Foundational Constructive Geometry

D. A. Ford
2023
email: mondford@slingshot.co.nz

An ideal constructor produces geometry from scratch, modelled through the bottom-up assembly of a graph-like lattice within a space that is defined, bootstrap-wise, by that lattice. Construction becomes the problem of assembling a homogeneous lattice in three-dimensional space; that becomes the problem of resolving geometrical frustration in quasicrystalline structure; achieved by reconceiving the lattice as a dynamical system. The resulting construction is presented as the introductory model sufficient to motivate the formal argument that it is a fundamental structure; based on which, it is proposed that where mathematics' numbers conventionally correspond to dimensionless points on the stateless number line, numbers more fundamentally correspond to an ordering of discrete objects constructed within the stateful number lattice. A second observation is that this fundamental lattice structure is helically configured with fractal character, which, as it relates to the geometry underlying spacetime, has relevance to questions in physics, particularly those involving wave-particle duality.

1. Introduction

Foundational constructive geometry[1] shares with the many forms of constructivism the criterion that objects and concepts must be accessible in terms of constructions that can be executed. Only finite operations over finite objects are demonstrably executable, consequently a geometry on this basis is expected to be discrete. However, while there is a well-established research area under the heading "constructive mathematics" and there is an interest in finitism in geometry generally, there is no comparable well-established program specifically under the heading "constructive geometry".

Mathematical constructivism in its various forms has a long history (introduced around the turn of the twentieth century), and has become increasingly relevant along with the mechanisation of computation – constructive mathematics is sometimes loosely defined as the part of mathematics that can be implemented on a computer. From within the incumbent classical mathematics tradition, however, constructive mathematics is still regarded with

[1] Disambiguated from the elementary sense in which it refers to constructions with a compass and ruler.

suspicion. Andrej Bauer refers (somewhat tongue in cheek, granted) to five stages of accepting constructive mathematics: those being denial, anger, bargaining and depression, before finally acceptance [1].

In a recent podcast Joscha Bach was asked (rather incredulously), “So real numbers don't exist?” He affirmed that they don't, adding, “For something to exist, it has to be implemented” [2, 01:14:03–01:14:18]. Elsewhere, Bach elaborates [3, 00:15:25–00:15:52]:

> What Gödel and Turing could show is that the assumption that you can have functions that run through infinitely many steps and give you a result leads to contradictions. And this was basically the constructivist turn in mathematics, the most important result of philosophy in the last century, and interestingly one that most philosophers did not understand and did not continue on.

J. A. Wheeler [4] is also on record as having said that real numbers don’t exist: “Then how can physics in good conscience go on using in its description of existence a number system [speaking of real numbers] that does not even exist?” And further on, “The continuum of natural numbers, Weyl taught us, is an illusion. It is an idealization. It is a dream. With numbers of ever increasing mathematical sophistication we can approach that infinity ever more closely; but we commit a folly if we think we can ever get there. That, in poor man’s language, is the inescapable lesson of Gödel's theorem and modern mathematical logic.”

Bauer refers to a “psychological agony” involved in having to “unlearn certain deeply ingrained intuitions and habits acquired during classical mathematical training”[1]. But for mathematical constructivists there is at least the comfort of an established approach to turn to, defined under its heading. This, however, does not extend to geometry. When Bach, clearly a proponent of constructive mathematics, outlines how to build geometry from scratch, it follows along conventional lines. Numbers are described as a subsequent labelling scheme combined with predecessor and successor operations, which gives the number line; and that can be folded into two-dimensions to produce the regular grid of a lattice; and that can be extended into three-dimensions [3, 1:29:29–1:35:34].

A conventional construction of geometry, however, typically builds the cubic lattice in which the unit cell has face diagonal length = $\sqrt{2}$ and body diagonal length = $\sqrt{3}$. Clearly, some distance relations with respect to vertices within that lattice have values for which we cannot compute the last digit and are therefore not constructible. But there is also a more basic problem in having to accept that the process of constructing geometry from first principles must involve an arbitrary instruction to proceed differently according to preferred directions through the lattice. This presents a problem also for those physical theories in which spatial discreteness is either implied or desired. Granular models of space with a fundamental length scale of the order of Planck length are not modelled well by a rectilinear lattice for the obvious reason of the inhomogeneous length scales mentioned that give preferred directions, implying a deformation of Lorentz invariance with respect to the lattice itself (without considering directions extra to the lattice).

The ideal lattice structure that would provide a useful substrate for such programs in physics, and more specifically for the constructive geometry that this project is interested in, is one in which there is vertex homogeneity (i.e., uniform degree) and edge-length homogeneity. This implies a three-dimensional lattice with a homogeneous structure where the vertices are distributed uniformly throughout the space such that all nearest neighbour vertices are the endpoints of unit length edges, without preferred directions. Constructing such a lattice does not, on the face of it, appear problematic. In two dimensions it is of course trivial to construct the lattice of equilateral triangles that gives a homogeneous tiling of the plane. And the naive intuition is (as it was for Aristotle) that this should extend up a dimension to a lattice of identical regular tetrahedra that tessellate three-dimensional space.

In a paper discussing Causal Dynamical Triangulation (CDT), titled, "Quantum Gravity, or the Art of Building Spacetime," [5] the authors ask, "What is more natural than constructing space from elementary geometric building blocks?" However, the next sentence adds, "It is not as easy as one might think [...]" In fact CDT does not construct the space, but rather, it is treated as an ensemble of geometrical objects described by numerical methods, typically Monte Carlo simulations. Loop quantum gravity (LQG) [6] also attempts to quantize spacetime and also proceeds without the assumption of a background space, but, again, this approach does not produce a constructible model. Here the geometry is conceived of in terms of abstract connections between points formalised as spin networks, from which space is theorised to follow as an emergent property.

A constructible model of a three-dimensional homogeneous lattice is not generally eschewed in favour of purely abstract (often hyperspatial) models on principle, but rather, it is because there is no commonly agreed constructible model. This is the problem that this article addresses. A first impulse may be to assume that a more complete or somehow more correct set of specifications will produce the desired lattice. The approach laid out here, however, recognises the problem to be overspecification.

The aim in this project is to bootstrap an intrinsic notion of the geometrical space, growing it from the bottom up, from first principles – to the extent that even the construction of the homogeneous lattice, while it becomes the problem that this article addresses, is not specified as an objective. Rather, this project's objective is effectively the null objective; it is the construction of the geometric structure that results from not imposing objectives; which plays out in the model development, broadly, as a Wheelerian concept of "organization that is no organization", toward producing a pregeometric structure that has no structure [4,7,8].

In this case, the construction self-organises to produce structure in the common three-dimensional space, with no obvious reason to adjust that, except that there is, of course, the obvious chokepoint that stymies all attempts to construct the homogeneous lattice in three-dimensional space – that is the problem of geometrical frustration [9]. This problem appears as far back as Sanskrit writings, 499 AD, in the form of the sphere-packing problem [10], which maps to the problem of packing identical regular tetrahedra. If there was a constructible tetrahedra-based homogeneous lattice in three-dimensional space, that geometrical structure would obviously be ubiquitous throughout mathematics.

This project's approach to resolving the problem can best be introduced by once again referencing Bach. In several interviews and discussions Bach has made the distinction between classical mathematics that is stateless, and constructive mathematics that is stateful (e.g., [2, 01:01:15–01:10:34]). Here, we extend that distinction to geometry and argue that classical geometry (in which there is no constructive solution to the problem of the homogeneous lattice in three-dimensional space) is a stateless model; in contrast to which this project introduces the concept of a stateful constructive geometry as a dynamically updating graph-like lattice that can also be conceived of as a type of asynchronous cellular automaton. It will also be argued that from the perspective of being embedded in what is empirically a stateful physical universe, the universe's background-running default geometry is a stateful system on top of which the proposed stateful lattice should be informationally inexpensive, or perhaps free, to construct.

It is understood from algorithmic information theory that an object generated from minimal information input can yet have a large information output, or, more generally, apparent complexity can arise from minimal rules, in which case it should not be surprising that the homogeneous lattice structure, constructed with minimal instructive information input, the maximally unbiased, maximum entropy structure that has no structure, is not necessarily a mundane object. A key result of this investigation is the bottom-up construction of the graph-like lattice as a dynamically updating, maximum entropy structure that proves to be helically configured in three-dimensional space, from which there is a waveform projection to the image plane – which introduces new geometric considerations with relevance to physics. As one example, under the current understanding the de Broglie–Bohm pilot wave theory is required to unparsimoniously impose a waveform character over top of the existing geometry of spacetime. The work outlined here, however, constructs the maximum entropy geometrical background structure of spacetime that is inherently waveform, thus potentially removing an objection that otherwise stands in the way of the pilot wave interpretation of quantum mechanics.

2. Construction of the Lattice

The task is to model the structure formed of fundamental elements in fundamental predecessor and successor relations through the analogous construction of the lattice. Fundamental elements are analogised with vertices. Fundamental predecessor and successor relations have their analogue in the positional relations of the vertices within the lattice.

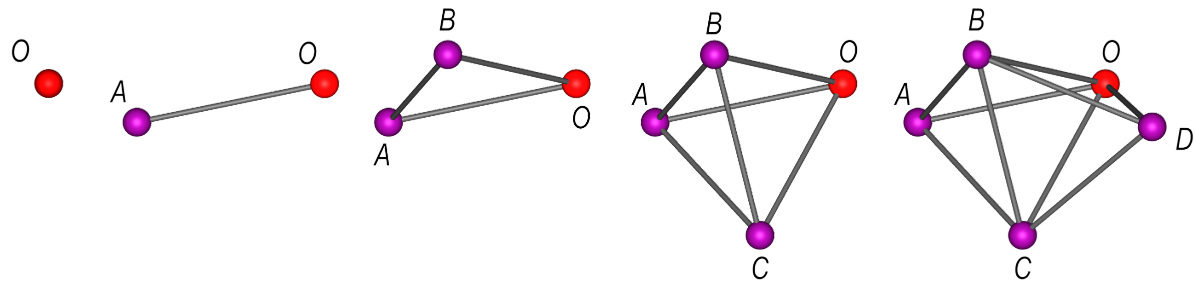
The first primitive is the vertex, beginning with origin vertex *O*. Successor vertex *A* is placed at some arbitrary position distinct from *O*. The graph-like lattice is defined to have vertices that are the endpoints of edges, and edges are the struts of the lattice. Importantly, the edge/strut is not a primitive. Any object would suffice. The edge/strut is merely a convenient construction object with which to model the positional relation between nearest neighbour vertices, it does not claim to represent the character of the space between vertices. Specifically, the edge/strut is not defined as the Euclidean line or any object that introduces a notion of infinitely many points.

The second primitive is the notion of congruence that enables that initial edge/strut object to be iterated over subsequent pairings of adjacent vertices throughout the construction – as such the edge/strut can be referred to as the unit length edge (without introducing any notion of magnitude). The graph-like lattice will become composed of strictly only those primitive objects that the construction is assembling – that is, vertices. From those vertices and the primitive notion of congruence the construction outlined in the following sections will bootstrap a space from the bottom up – there is no predefined background space.

As part of this program we imagine an ideal constructor who/that[2] while carrying out the construction is capable of recognising potential instructive information inputs, and who pre-emptively blocks those from entering the construction process. At this stage information is treated colloquially as semantic content. Instructive information (inputs) and descriptive information (outputs) will be recognized as ordinary language statements.

While there are no extrinsically originating construction rules imposed to manage the stepwise assembly of vertices, and construction proceeds without deference to any preconceived notion of spatial dimension, there is nevertheless an entropically driven control that arrives from the absence of control, from the absence of instructive information input that might otherwise impose specificity onto either elements of the construction (vertices), or relations between those elements (edges). The absence of specificity leaves the construction process to default toward producing the maximum entropy, spatially uniform distribution of vertices.

The maximally uninformed growth of constructive geometry's lattice is entropically biased toward producing the maximally homogeneous structure. Vertex homogeneity is in effect homogeneity of vertex degree, or number of edges incident (given that that is the only distinguishing feature available to a vertex – other than, obviously, the sequential labelling and location within the lattice). The requirement to default to homogeneity of degree acts as geometric entropy that selects for cyclic configurations resulting in cluster morphology wherein vertices arrange in a centrally symmetric configuration about the origin vertex *O* (geometric-statistical gravity).



(a) (b) (c) (d) (e)

Figure 1.

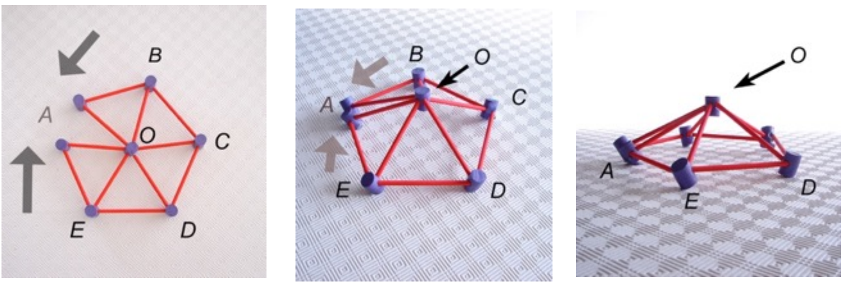
The unbiased constructor assembles the clustering configuration of vertices and edges until, ineluctably, four vertices and six connecting edges produce the regular tetrahedron *OABC*, as shown in Figure 1, (a) to (d); at which stage the structure implies (information output) the familiar three-dimensional space. There is very little empirical data with which to determine how many spatial dimensions there are (see related, [11]), however, it is reasonable to accept the observation that the lattice occupies precisely three spatial dimensions.

[2] Henceforth 'who'.

Figure 1 (e) shows tetrahedron *OABC* with successor vertex *D* (and associated edges) assembled to face *OBC,* producing new tetrahedron *OBCD.* The configuration shown in Figure 1 (e) can be viewed as tetrahedron *OABC* with tetrahedron *OBCD* glued at face *OBC*. Henceforth, lattice growth can equally be conceived of as the sequential addition of vertices (and associated edges), or as the accretion of regular tetrahedra glued at faces and sharing common central origin vertex *O*.

Figure 2. The icosahedral quasicrystal, IQC (constructed as a solid model).

The ideal constructor continues to assemble vertices and associated edges to the lattice. A shell of 12 vertices is filled in, which produces the cluster configuration that can also be conceived of as twenty regular tetrahedra that share central origin vertex *O.* The problem, of course, is that the dihedral angle of a tetrahedron, cos−1(1/3), is not a submultiple of 2π (although it is close to 2π/5), consequently gaps remain between tetrahedra, or, basically, not all vertices and edges in the outer shell of the graph-like lattice can connect and the structure is referred to as being geometrically frustrated. See Figure 2. This structure is known in materials science as the icosahedral quasicrystal*,* IQC, where all real IQCs are intermetallic compounds [12].



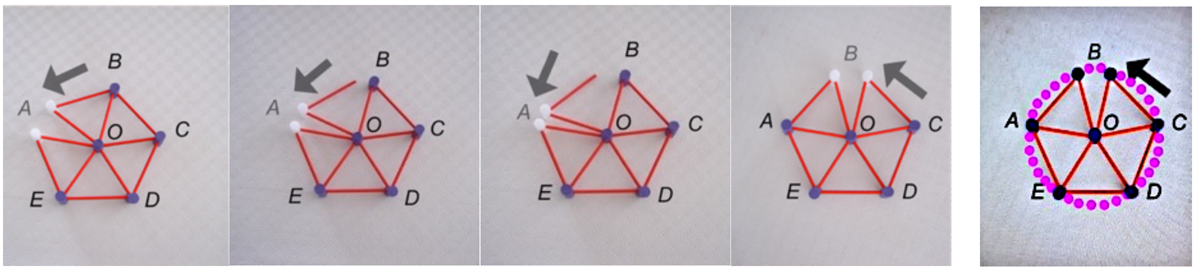
(a) (b) (c)

Figure 3.

Any attempt to continue construction of the lattice would go on to produce a global model of icosahedral quasicrystalline structure that is both geometrically frustrated and aperiodic in three dimensions. Unlike the structure analysis of conventional periodic crystals, it is not straightforward to find the correct mathematical formalism to express the ideal model. Most current approaches hypothesise that there is an ideal model that lives in some higher dimensional space that is only captured imperfectly in any projection to three-dimensional space [13]. Figure 3 shows a simplified, lower-dimensional illustration of one such model. A frustration gap at *A* is resolved by pulling the two vertices together (indicated with arrows). That action causes the two-dimensional structure to curve so that it becomes three-dimensional (Fig. 3 (b) and (c)). By dimensional analogy it is then possible to imagine curving a geometrically frustrated three-dimensional object (such as the IQC, Fig. 2) into a frustration-free structure in a four or higher dimensional space. There, is, however, no intuitive visualisation of any higher dimensional model (despite spurious examples) and in this setting it is definitely not possible to construct any higher dimensional lattice.

3. The Lattice as a Dynamically Updating System

This project investigates carrying out the construction in the common three-dimensional space, but where the lattice is now conceived of as a dynamically updating system. Again, a simplified illustration is first obtained by dropping down a dimension to the two-dimensional configuration shown in Figure 3 (a). The objective is again to resolve the geometrically frustrated structure, this time, however, the frustration problem is resolved by treating the configuration of vertices as a dynamically updating system. Figure 4, (a) to (d) shows vertices pulled together at *A* so that the frustration gap is closed – but it follows from that action that a neighbouring vertex-edge connection is pulled apart, causing a frustration gap to open there.



(a) (b) (c) (d) (e)

Figure 4.

3.1 Vertex creation and annihilation – the stateful system

The notion of vertices being dragged across a space to close the gap is merely a classical analogy of the causal mechanism that updates vertices in the system sequentially between the finite state in which the frustration gap is closed, referred to as vertex creation; and the state where the frustration gap is pulled open, referred to as vertex annihilation (borrowing the terminology, obviously, from particle physics: see also e.g., [14]. Remembering, of course, that edges don't exist but are merely convenient objects to analogise the uniform distance relations between pairwise adjacent vertices.

The ideal model of the lattice is now conceived of as a dynamically updating graph-like construction that is entropically driven toward a structure in which, optimally, there is vertex homogeneity (uniform degree) and edge-length homogeneity when averaged over the dynamical macrostate. Considering again the simplified system shown in Figure 4: (a) and (d) show the vertices and edges in a static snapshot that captures a microstate of the system. In each iteration there will be vertices and edges at some position such that they are separated by the frustration gap so that the system can be thought of as being in a state of tension as it tends toward equilibrium at the maximum entropy uniform distribution, inducing a domino effect that perpetuates the gap-closing, gap-opening action cyclically around the configuration, captured in Figure 4 (e) as the averaged, centrally symmetric macrostate.

3.2 Time and the stateful system

Traditionally, a dynamical system is defined to be evolving over time (where this can be continuous or discrete time). Here, however, the geometrically frustrated structure is a relational system in geometric disequilibrium, where spatially successive causal relations dictate that the update action is triggered and persists via the domino effect – as opposed to those updates being linked to the ticking of any external metronome. In this system, time can be investigated as an emergent property linked to the entropically driven sequential transition of states in the system.

3.3 Virtual vertices

It is hypothesised that over the entire, arbitrarily large, asynchronously updating, three-dimensional model of the graph-like lattice, the mechanics of the creation/annihilation events will produce vertex creation at random positions within delineated trajectories. Illustrating that concept with the simplified example shown in Figure 4, we can conceive of an animated model in which vertices and edges are clicking together and pulling apart, vertices created and annihilated in actions that, activated by the sequential domino effect, move cyclically around the configuration so that the positions that will be occupied by created vertices over multiple updates can be predicted to fill in a fuzzy band around the perimeter of the configuration (Fig. 4 (e). The geometrically frustrated, asymmetric configuration as it appeared in the static model, the microstate (Fig. 4 (a), becomes smoothed out in the dynamically updating system so that it produces, in the macrostate (Fig. 4 (e), an unfrustrated configuration with continuous rotational symmetry.

Figure 4: considering the interval *AB,* for example, the range of possible position that vertex creation and annihilation is restricted to, spanning that interval, is delineated by the arc *AB* that has unit length radius centred at *O*. For every static instance that captures a microstate of the system that includes *A* and *B,* there is that very basic geometry (arc *AB)* for the interval between those vertices that gives the positions that can probabilistically be occupied by vertices produced in some update of the system. The positional placeholders for these probabilistic

vertex positions are referred to as "virtual vertices." Arbitrarily many updates produce arbitrarily many positions that the virtual vertices occupy, and in principle it is possible to construct those.

If the simplified example of the six-vertex system shown in Figure 4 is thought of as part of a larger global lattice structure, and the mechanism whereby dynamical update produces vertices that fill in the intervals (as described above) applies throughout the structure, then over arbitrarily many updates a migration of the structure is expected so that vertex creation fills the entire global lattice space. On the basis of that simplified example, it is proposed that a typical microstate of the constructed lattice will show the vertices of one iteration of the lattice, constructed within the bootstrapped ambient background-cloud populated with virtual vertices that indicate probabilistic positions where vertex creation has occurred or will occur. There is the hierarchy of vertex ontology: (i) instantiated vertices; (ii) virtual vertices for which the trajectory of their positions can be drawn spanning the interval; (iii) the background cloud of virtual vertices that result from the migration of the global structure.

The development of the three-dimensional graph-like lattice operating as a dynamically updating system is, in principle, an intuitive and constructible model – we can reasonably conceptualise the lattice as a mechanical system in which vertices and edges are clicking together and pulling apart, analogising annihilated and created vertex states, and it follows that if we can conceptualise that, we can, in principle, construct it.

4. Constructing the Static Approximant

With constructive geometry's graph-like lattice now defined to be the asynchronously updating, non-deterministic dynamical system, there are three possible modes of construction:

i. Construct a working model of a representative portion of that dynamical system. This is possible in principle (as is required of constructive geometry).
ii. Construct a static model of the graph-like lattice, representing a specific microstate of that dynamical system – but as is the case for complicated dynamical systems generally, it is not possible, even in principle, to model the exact positions of all elements.
iii. The third mode, suitable for this introductory work, is to construct the static three-dimensional model of the graph-like lattice as an averaged approximant – essentially a static representation of the dynamical system as the averaged macrostate. The approximant lattice model attaches vertices imprecisely to the ends of edges so that they locate within a fuzzy range of position. The IQC that first appeared in Figure 2 as a solid model, is now, in Figure 5, constructed as the approximant graph-like lattice.

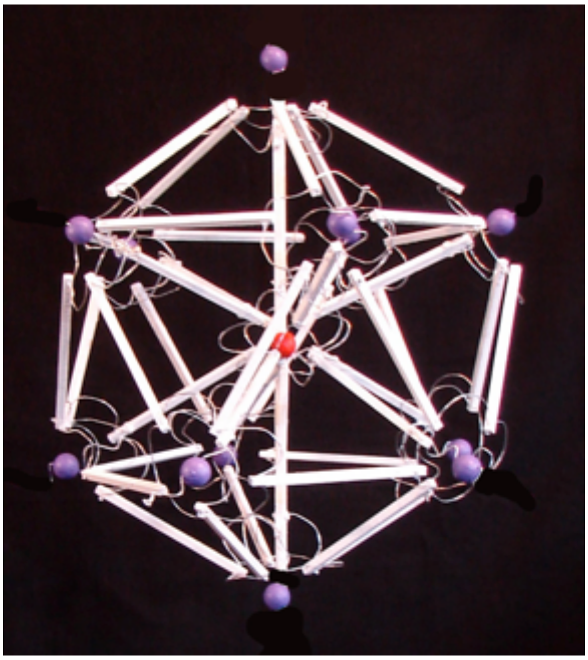


Figure 5. Approximant IQC

4.1 The IQC building block

Figure 5 shows the approximant IQC: thirteen vertices (one red + twelve purple); forty-two edges; twenty tetrahedra. It is recognised to be the fundamental building block for construction of the lattice. This is not an arbitrary designation (compared to, for example, Causal Dynamical Triangulation's building blocks that evolved from Euclidean quantum gravity's arbitrary decision that they should be triangular). In arguing for building block status for the IQC, there are the following observations (i.e., information outputs):

- In the bottom-up construction process, assembling the last of the 12 vertices into the first shell signals that a phase of construction has auto-terminated. It is observed (information output) that all possible vertex positions available within the shell are filled.
- It has been established that the maximally homogeneous lattice is necessarily a centrally symmetric configuration, of which the completed IQC is observed to be the first produced by the construction. Note: A conventional appraisal may have initially considered that the tetrahedron (or 3-simplex) is a superior candidate for building block status. However, unlike a group-theoretic interpretation where automorphisms include reflections and rotations about a central axis located at a place on an edge or on the background, in this project, with respect to a microstate of the system, the edges represent nothing real and there is no background that the lattice is constructed over; discrete vertices constitute the entirety of the space. Symmetries can only be decided with respect to an axis that coincides with a vertex, in which case the bottom-up construction produces no centrally symmetric configuration of vertices until the

complete IQC arrives – consequently there are no intermediary subunits that suggest themselves as building blocks.

- The IQC is the first configuration in which vertices have optimal degree homogeneity. The twelve peripheral vertices are uniformly degree 6. Only central origin vertex *O* (degree12) fails to conform, but this is remedied in the global context developed below.

4.2 Penetration twinned IQCs and the quasicrystaline pentakis dodecahedron (QPD)

The requirement for vertex degree homogeneity drives the construction process toward assembling IQC building blocks, not stacked as contiguous units, but in penetration twinned junction, as is found in natural crystallite compounds and in manmade nanoscale synthetic materials [15].

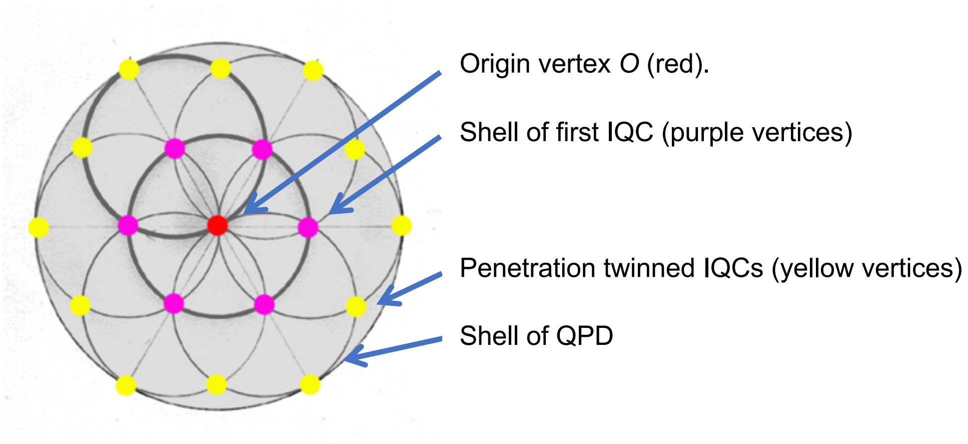


Figure 6. Penetration twinned IQCs produce the quasicrystalline pentakis dodecahedron (QPD). (Note: illustration, not true projection)

Figure 6 is a two-dimensional section cut through the lattice, showing the central origin vertex *O* and the first shell of vertices (purple) that make up the IQC. Each of those peripheral vertices (purple) are now identically the central origin vertex for additional IQC building blocks assembled to that first central IQC in interpenetrating formation with it and with each other, to form the second shell (yellow vertices).

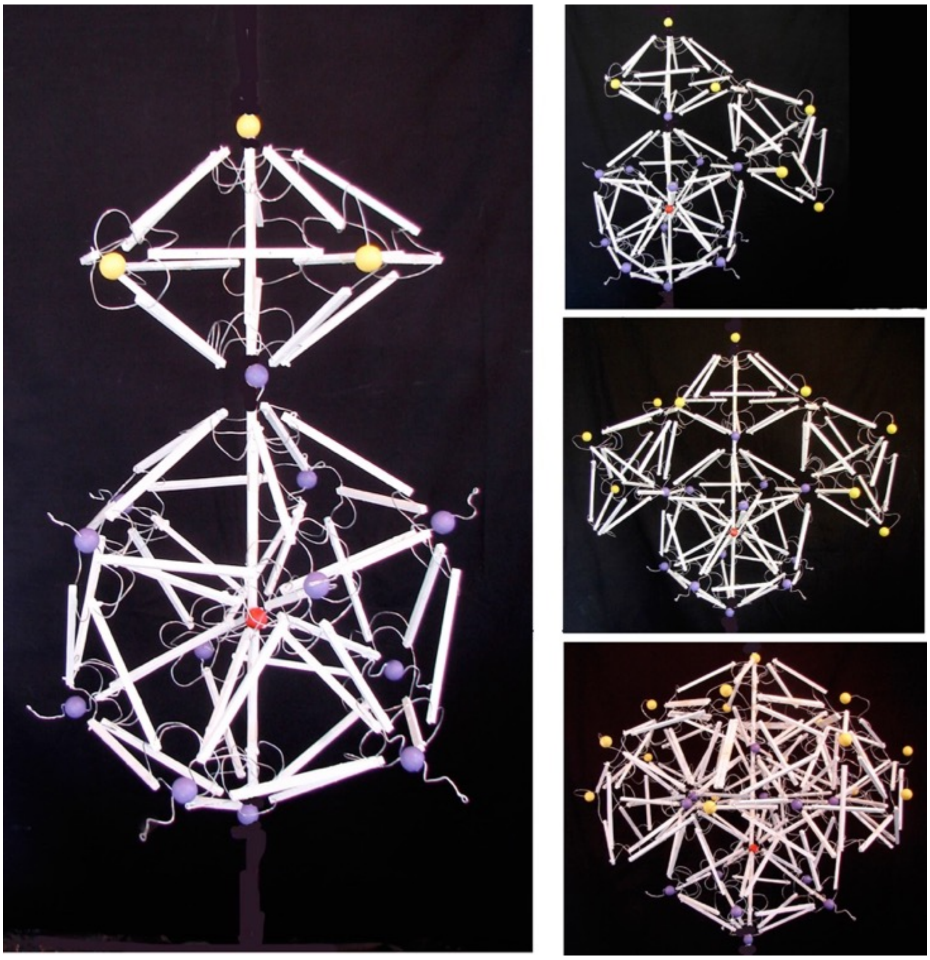


Figure 7. Assembly of penetration twinned IQCs.

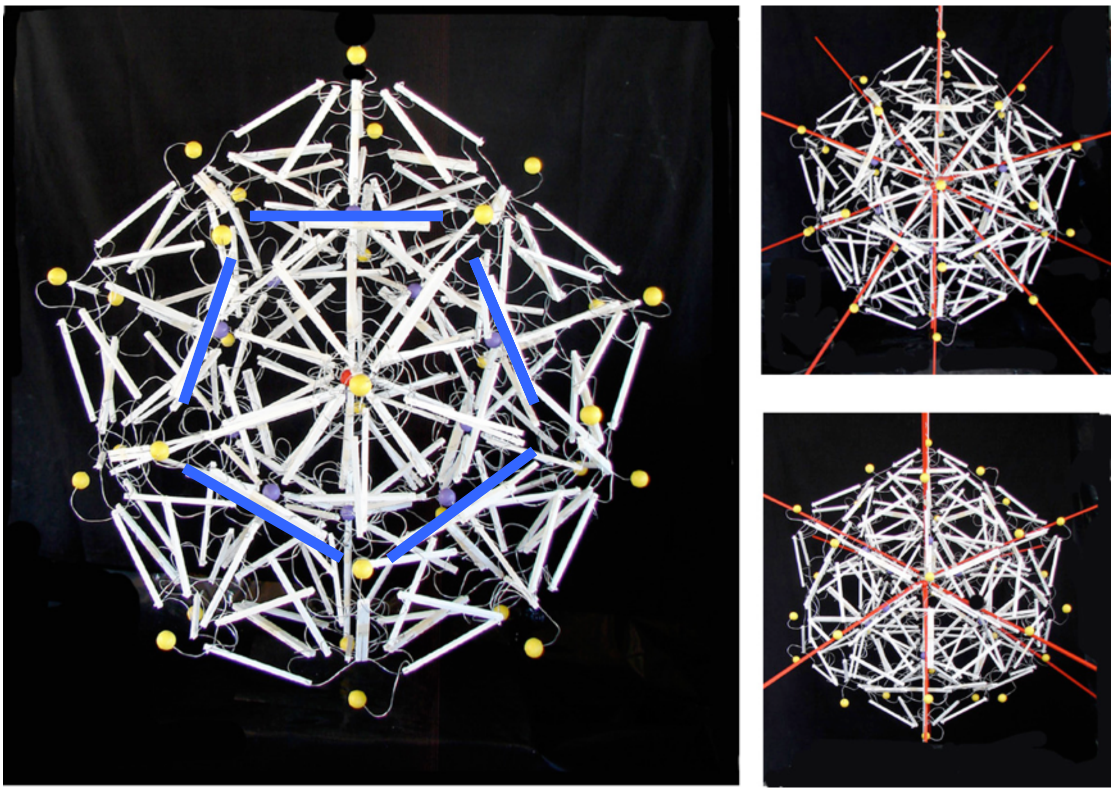


Figure 8. The quasicrystaline pentakis dodecahedron.

Figure 7 shows the actual construction of the three-dimensional approximant graph-like lattice. The penetration twinned IQCs are formed by attaching a pentagonal bipyramid (i.e., tetrahedra arranged in five-fold symmetry) to each of the IQC's twelve peripheral vertices. The Figure 7 thumbnails show pentagonal bipyramids being added, and Figure 8 shows the completion. The construction again reaches a stage where a shell (the second shell) has every available position occupied with a vertex (yellow) so that another phase of construction is observed (information output) to have auto-terminated. The lattice surface is now faceted with twelve pentagonal pyramids (Fig. 8, one is highlighted in blue). If an outer spherical shell was drawn around the lattice, the vertices at the apex of the pentagonal pyramids would all sit on that shell, however, the non-convexity of the construction has the effect that it does not appear spherical. In the Figure 8 thumbnails, the principal axes of rotational symmetry (five-fold and six-fold) are highlighted in red, as viewed from two different orientations.

The surface geometry of this new lattice structure corresponds to that of the deltahedral pentakis dodecahedron, shown in Figure 9 as a geometric solid. Because this project's graph-like lattice (Fig. 8) includes the interior struts it is consequently geometrically frustrated so that the lattice constructed here is referred to as the "quasicrystalline pentakis dodecahedron", or QPD. Components: Forty-five vertices. 204 edges. 110 tetrahedra. It remains, however, that although the penetration twin morphology effectively homogenises all vertex degree in the interior of the QPD, nonhomogeneous degree count will, of course, persist at the perimeter of any clustering structure that has a boundary/edge. This is resolved in subsection 5.2.

Figure 9. Deltahedral pentakis dodecahedron. The figure is slightly non-convex due to the height of the pentagonal pyramids that make up the surface.

4.3 Review of phases of lattice growth

Note: Vertex colours are keyed to Figures 6, 7, and 8.

- Phase I: Central origin vertex *O* (red).
- Phase II: The first shell is filled with twelve vertices (purple) creating the prototypical IQC building block (Figs 2 and 5).
- Phase III: The construction process assembles IQC building blocks (Figs 6 and 7). Each of the twelve vertices (purple) that make up the shell of the first IQC, becomes the

central origin vertex for an IQC that is in penetration twin formation with that first central IQC, and with each neighbouring IQC (yellow vertices), creating the QPD.

5. The QPD Lattice: Observations and Comments on the Construction Up to This Stage

5.1 Observations

(i) There is no allowable instructive information input that would inform the construction process when to terminate, or that specifies the cardinality of elements, or that arbitrarily determines the bounds of the structure. In principle the sequential assembly of vertices to the graph-like lattice is indefinitely extensible.
(ii) One aspect of the role of the ideal constructor is to block new instructive information from entering the system. Effectively, this means that from an early stage the total information available to the system is already in the system. It follows that there is some minimal structure beyond which no novel structure can appear in the ongoing construction. This observation, along with (i) above, implies that the growth of the lattice has fractal character.
(iii) There is the problem of persistent inhomogeneity of vertex degree count, first mentioned in subsection 4.2, where each phase of structure growth merely pushes the problem out to the new horizon. Appealing to the indefinitely extensible construction does not fundamentally resolve the problem, particularly considering that the fractal character implies that although construction is not terminated, we nevertheless have knowledge of the global structure, in that it resides also within the initial kernel structure.

5.2 Comments

The fractal character of the lattice entails that there is some minimal kernel structure that is iterated to produce the global model, and it follows that the problem of persistent inhomogeneity of vertex degree count must be resolved within that kernel structure. The IQC has been identified as the first building block. Now the lattice construction that forms the QPD operating as the dynamically updating system is identified as that minimal kernel structure that is identically the first vertex, and the complete global model. The global, homogeneous, maximum entropy lattice structure is now identified to be the stateful QPD, henceforth SQPD lattice.

First, while the presentation of the lattice in this article is limited to conventional images from the viewpoint of an extrinsic observer, it has also been noted that the limitation imposed on information available to the backgroundless bottom-up construction determines that the vertices of the SQPD lattice are all that there is – there is no meta-structure providing a location that supports any extrinsic observer viewpoint located anywhere other than on a vertex of the lattice. Consequently, the ideal observer is more correctly to be conceived of as an intrinsic observer with a viewpoint looking out from within the lattice structure.

Referring to Figure 6, consider that within the SQPD lattice the intrinsic ideal observer may be located at origin vertex *O* from which vantage point the lattice appears homogeneous and isotropic (as is required of the ideal model). If, however, the observer begins to move through that lattice shown in Figure 6, traversing vertices, after two steps that observer may arrive at the boundary from where the structure no longer appears isotropic. Confronted with this situation, topological models such as Poincaré dodecahedral space glue pairs of opposite faces, producing a closed 3-manifold within which a transitioning observer who exits one of those faces reappears through the opposite face. This, however, can only be understood abstractly, whereas this project is constrained to producing concepts that can in principle be demonstrated within the context of the constructible model.

Based on observations of the constructed lattice it is reasonable to hypothesize the following model:

i. The ideal lattice is a stateful system wherein vertices transition between states of creation and annihilation.
ii. The ideal intrinsic observer viewpoint necessarily occupies a position at a created vertex (as opposed to annihilated).
iii. The transition of the observer viewpoint from one position to another within the lattice is necessarily correlated with vertex creation, for which there is (at a causally prior location within the lattice) the associated vertex annihilation.
iv. The progress of the ideal intrinsic observer traversing vertices in the direction of the boundary has the effect that frustration gaps are pulled together such that vertices are created ahead, but that action also causes gaps to open behind such that vertices are annihilated. The observer's transition through the lattice induces a correlated churn of vertex creation/annihilation such that the observer viewpoint appears to drag the SQPD structure along with it, thus perpetually remaining at the centre with the boundary always on the horizon. Under this mechanism the global SQPD lattice structure is, for every intrinsic observer, homogeneous, isotropic and without edge or boundary.

The initial stages of the bottom-up construction began with primitive vertices and the notion of congruence but involved no concept of a straight-line or a ray. At this stage of the lattice construction, however, the underlying geometrical chassis of the lattice can now be conceived of as twelve rays that radiate outward from the central origin vertex *O*. These rays extend in alignment with the six major icosahedral symmetry axes that the IQC and QPD share (being mutually dual).

6. Fractal Structure

A first intuition may be that constructing a homogeneous lattice, starting from a central origin vertex, should result in an evenly dense distribution of vertices in multiple successive shells propagating out in onion-like layers. Approaches along these lines generally resolve the inevitable geometrical frustration problems by hypothesising hyperspatial models that are inaccessible to construction. In this project's specifically constructed lattice, however, from the boundary of the initial QPD, continuation of the existing assembly mode produces structure that does not fill in a subsequent shell, but instead, the structure seamlessly morphs (absent intervention of new instruction) into second fractal layer growth that reproduces the initiator QPD; and from there, self-similar fractal structure iterates indefinitely.

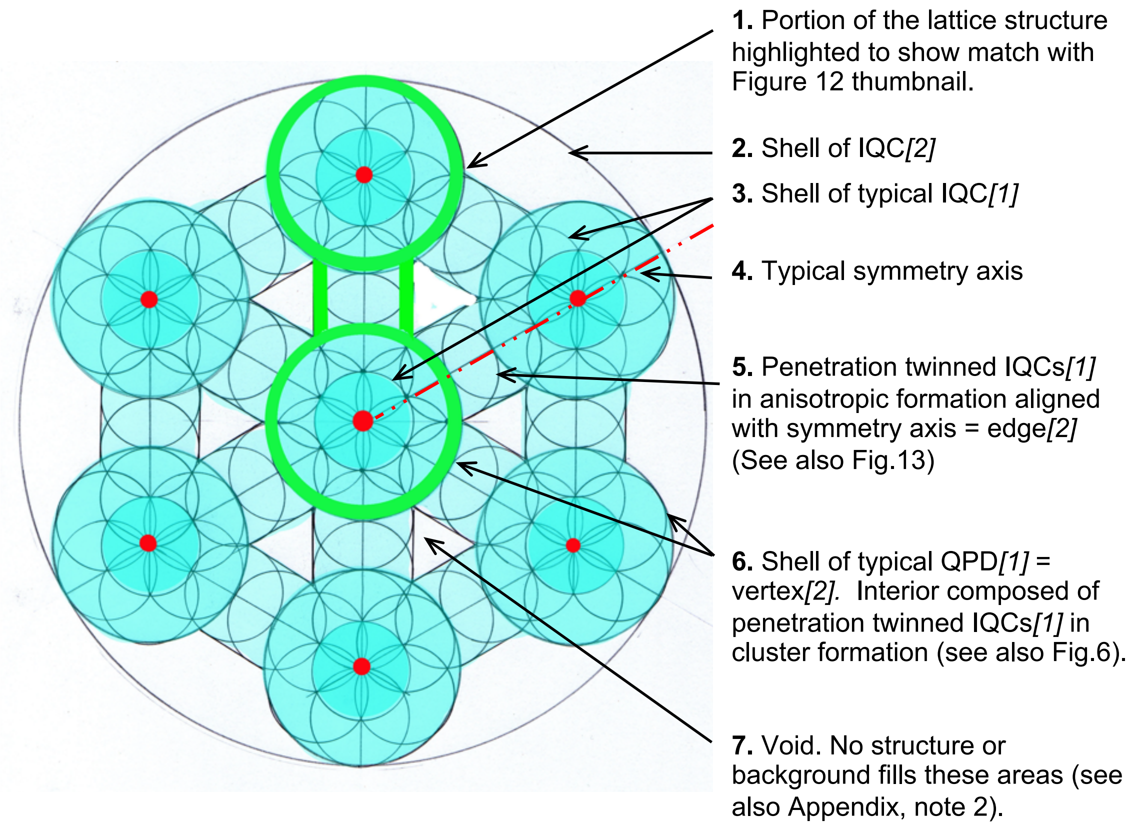


Figure 10

Figure 10 shows a two-dimensional section that cuts through the QPD and includes six of the twelve columns of stacked IQCs that radiate out (aligned with the icosahedral symmetry axes). In this diagram (and henceforth) the fractal layer number, or iteration number, is given in italicised square brackets. For example, the QPD in the first fractal layer is concomitantly the vertex in the second fractal layer, written as, "QPD*[1]* = vertex*[2]*".

6.1 Hierarchically nested fractal layers

Fractal Layer *[1]*:

The initiator input to Fractal Layer *[1]* = origin vertex *O* (under magnification it resolves to a QPD).

Phase I: Origin vertex *O[1]*. (See also growth phases, subsection 4.3.)

Phase II: Vertices and edges are assembled to form the IQC*[1]*. (Note 3, Fig. 10).

Phase III: Penetration twinned IQCs*[1]* are assembled to form the QPD*[1]*. (Note 6, Fig. 10).

Output from Fractal Layer *[1]* = QPD*[1]*.

Fractal Layer *[2]*:

Input to Fractal Layer *[2]* = QPD*[1]* = origin vertex *O[2]*.

The Fractal Layer *[2]* growth repeats the Phases I, II and III from above, except that now the QPDs*[1]* act as vertices*[2]* and the columns of IQCs*[1]* anisotropically stacked in penetration twinned morphology act as edges*[2]*.

Output from Fractal Layer *[2]* = QPD*[2]*

Fractal Layer *[3]*:

Input to Fractal Layer *[3]* = QPD*[2]* = origin vertex *O[3]*.

Again, lattice growth repeats Phases I, II and III.

Output from Fractal Layer *[3]* = QPD*[3]*

Fractal Layer *[4]*:

Input to Fractal Layer *[4]* = QPD*[3]* = origin vertex *O[4]*.

Lattice growth repeats Phases I, II and III.

Fractal Layer *[5]*:

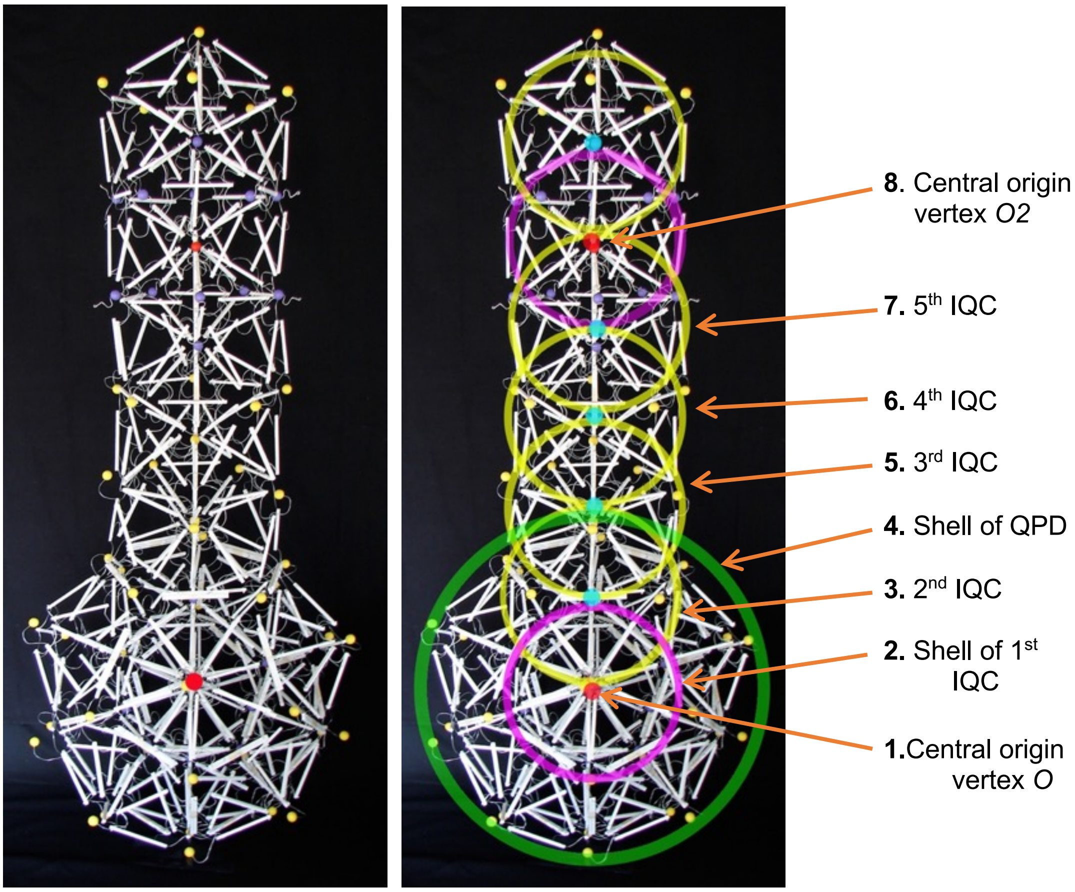
Hierarchically nested fractal growth continues…

6.2 Second fractal layer construction of the graph-like lattice

Having in the previous subsection discussed the fractal character of further lattice growth in general terms, now Figure 11 (a) shows the actual continued construction of the approximant graph-like lattice following on from Fig. 8. Note: the construction shown in Figure 11 is associated to one symmetry axis radiating from *O,* being, of course, typical of all twelve.

Figure 11 (a) is reproduced in Figure 11 (b) with features highlighted and notes attached. Note 8 indicates a second central origin vertex, labelled *O2.*

Figure 12 shows a continuation of the graph-like lattice construction that builds out a second QPD structure, formed around *O2*. The Figure 12 thumbnail shows the main image reduced in size, with the QPD/column/QPD structure outlined in green, which is concomitantly the fractal layer *[2]* vertex/edge/vertex (see also Figure 10, note 1).



(a) (b)

Figure 11

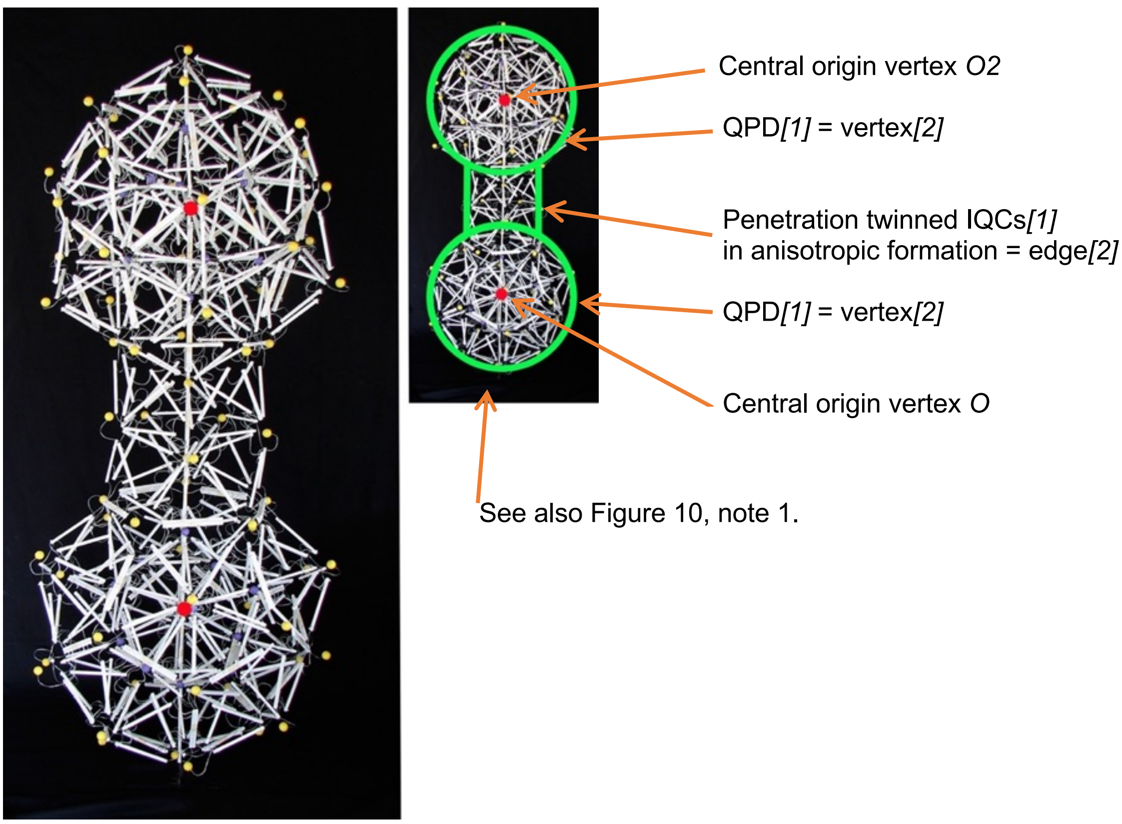
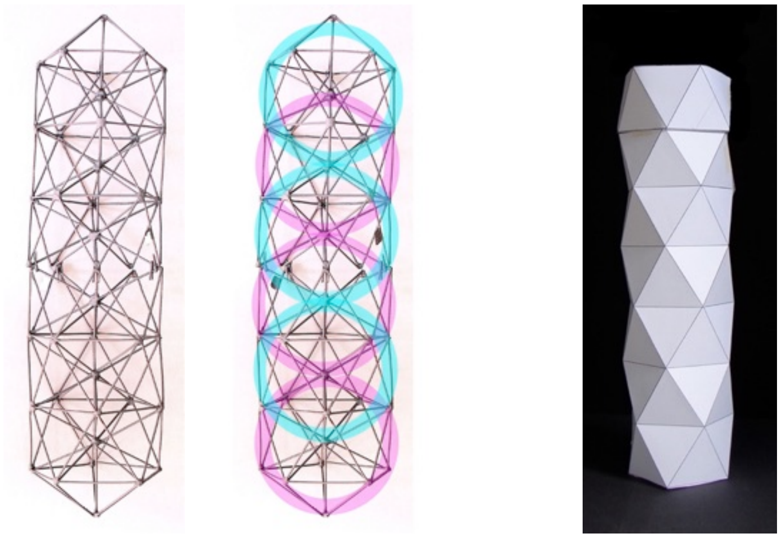


Figure 12



(*a*) (*b*) (*c*)

Figure 13. Anisotropic growth of penetration twinned icosahedra: (a) wire skeletal model; (b) twinned icosahedra are highlighted; (c) solid model.

7. Pathways Constructed Within the Lattice

The straight rays within the lattice that were referred to in subsection 5.2 are artifacts of Euclidean geometry that are overlayed onto the model – the straight rays are nowhere constructed (at least, not in the fundamental case, but are only constructible with large input of instructive information). To understand the fundamental distance relation between two remotely separated locations in the discrete graph-like lattice, it is meaningless to say that they are connected by a straight-line (introduced as a primitive and informally defined as the shortest distance between those two locations) or by any other arbitrary continuous geodesic. Rather, constructive geometry has available an information-theoretic approach that defines the fundamental pathway to be the route through the lattice, traversing discrete vertices, that is constructed with the minimum instructive information input. Constructive geometry offers, as a counterpart to the "line", the prospect of an extension in space that (contrastingly) does have a meaningful definition, and is constructible. An introductory overview of that extension, conceived of as a pathway through the lattice, is given in this section.

In the first instance, the fundamental pathway constructed with zero input of instructive information produces the trivial route that makes a circuit that traverses eleven vertices and returns to that pathway's origin. However, the pathway that is of interest is the case where a minimal information input instructs the ideal constructor to produce the fundamental pathway from some vertex *O* to some designated vertex *F* in the lattice, thus introducing directional bias.

First, referring again to the simplified example, Figure 4 (e), that diagram showed the two-dimensional graph-like structure tentatively modelled with arbitrary (i.e., not constructed) straight-line edges. Vertices *A,B,C,D,E* mark out the intervals around the perimeter. For each interval it is of course trivial to plot the locations to which vertex creation/annihilation is restricted. Those hypothetical vertex positions are plotted in Figure 4 (e), producing the interval arcs shown with pink (virtual) vertices.

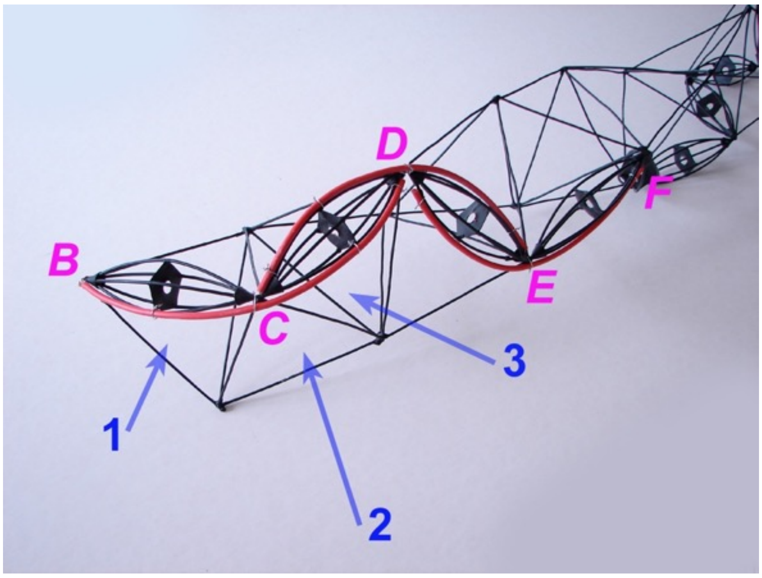


Figure 14.

The principle introduced in the Figure 4 two-dimensional example is applied to the three-dimensional lattice: The relevant arc located in the three-dimensional lattice requires two radii (i.e., it is a compound curve), for which the tetrahedron is the minimum geometric structure sufficient to anchor the two foci for those radii. Figure 14, note 1, shows a tetrahedron with arc *BC* constructed (highlighted in red). Arc *BC* can be continued smoothly to produce arc *CD,* which requires two more tetrahedra to anchor the radii (notes 2 and 3). The ideal constructor can then produce arc *CE* that passes through *D* (also highlighted in red). From there, arc *DF* that passes through *E,* and the associated tetrahedra, can be constructed. The minimal lattice construction that supports a directed pathway growth becomes a series of face-sharing tetrahedra that link together to form three intertwined helices that make up a Boerdijk-Coxeter helix (B-C helix) [16]. The minimum-instruction pathway forms a secondary helical coil that wraps around the B-C helix substructure; see Figure 15. Allowing that there is, first, an initiating instruction that imposes the directional bias, the pathway construction described above is the prototypical minimally instructed route that connects remotely separated vertices within the lattice structure.

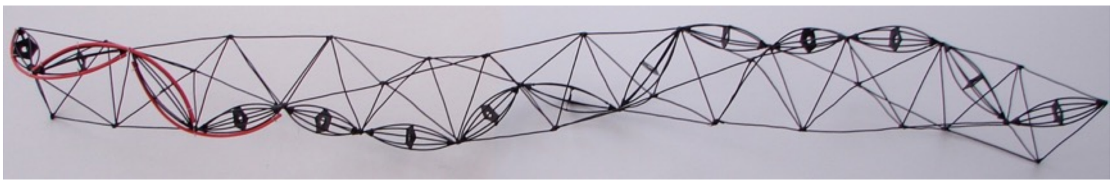


Figure 15. The Boerdijk-Coxeter helix

As described above, considered in isolation from the broader lattice, construction of the fundamental pathway has produced the B-C helix. The QPD lattice is now investigated for B-C helix structure: it is found that the columns formed from stacked penetration twinned IQCs (e.g., Figs 11, 12 and 13) can be decomposed into five strands of B-C helix. However, unlike any other stacking of Platonic solids, the B-C helix is not rotationally repetitive. In attempts to compact multiple B-C helix strands, face contact is always frustrated [17, 18]. However, novel to this project's approximant graph-like lattice construction, those B-C helix strands are also helically coiled about a central longitudinal symmetry axis so that face contact is effected. Five coiled-Boerdijk-Coxeter helix (C-B-C helix) strands compact in five-fold symmetry such that they make up an IQC column. See Figure 16, solid model, and Figure 17.

Figure 17, image (a) shows the graph-like lattice construction of the stacked penetration twinned IQC column. One of the C-B-C helices that contributes to the column has a red tube threaded through the centre of it, longitudinally, to identify it, and a second example is identified with a green tube. Image (b) shows all five C-B-C helix strands likewise identified with coloured tubes (noting that the coloured tubes do not represent an actual object of the construction, they merely highlight the shape of the C-B-C helix). Image (c) includes the QPDs constructed as a continuation from the IQC column (as was first shown in Fig. 12).

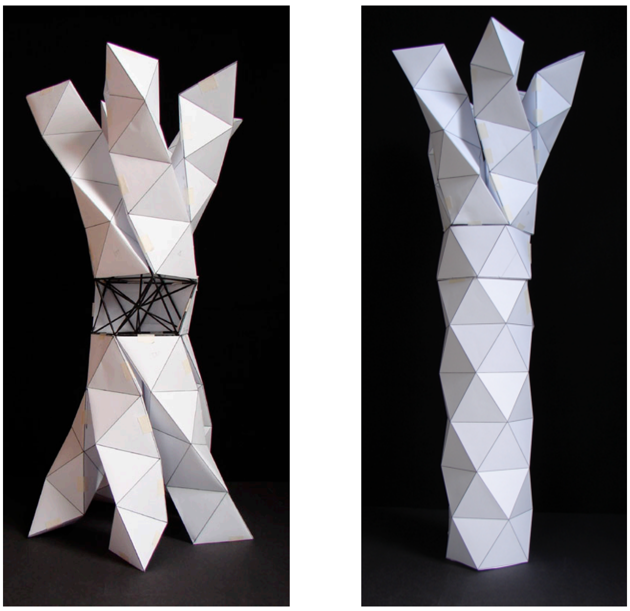


Figure 16.

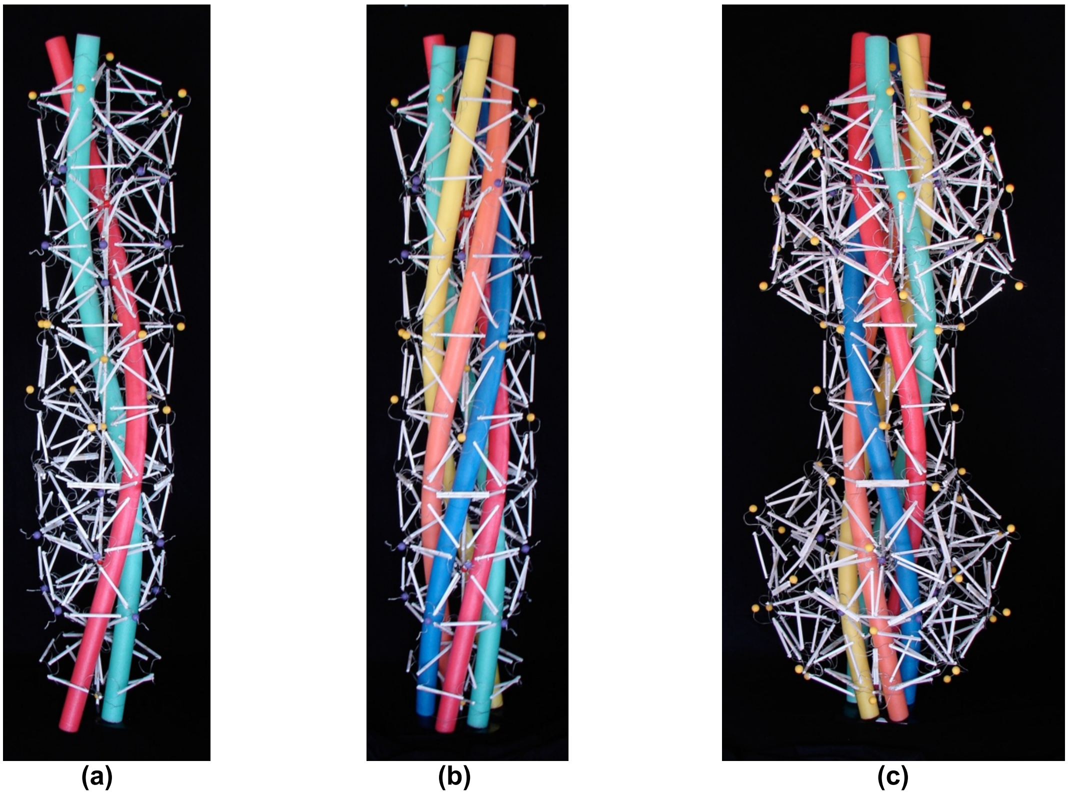


Figure 17.

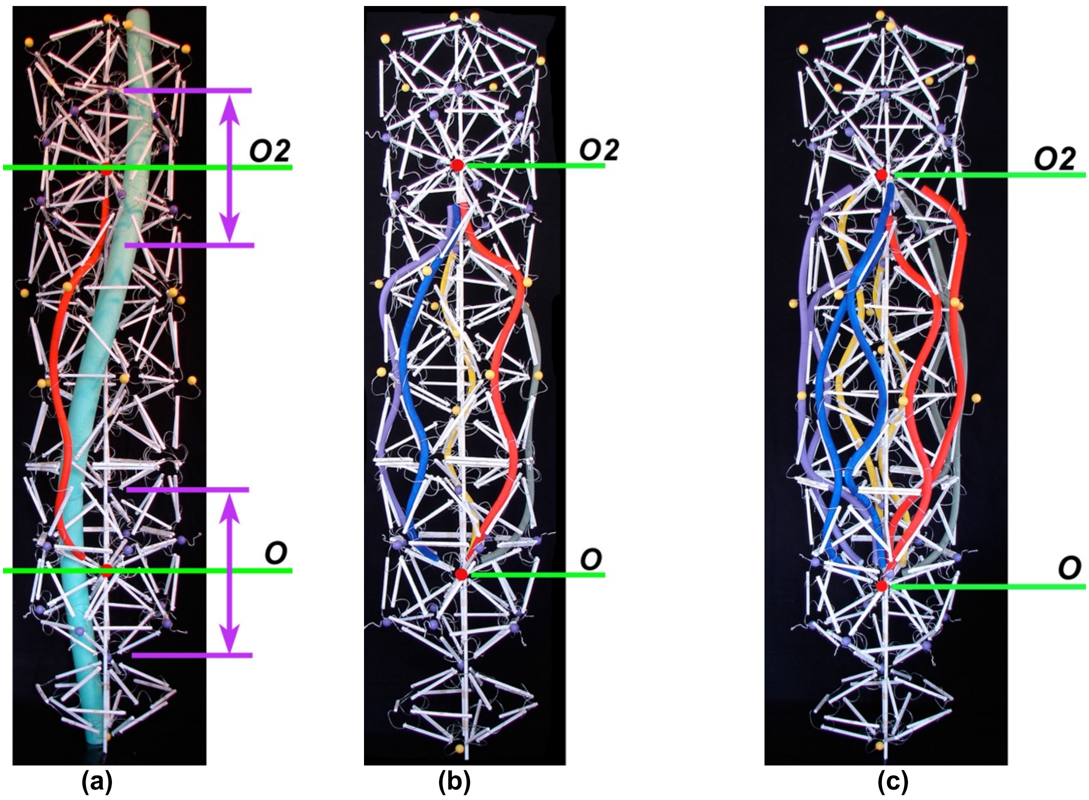


Figure 18.

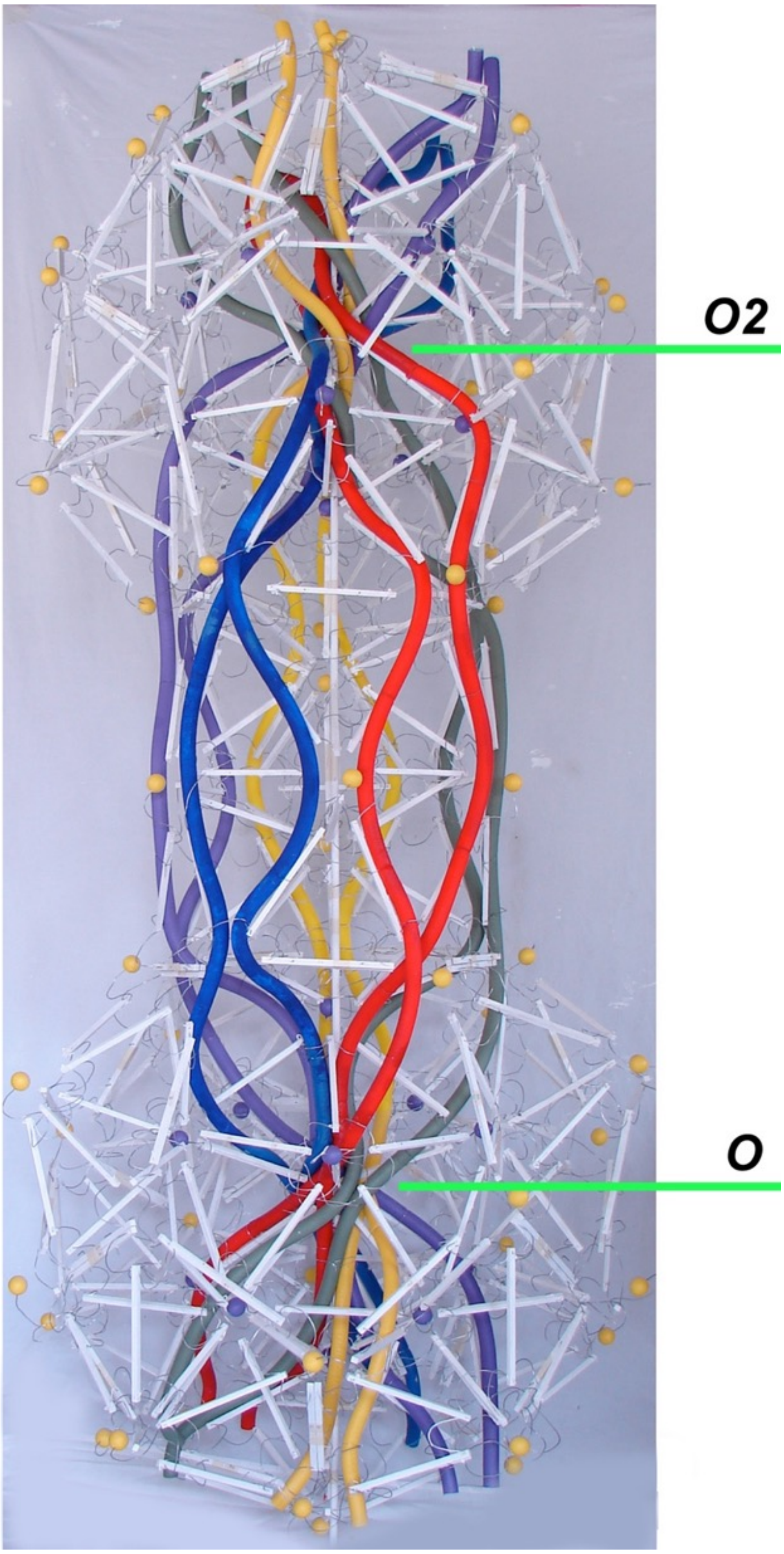


Figure 19.

Figure 18, image (a) shows one strand of C-B-C helix highlighted with the green tube. The prototypical fundamental pathway associated with that C-B-C helix is constructed, indicated with a red line. It begins at origin vertex *O* (indicated with green horizontal line). The pathway helically coils around the C-B-C helix (as first shown, Figs. 14 and 15), where that C-B-C helix is in turn helically coiled about the longitudinal axis; where that is one of the twelve axes that radiate from central origin vertex *O,* aligned with the icosahedral axes of symmetry.

To summarize: the fundamental pathway is helically coiled around the helical strand of tetrahedra (C-B-C helix), that is helically coiled about the symmetry axis. Duplicate copies of that pathway structure are arranged in five-fold symmetry about that axis. Figure 18 image (b) shows five colours of pathway that travel upward from origin vertex *O*. Image (c) fills in the five counterpart pathways that travel downward from origin vertex *O2.* Thus, all fundamental pathways that originate from *O* are constructed. See Figure 19. The construction shown from *O* to *O2* is one example that is typical of the twelve columns of IQCs that radiate outward from the origin vertex *O* (and from *O2, O3, O4...*) in alignment with the six fundamental axes of icosahedral symmetry.

Within the IQC column segment (which is essentially an interval*[2]*) shown in Figure 19, the ten fundamental pathways that travel from *O* to *O2* have reflectional symmetry – they originate from *O* located on the longitudinal axis, and they terminate at *O2* on that axis. This construction of the pathways through the lattice can be read as marking off prototypical segments of IQC columns that iterate throughout the lattice structure, where those segments have at their endpoints *O, O2, O3...* Importantly, it is only now, by understanding the fundamental pathways, that there is information originating from within the construction itself that signals where an edge/column terminates at the formation of an origin vertex/QPD. This supports the concept of constructive geometry's lattice as a minimally instructed, self-assembling structure.

8. Summary: The Constructed SQPD lattice

A key aspect of developing this theory of constructive geometry has been accepting that the homogeneous lattice should be conceived of as a dynamically updating system. It is possible to conceptualise an operational, naïve, animated mechanical model; in which case it is, in principle, possible to construct that model by employing vertices and edges that pull together and connect at some locations, triggering neighbouring vertices and edges to be pulled apart, analogising the process of vertex creation and annihilation. However, the asynchronous update feature means that for any static snapshot, not all vertices will be in a created state; some will be annihilated, and some will be in an ill-defined transition state, consequently the microstate of the system is not, in principle, constructible. There is no static model that has information entropy equivalent to that of the dynamical model (in effect, perhaps, a rather literal, mechanical, interpretation of the uncertainty principle).

Accepting those inherent limitations, this project has constructed the static averaged approximant as a representative base state of the dynamical graph-like lattice. The approximant model has produced a mapping of the space with fundamental, helically coiled pathways that explain where vertices may be found in causally subsequent (or previous) iterations of the lattice. Those locations are marked with virtual vertices. The assembly of the lattice is essentially the constructive method of deriving those pathways. Consider Figure 20, for example: proposed superior graphics will remove the preliminary straight-line edges and show only vertices and the ten helically coiled strands of pathway that delineate curved intervals composed of virtual vertices. That proposed superior representation, moreover, should not merely depict the construction shown in Figure 20 that relates to one symmetry axis, but ideally it should include the lattice of helically coiled pathways associated to all twelve rays radiating from *O* (and *O2, O3…*) in alignment with the six icosahedral symmetry axes, and the construction should continue, mapping the space through arbitrarily many fractal layers.

Encapsulated: Constructive geometry maps three-dimensional space with a dynamically updating, helically configured, icosahedrally ordered lattice with fractal character such that both the vertex and the global structure are the SQPD. It is hypothesised that continued updates of the lattice through cycles of vertex creation and annihilation cause a migration of the global structure that produces the arbitrarily dense and extended cloud of virtual vertices that make up the ambient virtual background that any static snapshot of the lattice is set within. Updates to the annihilation/creation status of vertices at the SQPD boundary have a causal linkage to the transitioning ideal intrinsic observer such that that observer remains always at the centre of the homogeneous and isotropic lattice structure.

8.1 Constructive geometry's unconventional geometric objects

The vertex: Introduced as a primitive in Section 2, the vertex was initially assumed to conform to the standard definition, "point where edges meet", with "point" defined as a "zero dimensional object that gives an exact location in space." However, the first principles, unbiased construction has revealed the lattice to have a self-similar fractal character such that both the global model and the first vertex are the SQPD, which does not have an exact location in space.

In cases where the lattice is calibrated against either physical objects, or mathematical objects, construction can be carried out to model the large or fine structure of the lattice to the degree that computational resources allow, but there is no resolution of the lattice, zooming in arbitrarily finely, such that the vertex resolves to the zero dimensional point. Every vertex resolves to the SQPD structure composed of vertices down through arbitrarily many hierarchically ordered fractal layers. There is no singularity.

The fundamental extension in SQPD space: A directed extension through the lattice that connects separated vertices, e.g., *O* and *F,* is considered for the case of (i) the dynamical model (ii) the microstate (iii) the static approximant: With respect to (i) the relation between *O* and *F* considered as a distance measurement or arithmetical operation is best understood from the perspective of the ideal intrinsic observer moving through the helically configured lattice as outlined in subsection 5.2. There, it was noted that the observer transition is causally correlated with vertex annihilation and creation updates. Any measurement or arithmetical operation occurs within the lattice operating as a stateful system and involves position/state update. (ii) In any microstate's ensemble of vertices, such as those that make up the directed extension through the lattice, there will be some indeterminacy around vertex creation/annihilation status which also reads as imprecision of vertex location. (iii) The static approximant model of the lattice represents the indeterminacy of state as an indeterminacy of position (within a range of position) averaged over all vertices. The lattice may be constructed to arbitrarily fine scale as computational recourses allow, however, precision of any distance measurement for an extension within the lattice remains limited by the non-zero size and indeterminacy of position of the vertex.

9. Conclusion and Outlook

This article has reported on an initial trial of a first principles bottom-up approach to constructing geometry from scratch. This has involved construction of the SQPD lattice as summarised in the previous section, including an initial sketching out of a novel theoretical framework for a foundational constructive geometry. It is intended that further work should build onto this introductory framework formal treatments, particularly information-theoretic treatments with the aim of showing quantitatively (perhaps in terms of Kolmogorov complexity [19]) that there is no structure that is constructible with less instructive information input (i.e., this is modelling a lattice that effectively self-assembles), and related to that, group-theoretic type arguments showing, via counting automorphisms, that the SQPD lattice is indeed the maximally symmetric/homogeneous lattice structure that is the prototypical structure-object with minimum structure-organization.

It is reasonable that in the universe of all examples of structure there should be at least one structure-object that has the minimum structure-organization – effectively a primordial informationally minimal substrate over top of which all other examples of structure involve layers of additional structure. Physics, typically, is a reductive process of stripping away structure toward an ever more fundamental substrate, arriving at an underlying stratum that (if the Standard Model is correct) is best described by quantum field theory, below which it is speculated there is a massless, timeless, purely informational field theory. Mathematics, on the other hand, can be characterised as building up a self-consistent formalism from an axiomatically minimal base. The degree and type of realism that should be assumed is a central debate, however, most agree that at its base, mathematics must make sense as a purely informational system of abstract relations.

SQPD space's stateful, helically configured, icosahedrally ordered lattice is not the model that geometric intuition first presents as a candidate for fundamental geometric structure. Geometric intuition, however, is conditioned not only by orthodoxy, but also biology. The way that the brain acts upon sensory data arriving from the external world determines our representations of the world – the insight perhaps associated most with Kant – and there is data compression involved in going from the noumena to phenomena; and in turn the reduced mental representations that we construct feed into the theories that we develop about the world.

Whether conceptualising the fabric of spacetime, vector spaces, fields, or any Euclidean or non-Euclidean space, there is at the basis a reflex to embed all examples of geometric structure ultimately within the framework of a stateless cubical lattice that maps a three-dimensional volume that resides firmly in the mind – not just reflecting learned habits of the mind, but also the deeper biological processes and structure by which visual representations are constructed in the consciousness.

It is not isomorphism with the architecture of the universe, rather it is isomorphism with the architecture of the brain's vision processing system that lies beneath our innate comfortableness with the static, planar spatial model that is supplied by standard Euclidean geometry. Although this project's development of constructive geometry arrives at the three-dimensional lattice operating as an asynchronously updating system – perhaps best modelled as a self-assembling (i.e., not superimposed over arbitrary grid) cellular automaton within which an intrinsic observer is an embedded calculating avatar managed by higher-level algorithms so that it traces pathways that essentially are computations – it nevertheless will remain necessary to reinterpret that through projections that produce the imprecise and compromised static two-dimensional representations that provide the intuitive human interface. This is indicative of a peculiar mismatch more broadly where the whole human project of accumulating data is the compromised task of putting on record determinate statements of fact that refer to static instances, or microstates, of the subject matter, the material universe that is, according to the third law of thermodynamics, thoroughly dynamical.

Traditionally, the two and three-dimensional Euclidean spaces are a model of Hilbert's axioms so that the subset, the real line, corresponds to a line in Euclidean geometry that is presumed to be composed of infinitely many points. Dedekind and Cantor both postulated that there is a one-to-one correspondence between all numbers and points on a line, and although Dedekind would have liked to jettison the crutch of the geometric number line in pursuit of arithmetizing the completeness of the real numbers, he was unable to do that.

There is the sense that tying mathematical objects in successor and predecessor relations to the static one-dimensional Euclidean line is the obvious, straightforward, and intuitive model – primarily on the basis that the line is considered an inherently fundamental geometric object; but there is no formal argument for that. In fact, linking the arithmetical property of an infinite successive ordering to its static, linear geometric counterpart, is, in the real line, only achieved at the cost of informationally expensive and unimplementable, largely philosophical, interpretations. As Weyl put it, "The introduction of numbers as coordinates by reference to the particular division scheme of the open one-dimensional continuum is an act of violence whose only practical vindication is the special calculatory manageability of the ordinary number continuum with its four basic operations" [20].

A key conclusion of this project is that, indeed, numbers described as a subsequent labelling scheme naturally organise linearly, but that linear extension should be constructed, not introduced as a primitive, in which case we arrive at the helically configured extension described in the previous section that is part of the dynamically updating number lattice. This is not an expected result. A comparison can be drawn with quantum field theory unexpectedly informing us that empty physical space is roiling with activity; now this project's investigation of constructive geometry is informing us that empty abstract space – the maximally entropic geometric ground state – is populated with fundamental geometric objects that are necessarily in a dynamical flux of creation and annihilation. In constructive geometry the vertex/point is not arbitrarily defined with the philosophical notion of zero size, rather it is constructed and has fractal structure, consequently there are no singularities. The interval between discrete vertices is also constructed and is understood in terms of discrete virtual vertices that are located probabilistically; consequently, populating the interval arbitrarily densely with discrete virtual vertices does not merely reproduce the infinities of the classical continuum – being a criticism of some discrete approaches.

To quote Butterfield and Isham [21]:

> Finally, we note that, from time to time, a few hardy souls have suggested that a full theory of quantum gravity may require changing the foundations of mathematics itself. A typical argument is that standard mathematics is based on set theory, and certain aspects of the latter (for example, the notion of the continuum) are grounded ultimately on our spatial perceptions. However, our perceptions probe only the world of classical physics – and hence we feed into the mathematical structures currently used in *all* domains of physics, ideas that are essentially classical in nature. The ensuing category error can be remedied only by thinking quantum theoretically from the very outset – in other words, we must look for "quantum analogies" of the categories of standard mathematics.

It was, of course, classical mathematics that was available to Schrödinger to develop his description of the quantum wave function, constructed as a function of the x axis with the incorporation of the complex numbers resulting in a helical path associated to a particle located in space and time. This project's investigation of constructive geometry, however, flips that scenario to say that the helical path produced by Schrödinger's wave function is describing something about the shape of the foundational underlying geometry, and it is in fact the x axis that has no foundational description and thus requires an informationally expensive construction within that fundamental, stateful, helically configured, geometric space. On this understanding it is not incumbent on pilot wave theory, for example, to construct a geometry of spacetime that explains the wavelike characteristics associated with the particle, and certainly there is no need to postulate wave-inducing forces, but rather, a helically configured spacetime is the default geometry.

This investigation of constructive geometry has highlighted the need to make the distinction between (*a*) the geometry that best aligns with the structure of the brain that is trying to process sensory inputs, and (*b*) the geometry that best aligns with the structure of the material universe that the senses are reporting on. This article has constructed an introductory model for (*b*). There are already indications that making clear that distinction between (*a*) and (*b*) will facilitate the shift from difficult and unimplementable, largely philosophical interpretations, toward intuitively accessible and constructible models with relevance to the fundaments of both mathematics and physics.

Appendix

Preliminary observations on the SQPD lattice in the context of fundamental physical structure:

1). Helicity is a characteristic of fundamental structure throughout the physical universe, from submicroscopic to cosmological scale. In some reference frame every known object in the physical universe, and in SQPD space, is carving out a helical trajectory (see also [22]).

2). A number of theories (e.g., [23]) propose that the distribution of matter in the universe, or the structure of the universe itself, is a fractal across a wide range of scales. However, the standard model of cosmology assumes that the large-scale structure of the universe is homogeneous and isotropic at all points. It may be reasonable to investigate whether those two characterisations can be reconciled in the SQPD lattice. Considering that (i) SQPD has homogeneous isotropic structure in the macrostate, and (ii) the microstate maps space with QPD vertices, being real objects linked with filaments of stacked IQCs constructed of virtual vertices and pockets of void within a background cloud also mapped by virtual vertices, but nothing real – more specifically, voids have a novel ontology; they are not filled with some background geometry given that SQPD claims to model the absolute background geometry.

When standard Euclidean 3-space (in which the entire volume is filled, with no points missing) is compared with SQPD space (that is honeycombed with a high percentage of the volume occupied by voids), and those two models are considered as alternative geometries of the physical universe in which all matter exists, they will obviously produce different results for the calculations of volume and expected total density. Some cosmological theories (e.g., [24]) suggest that the measurement anomalies in cosmology, conventionally attributed to dark energy, may be explained by a swiss-cheese cosmological model that is sympathetic to the sinuous and porous SQPD space that is riddled with voids (that perhaps can be thought of as "dark space").

3). Although current observational data are at this stage inconclusive regarding the global geometry of the universe, Poincaré dodecahedral space, for which there is some correlation with SQPD space, is one of only several models currently considered [25].

Acknowledgements

To follow.

References

[1] A. Bauer. "Five Stages of Accepting Constructive Mathematics", *Bulletin of the American Mathematical Society*, **54**, 2016 p. 481. www.ams.org/journals/bull/2017-54-03/S0273-0979-2016-01556-4/S0273-0979-2016-01556-4.pdf.

[2] Jaimungal, Curt. "Donald Hoffman Λ Joscha Bach on Consciousness & Gödel." YouTube, uploaded by Theories of Everything with Curt Jaimungal, 8 Feb. 2022, www.youtube.com/@TheoriesofEverything.

[3] "From Language to Consciousness (Guest: Joscha Bach)." YouTube, uploaded by The Computing Brain, 3 Nov. 2022, www.youtube.com/watch?v=ApHnqHfFWBk&t=427s.

[4] J.A. Wheeler, "Hermann Weyl and the Unity of Knowledge", *American Scientist*, **74**, 1986 pp. 366-75. http://www.weylmann.com/wheeler.shtml.

[5] J. Ambjorn, J. Jurkiewicz and R. Loll, "Quantum Gravity: the art of building spacetime", *Approaches to Quantum Gravity*, Cambridge: Cambridge University Press, 2006 pp. 341-59. arXiv.org/hep-th/0604212v1.

[6] C. Rovelli, "A New Look at Loop Quantum Gravity," *Classical and Quantum Gravity*, **28**, 2011. arXiv:1004.1780.

[7] J. A. Wheeler, "Pregeometry: Motivations and Prospects," in A. R. Marlow, *(ed.), Quantum Theory and Gravitation*, Academic Press, New York, 1980 pp. 1-11.

[8] J. A. Wheeler, "Information, Physics, Quantum: The Search For Links", *Proceedings III International Symposium on Foundations of Quantum Mechanics,* Tokyo, 1989 pp. 354-8. https://philarchive.org/rec/WHEIPQ.

[9] J. F. Sadoc and R. Mosseri, *Geometric Frustration*, Cambridge, UK: Cambridge University Press.

[10] Thomas C. Hales, "A proof of the Kepler conjecture", *Annals of Mathematics,* **162**, 2005 pp. 1065–1185. annals.math.princeton.edu/wp-content/uploads/annals-v162-n3-p01.pdf.

[11] K. Pardo, M. Fishbach, D. E. Holz and D. N. Spergel, "Limits on the number of spacetime dimensions from GW170817". (Jul 17, 2018) https://arxiv.org/abs/1801.08160.

[12] D. Shechtman, I. Blech, D. Gratias and J.W. Cahn, "Metallic phase with long-range orientational order and no translational symmetry", *Physical review letters*, **53**. 1984 p.1951.

[13] A. E. Madison and P. A. Madison, "Looking for Alternatives to the Superspace Description of Icosahedral Quasicrystals", *Proc. R. Soc. A,* **475**: 20180667, 2019. http://dx.doi.org/10.1098/rspa.2018.0667.

[14] F. Antonsen, "Models of Pregeometry", *Master's Thesis, University of Copenhagen, Niels Bohr Institute*, 1992 p. 291. https://inspirehep.net/files/b513827a30dccf8d3ad1ea8cd917263c.

[15] H. Hofmeister, "Shape variations and anisotropic growth of multiply twinned nanoparticles", *Zeitschrift für Kristallographie*, **224**, no. 11. 2009 pp. 528-38. https://doi.org/10.1524/zkri.2009.1034.

[16] H.S.M. Coxeter, *Regular Complex Polytopes*, Cambridge, UK: Cambridge University Press, 1974.

[17] B. R. Fuller, *Synergetics: Explorations in the Geometry of Thinking,* New York: Macmillan Publishing Co. Inc., 1975. https://fullerfuture.files.wordpress.com/2013/01/buckminsterfuller-synergetics.pdf.

[18] E. A. Lord and S. Ranganathan, "The γ-brass structure and the Boerdijk–Coxeter helix", *Journal of Non-Crystalline Solids*. 334&335. 2004 pp. 123–5.

[19] A. N. Kolmogorov, "Three Approaches to the Quantitative Definition of Information", *Problems of Information Transmission*, **1**, 1965 pp. 3–11. http://alexander.shen.free.fr/library/Kolmogorov65_Three-Approaches-to-Information.pdf.

[20] H. Weyl, *Philosophy of Mathematics and Natural Science*, Princeton: Princeton University Press, 1949 p. 90.

[21] J. Butterfield and C. J. Isham, "Spacetime and the philosophical challenge of quantum gravity", in C. Callender and N. Huggett (*eds*), *Physics Meets Philosophy at the Planck Scale,* Cambridge: Cambridge University Press, 2001 pp. 33-90.

[22] M. C. Parker and C. Jeynes, "Maximum Entropy (Most Likely) Double Helical and Double Logarithmic Spiral Trajectories in Space-Time", *Sci Rep* **9,** 10779, 2019. https://doi.org/10.1038/s41598-019-46765-w.

[23] J. Ambjorn, J. Jurkiewicz and R. Loll, R. "Reconstructing the Universe"*,* Physical Review D. **72**, 064014, 2005. arXiv:hep-th/0505154.

[24] J. R. Mureika and C. C. Dyer, "Review: Multifractal Analysis of Packed Swiss Cheese Cosmologies", *General Relativity and Gravitation.* **36** (1), 2004 pp. 151–84. arXiv:gr-qc/0505083.

[25] B. Roukema, Z. Buliński, A. Szaniewska and N. Gaudin, "A Test of the Poincare Dodecahedral Space Topology Hypothesis with the WMAP CMB Data", *Astronomy and Astrophysics.* **482** (3), 747, 2008. arxiv.org/abs/0801.1358.