Journal of Physics: Conference Series

PAPER • OPEN ACCESS

The Solution Cosmological Constant Problem. Quantum Field Theory in Fractal Space-Time with Negative Hausdorff-Colombeau Dimensions and Dark Matter Nature

To cite this article: Jaykov Foukzon *et al* 2019 *J. Phys.: Conf. Ser.* **1391** 012058

Journal of Physics: Conference Series **1391** (2019) 012058 doi:10.1088/1742-6596/1391/1/012058

The Solution Cosmological Constant Problem. Quantum Field Theory in Fractal Space-Time with Negative Hausdorff-Colombeau Dimensions and Dark Matter Nature

Jaykov Foukzon[1], **Elena R Men'kova**[2], **Alexander A Potapov**[3,4]

[1] Department of mathematics, Israel Institute of Technology, Haifa, 3200003, Israel
[2] All-Russian Research Institute for Optical and Physical Measurements, Moscow, 119361, Russia
[3] Kotelńikov Institute of Radioengineering and Electronics of the Russian Academy of Sciences, Moscow, 125009, Russia
[4] JNU-IREE Joint Laboratory of Fractal Method and Signal Processing, Department of Electronic Engineering, College of Information Science and Technology, Jinan University, Guangzhou, People's Republic of China

E-mail: jaykovfoukzon@list.ru

Abstract. The cosmological constant problem arises because the magnitude of vacuum energy density predicted by quantum field theory is about 120 orders of magnitude larger than the value implied by cosmological observations of accelerating cosmic expansion. We pointed out that the fractal nature of the quantum space-time with negative Hausdorff- Colombeau dimensions can resolve this tension. The canonical Quantum Field Theory is widely believed to break down at some fundamental high-energy cutoff Λ_* and therefore the quantum fluctuations in the vacuum can be treated classically seriously only up to this high-energy cutoff. In this paper we argue that Quantum Field Theory in fractal space-time with negative Hausdorff-Colombeau dimensions gives high-energy cutoff on natural way. We argue that there exists hidden physical mechanism which cancel divergences in canonical QED_4, QCD_4, Higher-Derivative - Quantum-Gravity, etc. In fact we argue that corresponding supermassive Pauli-Villars ghost fields really exists. It means that there exists the ghost-driven acceleration of the univers hidden in cosmological constant. In order to obtain desired physical result we apply the canonical Pauli-Villars regularization up to Λ_*. This would fit in the observed value of the dark energy needed to explain the accelerated expansion of the universe if we choose highly symmetric masses distribution between standard matter and ghost matter below that scale Λ_*,i.e., $f_{s.m}(\mu) \approx -f_{g.m}(\mu), \mu = mc, \mu \leq \mu_{\mathbf{eff}}, \mu_{\mathbf{eff}} < \Lambda_*/c$. The small value of the cosmological constant explaned by tiny violation of the symmetry between standard matter and ghost matter. Dark matter nature also explaned using a common origin of the dark energy and dark matter phenomena.

1. Introduction

One of the greatest challenges in modern physics is to reconcile general relativity and elementary particles physics into a unified theory. Perhaps the most dramatic clash between the two theories lies in the cosmological constant problem [1]-[6] and in the problem of the Dark (i.e., non-luminous and non-absorbing) Matter nature is, arguably, the most widely discussed topic in contemporary particle physics [7]. Naive predictions of vacuum energy from canonical quantum field theory predict a magnitude so high that the expansion of the Universe should have accelerated so quickly that no any structure could have formed. The predicted rate of acceleration resulting from vacuum energy is famously 120 orders of magnitude larger than what is observed. In order to avoid these difficultnes mentioned above we assume that:(i) physics of elementary particles essentially is separated into *low/high* energy ones, (ii) the standard notion of smooth spacetime is assumed to be altered at a high energy cutoff scale Λ_* and a new treatment based on quantum field theory (QFT) in a fractal space-time with negative dimension is used above that scale Λ_*. In this paper we argue that Quantum Field Theory in fractal space-time with negative Hausdorff-Colombeau dimensions [8], [9] gives high-energy cutoff on natural way. No one knows what dark energy is, but we need it to explain the discovered accelerated expansion of the Universe. The most elegant and natural solution is to identify dark energy with the energy of the quantum vacuum predicted by Quantum Field Theory, but the trouble is that QFT predicts the energy density of the vacuum to be orders of magnitude larger than the observed dark energy density: $\varepsilon_{\mathbf{de}} \approx 7.5 \times 10^{-27} kg/m^3$. Recall that it was stressed by Zeldovich [1] that quantum field theory generically demands that cosmological constant or, let us repeat, what is the same, vacuum energy is non-vanishing. Summing the zero-point energies of all normal modes of some quantum field of mass m up to a wave number cut-off $\Lambda_*/c^2 \gg m$, QFT yields [1], [5] a vacuum energy density

$$\varepsilon_{\mathbf{vac}}(p_*) \sim \int_0^{p_*} d^3p\sqrt{p^2+m^2} \simeq p_*^4. \tag{1.1}$$

If we take the Planck scale (i.e. the Planck mass) as a cut-off, the vacuum energy density $\varepsilon_{\mathbf{vac}}(p_*)$ is 10^{121} times larger than the observed dark energy density $\varepsilon_{\mathbf{de}}$. Several possible approaches to the problem of vacuum energy have been discussed in the contemporary literature, for the review see ref. [5]-[7], [10]. They can be roughly divided into four different groups: (1) Modification of gravity on large scales. (2) Anthropic principle. (3) Symmetry leading to $\varepsilon_{\mathbf{vac}} = 0$.(4) Adjustment mechanism, see [10]. (5) Hidden nonstandard dark matter sector and corresponding hidden symmetry leading to $\varepsilon_{\mathbf{vac}} \simeq 0$, see ref. [8], [9].

2. The formulation of the cosmological constant problem. Zel'dovich approach to cosmological constant problem by using Pauli-Villars regularization revisited

The cosmological constant problem arises at the intersection between general relativity and quantum field theory, and is regarded as a fundamental unsolved problem in modern physics. Remind that a peculiar and truly quantum mechanical feature of the quantum fields is that they exhibit zero-point fluctuations everywhere in space, even in regions which are otherwise 'empty' (i.e. devoid of matter and radiation). This vacuum energy density is believed to act as a contribution to the cosmological constant λ appearing in Einstein's field equations from 1917,

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R = \tfrac{8\pi G}{c^4}T'_{\mu\nu}, \tag{2.1}$$

where $R_{\mu\nu}$ and R refer to the curvature of space-time, $g_{\mu\nu}$ is the metric, $T'_{\mu\nu}$ the energy-momentum tensor,

$$T'_{\mu\nu} = T_{\mu\nu} + \tfrac{c^4\lambda}{8\pi G}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2.2}$$

where $T_{\mu\nu}$ is the energy-momentum tensor of matter. Thus $T'_{00} = T_{00} + \varepsilon_\lambda$, $T'_{\alpha\beta} = T_{\alpha\beta} + +\delta_{\alpha\beta}P_\lambda$, where

$$\varepsilon_\lambda = -P_\lambda = c^4\lambda/8\pi G. \tag{2.3}$$

Journal of Physics: Conference Series **1391** (2019) 012058 doi:10.1088/1742-6596/1391/1/012058

Remind that under Lorentz transformations $(\varepsilon_\Lambda, P_\Lambda) \to \varepsilon'_\Lambda, (\varepsilon_\Lambda, P_\Lambda) \to P'_\Lambda$ the quantities ε_Λ and P_λ are changes by the law

$$\varepsilon'_\lambda = \frac{\varepsilon_\lambda + \beta^2 P_\lambda}{1-\beta^2}, P'_\lambda = \frac{P_\lambda + \beta^2 \varepsilon_\lambda}{1-\beta^2}. \tag{2.4}$$

Thus for the quantities ε_λ and P_λ Lorentz invariance holds by Eq.(2.3) [1]. In review [5], Weinberg indicates that the first published discussion of the contribution of quantum fluctuations to the cosmological constant was a 1967 paper by Zel'dovich [6]. In his article [1] Zel'dovich emphasizes that zeropoint energies of particle physics theories cannot be ignored when gravitation is taken into account, and since he explicitly discusses the discrepancy between estimates of vacuum energy and observations, he is clearly pointing to a cosmological constant problem. As well known zeropoint energy density of scalar quantum field, etc.is divergent

$$\varepsilon_{\mathbf{vac}}(m) = \frac{2\pi c}{(2\pi\hbar)^3} \int_0^\infty \sqrt{p^2 + m^2c^2} p^2 dp = \infty. \tag{2.5}$$

In order to avoid difficultnes mentioned above, in article [1] Zel'dovich has applied canonical Pauli-Villars regularization [9], [11], [12] and formally has obtained a finite result (his formulas [1], Eqs. (VIII.12)-(VIII.13) p.228)

$$\varepsilon_{\mathbf{vac}} = -p_{\mathbf{vac}} = \frac{1}{8} \int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\, \mu^4 (\ln \mu)\, d\mu = \frac{c^4 \lambda}{8\pi G}, \tag{2.6}$$

where

$$\int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\, d\mu = \int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\, \mu^2 d\mu = \int\limits_0^{\mu_{\mathbf{eff}}} f(\mu)\, \mu^4 d\mu = 0. \tag{2.7}$$

Unfortunately the Eq.(2.6)-Eq.(2.7) give nothing in order to obtain desired small numerical values of the zero-point energy density $\varepsilon_{\mathbf{vac}}$. It is clear that aditional physical assumptions are needed. In his paper [1], Zel'dovich arrives at a zero-point energy (his formula [1], Eq.(IX.1))

$$\varepsilon_{\mathbf{vac}} = m \left(\frac{mc}{\hbar}\right)^3 \sim 10^{17} g/cm^3, \lambda \sim 10^{-10} cm^{-2}, \tag{2.8}$$

where m (the ultra-violet cut-of) is taken equal to the proton mass. Zel'dovich notes that since this estimate exceeds observational bounds by 46 orders of magnitude it is clear that "...such an estimate has nothing in common with reality".

In his paper [1], Zel'dovich wroted: "Recently A. D. Sakharov proposed a theory of gravitation, or, more precisely, a justification GR equations based on consideration of vacuum fluctuations. In this theory, the essential assumption is that there is some elementary length L or the corresponding limiting momentum $p_0 = \hbar/L$. Shorter lengths or for large impulses theory is not applicable. Sakharov gets the expression of gravitational constant G through L or p_0 (his formula [1], Eq.(IX.6))

$$G = \frac{c^3 L^2}{\hbar} = \frac{\hbar c^3}{p_0^2}. \tag{2.9}$$

This expression has been known since the days of Planck, but it was read "from right to left": gravity determines the length L and the momentum p_0. According to Sakharov, L and p_0 are primary. Substitute Eq. (IX. 6) in the expression Eq. (IX.4) (see [1]), we get

$$\rho_{\mathbf{vac}} = \frac{m^6 c^5}{p_0^2 \hbar^3}, \varepsilon_{\mathbf{vac}} = \frac{m^6 c^7}{p_0^2 \hbar^3}. \tag{2.10}$$

That is expressions that the first members (in the formulas [1], Eqs.(VIII.10)-(VIII. 11)) which are vanishing (with $p_0 \to \infty$). Thus, we can suggest the following interpretation of the cosmological constant: *there is a theory of elementary particles, which would give (according to the mechanism that has not been revealed at the present time) identically zero vacuum energy, if this theory was applicable infinitely, up to arbitrarily large momentum; there is a momentum p_0, beyond which the theory is nont applicable; along with other implications, modifying the theory gives different from zero vacuum energy; general considerations make it likely that the effect is portional p_0^{-2}. Clarification of the question of the importance for the theory of elementary particles".*

In contrast with Zel'dovich paper [1] we assume that:

(i) Poincaré group is deformed at some fundamental high-energy cutoff Λ_* [13]-[15] in accordance with the basis of the following deformed Poisson brackets

$$\begin{array}{c}\{x^\mu, x^\nu\} = \kappa^{-1}\left(x^\mu \eta^{0\nu} - x^\nu \eta^{\mu 0}\right), \{p^\mu, p^\nu\} = 0, \\ \{x^\mu, p^\nu\} = -\eta^{\mu\nu} + \kappa^{-1}\eta^{\mu 0} p^\nu\end{array} \tag{2.11}$$

where $\mu, \nu, = 0,1,2,3$, $\eta^{\mu\nu} = (+1,-1,-1,-1)$ and κ is a parameter identified as the ratio between the high-energy cutoff Λ_* and the light speed. The corresponding to (2.11) momentum transformation reads, (see ref. [14])

$$\begin{array}{c}p_0' = \frac{\gamma(p_0 - u p_x)}{1+(c\kappa)^{-1}[(\gamma-1)p_0 - \gamma u p_x]}, p_x' = \frac{\gamma\left(p_x - u p_0/c^2\right)}{1+(c\kappa)^{-1}[(\gamma-1)p_0 - \gamma u p_x]}, \\ p_y' = \frac{p_y}{1+(c\kappa)^{-1}[(\gamma-1)p_0 - \gamma u p_x]}, p_z' = \frac{p_z}{1+(c\kappa)^{-1}[(\gamma-1)p_0 - \gamma u p_x]}.\end{array} \tag{2.12}$$

and coordinate transformation reads, (see ref. [15])

$$\begin{array}{c}t' = \frac{\gamma\left(t - ux/c^2\right)}{1+(c\kappa)^{-1}[(\gamma-1)p_0 - \gamma u p_x]}, x' = \frac{\gamma(x - ut)}{1+(c\kappa)^{-1}[(\gamma-1)p_0 - \gamma u p_x]}, \\ y' = \frac{y}{1+(c\kappa)^{-1}[(\gamma-1)p_0 - \gamma u p_x]}, z' = \frac{z}{1+(c\kappa)^{-1}[(\gamma-1)p_0 - \gamma u p_x]},\end{array} \tag{2.13}$$

where $\gamma = \sqrt{1-u^2/c^2}$. It is easy to check that the energy $E = c\kappa$, identified as the high- energy cutoff Λ_*, is an invariant as it is also the case for the fundamental length $l_{\Lambda_*} = \hbar c/E = \hbar/\kappa$. Note that the transformation (2.12) defined in p-space and the transformation (2.13) defined in x-space becomes Lorentz for small energies and momenta and defines a large invariant energy $l_{\Lambda_*}^{-1}$.The high-energy cutoff Λ_* is preserved by the modified action of the Lorentz group [13]-[15]. Therefore the canonical quadratic invariant $\|p\|^2 = \eta^{ab} p_a p_b$ collapses at high-energy cutoff $\Lambda_*$and being replaced by the non-quadratic invariant:

$$\|p\|^2 = \frac{\eta^{ab} p_a p_b}{(1 + l_{\Lambda_*} p_0)}. \tag{2.14}$$

see ref. [13]-[14].

(ii) The canonical concept of Minkowski space-time collapses at a small distances $l_{\Lambda_*} = \Lambda_*^{-1}$ to fractal space-time with Hausdorff-Colombeau negative dimension and therefore the canonical Lebesgue measure d^4x being replaced by the Colombeau-Stieltjes measure with negative Hausdorff-Colombeau dimension D^- :

$$(d\eta\,(x,\varepsilon))_\varepsilon = (v_\varepsilon(s\,(x)) d^4x)_\varepsilon\,, \tag{2.15}$$

where

$$\begin{array}{c}(v_\varepsilon(s\,(x)))_\varepsilon = \left(\left(|s\,(x)|^{|D^-|} + \varepsilon\right)^{-1}\right)_\varepsilon, \\ s\,(x) = \sqrt{x_\mu x^\mu},\end{array} \tag{2.16}$$

see sect. 3 and ref. [8], [9].

(iii) The canonical concept of momentum space collapses at fundamental high-energy cutoff Λ_* to fractal momentum space with Hausdorff-Colombeau negative dimension and therefore the canonical Lebesgue measure $d^3\mathbf{k}$, where $\mathbf{k}=(k_x,k_y,k_z)$ being replaced by the Hausdorff-Colombeau measure

$$d^{D^+,D^-}\mathbf{k}=:\frac{\Delta(D^-)d^{D^+}\mathbf{k}}{\left(|\mathbf{k}|^{|D^-|}+\varepsilon\right)_\varepsilon}=\frac{\Delta(D^+)\Delta(D^-)p^{D^+-1}dp}{\left(p^{|D^-|}+\varepsilon\right)_\varepsilon}, \tag{2.17}$$

where $\Delta\left(D^{\pm}\right)=2\pi^{D^{\pm}/2}/\Gamma\left(D^{\pm}/2\right)$ and $p=|\mathbf{k}|=\sqrt{k_x+k_y+k_z}$ and where $D^+-|D^-|\leq-6$, see sect. 3 and ref. [8]. Hausdorff-Colombeau measure (2.7) avoid classical divergence (2.5) of the zeropoint energy $\varepsilon_{\mathbf{vac}}(m)$ and instead Eq. (2.5) one obtains

$$\varepsilon_{\mathbf{vac}}(m)=\int_0^{p_*}d^3p\sqrt{p^2+m^2}+\Delta\left(D^+\right)\Delta\left(D^-\right)\int_{p_*}^{\infty}dpp^2\frac{\sqrt{p^2+m^2}}{\left(p^{|D^-|}+\varepsilon\right)_\varepsilon}\simeq p_*^4. \tag{2.18}$$

3. Hidden ghost matter sector and corresponding nonstandard symmetry leading to $\varepsilon_{\mathbf{vac}}\simeq 0$. Dark matter nature

Dark matter is a hypothetical form of matter that is thought to account for approximately 85% of the matter in the universe, and about a quarter of its total energy density.The majority of dark matter is thought to be non-baryonic in nature, possibly being composed of some as-yet undiscovered subatomic particles. Its presence is implied in a variety of astrophysical observations, including gravitational effects that cannot be explained unless more matter is present than can be seen. For this reason, most experts think dark matter to be ubiquitous in the universe and to have had a strong influence on its structure and evolution. Dark matter is called dark because it does not appear to interact with observable electromagnetic radiation, such as light, and is thus invisible to the entire electromagnetic spectrum, making it extremely difficult to detect using usual astronomical equipment [16-18]

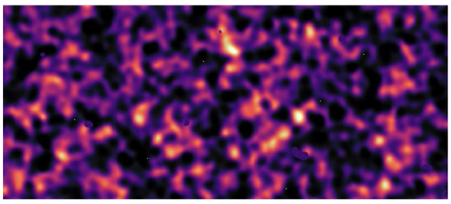

Figure 1. Dark matter map for a patch of sky based on gravitational lensing analysis [18], [19]

Analysis of a giant new galaxy survey, made with ESO's VLT Survey Telescope in Chile, suggests that dark matter may be less dense and more smoothly distributed throughout space than previously thought. An international team used data from the Kilo Degree Survey (KiDS) to study how the light from about 15 million distant galaxies was affected by the gravitational influence of matter on the largest scales in the Universe. The results appear to be in disagreement with earlier results from the Planck satellite. This map of dark matter in the Universe was obtained from data from the KiDS survey, using the VLT Survey Telescope at ESO's Paranal Observatory in Chile. It reveals an expansive web of dense (light) and empty (dark) regions. This image is one out of five patches of the sky observed by KiDS. Here the invisible dark matter is seen rendered in pink, covering an area of sky around 420 times the size of the full moon. This image reconstruction was made by analysing the light collected from over three million distant galaxies more than 6 billion light-years away. The observed galaxy images were warped by the gravitational pull of dark matter as the light travelled through the Universe. Some small dark regions, with sharp boundaries, appear in this image. They are the locations of bright stars and other nearby objects that get in the way of the observations of more distant galaxies and are hence masked out in these maps as no weak-lensing signal can be measured in these areas [16-18].

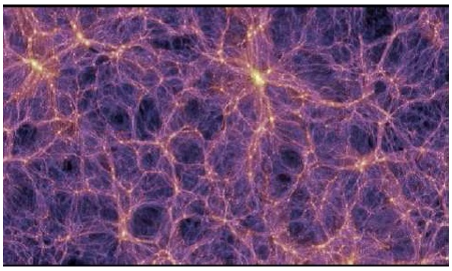

Figure 2. A simulation of the dark matter distribution in the universe 13.6 billion years ago

The luminous (light-emitting) components of the universe only comprise about 0.4% of the total energy. The remaining components are dark. Of those, roughly 3.6% are identified: cold gas and dust, neutrinos, and black holes. About 23% is dark matter, and the overwhelming majority is some type of gravitationally self-repulsive dark energy.

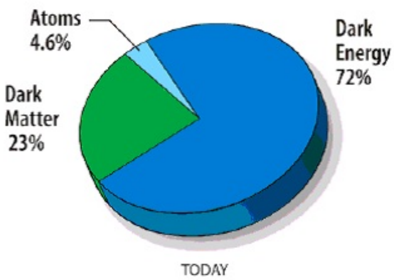

Figure 3. Matter and energy distribution in the universe today. The luminous (light-emitting) components of the universe only comprise about 0.4% of the total energy. The remaining components are dark

Remark 3.1. There is no candidate in the standard model of particle physics. In what way does dark matter extend the standard model?

Remark 3.2. In order to explain physical nature of dark matter sector we assume that main part of dark matter, i.e., $\simeq 23\% - 4.6\% = 18\%$ (see Fig.3) formed by supermassive ghost particles vith masess such that $mc^2 > \Lambda_*$.

Remark 3.3. In order to obtain QFT description of the dark component of matter in natural way we expand now the standard model of particle physics on a sector of ghost particles. QFT in a ghost sector developed in ref. [8], (see [8], sect. 3.1-3.4 and sect. 4.1-4.8).

4. Hausdorff-Colombeau measure and associated negative Hausdorff-Colombeau dimensions. Fractional Integration in negative dimensions

Let $\mu_H^{D^+}$ be a Hausdorff measure [20]-[21] and $X \subset R^n, D^+ < n$ is measurable set. Let $s(x)$ be a function $s : X \to R$ such that is symmetric with respect to some centre $x_0 \in X$, i.e. $s(x) = const$ for all x satisfying $d(x, x_0) = r$ for arbitrary values of r. Then the integral in respect to Hausdorff measure over n-dimensional metric space X is then given by [20]:

$$\int_X s(x) d\mu_H^{D^+} = \frac{2\pi^{D^+/2}}{\Gamma(D^+/2)} \int_0^\infty s(r)\, r^{D^+-1} dr. \tag{4.1}$$

The integral in RHS of the Eq. (4.1) is known in the theory of the Weyl fractional calculus where, the Weyl fractional integral $W^D f(x)$, is given by

$$W^{D^+} f(x) = \frac{1}{\Gamma(D^+)} \int_0^\infty (t-x)^{D^+-1} f(t)\, dt. \tag{4.2}$$

The notion of negative dimension in geometry and quantum physics was many developed,see [9], [22]-[32].

In order to extend the Weyl fractional integral (4.2) in negative dimensions we apply the Colombeau generalized functions [33]-[34] and Colombeau generalized numbers [35], [36].

Recall that Colombeau algebras $\mathcal{G}(\Omega)$ of the Colombeau generalized functions defined as follows [33]-[34]. Let Ω be an open subset of R^n. Throughout this paper, for elements of the space $C^\infty(\Omega)^{(0,1]}$ of sequences of smooth functions indexed by $\varepsilon \in (0,1]$ we shall use the canonical notation $(u_\varepsilon)_\varepsilon$ so $u_\varepsilon \in C^\infty(\Omega)$, $\varepsilon \in (0,1]$.

Definition 4.1. We set $\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega) / \mathcal{N}(\Omega)$, where

$$\begin{gathered}\mathcal{E}_M(\Omega) = \left\{ (u_\varepsilon)_\varepsilon \in C^\infty(\Omega)^{(0,1]} \middle| \forall K \subset\subset \Omega, \forall \alpha \in N^n \exists p \in N with \right. \\ \left. \sup_{x \in K} |u_\varepsilon(x)| = O(\varepsilon^{-p})\, as \varepsilon \to 0 \right\}, \\ \mathcal{N}(\Omega) = \left\{ (u_\varepsilon)_\varepsilon \in C^\infty(\Omega)^{(0,1]} \middle| \forall K \subset\subset \Omega, \forall \alpha \in N^n \forall q \in N \right. \\ \left. \sup_{x \in K} |u_\varepsilon(x)| = O(\varepsilon^{q})\, as \varepsilon \to 0 \right\}.\end{gathered} \tag{4.3}$$

Note that $\mathcal{G}(\Omega)$ is a differential algebra containing $D'(\Omega)$ as a linear subspace and $C^\infty(\Omega)$ as subalgebra. Equivalence classes of sequences $(u_\varepsilon)_\varepsilon$ will be denoted by $\mathbf{cl}[(u_\varepsilon)_\varepsilon]$.

Definition 4.2. Weyl fractional integral $\left(W_\varepsilon^{D^{-}-} f(x)\right)_\varepsilon$ in negative dimensions $D^- < 0$, $D^- \neq 0, -1, ..., -n, ..., n \in N$ is given by [8], [9]:

$$\begin{gathered} W^{D^-} f(x) = \frac{1}{\Gamma(D^-)} \left(\int_\varepsilon^\infty (t-x)^{D^- -1} f(t)\, dt \right)_\varepsilon \\ or \\ \left(W_\varepsilon^{D_-^-} f(x) \right)_\varepsilon = \frac{1}{\Gamma(D^-)} \left(\int_0^\infty \frac{1}{\varepsilon + (t-x)^{|D^-|+1}} f(t)\, dt \right)_\varepsilon, \end{gathered} \tag{4.4}$$

where $\varepsilon \in (0,1]$ and $\int_0^\infty |f(t)\, dt| < \infty$. Note that $\left(W_\varepsilon^{D^{-}-} f(x)\right)_\varepsilon \in \mathcal{G}(R)$. Thus in order to obtain apropriate extension of the Weyl fractional integral $W^{D^+} f(x)$ on the negative dimensions $D^- < 0$ the notion of the Colombeau generalized functions is essentially important. Thus in negative dimensions from Eq. (4.1) we formally obtain

$$\left(\int_X s(x) d\mu_{HC,\varepsilon}^{D^{-}-} \right)_\varepsilon = \frac{2\pi^{D^{-}-/2}}{\Gamma(D^-/2)} \left(\int_0^\infty \frac{s(r)}{\varepsilon + r^{|D^-|+1}} dr \right)_\varepsilon = \left(I_\varepsilon^{D_-^-} \right)_\varepsilon, \tag{4.5}$$

where $\varepsilon \in (0,1]$ and $D^- \neq 0, -2, ..., -2n, ..., n \in N$ and where $\left(\mu_{HC,\varepsilon}^{D_-^-}\right)_\varepsilon$ is apropriate generalized Colombeau outer measure. Namely Hausdorff-Colombeau outer measure, see [8], [9], sec. 6.1.

We apply throught this paper more general definition then definition (4.5):

$$\left(\int_X s(x)d\mu_{HC,\varepsilon}^{D^+,D^{--}}\right)_\varepsilon = \frac{4\pi^{D^+/2}\pi^{D^{--}/2}}{\Gamma(D^+/2)\Gamma(D^-/2)}\left(\int_0^\infty \frac{r^{D^+-1}s(r)}{\varepsilon + r^{|D^{--}|+1}}dr\right)_\varepsilon = \left(I_\varepsilon^{D^+,D_-}\right)_\varepsilon, \tag{4.6}$$

where $\varepsilon \in (0,1]$ and $D^+ \geq 1$, $D^- \neq 0,-2,...,-2n,...,n \in N$ and where $\left(\mu_{HC,\varepsilon}^{D^+,D^{---}}\right)_\varepsilon$ is apropriate generalized Colombeau outer measure. Namely Hausdorff-Colombeau outer measure. In ref. [9] (see [9], sec. 3.3) it has been proved that there exists Colombeau generalized measure $\left(d\mu_{HC,\varepsilon}^{D^+,D^{---}}\right)_\varepsilon$ and therefore Eq. (4.6) gives apropriate extension of the Eq. (4.1) on the negative Hausdorff-Colombeau dimensions.

Definition 4.3. We denote by $\widetilde{R}$ the ring of real,Colombeau generalized numbers. Recall that by definition $\widetilde{R} = \mathcal{E}_M(R)/\mathcal{N}(R)$ [35], [36] where

$$\begin{aligned}\mathcal{E}_M(R) &= \left\{ (x_\varepsilon)_\varepsilon \in R^{(0,1]} \middle| (\exists \alpha \in R)(\exists \varepsilon_0 \in (0,1])\,\forall \varepsilon \leq \varepsilon_0 \left[|x_\varepsilon| \leq \varepsilon^\alpha\right]\right\},\\ \mathcal{N}(R) &= \left\{ (x_\varepsilon)_\varepsilon \in R^{(0,1]} \middle| (\forall \alpha \in R)(\exists \varepsilon_0 \in (0,1])\,\forall \varepsilon \leq \varepsilon_0 \left[|x_\varepsilon| \leq \varepsilon^\alpha\right]\right\}.\end{aligned} \tag{4.7}$$

Notice that the ring $\widetilde{R}$ arises naturally as the ring of constants of the Colombeau algebras $\mathcal{G}(\Omega)$. Recall that there exists natural embedding $\widetilde{r} : R \hookrightarrow \widetilde{R}$ such that for all $r \in R, \widetilde{r} = (r_\varepsilon)_\varepsilon$ where $r_\varepsilon \equiv r$ for all $\varepsilon \in (0,1]$. We say that r is standard number and abbreviate $r \in R$ for short. The ring $\widetilde{R}$ can be endowed with the structure of a partially ordered ring: for $r, s \in \widetilde{R}$ $\eta \in R_+, \eta \leq 1$ we abbreviate $r \leq_{\widetilde{R},\eta} s$ or simply $r \leq_{\widetilde{R}} s$ if and only if there are representatives $(r_\varepsilon)_\varepsilon$ and $(s_\varepsilon)_\varepsilon$ with $r_\varepsilon \leq s_\varepsilon$ for all $\varepsilon \in (0,\eta]$. Colombeau generalized number $r \in \widetilde{R}$ with representative $(r_\varepsilon)_\varepsilon$ we abbreviate $\mathbf{cl}[(r_\varepsilon)_\varepsilon]$.

Definition 4.4. (i) Let $\delta \in \widetilde{R}$. We say that δ is infinite small Colombeau generalized number and abbreviate $\delta \approx_{\widetilde{R}} \widetilde{0}$ if there exists representative $(\delta_\varepsilon)_\varepsilon$ and some $q \in N$ such that $|\delta_\varepsilon| = O(\varepsilon^q)$ as $\varepsilon \to 0$. (ii) Let $\Delta \in \widetilde{R}$. We say that Δ is infinite large Colombeau generalized number and abbreviate $\Delta =_{\widetilde{R}} \widetilde{\infty}$ if $\Delta_{\widetilde{R}}^{-1} \approx_{\widetilde{R}} \widetilde{0}$. (iii) Let $R_{\pm\infty}$ be $R \cup \{\pm\infty\}$ We say that $\Theta \in \widetilde{R}_{\pm\infty}$ is infinite Colombeau generalized number and abbreviate $\Theta =_{\widetilde{R}} \pm\infty_{\widetilde{R}}$ if there exists representative $(\Theta_\varepsilon)_\varepsilon$ where $|\Theta_\varepsilon| = \infty$ for all $\varepsilon \in (0,1]$. Here we abbreviate $\mathcal{E}_M(R_{\pm\infty}) = \mathcal{E}_M(R \cup \{\pm\infty\})$, $\mathcal{N}(R_{\pm\infty}) = \mathcal{N}(R \cup \{\pm\infty\})$ and $\widetilde{R}_{\pm\infty} = \mathcal{E}_M(R_{\pm\infty})/\mathcal{N}(R_{\pm\infty})$.

Definition 4.5. (Standard Part Mapping). (i) The standard part mapping $\mathbf{st} : \widetilde{R} \to R$ is defined by the formula:

$$\mathbf{st}(x) = \sup\left\{r \in R | r \leq_{\widetilde{R}} x\right\}. \tag{4.8}$$

If $x \in \widetilde{R}$, then $\mathbf{st}(x)$ is called the standard part of x.

(ii) The mapping $\mathbf{st} : \widetilde{R} \to R \cup \{\pm\infty\}$ is defined by (i) and by $\mathbf{st}(x) = \pm\infty$ for $x \in \widetilde{R}$ and for $x \in \widetilde{R}_{\pm\infty}$, respectively.

Definition 4.6. The singular Hausdorff-Colombeau measure originate in Colombeau generalization of canonical Caratheodory's construction, which is defined as follows: for each metric space X, each set $F = \{E_i\}_{i\in N}$ of subsets E_i of X, and each Colombeau generalized function $(\zeta_\varepsilon(E, x, \breve{x}))_\varepsilon$, such that: (i) $0 \leq (\zeta_\varepsilon(E, x, \breve{x}))_\varepsilon$, (ii) $(\zeta_\varepsilon(E, \breve{x}, \breve{x}))_\varepsilon =_{\widetilde{R}} \widetilde{\infty}$, whenever $E \in F$, a preliminary Colombeau measure $(\phi_\delta(E, x, \breve{x}, \varepsilon))_\varepsilon$ can be constructed corresponding to $0 < \delta \leq +\infty$, and then a final Colombeau measure $(\mu_\varepsilon(E, x, \breve{x}))_\varepsilon$, as follows: for every subset $E \subset X$, the preliminary Colombeau measure $(\phi_\delta(E, x, \breve{x}, \varepsilon))_\varepsilon$ is defined by

$$\phi_\delta(E, x, \breve{x}, \varepsilon) = \sup_{\{E_i\}_{i\in N}}\left\{\sum\nolimits_{i\in N} \zeta_\varepsilon(E_i, x, \breve{x}) \,| E \subset \bigcup\nolimits_{i\in N} E_i, \mathbf{diam}(E_i) \leq \delta\right\}. \tag{4.9}$$

Since for all $\varepsilon \in (0,1] : \phi^{-}_{\delta_1}(E,x,\breve{x},\varepsilon) \geq \phi^{-}_{\delta_2}(E,x,\breve{x},\varepsilon)$ for $0 < \delta_1 < \delta_2 \leq +\infty$, the limit

$$(\mu_\varepsilon(E,x,\breve{x}))_\varepsilon = (\mu(E,x,\breve{x},\varepsilon))_\varepsilon =$$
$$\left(\lim_{\delta\to 0_+} \phi_\delta(E,x,\breve{x},\varepsilon)\right)_\varepsilon = \left(\inf_{\delta>0} \phi_\delta(E,x,\breve{x},\varepsilon)\right)_\varepsilon \tag{4.10}$$

exists for all $E \subset X$. In this context, $(\mu(E,x,\breve{x},\varepsilon))_\varepsilon$ can be called the result of Caratheodory's construction from $(\zeta_\varepsilon(E,x,\breve{x}))_\varepsilon$ on F and $(\phi_\delta(E,x,\breve{x},\varepsilon))_\varepsilon$ can be referred to as the size δ approximating Colombeau measure.

Definition 4.7. Let $(\zeta_\varepsilon(E_i,D^+,D^-,x,\breve{x}))_\varepsilon$ be

$$(\zeta_\varepsilon(E_i,D^+,D^-,x,\breve{x}))_\varepsilon = \begin{cases} \left(\frac{\Omega_1(D^+)\Omega_2(D^-)[\mathbf{diam}(E_i)]^{D^+}}{[d(x,\breve{x})]^{|D^-|}+\varepsilon}\right)_\varepsilon & if x \in E_i \\ 0 & if x \notin E_i \end{cases} \tag{4.11}$$

where $\varepsilon \in (0,1], \Omega_1(D^+), |\Omega_2(D^-)| > 0$. In particular, when F is the set of all (closed or open) balls in X,

$$\Omega_1(D^+) = \frac{2^{-D^+}\Gamma\left(\frac{1}{2}\right)^{D^+}}{\Gamma\left(1+\frac{D^+}{2}\right)} = \frac{2^{-D^+}\pi^{\frac{D^+}{2}}}{\Gamma\left(1+\frac{D^+}{2}\right)} \tag{4.12}$$

and

$$\Omega_2(D^-) = \frac{2^{-D^-}\pi^{\frac{D^-}{2}}}{\Gamma\left(1+\frac{D^-}{2}\right)}, \tag{4.13}$$

where $D^- \neq -2,-4,-6,...,-2(n+1),....$

Definition 4.8. The Hausdorff-Colombeau singular measure $(\mu_{HC}(E,D^+,D^-,x,\breve{x},\varepsilon))_\varepsilon$ of a subset $E \subset X$ with the associated Hausdorff-Colombeau dimension $\breve{D}^+(D^-) \in R_+, \quad D^- \in R_+$, which is defined by

$$\left(\mu_{HC}\left(E,\breve{D}^+,D^-,x,\breve{x},\varepsilon\right)\right)_\varepsilon =$$
$$\left(\lim_{\delta\to 0}\left[\sup_{\{E_i\}_{i\in N}}\left\{\sum\nolimits_{i\in N}\left(\zeta_\varepsilon\left(E_i,\breve{D}^+,D^-,x,\breve{x}\right)\right)_\varepsilon | E \subset \bigcup\nolimits_{i\in N} E_i, \forall i\left(\mathbf{diam}(E_i) < \delta\right)\right\}\right]\right)_\varepsilon,$$
$$\breve{D}^+ = \sup\left\{D^+ > 0 | \left(\mu_{HC}(E,D^+,D^-,x,\breve{x},\varepsilon)\right)_\varepsilon = \infty_{\widetilde{R}}\right\}. \tag{4.14}$$

The Colombeau-Lebesgue-Stieltjes integral over continuous functions $f : X \to R$ can be evaluated straightforward, but using the limit in sense of Colombeau generalized functions of infinitesimal covering diameter when $\{E_i\}_{i\in N}$ is a disjoined covering and $x_i \in E_i$:

$$\left(\int_X f(x)\, d\mu_{HC}(E,D^+,D^-,x,\breve{x},\varepsilon)\right)_\varepsilon =$$
$$\left(\lim_{diam(E_i)\to 0}\left[\sum\nolimits_{\cup E_i = X} f(x_i) \inf_{E_{ij} with \cup_j E_{ij} \supset E_i} \sum\nolimits_j \zeta_\varepsilon(E_i,D^+,D^-,x_i,\breve{x})\right]\right)_\varepsilon. \tag{4.15}$$

We assume now that X is metrically unbounded, i.e. for every $x \in X$ and $r > 0$ there exists a point y such that $d(x,y) > r$. The standard assumption that $\breve{D}^+$ and $\breve{D}^-$ is uniquely defined in all subsets E of X requires X to be regular (homogeneous, uniform) with respect to the measure, i.e. $\left(\mu_{HC}^{-}\left(B_r\left(\breve{x}\right),\breve{D}^{+},\breve{D}^{-},x,\breve{x},\varepsilon\right)\right)_{\varepsilon}=\left(\mu_{HC}^{-}\left(B_r\left(\breve{y}\right),\breve{D}^{+},\breve{D}^{-},x',\breve{y},\varepsilon\right)\right)_{\varepsilon}$, where $d(x,\breve{x})=d(x',\breve{y})$ for all elements $\breve{x},\breve{y},x,x'\in X$ and convex balls $B_r(\breve{x})$ and $B_r(\breve{y})$ of the form $B_r(\breve{x})=\{z|d(\breve{x},z)\leq r;\breve{x},z\in X\}$ and $B_r(\breve{y})=\{z|d(\breve{y},z)\leq r;\breve{y},z\in X\}$. In the limit $\mathbf{diam}\,(E_i)\to 0$, the infimum is satisfied by the requirement that the variation over all coverings $\{E_{ij}\}_{ij\in N}$ is replaced by one single covering E_i, such that $\cup_j E_{ij}\to E_i\ni x_i$. Therefore

$$\left(\int_X f\left(x\right)d\mu_{HC}\left(E,\breve{D}^{+},\breve{D}^{-},x,\breve{x},\varepsilon\right)\right)_{\varepsilon}= \left(\lim_{diam(E_i)\to 0}\sum_{\cup E_i=X} f\left(x_i\right)\zeta_{\varepsilon}\left(E_i,\breve{D}^{+},\breve{D}^{-},x_i,\breve{x}\right)\right)_{\varepsilon}. \tag{4.16}$$

Assume that $f\left(x\right)=f\left(r\right),r=\|r\|$ .The range of integration X may be parametrised by polar coordinates with $r=d(x,0)$ and angle ω. $\{E_{r_i,\omega_i}\}$ can be thought of as spherically symmetric covering around a centre at the origin. Thus

$$\left(\int_X f\left(r\right)d\mu_{HC}\left(E_x,\breve{D}^{+},\breve{D}^{-},x,\breve{x},\varepsilon\right)\right)_{\varepsilon}= \left(\lim_{diam(E_i)\to 0}\sum_{\cup E_i=X} f\left(r_i\right)\zeta_{\varepsilon}\left(E_i,\breve{D}^{+},\breve{D}^{-},x_i,\breve{x}\right)\right)_{\varepsilon}. \tag{4.17}$$

Notice that the metric set $X\subset R^n$ can be tesselated into regular polyhedra; in particular it is always possible to divide R^n into parallelepipeds of the form

$$\Pi_{i_1,...,i_n}=\{x=(x_1,...,x_n)\in X|x_j=(i_j-1)\,\Delta x_j+\gamma_j,0\leq\gamma_j\leq\Delta x_j,j=1,...,n\}. \tag{4.18}$$

For $n=2$ the polyhedra Π_{i_1,i_2} is shown in figure 4. Since X is uniform

$$\left(d\mu_{HC}\left(E,\breve{D}^{+},\breve{D}^{-},x,\breve{x},\varepsilon\right)\right)_{\varepsilon}=$$
$$\left(\lim_{diam\left(\Pi_{i_1,...,i_n}\right)\to 0}\zeta_{\varepsilon}\left(\Pi_{i_1,...,i_n},\breve{D}^{+},\breve{D}^{-},x,\breve{x}\right)\right)_{\varepsilon}=$$
$$\left(\lim_{diam\left(\Pi_{i_1,...,i_n}\right)\to 0}\Pi_{j=1}^{n}\left(\frac{\Delta x_j}{|x_j-\breve{x}_j|^{|\breve{D}^{-}|}+\varepsilon}\right)^{\frac{\breve{D}^{+}}{n}}\right)_{\varepsilon}:=$$
$$\left(\Pi_{j=1}^{n}\frac{d^{\frac{\breve{D}^{+}}{n}}x_j}{\left(|x_j-\breve{x}_j|^{|\breve{D}^{-}|}+\varepsilon\right)^{\frac{\breve{D}^{+}}{n}}}\right)_{\varepsilon}. \tag{4.19}$$

Notice that the range of integration X may also be parametrised by polar coordinates with $r=d(x,0)$ and angle Ω. $E_{r,\Omega}$ can be thought of as spherically symmetric covering around a centre at the origin (see figure 5 for the two-dimensional case). In the limit, the Colombeau generaliza function $\left(\zeta_{\varepsilon}\left(E_{r,\Omega},\breve{D}^{+},\breve{D}^{-},r,\breve{r}\right)\right)_{\varepsilon}$ is given by

$$\left(d\mu_{HC}\left(E_{r,\Omega},\breve{D}^{+},\breve{D}^{-},r,\breve{r},\Omega,\varepsilon\right)\right)_{\varepsilon}=$$
$$\left(\lim_{diam\left(\Pi_{i_1,...,i_n}\right)\to 0}\zeta_{\varepsilon}\left(E_{r,\Omega},\breve{D}^{+},\breve{D}^{-},r,\breve{r},\Omega\right)\right)_{\varepsilon}:=$$

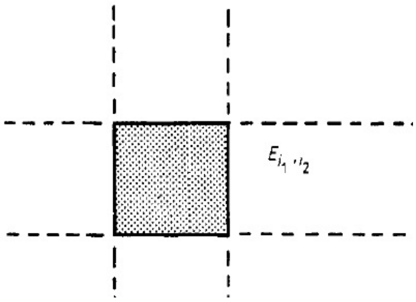

Figure 4. The polyhedra covering for n=2

$$\frac{d\Omega^{\breve{D}^+-1} r^{\breve{D}^+-1} dr}{\left((r-\breve{r})^{\left|\breve{D}^-\right|}+\varepsilon\right)_\varepsilon}. \tag{4.20}$$

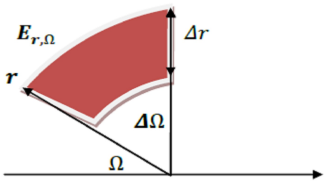

Figure 5. The spherical covering $E_{r,\Omega}$

When $f(x)$ is symmetric with respect to some centre $\breve{x} \in X$, i.e. $f(x) =$ constant for all x satisfying $d(x, \breve{x}) = r$ for arbitrary values of r, then change of the variable: $x \to z = x - \breve{x}$ can be performed to shift the centre of symmetry to the origin. The integral over metric space X is then given by formula

$$\left(\int_X f(x)\, d\mu_{HC}\left(E_x, \breve{D}^+, \breve{D}^-, x, \breve{x}, \varepsilon\right)\right)_\varepsilon = \frac{4\pi^{D^+/2}\pi^{D^-/2}}{\Gamma(D^+/2)\Gamma(D^-/2)}\left(\int_0^\infty \frac{r^{D^+-1} f(r)}{r^{|D^-|}+\varepsilon} dr\right)_\varepsilon. \tag{4.21}$$

The Colombeau integral defined in (4.17) satisfies the following conditions.(i) Linearity:

$$\begin{aligned} &\left(\int_X \left[a_1 f_1(x) + a_2 f_2(x)\right] d\mu_{HC}\left(E_x, \breve{D}^+, \breve{D}^-, x, \breve{x}, \varepsilon\right)\right)_\varepsilon = \\ &a_1\left(\int_X f_1(x)\, d\mu_{HC}\left(E_x, \breve{D}^+, \breve{D}^-, x, \breve{x}, \varepsilon\right)\right)_\varepsilon + \\ &a_2\left(\int_X f_2(x)\, d\mu_{HC}\left(E_x, \breve{D}^+, \breve{D}^-, x, \breve{x}, \varepsilon\right)\right)_\varepsilon. \end{aligned} \tag{4.22}$$

(ii) Translational invariance:

$$\begin{aligned} &\left(\int_X f(x + x_0)\, d\mu_{HC}\left(E_x, \breve{D}^+, \breve{D}^-, x, \breve{x} - x_0, \varepsilon\right)\right)_\varepsilon = \\ &\left(\int_X f(x)\, d\mu_{HC}\left(E_x, \breve{D}^+, \breve{D}^-, x, \breve{x}, \varepsilon\right)\right)_\varepsilon \end{aligned} \tag{4.23}$$

since

$$\left(d\mu_{HC}\left(E_{x-x_0},\breve{D}^+,\breve{D}^-,x-x_0,\breve{x}-x_0,\varepsilon\right)\right)_\varepsilon = \left(d\mu_{HC}\left(E_x,\breve{D}^+,\breve{D}^-,x,\breve{x},\varepsilon\right)\right)_\varepsilon . \tag{4.24}$$

5. Hausdorff-Colombeau measure and associated negative Hausdorff-Colombeau dimensions

During last 20 years the notion of negative dimension in geometry was many developed, see for example [9], [22]-[26]. Let Ω be an open subset of R^n, let X be metric space XR^n and let F be a set $F=\{E_i\}_{i\in N}$ of subsets E_i of X. Let $\zeta\left(E,x,\breve{x}\right)$ be a function $\zeta: F\times\Omega\times\Omega\to R$. Let $C_F^\infty(\Omega)$ be a set of the all functions $\zeta\left(E,x,\breve{x}\right)$ such that $\zeta\left(E,x,\breve{x}\right)\in C^\infty(\Omega\times\Omega)$ whenever $E\in F$. Throughout this paper, for elements of the space $C_F^\infty(\Omega\times\Omega)^{(0,1]}$ of sequences of smooth functions indexed by $\varepsilon\in(0,1]$ we shall use the canonical notations $\left(\zeta_\varepsilon\left(E,x,\breve{x}\right)\right)_\varepsilon$ and $(\zeta_\varepsilon)_\varepsilon$ so $\zeta_\varepsilon\in C_F^\infty(\Omega\times\Omega)$, $\varepsilon\in(0,1]$.

Definition 5.1. We set $\mathcal{G}_F(F,\Omega)=\mathcal{E}_M(F,\Omega)/\mathcal{N}(F,\Omega)$, where

$$\begin{aligned}\mathcal{E}_M(F,\Omega)=\Big\{\left(\zeta_\varepsilon\left(E,x,\breve{x}\right)\right)_\varepsilon\in C_F^\infty(\Omega\times\Omega)^{(0,1]}\Big|\,\forall K\subset\subset\Omega,\forall\alpha\in N^n\exists p\in N with\\ \sup\nolimits_{E\in F;x\in K}\left|\zeta_\varepsilon\left(E,x,\breve{x}\right)\right|=O\left(\varepsilon^{-p}\right)as\varepsilon\to0\Big\},\\ \mathcal{N}(F,\Omega)=\Big\{\left(\zeta_\varepsilon\left(E,x,\breve{x}\right)\right)_\varepsilon\in C_F^\infty(\Omega\times\Omega)^{(0,1]}\Big|\,\forall K\subset\subset\Omega,\forall\alpha\in N^n\forall q\in N\\ \sup\nolimits_{E\in F;x\in K}\left|\zeta_\varepsilon\left(E,x,\breve{x}\right)\right|=O\left(\varepsilon^{q}\right)as\varepsilon\to0\Big\}.\end{aligned} \tag{5.1}$$

Notice that $\mathcal{G}_F(F,\Omega)$ is a differential algebra. Equivalence classes of sequences $(\zeta_\varepsilon)_\varepsilon = \left(\zeta_\varepsilon\left(E,x,\breve{x}\right)\right)_\varepsilon$ will be denoted by $\mathbf{cl}\left[(\zeta_\varepsilon)_\varepsilon\right]$ or simply $\left[(\zeta_\varepsilon)_\varepsilon\right]$.

Definition 5.2. Any outer Colombeau metric measure on a set $X\subset R^n$ is a Colombeau generalized function $\left[(\phi_\varepsilon(E))_\varepsilon\right]\in\mathcal{G}_F(F,\Omega)$, $E\in F$ satisfies the following properties:

(i) Null empty set: The empty set has zero Colombeau outer measure

$$\left[(\phi_\varepsilon(\emptyset))_\varepsilon\right]=0. \tag{5.2}$$

(ii) Monotonicity: For any two subsets A and B of X

$$AB\left[(\phi_\varepsilon(A))_\varepsilon\right]\leq_{\widetilde{R}}\left[(\phi_\varepsilon(B))_\varepsilon\right]. \tag{5.3}$$

(iii) Countable subadditivity: For any sequence $\{A_j\}$ of subsets of X pairwise disjoint or not

$$\left[\left(\phi_\varepsilon\left(\cup_{j=1}^\infty A_j\right)\right)_\varepsilon\right]\leq_{\widetilde{R}}\left[\left(\textstyle\sum_{j=1}^\infty\phi_\varepsilon(A_j)\right)_\varepsilon\right]. \tag{5.4}$$

(iv) Whenever $d(A,B)=\inf\left\{d_n(x,y)\,|x\in A,y\in B\right\}>0$

$$\left[(\phi_\varepsilon(A\cup B))_\varepsilon\right]=\left[(\phi_\varepsilon(A))_\varepsilon\right]+\left[(\phi_\varepsilon(B))_\varepsilon\right], \tag{5.5}$$

where $d_n(x,y)$ is the usual Euclidean metric: $d_n(x,y)=\sqrt{\sum(x_i-y_i)^2}$.

Definition 5.3. We say that outer Colombeau metric measure $(\mu_\varepsilon)_\varepsilon$, $\varepsilon\in(0,1]$ is a Colombeau measure on σ-algebra of subests of $X\subset R^n$ if $(\mu_\varepsilon)_\varepsilon$ satisfies the following property:

$$\left[\left(\phi_\varepsilon\left(\cup_{j=1}^\infty A_j\right)\right)_\varepsilon\right]=\left[\left(\textstyle\sum_{j=1}^\infty\phi_\varepsilon(A_j)\right)_\varepsilon\right]. \tag{5.6}$$

Definition 5.4. If U is any non-empty subset of n-dimensional Euclidean space, R^n, the diamater $|U|$ of U is defined as

$$|U| = \sup\{d(x,y)\,|x,y\in U\}. \tag{5.7}$$

If $F \subseteq R^n$, and a collection $\{U_i\}_{i\in N}$ satisfies the following conditions:
(i) $|U_i| < \delta$ for all $i \in N$, (ii) $F \subseteq \cup_{i\in N} U_i$, then we say the collection $\{U_i\}_{i\in N}$ is a δ-cover of F.

Definition 5.5. If $F \subseteq R^n$ and $s, q, \delta > 0$, we define Hausdorff-Colombeau content:

$$\left(H_\delta^{s,q}(F,\varepsilon)\right)_\varepsilon = \left(\inf\left\{\sum_{i=1}^{\infty} \frac{|U_i|^s}{\|x_i\|^q+\varepsilon}\right\}\right)_\varepsilon \tag{5.8}$$

where the infimum is taken over all δ-covers of F and where $x_i = (x_{i,1}, ...x_{i,n}) \in U_i$ for all $i \in N$ and $\|x\|$ is the usual Euclidean norm:$\|x\| = \sqrt{\sum_{j=1}^{n} x_j^2}$.

Note that for $0 < \delta_1 < \delta_2 < 1, \varepsilon \in (0,1]$ we have

$$H_{\delta_1}^{s,q}(F,\varepsilon) \geq H_{\delta_2}^{s,q}(F,\varepsilon) \tag{5.9}$$

since any δ_1 cover of F is also a δ_2 cover of F, i.e. $H_{\delta_1}^{s,q}(F,\varepsilon)$ is increasing as δ decreases.

Definition 5.6. We define the (s,q)-dimensional Hausdorff-Colombeau (outer) measure as:

$$\left(H^{s,q}(F,\varepsilon)\right)_\varepsilon = \left(\delta \to 0 \lim H_\delta^{s,q}(F,\varepsilon)\right)_\varepsilon. \tag{5.10}$$

Theorem 5.1. For any δ-cover, $\{U_i\}_{i\in N}$of F, and for any $\varepsilon \in (0,1]$, $t > s$:

$$H^{t,q}(F,\varepsilon) \leq \delta^{t-s} H^{s,q}(F,\varepsilon). \tag{5.11}$$

Proof. Consider any δ-cover $\{U_i\}_{i\in N}$of F. Then each $|U_i|^{t-s} \leq \delta^{t-s}$ since $|U_i| \leq \delta$, so:

$$|U_i|^t = |U_i|^{t-s}|U_i|^s \leq \delta^{t-s}|U_i|^s. \tag{5.12}$$

From (5.12) it follows that

$$\frac{|U_i|^t}{\|x_i\|^q+\varepsilon} \leq \frac{\delta^{t-s}|U_i|^s}{\|x_i\|^q+\varepsilon} \tag{5.13}$$

and summing (5.12) over all $i \in N$ we obtain

$$\sum_{i=1}^{\infty} \frac{|U_i|^t}{\|x_i\|^q+\varepsilon} \leq \delta^{t-s} \sum_{i=1}^{\infty} \frac{|U_i|^s}{\|x_i\|^q+\varepsilon}. \tag{5.14}$$

Thus (5.11) follows by taking the infimum.

Theorem 5.2. (1) If $\left(H^{s,q}(F,\varepsilon)\right)_\varepsilon <_{\widetilde{R}} \infty_{\widetilde{R}}$, and if $t > s$, then $\left(H^{t,q}(F,\varepsilon)\right)_\varepsilon = 0_{\widetilde{R}}$.
(2) If $0_{\widetilde{R}} <_{\widetilde{R}} \left(H^{s,q}(F,\varepsilon)\right)_\varepsilon$, and if $t < s$, then $\left(H^{t,q}(F,\varepsilon)\right)_\varepsilon = \infty_{\widetilde{R}}$.

Proof. (1) The result follows from (5.11) after taking limits, since $\forall\varepsilon \in (0,1]$ by definitions it follows that $H^{s,q}(F,\varepsilon) < \infty$. (2) From (5.11) $\forall\varepsilon \in (0,1], \forall\delta > 0$ it follows that

$$\frac{1}{\delta^{s-t}} H^{s,q}(F,\varepsilon) \leq H^{t,q}(F,\varepsilon). \tag{5.15}$$

After taking limit $\delta \to 0$, we obtain $H^{t,q}(F,\varepsilon) = \infty$, since $\lim_{\delta\to 0}\left(\delta^{s-t}\right)^{-1} = \infty$ and $\lim_{\delta\to 0} H_\delta^{s,q}(F,\varepsilon) = H^{s,q}(F,\varepsilon) > 0$.

Definition 5.7. We define now the Hausdorff-Colombeau dimension $\dim_{HC}(F,q)$ of a set F (relative to $q>0$) as

$$\begin{gathered}\dim_{HC}(F,q)= \\ \sup\left\{s=s(q)\,|\,\left(H^{s,q}(F,\varepsilon)\right)_\varepsilon=\infty_{\widetilde{R}}\right\}=\inf\left\{s=s(q)\,|\,\left(H^{s,q}(F,\varepsilon)\right)_\varepsilon=0_{\widetilde{R}}\right\}.\end{gathered} \tag{5.16}$$

Remark 5.1. From Theorem 5.2 it follows that for any fixed $q=\breve{q}$: $\left(H^{s,\breve{q}}(F,\varepsilon)\right)_\varepsilon=0_{\widetilde{R}}$ or $\left(H^{s,\breve{q}}(F,\varepsilon)\right)_\varepsilon=\infty_{\widetilde{R}}$ everywhere except at a unique value s, where this value may be finite. As a function of $s, H^{s,\breve{q}}(F,\varepsilon)$ is decreasing function. Therefore, the graph of $H^{s,\breve{q}}(F,\varepsilon)$ will have a unique value where it jumps from $\infty_{\widetilde{R}}$ to $0_{\widetilde{R}}$.

Remark 5.2. Note that the graph of $\left(H^{s,\breve{q}}(F,\varepsilon)\right)_\varepsilon$ for a fixed $q=\breve{q}$ is

$$\left(H^{s,\breve{q}}(F,\varepsilon)\right)_\varepsilon=\begin{cases}\infty_{\widetilde{R}} & if\quad s<\dim_{HC}(F,\breve{q})\\ 0_{\widetilde{R}} & if\quad s>\dim_{HC}(F,\breve{q})\\ undetermined & if\quad s=\dim_{HC}(F,\breve{q})\end{cases} \tag{5.17}$$

Definition 5.8. We say that fractal $\subset R^n$ has a negative dimension relative to $\breve{q}>0$ iff $\dim_{HC}(F,\breve{q})-\breve{q}<0$.

6.Green's functions in spacetime with Hausdorff-Colombeau negative dimensions

We consider now as an example a self-interecting scalar field $(\varphi_\varepsilon(x))_\varepsilon\in\mathcal{G}\left(R_x^D\right)$ describing by the action

$$(S_\varepsilon)_\varepsilon=\int_{R_x^D}(dv_\varepsilon(x))_\varepsilon\left[\tfrac{1}{2}\left(\partial_\mu\varphi_\varepsilon\partial^\mu\varphi_\varepsilon\right)_\varepsilon-\tfrac{1}{2}m^2\left(\varphi_\varepsilon^2\right)_\varepsilon-\left(V_\varepsilon(\varphi_\varepsilon)\right)_\varepsilon\right], \tag{6.1}$$

where $(\varphi_\varepsilon(x))_\varepsilon,(V_\varepsilon(\varphi_\varepsilon(x)))_\varepsilon\in\mathcal{G}\left(R_x^D\right)$ and $(dv_\varepsilon(x))_\varepsilon=d^Dx\,(v_\varepsilon(x))_\varepsilon$ is the Colombeau-Stieltjes measure, where $(v_\varepsilon(x))_\varepsilon\in\mathcal{G}\left(R_x^D\right)$ and

$$(v_\varepsilon(x))_\varepsilon=(v_\varepsilon(x,N))_\varepsilon=\left(\left(|s(x)|^{|D^-|}+\varepsilon^N\right)^{-1}\right)_\varepsilon, N\gg1, s^2(x)=x_\mu x^\mu=x_ix^i-x_0^2, \tag{6.2}$$

i=1,...,D-1.

Remark 6.1. We will denote with $r=s_E(x)=\sqrt{x_ix^i+x_0^2}$ the Wick-rotated Lorentz invariant $s(x)$. Thus the Wick-rotated Lorentz invariant Colombeau-Stieltjes measure $(v_\varepsilon(x))_\varepsilon$ reads

$$\begin{gathered}\left(v_\varepsilon^E(x)\right)_\varepsilon=(v_\varepsilon(x,N))_\varepsilon= \\ \left(\left(|s_E(x)|^{|D^-|}+\varepsilon^N\right)^{-1}\right)_\varepsilon, N\gg1, s^2(x)=x_\mu x^\mu=x_ix^i+x_0^2.\end{gathered} \tag{6.3}$$

Corresponding to the action (6.1) generalized vacuum-to-vacuum amplitude in Hausdorff-Colombeau negative dimensions reads [9]:

$$\begin{gathered}(Z_M[J_\varepsilon])_\varepsilon=_{(\varphi_\varepsilon)_\varepsilon\in\widetilde{\mathcal{G}}(R_x^D)}\mathbf{D}\left[(\varphi_\varepsilon)_\varepsilon\right]\times \\ \exp\left[i\left(_{R_x^D}(dv_\varepsilon(x))_\varepsilon\left(12\left(\partial_\mu\varphi_\varepsilon\partial^\mu\varphi_\varepsilon\right)_\varepsilon-12m^2\left(\varphi_\varepsilon^2\right)_\varepsilon-\left(V_\varepsilon(\varphi_\varepsilon)\right)_\varepsilon+\varphi_\varepsilon J_\varepsilon\right)\right)_\varepsilon\right]= \\ _{(\varphi_\varepsilon)_\varepsilon\in\widetilde{\mathcal{G}}(R_x^D)}\mathbf{D}\left[(\varphi_\varepsilon)_\varepsilon\right]\times \\ \exp\left[i_{R_x^D}(dv_\varepsilon(x))_\varepsilon\left(12\left(\partial_\mu\varphi_\varepsilon\partial^\mu\varphi_\varepsilon\right)_\varepsilon-12m^2\left(\varphi_\varepsilon^2\right)_\varepsilon-\left(V(\varphi_\varepsilon)\right)_\varepsilon+(\varphi_\varepsilon)_\varepsilon(J_\varepsilon)_\varepsilon\right)\right]\end{gathered} \tag{6.4}$$

where $\widetilde{\mathcal{G}}\left(R_x^D\right) \subset \mathcal{G}\left(R_x^D\right)$ is a topological linear subspace of Colombeau algebra $\mathcal{G}\left(R_k^D\right)$, $\left(\varphi_\varepsilon(x)\right)_\varepsilon \in \widetilde{\mathcal{G}}\left(R_x^D\right)$ and $\left(J_\varepsilon(x)\right)_\varepsilon \in \widetilde{\mathcal{G}}\left(R_x^D\right)$ is a source.

Remark 6.2. Note that in (6.4) we integrate over an topological linear subspace $\widetilde{\mathcal{G}}\left(R_x^D\right) \subset \mathcal{G}\left(R_x^D\right)$ of Colombeau algebra $\mathcal{G}\left(R_x^D\right)$ but not over full Colombeau algebra $\mathcal{G}\left(R_x^D\right)$.

We will be write for short the expression $\left(Z\left[J_\varepsilon\right]\right)_\varepsilon$ in the following form

$$\begin{gathered}\left(Z_M\left[J_\varepsilon\right]\right)_\varepsilon=\mathbf{N}_M \int_{\widetilde{\mathcal{G}}\left(R_x^D\right)} \mathbf{D}\left[\left(\varphi_\varepsilon\right)_\varepsilon\right] \times \\ \exp\left[i\left\langle\frac{1}{2}\left(\partial_\mu \varphi_\varepsilon \partial^\mu \varphi_\varepsilon\right)_\varepsilon-\frac{1}{2} m^2\left(\varphi_\varepsilon^2\right)_\varepsilon-\left(V_\varepsilon\left(\varphi_\varepsilon\right)\right)_\varepsilon+\left(\varphi_\varepsilon\right)_\varepsilon\left(J_\varepsilon\right)\right\rangle_\nu\right],\end{gathered} \tag{6.5}$$

where $\mathbf{N}_M$ is a normalizing constant, the $\langle\ldots\rangle_v$ now means integration with nontrivial Colombeau-Stieltjes measure $d^D x\left(v_\varepsilon(x)\right)_\varepsilon$ over spacetime. The integrand in (6.5) is oscillatory and even path integrals are not well defined. There are two canonical ways to resolve this problem: (i) put in a convergence factor $\exp\left[-\frac{\epsilon}{2}\left\langle\left(\varphi_\varepsilon^2\right)_\varepsilon\right\rangle\right]$ with $\epsilon>0$, or (ii) define $\left(Z\left[J_\varepsilon\right]\right)_\varepsilon$ in Euclidean space by setting $x_0=i \bar{x}_0, d^D x=-i d^D \bar{x},\left(\partial_\mu \varphi_\varepsilon \partial^\mu \varphi_\varepsilon\right)_\varepsilon=-\left(\overline{\partial}_\mu \varphi_\varepsilon \overline{\partial}^\mu \varphi_\varepsilon\right)_\varepsilon$, where the bar denotes Euclidean space variables, $\overline{\partial}_\mu=\partial / \partial \bar{x}_\mu$. Then Eq. (6.5) becomes

$$\begin{gathered}\left(Z_E\left[J_\varepsilon\right]\right)_\varepsilon=\mathbf{N}_E \int_{\widetilde{\mathcal{G}}\left(R_x^D\right)} \mathbf{D}\left[\left(\varphi_\varepsilon\right)_\varepsilon\right] \times \\ \exp\left[-\left\langle\frac{1}{2}\left(\overline{\partial}_\mu \varphi_\varepsilon \overline{\partial}^\mu \varphi_\varepsilon\right)_\varepsilon+\frac{1}{2} m^2\left(\varphi_\varepsilon^2\right)_\varepsilon+\left(V_\varepsilon\left(\varphi_\varepsilon\right)\right)_\varepsilon-\left(\varphi_\varepsilon\right)_\varepsilon\left(J_\varepsilon\right)_\varepsilon\right\rangle_v\right],\end{gathered} \tag{6.6}$$

where for instance $\left(V_\varepsilon\left(\varphi_\varepsilon\right)\right)_\varepsilon=\sum_{k=3}^m c_k\left(\varphi_\varepsilon^k\right)_\varepsilon$. The exponent of the integrand is now negative definite for positive m and V_ε. In either case, the generating functional (6.6) is used to manufacture the Euclidean Green's functions which are the coefficients of the functional expansion

$$\left(Z\left[J_\varepsilon\right]\right)_\varepsilon=\sum_{N=0}^{\infty} \frac{(-1)^N}{N!}\left\langle\left(J_{1, \varepsilon} J_{2, \varepsilon} \ldots J_{N, \varepsilon} G^{(N)}(1,2, \ldots, N ; \varepsilon)\right)_\varepsilon\right\rangle_{\nu_1, \nu_2, \ldots, \nu_N}, \tag{6.7}$$

where $\langle\ldots\rangle_{\nu_1, \nu_2, \ldots, \nu_N}$. means integration with nontrivial Colombeau–Stieltjes measure $d^D x_1\left(\nu_\varepsilon^E\left(x_1\right)\right)_\varepsilon \times d^D x_2\left(\nu_\varepsilon^E\left(x_2\right)\right)_\varepsilon \times \ldots \times d^D x_N\left(\nu_\varepsilon^E\left(x_N\right)\right)_\varepsilon$ and therefore

$$\left(G^{(N)}(1,2, \ldots, N ; \varepsilon)\right)_\varepsilon=\frac{1}{(-1)^N} \frac{\delta}{\delta\left(J_{1, \varepsilon}^\nu\right)_\varepsilon} \frac{\delta}{\delta\left(J_{2, \varepsilon}^\nu\right)_\varepsilon} \cdots \frac{\delta}{\delta\left(J_{N, \varepsilon}^\nu\right)_\varepsilon}\left(Z\left[J_\varepsilon\right]\right)_\varepsilon\Bigg|_{\left(\mathbf{J}_\varepsilon^\nu\right)_\varepsilon=0}. \tag{6.8}$$

where

$$\left(\mathbf{J}_\varepsilon^\nu\right)_\varepsilon=\left(\left(J_{1, \varepsilon}^\nu\right)_\varepsilon,\left(J_{2, \varepsilon}^\nu\right)_\varepsilon, \ldots,\left(J_{N, \varepsilon}^\nu\right)_\varepsilon\right)$$

and $\left(J_{k, \varepsilon}^\nu\right)_\varepsilon=\left(J_\varepsilon(x) \nu_\varepsilon^E(x)\right)_\varepsilon$.

We evaluate now $\left(Z_E\left[J_\varepsilon\right]\right)_\varepsilon$ when $\left(V_\varepsilon\left(\varphi_\varepsilon\right)\right)_\varepsilon=0$. Let $\left(Z_{0, \varepsilon}^E\left[J_\varepsilon\right]\right)_\varepsilon$ be

$$\begin{gathered}\left(Z_{0, \varepsilon}^E\left[J_\varepsilon\right]\right)_\varepsilon=\mathbf{N}_M \int_{\widetilde{\mathcal{G}}\left(R_x^D\right)} \mathbf{D}\left[\left(\varphi_\varepsilon\right)_\varepsilon\right] \exp\left\{-\left(S_\varepsilon\left[\varphi_\varepsilon, J_\varepsilon\right]\right)_\varepsilon\right\}= \\ \mathbf{N}_M \int_{\widetilde{\mathcal{G}}\left(R_x^D\right)} \mathbf{D}\left[\left(\varphi_\varepsilon\right)_\varepsilon\right] \exp\left[-\left\langle\frac{1}{2}\left(\partial_\mu \varphi_\varepsilon \partial^\mu \varphi_\varepsilon\right)_\varepsilon+\frac{1}{2} m^2\left(\varphi_\varepsilon^2\right)_\varepsilon+\left(\varphi_\varepsilon\right)_\varepsilon\left(J_\varepsilon\right)\right\rangle_\nu\right].\end{gathered} \tag{6.9}$$

Remark 6.3. We assume now that $\left(\phi_\varepsilon(x)\right)_\varepsilon,\left(J_\varepsilon(x)\right)_\varepsilon \in \widetilde{\mathcal{G}}\left(R_x^D\right) \subset \mathcal{G}\left(R_x^D\right),\left(\varrho_{1, \varepsilon}(x)\right)_\varepsilon \in \mathcal{G}\left(R_x^D\right)$, $\left(\varrho_{2, \varepsilon}(k)\right)_\varepsilon \in \mathcal{G}\left(R_k^D\right)$ and introduce the D-dimensional Colombeau Fourier–Stieltjes transform $(\widetilde{\varphi}_\varepsilon(k))_\varepsilon\ (S)_\varrho\left[(\phi_\varepsilon(x))_\varepsilon\right](k)$, of the field $(\varphi_\varepsilon(x))_\varepsilon$ with weight $\{(\varrho_{1,\varepsilon}(x))_\varepsilon, (\varrho_{2,\varepsilon}(k))_\varepsilon\}$ using the following formal definitions: where the Colombeau-Fourier- Stieltjes transform $\left(\widetilde{G}_\varepsilon(k)\right)_\varepsilon \in \widetilde{\mathcal{G}}\left(R_k^D\right)$ of a function $(G_\varepsilon(x))_\varepsilon \in \widetilde{\mathcal{G}}\left(R_x^D\right)$ and its inverse are defined as

$$\left(\widetilde{G}_\varepsilon(k)\right)_\varepsilon = \left(\int d^D x \varrho_{1,\varepsilon}(x)\, G_\varepsilon(x)\, e^{-ik\cdot x}\right)_\varepsilon = \int \left(d^D x \varrho_{1,\varepsilon}(x)\right)_\varepsilon (G_\varepsilon(x))_\varepsilon\, e^{-ik\cdot x} \tag{6.10}$$

and

$$(G_\varepsilon(x))_\varepsilon = \frac{1}{(2\pi)^D}\left(\int d^D k \varrho_{2,\varepsilon}(k)\, \widetilde{G}_\varepsilon(k)\, e^{ik\cdot x}\right)_\varepsilon = \frac{1}{(2\pi)^D}\int \left(d^D k \varrho_{2,\varepsilon}(k)\right)_\varepsilon \left(\widetilde{G}_\varepsilon(k)\right)_\varepsilon e^{ik\cdot x}. \tag{6.11}$$

correspondingly, where $k\cdot x = k^0x^0 + \overrightarrow{k}\cdot\overrightarrow{x}$ and $\overrightarrow{k} = (k_1, ..., k_{D-1})$, $\overrightarrow{x} = (x_1, ..., x_{D-1})$.

Remark 6.4. In additional we assume that (i) for any $(G_\varepsilon(x))_\varepsilon \in \widetilde{\mathcal{G}}\left(R_x^D\right) : \|G_\varepsilon(x)\|_1 < \infty$, $\varepsilon \in (0,1]$ and (ii) $|[G_\varepsilon(x)]| \approx |k|^{D\left(1+\varepsilon^M\right)}, M \gg 1, \varepsilon \in (0,1]$.

Remark 6.5. Note that: (i) from (6.10)-(6.11) it follows that

$$\left(\widetilde{G}_\varepsilon(k)\right)_\varepsilon = \left(\varrho_{2,\varepsilon}^{-1}(k)\left[G_\varepsilon(x)\right](k)\right)_\varepsilon, (G_\varepsilon(x))_\varepsilon = \left(\varrho_{1,\varepsilon}^{-1}(x)^{-1}\left[\widetilde{G}_\varepsilon(k)\right](x)\right)_\varepsilon, \tag{6.12}$$

(ii) we choose now $\varrho_{2,\varepsilon}(k) = \left(\left(|\rho(k)|^{|D^-|} + \varepsilon\right)^{-1}\right)_\varepsilon, \rho_D(k) = \sqrt{\sum_{i=1}^D k_i^2}$

and

$\varrho_{1,\varepsilon}(x) = \left(\left(|r(x)|^{|D^-|} + \varepsilon^N\right)^{-1}\right)_\varepsilon, N \gg 1, r(x) = \sqrt{\sum_{i=1}^D x_i^2}.$

Remark 6.6. Note that $\varrho_{1,\varepsilon}(x) = \nu_\varepsilon^E(x,N)$ (see Remark 6.1, Eq.(6.3)) and therefore

$$\left(2^{-D}\pi^{-D}\int_{R_x^D} d^D x \nu_\varepsilon^E(x,N)\, e^{-ik\cdot x}\right)_\varepsilon \simeq \frac{\rho_D^{|D^-|}(k)}{\rho_D^D(k)} \simeq \rho_D^{|D^-|}(k)\,\delta^D(k), \tag{6.13}$$

since (i) $F\left[r^\lambda\right] = 2^{\lambda+D}\pi^{D/2}\Gamma\left(\frac{\lambda+D}{2}\right)\Gamma^{-1}\left(-\frac{\lambda}{2}\right)\rho_D^{\lambda-D}(k)$ and (ii) $\rho_D^{-D}(k) = 2^{-1}\Omega_D\delta^D(k)$, see [37].

Using now the definition of the D-dimensional Colombeau–Dirac distribution with nontrivial Colombeau-Stieltjes measure $\left(d^D k \rho_{2,\varepsilon}^{-1}(k)\right)_\varepsilon$ [10], [32]:

$$\left(\delta_{1/\rho_{2,\varepsilon}}(k)\right)_\varepsilon = \left(2^{-D}\pi^{-D}\int_{R_x^D} d^D x \nu_\varepsilon(x,N)\, e^{-ik\cdot x}\right)_\varepsilon \simeq \frac{\delta^D(k)}{\varrho_{2,\varepsilon}(k)}, \tag{6.14}$$

where $(v_\varepsilon(x))_\varepsilon = \left(\left(|s(x)|^{|D^-|} + \varepsilon^N\right)^{-1}\right)_\varepsilon, N \gg 1$, and the Colombeau-Fourier-Stieltjes transform (6.11), the exponent of the integrand in Eq. (6.9) is easily expressed in terms of the Colombeau- Fourier-Stieltjes transforms of $(\varphi_\varepsilon(x))_\varepsilon$ and $(J_\varepsilon(x))_\varepsilon$. Modulo the measure, we have followed exactly the same steps as in ordinary quantum field theory [38]. By using the Colombeau-Fourier-Stieltjes transform of the field $(\varphi_\varepsilon(x))_\varepsilon$ we obtain

$$\begin{gathered}
(S_\varepsilon[\varphi_\varepsilon, J_\varepsilon])_\varepsilon = \\
\frac{1}{2}\left(\int d^D x \nu_\varepsilon(x,N)\int \frac{d^D k_1 \varrho_{2,\varepsilon}(k_1)}{(2\pi)^D}\int \frac{d^D k_2 \varrho_{2,\varepsilon}(k_2)}{(2\pi)^D} e^{i(k_1+k_2)\cdot x}\left[-\widetilde{\varphi}_\varepsilon(k_1)(k_2^2+m^2)\widetilde{\varphi}_\varepsilon(k_2)\right.\right. \\
\left.\left. +\widetilde{J}_\varepsilon(k_1)\widetilde{\varphi}_\varepsilon(k_2) + \widetilde{J}_\varepsilon(k_2)\widetilde{\varphi}_\varepsilon(k_1)\right]\right)_\varepsilon = \\
\frac{1}{2}\left(\int \frac{d^D k_1 \varrho_{2,\varepsilon}(k_1)}{(2\pi)^D}\int \frac{d^D k_2 \varrho_{2,\varepsilon}(k_2)}{(2\pi)^D}\delta_{1/\rho_{2,\varepsilon}}(k_1+k_2)\left[-\widetilde{\varphi}_\varepsilon(k_1)(k_2^2+m^2)\widetilde{\varphi}_\varepsilon(k_2)\right.\right. \\
\left.\left. +\widetilde{J}_\varepsilon(k_1)\widetilde{\varphi}_\varepsilon(k_2) + \widetilde{J}_\varepsilon(k_2)\widetilde{\varphi}_\varepsilon(k_1)\right]\right)_\varepsilon = \\
\frac{1}{2}\left(\int \frac{d^D k_1 \varrho_{2,\varepsilon}(k_1)}{(2\pi)^D}\int \frac{d^D k_2 \varrho_{2,\varepsilon}(k_2)}{(2\pi)^D}\frac{(2\pi)^D\delta^D(k_1+k_2)}{\varrho_{2,\varepsilon}(k_1+k_2)}\left[-\widetilde{\varphi}_\varepsilon(k_1)(k_2^2+m^2)\widetilde{\varphi}_\varepsilon(k_2)\right.\right. \\
\left.\left. +\widetilde{J}_\varepsilon(k_1)\widetilde{\varphi}_\varepsilon(k_2) + \widetilde{J}_\varepsilon(k_2)\widetilde{\varphi}_\varepsilon(k_1)\right]\right)_\varepsilon = \\
\frac{1}{2}\left(\int \frac{d^D k \varrho_{2,\varepsilon}(-k)}{(2\pi)^D}\left[-\widetilde{\varphi}_\varepsilon(-k)(k_2^2+m^2)\widetilde{\varphi}_\varepsilon(k) + \widetilde{J}_\varepsilon(-k)\widetilde{\varphi}_\varepsilon(k) + \widetilde{J}_\varepsilon(k)\widetilde{\varphi}_\varepsilon(-k)\right]\right)_\varepsilon = \\
\frac{1}{2}\int \frac{d^D k \varrho_{2,\varepsilon}(-k)}{(2\pi)^D}\left[-\widetilde{\varphi}_\varepsilon(-k)(k_2^2+m^2)\widetilde{\varphi}_\varepsilon(k) + \frac{\widetilde{J}_\varepsilon(-k)\widetilde{J}_\varepsilon(k)}{(k_2^2+m^2)}\right].
\end{gathered} \tag{6.15}$$

Finally the integrand in Eq. (6.9) becomes

$$\left(\int d\varrho_{2,\varepsilon}(-k)\left[\widetilde{\varphi}^*_\varepsilon(k)\left(k^2+m^2\right)\widetilde{\varphi}^*(-k)-\widetilde{J}_\varepsilon(k)\left(k^2+m^2\right)^{-1}\widetilde{J}_\varepsilon(-k)\right]\right)_\varepsilon, \tag{6.16}$$

where

$$\left(\widetilde{\varphi}^*_\varepsilon(k)\right)_\varepsilon=\left(\widetilde{\varphi}_\varepsilon(k)\right)_\varepsilon+\left(k^2+m^2\right)\left(\widetilde{J}_\varepsilon(k)\right)_\varepsilon. \tag{6.17}$$

Note that

$$\int_{\widetilde{\mathcal{G}}\left(R^D_k\right)}\mathbf{D}\left[(\widetilde{\varphi}_\varepsilon)_\varepsilon\right]\exp\left[...\right]=\int_{\widetilde{\mathcal{G}}^*\left(R^D_k\right)}\mathbf{D}\left[(\widetilde{\varphi}^*_\varepsilon)_\varepsilon\right]\exp\left[...\right], \tag{6.18}$$

since the new variable $(\widetilde{\varphi}^*_\varepsilon)_\varepsilon$ differs from $(\widetilde{\varphi}_\varepsilon)_\varepsilon$ in function space by a constant, so that $\mathbf{D}\left[(\widetilde{\varphi}_\varepsilon)_\varepsilon\right]=\mathbf{D}\left[(\widetilde{\varphi}^*_\varepsilon)_\varepsilon\right]$. Putting it all together, we obtain

$$\begin{gathered}\left(Z^M_{0,\varepsilon}\left[J_\varepsilon\right]\right)_\varepsilon=\mathbf{N}_M\exp\left[\frac{1}{2}\left(\int d\varrho_{2,\varepsilon}(-k)\frac{\widetilde{J}_\varepsilon(k)\widetilde{J}_\varepsilon(-k)}{k^2+m^2}\right)_\varepsilon\right]\times\\ \int_{(\varphi^*_\varepsilon)_\varepsilon\in\widetilde{\mathcal{G}}^*(R^D_x)}\mathbf{D}\left[(\varphi^*_\varepsilon)_\varepsilon\right]\exp\left[-\left\langle\frac{1}{2}\left(\partial_\mu\varphi^*_\varepsilon\partial^\mu\varphi^*_\varepsilon\right)_\varepsilon+\frac{1}{2}m^2\left(\varphi^{*2}_\varepsilon\right)_\varepsilon\right\rangle_\nu\right].\end{gathered} \tag{6.19}$$

Thus

$$\left(Z^M_{0,\varepsilon}\left[J_\varepsilon\right]\right)_\varepsilon=\left(Z^M_{0,\varepsilon}\left[0\right]\right)_\varepsilon\exp\left[\frac{1}{2}\left(\int d\varrho_{2,\varepsilon}(-k)\frac{\widetilde{J}_\varepsilon(k)\widetilde{J}_\varepsilon(-k)}{k^2+m^2}\right)_\varepsilon\right]. \tag{6.20}$$

By adjusting $\mathbf{N}_M$, we can take $\left(Z^M_{0,\varepsilon}\left[0\right]\right)_\varepsilon=1$. The important thing is that we have succeeded in finding the explicit dependence of $\left(Z^M_{0,\varepsilon}\left[J_\varepsilon\right]\right)_\varepsilon$ on $(J_\varepsilon(x))_\varepsilon$. The use of the $(J_\varepsilon(x))_\varepsilon$ inverse Colombeau-Fourier-Stieltjes transform (6.10) yields

$$\begin{gathered}\left(\int_{R^D_k}d\varrho_{2,\varepsilon}(-k)\frac{\widetilde{J}_\varepsilon(k)\widetilde{J}_\varepsilon(-k)}{k^2-m^2+i\epsilon}\right)_\varepsilon=\\ \int_{R^D_k}\frac{(d\varrho_{2,\varepsilon}(-k))_\varepsilon}{(2\pi)^D}\int_{R^D_x}\left((d\varrho_{1,\varepsilon}(x))_\varepsilon\right)\int_{R^D_y}\left((d\varrho_{1,\varepsilon}(y))_\varepsilon\right)e^{ik\cdot(x-y)}\frac{(J_\varepsilon(x)J_\varepsilon(y))_\varepsilon}{k^2-m^2+i\epsilon}.\end{gathered} \tag{6.21}$$

and since such that $(\varrho_{2,\varepsilon}(-k))_\varepsilon=(\varrho_{2,\varepsilon}(k))_\varepsilon$, the free partition function reads

$$\begin{gathered}(Z_{0,\varepsilon}[J_\varepsilon])_\varepsilon=\left((Z_{0,\varepsilon}[0])_\varepsilon\right)\times\\ \exp\left[\frac{1}{2}\int_{R^D_x}\left((d\varrho_{1,\varepsilon}(x))_\varepsilon\right)\int_{R^D_y}\left((d\varrho_{1,\varepsilon}(y))_\varepsilon\right)\left(J_\varepsilon(x)\left((\Delta_F(x-y;\varepsilon))_\varepsilon\right)J_\varepsilon(y)\right)_\varepsilon\right]=\\ \exp\left[\frac{1}{2}\left\langle\left(J_\varepsilon(x)\left((\Delta_F(x-y;\varepsilon))_\varepsilon\right)J_\varepsilon(y)\right)_\varepsilon\right\rangle_{\rho_1(x),\rho_1(y)}\right],\end{gathered} \tag{6.22}$$

where

$$\left(\Delta_F(x-y;\varepsilon)\right)_\varepsilon=\frac{1}{(2\pi)^D}\int_{R^D_k}\left((d\varrho_{2,\varepsilon}(k))_\varepsilon\right)\frac{e^{ik\cdot(x-y)}}{k^2+m^2}. \tag{6.23}$$

Thus, we have recovered the usual definition of the propagator as the solution of the generalized Green equation

$$\left(+m^2\right)\Delta_F(x-y;\varepsilon))_\varepsilon=\left(\delta_{\varrho_{2,\varepsilon}}(x-y)\right)_\varepsilon. \tag{6.24}$$

7.The solution cosmological constant problem. Einstein-Gliner-Zel'dovich vacuum with tiny Lorentz invariance violation

Journal of Physics: Conference Series **1391** (2019) 012058 doi:10.1088/1742-6596/1391/1/012058

We will now briefly review the canonical assumptions that are made in the usual formulation of the cosmological constant problem.

7.1. The canonical assumptions

1.The physical dark matter.

Dark matter is a hypothetical form of matter that is thought to account for approximately 85% of the matter in the universe, and about a quarter of its total energy density. The majority of dark matter is thought to be non-baryonic in nature, possibly being composed of some as-yet undiscovered subatomic particles. Its presence is implied in a variety of astrophysical observations, including gravitational effects that cannot be explained unless more matter is present than can be seen. For this reason, most experts think dark matter to be ubiquitous in the universe and to have had a strong influence on its structure and evolution. The name dark matter refers to the fact that it does not appear to interact with observable electromagnetic radiation, such as light, and is thus invisible (or 'dark') to the entire electromagnetic spectrum, making it extremely difficult to detect using usual astronomical equipment. Because dark matter has not yet been observed directly, it must barely interact with ordinary baryonic matter and radiation. The primary candidate for dark matter is some new kind of elementary particle that has not yet been discovered, in particular, weakly-interacting massive particles (WIMPs), or gravitationally-interacting massive particles (GIMPs). Many experiments to directly detect and study dark matter particles are being actively undertaken, but none has yet succeeded.

2.The total effective cosmological constant.

The total effective cosmological constant $\lambda_{\mathbf{eff}}$ is on at least the order of magnitude of the vacuum energy density generated by zero-point fluctuations of the standard particle fields.

3.The Canonical QFT.

Canonical QFT is an effective field theory description of a more fundamental theory, which becomes significant at some high-energy scale Λ_*.

4.The vacuum energy-momentum tensor.

The vacuum energy-momentum tensor is Lorentz invariant.

5.The Moller-Rosenfeld approach to semiclassical gravity.

The Moller-Rosenfeld approach [39], [40] to semiclassical gravity by using an expectation value for the energy-momentum tensor is sound.

6.The Einstein equations.

The Einstein equations for the homogeneous Friedmann-Robertson-Walker metric accurately describe the large-scale evolution of the Universe.

Remark 7.1. Note that obviously there is a strong inconsistency between Assumptions **2** and **3**: the vacuum state cannot be Lorentz invariant if modes are ignored above some high-energy cutoff Λ_*, because a mode that is high energy in one reference frame will be low energy in another appropriately boosted frame. In this paper Assumption 3 is not used and this contradiction is avoided.

Remark 7.2. Note that also, Assumptions 1, 2 and 4 are modifed, which we denote as Assumptions $1'$, $2'$ and $4'$ respectively.

7.2. The modified assumptions

$1'$**.** The physical dark matter.

$2'$**.** The total effective cosmological constant $\lambda_{\mathbf{eff}}$ is on at least the order $|\mu_{\mathbf{eff}}|^{-n+5} \ln |\mu_{\mathbf{eff}}|$ of magnitude of the *renormalized* vacuum energy density generated by zero-point fluctuations of standard particle fields and ghost particle fields, (see [8] sec. 1.2 and [9]).

$3'$**.**The vacuum energy-momentum tensor is not Lorentz invariant.

7.3. The physical ghost matter and dark matter nature

In the contemporary quantum field theory, a ghost field, or gauge ghost is an unphysical state in a gauge theory. Ghosts are necessary to keep gauge invariance in theories where the local fields exceed a number of physical degrees of freedom. For example, in quantum electrodynamics, in order to maintain manifest Lorentz invariance, one uses a four component vector potential $A_\mu(x)$, whereas the photon has only two polarizations. Thus, one needs a suitable mechanism in order to get rid of the unphysical degrees of freedom.

Introducing fictitious fields, the ghosts, is one way of achieving this goal. Faddeev-Popov ghosts are extraneous fields which are introduced to maintain the consistency of the path integral formulation. Faddeev-Popov ghosts are sometimes referred to as "good ghosts". "Bad ghosts" represent another, more general meaning of the word "ghost" in theoretical physics: states of negative norm, or fields with the wrong sign of the kinetic term, such as Pauli-Villars ghosts, whose existence allows the probabilities to be negative thus violating unitarity.

In contrary with standard Assumption 1 in the case of the new approach introduced in this paper we assume that:

(1.a) The ghosts fields and ghosts particles with masses at a scale less then an fixed scale $m_{\mathbf{eff}}$ really exist in the universe and formed dark matter sector of the universe, in particular:

(1.b) these ghosts fields gives additive contribution to a full zero-point fluctuation, i.e. also to effective cosmological constant $\lambda_{\mathbf{eff}}$.

(1.c) Pauli-Villars renormalization of zero-point fluctuations (see [9], sec. 1.2) is no longer considered as an intermediate mathematical construct but obviously has rigorous physical meaning supported by assumption (4.a-4.b).

(2) The physical dark matter formed by ghosts particles.

(3) The standard model fields do not couple directly to the ghost sector in the ultraviolet region of energy at a scale less then an fixed large energy scale Λ_*, in particular.

(3.a) The "bad" ghosts fields with masses at a scale less then an fixed scale $m_{\mathbf{eff}}$, where $m_{\mathbf{eff}}c^2 \ll \Lambda_*$, cannot appear in any effective physical lagrangian which contain also the standard particles fields.

(4) The "bad" ghosts fields with masses at a scale m_*,where $m_*c^2 \gg \Lambda_*$ can appear in any effective physycal lagrangian which contain also the standard particles fields, in particular.

(4.a) Pauli-Villars finite renormalization with masses of ghosts fields at a scale m_* of the S-matrix in QFT (see [8], sec.3) is no longer considered as an intermediate mathematical construct but obviously has rigorous physical meaning supported by assumption (4.4).

Remark 7.3. We emphazize that in Universe standard matter coupled with a *physical* ghost matter has the equation of state [8]:

$$\varepsilon_{\mathbf{vac}}\left(\mu_{\mathbf{eff}}\right) = -p\left(\mu_{\mathbf{eff}}\right) = \tfrac{1}{8}\int\limits_0^{\mu_{\mathbf{eff}}} f\left(\mu\right)\mu^4\left(\ln\mu\right)d\mu = \tfrac{c^4\lambda_{\mathbf{vac}}}{8\pi G}, \tag{7.1}$$

where

$$|f\left(\mu\right)| = \begin{cases} O\left(\mu^{-n}\right), n>1 & \mu \leq \mu_{\mathbf{eff}} \\ 0 & \mu > \mu_{\mathbf{eff}} \end{cases} \tag{7.2}$$

$\mu_{\mathbf{eff}} = m_{\mathbf{eff}}c$ and therefore gives rise to a de Sitter phase of the universe even if bare cosmological constant $\lambda = 0$.

(5) In order to obtain QFT description of the dark component of matter in natural way we expand the standard model of particle physics on a sector of ghost particles, see ref. [10], sec. 2.3.2. QFT in a ghost sector developed in [10], sec.3.1-3.4 and sec.4.1-4.8.

7.4. Different contributions to $\lambda_{\mathbf{eff}}$.

The total effective cosmological constant $\lambda_{\mathbf{eff}}$ is on at least the order of magnitude of the vacuum energy density generated by zero-point fluctuations of *standard* particle fields.

Assumption 2 is well justified in the case of the traditional approach, because the contribution from zero-point fluctuations is on the order of 1 in Planck units and no other known contributions are as large thus, assuming no significant cancellation of terms (e.g. fine tuning of the bare cosmological constant λ), the total $\lambda_{\mathbf{eff}}$ should be at least on the order of the largest contribution [5], [8]

In contrary with standard Assumption 1 in the case of the new approach introduced in this paper we assume that:

(**1**) For simplisity though not necessary bare cosmological constant $\lambda = 0$.

(**2**) The total effective cosmological constant $\lambda_{\mathbf{eff}}$ depends only on mass distribution $f(\mu)$ and constant $\mu_{\mathbf{eff}} = m_{\mathbf{eff}}c$ but cannot depend on large energy scale $\sim \Lambda_*$

Remark 7.4. Note that in this subsection we pointed out that if bare cosmological constant $\lambda = 0$ the total cosmological constant $\lambda_{\mathbf{vac}}$ is on at least the order $|\mu_{\mathbf{eff}}|^{-n+5}$ of magnitude of the *renormalized* vacuum energy density generated by zero-point fluctuations of standard particle fields and ghost particle fields:

$$\begin{aligned}\varepsilon_{\mathbf{vac}}(\mu_{\mathbf{eff}}) &= \tfrac{1}{8}\int_0^{\mu_{\mathbf{eff}}} f(\mu)\,\mu^4(\ln\mu)\,d\mu + O\left(\Lambda_*^{-2}\right),\\ p_{\mathbf{vac}}(\mu_{\mathbf{eff}}) &= -\tfrac{1}{8}\int_0^{\mu_{\mathbf{eff}}} f(\mu)\,\mu^4(\ln\mu)\,d\mu + O\left(\Lambda_*^{-2}\right).\end{aligned} \tag{7.3}$$

7.5. Effective field theory and Lorentz invariance violation

To prevent the vacuum energy density from diverging, the traditional approach also assumes that performing a high-energy cutoff is acceptable. This type of regularization is a common step in renormalization procedures, which aim to eventually arrive at a physical, cutoff-independent result. However, in the case of the vacuum energy density, the result is inherently cutoff dependent, scaling quartically with the cutoff Λ_*.

Remark 7.5. By restricting to modes with particle energy a certain cutoff energy $\omega_{\mathbf{k}} \leq \Lambda_*$ a finite, regularized result for the energy density can be obtained. The result is proportional to Λ_*^4. Any other fields will contribute similarly, so that if there are n_b bosonic fields and n_f fermionic fields, the density scales with $(n_b - 4n_f)\,\Lambda_*^4$. Typically, the cutoff is taken to be near $= 1$ in Planck units (i.e. the Planck energy), so the vacuum energy gives a contribution to the cosmological constant on the order of at least unity according to Eq. (1.1).

Thus we see the extreme ne-tuning problem: the original cosmological constant λ must cancel this large vacuum energy density $\varepsilon_{\mathbf{vac}} \simeq 1$ to a precision of 1 in 10^{120} -but not completely- to result in the observed value $\lambda_{\mathbf{eff}} = 10^{-120}$ [5].

Remark 7.6. As it pointed out in this paper that a high-energy theory, i.e. QFT in fractal space-time with Hausdorff-Colombeau negative dimension would not display the zero-point fluctuations that are characteristic of QFT, and hence that the divergence caused by oscillations above the corresponding cutoff frequency is unphysical. In this case, the cutoff Λ_* is no longer an intermediate mathematical construct, but instead a physical scale at which the smooth, continuous behavior of QFT breaks down.

Poincaré group of the momentum space is deformed at some fundamental high-energy cutoff Λ_*. The canonical quadratic invariant $\|p\|^2 = \eta^{ab}p_a p_b$ collapses at high-energy cutoff Λ_* and being replaced by the non-quadratic invariant:

$$\|p\|^2 = \eta^{ab}p_a p_b(1 + l_{\Lambda_*}p_0). \tag{7.4}$$

Remark 7.7. In contrary with canonical approuch the total effective cosmological constant $\lambda_{\mathbf{eff}}$ depends only on mass distribution $f(\mu)$ and constant $\mu_{\mathbf{eff}} = m_{\mathbf{eff}}c$ but cannot depend on large energy scale $\sim \Lambda_*$.

7.6. Semiclassical Moller-Rosenfeld gravity

Remind that canonical Assumption 5 means that it is valid to replace the right-hand side of the Einstein equation $T_{\mu\nu}$ with its expectation $\langle T_{\mu\nu}\rangle$. It requires that either gravity is not in fact quantum, and the Moller-Rosenfeld approach is a complete description of reality, or at least a valid approximation in the weak field limit. The usual argument states that the vacuum state $|0\rangle$ should be locally Lorentz invariant so that observers agree on the vacuum state. This means that the expectation value of the energy-momentum tensor on the vacuum, $\langle 0|\widehat{T}_{\mu\nu}|0\rangle$, must be a scalar multiple of the metric tensor $g_{\mu\nu}$ which is the only Lorentz invariant rank $(0,2)$ tensor. By using Moller-Rosenfeld approach the Einstein field equations of general relativity, a term representing the curvature of spacetime $R_{\mu\nu}$ is related to a term describing the energy-momentum of matter $\langle 0|\widehat{T}_{\mu\nu}|0\rangle$, as well as the cosmological constant λ and metric tensor $g_{\mu\nu}$ reads:

$$R_{\mu\nu} - \tfrac{1}{2}R^{\upsilon}_{\upsilon}g_{\mu\nu} + \lambda g_{\mu\nu} = 8\pi\,\langle 0|\widehat{T}_{\mu\nu}|0\rangle\,. \tag{7.5}$$

The $\widehat{T}_{00}$ component is an energy density, we label $\langle 0|\widehat{T}_{\mu\nu}|0\rangle = \varepsilon_{\mathbf{vac}}$,so that the vacuum contribution to the right-hand side of Eq. (7.5) can be written as

$$8\pi\,\langle 0|\widehat{T}_{\mu\nu}|0\rangle = 8\pi\varepsilon_{\mathbf{vac}}g_{\mu\nu}. \tag{7.6}$$

Subtracting this from the right-hand side of Eq. (7.5) and grouping it with the cosmological constant term replaces with an "effective" cosmological constant [5]:

$$\lambda_{\mathbf{eff}} = \lambda + 8\pi\varepsilon_{\mathbf{vac}}. \tag{7.7}$$

Note that in flat spacetime, where $g_{\mu\nu} = diag(-1,+1,+1,+1)$, Eq. (7.6) implies $\varepsilon_{\mathbf{vac}} = -p_{\mathbf{vac}}$, where $p_{\mathbf{vac}} = \langle 0|\widehat{T}_{ii}|0\rangle$ for any $i = 1,2,3$ is the pressure. Obviously this implies that if the energy density is positive as is usually assumed, then the pressure must be negative, a conclusion which extends to any metric $g_{\mu\nu}$ with a $(-1,+1,+1,+1)$ signature.

Remark 7.8. In this paper we assume that the vacuum state $|0\rangle$ should be locally invariant under modified Lorentz boost (2.13) but not locally Lorentz invariant. Obviously this assumption violate Eq. (7.6). However modified Lorentz boosts (2.13) becomes Lorentz boosts for a sufficiently small energies and therefore in IR region one obtains in a good aproximation

$$8\pi\,\langle 0|\widehat{T}_{\mu\nu}|0\rangle \approx 8\pi\varepsilon_{\mathbf{vac}}g_{\mu\nu} \tag{7.8}$$

and

$$\lambda_{\mathbf{eff}} \approx \lambda + 8\pi\varepsilon_{\mathbf{vac}}. \tag{7.9}$$

Thus Moller-Rosenfeld approach holds in a good approximation.

7.7. Quantum gravity at energy scale $\Lambda \leq \Lambda_*$. Controlable violation of the unitarity condition

Gravitational actions which include terms quadratic in the curvature tensor are renormalizable. The necessary Slavnov identities are derived from Becchi-Rouet-Stora (BRS) transformations of the gravitational and Faddeev-Popov ghost fields. In general, non-gauge-invariant divergences do arise, but they may be absorbed by nonlinear renormalizations of the gravitational and ghost fields and of the BRS transformations [8], [9]. The generic expression of the action reads

$$I_{sym} = -d^4x\sqrt{-g}\left(\alpha R_{\mu\nu}R^{\mu\nu} - \beta R^2 + 2\kappa^{-2}R\right), \tag{7.10}$$

where the curvature tensor and the Ricci is defined by $R^{\lambda}_{\mu\alpha\nu} = \partial_\nu \Gamma^{\lambda}_{\mu\alpha}$ and $R_{\mu\nu} = R^{\lambda}_{\mu\lambda\nu}$ correspondingly, $\kappa^2 = 32\pi G$. The convenient definition of the gravitational field variable in terms of the contravariant metric density reads

$$\kappa h^{\mu\nu} = g^{\mu\nu}\sqrt{-g} - \eta^{\mu\nu}. \tag{7.11}$$

Analysis of the linearized radiation shows that there are eight dynamical degrees of freedom in the field. Two of these excitations correspond to the familiar massless spin-2 graviton. Five more correspond to a massive spin-2 particle with mass m_2. The eighth corresponds to a massive scalar particle with mass m_0. Although the linearized field energy of the massless spin-2 and massive scalar excitations is positive definite, the linearized energy of the massive spin-2 excitations is negative definite. This feature is characteristic of higher-derivative models, and poses the major obstacle to their physical interpretation.

In the quantum theory, there is an alternative problem which may be substituted for the negative energy. It is possible to recast the theory so that the massive spin-2 eigenstates of the free-fieid Hamiltonian have positive-definite energy, but also negative norm in the state vector space. These negative-norm states cannot be excluded from the physical sector of the vector space without destroying the unitarity of the **S** matrix. The requirement that the graviton propagator behave like p^{-4} for large momenta makes it necessary to choose the indefinite-metric vector space over the negative-energy states. The presence of massive quantum states of negative norm which cancel some of the divergences due to the massless states is analogous to the Pauli-Villars regularization of other field theories. For quantum gravity, however, the resulting improvement in the ultraviolet behavior of the theory is sufficient only to make it renormalizable, but not finite.

Remark 7.9. **(I)**The renormalizable models which we have considered in the papers [8], [9] many years mistakenly regarded only as constructs for a study of the ultraviolet problem of quantum gravity. The difficulties with unitarity appear to preclude their direct acceptability as canonical physical theories in locally Minkowski space-time. In canonical case they do have only some promise as phenomenological models.

(II) However, for their unphysical behavior may be restricted to *arbitrarily large energy scales* Λ_* mentioned above by an appropriate limitation on the renormalized masses m_2 and m_0. Actually, it is only the massive spin-two excitations of the field which give the trouble with unitarity and thus require a very large mass. The limit on the mass m_0 is determined only by the observational constraints on the static field.

Conclusion

We argue that a solution to the cosmological constant problem is to assume that there exists hidden physical mechanism which cancel divergences of the zero-point energy density in canonical QED_4, QCD_4, Higher-Derivative-Quantum-Gravity, etc. In fact we argue that corresponding supermassive Pauli-Villars ghost fields, etc. really exists [8], [9]. New theory of elementary particles which contains hidden ghost sector is proposed [9]. In accordance with Zel'dovich hypothesis [1] we suggest that physics of elementary particles is separated into low/high energy ones the standard notion of smooth spacetime is assumed to be altered at a high energy cutoff scale Λ_* and a new treatment based on QFT in a fractal spacetime with negative dimension is used above that scale. This would fit in the observed value of the dark energy needed to explain the accelerated expansion of the universe if we choose highly symmetric masses distribution below that scale Λ_*, i.e., $f_{s.m}(\mu) \approx -f_{g.m}(\mu), \mu \leq \mu_{\mathbf{eff}}, \mu_{\mathbf{eff}} c^2 < \Lambda_*$.

References

[1] Zel'dovich Ya B 1968 The cosmological constant and the theory of elementary particles *Sov. Phys. Usp.* **11** 381–393.

[2] Gliner É B 2002 Inflationary universe and the vacuumlike state of physical medium *Physics–Uspekhi* **45** 2, 213–220.
[3] Gliner É B 1965 *Zh. Eksp. Tear. Fiz.* **49** 542; *Sov. Phys. JETP* 1966 **22**, 378.
[4] Gliner É B, Dymnikova I G 1975 *Pis'ma Astron. Zh.* **15** 7, 4 ; *Sov. Astron. Lett.* 1975, **1** 93.
[5] Weinberg C S 1989 The cosmological constant problem *Rev. Mod. Phys.* **61** 1-23.
[6] Zel'dovich Y B 1967 'Cosmological Constant and Elementary Particles' *JETP Letters* **6** 316-317.
[7] Dolgov A D 1989 Proc. of the XXIVth Rencontre de Moriond; Series: Moriond Astrophysics Meetings, Les Arcs, France. ed. J. Adouse and J. Tran Thanh Van. p. 227.
[8] Foukzon F, Men'kova E R, Potapov A A 2019 The solution cosmological constant problem *Journal of Modern Physics* **10** 7. Paper ID 93134, 66 pages. DOI:10.4236/jmp.2019.107053 arXiv:1004.0451v4 [math.GM]
[9] Foukzon F, Men'kova E R, Potapov A A , Podosenov S A 2019 Quantum Field Theory in fractal spacetime with negative Hausdorff-Colombeau dimendions. The solution cosmological constant problem. arXiv:1004.0451v4 [math.GM]
[10] Cree S S, Davis T M, Ralph T C, Wang Q, Zhu Z, and Unruh W G 2018 Can the fluctuations of the quantum vacuum solve the cosmological constant problem? *Phys. Rev. D* **98** 063506. DOI:https://doi.org/10.1103/Phys.Rev.D.98.063506 https://arxiv.org/pdf/1805.12293.pdf
[11] Nagy K L 1966 State Vector Spaces with Indefinite Metric in Quantum Field Theory. Budapest, Akadémiai Kiadó.
[12] Bogoliubov N N and Shirkov D V 1984 *Introduction to the Theory of Quantized Fields.* John Wiley-Sons, New York.
[13] Maguejo J and Smolin L 2002 Lorentz invariance with an invariant energy scale *Phys. Rev. Lett.* **88** 190403. arXiv:hep-th/0112090
[14] Maguejo J and Smolin L 2003 Generalized Lorentz invariance with an invariant energy scale *Phys. Rev. D* **67** 044017 arXiv:gr-qc/0207085
[15] Bouda A, Foughali T 2012 On the Fock Transformation in Nonlinear Relativity *Mod. Phys. Lett. A* **27** 1250036 DOI: 10.1142/S0217732312500368 arXiv:1204.6397 [gr-qc]
[16] Stelle K S 1977 Renormalization-of-Higher-Derivative-Quantum-Gravity *Phys. Rev. D* **16** 953 DOI:https://doi.org/10.1103/Phys.Rev.D.16.953
[17] Bauer M, Plehn T *Yet Another Introduction to Dark Matter* Nov 2018 arXiv:1705.01987 [hep-ph]
[18] Lisanti M 2017 New Frontiers in Field and Strings. *Lectures on Dark Matter Physics.* Chapter 7 pp. 399-446 DOI:10.1142/9789813149441_0007 arXiv:1603.03797 [hep-ph]
[19] Hildebrandt H, Viola M 2016 *KiDS-450: Cosmological parameter constraints from tomographic weak gravitational lensing, MNRAS 000* 149. Preprint 28 October 2016.
[20] Svozil K 1987 *J. Phys. A: Math. Gen.* **20** 3861-3875.
[21] Falconer K J 1985 *The geometry of fractal sets* Cambridge University Press.1985.
[22] Manin Y I 2005 *The notion of dimension in geometry and algebra.* arXiv:math/0502016 [math.AG]
[23] Maslov V P 2006 *Negative Dimension in General and Asymptotic Topology* arXiv:math/0612543 [math.GM]
[24] Maslov V P 2007 General Notion of a Topological Space of Negative Dimension and Quantization of Its Density *Mathematical Notes* **81**, 1, 140–144. DOI: https://doi.org/10.4213/mzm3530
[25] Maslov V P 2006 Negative asymptotic topological dimension, a new condensate, and their relation to the quantized Zipf law *Mathematical Notes*, **80**6, 806–813. DOI: https://doi.org/10.4213/mzm3362
[26] Mandelbrot B B 1991 Random Multifractals: Negative Dimensions and the Resulting Limitations of the Thermodynamic Formalism,Proceedings *Mathematical and Physical Sciences* **434** 1890, Turbulence and Stochastic Process: Kolmogorov's Ideas 50 Years On (Jul. 8, 1991),79-88.
[27] Dunne G V, Halliday I G 1988 Negative dimensional oscillators *Nucl. Phys. B* **308** 589.
[28] Dunne G V 1989 Negative-dimensional groups in quantum physics *J. Phys. A* **22** 1719.
[29] Eyink G 1989 Quantum field-theory models on fractal spacetime. I. Introduction and overview *Commun. Math. Phys.* **125** 613-636.
[30] Eyink G 1989 Quantum field-theory models on fractal spacetime. II. Hierarchical propagators. *Commun. Math. Phys.* **126** 85-101.
[31] Calcagni G 2010 Quantum field theory, gravity and cosmology in a fractal universe. V. 2 arxiv.org/abs/1001.0571.
[32] Calcagni G 2010 Fractal universe and quantum gravity *Phys. Rev. Lett.* **104** 251301 https://arxiv.org/abs/0912.3142
[33] Colombeau J F 2000 *Elementary Introduction to New Generalized Functions,* eBook ISBN: 9780080872247 Imprint: North Holland Published Date: 1st April 2000 Page Count: 280.
[34] Colombeau J F 1984 *New Generalized Functions and Multiplication of Distributions,* North Holland, Amsterdam.
[35] Vernaeve H 2010 Ideals in the ring of Colombeau generalized numbers *Comm. Alg.* **38**, 6, 2199-2228.

arXiv:0707.0698 [math.RA]

[36] Foukzon J, Men'kova E R, Potapov A A, Podosenov S A. 2018 Was Polchinski wrong? Colombeau distributional Rindler space-time with distributional Levi-Cività connection induced vacuum dominance. Unruh effect revisited, *J. Phys.: Conf. Ser.* 1141012100

[37] Gel'fand I M and Shilov G E. 1964 *Generalized functions*, vol. I, Academic Press, New York U.S.A.

[38] Ramond P 2001 Field Theory: A Modern Primer (Frontiers in Physics Series, Vol. 74). Publisher: Westview Press; 2nd Edition. ISBN-13: 978-0201304503, ISBN-10: 0201304503

[39] Parker L and Toms D 2009 *Quantum Field Theory in Curved Spacetime: Quantized Fields and Gravity* Cambridge University Press, Cambridge, England.

[40] Birrell N and Davies P 1984 *Quantum Fields in Curved Space* Cambridge Monographs on Mathematical Physics. Cambridge University Press, Cambridge, England.