causality*. The condition states that the probability of the growth of a new element depends only on its past, and it is not affected by spacelike separated elements. The condition guarantees a form of locality to the dynamics of the theory by imposing that the probability of growth of a new element does not depend on the totality of the causal set.

relations have causal efficacy: "given an event, we take neither all events in its past lightcone to be its causes, nor all events in its future lightcone to be its effects" (Wüthrich 2019, p. 13). Again, this validates the idea that causal relations in causal set theory differ substantially from their everyday counterpart: "the causal relations of causal set theory of course differ from that attributed to events in our ordinary lives. Nevertheless, given the tight relationship between the fundamental relation of causal set theory and the causal structure of relativistic spacetimes, we take it to be legitimate to dub this relation a 'causal relation'" (Wüthrich 2019, p. 13).

One could suggest that an external time parameter can be associated to subsequent causal sets, thereby recovering the idea of a flow of time. This might be especially tempting in CSG, since each transition corresponds to the birth of a new element in the causal set. The partial order relation, as well as the so-called condition of internal temporality impose that if we call 'parent' and 'child' two elements of a causal set, then no 'child' element can be born before the 'parent'. In other words: "[...] each element is born either to future of, or unrelated to, all existing elements; that is, no element can arise to the past of an existing element" (Rideout and Sorkin 1999, p. 9). We could interpret this process of progressive accretion of a causal set as the manifestation of the 'phenomenological time', and we could do so by imposing a label to each of the subsequent causal sets. To each causal history of the universe —that is, to each sequence of subsequent causal sets— we would have a unique labeling, that is, a sequence of natural numbers that externally tracks the succession of causal sets. However, the order of growth of two events that are spacelike with respect to one another is pure gauge, and thus carries no physical meaning. In addition, different causal histories (or path) leading to the same causal set are also gauge invariant, and that is because: "We want the physics to be independent of labeling, so different paths in $\mathcal{P}$ leading to the same [causal set] should be regarded as representing the same (partial) universe, the distinction between them being pure gauge" (Rideout and Sorkin 1999, p. 5).

But then, if the time order in the causal tree is gauge invariant, how should we recover a more traditional phenomenological passage of time? For example, (Dowker 2014) emphasizes how in general relativity each physical system is locally associated with a world-line that is causally ordered. Then, in causal set theory: "[a] sequential growth model is a model of a physical world which becomes in a manner compatible with the lack of a physical global time" (Bento, Dowker, and Zalel 2022, p. 8). In this sense, causal relations constitute the local ordering of events in a given causal history, and time plays the role of an independent (and un-physical) parameter used to keep track of the different positions of the events with respect to each other within the causal set.

The physical passage of time, as mentioned earlier, should emerge as the resulting effect of the stochastic birthing of successive events. But, because of the invariance under label transformation, we would need a (yet unspecified) process of cancellation of the many gauge invariant causal histories to give rise to an actual history of the universe at the classical limit. In this sense, the analogy with the sum-over-histories approach becomes salient, since, there, a single classical history emerges at the classical limit from the mutual cancellation of the many possible histories of the quantum ensemble. Alas, for the analogy to work, we need a proper quantum dynamics for causal set theory that would dictate the process of cancellation of the causal histories. What remains is that causal set theory does not admit an external time that tracks the evolution of a causal set and, independently of how such a phenomenological time will be recovered, it is safe to conclude that time is an emergent property of the theory.

3.1 Causal Theory of Spacetime and Fundamental Causation

If we accept that the passage of time is an emergent property in causal set theory, then we might be able to decouple the partial order relation (causal relation) from the notion of time. The issue is not new in the literature, as, for example, (Russell and Slater 2022, p. 381) expressed it in terms of reduction and fundamentality: "can time be derived from causality, or must we retain temporal order as fundamental, and distinguish cause and effect as the earlier and later terms in a causal relation?" An affirmative answer is defended by the causal theory of spacetime (CTS), which maintains that the structure of spacetime can be recovered from causal structures. Again, the idea is supported by Hawking and Malament's causal metric hypothesis, but also by authors such as H. Reichenbach and B.V. Fraassen. Most recently, (Baron and Le Bihan 2023) have revitalized the debate and defended a version of the causal theory of spacetime for which spacetime relations are grounded on causal relations.

Baron and Le Bihan (2023) present two approaches to the causal theory of spacetime: the identity theory and the non-identity theory. The former comes in two flavors: strong and weak, which are distinguished based on whether spacetime-relations are characterized as causal relations *a priori* or *a posteriori* respectively.

The *a priori* approach seeks to define spatiotemporal relations as causal relations, but (Baron and Le Bihan 2023) object that we can imagine a world in which spacetime structure exists and there are no causal relations. This scenario implies that the universe, or a subset thereof, would be entirely idle with no dynamics. Conversely, we can also imagine "worlds with causation that do not feature relativistic spacetimes, where space and time are different manifolds (as in Newtonian worlds). These worlds may be physically impossible or even metaphysically impossible, but they are certainly not conceptually incoherent" (Baron and Le Bihan 2023, p. 3). They conclude: "[a]ssuming that, if a can be conceived of without b, then the identity of a and b is not a priori, there does not seem to be a conceptual link between causation and relativistic spacetime of the right kind" (*ibid*).

Another objection, originally raised by (Smart 1969) and recalled by (Baron and Le Bihan 2023) is that we should not identify clear terms with terms that are less clear: "To elucidate the concept of space-time in terms of the concept of causal connectedness seems to be to elucidate the comparatively clear by reference to the comparatively unclear" (Smart 1969, p. 394). Surely, reply (Baron and Le Bihan 2023), the past decades have brought more clarity to the notion of cause, especially in philosophy of science. However: "[w]hile we have developed theories of causation that have some level of precision, what we take to be the most precise of these —the interventionist account coupled to the structural equation framework— typically foregoes any reductive ambitions and takes causation to be an unanalysed primitive. Arguably, causation is still less well understood than spacetime" (Baron and Le Bihan 2023, p. 3).

Perhaps, an even more precise notion of causation could be based on the idea of functional dependence, especially with respect to the use of hyperbolic partial differential equations as best representations of Humean causation (Smith 2000). Since it is not my intention here to discuss new possible definition of causation in science, I shall simply accept that the notion of spacetime is better defined than that of causation —as also supported by both Baron and Smart.

With respect to the weak approach to the non-identity theory, the main objection also comes from (Smart 1969, p. 394): "It is difficult to see how the causal theory of time is applicable to theories which allow for the existence of events which are neither causes nor effects of other events". The objection, which is dubbed by (Baron and Le Bihan 2023) as the problem of causal indolence, is indeed severe for it applies to both the strong and weak approach. Because of the identification of spacetime relations with causal relations: "any spatiotemporal relation must be a causal connection, which means that there cannot be any entities that are causally idle so long as they bear spatiotemporal relations to other entities (which they must if they are located in spacetime)" (Baron and Le Bihan 2023, p. 4). The causal indolence objection consists of three different sub-problems: (1) there might be spacetime regions that are free of matter and energy, (2) the existence of timelike connected events that are causally disconnected, and (3) the existence of spacelike events that would need superluminal signals to be causally connected.

The first sub-problem resembles in kind the problem of the strong approach to the identity theory, which I have addressed above. The second sub-problem, is not clear in that there cannot be timelike connected events that are not causally connected. Everything that is in the past-lightcone of a given system constitutes a cause, independently of how weak that connection might be. Perhaps the strongest case for such an argument is to conceive of the Big Bang as a universal common cause for the entire universe. Alternatively, we can consider the formation of our sun as something in our past lightcone and thus as a cause of my writing this paper. With respect to the third problem —that is, the existence of spacelike separated events requiring faster-than-light signals to be causally connected— (Grünbaum 1973) already suggested to trade causal connections with causal connectability. However, the trade-off is not a viable solution since, when it comes to spacelike events, a causal connection is not even a possibility unless we reject special relativity. Therefore, suggest (Baron and Le Bihan 2023), one needs to look at another way to address the problem of causal indolence.

The suggested solution is to give up on the identity approach and characterize the relationship between spacetime and causal relations as an ontological dependence. The suggestion is that spacetime relations are grounded in causal relations, thereby making the latter more fundamental than the former: "our view is that spatiotemporal relations are grounded in a pattern of more fundamental causal relations between events" (Baron and Le Bihan 2023, p. 10). This would constitute a solution to the problem of causal indolence because spacelike connections between two events would be grounded in the absence of a causal relation: "in particular, if two physical events are not linked by a fundamental causal relation, then that grounds a spacelike connection at the spatiotemporal level" (Baron and Le Bihan 2023, p. 11). The account is further refined by adding that being part of a causal structure is a necessary requirement for any spatiotemporal connection, in such a way that all events in a causal set would be in some causal connection with some other events, thereby avoiding isolated elements of the set.[7] One more addition to the non-identity approach is that the total causal structure of a given causal set is rule-governed, that is, the physical laws dictate both what events are causally connected, and also what events can be and cannot be causally connected.

In sum, the non-identity approach establishes a relation of grounding between spacetime and causal relations in causal set theory. Instead of saying that spacetime relations can be expressed in terms of causal relations, (Baron and Le Bihan 2023) set forth a form of ontological dependence such that causal relations become more fundamental than spatiotemporal ones. Then, the explanation of the causal connectability between events is due to the total causal structure and to the rules that dictate the possible causal connections within such a structure. However, the total causal structure, as seemingly interpreted by (Baron and Le Bihan 2023), corresponds to the causal configuration of all events of the universe, i.e., to a causal set. In addition, the introduction of physics laws that dictate what events can be causally connected adds one additional layer to the ontology of causal set theory, and brings about the difficulty of explaining what such rules would consist of.

It remains that causal relations ought to be more fundamental than spacetime relations, but this conclusion is independent from the use of the non-identity theory and becomes evident from more general considerations. For example, the fact that the structure of relativistic spacetime can be expressed in terms of causal relations (up to a conformal factor) is but one of the pillars on which causal set theory stands. The fundamental discreteness of the theory is not required for the validity of the metric recovery theorem, and the use of the counting measure to define a natural scale for volume is a strategy that overcomes the *up to conformal factor* condition. What

[7]Notably, one might advance the objection for which mathematical objects might be in a spacelike relation with physical events, but a proper response would involve tackling the debate on the existence of mathematical entities, which I shall not do it here.

follows is that the theory, by identifying discreteness as one of its fundamental axioms, naturally requires a mechanism for recovering the continuum of relativistic spacetime.[8] It is then natural to assume that the fundamental objects and relations of the theory should be more fundamental than the continuum spacetime relations proper of relativity theories. In other words: if discreteness and partial ordering give rise to spacetime relations, it is only natural that the partial order relations and discrete entities should be more fundamental than the relativistic spacetime relations.

4 Causation as Constraint

Thus far, along with (Baron and Le Bihan 2023), I argued that causal relations are more fundamental than spacetime relations. Yet, this does not help us clarify the former without any use of the latter. The problem was already mentioned by (Smart 1969), in that we should not be explaining a clear concept starting from a less clear one. One possibility is to provide a clear account of causation in physics to explain well-defined spatiotemporal relations with well-defined causal relations. Completing the task is no small feat, and many philosophers have already tried. Here, without a pretense of completeness, I will review some of such attempts with a special focus on the interventionist account.

Regularity theories of causation, see: (Andreas and Guenther 2021), reduce causation to instances of specific pattern of succession (this is in contrast to notions of causal efficacy and causal power). Some fundamental principles of the regularity theories are: the constant conjunction of the same types of events, the contiguity in time and space, the asymmetry between causes and effects. The central idea is that a given event A that is lawfully followed by an event B can be considered as the cause of B. One of the objections to this cluster of theories is that modern science is based on repeatability, and that a single instance of one event following another should not necessarily constitute a causal connection. One could respond that any lawful connection implies that causal connections require repeatability. Yet, this might constitute too strong of a constraint since it remains that the correlation between two events does not always constitute an instance of causation. A possible solution would be to provide an underlying mechanism that explains

[8]Notably, this is a difficulty proper of almost every theory of quantum gravity, and not only of causal set theory.

the relation of causation between the two events, but such mechanisms are not always possible —for example, the speed of light is a causal constraint that does not have an underlying mechanism.

Another approach to causation is represented by the counterfactual theories, originally developed by (Lewis 1973). These theories analyze causal relations in terms of counterfactuals of the form: 'if I were you, I would accept the job'. An objection to such theories is that they do not always distinguish counterfactuals from causal relations, as made evident by the example above. The fact that *I am not you* can hardly count as a cause for not accepting the job.

The process account of causation focuses on the perduration and progression of phenomena, rather than on their instantaneous occurrence. Dowe (2000), Salmon (1998), and others, tie the notion of causation to a process of transfer of a conserved quantity between two systems. For example, the transition of a ball through a gaseous medium is a causal process in that we could easily mark the ball with, say, a Sharpie, and observe that the mark is transmitted throughout the entire process. Further refinements of the theory will individuate as markers some conserved physical quantities such as momentum or energy (see, for example: (Dowe 2000) and (Salmon 1997)). However, the account seems to fail at distinguishing which events or factors constitute a cause, in that the transmission of a marker does not guarantee that the marker is also a cause for a given event. For example, (Woodward 2005, p. 357) points out that "the feature that makes a process causal (transmission of some conserved quantity or other) tells us nothing about which features of the process are causally or explanatorily relevant to the outcome we want to explain". In addition, the account seems to fail at explaining the lack of those marks as a cause for a given event. For example, statements such as: "I killed the plant by not watering it" (Beebee 2004) are considered cases of causation by omission, which express causation without the transmission of any mark.

The interventionist account proposed by (Woodward 2005) suggests that the proper way of characterizing causation is in terms of manipulations of some variables within a given causal structure. Causes are factors that when intervened upon produce a change in a system that was otherwise unexpected. The interventionist approach treats causation as a primitive where: X is a total cause of Y if and only if under some intervention on X there is an associated change in Y. The concept of changing a variable within the system can be interpreted modally, so that interventions need not be taking place in spacetime —this is because the variation of the values of X and Y can be intended as being within the space of possibilities defined by the theory. In addition, the manipulation of one of the variables does not need to add an external factor to the system. Indeed, (Frisch 2014) and (Pearl 2009) suggest that the variable itself is the cause, and that the intervention amounts to changing the value of the variable. The interventionist accounts applies well to causal set theory in that, for example, tweaking the value of a variable corresponds to the addition of an element to the causal set —where the modal aspects of the interventionist account consists of the many possible ways in which one element can be added to a given set.

There are two possible issues with the interventionist account though. The first one is already pointed out by (Ben-Menahem 2018) and maintains that the manipulation of a given variable to explain causal connections might be too limited at times. For example, many laws of physics that seem to convey causal connections are expressed in terms of hyperbolic differential equations. Even without entering the debate on what constitutes a physical law, we can simply consider the 1-d wave equation $u_{tt} - c^2 u_{xx} = 0$ which can be used to model a vast number of physical phenomena (plucked strings, vibrations of elastic beams, springs, and others). The mathematical form of those equations requires the input of some additional data (for example: position $u(x, 0) = f(x)$ and momentum $u_t(x, 0) = g(x)$) to obtain a well-defined solution. In general, these data are applied to the dynamical equation to obtain a solution at a subsequent instant of time.

The interventionist approach seems to work fine with phenomena modeled by such equations, since the manipulation of the initial conditions determines a change in the solution of the equation at a subsequent instant of time. But, the account is also limited in that it is not always the case that the initial conditions required by the mathematical formalism are the sole cause for a given event. For example, the initial conditions of a projectile will determine when and where it will hit the target, but it is also easy to imagine how heavy rain and hail might affect its trajectory. Weather forecast is (typically) not accounted for by the initial conditions that describe the projectile, yet the latter can be considered as a con-cause for the projectile's final trajectory. This is to say that the initial conditions required by the mathematical formalism are necessary, yet not always sufficient to account for the causes of a physical system.

Perhaps, one could add to the causal history of the system the extra causes (e.g., heavy rain and hail) that contribute to the corresponding effect. Then, to assess whether the new variables play the role of con-causes, one can verify that by changing the value of the variables corresponds a change in the phenomenon. In some cases though, the two variables X and Y —respectively the supposed cause and effect— could be related by association laws, namely laws that express mutual functional dependence between variables. In these cases, an intervention on X would produce a change in Y, but the other way around would also hold, thereby conflicting with the asymmetric character of causal relations.[9]

Another objection is raised in (Blanchard 2023) and is based on the difficulty of addressing the causal efficacy of wholes and parts in the interventionist account. With respect to composite objects, the change of value of a variable due to intervention ought to either keep the variables associated to the individual parts fixed, or to allow for a change of behavior of the individual parts. However "[t]he first strategy runs the risk of making wholes causally excluded by their parts, whereas the second strategy is in danger of mistakenly ascribing to composite objects causal abilities that properly belong to their parts only" (Blanchard 2023, p. 20).

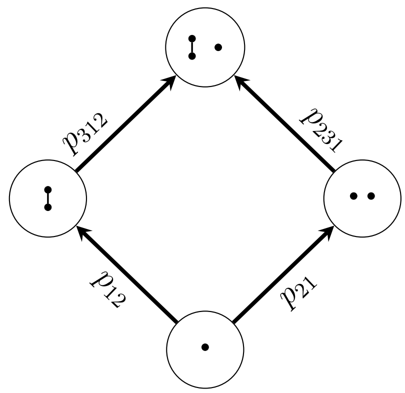
The discussion on the interventionist account and modifications thereof is far from settled. Nonetheless, since it has been considered as a good fit for causal set theory, I contend that it falls short of giving a proper explanation of the causal relations. For example, consider the simplified causal set tree below. The variable subject to manipulation is the last event of the given node (causal set) from which the new event is born. Because of transitivity, the formation of a causal connection between the two events on the left-branch, that is the transition $\bullet \rightarrow$ ⁚, can be considered as the cause for the top causal set ⁚ $\bullet$. However, the same goes for the right-branch of the tree, that is, the transition $\bullet \rightarrow \bullet\bullet$ can be considered as the cause for the same top causal set. The interventionist account does not distinguish between the two branches as possible different causal transitions.[10] One could argue that both causal sets ⁚ and $\bullet\bullet$ (causally) contribute to the top one, but this would conflict with the fact that, while the different nodes are part of the kinematic scheme of the theory, they are not physical realizations.[11] The

[9] The argument was raised and commented in (Kistler 2013).

[10] That the two branches are physically indistinguishable is a consequence of gauge invariance. Nonetheless, this applies only to the physical content of the theory, and not to its modal aspects.

[11] One could also reply that, as mathematical possibilities, every possible causal set contributes causally to the realization of a subsequent causal set. This would be similar

same argument runs against the counterfactual approach, for counterfactual statements would not be able to distinguish the cause of the realization of the top-node.



Perhaps, to solve the impasse of characterizing causal relations, we can focus on the order relation among the events of the causal set. The binary relation $\prec$ is characterized as a partial order —this is evident also in other reviews of causal set theory such as (Wüthrich 2012) and (Dowker 2006). There, a causal set is defined by the pair $\langle E, \leq \rangle$ where E is the set of elements and $\leq$ is the binary relation of partial ordering between those elements. For example, (Wüthrich 2012) defines the relation $\leq$ as inducing a partial order on a set $\mathcal{C}$, that is, the relation is transitive, reflexive and antisymmetric. Similarly, (Dowker 2006) describes causal sets as partially ordered sets with a relation of precedence that satisfies: transitivity, non-circularity, and finiteness. Another example is (Surya 2019), who defines a causal set as a set with an order relation $\prec$ that is acyclic, transitive, and locally finite. Notably, (Surya 2019, p. 12) defines the property of local finiteness as: $\forall x, y \in \mathcal{C}, |\mathbf{I}[x, y]| < \infty$, where $|\mathbf{I}[x, y]| \equiv Fut(x) \cap Past(y)$. The definition makes use of temporal parlance in the terms 'past' and 'future', but one can easily avoid them by giving a definition that uses the notion of cardinality —as we mentioned in section 2. Thus, causation can be defined in terms of ordering between elements, and without making use of any spatiotemporal terms. This suggests that we should be able to give an account of the causal relation of causal set theory using an a-temporal approach to causation.

to saying that the possible trajectories individuated by a path integral causally contribute to the total transition amplitude. This approach would require the backing of some form of mathematical realism, and we thus leave it to later works.

Most recently, (Ben-Menahem 2018) defended a pluralist account of causation which has roots in the works by (Cartwright 1984), (Cartwright 2004), (Godfrey-Smith 2009), and others. Ben-Menahem (2018) characterizes causation in terms of constraints, similarly to how the lightcones of special relativity constrain the possible causal interactions for a given system. While causation "appears to be the basis for the very structure of spacetime" (Ben-Menahem 2018, p. 28), the pluralistic account consists of approaching the notion of cause as a cluster of irreducible constraints imported from our theories. The view opposes the tradition of searching for a single notion of cause that can be applied to all sciences, and suggests that causal notions provide explanations and descriptions of possible changes in our physical world: "causal notions and constraints, I suggest, are employed to describe, predict, and explain change. They tell us which physical processes and changes in the physical world are possible, and which are not" (Ben-Menahem 2018, p. 14).

We can thus interpret the partial order relation of causal set theory as a constraint over the possible growths in the kinematic space defined by the theory. This interpretation is compatible with the conclusions of the previous section —for which causal relations are more fundamental than spatiotemporal relations— and frees the concept of causation from temporal connotations. The a-temporal partial order between the elements of a given causal set is then embedded within the total kinematic space, and partakes to the dynamics that assigns a probability amplitude to each possible partially ordered growth. It remains that a quantum model for the interference between the different nodes has not been developed yet, but the underlying intuition is to follow the analogy with the sum-over-histories account to suppress the un-physical causal sets (Carlip, Carlip, and Surya 2023).

In sum, the interpretation of causal relations as constraints fits well with the three main claims of this contribution. The two levels of the theory, classical and quantum, are compatible with the constraint interpretation of the causal relation. Indeed, the elements of the individual sets are sorted by the partial order relation —whose definition can vary slightly depending on the formulation of the theory. The causal relation acts as a super-selection rule over all the possible growths by limiting those that would violate the conditions necessary to the recovery of relativistic metric. The second claim was that physical time in causal set theory is derived from the more fundamental causal relation. Again, this is also compatible with the constraint interpretation, for there is no addition of temporal terms and no changes in how physical time would emerge starting from causal sets. Finally, the constraint interpretation, under the pluralist view suggested by (Ben-Menahem 2018), applies to the case of causal set theory, and it does not aim at becoming a general interpretation of causal relations across different disciplines. In addition, partial order relations are as clear as the spatiotemporal relations of relativity theory, thereby sidestepping the objection raised by (Smart 1969).

5 Conclusions

In this contribution I have offered a brief overview of the main axioms of causal set theory, with a focus on the analogy with the sum-over-histories account, and on the relationship between the nodes representing individual causal sets and the causal tree representing the multiplicity of possible growths. I have then introduced the Malament-Hawking theorem, which plays a central role in relating causal set theory to relativistic spacetime and its causal structure. After the review of the theory, I discussed the role of the time parameter in terms of bookkeeping device for the growth of new elements of a given set with respect to other events. What emerged is that it is possible to consider the causal relations as more fundamental than spacetime relation. Finally, after reviewing some accounts of causation in the context of philosophy of science, I suggested that starting from a pluralistic account of causation, we can interpret the relation of partial order as a constraint on the possible growths defined by the theory.

References

Andreas, Holger and Mario Guenther (2021). "Regularity and Inferential Theories of Causation". In: *The Stanford Encyclopedia of Philosophy*. Ed. by Edward N. Zalta. Fall 2021. Metaphysics Research Lab, Stanford University.

Baron, Sam and Baptiste Le Bihan (2023). "Causal theories of spacetime". In: *Noûs*.

Beebee, H (2004). "Causing and Nothingness". In: *Causation and Counterfactuals*. MIT Press, pp. 291–308.

Ben-Menahem, Yemima (2018). *Causation in science*. Princeton University Press.

Bento, Bruno Valeixo, Fay Dowker, and Stav Zalel (2022). “If time had no beginning: growth dynamics for past-infinite causal sets”. In: *Classical and Quantum Gravity* 39.4, p. 045002.

Blanchard, Thomas (2023). “The causal efficacy of composites: a dilemma for interventionism”. In: *Philosophical Studies* 180.9, pp. 2685–2706.

Bombelli, Luca (2000). “Statistical Lorentzian geometry and the closeness of Lorentzian manifolds”. In: *Journal of Mathematical Physics* 41.10, pp. 6944–6958.

Bombelli, Luca, Johan Noldus, and Julio Tafoya (2012). “Lorentzian manifolds and causal sets as partially ordered measure spaces”. In: *Withdrawn manuscript*.

Bombelli, Luca et al. (1987). “Space-time as a causal set”. In: *Physical review letters* 59.5, p. 521.

Carlip, P, S Carlip, and S Surya (2023). “Path integral suppression of badly behaved causal sets”. In: *Classical and Quantum Gravity* 40.9, p. 095004.

Cartwright, Nancy (1984). *How the laws of physics lie*.

— (2004). “Causation: One word, many things”. In: *Philosophy of Science* 71.5, pp. 805–819.

Dowe, Phil (2000). *Physical Causation*. Cambridge Studies in Probability, Induction and Decision Theory. Cambridge University Press. DOI: 10.1017/CBO9780511570650.

Dowker, Fay (2005). “Causal sets and the deep structure of spacetime”. In: *A. Ashtekar (ed.)* 100, pp. 445–467.

— (2006). “Causal sets as discrete spacetime”. In: *Contemporary Physics* 47.1, pp. 1–9.

— (2014). “The birth of spacetime atoms as the passage of time”. In: *Annals of the New York Academy of Sciences* 1326.1, pp. 18–25.

Dribus, Benjamin F (2013). “On the axioms of causal set theory”. In: *arXiv preprint arXiv:1311.2148*.

— (2017). *Discrete Causal Theory*. Springer.

Forgione, Marco (2020). “Path Integrals and Holism”. In: *Foundations of Physics* 50, pp. 799–827.

Frisch, Mathias (2014). *Causal reasoning in physics*. Cambridge University Press.

Godfrey-Smith, Peter (2009). “Causal pluralism”. In.

Grünbaum, Adolf (1973). “The causal theory of time”. In: *Philosophical Problems of Space and Time*, pp. 179–208.

Hawking, Stephen W, Andrew R King, and Patrick J McCarthy (1976). “A new topology for curved space–time which incorporates the causal, differential, and conformal structures”. In: *Journal of mathematical physics* 17.2, pp. 174–181.

Henson, Joe (2009). “The causal set approach to quantum gravity”. In: *Approaches to quantum gravity: Toward a new understanding of space, time and matter*, pp. 393–413.

Kistler, Max (2013). “The interventionist account of causation and non-causal association laws”. In: *Erkenntnis* 78, pp. 65–84.

Lewis, David (1973). “Causation”. In: *The journal of philosophy* 70.17, pp. 556–567.

Malament, David B (1977). “The class of continuous timelike curves determines the topology of spacetime”. In: *Journal of mathematical physics* 18.7, pp. 1399–1404.

Myrheim, Jan (1978). *Statistical geometry*. Tech. rep.

Noldus, Johan (2004). “A Lorentzian Gromov–Hausdorff notion of distance”. In: *Classical and Quantum Gravity* 21.4, p. 839.

Pearl, Judea (2009). *Causality*. Cambridge university press.

Rideout, David Porter and Rafael D Sorkin (1999). “Classical sequential growth dynamics for causal sets”. In: *Physical Review D* 61.2, p. 024002.

Russell, Bertrand and John G Slater (2022). *The analysis of matter*. Routledge.

Salmon, Wesley C (1997). “Causality and explanation: A reply to two critiques”. In: *Philosophy of Science* 64.3, pp. 461–477.

— (1998). *Causality and explanation*. Oxford University Press.

Smart, John JC (1969). “Causal theories of time”. In: *The Monist*, pp. 385–395.

Smith, Sheldon R (2000). “Resolving Russell’s anti-realism about causation: The connection between causation and the functional dependencies of mathematical physics”. In: *The Monist* 83.2, pp. 274–295.

Sorkin, Rafael D (2005). “Causal sets: Discrete gravity”. In: *Lectures on quantum gravity*. Springer, pp. 305–327.

Surya, Sumati (2019). “The causal set approach to quantum gravity”. In: *Living Reviews in Relativity* 22, pp. 1–75.

Wallden, Petros (2013). “Causal sets dynamics: review & outlook”. In: *Journal of Physics: Conference Series*. Vol. 453. 1. IOP Publishing, p. 012023.

Woodward, James (2005). *Making things happen: A theory of causal explanation*. Oxford university press.

Wüthrich, Christian (2012). “The structure of causal sets”. In: *Journal for general philosophy of science* 43, pp. 223–241.

— (2019). “Spacetime from causality: causal set theory”. In: *The Emergence of Spacetime in Quantum Theories of Gravity. Oxford University Press, Oxford*.