BOOK. « EPRB Paradox Resolution. A New Quantum Mechanical Formalism Based on the Probability Representation of Quantum States. Bell inequalities revisited»

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PREFACE

This book is devoted to the presentation of the new quantum mechanical formalism based on the probability representation of quantum states.

In the 20s and 30s it became evident that some properties in quantum mechanics can be assigned only to the quantum mechanical system, but not necessarily to its constituents. This led Einstein, Podolsky and Rosen (EPR) to their remarkable 1935 paper where they concluded that quantum mechanics is not a complete theory of nature (EPR paradox).

In order to avoid the contradiction which arises from instantaneous action at a distance mentioned above we introduce an extension of the canonical relativity by using measure algebra of physical events in Minkowski space-time. The canonical QM formalism is extended by additional new postulate of EPRB nonlocality for continuous and discrete observables, chapter I. The postulate of EPRB nonlocality is supported by new quantum mechanical formalism based on the probability representation of quantum states. Chapter II is devoted to the new quantum mechanical formalism based on the probability representation of quantum states. Chapter III is devoted to the Einstein's 1927 gedanken experiment resolution. Chapter IV is devoted to the EPR paradox resolution. Chapter V is devoted to the EPR-B paradox resolution. Chapter VI is devoted to the Schrödinger's cat (measured spin) paradox resolution. Chapter VII is devoted to the Bell inequalities revisited.

Remind that the canonical arguments which were presented by many authors, namely, that violations of Bell type inequalities signal us that the classical Kolmogorovian model of probability is inapplicable to quantum phenomena. We claimed that the canonical assumption, under which Bell type inequalities were derived, is not supported by real physical nature of the EPRB experiments. The fundamental physical nature violations of the canonical Bell type inequalities explained by Postulate of EPR-Nonlocality and Heisenberg noise-disturbance uncertainty relations.

INTRODUCTION

I.1. Bell's type inequalities violations

The canonical argument which were presented by many authors, namely, that violations of Bell type inequalities [1], [2] signal us that the classical model of probability [3] (Kolmogorov, 1933) is inapplicable to quantum phenomena. It well known that any attempt to explain to these violations by some additional value e.g., to philosophize about (non)locality and (un)reality, is not helpful [4], [5], [6].

Remind that one of the Bell's assumptions in the original derivation of his inequalities was the hypothesis of locality, i.e., of the absence of the influence of two remote measuring instruments on one another. That is why violations of these inequalities observed in experiments are often interpreted as a manifestation of the nonlocal nature of quantum mechanics, or a refutation of local realism.

Let $\Im = (\Omega, \mathcal{F}, \mathbf{P})$ be a Kolmogorov probability space. For two random variables $A,\ B : \Omega \to \mathbb{R}$ we set

$$E(AB) = \int_{\Omega} A(\omega)B(\omega)d\mathbf{P}(\omega) \qquad (1.1)$$

Theorem *I*.1.1. (CHSH-inequality). Let $A_i, B_j, i;j = 1,2$, be random variables with values in $\{-1,1\}$. Then the corresponding combination of correlation satisfies the CHSH-inequality:

$$S = E(A_1B_1) + E(A_1B_2) + E(A_2B_1) - E(A_2B_2) \leq 2. \qquad (1.2)$$

Theorem *I*.1.2. (Bell's no-go theorem [2]). Bell's no-go theorem says that Bell type inequalities, e.g., the CHSH-inequality (1.2), which are derived in Kolmogorovian model of probability are violated for correlations calculated in the quantum probability model.

Remark ***I*****.1.1.** By using Bell's no-go theorem many authors were concluded that the Kolmogorovian model of probability has to be rejected in general as inapplicable to these correlations in any cases, see for example [4]-[8].

In papers [7]-[9] authors claimed that the fair sampling assumption is not supported by real EPRB experiments. In papers [8]-[9] complete, probability spaces consistent with EPR-Bohm-Bell experimental data by taking into account random choice of settings, were obtained.

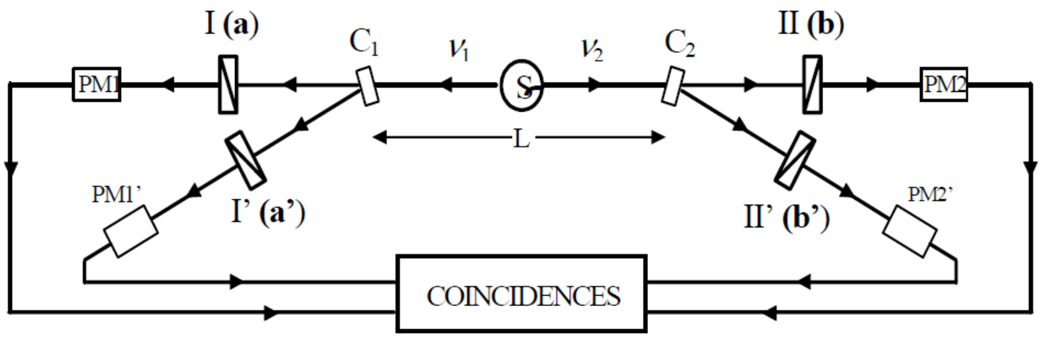

Fig. *I*.1.1. Timing-experiment with optical switches(C_1 and C_2).
Adopted from [10]

The switch C_1 followed by the two polarizers in orientations a and a' is equivalent to a single polarizer switched between the orientations a and a'. A switching occurs approximatively each 10 ns. A similar setup, independently driven, is implemented on the second side. In experiment, the distance L between the switches was large enough (13 m) that the time of travel of a signal between the switches at the velocity of light (43 ns) was significantly larger than the delay between two switchings (about 10 ns) and the delay between the emission between the two photons (5 ns average).

Let us consider now an experiment taking into account random choice of settings [9].

(a) There is a source of entangled photons.

(b) There are 4 PBSs and corresponding pairs of detectors for each PBS, totally 8 detectors. PBSs are labeled as $i = 1,2$ (at the left-hand side, LHS) and $j = 1,2$ (at the right-hand side, RHS).
(c) Directly after source there are 2 distribution devices, one at LHS and one at RHS. At each instance of time, $t = 0,\tau,2\tau\ldots$ each device opens the port to only one (of two) optical fibers going to the corresponding two PBSs. For simplicity, we suppose that each pair of ports $(i,j),(1,1),(1,2),(2,1),(2,2),$ can be opened with equal probabilities $\mathbf{P}(i,j) = 1/4.$
We introduce the observables measured in this experiment.
They are modifications of the polarization observables $a_{\theta_i}, i = 1,2,$ and $b_{\theta'_j}, j = 1,2,$
We define the "LHS-observables":
(1) $A_i = \pm 1, i = 1,2$ if the corresponding (up or down) detector is coupled to i-th PBS (at LHS) fires and the i-th channel is open,
(2) $A_i = 0$ if the i-th channel (at LHS) is blocked,
(3) in the same way we define the "RHS-observables": $B_j = 0,\pm 1,$ corresponding to PBSs $j = 1,2.$
Remark *I*.1.2. [9]. Thus unification of 4 incompatible experiments of the CHSH-test into a single experiment modifies the range of values of polarization observables for each of 4 experiments; the new value, zero, is added to reflect the random choice of experimental settings. We emphasize that this value has no relation to the efficiency of detectors. In this model we assume that detectors have 100% efficiency. The observables take the value zero when the optical bers going to the corresponding PBSs are blocked.
Let $(\Omega,\mathcal{F},\mathbf{P})$ be an arbitrary probability space and let $\Omega_0 \subset \Omega, \Omega_0 \in \mathcal{F}, \mathbf{P}(\Omega_0) > 0$ and ξ let be arbitrary random variable $\xi : \Omega \to \mathbb{R}.$ Then the conditional expectation of the random variable $\xi(\omega)$ conditioned to the event Ω_0 is defined as follows:

$$E(\xi|\Omega_0) = \int \xi(\omega) d\mathbf{P}_{\Omega_0}(\omega), \tag{1.3}$$

where the conditional probability $\mathbf{P}_{\Omega_0}(\omega)$ is defined by Bayes formula

$$\mathbf{P}_{\Omega_0}(X) = \mathbf{P}(X|\Omega_0) = \frac{\mathbf{P}(X \cap \Omega_0)}{\mathbf{P}(\Omega_0)} \tag{1.4}$$

Let us now consider the sample space of points Ω :

$$\Omega = \{(\epsilon_1, 0, \epsilon'_1, 0), (\epsilon_1, 0, 0, \epsilon'_2), (0, \epsilon_2, \epsilon'_1, 0), (0, \epsilon_2, 0, \epsilon'_2)\}, \tag{1.5}$$

where $\epsilon, \epsilon' \in \{-1, 1\}$.
We define the following probability measure on Ω :

$$\begin{aligned} \mathbf{P}(\epsilon_1, 0, \epsilon'_1, 0) = \tfrac{1}{4} p_{11}(\epsilon_1, \epsilon'_1), \mathbf{P}(\epsilon_1, 0, 0, \epsilon'_2) = \tfrac{1}{4} p_{12}(\epsilon_1, \epsilon'_2) \\ \mathbf{P}(0, \epsilon_2, \epsilon'_1, 0) = \tfrac{1}{4} p_{21}(\epsilon_2, \epsilon'_1), \mathbf{P}(0, \epsilon_2, 0, \epsilon'_2) = \tfrac{1}{4} p_{22}(\epsilon_2, \epsilon'_2) \end{aligned} \tag{1.6}$$

where p_{ij} is any collection of probabilities, i.e.,

$$p_{ij} > 0, \textstyle\sum_{\epsilon, \epsilon'} p_{ij}(\epsilon, \epsilon') = 1, \epsilon, \epsilon' \in \{-1, 1\}. \tag{1.7}$$

We define random variables $A_i(\omega), B_j(\omega)$:

$$\begin{aligned} A_1(\epsilon_1, 0, \epsilon'_1, 0) = A_1(\epsilon_1, 0, 0, \epsilon'_2) = \epsilon_1, A_2(0, \epsilon_2, \epsilon'_1, 0) = A_2(0, \epsilon_2, 0, \epsilon'_2) = \epsilon_2, \\ B_1(\epsilon_1, 0, \epsilon'_1, 0) = B_1(\epsilon_1, 0, 0, \epsilon'_2) = \epsilon'_1, B_2(0, \epsilon_2, \epsilon'_1, 0) = B_2(0, \epsilon_2, 0, \epsilon'_2) = \epsilon'_2 \end{aligned} \tag{1.8}$$

and we put these variables equal to zero in other points. We define the random variables which are responsible for selections of pairs of ports to PBSs. For the device at LHS:

$$\eta_L(\epsilon_1, 0, 0, \epsilon'_2) = \eta_L(\epsilon_1, 0, \epsilon'_1, 0) = 1, \eta_L(0, \epsilon_2, 0, \epsilon'_2) = \eta_L(0, \epsilon_2, \epsilon'_1, 0) = 2. \tag{1.9}$$

For the device at RHS:

$$\eta_R(\epsilon_1,0,\epsilon_1',0) = \eta_R(0,\epsilon_2,\epsilon_1',0) = 1, \eta_R(0,\epsilon_2,0,\epsilon_2') = \eta_R(\epsilon_1,0,0,\epsilon_2') = 2. \quad (1.10)$$

We choose now $\Omega_0 \triangleq \Omega_{ij} = \{\omega | \eta_L(\omega) = i, \eta_R(\omega) = j\}$. We set

$$E_C(A_iB_j) = E(A_iB_j|\eta_L = i, \eta_R = j) = \int_\Omega A_i(\omega)B_j(\omega)d\mathbf{P}_{\Omega_{ij}}(\omega), \qquad (1.11)$$

and

$$\begin{gathered} S_C = E(A_1B_1|\eta_L = 1, \eta_R = 1) + E(A_1B_2|\eta_L = 1, \eta_R = 2) + \\ E(A_2B_1|\eta_L = 2, \eta_R = 1) - E(A_2B_2|\eta_L = 2, \eta_R = 2). \end{gathered} \qquad (1.12)$$

Theorem *I*.1.3. [9]. (CHSH-inequality for conditional correlations.) Let $A_i, B_j, i,j = 1,2$, be random variables defined by Eq. (1.8). Then the corresponding combination of conditional correlation S_C satisfies the inequality:

$$S_C \leq 4. \qquad (1.13)$$

However in papers [9], [10], [11], Bell's type inequality were derived in its traditional form, without resorting to the hypothesis of locality and hidden-variable theory the only assumption being that the probability distributions are nonnegative. The starting point of these papers that is a recognition of the existence of a positive-definite probability distribution function.

Let A, A', B, B' be random variables with values in the set $\{-1,+1\}$, i.e.,

$$A = \pm 1, A' = \pm 1, B = \pm 1, B' = \pm 1. \qquad (1.14)$$

Assume that there exists joint probability distribution function $P(A,A',B,B')$ of A, A', B, B' defining probabilities for each possible set of outcomes such that:

$$\sum_{A,A',B,B'} P(A,A',B,B') = 1, P(A,A',B,B') \geq 0, \tag{1.15}$$

and

$$P(A,A',B,B') - P(-A,A',B,B') = P(A',B,B'). \tag{1.16}$$

The main point of papers [9], [10], [11] also be a recognition of the existence of a positive-definite probability distribution function without any reference to the full classical Kolmogorovian model of probability [3].
We abbreviate now for short [11]:

$$\begin{gathered} P_1 = P_{A,A',B,B'}(++++) = P(A = 1, A' = 1, B = 1, B' = 1), \\ P_2 = P_{A,A',B,B'}(+++-) = P(A = 1, A' = 1, B = 1, B' = -1), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ P_{16} = P_{A,A',B,B'}(----) = P(A = -1, A' = -1, B = -1, B' = -1). \end{gathered} \tag{1.17}$$

For the quantities $\langle AB\rangle, \langle A'B\rangle, \langle AB'\rangle$ and $\langle A'B'\rangle$ using Eq. (1.17) one obtains the representatives

$$\langle AB\rangle = P_{AB}(++) + P_{AB}(--) - P_{AB}(+-) - P_{AB}(-+), \tag{1.18}$$

etc.,

where

$$\begin{gathered} P_{AB}(++) = P_{A,A',B,B'}(++++) + P_{A,A',B,B'}(+++-) + \\ P_{A,A',B,B'}(+-++) + P_{A,A',B,B'}(+-+-), \end{gathered} \tag{1.19}$$

etc.

Theorem *I*.1.4. [11]. Assume that there exists joint probability distribution function $P(A,A',B,B')$ of A,A',B,B' such that Eq. (1.15) - Eq.(1.16) is satisfied. Then Bell inequality of the form

$$|\langle AB\rangle + \langle A'B\rangle + \langle AB'\rangle - \langle A'B'\rangle| \leq 2. \qquad (1.20)$$

holds.

Remark I.1.3. We claim that the even general assumption given by Eqs.(1.15) - (1.16) is not supported by fundamental physical nature of the EPRB experiments. This fundamental physical nature formally explained by (i) Postulate of EPR-Nonlocality (see subsection I.4) and by (ii) Heisenberg noise-disturbance uncertainty relation, (see Appendix A).

Remind that in a typical Bell experiment, (see Fig. I.1.2), two particles A and B which may have previously interacted - for instance they may have been produced by a common source - are now spatially separated and are each measured by one of two distant observers, Alice and Bob.

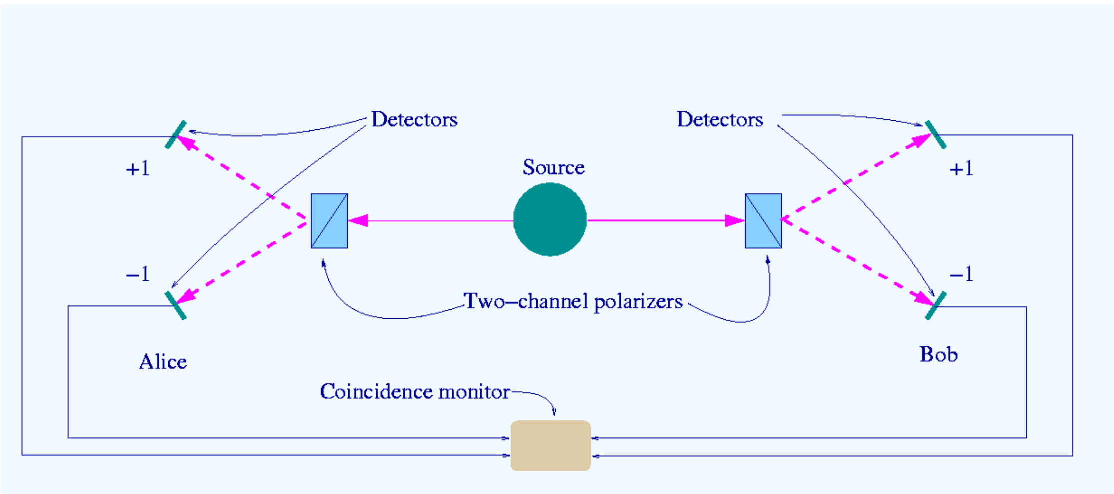

Fig. I.1.2. Scheme of a "two-channel" Bell experiment

Alice may choose one out of several possible measurements to perform on her system and we let x denote her measurement choice. For instance, x may refer to the position of a knob on her measurement apparatus. Similarly, we let y denote Bob's measurement choice. Once the measurements are performed, they yield random outcomes $a(\omega)$ and $b(\omega)$ on the two systems. The actual values assigned to the measurement choices x,y and outcomes $a(\omega), b(\omega)$ are purely conventional; they are mere macroscopic labels distinguishing the different possibilities. These outcomes $a(\omega)$ and $b(\omega)$ are thus in general governed by a Kolmogorovian probability distribution $p(ab|xy),$ which can of course depend on the particular experiment being performed. The assumption of locality implies that we should be able to identify a set of past factors, described by some variables $\lambda,$ having a joint causal influence on both outcomes, and which fully account for the dependence between $a(\omega)$ and $b(\omega).$ Once all such factors have been taken into account, the residual indeterminacies about the outcomes must now be decoupled, that is, the Kolmogorovian probabilities for $a(\omega)$ and $b(\omega)$ should factorize:

$$p(ab|xy,\lambda) = p(a|x,\lambda)p(b|y,\lambda). \tag{1.21}$$

The different values of λ across the runs should thus be characterized by a probability distribution $q(\lambda).$ Combined with the above factorability condition, we can thus write

$$p(ab|xy) = \int_{\Lambda} d\lambda q(\lambda)p(a|x,\lambda)p(b|y,\lambda), \tag{1.22}$$

where we also implicitly assumed that the measurements x and y can be freely chosen in a way that is independent of λ, i.e., that $q(\lambda|x,y) = q(\lambda)$. Let us consider for simplicity an experiment where there are only two measurement choices per observer $x,y \in \{0,1\}$ and where the possible outcomes take also two values labelled $a,b \in \{-1,+1\}.$ Let $\langle a_x b_y \rangle$ be the expectation value of the product ab for given measurement choices (x,y) :

$$\langle a_x b_y \rangle = \sum_{a,b} abp(ab|xy). \tag{1.23}$$

Consider the following expression

$$S = \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle, \tag{1.24}$$

which is a function of the probabilities $p(ab|xy)$. If these probabilities satisfy the locality decomposition (1.21), we necessarily have that

$$S = \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle \leq 2, \tag{1.25}$$

which is known as the Clauser-Horne-Shimony-Holt (CHSH) inequality [12].

Remark *I*.1.4. Note that particles A and B in real EPRB experiments cannot collapse simultaneously. Taking into account postulate of EPR-Nonlocality (see subsection *I*.4) and Heisenberg noise-disturbance uncertainty relation we obtain that random outcomes a and b mentioned above in real EPRB experiments they is not simply random variables $a(\omega)$ and $b(\omega)$ but time dependent random functions $a_{t_1}(\omega)$ and $b_{t_2}(\omega)$.

Remark *I*.1.5. These time dependent outcomes $a_{t_1}(\omega)$ and $b_{t_2}(\omega)$ are thus in general governed by a time dependent Kolmogorovian joint probability distribution $p(a,t_1;b,t_2|x_{t_1}y_{t_2})$, which can of course depend on the particular experiment being performed. By repeating the experiment a sufficient number of times and collecting the observed data, one can get a fair estimate of such time dependent Kolmogorovian joint probabilities distribution. The assumption of locality implies that we should be able to identify a set of past factors, described by some variables λ, having a joint causal influence on both outcomes $a_{t_1}(\omega)$ and $b_{t_2}(\omega)$, and which fully account for the dependence between a_{t_1} and b_{t_2}. Once all such factors have been taken into account, the residual indeterminacies about the outcomes must now be decoupled, that is, the time dependent Kolmogorovian joint probabilities for a and b should factorize:

$$p(a,t_1;b,t_2|xy,\lambda) = p(a,t_1|x,\lambda)p(b,t_2|y,\lambda). \tag{1.26}$$

The variable λ will not necessarily be constant for all runs of the experiment, even if the procedure which prepares the particles to be measured is held fixed, because λ may involve physical quantities that are not fully controllable. The different values of λ across the runs should thus be characterized by a probability distribution $q(\lambda,t_1,t_2)$. Combined with the above factorability condition, we can thus write instead (1.22)

$$p(a,t_1;b,t_2|xy) = \int_\Lambda d\lambda q(\lambda,t_1,t_2)p(a,t_1|x,\lambda)p(b,t_2|y,\lambda), \tag{1.27}$$

where we also implicitly assumed that the measurements x_{t_1} and y_{t_2} can be freely chosen in a way that is independent of λ, i.e., that $q(\lambda,t_1,t_2|x_{t_1},y_{t_2}) = q(\lambda,t_1,t_2)$.
Let us consider for simplicity an experiment where there are only two measurement choices per observer $x_{t_1},y_{t_2} \in \{0,1\}$ and where the possible outcomes take also two values labelled $a_{t_1},b_{t_2} \in \{-1,+1\}$. Let $\langle a_{t_1x_{t_1}}b_{t_2y_{t_2}}\rangle$ be the expectation value of the product $a_{t_1}b_{t_2}$ for given measurement choices (x_{t_1},y_{t_2}) :

$$\langle a_{t_1x_{t_1}}b_{t_2y_{t_2}}\rangle = \sum_{a,b} abp(ab,t_1,t_2|x_{t_1}y_{t_2}). \tag{1.28}$$

We assume now that $(t_1,t_2) \in \mathbb{Z}^2_{+\delta}, \mathbb{Z}_{+\delta} = \mathbb{Z}_+ \times \delta, 0 < \delta \ll 1,$

$$\begin{gathered} p(ab,t_1,t_2|x_{t_1}y_{t_2}) = p(ab,t_1-t_2|x_{t_1}y_{t_2}) \Leftrightarrow (|t_1-t_2| = \delta) \wedge (|t_1-t_2| = 0), \\ q(\lambda,t_1,t_2) = q(\lambda,t_1-t_2) \Leftrightarrow (|t_1-t_2| = \delta) \wedge (|t_1-t_2| = 0) \end{gathered} . \tag{1.29}$$

Thus

$$\left\langle a_{t_1 x_{t_1}} b_{t_2 y_{t_2}} \right\rangle = \sum_{a,b} abp(ab, t_1 - t_2 | x_{t_1} y_{t_2}). \tag{1.30}$$

We denote

$$\left\langle a_{t_1 x_{t_1}} b_{t_2 y_{t_2}} \right\rangle \triangleq \langle a_x b_y \rangle_{=} \tag{1.31}$$

iff $|t_1 - t_2| = 0$,

$$\left\langle a_{t_1 x_{t_1}} b_{t_2 y_{t_2}} \right\rangle \triangleq \langle a_x b_y \rangle_{>} \tag{1.32}$$

iff $|t_1 - t_2| = \delta$ and $t_1 > t_2$,

$$\left\langle a_{t_1 x_{t_1}} b_{t_2 y_{t_2}} \right\rangle \triangleq \langle a_x b_y \rangle_{<} \tag{1.33}$$

iff $|t_1 - t_2| = \delta$ and $t_1 < t_2$,

$$\langle a_x b_y \rangle \triangleq \langle a_x b_y \rangle_{=} + \langle a_x b_y \rangle_{>} + \langle a_x b_y \rangle_{<}. \tag{1.34}$$

Consider now the following expression

$$S = \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle, \tag{1.35}$$

which is a function of the probabilities $p(ab, t_1, t_2 | x_{t_1} y_{t_2})$. If these probabilities satisfy the locality decomposition (1.26) and Eq. (1.28), we necessarily have that

$$S = S_{=} + S_{>} + S_{<} = \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle \leq 6, \tag{1.36}$$

where

$$
\begin{aligned}
S_{=} &= \langle a_0 b_0 \rangle_{=} + \langle a_0 b_1 \rangle_{=} + \langle a_1 b_0 \rangle_{=} - \langle a_1 b_1 \rangle_{=},\\
S_{>} &= \langle a_0 b_0 \rangle_{>} + \langle a_0 b_1 \rangle_{>} + \langle a_1 b_0 \rangle_{>} - \langle a_1 b_1 \rangle_{>}, \qquad (1.37)\\
S_{>} &= \langle a_0 b_0 \rangle_{<} + \langle a_0 b_1 \rangle_{<} + \langle a_1 b_0 \rangle_{<} - \langle a_1 b_1 \rangle_{<}.
\end{aligned}
$$

Remark *I*.1.6. Note that in contrast with CHSH-inequality for conditional correlations **(**1.13) the inequality **(**1.36) is obtained without any references to random choice of settings x_{t_1} and y_{t_2}.

I.2. A new quantum mechanical formalism based on the probability representation of quantum states

Definition *I*.2.1. In probability theory, the **sample space** (observation space) of an experiment or random trial is the set of all possible outcomes or results of that experiment. A sample space is usually denoted using set notation, and the possible outcomes are listed as elements in the set. It is common to refer to a sample space by the label Ω.

Remark *I*.2.1. A well-defined sample space (observation space) is one of three basic elements in a probabilistic model (a probability space $\Theta = \{\Omega, \Sigma, \mathbf{P}\}$); the other two are a well-defined set of possible events (a sigma-algebra Σ) and a probability assigned to each event (a probability measure function $\mathbf{P}$).

A simple example of a sample phase space and corresponding probability space closer to our Stern-Gerlach experiment is a coin toss. Consider 1000 coin tosses. If the coin is tossed without bias, you will find close to 500 heads and 500 tails, corresponding to prob $_{heads} = 0.5$ and prob$_{tails} = 0.5.$ Here the sample space consists of the 1000 detailed trajectories of the toss, which your eye cannot follow, but which if analyzed by a very fast computer could predict which toss would give a head and which a tail (Fig. *I*.2.1).

Again, the probabilities are just reflections of our ignorance of the details, but the details are there. So we have the questions - are there hidden details underlying the probabilities in quantum mechanics? Is there a hidden sample space and corresponding probability space?

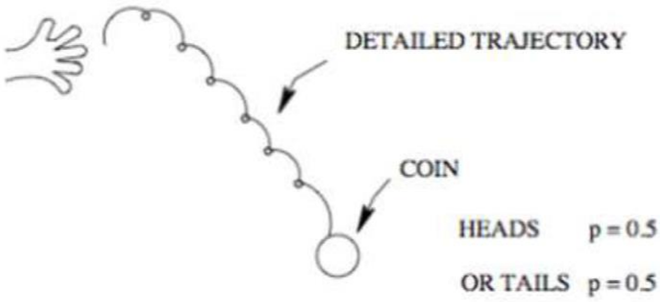

Fig. *I*.2.1. A sample space. Trajectories in a coin toss.
Adapted from [7]

In the de Broglie-Bohm interpretation: a particle has an initial position and follows a path whose velocity at each instant is given by a classical equation. On the basis of this assumption we conduct a simulation experiment by drawing random initial positions of the electrons in the initial wave packet ("quantum equilibrium hypothesis").

Fig. *I*.2.2 shows, after its initial starting position, 100 possible quantum trajectories of an electron passing through one of the two slits: We have not represented the paths of the electron when it is stopped by the first screen. Fig. *I*.2.3 shows a close-up of these trajectories just after they leave their slits.

Remark *I*.2.2. The different trajectories explain both the impact of electrons on the detection screen and the interference fringes. This is the simplest and most natural interpretation to explain the impact positions: "The position of an impact is simply the position of the particle at the time of impact." This was the view defended by Einstein at the Solvay Congress of 1927. The position is the only measured variable of the experiment.

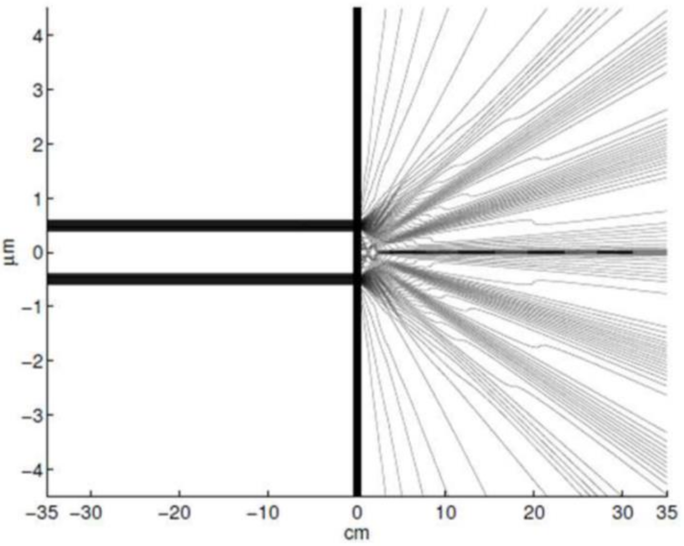

Fig. *I*.2.2. A sample space in Bohmian QM. 100 electron trajectories for the Jönsson experiment. Adapted from [8]

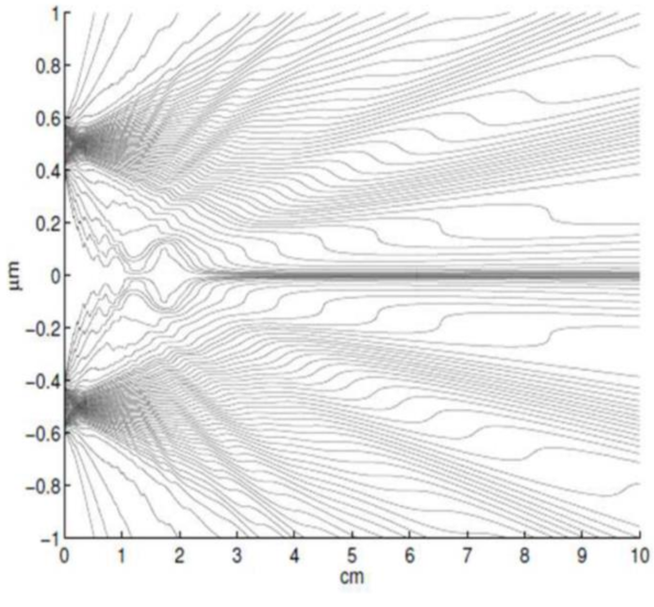

Fig. *I*.2.3. Close-up on the 100 trajectories of the electrons just after the slits. Adapted from [8]

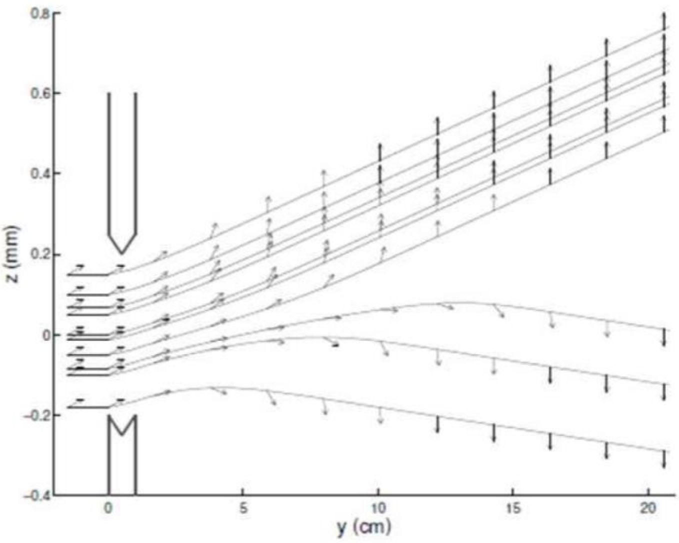

Fig. I.2.4. Ten silver atom trajectories within initial spin orientation $\theta_0 = \pi/3$ and initial position z_0; arrows represent the spin orientation $\theta(z,t)$ along the trajectories. Adapted from [8]

Fig. $\boldsymbol{I}$.2.4 presents, for a silver atom with the initial spinor orientation ($\theta_0 = \pi/3$, ϕ_0=0), a plot in the (Oyz) plane of a set of 10 trajectories whose initial position z_0 has been randomly chosen from a Gaussian distribution with standard deviation σ_0. The spin orientations $\theta(z,t)$ are represented by arrows.

Now let us consider a mixture of pure states where the initial orientation (θ_0, ϕ_0) from the spinor has been randomly chosen. These are the conditions of the initial Stern and Gerlach experiment. Fig. I.2.5 represents a simulation of 10 quantum trajectories of silver atoms from which the initial positions z_0 are also randomly chosen.

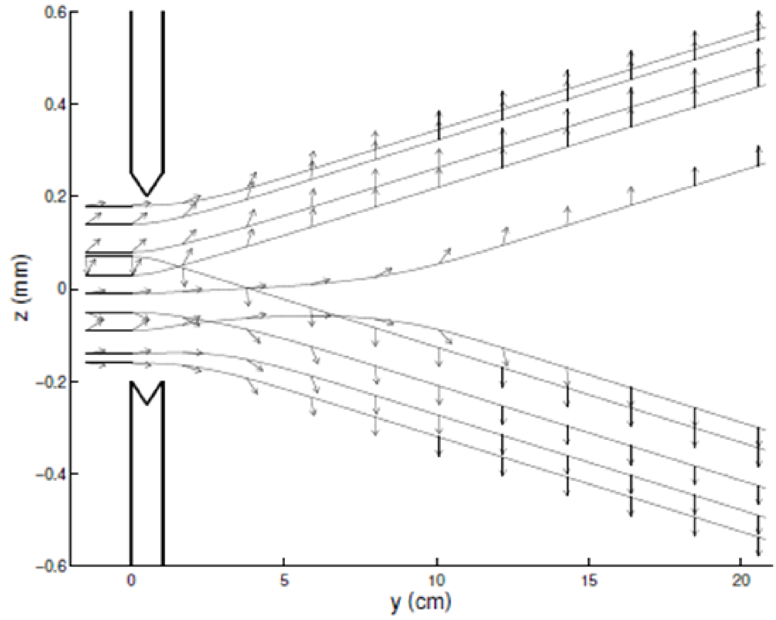

Fig. I.2.5. Ten silver atom trajectories where the initial spin orientation (θ_0, ϕ_0) has been randomly chosen; arrows represent the spin orientation $\theta(z,t)$ along the trajectories. Adapted from [8]

Definition I.2.2. A probability space consists of three parts:

1. A sample space (observation space) $\Omega,$ which is the set of all possible single outcomes $\omega \in \Omega.$
2. A set of events $\Sigma,$ where each event is a set containing $\varnothing$ or more outcomes.
3. The assignment of probabilities to the events; that is, a function $\mathbf{P}$ from events to probabilities.

An outcome is the result of a single execution of the model. Since individual outcomes might be of little practical use, more complex events are used to characterize groups of outcomes. The collection of all such events is a σ-algebra Σ. Finally, there is a need to specify each event's likelihood of happening. This is done using the probability measure function, $\mathbf{P} : \boldsymbol{\Sigma} \to [0,1].$

Remark I.2.3. Note that:

(i) In conventional quantum mechanics we dealing with a probabilities without any probability space $\Theta = \{\Omega, \Sigma, \mathbf{P}\}.$

(ii) However a wave function ψ in quantum mechanics is a description of the quantum state $|\psi\rangle$ of a quantum system Ξ. The wave function is a complex-valued probability amplitude, and the probabilities for the possible results of measurements of an observable $Q = Q_{\Xi}$ (represented by operator $\widehat{Q}$) made on the quantum system Ξ in state $|\psi\rangle$ can be derived from a wave function ψ.

(iii) From (ii) it follows that there exists an probability space $\Theta_{\Xi} = \{\Omega_{\Xi}, \Sigma_{\Xi}, \mathbf{P}_{\Xi}\}$ and random variable $Q_{\widehat{Q}|\psi\rangle} : \Omega_{\Xi} \to E_{\Xi}$, i.e. $X_{\widehat{Q}|\psi\rangle}$ is a measurable function from the set of possible outcomes Ω_{Ξ} to some set E_{Ξ}.

Example. We now, consider as an example, the simple case of a non-relativistic single particle, without spin, in one spatial dimension. Note that:

(i) The state of such a particle is completely described by its position-space wave function, $\psi(x)$ where x is the position of a particle. This is a complex-valued function of real variable x. For one spinless particle in $1D$, if the wave function is interpreted as a probability amplitude, the square modulus of the wave function, the positive real number

$$|\psi(x)|^2 = \psi^*(x)\psi(x) = \rho(x) \tag{2.1}$$

is interpreted as the probability density that the particle is at x.

(ii) If the particle position is measured, its location cannot be determined from the wave function, but is described by a probability distribution. The probability that its position x will be in the interval $a \leq x \leq b$ is the integral of the density over this interval:

$$P(a \leq x \leq b) = \int_a^b |\psi(x)|^2 dx. \tag{2.2}$$

This leads to the normalization condition

$$\int_{-\infty}^{\infty} |\psi(x)|^2 dx = 1 \tag{2.3}$$

because if the particle is measured, there is 100% probability that it will be somewhere.

(iii) Assume that particle is in state $|\psi\rangle$. From a statement (ii) it follows that the coordinate x of the particle wave function, $\psi(x)$ under measurement by a measuring device is a random variable $x_\psi(\omega) \triangleq X_{\hat{x}|\psi\rangle}(\omega),\ X_{\hat{x}|\psi\rangle}\Omega_{|\psi\rangle} \to E_{|\psi\rangle}$ which is well defined on a probability space

$$\Theta_{|\psi\rangle} = \{\Omega_{|\psi\rangle}, \Sigma_{|\psi\rangle}, \mathbf{P}\}. \tag{2.4}$$

(iv) However in conventional quantum mechanics as mentioned above such probability space $\Theta_{|\psi\rangle} = \{\Omega_{|\psi\rangle}, \Sigma_{|\psi\rangle}, \mathbf{P}\}$ is missing.

Remark *I*.2.4. For a given system, the set of all possible normalizable wave functions (at any given time) forms an abstract mathematical vector space, meaning that it is possible to add together different wave functions, and multiply wave functions by complex numbers. Note that:

(i) Technically, because of the normalization condition, wave functions form a projective space $\mathbf{H}_p$ rather than an ordinary infinite-dimensional vector space $\mathbf{H}$. Also $\mathbf{H}$ is a Hilbert space, because the inner product of two wave functions ψ_1 and ψ_2 can be defined as the complex number

$$\langle\psi_1|\psi_2\rangle = \int_{-\infty}^{\infty} \overline{\psi}_1(x)\psi_2(x)dx. \tag{2.5}$$

(ii) $\mathbf{H}_p = \mathbf{S}^\infty \subsetneqq \mathbf{H}$.

(iii) The all values of the wave function $\psi(x)$ are components of a vector $|\psi\rangle$. There are uncountably infinitely many of them and integration is used in place of summation. In Bra-ket notation, this vector is written

$$|\psi\rangle = \int_{-\infty}^{\infty} dx\psi(x)|x\rangle, \qquad (2.6)$$

where $\langle x'|x\rangle = \delta(x' - x)$.
Let us consider QM system which consists of one particle with a wave function $\psi(x)$, $x \in [a,b]$, such that supp $(\psi(x)) \subseteq [a,b]$ and $\int_{\mathbb{R}} |\psi(x)|^2 dx = 1$. We go to construct now corresponding probability space $\Theta_{|\psi\rangle} = \{\Omega_{|\psi\rangle}, \Sigma_{|\psi\rangle}, \mathbf{P}\}$. In one dimension, the position x of such a particle can range over the values $a \leq x \leq b$. Consider now the measurement of coordinate of such QM particle. Obviously a sample space for such a coordinate measurement is $\Omega_{|\psi\rangle} = \Omega_{a,b} = [a,b]$. Note that in practice observable x is measured to an accuracy δx determined by the measuring device. Thus

$$\forall x \forall \delta x_1 \forall \delta x_2 [(x - \delta x_1, x + \delta x_2) \subseteq [a,b] \Rightarrow (x - \delta x_1, x + \delta x_2) \in \Sigma_{|\psi\rangle}]$$

and therefore σ-algebra $\Sigma_{a,b} = B([a,b])$ is the Borel algebra on the set $[a,b]$. Let B^{∞} be the Borel algebra $B^{\infty} = \cup_{a<b} \Sigma_{a,b}$, we choose the probability measure $\mathbf{P}_{B^{\infty}} : B^{\infty} \to [0,1]$ of the form

$$\mathbf{P}_{B^{\infty}}(A) = \int_A \sigma(x) d\mu(x), \qquad (2.7)$$

where $A \in B^{\infty}, \int_{\mathbb{R}} \sigma(x) d\mu(x) = 1$ and $d\mu(x)$ is the Lebesgue measure.
Definition *I*.2.3. We choose the probability measure $\mathbf{P}_{|\psi\rangle} : B^{\infty} \to [0,1]$ corresponding to a wave function $\psi(x), \|\psi(x)\|_2^2 = 1$, in the following form:

$$\mathbf{P}_{|\psi\rangle}(A) = \int_A |\psi(x)|^2 d\mu(x), \qquad (2.8)$$

where $A \in B^\infty$ and $d\mu(x)$ is the Lebesgue measure.

Definition *I*.2.4. A random variable $X_{|\psi\rangle} : \Omega_{|\psi\rangle} \to E_{|\psi\rangle}$ is a measurable function from the set of possible outcomes Ω to some set $E_{|\psi\rangle}$. The technical axiomatic definition requires $\Omega_{|\psi\rangle}$ to be a probability space and $E_{|\psi\rangle}$ to be a measurable space. Note that although $X_{|\psi\rangle}$ is usually a real-valued function $X_{|\psi\rangle} : \Omega_{|\psi\rangle} \to [a,b]$, it does not return a probability. The probabilities of different outcomes or sets of outcomes (events) in our case are already given by the probability measure $\mathbf{P}_{|\psi\rangle}$ with which $\Omega_{|\psi\rangle}$ is equipped above.

Definition *I*.2.5. (Real-valued random variables.) In a case mentioned above the observation space is a set $[a,b]$. Recall, $\{\mathbb{R}, B^\infty, \mathbf{P}_{B^\infty}\}$ is the probability space. For real observation space, the function $X_{|\psi\rangle} : \Omega_{a,b} \to [a,b]$ is a real-valued random variable, i.e. $\forall a \forall b (a < b)[\{\omega | a \leq X_{|\psi\rangle}(\omega) \leq b\} \in \Sigma_{a,b}]$.

Definition *I*.2.6. Let $|\psi\rangle \in \mathbf{H}$. We define now a signed measure $\mathbf{P}_{|\psi_\mathbf{A}\rangle} : B^\infty \to \mathbb{R}$ by formula

$$\mathbf{P}_{|\psi_\mathbf{A}\rangle}(A) = \int_A x p_{|\psi_\mathbf{A}\rangle}(x) d\mu(x), \qquad (2.9)$$

where $p_{|\psi_\mathbf{A}\rangle}(x) = |\langle x | \psi_\mathbf{A}\rangle|^2$.

Remark *I*.2.5. We assume now that $(\Omega, \mathcal{F}, \mathbf{P}) = (\mathbb{R}, B^\infty, \mathbf{P}_{B^\infty})$ and $\mathbf{P}_{|\psi\rangle} \ll \mathbf{P}_{\mathbf{B}^\infty}$, i.e. $\mathbf{P}_{|\psi\rangle}$ is absolutely continuous with respect to $\mathbf{P}$. By Radon-Nicodym theorem there exists a random variable $X_{|\psi\rangle}(\omega) : \Omega \to \mathbb{R}$ such that for any $A \in B^\infty$:

$$\mathbf{P}_{|\psi\rangle}(A) = \int_A X_{|\psi\rangle}(\omega) d\mathbf{P}_{B^\infty}(\omega), \qquad (2.10)$$

Using Eq. (2.10) we define random variable $X_{|\psi\rangle} : \Omega_{|\psi\rangle} \to \mathbb{R}$ by formula

$$X_{|\psi\rangle}(\omega) = \frac{d\mathbf{P}_{|\psi\rangle}}{d\mathbf{P}_{B^\infty}}. \qquad (2.11)$$

Definition *I*.2.7. The cumulative distribution function of a real-valued random variable $X_{|\psi\rangle}(\omega)$ is the function given by

$$F_{X_{|\psi\rangle}}(x) = \mathbf{P}(\omega \in B^\infty | X_{|\psi\rangle}(\omega) \leq x), \qquad (2.12)$$

where the right-hand side represents the probability that the random variable $X_{|\psi\rangle}(\omega)$ takes on a value less than or equal to x. The probability that $X_{|\psi\rangle}(\omega)$ lies in the semi-closed interval $(a_1, b_1] \subset [a, b]$, where $a_1 < b_1$, is therefore $\mathbf{P}_{|\psi\rangle}(a_1 < X_{|\psi\rangle} \leq b_1) = F_{X_{|\psi\rangle}}(b_1) - F_{X_{|\psi\rangle}}(a_1)$.
The CDF of any continuous random variable $X_{|\psi\rangle} = X_{|\psi\rangle}(\omega)$ can be expressed as the integral of its probability density function $p_{X_{|\psi\rangle}}(x)$ as follows:

$$F_{X_{|\psi\rangle}}(x) = \int_{-\infty}^{x} p_{X_{|\psi\rangle}}(t)dt = \int_{-\infty}^{x} |\psi(t)|^2 dt. \qquad (2.13)$$

From Eq. (2.9) - Eq. (2.10) we obtain

$$\mathbf{E}[X_{|\psi\rangle}(\omega)] = \int_{\Omega_{a,b}} X_{|\psi\rangle}(\omega) d\mathbf{P} = \int_{-\infty}^{+\infty} x p_{X_{|\psi\rangle}}(x)dx. \qquad (2.14)$$

Using canonical QM-abbreviation

$$|\psi\rangle = \int_{-\infty}^{+\infty} |x\rangle\langle x|\psi\rangle dx, \qquad (2.15)$$

where $\langle x|\psi\rangle = \psi(x), |\psi\rangle \in \mathbf{S}^{\infty} \subsetneq \mathbf{H},$ from Eq. (2.14) - Eq. (2.15) we obtain

$$\langle\psi|\widehat{x}|\psi\rangle = \int\limits_{\mathbb{R}} X_{|\psi\rangle}(\omega)d\mathbf{P} = \int\limits_{-\infty}^{+\infty} xp_{X_{|\psi\rangle}}(x)dx. \tag{2.16}$$

where $\widehat{x}|\psi\rangle = x|\psi\rangle$.

Remark *I*.2.6. We assume now that:

(i) for any $\psi(x) \in \mathbf{H}$: (a) $\int\limits_{\mathbb{R}} |X_{|\psi\rangle}|(\omega)d\mathbf{P} < \infty,$ (b) $\int\limits_{\mathbb{R}} X^2_{|\psi\rangle}(\omega)d\mathbf{P} < \infty,$

(ii) for any $\psi(x) \in \mathbf{H}$: $X_{|\psi\rangle} \in \mathcal{L}_{1,2}(d\mathbf{P}) = \mathcal{L}_1(d\mathbf{P}) \cap \mathcal{L}_2(d\mathbf{P}).$

Definition *I*.2.8. We will write the Eq. (1.16) in the following form

$$\langle\psi|\widehat{x}|\psi\rangle = \int\limits_{\mathbb{R}} X_{\widehat{x}|\psi\rangle}(\omega)d\mathbf{P} = \int\limits_{-\infty}^{+\infty} xp_{X_{\widehat{x}|\psi\rangle}}(x)dx, \tag{2.17}$$

where $\widehat{x}|\psi\rangle = x|\psi\rangle$. This form remind that continuous random variable $X_{|\widehat{x}\psi\rangle} = X_{|\widehat{x}\psi\rangle}(\omega)$ corresponds to the coordinate of a particle with a state vector $|\psi\rangle$.

Remark *I*.2.7. We assume that particle A is initially in the state $|\psi_{\mathbf{A}}\rangle \in \mathbf{H}$. We assume now that: if on performing a measurement of $\widehat{x}$ on particle $\mathbf{A}$ with an accuracy $\delta x,$ and the result is obtained in the range $(x_{\mathbf{A}} - \delta x, x_{\mathbf{A}} + \delta x)$ at instant $t,$ then unconditional measure $\mathbf{P}_{B^{\infty}}$ immediately after the measurement at instant t collapses to conditional measure $\mathbf{P}_{\Sigma_{|\psi_{\mathbf{A}}\rangle}}(X|\Sigma_{|\psi_{\mathbf{A}}\rangle}(x_{\mathbf{A}},\delta x)),$ where $X \in B^{\infty}$:

$$\mathbf{P}_{\Sigma_{|\psi_{\mathbf{A}}\rangle}}\left(X \mid \Sigma_{|\psi_{\mathbf{A}}\rangle}(x_{\mathbf{A}},\delta x)\right) = \frac{\mathbf{P}_{B^{\infty}}\left(X \wedge \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x)\right)}{\mathbf{P}_{B^{\infty}}\left(\Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x)\right)}, \tag{2.18}$$

and where $\Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x) \triangleq \{\omega | x_{\mathbf{A}} - \delta x \leq X_{|\psi_{\mathbf{A}}\rangle}(\omega) \leq x_{\mathbf{A}} + \delta x\}$.

Remark *I*.2.8. (i) From Eq. (2.18) it follows that unconditional probability density function $p_{\mathbf{A}}(x) = |\langle x|\psi_{\mathbf{A}}\rangle|^2$ immediately after the measurement at instant t collapses to the following conditional probability density function

$$p_{\mathbf{A}}\left(x|\Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x)\right) = \begin{cases} \dfrac{p_{\mathbf{A}}(x)}{\mathbf{P}_{B^{\infty}}\left(\Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x)\right)} & \Leftrightarrow \ x \in \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x) \\ 0 & \Leftrightarrow \ x \notin \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x) \end{cases} \quad (2.19)$$

(ii) From Eq. (2.18) it follows that immediately after the measurement on particle $\mathbf{B}$ at instant t a wave function $\psi_{\mathbf{A}}(x) = \langle x|\psi_{\mathbf{A}}\rangle$ collapses to the following wave function

$$\psi_{\mathbf{A}}^{\mathbf{coll}}(x) = \begin{cases} \dfrac{\psi_{\mathbf{A}}(x)}{\sqrt{\mathbf{P}_{B^{\infty}}\left(\Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x)\right)}} & \Leftrightarrow \ x \in \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x) \\ 0 & \Leftrightarrow \ x \notin \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x). \end{cases} \quad (2.20)$$

Remark *I*.2.9. Note that in contrast with the usual 'Copenhagen' interpretation in quantum mechanical formalism based on the probability representation of quantum states, coordinate of a particle with a state vector $|\psi\rangle$ that continuous random variable $X_{|\hat{x}\psi\rangle} = X_{|\hat{x}\psi\rangle}(\omega)$ and such a random variable in contrast with a state vector $|\psi\rangle$ does not collapses under the measurement. However unconditional measure $\mathbf{P}_{B^{\infty}}$ immediately after the measurement at instant t collapses to conditional measure $\mathbf{P}_{B^{\infty}}(X|\Sigma_{|\psi_{\mathbf{A}}\rangle}(x_{\mathbf{A}},\delta x))$ (is given by Eq.(2.18)) on whole probability space $\Theta = \{\Omega, B^{\infty}, \mathbf{P}_{B^{\infty}}\}$.

Remark *I*.2.10. Thus immediately after measurement on any particle $\mathbf{A}$ probability space $\Theta = \{\Omega, B^{\infty}, \mathbf{P}_{B^{\infty}}\}$ collapses to probability space $\Theta_{\Sigma_{|\psi_{\mathbf{A}}\rangle}} = \left\{\Omega, B^{\infty}, \mathbf{P}_{B^{\infty}}\left(X \,\middle|\Sigma_{|\psi_{\mathbf{A}}\rangle}(x_{\mathbf{A}},\delta x)\right)\right\}$

$$\{\Omega, B^{\infty}, \mathbf{P}_{B^{\infty}}\} \overset{\textbf{collaps}}{\Rightarrow} \left\{\Omega, B^{\infty}, \mathbf{P}_{\Sigma_{|\psi_{\mathbf{A}}\rangle}}\left(X \left|\Sigma_{|\psi_{\mathbf{A}}\rangle}(x_{\mathbf{A}}, \delta x)\right)\right\}. \quad (2.21)$$

I.3. EPR Paradox resolution by using Postulate of EPR-Nonlocality

In the 20s and 30s it became evident that some properties in quantum mechanics can be assigned only to the quantum mechanical system, but not necessarily to its constituents. This led Einstein, Podolsky and Rosen (EPR) to their remarkable 1935 paper where they concluded that quantum mechanics is not a complete theory of nature (EPR paradox).
The conclusion was derived from some common sense requirements that EPR postulated:

1. Completeness: Each element of realism should have its correspondence in a theory.
2. Realism: If a property can be assigned to a physical system with certainty then there exists an element of realism that corresponds to this property.
3. Locality: Measurements of different elements of realism in spatially separated systems can not influence each other.

In original paper [13], Einstein, Podolsky and Rosen describe two particles A and B with perfectly correlated position

$$x_{\mathbf{B}} = x_{\mathbf{A}} + x_0 \quad (3.1)$$

and perfectly anti-correlated momentum

$$p_{\mathbf{B}} = -p_{\mathbf{A}}, \quad (3.2)$$

see Fig. *I*.3.1.

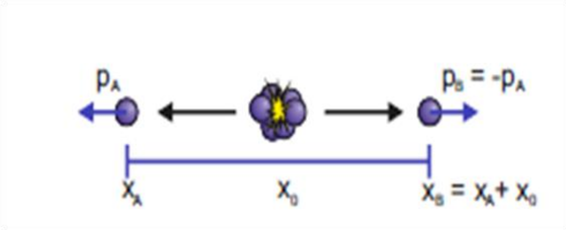

Fig. I.3.1. Particles A and B with correlated position and anti-correlated momentum

EPR originally argued as follows. Consider two spatially separated subsystems at A and B. EPR considered two observables $\hat{x}$ (the position) and $\hat{p}$ (momentum) for subsystem A, where $\hat{x}$ and $\hat{p}$ do not commute, so that $C \neq 0$

$$[\hat{x}, \hat{p}] = 2C. \tag{3.3}$$

Suppose now that one may predict with certainty the result of measurement $\hat{x}$ based on the result of a measurement performed at B. Also, for a different choice of measurement at B, suppose one may predict the result of measurement $\hat{p}$ at A. Such correlated systems are predicted by quantum theory. Assuming local realism EPR deduce the existence of an element of reality, $\tilde{x}$, for the physical quantity $\hat{x}$ and also an element of reality, $\tilde{p}$, for $\hat{p}$. Local realism implies the existence of two hidden variables $\tilde{x}$ and $\tilde{p}$ that simultaneously predetermine, with no uncertainty, the values for the result of an $\hat{x}$ or $\hat{p}$ measurement on subsystem A, should it be performed. This hidden variable state for the subsystem A alone is not describable within quantum mechanics, since simultaneous eigenstates of $\hat{x}$ and $\hat{p}$ do not exist. Hence, EPR argued, if quantum mechanics is to be compatible with local realism, we must regard quantum mechanics to be incomplete. In the idealized entangled state proposed by EPR,

$$|EPR\rangle = \int_{-\infty}^{\infty} |x, x\rangle dx = \int_{-\infty}^{\infty} |p, p\rangle dp \tag{3.4}$$

the positions and momenta of the two particles are perfectly correlated. Note that: this state is non-normalizable and cannot be realized in the laboratory. When coordinates x^A and p^A are measured in independent realizations of the same state, the correlations allow for an exact prediction of x^B and p^B. EPR assumed that such exact predictions necessitate an element of reality which predetermines the outcome of the measurement. Quantum mechanics however prohibits the exact knowledge of two noncommuting variables like x^B and p^B, since their measurement uncertainties are subject to the Heisenberg relation

$$\Delta x^B \Delta p^B \geq \hbar/2. \tag{3.5}$$

Remark *I*.3.1. A most critical component of the EPR argument was the principle of EPR-locality. Indeed, one may regard the EPR paradox as a statement of the mutual incompatibility of EPR-locality, entanglement, and completeness.

We accept now the following postulate:

The postulate of nonlocality for continuous variables.

The Heisenberg uncertainty relations

$$\Delta x^{\mathbf{A}} \Delta p^{\mathbf{A}} \geq 1 \tag{3.6}$$

cannot be violated in any cases:

(i) according to quantum mechanics, the Heisenberg uncertainty relations (1.3.4) cannot be violated if the coordinate $x^{\mathbf{A}}$ and momentum $p^{\mathbf{A}}$ of the particle $\mathbf{A}$ are measured directly by measurements performed on the particle $\mathbf{A}$,

(ii) the Heisenberg uncertainty relations (3.8) cannot be violated even if the coordinate $x^{\mathbf{A}}$ and momentum $p^{\mathbf{A}}$ of the particle $\mathbf{A}$ are measured indirectly, i.e. by using measurement on particle $\mathbf{B}$, as required in EPR gedanken experiment,

(iii) in any cases true coordinate $x^{\mathbf{A}}$ and momentum $p^{\mathbf{A}}$ of the particle **A** cannot be predicted simultaneously with a sufficiently small uncertainty $\Delta x^{\mathbf{A}}$ and $\Delta p^{\mathbf{A}}$ such that the Reid's inequality [14]:

$$\Delta x^{\mathbf{A}} \Delta p^{\mathbf{A}} < 1 \tag{3.7}$$

based on local realism cannot be satisfied, i.e., always

$$\Delta x^{\mathbf{A}} \Delta p^{\mathbf{A}} \nless 1. \tag{3.8}$$

Remark *I*.3.2. Obviously under postulate of nonlocality EPR paradox disappears. However postulate of nonlocality is supported by quantum mechanical formalism based on the probability representation of quantum states.

Using probability representation Eq. (2.11) of quantum states $|\psi_{\mathbf{A}}\rangle$ and $|\psi_{\mathbf{B}}\rangle$ from Eq. (3.4) we obtain

$$X_{|\psi_{\mathbf{B}}\rangle}(\omega) = X_{|\psi_{\mathbf{A}}\rangle}(\omega) + x_0, \text{a.s.} \tag{3.9}$$

We assume that particle A is initially in the state $|\psi_{\mathbf{A}}\rangle \in \mathbf{H}$. We assume now that a measurement of $\hat{x}$ performing on particle $\mathbf{B}$ with an accuracy $\delta x,$ and the result is obtained in the range $(x_{\mathbf{B}} - \delta x, x_{\mathbf{B}} + \delta x)$ at instant t. Then from Eq.(3.9) we obtain

$$\begin{gathered}\{\omega | x_{\mathbf{B}} - \delta x \leq X_{|\psi_{\mathbf{B}}\rangle}(\omega) \leq x_{\mathbf{B}} + \delta x\} = \\ \{\omega | (x_{\mathbf{B}} - x_0) - \delta x \leq X_{|\psi_{\mathbf{B}}\rangle}(\omega) - x_0 \leq (x_{\mathbf{B}} - x_0) + \delta x\} = \\ \{\omega | x_{\mathbf{A}} - \delta x \leq X_{|\psi_{\mathbf{A}}\rangle}(\omega) \leq x_{\mathbf{A}} + \delta x\}.\end{gathered} \tag{3.10}$$

We let now for short

$$\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x) \triangleq \{\omega | x_{\mathbf{B}} - \delta x \leq X_{|\psi_{\mathbf{B}}\rangle}(\omega) \leq x_{\mathbf{B}} + \delta x\},$$
$$\Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x) \triangleq \{\omega | x_{\mathbf{A}} - \delta x \leq X_{|\psi_{\mathbf{A}}\rangle}(\omega) \leq x_{\mathbf{A}} + \delta x\}. \quad (3.11)$$

Then from Eq. (3.10) - Eq. (3.11) we obtain

$$\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x) = \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x). \quad (3.12)$$

Since event $\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x) \in \Omega$ was occurred by performing a measurement on particle $\mathbf{B}$, then unconditional measure $\mathbf{P}_{B^{\infty}}$ immediately after measurement at instant t collapses to conditional measure $\mathbf{P}_{\Sigma_{|\psi_{\mathbf{B}}\rangle}}(X | \Sigma_{|\psi_{\mathbf{B}}\rangle}(x_{\mathbf{B}},\delta x))$, where $X \in B^{\infty}$ and therefore from Eq.(2.18) we obtain

$$\mathbf{P}_{\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}}\left(\Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x) \,\middle|\, \Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x)\right) = \frac{\mathbf{P}_{B^{\infty}}\left(\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x) \cap \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x)\right)}{\mathbf{P}_{B^{\infty}}\left(\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x)\right)} =$$
$$\frac{\mathbf{P}_{B^{\infty}}\left(\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x) \cap \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x)\right)}{\mathbf{P}_{B^{\infty}}\left(\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x)\right)} = \frac{\mathbf{P}_{B^{\infty}}\left(\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x)\right)}{\mathbf{P}_{B^{\infty}}\left(\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x)\right)} = 1. \quad (3.13)$$

Therefore since event $\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x) \in \Omega$ was occurred by performing a measurement on particle $\mathbf{B}$ immediately after measurement event $\Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x)$ occurs with a probability = 1.
From Eq. (3.13) it follows that unconditional probability density function $p_{\mathbf{A}}(x) = |\langle x|\psi_{\mathbf{A}}\rangle|^2$ immediately after measurement at instant t collapses to the following conditional probability density function

$$p_{\mathbf{A}}\left(x | \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x)\right) = \begin{cases} \dfrac{p_{\mathbf{A}}(x)}{\mathbf{P}_{B^{\infty}}\left(\Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x)\right)} & \Leftrightarrow \quad x \in \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x) \\ 0 & \Leftrightarrow \quad x \notin \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x) \end{cases} \quad (3.14)$$

a wave function $\psi_{\mathbf{A}}(x) = \langle x|\psi_{\mathbf{A}}\rangle$ collapses to the following wave function

$$\psi_{\mathbf{A}}^{\mathbf{coll}}(x) = \begin{cases} \dfrac{\psi_{\mathbf{A}}(x)}{\sqrt{\mathbf{P}_{B^{\infty}}\left(\Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x)\right)}} & \Leftrightarrow \quad x \in \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x) \\ 0 & \Leftrightarrow \quad x \notin \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x). \end{cases} \tag{3.15}$$

From Theorem C.1, (see Appendix C) it follows that a wave function $\psi_{\mathbf{A}}(p) = \langle p|\psi_{\mathbf{A}}\rangle$ collapses in accordance with Heisenberg uncertainty relations (3.6).

I.4. EPR-B Paradox resolution and Postulate of EPR-B-Nonlocality

Entangled states are very interesting states because they exhibit correlations that have no classical analog. They are of particular importance in quantum computation and quantum information. As an example let us take the entangled bi-partite pure state:

$$\psi_1 = \tfrac{1}{\sqrt{2}}(|01\rangle_{AB} + |10\rangle_{AB}) \in H_A \otimes H_B. \tag{4.1}$$

Obviously this state can not be decomposed as a simple product state $|q\rangle_A|p\rangle_B, q,p \in \{0,1\}$.

Remark *I*.4.1. Note that ψ_1 mentioned above is one of the so-called four Bell states:

$$\begin{aligned} \psi_1 &= \tfrac{1}{\sqrt{2}}(|01\rangle_{AB} + |10\rangle_{AB}), \psi_2 = \tfrac{1}{\sqrt{2}}(|01\rangle_{AB} - |10\rangle_{AB}), \\ \psi_3 &= \tfrac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB}), \psi_4 = \tfrac{1}{\sqrt{2}}(|00\rangle_{AB} - |11\rangle_{AB}). \end{aligned} \tag{4.2}$$

They form a convenient basis of bi-partite quantum states of two-dimensional Hilbert spaces. The state is maximally entangled, i.e. when we trace over the state B then the reduced density operator ρ_A of the system will be a multiple of the identity operator. This means that if we measure in system A in any basis the result will be completely random (0 or 1 with equal probability $1/2$).

Remark *I*.4.2. However, there is a perfect correlation: Whenever we measure with certainty 1 in system A then we will measure 0 in system B with certainty and vice versa.

Remark *I*.4.3. However, despite the randomness, the choice of basis for measurement in system A clearly has a nonlocal effect on the state of the system B: it gives it a definite orientation in the basis $\{|0\rangle_B, |1\rangle_B\}$, which it did not have before the measurement.

Remark *I*.4.4. Obviously the process described above is nonlocal: the state changes instantly even though the systems A and B could be space-like separated. We are accustomed to saying that this sort of instantaneous action at a distance is forbidden by canonical relativity.

Remark *I*.4.5. In order to avoid the contradiction which arises from instantaneous action at a distance mentioned above we introduce an extension of the canonical relativity by using measure algebra of physical events in Minkowski space-time, see Chapter III.2.

Definition *I*.4.1. A measure algebra $\mathcal{F} = (\mathbf{B}, \mathbf{P})$ with a probability measure $\mathbf{P}$, is a Boolean algebra $\mathbf{B}$ with a countably additive probability measure.

Definition *I*.4.2. (i) A measure algebra of physical events $\mathcal{F}^{ph} = (\mathbf{B}, \mathbf{P})$ with a probability measure $\mathbf{P}$, is a Boolean algebra of physical events $\mathbf{B}$ with an countably additive probability measure.

(ii) A Boolean algebra of physical events can be formally defined as a set $\mathbf{B}$ of elements $a, b, \ldots$ with the following properties:

1. $\mathbf{B}$ has two binary operations, $\wedge$ (logical AND, or "wedge") and $\vee$ (logical OR, or "vee"), which satisfy:

the idempotent laws: (1) $a \wedge a = a \vee a = a,$
the commutative laws: (2) $a \wedge b = b \wedge a,$ (3) $a \vee b = b \vee a,$
and the associative laws: (4) $a \wedge (b \wedge c) = (a \wedge b) \wedge c,$
(5) $a \vee (b \vee c) = (a \vee b) \vee c.$

2. The operations satisfy the absorption law:
(6) $a \wedge (a \vee b) = a \vee (a \wedge b) = a.$

3. The operations are mutually distributive
(7) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c),$ (8) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c).$

4. $\mathbf{B}$ contains universal bounds $\mathbf{0}$ and $\mathbf{1}$ which satisfy

(9) $\mathbf{0} \wedge a = \mathbf{0},$ (10) $\mathbf{0} \vee a = a,$ (11) $\mathbf{1} \wedge a = a,$ (12) $\mathbf{1} \vee a = \mathbf{1}.$

5. $\mathbf{B}$ has a unary operation $\neg a$ (or a') of complementation (logical negation), which obeys the laws:
(13) $a \wedge \neg a = \mathbf{0},$ (14) $a \vee \neg a = \mathbf{1}$.
All properties of negation including the laws below follow from the above two laws alone.

6. Double negation law: $\neg(\neg a) = a$.

7. De Morgan's laws: (i) $\neg a \wedge \neg b = \neg(a \vee b),$ (ii) $\neg a \vee \neg b = \neg(a \wedge b).$

8. Operations composed from the basic operations include the following important examples:
The first operation, $a \to b$ (logical material implication):
(i) $a \to b \triangleq \neg a \vee b.$
The second operation, $a \oplus b,$ is called exclusive. It excludes the possibility of both a and b
(ii) $a \oplus b \triangleq (a \vee b) \wedge \neg(a \wedge b).$
The third operation, the complement of exclusive or, is equivalence or Boolean equality:
(iii) $a \equiv b \triangleq \neg(a \oplus b)$.

9. $\mathbf{B}$ has a unary predicate $\mathbf{Occ}(a)$, which meant that event a has occurred, and which obeys the laws:
(i) $\mathbf{Occ}(a \wedge b) \Leftrightarrow \mathbf{Occ}(a) \wedge \mathbf{Occ}(b),$ (ii) $\mathbf{Occ}(a \vee b) \Leftrightarrow \mathbf{Occ}(a) \vee \mathbf{Occ}(b),$
(iii) $\mathbf{Occ}(\neg a) \Leftrightarrow \neg\mathbf{Occ}(a).$

Definition I.4.3. (i) Let $\mathbf{B}$ be a Boolean algebra of physical events. A Boolean algebra $\mathbf{B}_{M_4}$ of physical events in Minkowski spacetime $M_4 = \mathbb{R}^{1,3}$ that is cartesian product $\mathbf{B}_{M_4} = \mathbf{B} \times M_4$.
(ii) Let $\mathbf{B}_{M_4}$ be a Boolean algebra of physical events in Minkowski spacetime. A measure algebra of physical events $\mathcal{F}_{M_4}^{ph} = (\mathbf{B}_{M_4}, \mathbf{P})$ in Minkowski spacetime that is a Boolean algebra $\mathbf{B}_{M_4}$ with a probability measure $\mathbf{P}$ such that

$$\forall A\left(A \in \mathcal{F}_{M_4}^{ph}\right)\forall B\left(B \in \mathcal{F}_{M_4}^{ph}\right)[A \equiv B \Rightarrow \mathbf{P}(A) = \mathbf{P}(B)],$$
$$\forall A[\mathbf{P}(A) = \mathbf{P}(A^{Oc})]. \tag{4.3}$$

(iii) Let $\mathbf{B}_{M_4}$ be Boolean algebra of the all physical events in Minkowski spacetime and let $\mathcal{F}_{M_4}^{ph}$ be an measure algebra $\mathcal{F}_{M_4}^{ph} = (\mathbf{B}_{M_4}, \mathbf{P})$ with a probability measure $\mathbf{P}$. We denote such physical events by $A(\mathbf{x}), B(\mathbf{x}), \ldots$ etc., where $\mathbf{x} = (t, x_1, x_2, x_3) \in M_4$ or $A, B, \ldots$ etc.
(iv) We will write for a short $A^{Oc}(\mathbf{x}), B^{Oc}(\mathbf{x}), \ldots$ etc., instead $\mathbf{Occ}(A(\mathbf{x})), \mathbf{Occ}(B(\mathbf{x})), \ldots$ etc.
Remark I.4.6. Note that Boolean algebra $\mathbf{B}_{M_4}$ of physical events in Minkowski spacetime obviously contains a pairs $\{A(\mathbf{x}_1), B(\mathbf{x}_2)\}$ of a Boolean equivalent events $A(\mathbf{x}_1)$ and $B(\mathbf{x}_2)$ such that

$$\forall \mathbf{x}_1 \forall \mathbf{x}_2 [A(\mathbf{x}_1) \equiv B(\mathbf{x}_2)], \tag{4.4}$$

i.e., Boolean equality $A(\mathbf{x}_1) \equiv B(\mathbf{x}_2)$ always holds even the events $A(\mathbf{x}_1)$ and $B(\mathbf{x}_2)$ are space-like separated.

Definition I.4.4. A probability measure $\mathbf{P}$ on a measure space (Ω, Σ) gives a probability measure algebra $\mathcal{F} = (\Omega, \Sigma, \mathbf{P}_\Sigma)$ on the Boolean algebra of measurable sets modulo null sets, i.e., sets $\mathbf{P}_\Sigma$ measure zero.
Remark I.4.7. We assume now that a measure algebra

$\mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4}, \mathbf{P})$ admit a representation $\Re[\cdot] : \mathcal{F}^{ph}_{M_4} \to (\Omega, \Sigma, \mathbf{P}_\Sigma)$ of the measure algebra $\mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4}, \mathbf{P})$ in a probability measure algebra $\mathcal{F} = (\Omega, \Sigma, \mathbf{P}_\Sigma)$, such that (ii) $\mathbf{P}_\Sigma(X) = \mathbf{P}(\Re^{-1}[X])$ for any $X \in \Sigma$.

Definition *I*.4.5. Given two events $A(\mathbf{x}_1)$ and $B(\mathbf{x}_2)$ from the measure algebra $\mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4}, \mathbf{P})$ the conditional probability of $A(\mathbf{x}_1)$ given $B(\mathbf{x}_2)$ is defined as the quotient of the probability of the joint of events $A(\mathbf{x}_1)$ and $B(\mathbf{x}_2)$, and the probability of B :

$$\mathbf{P}(A(\mathbf{x}_1)|B(\mathbf{x}_2)) = \frac{\mathbf{P}(A(\mathbf{x}_1) \wedge B(\mathbf{x}_2))}{\mathbf{P}(B(\mathbf{x}_2))} \tag{4.5}$$

where $\mathbf{P}(B(\mathbf{x}_2)) \neq 0$.

Remark *I*.4.8. Assume that: (i) $B^{Oc}(\mathbf{x}_2)$, then since event $B(\mathbf{x}_2)$ is occurred, unconditional probability measure $\mathbf{P}$ on algebra $\mathbf{B}_{M_4}$ collapses to the conditional probability measure

$$\mathbf{P}_{B^{Oc}}(A(\mathbf{x}_1)) \triangleq \mathbf{P}(A(\mathbf{x}_1)|B^{Oc}(\mathbf{x}_2)) = \mathbf{P}(A(\mathbf{x}_1)|B(\mathbf{x}_2)) = \frac{\mathbf{P}(A(\mathbf{x}_1) \wedge B(\mathbf{x}_2))}{\mathbf{P}(B(\mathbf{x}_2))} \tag{4.6}$$

and measure algebra $\mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4}, \mathbf{P})$ collapses to the measure algebra $\mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4}, \mathbf{P}_{B^{Oc}})$.

(ii) $B^{Oc}(\mathbf{x}_2)$ and $A(\mathbf{x}_1) \equiv B(\mathbf{x}_2)$, then

$$\mathbf{P}_{B^{Oc}}(A(\mathbf{x}_1)) = \mathbf{P}(A(\mathbf{x}_1)|B^{Oc}(\mathbf{x}_2)) = \frac{\mathbf{P}(A(\mathbf{x}_1) \wedge B(\mathbf{x}_2))}{\mathbf{P}(B(\mathbf{x}_2))} = \frac{\mathbf{P}(B(\mathbf{x}_2))}{\mathbf{P}(B(\mathbf{x}_2))} = 1, \tag{4.7}$$

i.e., $\mathbf{P}_{B^{Oc}}(A(\mathbf{x}_1)) = 1$ and therefore, since event $B(\mathbf{x}_2)$ is occurred in point $\mathbf{x}_2$, the event $A(\mathbf{x}_1)$ is occurred with probability 1 in point $\mathbf{x}_2$ even points $\mathbf{x}_1$ and $\mathbf{x}_2$ are space-like separated.

Remind that Bohm [15] considered two spatially-separated spin-$1/2$ particles at A and B produced in an entangled singlet state (often referred to as the EPR-Bohm state or the Bell-state):

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\left|\tfrac{1}{2}\right\rangle_A\left|-\tfrac{1}{2}\right\rangle_B - \left|-\tfrac{1}{2}\right\rangle_A\left|\tfrac{1}{2}\right\rangle_B\right) \tag{4.8}$$

Here $|\pm\frac{1}{2}\rangle_A$ are eigenstates of the spin operator $\widehat{J}_z^A$, and we use $\widehat{J}_z^A$, $\widehat{J}_x^A$, $\widehat{J}_y^A$ to define the spin-components measured at location A. The spin-eigenstates and measurements at B are defined similarly. By considering different quantization axes, one obtains different but equivalent expansions of $|\psi\rangle$ in Eq. (4.4), just as EPR suggested.

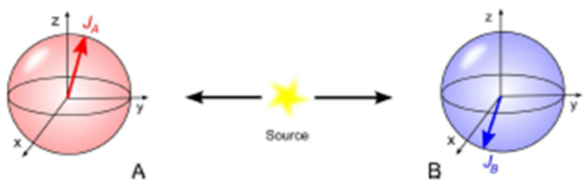

Fig. I.4.1. The Bohm gedanken EPR experiment. Two spin-1/2 particles prepared in a singlet state from the source into spatially separated regions A and B, and give anti-correlated outcomes for J_θ^A and J^B, where θ is x, y or z

Bohm's paradox is based on the existence, for Eq. (4.1), of a maximum anti-correlation between not only $\widehat{J}_z^A$ and $\widehat{J}_z^B$, but $\widehat{J}_y^A$ and $\widehat{J}_y^B$, and also $\widehat{J}_x^A$ and $\widehat{J}_x^B$. An assumption of local realism would lead to the conclusion that the three spin components of particle A were simultaneously predetermined, with absolute definiteness. Since no such quantum description exists, this is the situation of an EPR paradox.
We accept now the following postulate:

The postulate of nonlocality for observables with discrete values.
The Heisenberg spin uncertainty relations

$$\Delta J_x^{\mathbf{A}} \Delta J_y^{\mathbf{A}} \geq \left|\langle J_z^{\mathbf{A}} \rangle\right|/2, \Delta J_x^{\mathbf{A}} \Delta J_z^{\mathbf{A}} \geq \left|\langle J_y^{\mathbf{A}} \rangle\right|/2, \Delta J_z^{\mathbf{A}} \Delta J_y^{\mathbf{A}} \geq \left|\langle J_x^{\mathbf{A}} \rangle\right|/2 \qquad (4.9)$$

cannot be violated in any cases:

(i) if the three spin components of the particle **A** are measured directly by measurements performed on the particle **A,**

(ii) and even if some spin components of the particle **A** are measured indirectly as required in Bohm gedanken EPR experiment.

Remark *I*.4.11. Obviously under postulate of nonlocality EPR paradox disappears.

However postulate of nonlocality supported by quantum mechanical formalism based on the probability representation of quantum states, see Chapter V.3.

Chapter I

THE POSTULATE OF EPR-B NONLOCALITY

I.1. The EPR paradox

In 1935, Einstein, Podolsky and Rosen (EPR) originated the famous EPR paradox [6]. This argument concerns two spatially separated particles which have both perfectly correlated positions and momenta, as is predicted possible by quantum mechanics. The EPR paper spurred investigations into the nonlocality of quantum mechanics, leading to a direct challenge of the philosophies taken for granted by most physicists. The EPR conclusion was based on the assumption of local realism, and thus the EPR argument pinpoints a contradiction between local realism and the completeness of quantum mechanics.

I.2. Einstein's 1927 gedanken experiment

Einstein never accepted orthodox quantum mechanics because he did not believe that its nonlocal collapse of the wave function could be real. When he first made this argument in 1927 [7], he considered just a single particle. The particle's wave function was diffracted through a tiny hole so that it `dispersed' over a large hemispherical area before encountering a screen of that shape covered in photographic film. Since the film only ever registers the particle at one point on the screen, orthodox quantum mechanics must postulate a `peculiar mechanism of action at a distance, which prevents the wave... from producing an action in two places on the screen’. That is, according to the theory, the detection at one point must instantaneously collapse the wave function to nothing at all other points.

Remark 1.2.1. It was only in 2010, nearly a century after Einstein's original proposal, that a scheme to rigorously test Einstein's `spooky action at a distance' [7], [8] using a single particle (a photon), as in his original conception, was conceived [9]. In this scheme, Einstein's 1927 gedanken experiment is simplified so that the single photon is split into just two wave packets, one sent to a laboratory supervised by Alice and the other to a distant laboratory supervised by Bob. However, there is a key difference, which enables demonstration of the nonlocal collapse experimentally: rather than simply detecting the presence or absence of the photon, homodyne detection is used. This gives Alice the power to make different measurements, and enables Bob to test (using tomography) whether Alice's measurement choice affects the way his conditioned state collapses, without having to trust anything outside his own laboratory.

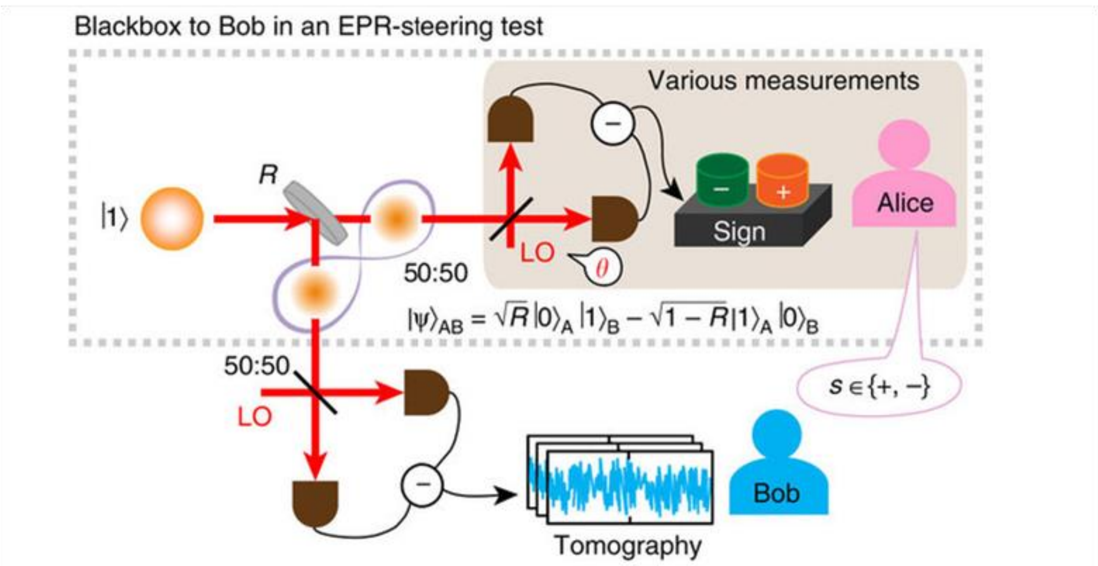

Fig. 1.2.1. Simplified version of Einstein's original gedanken experiment. Adapted from [10]

A single photon is incident on a beam splitter of reflectivity R and then subjected to homodyne measurements at two spatially separated locations.

Alice is trying to convince Bob that she can steer his portion of the single photon to different types of local quantum states by performing various measurements on her side. She does this by using different values of her θ LO phase, and extracting only the sign $s \in \{+,-\}$ of the quadrature she measures. Meanwhile, Bob scans his LO and performs full quantum-state tomography to reconstruct his local quantum state. He reconstructs unconditional and conditional local quantum states to test if his portion of the single photon has collapsed to different states according to Alice's LO setting θ, and result s see Fig. 1.2.1.

The key role of measurement choice by Alice in demonstrating `spooky action at a distance' was introduced in the famous Einstein-Podolsky-Rosen (EPR) paper [6] of 1935. In its most general form, this phenomenon has been called EPR-steering, to acknowledge the contribution and terminology of Schrödinger [11], who talked of Alice `steering' the state of Bob's quantum system.

From a quantum information perspective, EPR-steering is equivalent to the task of entanglement verification when Bob (and his detectors) can be trusted but Alice (or her detectors) cannot. This is strictly harder than verifying entanglement with both parties trusted [12], but strictly easier than violating a Bell inequality [13], where neither party is trusted [12].

Remark 1.2.2. A recent experimental test of entanglement for a single photon via an entanglement witness has no efficiency loophole [14], however, it demonstrates a weaker form on nonlocality than EPR-steering. In [10], it was demonstrated experimentally that there exists Einstein's elusive `spooky action at a distance' for a single particle without opening the efficiency loophole without claim to have closed the separation loophole. That is the one-sided device-independent verification of spatial-mode entanglement for a single photon.

I.3. The continuous variable EPR paradox. EPR-Reid's criteria

We remind that EPR treated the case of a non-factorizable pure state $|\psi\rangle$ which describes the results for measurements performed on two spatially separated systems at A and B (Fig. 1.3.1). Non-factorizable means entangled, that is, we cannot express $|\psi\rangle$ as a simple product $|\psi\rangle = |\psi\rangle_A \otimes |\psi\rangle_B$, where $|\psi\rangle_A$ and $|\psi\rangle_B$ are quantum states for the results of measurements at A and B, respectively.

Fig.1.3.1. The original EPR gedanken experiment. Two particles move from the source into spatially separated regions *A* and *B*, and yet continue to have maximally correlated positions: $x_A+x_0=x_B$ and anti-correlated momenta: $p_A=-p_B$. Adapted from [15]

In the first part of their paper, EPR point out in a general way the problematic aspects of such entangled states. The key issue is that one can expand $|\psi\rangle$ in terms of more than one basis, that corresponds to different experimental settings, which we parametrize by ϕ. Let us consider the state

$$|\psi\rangle = \int dx |\psi_x\rangle_{\phi,A} \otimes |u_x\rangle_{\phi,B}\,, \tag{1.3.1}$$

where the x eigenvalue could be continuous or discrete. The parameter setting ϕ at the detector B is used to define a particular orthogonal measurement basis $|u_x\rangle_{\phi,B}$. On measurement at B, that projects out a wave-function $|\psi_x\rangle_{\phi,A}$ at A, the process called reduction of the wave packet.

Remark 1.3.1. The locality assumption postulates no action-at-a-distance, so that measurements at a location B cannot immediately disturb the system at a spatially separated A location.

Remark 1.3.2. The problematic issue is that different choices of measurements ϕ at B will cause reduction of the wave packet at A in more than one possible way. EPR states that, as a consequence of two different measurements at B, the second system may be left in states with two different wave functions. Yet, no real change can take place in the second system in consequence of anything that may be done to the first system.

The problem was established by EPR by a specific example, shown in Fig.1.3.1. EPR considered two spatially separated subsystems, at A and B, each with two observables $\hat{x}$ and $\hat{p}$ where $\hat{x}$ and $\hat{p}$ are non-commuting quantum operators, with commutator

$$[\hat{x},\hat{p}] = \hat{x}\hat{p} - \hat{p}\hat{x} = 2C \neq 0. \tag{1.3.2}$$

The results of the measurements $\hat{x}$ and $\hat{p}$ are denoted x and p respectively, and this convention we follow throughout the paper. We note that EPR assumed a continuous variable spectrum and considered wave function ψ defined in a position representation by

$$\psi(x,x^B) = \int e^{(ip/\hbar)(x-x^B-x_0)}dp, \tag{1.3.3}$$

where x_0 is a constant implying space-like separation. Here the pairs x and p refer to the results for position and momentum measurements at A, while x^B and p^B denote the position and momentum measurements at B. We leave off the superscript for A system, to emphasize the inherent asymmetry that exists in the EPR argument, where one system A is steered by the B other.

Remark 1.3.3. According to canonical quantum mechanics, one can predict with certainty that a measurement $\hat{x}$ will give result $x^B + x_0$, if a measurement $\hat{x^B}$, with result x^B, was already performed at B. One may also predict with certainty the result of $\hat{p}$ measurement, for a different choice of measurement at B. If the momentum at B is measured to be p, then the result for $\hat{p}$ is $-p$. These predictions are made without disturbing the second system at A, based on the assumption, implicit in the original EPR paper, of locality.

Remark 1.3.4. The locality assumption can be strengthened if the measurement events at A and B are causally separated (such that no signal can travel from one event to the other, unless faster than the speed of light).

Remark 1.3.5. The remainder of the EPR argument may be summarized as follows. Assuming local realism, one deduces that both the measurement outcomes, for x and p at A, are predetermined. The perfect correlation of x with $x^B + x_0$ implies the existence of an element of reality for the $\hat{x}$ measurement. Similarly, the correlation of p with $-p^B$ implies an element of reality for $\hat{p}$. Although not mentioned by EPR, it will prove useful to mathematically represent the elements of reality for $\hat{x}$ and $\hat{p}$ by the respective variables μ_x^A and μ_p^A, whose possible values are the predicted results of the measurement.

Remark 1.3.6. To continue the argument, *local realism* implies the existence of two elements of reality, μ_x^A and μ_p^A, that *simultaneously* predetermine, with absolute definiteness, the results for measurement x or p at A. These elements of reality for the localized subsystem A are not themselves consistent with quantum mechanics. Simultaneous determinacy for both the position and momentum is not possible for any quantum state. Hence, assuming the validity of local realism, one concludes quantum mechanics to be incomplete or even inconsistent!

Remark 1.3.7. We claim that any assumption of local realism is completely wrong.
Such a claim meant as minimum the weak postulate of nonlocality.

I.3.1. The weak postulate of nonlocality for continuous variables

The Heisenberg uncertainty relations

$$\Delta x^{\mathbf{A}} \Delta p^{\mathbf{A}} \geq 1 \quad (1.3.4)$$

cannot be violated in any cases:
(i) of course, according to quantum mechanics, the Heisenberg uncertainty relations (1.3.4) cannot be violated if the coordinate $x^{\mathbf{A}}$ and momentum $p^{\mathbf{A}}$ of the particle $\mathbf{A}$ are measured directly by measurements performed on the particle $\mathbf{A}$,
(ii) the Heisenberg uncertainty relations (1.3.4) cannot be violated even if the coordinate $x^{\mathbf{A}}$ and momentum $p^{\mathbf{A}}$ of the particle $\mathbf{A}$ are measured indirectly, i.e. by using measurement on particle $\mathbf{B}$, as required in EPR gedanken experiment,
(iii) in any cases true coordinate $x^{\mathbf{A}}$ and momentum $p^{\mathbf{A}}$ of the particle **A** cannot be predicted simultaneously with a sufficiently small uncertainty $\Delta x^{\mathbf{A}}$ and $\Delta p^{\mathbf{A}}$ such that the Reid's inequality [16]:

$$\Delta x^{\mathbf{A}} \Delta p^{\mathbf{A}} < 1 \quad (1.3.5)$$

based on local realism would be satisfied, i.e., always

$$\Delta x^{\mathbf{A}} \Delta p^{\mathbf{A}} \not< 1. \qquad (1.3.6)$$

We claim strictly stronger assumptions of nonlocality than mentioned above.

I.3.2. The strong postulate of nonlocality for continuous variables

Let $|\psi_t^x\rangle_{\mathbf{A}}$ and $|\psi_t^x\rangle_{\mathbf{B}}$ be a state vector in x-representation at instant t of the particle **A** and particle **B** correspondingly.

Let $|\psi_t^p\rangle_{\mathbf{A}}$ and $|\psi_t^p\rangle_{\mathbf{B}}$ be a state vector in p-representation at instant t of the particle **A** and particle **B** correspondingly.

Let $\psi_t^{\mathbf{A}}(x) = \langle x|\psi_t^x\rangle_{\mathbf{A}}, \psi_t^{\mathbf{B}}(x) = \langle x|\psi_t^x\rangle_{\mathbf{B}}$ be a wave functions in x-representation of the particle **A** and particle **B** correspondingly.

Let $\psi_t^{\mathbf{A}}(p) = \langle p|\psi_t^p\rangle_{\mathbf{A}}, \psi_t^{\mathbf{B}}(p) = \langle p|\psi_t^p\rangle_{\mathbf{B}}$ be a wave functions in p-representation of the particle **A** and particle **B** correspondingly.

Let $\psi_t^{\mathbf{A/B}}(x_{\mathbf{A}}, x_{\mathbf{B}})$ be corresponding two-particle wave function in x-representation and let $\psi_t^{\mathbf{A/B}}(p_{\mathbf{A}}, p_{\mathbf{B}})$ be corresponding two-particle wave function in p-representation.

We claim that:

(i) whenever a measurement of the coordinate x of a particle **B** is performed at instant t with result $\bar{x}^{\mathbf{B}} \in [x^{\mathbf{B}} - \varepsilon, x^{\mathbf{B}} + \varepsilon], \varepsilon \ll 1,$ then:

(a) according to quantum mechanics a state vector $|\psi_t^x\rangle_{\mathbf{B}}$ collapses at instant t to the state vector

$$\left|\psi^x_{t,\delta,\varepsilon,x^{\mathbf{B}}}\right\rangle_{\mathbf{B}} \sim \widehat{L}^{\mathbf{B}}_{x^{\mathbf{B}}}(\delta,\varepsilon)|\psi_t^x\rangle_{\mathbf{B}} \qquad (1.3.7)$$

given by law (1.2.20), where $\widehat{L}^{\mathbf{B}}_{\mathbf{x}^{\mathbf{B}}}(\delta,\varepsilon)$ is a norm-reducing, positive, self-adjoint, linear operator in the 2 -particle non projective Hilbert space $\mathbf{H}$, representing the localization of particle $\mathbf{B}$ around the point $x^{\mathbf{B}}$, (see subsection II.2.);

(b) according to postulate of nonlocality a state vector $|\psi^x_t\rangle_{\mathbf{A}}$ immediately collapses at instant t to the state vector

$$\left|\psi^x_{t,\delta,\varepsilon,x^{\mathbf{A}}}\right\rangle_{\mathbf{A}} \sim \widehat{L}^{\mathbf{A}}_{x^{\mathbf{B}}+x_0}(\delta,\varepsilon)|\psi^x_t\rangle_{\mathbf{A}} \tag{1.3.8}$$

given by law (1.2.20) and this is true independent of the distance in Minkowski space-time $M_4 = \mathbb{R}^{1,3}$ that separates the particles. Thus

$$|\psi^x_t\rangle_{\mathbf{B}} \overset{\text{collapse}}{\longrightarrow} \left|\psi^x_{t,\delta,\varepsilon,x^{\mathbf{B}}}\right\rangle_{\mathbf{B}} \Rightarrow |\psi^x_t\rangle_{\mathbf{A}} \overset{\text{collapse}}{\longrightarrow} \left|\psi^x_{t,\delta,\varepsilon,x^{\mathbf{B}}+x_0}\right\rangle_{\mathbf{A}} \tag{1.3.9}$$

(ii) under conditions given by Eq. (1.3.7) - Eq. (1.3.9) two-particle wave function $\psi^{\mathbf{A/B}}_t(x_{\mathbf{A}},x_{\mathbf{B}})$ collapses at instant t by law

$$\psi^{\mathbf{A/B}}_t(x_{\mathbf{A}},x_{\mathbf{B}}) \overset{\text{collapse}}{\longrightarrow} \widehat{L}^{\mathbf{A}}_{x^{\mathbf{B}}+x_0}\widehat{L}^{\mathbf{B}}_{x^{\mathbf{B}}}(\delta,\varepsilon)\psi^{\mathbf{A/B}}_t(x_{\mathbf{A}},x_{\mathbf{B}}) \tag{1.3.10}$$

(iii) whenever a measurement of the momentum $p^{\mathbf{B}}$ of a particle **B** is performed at instant t with result $\bar{p}^{\mathbf{B}} \in [p^{\mathbf{B}}-\varepsilon, p^{\mathbf{B}}+\varepsilon], \varepsilon \ll 1,$ then:

(a) according to quantum mechanics a state vector $|\psi^p_t\rangle_{\mathbf{B}}$ collapses at instant t to the state vector

$$\left|\psi^p_{t,\delta,\varepsilon,p^{\mathbf{B}}}\right\rangle_{\mathbf{B}} \sim \widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon)|\psi^p_t\rangle_{\mathbf{B}}, \tag{1.3.11}$$

where $\widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon)$ is a norm-reducing, positive, self-adjoint, linear operator in the 2 -particle non projective Hilbert space $\mathbf{H}$, representing the localization of momentum of the particle $\mathbf{B}$

around the value $p^{\mathbf{B}}$. The localization operators $\widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon)$ have been chosen to have the following form:

$$\widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon) = \left(\frac{1}{\delta\pi_\delta}\right)^{3/4} \exp\left[-\frac{1}{2\delta}(\widehat{p} - p^{\mathbf{B}})^2\right] \qquad (1.3.12)$$

where $\delta \in (0,1]$ and $\lim_{\delta\to 0}\pi_\delta = \pi$.

(b) according to postulate of nonlocality a state vector $|\psi^p_t\rangle_{\mathbf{A}}$ immediately collapses at instant t to the state vector

$$\left|\psi^p_{t,\delta,\varepsilon,x^{\mathbf{A}}}\right\rangle_{\mathbf{A}} \sim \widehat{L}^{\mathbf{A}}_{-p^{\mathbf{B}}}(\delta,\varepsilon)|\psi^p_t\rangle_{\mathbf{A}} \qquad (1.3.13)$$

and this is true independent of the distance in Minkowski space-time $M_4 = \mathbb{R}^{1,3}$ that separates the particles. Thus

$$|\psi^p_t\rangle_{\mathbf{B}} \overset{\text{collapse}}{\longrightarrow} \left|\psi^p_{t,\delta,\varepsilon,p^{\mathbf{B}}}\right\rangle_{\mathbf{B}} \Rightarrow |\psi^p_t\rangle_{\mathbf{A}} \overset{\text{collapse}}{\longrightarrow} \left|\psi^p_{t,\delta,\varepsilon,-p^{\mathbf{B}}}\right\rangle_{\mathbf{A}} \qquad (1.3.14)$$

(iv) under conditions given by Eq. (1.3.11) - Eq. (1.3.13) two-particle wave function $\psi^{\mathbf{A/B}}_t(p_{\mathbf{A}},p_{\mathbf{B}})$ collapses at instant t by the law

$$\psi^{\mathbf{A/B}}_t(p_{\mathbf{A}},p_{\mathbf{B}}) \overset{\text{collapse}}{\longrightarrow} \widehat{L}^{\mathbf{A}}_{-p^{\mathbf{B}}}\widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon)\psi^{\mathbf{A/B}}_t(p_{\mathbf{A}},p_{\mathbf{B}}). \qquad (1.3.15)$$

Remark 1.3.8. Let $p^{\mathbf{A}}_t$ and $p^{\mathbf{B}}_t$ be the momentum at instant t of the particle **A** and particle **B** correspondingly. Note that whenever a measurement of the coordinate x of a particle **B** is performed at instant t with the accuracy $\varepsilon_{x^{\mathbf{B}}} \ll 1$ then:

(i) immediately after this measurement the momentum $p^{\mathbf{B}}_t$ at instant t changed according to quantum mechanics by the Heisenberg uncertainty relations (1.3.4);

(ii) immediately after this measurement the momentum $p^{\mathbf{A}}_t$ at instant

t changed according to postulate of nonlocality by the Heisenberg uncertainty relations (1.3.4).

Remark 1.3.9. Let $x_t^{\mathbf{A}}$ and $x_t^{\mathbf{B}}$ be the coordinate at instant t of the particle **A** and particle **B** correspondingly. Note that whenever a measurement of the momentum p of a particle **B** is performed at instant t with the accuracy $\varepsilon_{p^{\mathbf{B}}} \ll 1$ then:

(i) immediately after this measurement the coordinate $x_t^{\mathbf{B}}$ at instant t changed according to quantum mechanics by the Heisenberg uncertainty relations (1.3.4);

(ii) immediately after this measurement the momentum $x_t^{\mathbf{A}}$ at instant t changed according to postulate of nonlocality by the Heisenberg uncertainty relations (1.3.4).

Remark 1.3.10. Schrödinger [11] pointed out that the EPR two-particle wave function in Eq. (1.3.3) was verschränkten - which he later translated as entangled - i.e., not of the separable form $\psi_A\psi_B$. Schrödinger considered as a possible resolution of the paradox that this entanglement degrades as the particles separate spatially, so that EPR correlations would not be physically realizable.

Definition 1.3.1. Quantum inseparability (entanglement) for a general mixed quantum state is defined as the failure of

$$\widehat{\rho} = \int d\lambda P(\lambda)\widehat{\rho}_{\lambda}^{\mathbf{A}} \otimes \widehat{\rho}_{\lambda}^{\mathbf{B}}, \tag{1.3.16}$$

where $\int d\lambda P(\lambda) = 1$ and $\widehat{\rho}$ is the density operator. Here λ is a discrete or continuous label for component states, and $\widehat{\rho}_{\lambda}^{\mathbf{A}}$ and $\widehat{\rho}_{\lambda}^{\mathbf{B}}$ correspond to density operators that are restricted to the Hilbert spaces $\mathbf{A}$ and B respectively.

Remark 1.3.11. The definition of inseparability extends beyond that of the EPR situation, in that one considers a whole spectrum of measurement choices, parametrized by θ for those performed on A system, and by ϕ for those performed on B. We use canonical notation $\hat{x}_\theta^A$ and $\hat{x}_\phi^B$ to describe all measurements at A and B. Denoting the eigenstates of $\hat{x}_\theta^A$ by $|x_\theta^A\rangle$, we define $P_Q(x_\theta^A|\theta,\lambda) = \langle x_\theta^A|\widehat{\rho}_\lambda^A|x_\theta^A\rangle$ and $P_Q(x_\phi^B|\phi,\lambda) = \langle x_\phi^B|\widehat{\rho}_\lambda^B|x_\phi^B\rangle$, which are the localized probabilities for observing results x_θ^A and x_ϕ^B respectively. The separability condition (1.3.9) then implies that joint probabilities $P(x_\theta^A, x_\phi^B)$ are given as [16]:

$$P(x_\theta^A, x_\phi^B) = \int d\lambda P(\lambda) P_Q(x_\theta^A|\lambda) P_Q(x_\phi^B|\lambda)\,. \qquad (1.3.17)$$

Remark 1.3.12. We note the canonical restriction

$$\Delta^2(x^A|\lambda)\Delta^2(p^A|\lambda) \geq 1 \qquad (1.3.18)$$

where $\Delta^2(x^{\mathbf{A}}|\lambda)$ and $\Delta^2(p^{\mathbf{A}}|\lambda)$ are the variances of $P_Q(x_\theta^{\mathbf{A}}|\theta,\lambda)$ for the choices θ corresponding to position x and p momentum, respectively. Thus,

$$\Delta^2(x^{\mathbf{A}}|\lambda)\Delta^2(p^{\mathbf{A}}|\lambda) < 1 \qquad (1.3.19)$$

is an EPR criterion, meaning that this would imply an EPR "paradox".

Remark 1.3.13. Note that the original EPR state of Eq. (1.3.3) is not separable. Suppose that, based on a result $x^{\mathbf{B}}$ for the measurement at $\mathbf{B}$, an estimate $x_{\mathbf{est}}(x_{\mathbf{B}})$ is made of the result x at $\mathbf{A}$. We may define the average error $\Delta_{\mathbf{inf}}x$ of this inference as the root mean square (RMS) of the deviation of the estimate from the actual value, so that [16-18]:

$$\Delta_{\mathbf{inf}}^2 x = \int dx dx^{\mathbf{B}} P_t(x, x^{\mathbf{B}})(x - x_{\mathbf{est}}(x^B))^2 \ . \tag{1.3.20}$$

An inference variance $\Delta_{\mathbf{inf}}^2 p$ is defined similarly, i.e.

$$\Delta_{\mathbf{inf}}^2 p = \int dp dp^{\mathbf{B}} P_t(p, p^{\mathbf{B}})(p - p_{\mathbf{est}}(p^B))^2 \ . \tag{1.3.21}$$

Remark 1.3.14. Let $\psi_t^{\mathbf{A/B}}(x_{\mathbf{A}}, x_{\mathbf{B}})$ be corresponding two-particle wave function in x -representation and let $\psi_t^{\mathbf{A/B}}(p_{\mathbf{A}}, p_{\mathbf{B}})$ be corresponding two-particle wave function in p -representation. Note that:
(i) $P_t(x, x^{\mathbf{B}})$ is the joint probability of obtaining an outcome x at $\mathbf{A}$ and $x^{\mathbf{B}}$ at $\mathbf{B}$ at t instant is of the form

$$P_t(x, x^{\mathbf{B}}) \sim \left| \psi_t^{\mathbf{A/B}}(x_{\mathbf{A}}, x_{\mathbf{B}}) \right|^2, \tag{1.3.22}$$

(ii) $P_t(p, p^{\mathbf{B}})$ is the joint probability of obtaining an outcome p at $\mathbf{A}$ and $p^{\mathbf{B}}$ at $\mathbf{B}$ at t instant is of the form

$$P_t(p, p^{\mathbf{B}}) \sim \left| \psi_t^{\mathbf{A/B}}(p_{\mathbf{A}}, p_{\mathbf{B}}) \right|^2. \tag{1.3.23}$$

The best estimate, which minimizes $\Delta_{\mathbf{inf}} x$, is given by choosing $x_{\mathbf{est}}$ for each $x^{\mathbf{B}}$ to be the mean $\langle x | x^{\mathbf{B}} \rangle$ of the conditional distribution $P_t(x | x^{\mathbf{B}})$. This is seen upon noting that for each $x^{\mathbf{B}}$ result, we can define the RMS error in each estimate as

$$\Delta_{\mathbf{inf}}^2(t, x | x^{\mathbf{B}}) = \int dx P_t(x | x^{\mathbf{B}})(x - x_{\mathbf{est}}(x^{\mathbf{B}}))^2 \ . \tag{1.3.24}$$

The average error in each inference is minimized for $x_{est} = \langle x | x^{\mathbf{B}} \rangle$, when each $\Delta_{\mathbf{inf}}^2(t, x | x^{\mathbf{B}})$ becomes the variance $\Delta^2(t, x | x^B)$ of $P_t(x | x^B)$. We thus define the minimum inference error $\Delta_{\mathbf{inf}} x$ for position, averaged over all possible values of $x^{\mathbf{B}}$, as

$$V^{x}_{\mathbf{A|B}} = \min\left(\Delta^2_{\mathbf{inf}}x\right) = \int dx^B P_t(x^B)\Delta^2(t,x|x^B)\,, \qquad (1.3.25)$$

where $P(x^{\mathbf{B}})$ is the probability density for a result $x^{\mathbf{B}}$ upon measurement of $\hat{x}^{\mathbf{B}}$. This minimized inference variance is the average of the individual variances for each outcome at $\mathbf{B}$. Similarly, we can define a minimum inference variance $V^{p}_{A|B}$ for momentum, i.e.

$$V^{p}_{\mathbf{A|B}} = \min\left(\Delta^2_{\mathbf{inf}}p\right) = \int dp^B P_t(p^B)\Delta^2(t,p|p^B)\,. \qquad (1.3.26)$$

Remark 1.3.15. Let $\psi^{\mathbf{A/B}}_t(x_{\mathbf{A}},x_{\mathbf{B}})$ be corresponding two-particle wave function in x -representation and let $\psi^{\mathbf{A/B}}_t(p_{\mathbf{A}},p_{\mathbf{B}})$ be corresponding two-particle wave function in p -representation. Note that:

(i) according to local realism the conditional distributions densities $P_{\mathbf{loc}}(x|x^B)$ and $P_{\mathbf{loc}}(p|p^B)$ vary given by formulae

$$P_{\mathbf{loc}}(x|x^B) \sim \widehat{L}^{\mathbf{B}}_{x^{\mathbf{B}}}(\delta,\varepsilon)\psi^{\mathbf{A/B}}_t(x,x_{\mathbf{B}}) \qquad (1.3.27)$$

and

$$P_{\mathbf{loc}}(p|p^B) \sim \widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon)\psi^{\mathbf{A/B}}_t(p_{\mathbf{A}},p_{\mathbf{B}}). \qquad (1.3.28)$$

(ii) distribution densities $P_{\mathbf{loc}}(t,x^{\mathbf{B}})$ and $P_{\mathbf{loc}}(t,p^B)$ are given by formulae

$$P_{\mathbf{loc}}(t,x^B) = \int dx P_{\mathbf{loc}}(t,x|x^B) \qquad (1.3.29)$$

and

$$P_{\mathbf{loc}}(t,p^B) = \int dp P_{\mathbf{loc}}(t,p|p^B). \qquad (1.3.30)$$

Remark 1.3.16. Let $\psi_t^{\mathbf{A/B}}(x_{\mathbf{A}},x_{\mathbf{B}})$ be corresponding two-particle wave function in x-representation and let $\psi_t^{\mathbf{A/B}}(p_{\mathbf{A}},p_{\mathbf{B}})$ be corresponding two-particle wave function in p-representation. Note that:

(i) according to postulates of nonlocality the conditional distributions densities $P_{\mathbf{n.loc}}(t,x|x^B)$ and $P_{\mathbf{n.loc}}(t,p|p^B)$ are given by formulae

$$P_{\mathbf{n.loc}}(t,x|x^B) = \widehat{L}^{\mathbf{A}}_{x^{\mathbf{B}}+x_0}\widehat{L}^{\mathbf{B}}_{x^{\mathbf{B}}}(\delta,\varepsilon)\psi_t^{\mathbf{A/B}}(x,x_{\mathbf{B}}) \qquad (1.3.31)$$

and

$$P_{\mathbf{n.loc}}(t,p|p^B) \sim \widehat{L}^{\mathbf{A}}_{-p^{\mathbf{B}}}\widehat{L}^{\mathbf{B}}_{p^{\mathbf{B}}}(\delta,\varepsilon)\psi_t^{\mathbf{A/B}}(p,p_{\mathbf{B}}), \qquad (1.3.32)$$

see Eq. (1.3.10) and Eq. (1.3.15) respectively.

(ii) distributions $P_{\mathbf{n.loc}}(t,x^{\mathbf{B}})$ and $P_{\mathbf{n.loc}}(t,p^B)$ are given by formulae

$$P_{\mathbf{n.loc}}(t,x^B) = \int dx P_{\mathbf{n.loc}}(t,x|x^B) \qquad (1.3.33)$$

and

$$P_{\mathbf{n.loc}}(t,p|^B) = \int dp P_{\mathbf{n.loc}}(t,p|p^B) \qquad (1.3.34)$$

Thus we can define corresponding RMS errors as

$$\begin{aligned}\Delta^2_{\mathbf{loc.inf}}(t,x|x^{\mathbf{B}}) &= \int dx P_{\mathbf{loc}}(t,x|x^{\mathbf{B}})(x - x_{\mathbf{est}}(x^{\mathbf{B}}))^2 \\ \Delta^2_{\mathbf{loc.inf}}(t,p|p^{\mathbf{B}}) &= \int dx P_{\mathbf{loc}}(t,p|p^{\mathbf{B}})(p - x_{\mathbf{est}}(p^{\mathbf{B}}))^2\end{aligned} \qquad (1.3.35)$$

and

$$\begin{aligned}\Delta^2_{\mathbf{n.loc.inf}}(t,x|x^{\mathbf{B}}) &= \int dx P_{\mathbf{loc}}(t,x|x^{\mathbf{B}})(x - x_{\mathbf{est}}(x^{\mathbf{B}}))^2, \\ \Delta^2_{\mathbf{n.loc.inf}}(t,p|p^{\mathbf{B}}) &= \int dx P_{\mathbf{loc}}(t,p|p^{\mathbf{B}})(p - x_{\mathbf{est}}(p^{\mathbf{B}}))^2\end{aligned} \qquad (1.3.36)$$

respectively. We thus define the minimum inference error $\Delta_{\mathbf{inf}}x$ for position, averaged over all possible values of $x^{\mathbf{B}}$ and $p^{\mathbf{B}}$ as

$$\begin{aligned}\min\left(\Delta^2_{\mathbf{loc.inf}}x\right) &= \int dx^B P_{\mathbf{loc}}(t,x^B)\Delta^2_{\mathbf{loc.}}(t,x|x^B), \\ \min\left(\Delta^2_{\mathbf{loc.inf}}p\right) &= \int dp^B P_{\mathbf{loc}}(t,p^B)\Delta^2_{\mathbf{loc.}}(t,p|p^B)\end{aligned} \qquad (1.3.37)$$

and

$$\min\left(\Delta^2_{\mathbf{n.loc.inf}} x\right) = \int dx^B P_{\mathbf{n.loc}}(t, x^B) \Delta^2_{\mathbf{n.loc.}}(t, x | x^B),$$
$$\min\left(\Delta^2_{\mathbf{n.loc.inf}} p\right) = \int dp^B P_{\mathbf{n.loc}}(t, p^B) \Delta^2_{\mathbf{n.loc.}}(t, p | p^B). \qquad (1.3.38)$$

respectively. From Eq. (1.3.37) and Eq. (1.3.38) we obtain the EPR-nonlocality criteria

$$\left| \min\Delta^2_{\mathbf{loc.inf.}} x - \min \Delta^2_{\mathbf{n.loc.inf.}} x \right| =$$
$$\left| \int dx^B [P_{\mathbf{loc}}(t, x^B) \Delta^2_{\mathbf{loc.}}(t, x | x^B) - P_{\mathbf{n.loc}}(t, x^B) \Delta^2_{\mathbf{n.loc.}}(t, x | x^B)] \right| > 0,$$
$$\left| \min\Delta^2_{\mathbf{loc.inf.}} p - \min\Delta^2_{\mathbf{n.loc.inf.}} p \right| = \qquad (1.3.39)$$
$$\left| \int dp^B [P_{\mathbf{loc}}(t, p^B) \Delta^2_{\mathbf{loc.}}(t, p | p^B) - P_{\mathbf{n.loc}}(t, p^B) \Delta^2_{\mathbf{n.loc.}}(t, p | p^B)] \right| > 0$$

and

$$\left| \left(\min\Delta^2_{\mathbf{loc.inf.}} x\right)\left(\min\Delta^2_{\mathbf{loc.inf.}} p\right) - \left(\min \Delta^2_{\mathbf{n.loc.inf.}} x\right)\left(\left(\Delta^2_{\mathbf{n.loc.inf}} p\right)\right) \right| > 0. \qquad (1.3.40)$$

I.4. The EPR-Bohm paradox. Reid's criteria for EPR-Bohm paradox

Bohm [19]-[20] considered two spatially-separated spin-$1/2$ particles at A and B produced in an entangled singlet state (often referred to as the EPR-Bohm state or the Bell-state):

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(\left|\frac{1}{2}\right\rangle_A \left|-\frac{1}{2}\right\rangle_B - \left|-\frac{1}{2}\right\rangle_A \left|\frac{1}{2}\right\rangle_B \right) \qquad (1.4.1)$$

Here $|\pm\frac{1}{2}\rangle_A$ are eigenstates of the $\hat{J}^A_z$ spin operator, and we use $\hat{J}^A_z$, $\hat{J}^A_x$, $\hat{J}^A_y$ to define the spin-components measured at location A. The spin-eigenstates and measurements at B are defined similarly. By considering different quantization axes, one obtains different but equivalent expansions of $|\psi\rangle$ in Eq. (1.3.1), just as EPR suggested.

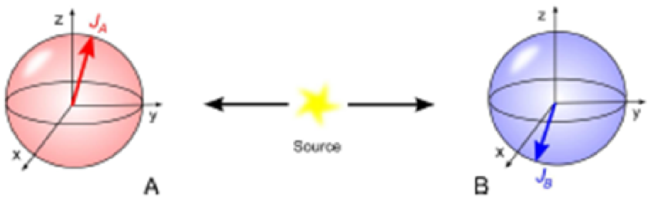

Fig. 1.4.1. The Bohm gedanken EPR experiment. Two spin-1/2 particles prepared in a singlet state from the source into spatially separated regions A and B, and give anti-correlated outcomes for J_θ^A and J^B, where θ is x, y or z. Adapted from [16]

Bohm's paradox is based on the existence, for Eq. (1.9.1), of a maximum anti-correlation between not only $\hat{J}_z^A$ and $\hat{J}_z^B$, but $\hat{J}_y^A$ and $\hat{J}_y^B$, and also $\hat{J}_x^A$ and $\hat{J}_x^B$. An assumption of local realism would lead to the conclusion that the three spin components of particle A were simultaneously predetermined, with absolute definiteness. Since no such quantum description exists, this is the situation of an EPR paradox.

Remark 1.4.1. Bohm's paradox is based on the existence, for Eq.(1.4.1), of a maximum anti-correlation between not only $\hat{J}_z^A$ and $\hat{J}_z^B$, but $\hat{J}_y^A$ and $\hat{J}_y^B$, and also $\hat{J}_x^A$ and $\hat{J}_x^B$.

Remark 1.4.2. Note that an assumption of local realism would lead to the conclusion that the three spin components of particle A were simultaneously predetermined, with absolute definiteness. Since no such quantum description exists, this is the situation of an EPR paradox.

Remark 1.4.3. Criteria sufficient to demonstrate Bohm's EPR paradox can be derived using Reid's canonical inferred uncertainty approach [16]. Using the Heisenberg spin uncertainty relation

$$\Delta J_x^{\mathbf{A}} \Delta J_y^{\mathbf{A}} \geq |\langle J_z^{\mathbf{A}} \rangle|/2, \quad (1.4.2)$$

one obtains the following canonical spin-EPR criterion that is useful for the Bell state given by Eq. (1.4.1)

$$\Delta_{\mathbf{inf}} J_x^{\mathbf{A}} \Delta_{\mathbf{inf}} J_y^{\mathbf{A}} < \frac{1}{2} \sum_{J_z^B} P(J_z^{\mathbf{B}}) \left| \langle J_z^{\mathbf{A}} \rangle_{J_z^{\mathbf{B}}} \right|. \quad (1.4.3)$$

Here $\langle J_z^{\mathbf{A}} \rangle_{J_z^{\mathbf{B}}}$ is the mean of the conditional $P(J_z^{\mathbf{A}} | J_z^{\mathbf{B}})$ distribution. Calculations for Eq. (1.4.1) including the effect of detection efficiency η reveals this EPR criterion to be satisfied for $\eta > 0.62.$ The concept of spin-EPR has been experimentally tested in the continuum limit with purely optical systems for states with $\langle J_z^A \rangle \neq 0.$ In this case the EPR criterion linked closely to definition of spin squeezing

$$\Delta_{\mathbf{inf}} J_x^A \Delta_{\mathbf{inf}} J_y^A < \left| \langle J_z^A \rangle \right|. \quad (1.4.4)$$

Remark 1.4.4. We claim that any assumption of local realism is completely wrong. The three spin components of particle A were simultaneously predetermined, does not with absolute definiteness but only with uncertainties which required by Heisenberg spin uncertainty relations (1.4.5). Such claim meant as minimum the weak postulate of nonlocality.

I.4.1. The weak postulate of nonlocality Heisenberg spin uncertainty relations

The Heisenberg spin uncertainty relations

$$\Delta J_x^{\mathbf{A}} \Delta J_y^{\mathbf{A}} \geq \left| \langle J_z^{\mathbf{A}} \rangle \right| / 2, \Delta J_x^{\mathbf{A}} \Delta J_z^{\mathbf{A}} \geq \left| \langle J_y^{\mathbf{A}} \rangle \right| / 2, \Delta J_z^{\mathbf{A}} \Delta J_y^{\mathbf{A}} \geq \left| \langle J_x^{\mathbf{A}} \rangle \right| / 2 \quad (1.4.5)$$

does not violate in any cases:
(i) if the three spin components of the particle **A** are measured directly by measurements performed on the particle **A,**
(ii) and even if some spin components of the **A** particle are

measured indirectly as required in Bohm gedanken EPR experiment.
Think of the following situation: a particle with zero spin decays into two particles (**A** and **B**), each with $1/2$-spin. Due to the fact that spin angular momentum must be conserved during the decay, if initially the total spin angular momentum was zero, then after the decaying process it must still be zero. Therefore, particles **A** and **B** have opposite spin. Take as an example the dissociation of an exited hydrogen molecule into two hydrogen atoms. If the decaying mechanism does not change total angular momentum, then the spins on the hydrogen atoms will be anti-correlated.
Remark 1.4.5. Whenever a measurement of the spin of **A** is found to be positive with respect of the z-axis (we shall note this state as $|\uparrow\rangle_z$, then, under local realism, we could infer that the spin of the **B** particle must be negative $|\downarrow\rangle_z$, and this is true independent of the distance that separates the particles. The spin of these particles are then entangled.
Remark 1.4.6. We claim again that any assumption of local realism is completely wrong.

I.4.2. The strong postulate of nonlocality

Let $|\psi_t\rangle_{\mathbf{A}}$ and $|\psi_t\rangle_{\mathbf{B}}$ are states at instant t of the particle **A** and particle **B** correspondingly.
Let $|\uparrow\rangle_{z,\mathbf{A/B}}$ be eigenstates of the spin operator $\mathbf{S}^z_{\mathbf{A/B}}$:

$$\mathbf{S}^z_{\mathbf{A/B}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.4.6}$$

We claim that:
(i) whenever a measurement of the spin of a particle **A** is performed at instant $t_1 \geq t$ and particle **A** is found in the state $|\uparrow\rangle_z$, i.e., a state $|\psi_{t_1}\rangle_A$ collapses at instant t_1 to the state $|\uparrow\rangle_{z,A}$ with respect of the

Heisenberg spin uncertainty relations (1.4.5), then a state $|\psi_{t_1}\rangle_B$ immediately collapses at instant t_1 to the state $|\downarrow\rangle_{z,B}$ with respect of the Heisenberg spin uncertainty relations (1.4.5), and this is true independent of the distance in Minkowski space-time that separates the particles:

$$|\psi_{t_1}\rangle_A \stackrel{\text{collapse}}{\rightarrow} |\uparrow\rangle_{z,A} \Rightarrow |\psi_{t_1}\rangle_B \stackrel{\text{collapse}}{\rightarrow} |\downarrow\rangle_{z,B} \tag{1.4.7}$$

(ii) whenever a measurement of the spin of a particle **A** is performed at instant $t_1 \geq t$ and particle **A** is found in the state $|\downarrow\rangle_z$, i.e., a state $|\psi_{t_1}\rangle_A$ collapses at instant t_1 to the state $|\downarrow\rangle_{z,A}$ with respect of the Heisenberg spin uncertainty relations (1.4.5), then a state $|\psi_{t_1}\rangle_B$ immediately $|\psi_t\rangle_B$ collapses at instant t_1 to the state $|\uparrow\rangle_{z,B}$ with respect of the Heisenberg spin uncertainty relations (1.4.5), and this is true independent of the distance in Minkowski space-time that separates the particles:

$$|\psi_{t_1}\rangle_A \stackrel{\text{collapse}}{\rightarrow} |\downarrow\rangle_{z,A} \Rightarrow |\psi_{t_1}\rangle_B \stackrel{\text{collapse}}{\rightarrow} |\uparrow\rangle_{z,B} \tag{1.4.8}$$

Note that, we can not predict which spin will be positive (or negative) with respect of the z-axis, so the state that describes the spins of the particles could be for instance the spin singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\downarrow\uparrow\rangle - |\uparrow\downarrow\rangle) \tag{1.4.9}$$

We have 50% probability for the spin of particle **A** to be positive (and the spin of **B** negative) and 50% probability of it being the other way around.

Remark 1.4.7. So far we have assumed that we are performing a measurement along the z -axis, but measurements are not restricted to this particular election, we could measure for instance the spin of particle **A** along the $\mathbf{a}$ -axis and the spin of **B** along the $\mathbf{b}$ -axis. Let us see what happens if we decide to measure the spin along the x-axis: $\mathbf{a} = \mathbf{b} = x$. As it known for $1/2$-spins, the spin operator $\mathbf{S}^x_{\mathbf{A/B}}$ can be represented by the 2×2 Hermitian matrix

$$\mathbf{S}^x_{\mathbf{A/B}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad (1.4.10)$$

By performing a change of basis we can rewrite the state $|\psi\rangle$ in terms of the eigenstates of the spin operator $\mathbf{S}^x_{\mathbf{A/B}}$:

$$|u\rangle = \tfrac{1}{\sqrt{2}}(|\downarrow\rangle + |\uparrow\rangle), |v\rangle = \tfrac{1}{\sqrt{2}}(|\downarrow\rangle - |\uparrow\rangle), \qquad (1.4.11)$$

and using Eq. (1.4.10), we can rewrite the state $|\psi\rangle$ as

$$|\psi\rangle = \tfrac{1}{\sqrt{2}}(|vu\rangle - |uv\rangle). \qquad (1.4.12)$$

The strong postulate of nonlocality in this case takes the form similarly mentioned above. Just like before, by choosing to measure the spin of **A** along the x-axis we can determine its value and infer the value of the spin of particle $\widetilde{\mathbf{B}}$ $\left(\widetilde{\mathbf{B}} \neq \mathbf{B}\right)$ in the state $|\psi\rangle_{x,\widetilde{\mathbf{B}}} = |u\rangle_{x,\mathbf{B}} \neq |\psi\rangle_{x,\mathbf{B}}$ without the need to measure it (and vice versa).

Furthermore, it turns out that this is the case independent of the election of the axis we choose to measure! (Provided that $\mathbf{a} = \mathbf{b} = v$). This is exactly the same situation such that a simple choice of the axis along which to measure the spin **A** allow us to establish the value of the spin of **B** along this same axis without the need to measure it. And this is also the case (as we already saw) for physical properties described by non-commuting operators ($\mathbf{S}^x$ and $\mathbf{S}^z$ do not commute).

Chapter II

A NEW QUANTUM MECHANICAL FORMALISM BASED ON THE PROBABILITY REPRESENTATION OF QUANTUM STATES AND OBSERVABLES

II.1. Generalized Postulates for Continuous Valued Observables

Suppose we have an n -dimensional physical quantum system.

I. Then we claim the following:

Q.I.1. Any given n -dimensional quantum system is identified by a set $\mathbf{Q}$:

$$\mathbf{Q} \triangleq \langle \mathbf{H}, \mathfrak{I}, \mathfrak{R}, \mathcal{L}_{2,1}, \mathfrak{I}^*(\mathbf{H}), \mathbf{G}, |\psi_t\rangle \rangle$$

where:

(i) $\mathbf{H}$ that is some infinite-dimensional complex Hilbert space,
(ii) $\mathfrak{I} = (\Omega, \mathcal{F}, \mathbf{P})$ that is complete probability space,
(iii) $\mathfrak{R} = (\mathbb{R}^n, \Sigma)$ that is measurable space,
(iv) $\mathcal{L}_{2,1}(\Omega)$ that is complete space of complex valued random variables $X : \Omega \to \mathbb{C}^n$ such that

$$\int_\Omega \|X(\omega)\| d\mathbf{P} < \infty, \int_\Omega \|X(\omega)\|^2 d\mathbf{P} < \infty \qquad (2.1.1)$$

(v) $\mathbf{G} : \mathbf{C}^*(\mathbf{H}) \times \mathbf{H} \to \mathcal{L}_{2,1}(\Omega)$ that is one to one correspondence such that

$$\left|\langle\psi|\widehat{Q}|\psi\rangle\right| = \int_{\Omega}\left(\mathbf{G}\left[\widehat{Q},|\psi\rangle\right](\omega)\right)d\mathbf{P} = \mathbf{E}_{\Omega}\left(\mathbf{G}\left[\widehat{Q}|\psi\rangle\right](\omega)\right),$$
$$\mathbf{G}\left[\widehat{\mathbf{1}},|\psi\rangle\right](\omega) = 1 \tag{2.1.2}$$

for any $|\psi\rangle \in \mathbf{H}$ and for any Hermitian adjoint operator $\widehat{Q} : \mathbf{H} \to \mathbf{H}$ such that $\hat{Q} \in \mathfrak{I}^*(\mathbf{H}) \subsetneqq C^*(\mathbf{H})$, where $C^*(\mathbf{H})$ is C^* - algebra of the Hermitian adjoint operators in $\mathbf{H}$ and $\mathfrak{I}^*(\mathbf{H})$ an commutative subalgebra of $C^*(\mathbf{H})$.

(vi) $|\psi_t\rangle$ is a continuous vector function $|\psi_t\rangle : \mathbb{R}_+ \to \mathbf{H}$ which represented the evolution of the quantum system $\mathbf{Q}$.

Q.I.2. For any $|\psi_1\rangle, |\psi_2\rangle \in \mathbf{H}$ and for any Hermitian operator $\widehat{Q} : \mathbf{H} \to \mathbf{H}$ such that

$$\left\langle \psi_1 \middle| \widehat{Q} \middle| \psi_2 \right\rangle = \left\langle \psi_2 \middle| \widehat{Q} \middle| \psi_1 \right\rangle = 0 \tag{2.1.3}$$

the equality holds

$$\mathbf{G}\left[\widehat{Q}(|\psi_1\rangle + |\psi_2\rangle)\right](\omega) = \mathbf{G}\left[\widehat{Q}|\psi_1\rangle\right](\omega) + \mathbf{G}\left[\widehat{Q}|\psi_2\rangle\right](\omega). \tag{2.1.4}$$

Definition 2.1.1. A random variable $X : \Omega \to E$ is a measurable function from the set of possible outcomes Ω to some E set.

Definition 2.1.2. Given a probability space $\mathfrak{I} = (\Omega, \mathcal{F}, \mathbf{P})$ and a measurable space $\mathfrak{R} = (\mathbb{R}^n, \Sigma)$, any $\mathbb{R}^n$-valued stochastic process that is a collection of $\mathbb{R}^n$-valued random variables on Ω, indexed by a totally ordered set T ("time"). That is, a stochastic process $X_t(\omega)$ is a collection $\{X_t(\omega) | t \in T\}$, where each $X_t(\omega)$ is an $\mathbb{R}^n$-valued random variable on Ω. The space $\mathbb{R}^n$ is then called the state space of the process.

Q.I.3. Suppose that the evolution of the quantum system is represented by continuous vector function $|\psi_t\rangle : \mathbb{R}_+ \to \mathbf{H}$. Then any process of continuous measurements on measuring observable $\widehat{Q}$

for the system in state $|\psi_t\rangle$ one can to describe by an continuous $\mathbb{R}^n$-valued stochastic process

$$X_t(\omega) = X_t\left(\omega; \left|\widehat{Q}\psi_t\right\rangle\right) \triangleq X_{\left|\widehat{Q}\psi_t\right\rangle}(\omega)$$

given on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and the measurable space $(\mathbb{R}^n, \Sigma)$.

Remark 2.1.1. We assume now for short but without loss of generality that $n = 1$.

Remark 2.1.2. Let $X(\omega)$ be random variable $X(\omega) \in \mathcal{L}_{2,1}(\Omega)$ such that $X(\omega) = \mathbf{G}[|\psi\rangle](\omega)$, then we denote such random variable by $X_{|\psi\rangle}(\omega)$. The probability density of $X_{|\psi\rangle}(\omega)$ random variable we denote by $p_{|\psi\rangle}(q), q \in \mathbb{R}$.

Definition 2.1.3. The *classical pure states* correspond to vectors $\mathbf{v} \in \mathbf{H}$ of $\|\mathbf{v}\| \equiv 1$ norm. Thus the set of all classical pure states corresponds to the unit sphere $\mathbf{S}^\infty \subset \mathbf{H}$ in a $\mathbf{H}$ Hilbert space.

Definition 2.1.4. The projective Hilbert space $P(\mathbf{H})$ of a complex Hilbert space $\mathbf{H}$ is the set of equivalence classes $[\mathbf{v}]$ of vectors $\mathbf{v}$ in $\mathbf{H}$, with $\mathbf{v} \neq \mathbf{0}$, for the equivalence relation given by $\mathbf{v} \sim_P \mathbf{w} \Leftrightarrow \mathbf{v} = \lambda \mathbf{w}$ for some non-zero complex number $\lambda \in \mathbb{C}$. The equivalence classes for the relation $\sim_P$ are also called rays or projective rays.

Remark 2.1.3. The physical significance of the projective Hilbert space $P(\mathbf{H})$ is that in canonical quantum theory, the $|\psi\rangle$ and $\lambda|\psi\rangle$ states represent the same physical state of the quantum system, for any $\lambda \neq 0$. It is conventional to choose a state $|\psi\rangle$ from the ray $[|\psi\rangle]$ so that it has unit norm $\sqrt{\langle\psi|\psi\rangle} = 1$.

Remark 2.1.4. In contrast with canonical quantum theory we have used instead contrary to $\sim_P$ equivalence relation $\sim_Q$, a $\mathbf{H}$ Hilbert space, see Definition 2.1.7.

Definition 2.1.5. The *non-classical pure states* correspond to the vectors $\mathbf{v} \in \mathbf{H}$ of a $\|\mathbf{v}\| \neq 1$ norm. Thus the set of all non-classical pure states corresponds to the set $\mathbf{H} \backslash \mathbf{S}^{\infty} \subset \mathbf{H}$ in the $\mathbf{H}$ Hilbert space.

Suppose we have an observable Q of a quantum system that is found through an exhaustive series of measurements, to have a set $\mathfrak{I}$ of values $q \in \mathfrak{I}$ such that $\mathfrak{I} = \cup_{i=1}^{m} (\theta_1^i, \theta_2^i), m \geq 2,$ $(\theta_1^i, \theta_2^i) \cap \left(\theta_1^j, \theta_2^j\right) = \varnothing, i \neq j.$ Note that in practice any observable Q is measured to an accuracy δq determined by the measuring device. We represent now by $|q\rangle$ the idealized state of the system in the limit $\delta q \to 0,$ for which the observable definitely has the value $q.$

II. Then we claim the following:

Q.II.1. The states $\{|q\rangle : q \in \mathfrak{I}\}$ form a complete set of δ-function normalized basis states for the state space $\mathbf{H}_{\mathfrak{I}}$ of the system. That the states $\{|q\rangle : q \in \mathfrak{I}\}$ form a complete set of basis states means that any state $|\psi[\mathfrak{I}]\rangle \in \mathbf{H}_{\mathfrak{I}}$ of the system can be expressed as:

$$|\psi[\mathfrak{I}]\rangle = \int_{\mathfrak{I}} c_{\psi[\mathfrak{I}]}(q) dq, \tag{2.1.5}$$

where supp $(c_{\psi[\mathfrak{I}]}(q)) \subseteq \mathfrak{I}$ and while δ-function normalized means that $\langle q|q'\rangle = \delta(q - q')$ from which follows $c_{\psi[\mathfrak{I}]}(q) = \langle q|\psi[\mathfrak{I}]\rangle$ so that

$$|\psi[\mathfrak{I}]\rangle = \int_{\mathfrak{I}} |q\rangle\langle q|\psi[\mathfrak{I}]\rangle dq. \tag{2.1.6}$$

The completeness condition can then be written as

$$\int_{\mathfrak{I}} |q\rangle\langle q| dq = \hat{\mathbf{1}}_{\mathbf{H}_{\mathfrak{I}}}. \quad (2.1.7)$$

Q.II.2. For the system in state $|\psi[\mathfrak{I}]\rangle$ the probability $P(q, q+dq; |\psi[\mathfrak{I}]\rangle)$ of obtaining the result $q \in \mathfrak{I}$ lying in the range $(q, q+dq) \subset \mathfrak{I}$ on measuring observable Q is given by

$$P(q, q+dq; |\psi[\mathfrak{I}]\rangle) = p_{|\psi[\mathfrak{I}]\rangle}(q) dq \quad (2.1.8)$$

for any $|\psi[\mathfrak{I}]\rangle \in \mathbf{H}_{\mathfrak{I}}$.

Remark 2.1.5. Note that in general case $p_{|\psi[\mathfrak{I}]\rangle}(q) \neq |c_{\psi[\mathfrak{I}]}(q)|^2$.

Q.II.3. The observable $Q_{\mathfrak{I}}$ is represented by a Hermitian operator $\hat{Q}_{\mathfrak{I}} : \mathbf{H}_{\mathfrak{I}} \to \mathbf{H}_{\mathfrak{I}}$ whose eigenvalues are the possible results $\{q : q \in \mathfrak{I}\}$, of a measurement of $Q_{\mathfrak{I}}$, and the associated eigenstates are the states $\{|q\rangle : q \in \mathfrak{I}\}$, i.e. $\hat{Q}_{\mathfrak{I}}|q\rangle = q|q\rangle, q \in \mathfrak{I}$.

Remark 2.1.6. Note that the spectral decomposition of the operator $\hat{Q}_{\mathfrak{I}}$ is then

$$\hat{Q}_{\mathfrak{I}} = \int_{\mathfrak{I}} q|q\rangle\langle q| dq. \quad (2.1.9)$$

Definition 2.1.6. A connected set in $\mathbb{R}$ is a set $X \subset \mathbb{R}$ that cannot be partitioned into two nonempty subsets which are open in the relative topology induced on the set. Equivalently, it is a set which cannot be partitioned into two nonempty subsets such that each subset has no points in common with the set closure of the other.

Definition 2.1.7. The *well localized pure states* $|\psi[\Theta]\rangle$ *with a support* $\Theta = (\theta_1, \theta_2)$ correspond to vectors of the norm 1 and such that: supp $(c_{\psi[\Theta]}(q)) = \Theta$ is a connected set in $\mathbb{R}$. Thus the set of all well localized pure states corresponds to the unit sphere $\mathbf{S}^{\infty}_{\Theta} \subsetneqq \mathbf{S}^{\infty} \subset \mathbf{H}$ in the Hilbert space $\mathbf{H}_{\Theta} \subsetneqq \mathbf{H}$.

Suppose we have an observable Q_{Θ} of a system that is found through an exhaustive series of measurements, to have a continuous range of values $q : \theta_1 < q < \theta_2$.

III. Then we claim the following:

Q.III.1. For the system in well localized pure state $|\psi[\Theta]\rangle$ such that:

(i) $|\psi[\Theta]\rangle \in \mathbf{S}^{\infty}_{\Theta}$ and

(ii) supp $(c_{\psi[\Theta]}(q)) \triangleq \{q|c_{\psi[\Theta]}(q) \neq 0\}$ is a connected set in $\mathbb{R}$, then the probability $P(q,q+dq;|\psi[\Theta]\rangle)$ of obtaining the result q lying in the range $(q,q+dq)$ on measuring observable Q_{Θ} is given by

$$P(q,q+dq;|\psi[\Theta]\rangle) = |\langle q|\psi[\Theta]\rangle|^2 dq = |c_{\psi[\Theta]}(q)|^2 dq. \qquad (2.1.10)$$

Q.III.2. $p_{|\psi[\Theta]\rangle}(q)dq = |\langle q|\psi[\Theta]\rangle|^2 dq = |c_{\psi[\Theta]}(q)|^2 dq.$

Q.III.3. Let $|\psi[\Theta_1]\rangle$ and $|\psi[\Theta_2]\rangle$ be well localized pure states with $\Theta_1 = (\theta^1_1, \theta^1_2)$ and $\Theta_2 = (\theta^2_1, \theta^2_2)$ correspondingly. Let $X_1(\omega) = X_{|\psi[\Theta_1]\rangle}(\omega)$ and $X_2(\omega) = X_{|\psi[\Theta_2]\rangle}(\omega)$ correspondingly. Assume that $\overline{\Theta}_1 \cap \overline{\Theta}_2 = \varnothing$ (here the closure of $\Theta_i, i = 1,2$ is denoted by $\overline{\Theta}_i, i = 1,2$) then random variables $X_1(\omega)$ and $X_2(\omega)$ are independent.

Q.III.4. If the system is in well localized pure state $|\psi[\Theta]\rangle$ the state $|\psi[\Theta]\rangle$ described by a wave function $\psi(q,\Theta) = \langle q||\psi[\Theta]\rangle$ and the value of observable Q_{Θ} is measured once each on many identically prepared system, the average value of all the measurements will be

$$\langle Q_\Theta \rangle = \frac{\int\limits_\Theta q|\psi(q,\Theta)|^2 dq}{\int\limits_\Theta |\psi(q,\Theta)|^2 dq}. \tag{2.1.11}$$

The completeness condition can then be written as $\int_\Theta |q\rangle\langle q| dq = \widehat{\mathbf{1}}_{\mathbf{H}_\Theta}$. Completeness means that for any state $|\psi[\Theta]\rangle \in \mathbf{S}^\infty_\Theta$ it must be the case that $\int_\Theta |\langle q|\psi[\Theta]\rangle|^2 dq \neq 0,$ i.e. there must be a non-zero probability to get some result on measuring observable Q_Θ.

Q.III.5. (von Neumann measurement postulate) Assume that (i) $|\psi\rangle \in \mathbf{S}^\infty_\Theta$ and (ii) supp $(c_\psi(q)) = \Theta$ is a connected set in $\mathbb{R}$. Then if on performing a measurement of Q_Θ with an accuracy $\delta q,$ the result is obtained in the $\left(q - \frac{1}{2}\delta q, q + \frac{1}{2}\delta q\right)$ range, then the system will end up in the state

$$\frac{\widehat{P}(q,\delta q)|\psi[\Theta]\rangle}{\sqrt{\langle\psi|\widehat{P}(q,\delta q)|\psi[\Theta]\rangle}} = \frac{\int_{|q-q'|\leq\delta q/2} |q'\rangle\langle q'||\psi[\Theta]\rangle dq'}{\sqrt{\int_{|q-q'|\leq\delta q/2} |\langle q'||\psi[\Theta]\rangle|^2 dq'}}. \tag{2.1.12}$$

IV. We claim the following:

Q.IV.1. For the system in state $|\psi^a[\Theta]\rangle = a|\psi[\Theta]\rangle \in \mathbf{H}_\Theta,$ where:

(i) $|\psi[\Theta]\rangle \in \mathbf{S}^\infty_\Theta, |a| \neq 1,$

(ii) supp $(c_{\psi[\Theta]}(q))$ is a connected set in $\mathbb{R}$ and

(iii) $|\psi[\Theta]\rangle = \int_{\theta_1}^{\theta_2} c_{\psi[\Theta]}(q)|q\rangle dq$

$$\mathbf{G}\left[\widehat{Q}_{\Theta}|\psi^{a}[\Theta]\rangle\right] = |a|^{2}\mathbf{G}\left[\widehat{Q}_{\Theta}|\psi[\Theta]\rangle\right]. \qquad (2.1.13)$$

Q.IV.2. Assume that the system in state $|\psi^{a}[\Theta]\rangle = a|\psi[\Theta]\rangle \in \mathbf{H}_{\Theta},$ where

(i) $|\psi[\Theta]\rangle \in \mathbf{S}_{\Theta}^{\infty},\ \ |a| \neq 1,$

(ii) supp $(c_{\psi[\Theta]}(q))$ is a connected set in $\mathbb{R}$ and

(iii) $|\psi[\Theta]\rangle = \int_{\theta_1}^{\theta_2} c_{\psi[\Theta]}(q)|q\rangle dq.$

Then if the system is in state $|\psi^{a}[\Theta]\rangle$ described by a wave function $\psi^{a}(q;\Theta) = \langle q||\psi^{a}[\Theta]\rangle$ and the value of observable Q_{Θ} is measured once each on many identically prepared system, the average value of all the measurements will be

$$\langle Q_{\Theta}\rangle = \int_{\Theta} q|\psi^{a}(q;\Theta)|^{2}dq. \qquad (2.1.14)$$

Q.IV.3. The probability $P(q,q+dq;|\psi^{a}[\Theta]\rangle)dq$ of obtaining the result q lying in the range $(q,q+dq)$ on measuring Q_{Θ} is

$$P(q,q+dq;|\psi^{a}[\Theta]\rangle)dq = |a|^{-2}\left|c_{\psi[\Theta]}(q|a|^{-2})\right|^{2}dq. \qquad (2.1.15)$$

Remark 2.1.7. Note that Q.IV.3 immediately follows from Q.IV.1 and Q.III.2.

Q.IV.4. (Generalized von Neumann measurement postulate)
If on performing a measurement of observable Q_{Θ} with an accuracy $\delta q,$ the result is obtained in the $\left(q-\frac{1}{2}\delta q, q+\frac{1}{2}\delta q\right)$ range, then the system immediately after measurement will end up in the state

$$\frac{\widehat{P}(q,\delta q)|\psi^a[\Theta]\rangle}{\sqrt{\langle\psi|\widehat{P}(q,\delta q)|\psi[\Theta]\rangle}} = \frac{\int_{|q-q'|\leq\delta q/2}|q'\rangle\langle q'||\psi^a[\Theta]\rangle dq'}{\sqrt{\int_{|q-q'|\leq\delta q/2}|\langle q'||\psi[\Theta]\rangle|^2 dq'}} = \tag{2.1.16}$$

$$\frac{a\int_{|q-q'|\leq\delta q/2}|q'\rangle\langle q'||\psi[\Theta]\rangle dq'}{\sqrt{\int_{|q-q'|\leq\delta q/2}|\langle q'||\psi[\Theta]\rangle|^2 dq'}} \in \mathbf{H}_\Theta.$$

V. We claim the following:

Q.V.1. Let $|\Psi^{a_1,a_2}[\Theta_1,\Theta_2]\rangle = |\psi_1^{a_1}[\Theta_1]\rangle + |\psi_2^{a_2}[\Theta_2]\rangle \in \mathbf{H}_{1,2} \triangleq \mathbf{H}_{\Theta_1} \oplus \mathbf{H}_{\Theta_2} \subsetneqq \mathbf{H}$, where

(i) $|\psi_i^{a_i}[\Theta_i]\rangle = a_i|\psi_i[\Theta_i]\rangle \in \mathbf{H}_{\Theta_i}, |\psi_i\rangle = |\psi_i[\Theta_i]\rangle \in \mathbf{S}^\infty_{\Theta_i}, |a_i| \neq 1, i = 1,2;$

(ii) supp $(c_{\psi_i[\Theta_i]}(q)), i = 1,2$ are the connected sets in $\mathbb{R}$;

(iii) $\left(\mathrm{supp}(c_{\psi_1[\Theta_1]}(q))\right) \cap \left(\mathrm{supp}(c_{\psi_2[\Theta_2]}(q))\right) = \varnothing$ and

(iv) $|\psi_i[\Theta_i]\rangle = \int_{\theta_1}^{\theta_2} c_{\psi_i[\Theta_i]}(q)|q\rangle dq, i = 1,2.$

Then if the system is in a state $|\Psi^{a_1,a_2}[\Theta_1,\Theta_2]\rangle$ described by a wave function $\Psi^{a_1,a_2}(q;\Theta_1,\Theta_2) = \langle q||\Psi^{a_1,a_2}[\Theta_1,\Theta_2]\rangle, q \in \Theta_1 \cup \Theta_2$ and the value of observable Q_{Θ_1,Θ_2} is measured once each on many identically prepared system, the average value of all the measurements will be

$$\langle Q_{\Theta_1,\Theta_2}\rangle = \int\limits_{\Theta_1\cup\Theta_2} q|\Psi^{a_1,a_2}(q;\Theta_1,\Theta_2)|^2 dq. \tag{2.1.17}$$

Q.V.2. The probability of getting a result q with an accuracy δq such that $\left(q-\frac{1}{2}\delta q, q+\frac{1}{2}\delta q\right) \in supp(c_{\psi_1}(q))$ or $\left(q-\frac{1}{2}\delta q, q+\frac{1}{2}\delta q\right) \in supp(c_{\psi_2}(q))$ given by

$$\int_{|q-q'|\leq\delta q/2}\left[\left(|\langle q'||\psi_1^{a_1}[\Theta_1]\rangle|^2\right) * \left(|\langle q'||\psi_2^{a_2}[\Theta_2]\rangle|^2\right)\right]dq'. \tag{2.1.18}$$

Remark 2.1.8. Note that Q.IV.3 immediately follows from Q.III.3.

Q.V.3. Assume that the system is initially in the state $|\Psi^{a_1,a_2}[\Theta_1,\Theta_2]\rangle$. If on performing a measurement of Q_{Θ_1,Θ_2} with an accuracy δq, the result is obtained in the $\left(q-\frac{1}{2}\delta q, q+\frac{1}{2}\delta q\right)$ range, then the state of the system immediately after measurement given by

$$\frac{\widehat{P}(q_i,\delta q)|\Psi^{a_1,a_2}[\Theta_1,\Theta_2]\rangle}{\sqrt{\langle\psi|\widehat{P}(q_i,\delta q)|\psi\rangle}} =$$
$$\frac{\int_{|q_i-q'|\leq\delta q/2}(|q'\rangle\langle q'||\psi_1^{a_1}[\Theta_1]\rangle+|q'\rangle\langle q'||\psi_2^{a_2}[\Theta_2]\rangle)dq'}{\sqrt{\int_{|q_i-q'|\leq\delta q/2}\left[|\langle q'||\psi_1[\Theta_1]\rangle|^2+|\langle q'||\psi_2[\Theta_2]\rangle|^2\right]dq'}} = \tag{2.1.19}$$
$$\frac{\int_{|q_i-q'|\leq\delta q/2}(a_1|q'\rangle\langle q'||\psi_1[\Theta_1]\rangle+a_2|q'\rangle\langle q'||\psi_2[\Theta_2]\rangle)dq'}{\sqrt{\int_{|q_i-q'|\leq\delta q/2}\left[|\langle q'||\psi_1[\Theta_1]\rangle|^2+|\langle q'||\psi_2[\Theta_2]\rangle|^2\right]dq'}} \in \mathbf{H}_{\Theta_i},$$
$$q_i \in \Theta_i, i=1,2.$$

Definition 2.1.8. Let $\mathbf{H}_{1,2}$ be $\mathbf{H}_{1,2} \triangleq \mathbf{H}_{\Theta_1} \oplus \mathbf{H}_{\Theta_2}$.

Definition 2.1.9. Let $|\psi^a\rangle$ be a state $|\psi^a\rangle = a|\psi\rangle$, where $|\psi\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and $|\psi\rangle = \int_{\theta_1}^{\theta_2} c_\psi(q)|q\rangle dq$. Let $|\psi_a\rangle$ be an state such that $|\psi_a\rangle \in \mathbf{S}^\infty$. States $|\psi^a\rangle$ and $|\psi_a\rangle$ is a Q -equivalent: $|\psi^a\rangle \sim_Q |\psi_a\rangle$ iff

$$P(q,q+dq;|\psi^a\rangle) = |a|^{-2}\left|c_\psi(q|a|^{-2})\right|^2 dq = P(qq+dq;|\psi_a\rangle)dq \tag{2.1.20}$$

Q.V.4. For any state $|\psi^a\rangle = a|\psi\rangle$, where $|\psi\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and $|\psi\rangle = \int_{\theta_1}^{\theta_2} c_\psi(q)|q\rangle dq$ there exists a state $|\psi_a\rangle \in \mathbf{S}^\infty$ such that: $|\psi^a\rangle \sim_Q |\psi_a\rangle$.

Definition 2.1.10. Let $|\psi^a\rangle$ be a state $|\psi^a\rangle = a|\psi\rangle$, where $|\psi\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and $|\psi\rangle = \int_{\theta_1}^{\theta_2} c_\psi(q)|q\rangle dq$. Let $|\psi_a\rangle$ be a state such that $|\psi_a\rangle \in \mathbf{S}^\infty$. States $|\psi^a\rangle$ and $|\psi_a\rangle$ is a $\widehat{Q}$-equivalent ($|\psi^a\rangle \sim_{\widehat{Q}} |\psi_a\rangle$) iff:

$\langle\psi^a|\widehat{Q}|\psi^a\rangle = \langle\psi_a|\widehat{Q}|\psi_a\rangle$.

Q.VI. For any state $|\psi^a\rangle = a|\psi\rangle$, where $|\psi\rangle \in \mathbf{S}^\infty, |a| \neq 1$ and $|\psi\rangle = \int_{\theta_1}^{\theta_2} c_\psi(q)|q\rangle dq$ there exists a state $|\psi_a\rangle \in \mathbf{S}^\infty$ such that: $|\psi^a\rangle \sim_{\widehat{Q}} |\psi_a\rangle$

II.2. The nonclassical collapse models with spontaneous localizations based on generalized measurement postulates

The nonclassical collapse models attempt to overcome the difficulties that standard quantum mechanics meets in accounting for the measurement (or macro-objectification) problem, an attempt based on the consideration of nonlinear and nonlocal stochastic modifications of the Schrödinger equation.

The proposed new nonlocal dynamics is characterized by the feature of not contradicting any known fact about microsystems and of accounting, on the basis of a unique, universal dynamical principle, for wave packet reduction and for the classical behavior of macroscopic systems.

II.2.1. Quantum Mechanics with Nonclassical Spontaneous Localizations

Quantum Mechanics with Nonclassical Spontaneous Localizations is based on the following assumptions:

(**1**) Each particle of a system of n distinguishable particles experiences, with a mean rate λ_i, a sudden spontaneous localization process.
(**2**) In the time interval between two successive spontaneous

processes the system evolves according to the usual Schrödinger equation.

(**3**) Let $|\psi\rangle_{cl}$ be the classical pure state correspond to a vector $|\psi\rangle_{cl} \in \mathbf{S}^{\infty} \subsetneqq \mathbf{H}$ in a nonprojective Hilbert space $\mathbf{H}$, see Subsection II.1, Def. 2.1.1-2.1.2. Then the sudden spontaneous process is a localization given by:

$$|\psi\rangle_{cl} \overset{\delta,\varepsilon\text{-}\mathbf{localization}}{\longrightarrow} \frac{|\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl}}{\| \, |\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl} \|}, \mathbf{x} \in \mathbb{R}^3, \delta \in (0,1], \varepsilon \ll 1, \quad (2.2.1)$$

where

$$|\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl} = \widehat{L}^{i}_{\mathbf{x}}(\delta,\varepsilon)|\psi\rangle_{cl}. \quad (2.2.2)$$

Here $\widehat{L}^{i}_{\mathbf{x}}(\delta,\varepsilon)$ is a norm-reducing, positive, self-adjoint, linear operator with a symbol $L^{i}_{\mathbf{x}}(\delta,\varepsilon)$ in the n -particle nonprojective $\mathbf{H}$ Hilbert space, representing the localization of particle i around the point $\mathbf{x}$.

Definition 2.2.1. Such localization as mentioned above is called δ,ε-localization or δ,ε -collapse of the state $|\psi\rangle_{cl}$.

(**4**) The probability density $p_i(\mathbf{x},\delta,\varepsilon)$ for the occurrence of the localization at point $\mathbf{x}$ is assumed to be

$$p_i(\mathbf{x},\delta,\varepsilon) = \frac{\| \, |\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl} \|^2}{\iiint\limits_{\mathbb{R}^3} \| \, |\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl} \|^2 d^3x}. \quad (2.2.3)$$

(**5**) Let $|\psi\rangle_{n.cl}$ be the nonclassical pure state correspond to a vector

$$|\psi^{\zeta}\rangle = \zeta|\psi\rangle \in \mathbf{H} \backslash \mathbf{S}^{\infty},$$

where $|\psi\rangle \in \mathbf{S}^{\infty}, |\zeta| \neq 1,$ see subsection II.1, Def. 2.1.10. Then the sudden spontaneous process is a localization given by:

$$|\psi\rangle_{n.cl} \overset{\delta,\varepsilon\text{–localization}}{\longrightarrow} \frac{\zeta|\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{n.cl}}{\big\| |\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{n.cl}\big\|}, \mathbf{x} \in \mathbb{R}^3, \tag{2.2.4}$$

where

$$|\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{n.cl} = \widehat{L}^{i}_{\mathbf{x}}(\delta,\varepsilon)|\psi\rangle_{n.cl}. \tag{2.2.5}$$

Definition 2.2.2. Such localization as mentioned above is called δ,ε-localization or δ,ε-collapse of the state $|\psi\rangle_{n.cl}$.

(**6**) The probability density $p_i(\mathbf{x},\boldsymbol{\zeta},\delta,\varepsilon,)$ for the occurrence of the localization at point $\mathbf{x} \in \mathbb{R}^3$ in accordance to postulate Q.IV.3 (see Subsection II.1, Eq.(2.1.15)) is assumed to be

$$p_i(\mathbf{x},\boldsymbol{\zeta},\delta,\varepsilon,) = \frac{|\zeta|^{-6}\Big\| \big|\psi^{i}_{\delta,\varepsilon,|\zeta|^{-2}\mathbf{x}}\big\rangle_{n.cl}\Big\|^2}{\iiint\limits_{\mathbb{R}^3}\big\| |\psi^{i}_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl}\big\|^2 d^3x}. \tag{2.2.6}$$

(**7**) The localization operators $\widehat{L}^{i}_{\mathbf{x}}(\delta,\varepsilon)$ have been chosen to have the form:

$$\widehat{L}^{i}_{\mathbf{x}}(\delta,\varepsilon) = \begin{cases} \left(\frac{1}{\delta\pi_\delta}\right)^{3/4} \exp\left[-\frac{1}{2\delta}(\widehat{\mathbf{q}}_i - \mathbf{x})^2\right] \text{ iff } \|\mathbf{q}_i - \mathbf{x}\| \leq \varepsilon \ll 1, \\ 0 \text{ iff } \|\mathbf{q}_i - \mathbf{x}\| > \varepsilon. \end{cases} \tag{2.2.7}$$

Here $\delta \in (0,1]$ $\int d^3x[L^{i}_{\mathbf{x}}(\delta,\varepsilon)]^2 = 1$ and $\lim_{\delta\to 0}\pi_\delta = \pi$.

Remark 2.2.1. In one dimension case it follows that

$$\widehat{L}^{i}_{x}(\delta,\varepsilon) = \begin{cases} \left(\frac{1}{\delta\pi_\delta}\right)^{1/4} \exp\left[-\frac{1}{2\delta}(\widehat{q}_i - x)^2\right] \text{ iff } |q_i - x| \leq \varepsilon \ll 1, \\ 0 \text{ iff } |q_i - x| > \varepsilon. \end{cases} \tag{2.2.8}$$

Remark 2.2.2. Note that from Eq. (2.2.3) and Eq. (2.2.7) it follows that a probability density $p_i(\mathbf{x},\boldsymbol{\zeta},\delta,\varepsilon,)$ for the occurrence of the localization inside sphere $S(\mathbf{x},\varepsilon) = \{\mathbf{q}_i \in \mathbb{R}^3 | \|\mathbf{q}_i - \mathbf{x}\| \leq \varepsilon\}$ is given by

$$p_i(\mathbf{x},\delta,\varepsilon) = \frac{\| |\psi^i_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl} \|^2}{\Omega(\delta,\varepsilon)}, \Omega(\delta,\varepsilon) = \iiint\limits_{\mathbb{R}^3} \| |\psi^i_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl} \|^2 d^3x,$$

$$\| |\psi^i_{\delta,\varepsilon,\mathbf{x}}\rangle_{cl} \|^2 = \left(\frac{1}{\delta\pi_\delta}\right)^{3/2} \int\limits_{\|\mathbf{q}_i - \mathbf{x}\| \leq \varepsilon} d^3q_i \psi^i(\mathbf{q}_i) \exp\left[-\frac{1}{\delta}(\mathbf{q}_i - \mathbf{x})^2\right], \quad (2.2.9)$$

$$\psi^i(q_i) = \langle q_i \| \psi^i \rangle_{cl},$$

and therefore

$$p_i(\mathbf{x},\varepsilon) = \lim_{\delta\to 0} p_i(\mathbf{x},\delta,\varepsilon) =$$

$$\lim_{\delta\to 0} \Omega^{-1}(\delta,\varepsilon)\left(\frac{1}{\delta\pi_\delta}\right)^{3/2} \int\limits_{\|\mathbf{q}_i - \mathbf{x}\| \leq \varepsilon} d^3q_i \psi^i(\mathbf{q}_i) \exp\left[-\frac{1}{\delta}(\mathbf{q}_i - \mathbf{x})^2\right] = \psi^i(\mathbf{x}). \quad (2.2.10)$$

Remark 2.2.3. In one dimension case it follows that a probability density $p_i(x,\delta,\varepsilon,)$ for the occurrence of the localization inside interval $[x-\varepsilon, x+\varepsilon]$ is given by

$$p_i(x,\delta,\varepsilon) = \| |\psi^i_{\delta,\varepsilon,x}\rangle_{cl} \|^2 = \left(\frac{1}{\delta\pi_\delta}\right)^{1/2} \int\limits_{|q_i - x| \leq \varepsilon} d^3q_i \psi^i(q_i) \exp\left[-\frac{1}{\delta}(q_i - x)^2\right],$$

$$\psi^i(q_i) = \langle q_i \| \psi^i \rangle_{cl}, \quad (2.2.11)$$

and therefore

$$p_i(x,\varepsilon) = \lim_{\delta\to 0} p_i(x,\delta,\varepsilon) =$$

$$= \lim_{\delta\to 0} \Omega^{-1}(\delta,\varepsilon) \left(\frac{1}{\delta\pi_\delta}\right)^{1/2} \int\limits_{|q_i - x| \leq \varepsilon} dq_i \psi^i(q_i) \exp\left[-\frac{1}{\delta}(q_i - x)^2\right] = \psi^i(x). \quad (2.2.12)$$

II.2.2. The generalization of nonclassical collapse models

(**1**) Let $|\psi_t\rangle_{cl}, t \in [0,T]$ be the classical pure states correspond to a vector-function $|\psi_t\rangle_{cl} : [0,T] \times \mathbf{S}^\infty \to \mathbf{S}^\infty$ such that

$$|\psi_t\rangle_{cl} \in \mathbf{S}^{\infty} \subsetneqq \mathbf{H}\ , t \in [0,T],$$

where is a nonprojective $\mathbf{H}$ Hilbert space, see Subsection II.1, Def.2.1.1-2.1.2. Then the sudden spontaneous process is the localization along classical trajectory $\mathbf{x}_t : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ given by:

$$|\psi_t\rangle_{cl} \overset{\delta,\varepsilon,\mathbf{x}_t\text{–localization}}{\longrightarrow} \frac{|\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl}}{\| |\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl} \|}, \quad (2.2.13)$$
$$\delta \in (0,1], \varepsilon \ll 1, \mathbf{x}_t \in \mathbb{R}^3, t \in [0,T],$$

where

$$|\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl} = \widehat{L}^i_{\mathbf{x}_t}(\delta,\varepsilon)|\psi_t\rangle_{cl}. \quad (2.2.14)$$

Here $\widehat{L}^i_{\mathbf{x}_t}(\delta,\varepsilon)$ is a norm-reducing, positive, self-adjoint, linear operator with a symbol $L^i_{\mathbf{x}_t}(\delta,\varepsilon)$ in the n-particle nonprojective $\mathbf{H}$ Hilbert space, representing the localization of particle i at each instant $t \in [0,T]$ around the point $\mathbf{x}_t$.

Definition 2.2.3. Such localization as mentioned above is called $\delta,\varepsilon,\mathbf{x}_t$-localization or $\delta,\varepsilon,\mathbf{x}_t$-collapse of the state $|\psi_t\rangle_{cl}$.

(**2**) The probability density $p_i(t,\mathbf{x}_t,\delta,\varepsilon)$ for the occurrence of the localization at point $\mathbf{x}_t$ at instant t is assumed to be

$$p_i(t,\mathbf{x}_t,\delta,\varepsilon) = \frac{\| |\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl} \|^2}{\Omega(t,\delta,\varepsilon)}, \Omega(t,\delta,\varepsilon) = \iiint_{\mathbb{R}^3} \| |\psi^i_{\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl} \|^2 d^3x. \quad (2.2.15)$$

(**3**) Let $|\psi_t\rangle_{n.cl}$ be the nonclassical pure state corresponding to a vector-function $|\psi^{\zeta}_t\rangle = \zeta|\psi_t\rangle \in \mathbf{H} \backslash \mathbf{S}^{\infty}$, where $|\psi_t\rangle \in \mathbf{S}^{\infty}, |\zeta| \neq 1, t \in [0,T]$, see Subsection II.1, Def.2.1.10.

Then the sudden spontaneous process is the localization along classical trajectory $\mathbf{x}_t : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ given by:

$$|\psi_t\rangle_{n.cl} \overset{\delta,\varepsilon,\mathbf{x}_t\text{-}\mathbf{localization}}{\longrightarrow} \frac{\zeta|\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{n.cl}}{\| |\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{n.cl} \|}, \quad \mathbf{x}_t \in \mathbb{R}^3, t \in [0,T] \tag{2.2.16}$$

where

$$|\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{n.cl} = \widehat{L}^i_{\mathbf{x}_t}(\delta,\varepsilon)|\psi_t\rangle_{n.cl}. \tag{2.2.17}$$

Definition 2.2.4. Such localization as mentioned above is called $\delta,\varepsilon,\mathbf{x}_t$-localization or $\delta,\varepsilon,\mathbf{x}_t$-collapse of the state $|\psi\rangle_{n.cl}$.

(**4**) The probability density $p_i(t,\mathbf{x}_t,\zeta,\delta,\varepsilon,)$ for the occurrence of the localization at point $\mathbf{x}_t \in \mathbb{R}^3$ at instant $t \in [0,T]$ in accordance to postulate Q.IV.3 (see Subsection **II**.1, Eq. (2.1.14)) is assumed to be

$$p_i(t,\mathbf{x}_t,\zeta,\delta,\varepsilon,) = \frac{|\zeta|^{-6} \left\| \left|\psi^i_{t,\delta,\varepsilon,|\zeta|^{-2}\mathbf{x}_t}\right\rangle_{n.cl} \right\|^2}{\Omega(t,\delta,\varepsilon)}, \Omega(t,\delta,\varepsilon) = \iiint_{\mathbb{R}^3} \| |\psi^i_{\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl} \|^2 d^3x. \tag{2.2.18}$$

(**5**) The localization operators $\widehat{L}^i_{\mathbf{x}_t}(\delta,\varepsilon)$ have been chosen to have the form:

$$\widehat{L}^i_{\mathbf{x}_t}(\delta,\varepsilon) = \begin{cases} \left(\frac{1}{\delta\pi_\delta}\right)^{3/4} \exp\left[-\frac{1}{2\delta}(\widehat{\mathbf{q}}_i - \mathbf{x}_t)^2\right] \text{ iff } \|\mathbf{q}_i - \mathbf{x}_t\| \leq \varepsilon \ll 1, \\ 0 \text{ iff } \|\mathbf{q}_i - \mathbf{x}_t\| > \varepsilon. \end{cases} \tag{1.2.19}$$

Here $\delta \in (0,1]$ and $\lim_{\delta\to 0} \pi_\delta = \pi$.

Remark 2.2.4. In one dimension case it follows that

$$\widehat{L}^i_{x_t}(\delta,\varepsilon) = \begin{cases} \left(\frac{1}{\delta\pi_\delta}\right)^{1/4} \exp\left[-\frac{1}{2\delta}(\widehat{q}_i - x_t)^2\right] \text{ iff } |q_i - x_t| \leq \varepsilon \ll 1, \\ 0 \text{ iff } |q_i - x_t| > \varepsilon. \end{cases} \tag{2.2.20}$$

Remark 2.2.5. Note that from Eq. (2.2.18) and Eq. (2.2.19) it follows that a probability density $p_i(t, \mathbf{x}_t, \boldsymbol{\zeta}, \delta, \varepsilon,)$ for the occurrence of the localization at instant t inside sphere $S(\mathbf{x}_t, \varepsilon) = \{\mathbf{q}_i \in \mathbb{R}^3 | \|\mathbf{q}_i - \mathbf{x}_t\| \leq \varepsilon\}$ is given by

$$p_i(t, \mathbf{x}_t, \delta, \varepsilon) = \frac{\| |\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl} \|^2}{\Omega(t, \delta, \varepsilon)}$$
$$\| |\psi^i_{t,\delta,\varepsilon,\mathbf{x}_t}\rangle_{cl} \|^2 = \left(\frac{1}{\delta \pi_\delta}\right)^{3/2} \int\limits_{\|\mathbf{q}_i - \mathbf{x}_t\| \leq \varepsilon} d^3 q_i \psi^i_t(\mathbf{q}_i) \exp\left[-\frac{1}{\delta}(\mathbf{q}_i - \mathbf{x}_t)^2\right], \tag{2.2.21}$$
$$\psi^i_t(q_i) = \langle q_i | |\psi^i_t\rangle_{cl},$$

and therefore

$$p_i(t, x, \varepsilon) = \lim_{\delta \to 0} p_i(t, x, \delta, \varepsilon) =$$
$$= \lim_{\delta \to 0} \Omega^{-1}(t, \delta, \varepsilon) \left(\frac{1}{\delta \pi_\delta}\right)^{1/2} \int\limits_{|q_i - x| \leq \varepsilon} dq_i \psi^i(q_i) \exp\left[-\frac{1}{\delta}(q_i - x_t)^2\right] = \psi^i(x_t). \tag{2.2.22}$$

Chapter III

III. EINSTEIN’S 1927 GEDANKEN EXPERIMENT REVISITED

III.1. Single-photon space-like antibunching

During the famous 5-th Solvay conference in 1927, Einstein [7] considered a single particle which, after diffraction in a pin-hole encounters a detection plate (e.g. in the case of photons, a photographic plate), see Fig 3.1.1.

We simplify this thought experiment, though keeping the essence, by replacing the detection plate by two detectors. Einstein noted that there is no a question that only one of them can detect the particle, otherwise energy would not be conserved. However, he was deeply concerned about the situation in which the two detectors are space-like separated, as this prevents - according to relativity - any possible coordination among the detectors: It seems to me, Einstein continued, that this difficulty cannot be overcome unless the description of the process in terms of the Schrödinger wave is supplemented by some detailed specification of the localization of the particle during its propagation. I think M. de Broglie is right in searching in this direction.
But what happened to Einstein's original Gedanken experiment?
This simple - with today's technology - experiment had been done originally by T. Guerreiro, B. Sanguinetti, H. Zbinden N. Gisin, and A. Suarez, see [21]. This experiment consists in verifying that when a single photon is thrown at a beam splitter, it is detected in only one arm, i.e. the probability $\mathbf{P}_{A\wedge B}$ of getting a coincidence between the two detectors A and B is much smaller than the product of the probabilities of detection on each side $\mathbf{P}_A \times \mathbf{P}_B,$ as would be expected in the case of uncorrelated events.

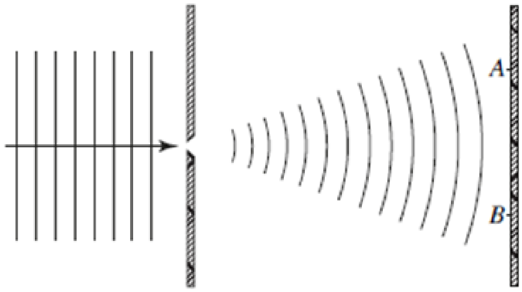

Fig. 3.1.1. Einstein’s 1927 gedanken experiment. A and B are points on the photographic plate, for which the events of detection can be space-like separated from each other. Adapted from [7]

The experimental setup is shown in Fig. 3.1.2 and consists of a source of heralded single photons which is coupled into a single mode fiber and injected into a fiber beam splitter (BS). Each of the two outputs of the beam splitter goes to a single photon detector (IDQ ID200), A detector being close to the source and B detector being separated by a distance of approximately 10 meters.

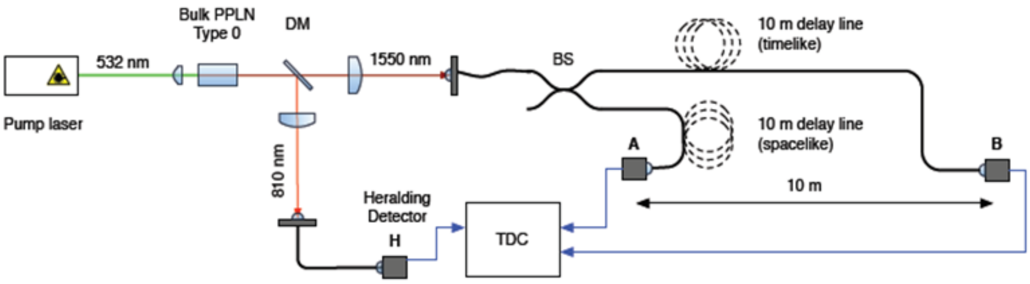

Fig. 3.1.2. Experimental setup: photon pairs regenerated by Spontaneous Parametric Down Conversion at the wavelengths of 1550 nm and 810 nm. These pairs are splitted by a dichroic mirror (DM), and 810 nm photon is sent to detector D, used to herald the presence of the 1550 nm photon that follows to the beam splitter (BS). Arbitrary electronic delays were applied before TDC to ensure the coincidence peak would remain on scale. Adapted from [21]

If we ensure that the fiber lengths before each detector are equal by inserting a 10 m (50 ns) fiber delay loop before detector A, the detections will happen simultaneously in some reference frame, thus being space-like separated (a signal would take 33 ns to travel between the two detectors at the speed of light; simultaneity of detection is guaranteed to within 1ns by the matched length of fiber both before and inside the detectors). It is also possible to make the detections time-like separated by removing the 10m delay line from detector A and adding it to detector B.

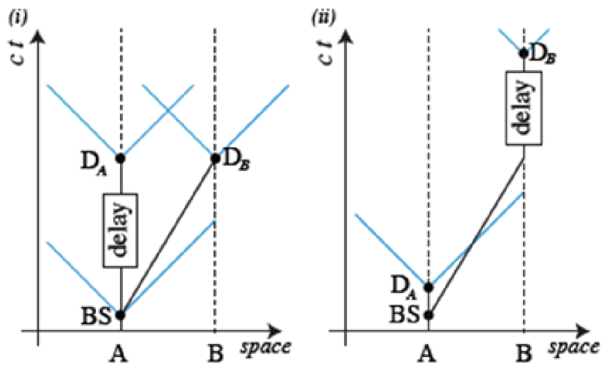

Fig. 3.1.3. Spacetime diagrams for spacelike (i) and timelike (ii) configurations. A and B represent the locations of the detectors. Adapted from [21]

First, one measures the probabilities of detecting a photon at detector A or at detector B given that a heralding photon has been detected at H. We denote R_{HA} the total number of coincident counts at detector H and detector A during the time of measurement, and $R_{H(A)}$ the total number of counts at detector H alone during the same measurement; R_{HB} and $R_{H(B)}$ denote similar quantities for the measurement with H and B.
Next we measure the probability of detectors A and B clicking at the same time, again given a heralding signal. R_{HAB} denotes the number

of triple coincident counts at the detectors H, A and B, and $R_{H(AB)}$ denotes the total number of counts at detector H alone during the same measurement. All these quantities are measured directly for both a space-like configuration and a time-like configuration.

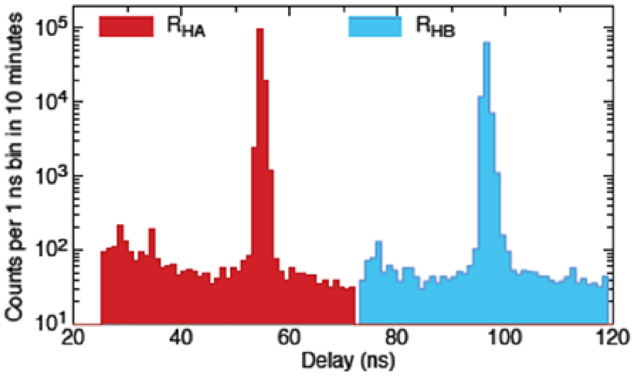

Fig. 3.1.4. Coincidences between the heralding detector and each of the detectors A (red) and B (blue) with spacelike separation, measured in a window of 1 ns during a time period of 10 minutes. R_{HA}=9.49×104/10 min, R_{HB}=6.39×104/10 min. The noise is on average: R_N=50/10 min. Adapted from [21]

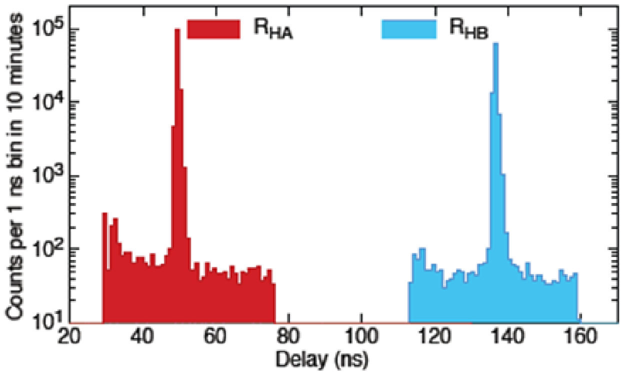

Fig. 3.1.5. Coincidences between the heralding detector and each of the detectors A (red) and B (blue) with timelike separation, measured in a window of 1 ns during a time period of 10 minutes. R_{HA}=9.90×104/10 min, R_{HB}=6.22×104/10 min. Adapted from [21]

The raw TDC data is shown in Figures 3.1.4-3.1.5 and the results are summarized in Table 3.1.1.

Table 3.1.1.

Summary of results. Values obtained for the different counting rates and corresponding probabilities defined in the text, measured with spacelike and timelike separation. Adapted from [21]

Spacelike separation

R_{HA}	$R_{H(A)}$	$P_A^{SL} = R_{HA}/R_{H(A)}$
$(94.8 \pm 0.3) \cdot 10^3$	$(5570 \pm 2) \cdot 10^3$	$(1.703 \pm 0.006) \cdot 10^{-2}$
R_{HB}	$R_{H(B)}$	$P_B^{SL} = R_{HB}/R_{H(B)}$
$(63.8 \pm 0.2) \cdot 10^3$	$(5860 \pm 2) \cdot 10^3$	$(1.090 \pm 0.004) \cdot 10^{-2}$
R_{HAB}	$R_{H(AB)}$	$P^{SL}(1,1) = R_{HAB}/R_{H(AB)}$
4 ± 2	$(17145 \pm 4) \cdot 10^3$	$(2.3 \pm 1.2) \cdot 10^{-7}$
R_{HN}	$R_{H(N)}$	$P_N^{SL} = R_{HN}/R_{H(N)}$
50 ± 7	$(5500 \pm 2) \cdot 10^3$	$(9.0 \pm 1.3) \cdot 10^{-6}$

Timelike separation

R_{HA}	$R_{H(A)}$	$P_A^{TL} = R_{HA}/R_{H(A)}$
$(99.0 \pm 0.3) \cdot 10^3$	$(6130 \pm 2) \cdot 10^3$	$(1.616 \pm 0.005) \cdot 10^{-2}$
R_{HB}	$R_{H(B)}$	$P_B^{TL} = R_{HB}/R_{H(B)}$
$(62.2 \pm 0.2) \cdot 10^3$	$(6100 \pm 2) \cdot 10^3$	$(1.019 \pm 0.004) \cdot 10^{-2}$
R_{HAB}	$R_{H(AB)}$	$P^{TL}(1,1) = R_{HAB}/R_{H(AB)}$
4 ± 2	$(18345 \pm 4) \cdot 10^3$	$(2.2 \pm 1.1) \cdot 10^{-7}$

The number of counts given by detector noise and two-photon events can be estimated by looking at the counts away from the peak. As an example, for the space-like configuration (Figure 3.1.4) in a window of 1 ns the noise rate is on average R_{HN} = 50/10 for a 10 minutes integration time [21]. This corresponds to a noise probability $\mathbf{P}_N = 9 \cdot 10^{-6}$ (1.310^{-6}). From the values in Table 1 one derives the following probability values for spacelike separation:

$$\mathbf{P}_A^{SL} \cdot P_B^{SL} = 1.86 \pm 0.01 \cdot 10^{-4}, \quad \mathbf{P}_{A \wedge B}^{SL} = 0.002 \pm 0.001 \cdot 10^{-4}. \qquad (3.1.1)$$

For timelike separation one derives the values:

$$\mathbf{P}_A^{TL} \cdot P_B^{TL} = 1.65 \pm 0.01 \cdot 10^{-4},$$
$$\mathbf{P}_{A \wedge B}^{TL} = 0.002 \pm 0.001 \cdot 10^{-4}. \quad (3.1.2)$$

For the probability $\mathbf{P}_N^{SL}$ that A and B detect photons coming from different pairs (noise) one derives the value:

$$\mathbf{P}_N^{SL}(1,1) = \mathbf{P}_N^{SL} \cdot \mathbf{P}_A^{SL} + \mathbf{P}_N^{SL} \cdot \mathbf{P}_B^{SL} \approx$$
$$0.0025 \pm 0.0026 \cdot 10^{-4} \quad (3.1.3)$$

III.2. The measure algebra of physical events in Minkowski space-time

Definition 3.2.1. [22]. A measure algebra $\mathcal{F} = (\mathbf{B}, \mathbf{P})$ with a probability measure $\mathbf{P}$, is a Boolean algebra $\mathbf{B}$ with a countably additive probability measure.

Definition 3.2.2. (i) A measure algebra of physical events $\mathcal{F}^{ph} = (\mathbf{B}, \mathbf{P})$ with a probability measure $\mathbf{P}$, is a Boolean algebra of physical events $\mathbf{B}$ with a countably additive probability measure.
(ii) A Boolean algebra of physical events can be formally defined as a set $\mathbf{B}$ of elements $a, b, \ldots$ with the following properties:
1. $\mathbf{B}$ has two binary operations, $\wedge$ (logical AND, or "wedge") and $\vee$ (logical OR, or "vee"), which satisfy:
the idempotent laws:(1) $a \wedge a = a \vee a = a$,
the commutative laws:(2) $a \wedge b = b \wedge a$,
(3) $a \vee b = b \vee a$,
and the associative laws:
(4) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$,
(5) $a \vee (b \vee c) = (a \vee b) \vee c$.
2. The operations satisfy the absorption law:
(6) $a \wedge (a \vee b) = a \vee (a \wedge b) = a$.
3. The operations are mutually distributive

(7) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

(8) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$.

4. $\mathbf{B}$ contains universal bounds $\mathbf{0}$ and $\mathbf{1}$ which satisfy

(9) $\mathbf{0} \wedge a = \mathbf{0}$,

(10) $\mathbf{0} \vee a = a$,

(11) $\mathbf{1} \wedge a = a$,

(12) $\mathbf{1} \vee a = \mathbf{1}$.

5. $\mathbf{B}$ has an unary operation $\neg a$ (or a') of complementation (logical negation), which obeys the laws:

(13) $a \wedge \neg a = \mathbf{0}$,

(14) $a \vee \neg a = \mathbf{1}$.

All properties of negation including the laws below follow from the above two laws alone.

6. Double negation law: $\neg(\neg a) = a$.

7. De Morgan's laws: (i) $\neg a \wedge \neg b = \neg(a \vee b)$,

(ii) $\neg a \vee \neg b = \neg(a \wedge b)$.

8. Operations composed from the basic operations include the following important examples:

The first operation, $a \to b$ (logical material implication):

(i) $a \to b \triangleq \neg a \vee b$.

The second operation, $a \oplus b$, is called exclusive. It excludes the possibility of both a and b

(ii) $a \oplus b \triangleq (a \vee b) \wedge \neg(a \wedge b)$.

The third operation, the complement of exclusive or, is equivalence or Boolean equality:

(iii) $a \equiv b \triangleq \neg(a \oplus b)$.

9. $\mathbf{B}$ has a unary predicate $\mathbf{Occ}(a)$, which meant that event a has occurred, and which obeys the laws:

(i) $\mathbf{Occ}(a \wedge b) \Leftrightarrow \mathbf{Occ}(a) \wedge \mathbf{Occ}(b)$,

(ii) $\mathbf{Occ}(a \vee b) \Leftrightarrow \mathbf{Occ}(a) \vee \mathbf{Occ}(b)$,

(iii) $\mathbf{Occ}(\neg a) \Leftrightarrow \neg\mathbf{Occ}(a)$.

Remark 3.2.1. A probability measure $\mathbf{P}$ on a measure space (Ω, Σ) gives the probability measure algebra $\mathcal{F} = (\Omega, \Sigma, \mathbf{P})$ on the Boolean algebra of measurable sets modulo null sets.

Definition 3.2.3. (i) Let $\mathbf{B}$ be a Boolean algebra of physical events. A Boolean algebra $\mathbf{B}_{M_4}$ of physical events in Minkowski space-time $M_4 = \mathbb{R}^{1,3}$ that is Cartesian product $\mathbf{B}_{M_4} = \mathbf{B} \times M_4$.

(ii) Let $\mathbf{B}_{M_4}$ be a Boolean algebra of physical events in Minkowski space-time. A measure algebra of physical events $\mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4}, \mathbf{P})$ in Minkowski space-time that is a Boolean algebra $\mathbf{B}_{M_4}$ with a probability measure $\mathbf{P}$.

(iii) Let $\mathbf{B}_{M_4}$ be Boolean algebra of the all physical events in Minkowski space-time and let $\mathcal{F}^{ph}_{M_4}$ be a measure algebra $\mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4}, \mathbf{P})$ with a probability measure $\mathbf{P}$. We denote such physical events by $A(\mathbf{x}), B(\mathbf{x}), \ldots$ etc., where $\mathbf{x} = (t, x_1, x_2, x_3) \in M_4$ or $A, B, \ldots$ etc.

(iv) We will write for a short $A^{Oc}(\mathbf{x}), B^{Oc}(\mathbf{x}), \ldots$ etc., instead $\mathbf{Occ}(A(\mathbf{x})), \mathbf{Occ}(B(\mathbf{x})), \ldots$ etc.

Definition 3.2.4. Let $\mathbf{Aut}_{\mathbf{P}}(\mathbf{B}_{M_4})$ be a set of the all measure-preserving automorphism of $\mathbf{B}_{M_4}$. This is a group, being a subgroup of the group $\mathbf{Aut}(\mathbf{B}_{M_4})$ of all Boolean automorphism of $\mathbf{B}_{M_4}$. Let $P^{\uparrow}_{+}$ be Poincaré group.

Remark 3.2.2. We assume now that: any element $\Theta = (\Lambda, a) \in P^{\uparrow}_{+}$ induced an element $\widetilde{\Theta} \in \mathbf{Aut}_{\mathbf{P}}(\mathbf{B}_{M_4})$ by formula $\widetilde{\Theta} = \Theta[A(\mathbf{x})] = A(\Lambda\mathbf{x} + \mathbf{a}) \in \mathbf{B}_{M_4}$.

Definition 3.2.5. Given two events A and B from the algebra

$\mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4}, \mathbf{P})$ the conditional probability of A given B is defined as the quotient of the probability of the joint of events A and B, and the probability of B :

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \wedge B)}{\mathbf{P}(B)} = \frac{\mathbf{P}_{A \wedge B}}{\mathbf{P}_B} = \mathbf{P}_{A|B}, \tag{3.2.1}$$

where $\mathbf{P}(B) \neq 0$.

Definition 3.2.6. (i) Events A and B from the algebra $\mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4}, \mathbf{P})$ are defined to be statistically independent or uncorrelated iff

$$\mathbf{P}_{A \wedge B} = \mathbf{P}_A \cdot \mathbf{P}_B, \tag{3.2.2}$$

where $\mathbf{P}_B \neq 0$, then this is equivalent to the statement that $\mathbf{P}_{A|B} = \mathbf{P}_A$. Similarly if $\mathbf{P}_A$ is not zero, then $\mathbf{P}_{B|A} = \mathbf{P}_B$ is also equivalent.

(ii) Events A and B from the algebra $\mathcal{F} = (\mathbf{B}_{M_4}, \mathbf{P})$ are defined to be statistically almost independent or almost uncorrelated iff

$$\begin{gathered} \mathbf{P}_{A \wedge B} \approx \mathbf{P}_A \cdot \mathbf{P}_B, \\ \mathbf{P}_{A \wedge B} = \mathbf{P}_A \cdot \mathbf{P}_B - \delta(A,B), 0 < \delta(A,B) \ll \mathbf{P}_A \cdot \mathbf{P}_B. \end{gathered} \tag{3.2.3}$$

Remark 3.2.3. Note that

$$\mathbf{P}_{A \vee B} = \mathbf{P}_A + \mathbf{P}_B - \mathbf{P}_{A \wedge B}. \tag{3.2.4}$$

Although mathematically equivalent, this may be preferred philosophically; under major probability interpretations such as the subjective theory, conditional probability is considered a primitive entity. Further, this "multiplication axiom" introduces the symmetry with the summation axiom for mutually exclusive events, i.e.

$$\mathbf{P}_{A \vee B} = \mathbf{P}_A + \mathbf{P}_B - \not{\mathbf{P}}_{A \wedge B}. \tag{3.2.5}$$

Definition 3.2.7. (i) Events $A_1, A_2, \ldots, A_n \in \mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4}, \mathbf{P})$ are said to be exactly mutually exclusive if the occurrence of any one of them implies the non-occurrence of the remaining $n-1$ events. Therefore, two mutually exclusive events cannot both occur. Formally said, the conjunction of each two of them is $\mathbf{0}$ (the null event): $A \wedge B = \mathbf{0}$. In consequence, exactly mutually exclusive events A and B have the property:

$$\mathbf{P}(A \wedge B) = 0. \tag{3.2.6}$$

(ii) Events $A_1, A_2, \ldots, A_n \in \mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4}, \mathbf{P})$ are said to be almost mutually exclusive if $A_1, A_2, \ldots, A_n$ have the property:

$$\begin{gathered} \mathbf{P}(A_1 \wedge A_2 \wedge \ldots \wedge A_n) \approx 0, \\ \mathbf{P}(A_1 \wedge A_2 \wedge \ldots \wedge A_n) \ll \mathbf{P}(A_1) \cdot \mathbf{P}(A_2) \cdots \mathbf{P}(A_n). \end{gathered} \tag{3.2.7}$$

In consequence, almost mutually exclusive events A and B have the property:

$$\begin{gathered} \mathbf{P}(A \wedge B) \approx 0, \\ \mathbf{P}(A \wedge B) \ll \mathbf{P}(A) \cdot \mathbf{P}(B). \end{gathered} \tag{3.2.8}$$

Remark 3.2.4. Let A^{ph}, B^{ph} be events such that detectors A, B detect photon at an instants t_1 and t_2 correspondingly. Note that (3.1.1) and (3.1.2) show that whether the separation between the detectors is timelike or spacelike, the number of coincidences is three orders of magnitude smaller than what would be expected had the events been statistically almost uncorrelated, i.e., $\mathbf{P}_{A \wedge B} \approx \mathbf{P}_A \cdot \mathbf{P}_B$, see Def.3.2.6 (ii).

Remark 3.2.5. Let A^{ph}, B^{ph} be events such that detectors A, B detect photon at an instants t_1 and t_2 correspondingly. Note that:

(i) from Eq. (3.1.1) probability value for spacelike separation follows:

$$\mathbf{P}^{SL}_{A^{ph}\wedge B^{ph}} = 0.002 \pm 0.001 \cdot 10^{-4} \neq 0, \qquad (3.2.9)$$

(ii) from Eq. (3.1.2) probability value for timelike separation follows:

$$\mathbf{P}^{TL}_{A^{ph}\wedge B^{ph}} = 0.002 \pm 0.001 \cdot 10^{-4} \neq 0. \qquad (3.2.10)$$

Therefore in both cases the property (3.2.6) are violated, i.e. $\mathbf{P}_{A^{ph}\wedge B^{ph}} \neq 0$ but however in both cases the property (3.2.8) is satisfied

$$\begin{aligned} 0.002 \pm 0.001 \cdot 10^{-4} &= \mathbf{P}^{SL}_{A^{ph}\wedge B^{ph}} \ll \mathbf{P}^{SL}_{A^{ph}} \cdot \mathbf{P}^{SL}_{B^{ph}} = 1.86 \pm 0.01 \cdot 10^{-4}, \\ 0.002 \pm 0.001 \cdot 10^{-4} &= \mathbf{P}^{TL}_{A^{ph}\wedge B^{ph}} \ll \mathbf{P}^{TL}_{A^{ph}} \cdot \mathbf{P}^{TL}_{B^{ph}} = 1.65 \pm 0.01 \cdot 10^{-4} \end{aligned} \qquad (3.2.11)$$

and therefore in both cases the events A^{ph}, B^{ph} are almost mutually exclusive events.

III.2.1. Beamsplitter transformation

A beamsplitter is the most simple way to mix two modes, see Fig.3.2.1. From classical electrodynamics, one gets the following amplitudes for the outgoing modes:

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix}_{\mathbf{in}} \mapsto \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}_{\mathbf{out}} = \begin{pmatrix} t & r \\ r' & t' \end{pmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}_{\mathbf{in}}. \qquad (3.2.12)$$

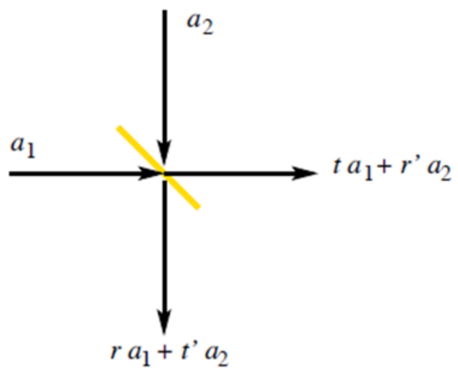

Fig. 3.2.1. Mixing of two models by a beamsplitter

The recipe for quantization is now: `replace the classical amplitudes by annihilation operators'.
If the outgoing modes are still to be useful for the quantum theory, they have to satisfy the commutation relations:

$$[A_i(out), A_j(out)] = \delta_{ij} \tag{3.2.13}$$

These conditions give constraints on the reflection and transmission amplitudes, for example $|t'|^2 + |r'|^2 = 1$.

We are now looking for an unitary operator $\mathbf{S}$ [the $\mathbf{S}$ -matrix] that implements this beamsplitter transformation in the following sense:

$$A_i = \mathbf{S}^\dagger a_i \mathbf{S}, i = 1, 2. \tag{3.2.14}$$

Let us start from the general transformation (summation over double indices)

$$a_i \longmapsto A_i = B_{ij} a_i, \vec{a} \longmapsto \vec{A} = \mathbf{B}\vec{a} \tag{3.2.15}$$

where we have introduced the matrix and vector notation.
Using this $\mathbf{S}$ -matrix one can also compute the transformation of the states: $|out\rangle = \mathbf{S}|in\rangle$. For the unitary transformation, we make the ansatz

$$\mathbf{S}(\theta) = \exp(i\theta J_{ki} a_k^\dagger a_i) \tag{3.2.16}$$

with J_{kl} a Hermitian matrix (ensuring unitarity). The action of this unitary on the photon mode operators is now required to reduce to

$$a_i \longmapsto A_i(\theta) = \mathbf{S}^\dagger(\theta) a_i \mathbf{S}(\boldsymbol{\theta}) = B_{ij} a_j. \tag{3.2.17}$$

We compute this `operator conjugation' by using a differential equation:

$$\frac{dA_i(\theta)}{d\theta} = iJ_{ki}A_i(\theta). \tag{3.2.18}$$

This is a system of linear differential equations with constant coefficients, so that one obtains a solution

$$\vec{A}(\theta) = \exp(i\theta\mathbf{J}). \tag{3.2.19}$$

We thus conclude that the so-called generator $\mathbf{J}$ of the beam splitter matrix is fixed by equation

$$\mathbf{B} = \exp(i\theta\mathbf{J}). \tag{3.2.20}$$

If the transformation $\mathbf{B}$ is part of a continuous group and depends on θ as a parameter, we can expand it around unity. Doing the same for the matrix exponential, we get

$$\mathbf{B} = 1 + i\theta\mathbf{J} + \dots \tag{3.2.21}$$

Equation (3.2.21) explains the name generator for the $\mathbf{J}$ matrix: it actually generates a subgroup of matrices $\mathbf{B} = \mathbf{B}(\theta)$ parametrized by the θ angle. The unitary transformation we are looking for is thus determined via the same $\mathbf{J}$ generator. For the two-mode beam splitter, an admissible transformation is given by

$$\mathbf{B}(\theta) = \begin{pmatrix} t & r \\ r' & t' \end{pmatrix} = \begin{pmatrix} \cos\theta & i\sin\theta \\ i\sin\theta & \cos\theta \end{pmatrix} \tag{3.2.22}$$

The factor i is just put for convenience so that the reflection amplitudes are the same for both sides, $r = r'$, as expected by symmetry. Expanding for small θ, the generator is

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1 \tag{3.2.23}$$

and so that the unitary operator for this beamsplitter is

$$S(\theta) = \exp[i\theta(a_1^\dagger a_2 + a_2^\dagger a_1)]. \tag{3.2.24}$$

herefore, the effective Hamiltonian of the beam splitter is given by

$$\widehat{H}_{\mathbf{eff}} = a_1^\dagger a_2 + a_2^\dagger a_1. \tag{3.2.25}$$

III.2.2. Splitting a two-photon state

Let us consider two single photon states $|in\rangle = |1,1\rangle$ incident on the beam splitter such that mentioned above. Then

$$\begin{gathered} |\psi(\theta)\rangle = |out\rangle = \mathbf{S}|in\rangle = \mathbf{S}a_1^\dagger S^\dagger S a_2^\dagger S^\dagger S|0,0\rangle = \\ (a_1^\dagger \cos\theta + i a_2^\dagger \sin\theta)(a_2^\dagger \cos\theta + i a_1^\dagger \sin\theta)|0,0\rangle = \\ (|2,0\rangle - |0,2\rangle)\frac{\sin\theta}{\sqrt{2}} + |1,1\rangle\cos\theta. \end{gathered} \tag{3.2.26}$$

Let $\mathbf{H}$ be a complex Hilbert space such that

$$\begin{gathered} \forall\theta[|\psi(\theta)\rangle_{cl} \in \mathbf{H}], \\ \forall\theta\forall\delta(\delta \in (0,1])\forall\varepsilon(\varepsilon \in (0,1])\left[\,\left|\psi^i_{\delta,\varepsilon,\mathbf{x}}(\theta)\right\rangle_{cl} \in \mathbf{H}\right], \\ \left|\psi^i_{\delta,\varepsilon,\mathbf{x}}(\theta)\right\rangle_{cl} = L^i_{\mathbf{x}}(\delta,\varepsilon)|\psi(\theta)\rangle_{cl}. \end{gathered} \tag{3.2.27}$$

By postulate Q.I.1 (see section II.1) quantum system with Hamiltonian given by Eq. (3.2.25) is identified with a set

$\Xi \triangleq \left\langle \mathbf{H}, \widehat{H}_{\mathbf{eff}}, \Im, \Re, \mathcal{L}_{2,1}(\Omega), \mathbf{G}, |\psi_t\rangle \right\rangle,$ where

(i) $\mathbf{H}$ that is a complex Hilbert space defined above,

(ii) $\Im = (\Omega, \mathcal{F}, \mathbf{P})$ that is complete probability space,

(iii) $\Re = (\mathbb{R}^n, \Sigma)$ that is measurable space,

(iv) $\mathcal{L}_{2,1}(\Omega)$ that is complete space of random variables $X : \Omega \to \mathbb{R}^n$ such that

$$\int_{\Omega} \|X(\omega)\| d\mathbf{P} < \infty, \int_{\Omega} \|X(\omega)\|^2 d\mathbf{P} < \infty, \tag{3.2.28}$$

(v) $\mathbf{G} : \mathbf{H} \to \mathcal{L}_{2,1}(\Omega)$ that is one to one correspondence such that

$$\left|\langle\psi|\widehat{Q}|\psi\rangle\right| = \int_{\Omega} \left(\mathbf{G}\left[\widehat{Q}|\psi\rangle\right](\omega)\right) d\mathbf{P} = \mathbf{E}_{\Omega}\left[\mathbf{G}\left[\widehat{Q}|\psi\rangle\right](\omega)\right] \tag{3.2.29}$$

for any $|\psi\rangle \in \mathbf{H}$ and for any Hermitian operator $\widehat{Q} : \mathbf{H} \to \mathbf{H}$,

(vi) $|\psi_t\rangle$ is a continuous vector function $|\psi_t\rangle : \mathbb{R}_+ \to \mathbf{H}$ which represented the canonical evolution of the quantum system Ξ.

Remark 3.2.6. Note that $\Im^{ph}_{M_4} = \mathcal{F} \times M_4 = (\Omega, \Sigma, \mathbf{P}) \times M_4,$ where $\mathcal{F}$ is a probability measure algebra $\mathcal{F} = (\Omega, \Sigma, \mathbf{P})$ on the Boolean algebra of measurable sets modulo null sets, see Remark 3.2.1.

Let $\mathbf{B}_{M_4}$ be Boolean algebra of the all physical events in Minkowski space-time M_4 and let $\widetilde{\mathcal{F}}_{M_4}$ be a measure algebra $\widetilde{\mathcal{F}}_{M_4} = \left(\mathbf{B}_{M_4}, \widetilde{\mathbf{P}}\right)$ with a probability measure $\widetilde{\mathbf{P}},$ see Definition 3.2.2 (7).

We assume now that there exists subalgebra $\mathcal{F}^{\#}_{M_4} \subsetneq \widetilde{\mathcal{F}}_{M_4}$ and isomorphism $\lambda[\cdot] : \mathcal{F}^{\#}_{M_4} \mapsto \Im^{ph}_{M_4}$ such that for any event $A(\mathbf{x}) \in \mathcal{F}^{\#}_{M_4}, \mathbf{x} = (t, x_1, x_2, x_3) \in M_4$ (see Definition 3.2.2):

$$\lambda[A(\mathbf{x})] = \lambda[A](\mathbf{x}),$$
$$\widetilde{\mathbf{P}}(A(\mathbf{x})) = \mathbf{P}(\lambda[A](\mathbf{x})) \triangleq \mathbf{P}(A_\lambda(\mathbf{x})). \quad (3.2.30)$$

Proposition 3.2.1. Suppose that A and B are events in measure algebra $\widetilde{\mathcal{F}}_{M_4} = \left(\mathbf{B}_{M_4}, \widetilde{\mathbf{P}}\right)$.

Then following properties is satisfied:

$$\begin{aligned}
&1. \mathbf{P}(A|B) > \mathbf{P}(A) \Leftrightarrow \mathbf{P}(B|A) > \mathbf{P}(B) \Leftrightarrow \mathbf{P}(A \wedge B) > \mathbf{P}(A)\mathbf{P}(B)\\
&2. \mathbf{P}(A|B) < \mathbf{P}(A) \Leftrightarrow \mathbf{P}(B|A) < \mathbf{P}(B) \Leftrightarrow \mathbf{P}(A \wedge B) < \mathbf{P}(A)\mathbf{P}(B)\\
&3. \mathbf{P}(A|B) = \mathbf{P}(A) \Leftrightarrow \mathbf{P}(B|A) = \mathbf{P}(B) \Leftrightarrow \mathbf{P}(A \wedge B) = \mathbf{P}(A)\mathbf{P}(B)
\end{aligned} \quad (3.2.31)$$

Proposition 3.2.2. Suppose that A and B are events in measure algebra $\mathfrak{I}_{M_4} = (\Omega, \mathcal{F}, \mathbf{P})$.
Then following properties is satisfied:

$$\begin{aligned}
&1. \mathbf{P}(A|B) > \mathbf{P}(A) \Leftrightarrow \mathbf{P}(B|A) > \mathbf{P}(B) \Leftrightarrow \mathbf{P}(A \cap B) > \mathbf{P}(A)\mathbf{P}(B)\\
&2. \mathbf{P}(A|B) < \mathbf{P}(A) \Leftrightarrow \mathbf{P}(B|A) < \mathbf{P}(B) \Leftrightarrow \mathbf{P}(A \cap B) < \mathbf{P}(A)\mathbf{P}(B)\\
&3. \mathbf{P}(A|B) = \mathbf{P}(A) \Leftrightarrow \mathbf{P}(B|A) = \mathbf{P}(B) \Leftrightarrow \mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B)
\end{aligned} \quad (3.2.32)$$

Definition 3.2.8. In case (1), A and B are said to be positively correlated.
Intuitively, the occurrence of either event means that the other event is more likely.

In case (2), A and B are said to be negatively correlated.
Intuitively, the occurrence of either event means that the other event is less likely.

In case (3), A and B are said to be uncorrelated or independent.
Intuitively, the occurrence of either event does not change the probability of the other event.

Remark 3.2.7. Suppose that A and B are events in measure algebra

$\Im_{M_4} = (\Omega, \mathcal{F}, \mathbf{P})$.

Note from the result above that if $A \subseteq B$ or $B \subseteq A$ then A and B are positively correlated. If A and B are disjoint then A and B are negatively correlated.

Proposition 3.2.3. Suppose that A and B are events in measure algebra $\widetilde{\mathcal{F}}_{M_4} = (\mathbf{B}_{M_4}, \widetilde{\mathbf{P}})$. Then:
(i) A and B have the same correlation (positive, negative, or zero) as $\neg A$ and $\neg B$.
(ii) A and B have the opposite correlation as A and $\neg B$ (that is, positive-negative, negative-positive, or zero-zero).

Proposition 3.2.4. Suppose that A and B are events in measure algebra $\Im_{M_4} = (\Omega, \mathcal{F}, \mathbf{P})$.
Then: (i) A and B have the same correlation (positive, negative, or zero) as A^c and B^c.
(ii) A and B have the opposite correlation as A and B^c (that is, positive-negative, negative-positive, or zero-zero).

Definition 3.2.9. Let $A(\mathbf{x}_1) = A(t_1, \mathbf{r}_1)$ and $B(\mathbf{x}_2) = B(t_2, \mathbf{r}_2)$ be the events $A(\mathbf{x}_1) \in \mathcal{F}^{\#}_{M_4}$ which occur at instant t_1 and $B(\mathbf{x}_2) \in \mathcal{F}^{\#}_{M_4}$ at instant t_2 correspondingly.
Let $\mathbf{x}_{1,2}$ be a vector:

$$\mathbf{x}_{1,2} = \{c(t_1 - t_2), \mathbf{r}_1 - \mathbf{r}_2\} = (ct_{1,2}, \mathbf{r}_{1,2}), t_{1,2} = t_1 - t_2, \mathbf{r}_{\mathbf{1,2}} = \mathbf{r}_1 - \mathbf{r}_2.$$

Vectors $\mathbf{x}_{1,2} = (ct_{1,2}, \mathbf{r}_{1,2})$ are classified according to the sign of $c^2 t_{1,2}^2 - \mathbf{r}_{1,2}^2$. A vector is

(i) timelike if $c^2 t_{1,2}^2 > \mathbf{r}_{\mathbf{1,2}}^{\mathbf{2}}$,

(ii) spacelike if $c^2 t_{1,2}^2 < \mathbf{r}_{\mathbf{1,2}}^{\mathbf{2}}$, and null or lightlike if

(iii) $c^2 t_{1,2}^2 = \mathbf{r}_{1,2}^2$.

Pairs of events $\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\} \in \mathcal{F}_{M_4}^{\#} \times \mathcal{F}_{M_4}^{\#}$ are classified according to the sign of $c^2 t_{1,2}^2 - \mathbf{r}_{1,2}^2$:

(i) a pair $\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\}$ is timelike separated if $c^2 t_{1,2}^2 > \mathbf{r}_{1,2}^2$, and we denoted such pairs by $\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\}_{\mathbf{t.l.s.}}$;

(ii) a pair $\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\}$ is spacelike separated if $c^2 t_{1,2}^2 < \mathbf{r}_{1,2}^2$; and we denoted such pairs by $\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\}_{\mathbf{s.l.s.}}$;

(iii) a pair $\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\}$ is null or lightlike separated if $c^2 t_{1,2}^2 = \mathbf{r}_{1,2}^2$, and we denoted such pairs by $\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\}_{\mathbf{l.l.s.}}$.

Definition 3.2.10. (i) Let $[\mathcal{F}_{M_4}^{\#},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{t.l.s.}}$ be a set of the all timelike separated pairs $\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\}_{\mathbf{t.l.s.}}$ which are corresponding to a given vector $\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\} \in M_4 \times M_4$, i.e.,

$$\begin{gathered}[\mathcal{F}_{M_4}^{\#},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}} = \\ \{\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\} \in \mathcal{F}_{M_4}^{\#} \times \mathcal{F}_{M_4}^{\#} \,|\, c^2 t_{1,2}^2 > \mathbf{r}_{1,2}^2\}.\end{gathered} \quad (3.2.33(a))$$

(ii) Let $[\mathcal{F}_{M_4}^{\#},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}}$ be a set of the all spacelike separated pairs $\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\}_{\mathbf{s.l.s.}}$ which is corresponding to a given vector $\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\} \in M_4 \times M_4$, i.e.,

$$\begin{gathered}[\mathcal{F}_{M_4}^{\#},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}} = \\ \{\{A(t_1,\mathbf{r}_1),B(t_2,\mathbf{r}_2)\} \in \mathcal{F}_{M_4}^{\#} \times \mathcal{F}_{M_4}^{\#} \,|\, c^2 t_{1,2}^2 < \mathbf{r}_{1,2}^2\}.\end{gathered} \quad (3.2.33(b))$$

Remark 3.2.8. Let $[\mathcal{F}_{M_4}^{\#},\{(t,\mathbf{r}_1),(t,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}}$ be a set of all pairs $\{A(t,\mathbf{r}_1),B(t,\mathbf{r}_2)\}$, which corresponds to a given vector $\{(t,\mathbf{r}_1),(t,\mathbf{r}_2)\} \in M_4 \times M_4$, $\mathbf{r}_1 \neq \mathbf{r}_2$, i.e.,

$$[\mathcal{F}^{\#}_{M_4}, \{(t,\mathbf{r}_1),(t,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}} =$$
$$\{\{A(t,\mathbf{r}_1),B(t,\mathbf{r}_2)\} \in \mathcal{F}^{\#}_{M_4} \times \mathcal{F}^{\#}_{M_4} \,|\, 0 < \mathbf{r}^2_{1,2}\}, \quad (3.2.34)$$
$$\mathbf{r}_{1,2} = \mathbf{r}_1 - \mathbf{r}_2.$$

Such pairs obviously are spacelike separated. Note that

$$\forall t \forall \mathbf{r}_1 \forall \mathbf{r}_2 (\mathbf{r}_1 \neq \mathbf{r}_2) \left\{ [\mathcal{F}^{\#}_{M_4}, \{(t,\mathbf{r}_1),(t,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}} \neq \varnothing \right\}. \quad (3.2.35)$$

Definition 3.2.11. Let $A^{t_1} \triangleq A(\mathbf{x}_1) = A(t_1, x_A)$ and $B^{t_2} \triangleq B(\mathbf{x}_2) = B(t_2, x_B)$ be a symbols such that A^{t_1} and B^{t_2} represent there is detection events $A(\mathbf{x}_1) \in \mathcal{F}^{\#}_{M_4}$ at instant t_1 and $B(\mathbf{x}_2) \in \mathcal{F}^{\#}_{M_4}$ at instant t_2 correspondingly, where symbols x_A and x_B represent the locations of the detectors A and B correspondingly (see Fig. 3.1.3). We assume that

$$\{A^{t_1}, B^{t_2}\} \in [\mathcal{F}^{\#}_{M_4}, \{(t_1,x_A),(t_2,x_B)\}]_{\mathbf{s.l.s.}}. \quad (3.2.36)$$

Remark 3.2.9. We assume now without loss of generality that $t_1 = t_2 = t,$ note that such assumption valid by properties: $A(\Lambda\mathbf{x}_1) \in \mathcal{F}^{\#}_{M_4}$ and $B(\Lambda\mathbf{x}_2) \in \mathcal{F}^{\#}_{M_4},$ required above, see Remark 3.2.2.

III.3. Einstein's 1927 gedanken experiment explained

In classical case considered by A. Einstein in his 1927 gedanken experiment, by postulates of canonical QM, both events $A^t \in \mathcal{F}^{\#}_{M_4}$ and $B^t \in \mathcal{F}^{\#}_{M_4}$ cannot occur simultaneously, i.e. that is mutually exclusive events with a probability $= 1,$ and therefore $A^t \wedge B^t = \mathbf{0}.$ Such exactly mutually exclusive events have the property:

$$\widetilde{\mathbf{P}}(A^t \wedge B^t) = 0, \quad (3.3.1)$$

see Definition 3.2.6.

We remind that the probability density $p^{\mathbf{ph}}(x,\delta,\varepsilon)$ for the occurrence of a photon localization at point x is assumed to be

$$p^{\mathbf{ph}}(x,\delta,\varepsilon) = \left\| \left|\psi^{\mathbf{ph}}_{\delta,\varepsilon,x}(\theta)\right\rangle_{cl} \right\|^2, \quad \delta \in (0,1], \varepsilon \in (0,1], \tag{3.3.2}$$

where

$$\left|\psi^{\mathbf{ph}}_{\delta,\varepsilon,x}\right\rangle_{cl} = L_x(\delta,\varepsilon)\left|\psi^{\mathbf{ph}}(\theta)\right\rangle_{cl}. \tag{3.3.3}$$

and where the localization operators $L_{\mathbf{x}}(\delta,\varepsilon)$ have been chosen to have the form:

$$\widehat{L}_x(\widehat{q},\delta,\varepsilon) = \begin{cases} \left(\frac{1}{\delta\pi_\delta}\right)^{1/4} \exp\left[-\frac{1}{2\delta}(\widehat{q}-x)^2\right] \text{ iff } |\widehat{q}-x| \leq \varepsilon \ll 1, \\ 0 \text{ iff } |\widehat{q}-x| > \varepsilon. \end{cases} \tag{3.3.4}$$

see subsection II.2.1.

Remark 3.3.1. Note that: (i) from (3.2.27) it follows that $\left|\psi^{\mathbf{ph}}_{\delta,\varepsilon,\mathbf{x}}(\theta)\right\rangle_{cl} \in \mathbf{H}$,

(ii) from (3.3.3) and (3.3.4) where $\delta \ll 1$ it follows that

$$\begin{gathered} p^{\mathbf{ph}}(x,\delta,\varepsilon) = \left\| \left|\psi^{\mathbf{ph}}_{\delta,\varepsilon,x}(\theta)\right\rangle_{cl} \right\|^2 = \int dq \left\langle \psi^{\mathbf{ph}}_{\delta,\varepsilon,x}(\theta) \middle| |q\rangle\langle q| \middle| \psi^{\mathbf{ph}}_{\delta,\varepsilon,x}(\theta) \right\rangle = \\ \int dq \left\langle \widehat{L}_x(\widehat{q},\delta,\varepsilon)\psi^{\mathbf{ph}}(\theta) \middle| |q\rangle\langle q| \middle| \widehat{L}_x(\widehat{q},\delta,\varepsilon)\psi^{\mathbf{ph}}(\theta) \right\rangle = \\ \int dq L_x^2(q,\delta,\varepsilon) \langle \psi^{\mathbf{ph}}(\theta) \left| |q\rangle\langle q| \right| \psi^{\mathbf{ph}}(\theta) \rangle = \left\| \langle x| \left|\psi^{\mathbf{ph}}(\theta)\right\rangle_{cl} \right\|^2 + O(\delta) \asymp \\ \asymp \left\| \langle x| \left|\psi^{\mathbf{ph}}(\theta)\right\rangle_{cl} \right\|^2, \\ \delta \ll 1, \varepsilon \in (0,1], \end{gathered} \tag{3.3.5}$$

It follows from postulate Q.I.3 that there exists unique random variable $X\left(\omega; \left||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right\rangle\right)$ given on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and a measurable space $(\mathbb{R}^n, \Sigma)$ by formula

$$X\left(\omega; \left||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right\rangle\right) \triangleq X_{|\psi^{\mathbf{ph}}(\theta)\rangle}(\omega) = \mathbf{G}\left[\left||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right\rangle\right] \tag{3.3.6}$$

The probability density of random variable $X_{|\psi^{\mathbf{ph}}(\theta)\rangle}(\omega)$ we denote by $p_{|\psi^{\mathbf{ph}}(\theta)\rangle}(q), q \in \mathbb{R}$.

Remark 3.3.2. From postulate Q.II.2 (see subsection II.1) it follows that for the system in state $\left||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right\rangle$ the probability $P\left(q, q+dq; \left||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right\rangle\right)$ of obtaining the result q lying in the range $(q, q+dq)$ on measuring observable $\hat{q}$ is given by

$$P\left(q, q+dq; \left||\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right\rangle\right) = p_{|\psi^{\mathbf{ph}}(\theta)\rangle_{cl}}(q)dq = \left|c_{|\psi^{\mathbf{ph}}(\theta)_{cl}\rangle}(q)\right|^2 = \left|\langle q|\psi^{\mathbf{ph}}(\theta)\rangle_{cl}\right|^2 \tag{3.3.7}$$

Now we go to explain Einstein's 1927 gedanken experiment. Let $A^{ph}(t, x_A)$ and $B^{ph}(t, x_A)$ be events such that detectors A, B detect photon at an instant t correspondingly. By properties (3.2.31) we obtain

$$\begin{aligned} \mathbf{P}\left(A^{ph}_{\lambda}(t, x_A)\right) &\triangleq \mathbf{P}(\lambda[A^{ph}](t, x_A)) = \widetilde{\mathbf{P}}(A^{ph}(t, x_A)), \\ \mathbf{P}\left(B^{ph}_{\lambda}(t, x_B)\right) &\triangleq \mathbf{P}(\lambda[B^{ph}](t, x_B)) = \widetilde{\mathbf{P}}(B^{ph}(t, x_B)). \end{aligned} \tag{3.3.8}$$

Note that

$$\begin{gathered} A^t \triangleq A^{ph}_{\lambda}(t, x_A) = \left\{\omega | x_A - \epsilon \leq X_{|\psi^{\mathbf{ph}}(\theta)\rangle}(\omega) \leq x_A - \epsilon\right\}, \\ B^t \triangleq B^{ph}_{\lambda}(t, x_B) = \left\{\omega | x_B - \epsilon \leq X_{|\psi^{\mathbf{ph}}(\theta)\rangle}(\omega) \leq x_B - \epsilon\right\}, \\ \epsilon \in (0, \gamma], \gamma \ll 1, \end{gathered} \tag{3.3.9}$$

where a small parameter $\epsilon \ll |x_A - x_B|$ is dependent on the measuring device. Thus by general definition of random variable one obtains directly

$$A_\lambda^{ph}(t, x_A) \cap B_\lambda^{ph}(t, x_B) = \varnothing \tag{3.3.10}$$

and therefore

$$\mathbf{P}\left(A_\lambda^{ph}(t, x_A) \cap B_\lambda^{ph}(t, x_B)\right) = 0 \tag{3.3.11}$$

The property (3.3.11) follows directly from (3.3.8).

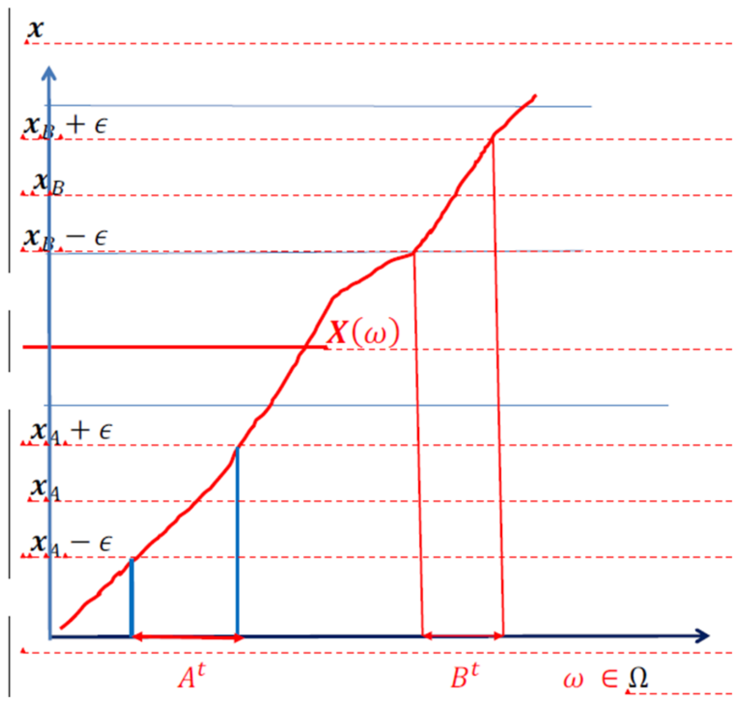

Fig. 3.3.1. The plot of the random variable $X_{|\psi^{\mathbf{ph}}(\theta)\rangle}(\omega)$.

$$A^t = A_\lambda^{ph}(t, x_A), B^t = B_\lambda^{ph}(t, x_B), A^t \cap B^t = \varnothing$$

Remark 3.3.3. Let $[\mathcal{F}^{\#}_{M_4}, \{(t,x_A),(t,x_B)\}]_{\mathbf{s.l.s.}}$ be a set of the all pairs $\{A(t,x_A),B(t,x_B)\}$ which is corresponding to a given vector $\{(t,x_A,0,0),(t,x_B,0,0)\} \in M_4 \times M_4,\ \ x_A \neq x_B,$ i.e.,

$$[\mathcal{F}^{\#}_{M_4}, \{(t,x_A),(t,x_B)\}]_{\mathbf{s.l.s.}} = \\ \{\{A(t,x_A),B(t,x_B)\} \in \mathcal{F}^{\#}_{M_4} \times \mathcal{F}^{\#}_{M_4} \,|\, 0 < (x_A - x_B)^2\}. \tag{3.3.12}$$

Such pairs obviously are spacelike separated. Note that

$$\forall t \forall x_A \forall x_B (x_A \neq x_B) \Big\{ [\mathcal{F}^{\#}_{M_4}, \{(t,x_A),(t,x_B)\}]_{\mathbf{s.l.s.}} \neq \varnothing \Big\}. \tag{3.3.13}$$

Now we will go to explain non zero result $\widetilde{\mathbf{P}}(A^t \wedge B^t) \neq 0$ given above by (3.1.1) and (3.1.2):

$$\widetilde{\mathbf{P}}^{TL}_{A^t \wedge B^t} = 0.002 \pm 0.001 \cdot 10^{-4}, \mathbf{P}^{TL}_{A^t} \cdot P^{TL}_{B^t} = 1.65 \pm 0.01 \cdot 10^{-4}, \\ \widetilde{\mathbf{P}}^{SL}_{A^t \wedge B^t} = 0.002 \pm 0.001 \cdot 10^{-4}. \mathbf{P}^{SL}_{A^t \wedge B^t} = 0.002 \pm 0.001 \cdot 10^{-4}. \tag{3.3.14}$$

We consider this problem in general case.

Remark 3.3.4. Note that:

(i) a probability density $p(x,\delta_A,\epsilon,)$ for the occurrence of the localization inside interval $[x-\epsilon, x+\epsilon]$ in arm with detector A (see Fig.3.1.2.) is given by formula

$$p(x,\delta_A,\epsilon) = \frac{\| |\psi_{\delta_A,\epsilon,x}\rangle_{cl} \|^2}{\Delta(\delta_A,\epsilon)}, \tag{3.3.15}$$

where

$$\| |\psi_{\delta_A,\epsilon,x}\rangle_{cl} \|^2 = \left(\frac{1}{\delta_A \pi_{\delta_A}}\right)^{1/2} \int\limits_{|q-x|\leq\epsilon} dq |\psi(q)|^2 \exp\left[-\frac{1}{\delta_A}(q-x)^2\right],$$

$$\psi(q) = \langle q \| \psi \rangle, \tag{3.3.16}$$

$$\Delta(\delta_A, \epsilon) = \int\limits_{-\infty}^{\infty} \| |\psi_{\delta_A,\epsilon,x}\rangle_{cl} \|^2 dx,$$

and where parameter δ_A depends on arm with detector A.

(ii) a probability density $p(x, \delta_B, \epsilon,)$ for the occurrence of the localization inside interval $[x - \epsilon, x + \epsilon]$ in arm with detector B (see Fig.3.1.2) is given by formula

$$p(x, \delta_B, \epsilon) = \frac{\| |\psi_{\delta_B,\epsilon,x}\rangle_{cl} \|^2}{\Delta(\delta_B, \epsilon)}, \tag{3.3.17}$$

where

$$\| |\psi_{\delta_B,\epsilon,x}\rangle_{cl} \|^2 = \left(\frac{1}{\delta_B \pi_{\delta_B}}\right)^{1/2} \int\limits_{|q-x|\leq\epsilon} dq |\psi(q)|^2 \exp\left[-\frac{1}{\delta_B}(q-x)^2\right],$$

$$\psi(q) = \langle q \| \psi \rangle, \tag{3.3.18}$$

$$\Delta(\delta_B, \epsilon) = \int\limits_{-\infty}^{\infty} \| |\psi_{\delta_B,\epsilon,x}\rangle_{cl} \|^2 dx,$$

and where parameter δ_B depends on arm with detector B.

Remark 3.3.5. Note that parameter δ in formula (3.3.18) of course depends on measurement device and there no exist two equivalent devices such that $\delta_A = \delta_B$.

We assume now that

$$\delta_A \simeq \delta_B \ll 1,$$

$$0 < |\delta_A - \delta_B|, \tag{3.3.19}$$

$$\int_{-\infty}^{\infty} \left[|\psi(x)|^2\right]'' dx < \infty,$$

From Eq. (3.3.16) and Eq. (3.3.19) by using Laplace approximation, we obtain:

$$
\begin{aligned}
&\| |\psi_{\delta_A,\epsilon,x}\rangle_{cl} \|^2 = \left(\frac{1}{\delta_A \pi_{\delta_A}}\right)^{1/2} \int\limits_{|q-x|\leq\epsilon} dq |\psi(q)|^2 \exp\left[-\frac{1}{\delta_A}(q-x)^2\right] \approx \\
&\approx |\psi(x)|^2 + \delta_A O\left(\left[|\psi(x)|^2\right]''\right) = |\psi(x)|^2 + \delta_A c_1^A \left[|\psi(x)|^2\right]'', \\
&\Delta(\delta_A,\epsilon) = \int\limits_{-\infty}^{\infty} \| |\psi_{\delta_A,\epsilon,x}\rangle_{cl} \|^2 dx = 1 + c_2^A \delta_A, c_2^A = O\left(\int_{-\infty}^{\infty}\left[|\psi(x)|^2\right]'' dx\right).
\end{aligned} \tag{3.3.20}
$$

From Eq. (3.3.18) and Eq. (3.3.19) by using Laplace approximation, we obtain:

$$
\begin{aligned}
&\| |\psi_{\delta_B,\epsilon,x}\rangle_{cl} \|^2 = \left(\frac{1}{\delta_B \pi_{\delta_B}}\right)^{1/2} \int\limits_{|q-x|\leq\epsilon} dq |\psi(q)|^2 \exp\left[-\frac{1}{\delta_B}(q-x)^2\right] \approx \\
&|\psi(x)|^2 + \delta_B O\left(\left[|\psi(x)|^2\right]''\right) = |\psi(x)|^2 + \delta_B c_1^B \left[|\psi(x)|^2\right]'', \\
&\Delta(\delta_B,\epsilon) = \int\limits_{-\infty}^{\infty} \| |\psi_{\delta_B,\epsilon,x}\rangle_{cl} \|^2 dx = 1 + c_2^B \delta_B, c_2^B = O\left(\int_{-\infty}^{\infty}\left[|\psi(x)|^2\right]'' dx\right).
\end{aligned} \tag{3.3.21}
$$

From Eq. (3.3.15) and Eq. (3.3.17) we obtain

$$
p(x,\delta_A,\epsilon) = \frac{\| |\psi_{\delta_A,\epsilon,x}\rangle_{cl} \|^2}{\Delta(\delta_A,\epsilon)} = \frac{|\psi(x)|^2 + \delta_A c_1^A \left[|\psi(x)|^2\right]''}{1 + c_2^A \delta_A,}. \tag{3.3.22a}
$$

From Eq. (2.2.54) and Eq. (2.2.57) we obtain

$$
p(x,\delta_B,\epsilon) = \frac{\| |\psi_{\delta_B,\epsilon,x}\rangle_{cl} \|^2}{\Delta(\delta_B,\epsilon)} = \frac{|\psi(x)|^2 + \delta_B c_1^B \left[|\psi(x)|^2\right]''}{1 + c_2^B \delta_B,}. \tag{3.3.22b}
$$

Definition 3.3.1. We define now the probability measures $\mathbf{P}_{|\psi_{\delta_A,\epsilon,x}\rangle}(A^t)$ and $\mathbf{P}_{|\psi_{\delta_B,\epsilon,x}\rangle}(A^t)$ by formulae

$$\mathbf{P}_{|\psi_{\delta_A,\epsilon,x}\rangle}(A^t) = \int_{A^t} p(x,\delta_A,\epsilon)d\mu(x),$$
$$\mathbf{P}_{|\psi_{\delta_B,\epsilon,x}\rangle}(A^t) = \int_{A^t} p(x,\delta_B,\epsilon)d\mu(x), \tag{3.3.23}$$

where $A^t \in \Sigma_{a,b}$ and $d\mu(x)$ is the Lebesgue measure and $\Sigma_{a,b} = B([a,b])$ is the Borel algebra on a set $[a,b]$.

Definition 3.3.2. We assume now that $\mathbf{P}_{|\psi_{\delta_A,\epsilon,x}\rangle} \ll \mathbf{P}$ and $\mathbf{P}_{|\psi_{\delta_B,\epsilon,x}\rangle} \ll \mathbf{P}$, i.e. $\mathbf{P}_{|\psi\rangle}$ is absolutely continuous with respect to $\mathbf{P}$. By Radon-Nicodym theorem we obtain for any $A^t \in \Sigma_{a,b}$:

$$\mathbf{P}_{|\psi_{\delta_A,\epsilon,x}\rangle}(A^t) = \int_{A^t} X_{|\psi_{\delta_A,\epsilon,x}\rangle}(\omega)d\mathbf{P},$$
$$X_{|\psi_{\delta_A,\epsilon,x}\rangle}(\omega) = \frac{d\mathbf{P}_{|\psi_{\delta_A,\epsilon,x}\rangle}}{d\mathbf{P}},$$
$$\mathbf{P}_{|\psi_{\delta_A,\epsilon,x}\rangle}(A^t) = \int_{A^t} X_{|\psi_{\delta_A,\epsilon,x}\rangle}(\omega)d\mathbf{P},$$
$$X_{|\psi_{\delta_A,\epsilon,x}\rangle}(\omega) = \frac{d\mathbf{P}_{|\psi_{\delta_A,\epsilon,x}\rangle}}{d\mathbf{P}}. \tag{3.3.24}$$

We write below for a short

$$X_1(\omega) \triangleq X_{|\psi_{\delta_A,\epsilon,x}\rangle}(\omega), X_2(\omega) \triangleq X_{|\psi_{\delta_B,\epsilon,x}\rangle}(\omega). \tag{3.3.25}$$

Remark 3.3.6. We assume now without loss of generality that

$$X_2(\omega) - X_1(\omega) \geq 0 \text{ a.s.} \tag{3.3.26}$$

see Fig. 3.3.1.

Let us consider now the quantity

$$\eta_{1,2} = \int_{\Omega} |X_1(\omega) - X_2(\omega)| d\mathbf{P} = \int_{\Omega} [X_2(\omega) - X_1(\omega)] d\mathbf{P}. \quad (3.3.27)$$

We assume now that

$$\int_{-\infty}^{\infty} x|\psi(x)|^2 dx < \infty, \int_{-\infty}^{\infty} x\left[|\psi(x)|^2\right]'' dx < \infty, \quad (3.3.28)$$

From Eq. (3.3.27) by using Eq. (3.3.21) and Eq. (3.3.22) we obtain

$$\begin{aligned}
&\eta_{1,2} = \\
&\int_{\mathbb{R}} xp(x,\delta_B,\epsilon)dx - \int_{\mathbb{R}} xp(x,\delta_A,\epsilon)dx = \frac{1}{1 + c_2^B \delta_B} \int_{\mathbb{R}} x\left[|\psi(x)|^2 + \delta_B c_1^B \left[|\psi(x)|^2\right]''\right]dx - \\
&- \frac{1}{1 + c_2^A \delta_A} \int_{\mathbb{R}} x\left[|\psi(x)|^2 + \delta_A c_1^A \left[|\psi(x)|^2\right]''\right]dx \simeq \\
&(1 - c_2^B \delta_B) \int_{\mathbb{R}} x\left[|\psi(x)|^2 + \delta_B c_1^B \left[|\psi(x)|^2\right]''\right]dx - \\
&-(1 - c_2^A \delta_A) \int_{\mathbb{R}} x\left[|\psi(x)|^2 + \delta_A c_1^A \left[|\psi(x)|^2\right]''\right]dx = \\
&\delta_B c_1^B \int_{\mathbb{R}} x\left[|\psi(x)|^2\right]'' dx - c_2^B \delta_B \int_{\mathbb{R}} x|\psi(x)|^2 dx - \delta_B^2 c_1^B c_2^B \int_{\mathbb{R}} x\left[|\psi(x)|^2\right]'' dx - \\
&-\delta_A c_1^A \int_{\mathbb{R}} x\left[|\psi(x)|^2\right]'' dx + c_2^A \delta_A \int_{\mathbb{R}} x|\psi(x)|^2 dx + \delta_A^2 c_1^A c_2^A \int_{\mathbb{R}} x\left[|\psi(x)|^2\right]'' dx = \\
&(\delta_B c_1^B - \delta_A c_1^A - \delta_B^2 c_1^B c_2^B + \delta_A^2 c_1^A c_2^A) \int_{\mathbb{R}} x\left[|\psi(x)|^2\right]'' dx + (c_2^A \delta_A - c_2^B \delta_B) \int_{\mathbb{R}} x|\psi(x)|^2 dx \simeq \\
&\alpha_1 (c_2^A \delta_A - c_2^B \delta_B) + \alpha_2 (\delta_B c_1^B - \delta_A c_1^A),
\end{aligned} \quad (3.3.29)$$

where

$$\alpha_1 = \int_{\mathbb{R}} x|\psi(x)|^2 dx, \alpha_2 = \int_{\mathbb{R}} x\left[|\psi(x)|^2\right]'' dx. \quad (3.3.30)$$

Lemma 3.3.1. Let $(\Omega, \Sigma, \mathbf{P})$ be a measure space, and let f be a real-valued measurable function defined on Ω.
Then for any real number $t > 0$:

$$\mathbf{P}\{\omega \in \Omega \| f(\omega)| \geq t\} \leq \frac{1}{t} \int_{|f(\omega)| \geq t} \left|f(\omega)\right| d\mathbf{P}(\omega). \quad (3.3.31)$$

From inequality (3.3.31) and Eq. (3.3.29) we obtain

$$\begin{aligned}&\mathbf{P}\left\{\omega \in \Omega : |X_1(\omega) - X_2(\omega)| \geq t\right\} \leq \frac{1}{t} \int\limits_{|X_1(\omega) - X_2(\omega)| \geq t} [|X_1(\omega) - X_2(\omega)|] d\mathbf{P}(\omega) \\ &< \frac{1}{t^2} \int\limits_{\Omega} [X_1(\omega) - X_2(\omega)] = \frac{\eta_{1,2}}{t} \simeq \frac{\alpha_1(c_2^A \delta_A - c_2^B \delta_B) + \alpha_2(\delta_B c_1^B - \delta_A c_1^A)}{t}.\end{aligned} \tag{3.3.32}$$

We define now

$$\begin{aligned}A^t &= A_\lambda^{ph}(t, x_A) = \{\omega | x_A - \epsilon \leq X_1(\omega) \leq x_A - \epsilon\}, \\ B^t &= B_\lambda^{ph}(t, x_B) = \{\omega | x_B - \epsilon \leq X_2(\omega) \leq x_B - \epsilon\},\end{aligned} \tag{3.3.33}$$

and chose in (3.3.31) number $t = x_B - x_A \gg 1$.

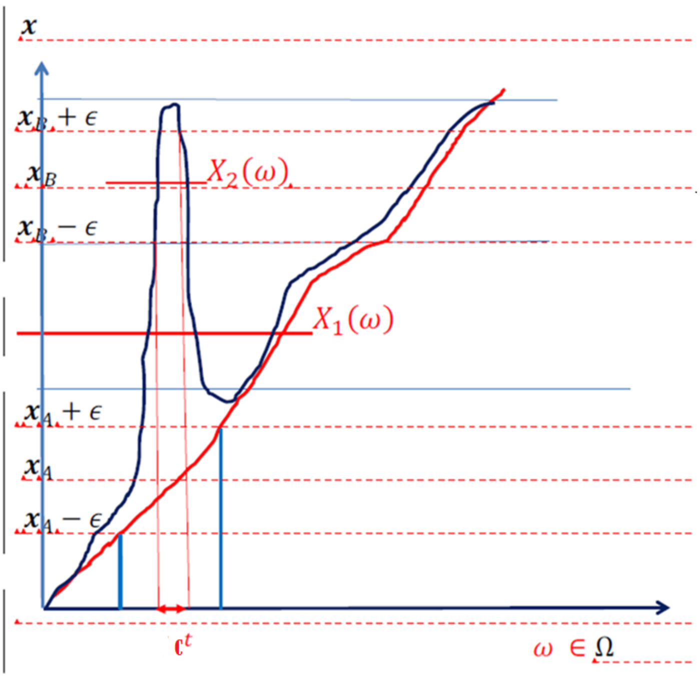

Fig. 3.3.2. The plot of the random variables $X_1(\omega)$ and $X_2(\omega)$.
$A^t = A_\lambda^{ph}(t, x_A), B^t = B_\lambda^{ph}(t, x_B), A^t \cap B^t \subseteq C^t$

Note that

$$\mathbf{P}(A^t \cap B^t) \leq \mathbf{P}(C^t), \tag{3.3.34}$$

see Fig. 3.3.2. From Eq. (3.3.32) - Eq. (3.3.34) it follows that

$$\mathbf{P}(A^t \cap B^t) < \frac{\alpha_1(c_2^A \delta_A - c_2^B \delta_B) + \alpha_2(\delta_B c_1^B - \delta_A c_1^A)}{(x_B - x_A)^2} \ll 1. \tag{3.3.35}$$

Chapter IV

THE ERP PARADOX RESOLUTION

IV.1. The relaxed locality principle

The Special Theory of Relativity limits the speed at which any physical influences and any real information can travel to the speed of light, c.

The Einstein's principle of locality (EPL): any effects do not propagate faster than the speed of light, i.e. speed of light is a limiting factor.

The principle of locality claimed that:

(**i**) Any physical event $\mathbf{A}(t_1,\mathbf{r}_1)$ which has occurred at point $A(t_1,\mathbf{r}_1) \in M_4$ (see Chapt. III, Definition 3.2.9) cannot cause (by physical interaction) a physical event $\mathbf{B}(t_2,\mathbf{r}_2)$ (result) which has occurred at point $B(t_2,\mathbf{r}_2) \in M_4$ in a time less than $T = D/c,$ where $D,$ is the distance between the points.

(**ii**) Any physical event $\mathbf{A}(t,\mathbf{r}_1)$ which has occurred at point $A(t,\mathbf{r}_1) \in M_4$ cannot cause a simultaneous physical event $\mathbf{B}(t,\mathbf{r}_2)$ (result) which has occurred at another point $B(t,\mathbf{r}_2) \in M_4$.

(**iii**) Any real physical information about physical event $\mathbf{A}(t_1,\mathbf{r}_1)$ at point $A(t_1,\mathbf{r}_1)$ cannot be obtained by observer at point $B(t_2,\mathbf{r}_2)$ in a time less than $T = D/c,$ where $D,$ is the distance between the points.

Definition 4.1.1. Let $[\mathcal{F}^{\#}_{M_4}, \{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]^{\Rightarrow}_{\mathbf{t.l.s.}}$ be a set of the all timelike separated pairs of events $\{\mathbf{A}(t_1,\mathbf{r}_1),\mathbf{B}(t_2,\mathbf{r}_2)\}_{\mathbf{t.l.s.}} \in [\mathcal{F}^{\#}_{M_4}, \{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{t.l.s.}},$ (see Chapt. III, Definition 3.2.10(i)) such that $t_2 > t_1$ and $\mathbf{A}^{Oc}(t_1,\mathbf{r}_1) \Rightarrow \mathbf{B}^{Oc}(t_2,\mathbf{r}_2).$

Note that $[\mathcal{F}^{\#}_{M_4}, \{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]^{\Rightarrow}_{\mathbf{t.l.s.}} \subsetneq [\mathcal{F}^{\#}_{M_4}, \{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{t.l.s.}}.$

Remark 4.1.1. Note that the claim (i) obviously meant that

$$\forall(t_1 > t_2)\forall\mathbf{A}(t_1,\mathbf{r}_1)\forall\mathbf{B}(t_2,\mathbf{r}_2)\Big\{[\mathbf{A}^{Oc}(t_1,\mathbf{r}_1) \Rightarrow \mathbf{B}^{Oc}(t_2,\mathbf{r}_2)] \Leftrightarrow \{\mathbf{A}(t_1,\mathbf{r}_1),\mathbf{B}(t_2,\mathbf{r}_2)\} \in [\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{t.l.s.}}\Big\}. \quad (4.1.1)$$

Remark 4.1.2. In spacetime diagram, see Fig. 4.1.1, the interval s^2_{AB} is "time-like", i.e. there is a frame of reference in which events **A** and **B** occur at the same location in space, separated only by occurring at different times. If **A** precedes **B** in that frame, then **A** precedes **B** in all frames. It is hypothetically possible for matter (or information) to travel from A to B, so there can be a causal relationship (with **A** the cause and **B** the effect).

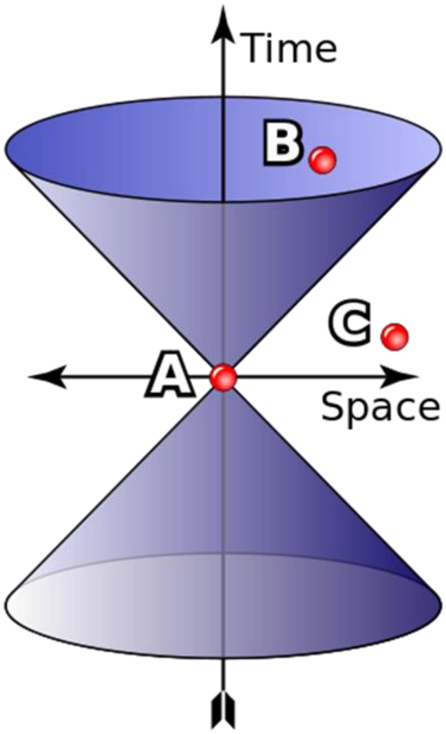

Fig. 4.1.1. Spacetime diagram

Remark 4.1.3. Note that:

(i) the interval s^2_{AC} in the diagram, see Fig. 4.1.1, is "space-like"; i.e. there is a frame $\mathcal{F}_t$ of reference in which events $\mathbf{A}(t,\mathbf{r}_1)$ and $\mathbf{C}(t,\mathbf{r}_2)$ occur simultaneously at instant t, separated only in space. There are also frames in which **A** precedes **C** and frames in which **C** precedes **A.**

(ii) If it were possible for a cause-and-effect relationship to exist between events **A** and **C**, then paradoxes of causality would result. For example, if **A** was the cause, and **C** the effect, then there would be frames of reference in which the effect preceded the cause. Although this in itself will not give rise to a paradox, one can show that faster than light signals can be sent back into one's own past. A causal paradox can then be constructed by sending the signal if and only if no signal was received previously.

(iii) Obviously there exist space-like separated pairs of physical events $\{\mathbf{A}(t,\mathbf{r}_1),\mathbf{B}(t,\mathbf{r}_2)\}_{\mathbf{s.l.s.}}$ such that the events $\mathbf{A}(t,\mathbf{r}_1)$ and $\mathbf{C}(t,\mathbf{r}_2)$ always occur only simultaneously at any instant t i.e.,

$$\mathbf{A}^{Oc}(t,\mathbf{r}_1) \Leftrightarrow \mathbf{C}^{Oc}(t,\mathbf{r}_2). \tag{4.1.2}$$

Example 4.1.1. Let us consider two synchronized clock A and B which at rest on given inertial frame F_I. Assume that clock A at rest in point $\mathbf{r}_1$ and clock B at rest in point $\mathbf{r}_2$ correspondingly.

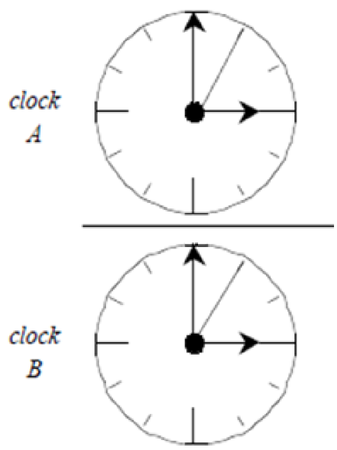

Fig. 4.1.2. Clock A and clock B
which at rest on given inertial frame F_I.

Let $\mathbf{A}(t,\mathbf{r}_1)$ be the event which consists that time on clock A is t at time t according to clock A and let $\mathbf{B}(t,\mathbf{r}_1)$ be the event which consists that time on clock B is t at time t according to clock B. It is clear that $\mathbf{A}^{Oc}(t,\mathbf{r}_1) \Leftrightarrow \mathbf{B}^{Oc}(t,\mathbf{r}_2)$.

Definition 4.1.2. Let $[\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]^{\Leftrightarrow}_{\mathbf{s.l.s.}}$ be a set of the all spacelike separated pairs of events $\{\mathbf{A}(t_1,\mathbf{r}_1),\mathbf{B}(t_2,\mathbf{r}_2)\}_{\mathbf{s.l.s.}} \in [\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}}$, (see Chapt. III, Definition 3.2.10 (ii)) such that

$$\mathbf{A}^{Oc}(t_1,\mathbf{r}_1) \Leftrightarrow \mathbf{B}^{Oc}(t_2,\mathbf{r}_2). \tag{4.1.3}$$

Remark 4.1.4. Note that the condition (4.1.3) does not violate the Einstein's principle of locality and gives only an additional properties of the algebra $\mathcal{F}^{\#}_{M_4}$.

Remark 4.1.5. Note that from (4.1.3) it follows that

$$[\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]^{\Leftrightarrow}_{\mathbf{s.l.s.}} \subsetneq [\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]_{\mathbf{s.l.s.}}.$$

On the basis of this Gedanken experiment, which is also realized by photons, the EPR-paradox can be derived if the following two principles are taken as postulates.

1. The principle of reality R :

If the value A_i of an observable A can be determined without altering the quantum system S, then any property $P(A_i)$ which corresponds to this value of A pertains to the system S.

2. The principle of locality L :

2.1. If two quantum systems S_1 and S_2 cannot interact with each other, then a measurement with respect to one system cannot alter the other system and therefore we can assume the existence of state vectors $|S_1\rangle$ and $|S_2\rangle$.

2.2. Let $\hat{x}_1$ and $\hat{x}_2$ be two observables measured with respect to systems S_1 and S_2 mentioned above. Then by result of measurement of the quantity $\bar{x}_2 = \langle S_2|\hat{x}_2|S_2\rangle$ at instant t, it is impossible to get any information on result of measurement of the quantity $\bar{x}_1 = \langle S_1|\hat{x}_1|S_1\rangle$ at the same instant t.

We assume now the relaxed principle of locality. Intuitively this principle says that for even spacelike separated entangled quantum systems S_1 and S_2 any measurement at instant t with respect to system S_1 always immediately alter the other system S_2 at the same instant t. But no additional information about the system S_1 can be found out upon measurement on the system S_2 except the canonical information which can be predicted by using correlation relations which follows from concrete type of entanglement.

3. The relaxed principle of locality L_{rel} :

3.1. Any spacelike separated quantum systems $\mathbf{S}_1$ and $\mathbf{S}_2$ cannot interact with each other and therefore we can assume the existence of state vectors $|\mathbf{S}_1\rangle$ and $|\mathbf{S}_2\rangle$ correspondingly.

3.2. Let $\mathbf{S}_{1(2)}(t,\mathbf{r}_1)$ and $\mathbf{S}_{2(1)}(t,\mathbf{r}_2)$ be two spacelike separated entangled quantum systems located in points $(t,\mathbf{r}_1)$ and $(t,\mathbf{r}_2)$ correspondingly.

(i) Assume that a state vector $|\mathbf{S}_{1(2)}(t,\mathbf{r}_1)\rangle$ suddenly collapses at instant t to state vector $\left|\mathbf{S}^{\mathbf{s\text{-}col}}_{1(2)}(t,\mathbf{r}_1)\right\rangle$:

$$|\mathbf{S}_{1(2)}(t,\mathbf{r}_1)\rangle \overset{\mathbf{s\text{-}collapse}}{\rightarrow} \left|\mathbf{S}^{\mathbf{s\text{-}col}}_{1(2)}(t,\mathbf{r}_1)\right\rangle, \qquad (4.1.4)$$

then a state vector $|\mathbf{S}_{2(1)}(t,\mathbf{r}_2)\rangle$ immediately collapses to state vector $\left|\mathbf{S}^{\mathbf{col}}_{2(1)}(t,\mathbf{r}_2)\right\rangle$:

$$|\mathbf{S}_{2(1)}(t,\mathbf{r}_2)\rangle \overset{\mathbf{collapse}}{\rightarrow} \left|\mathbf{S}^{\mathbf{col}}_{2(1)}(t,\mathbf{r}_2)\right\rangle \qquad (4.1.5)$$

(ii) Assume that a state vector $|\mathbf{S}_{1(2)}(t,\mathbf{r}_1)\rangle$ after measurement immediately collapses at instant t to state vector $\left|\mathbf{S}^{\mathbf{m\text{-}col}}_{1(2)}(t,\mathbf{r}_1)\right\rangle$:

$$|\mathbf{S}_{1(2)}(t,\mathbf{r}_1)\rangle \overset{\mathbf{m\text{-}collapse}}{\rightarrow} \left|\mathbf{S}^{\mathbf{m\text{-}col}}_{1(2)}(t,\mathbf{r}_1)\right\rangle, \qquad (4.1.6)$$

then a state vector $|\mathbf{S}_{2(1)}(t,\mathbf{r}_2)\rangle$ immediately collapses to state vector

$\left|\mathbf{S}^{\mathbf{col}}_{2(1)}(t,\mathbf{r}_2)\right\rangle$:

$$|\mathbf{S}_{2(1)}(t,\mathbf{r}_2)\rangle \overset{\text{col}\mathbf{lapse}}{\longrightarrow} \left|\mathbf{S}^{\mathbf{col}}_{2(1)}(t,\mathbf{r}_2)\right\rangle \qquad (4.1.7)$$

(iii) Let $\mathbf{S}^{\mathbf{s\text{-}col}}_{1(2)}(t,\mathbf{r}_1)$ and $\mathbf{S}^{\mathbf{col}}_{2(1)}(t,\mathbf{r}_2)$ be a physical events defined by formulae (4.1.4) and (4.1.5) correspondingly, then

$$\mathbf{Occ}\left[\mathbf{S}^{\mathbf{s\text{-}col}}_{1(2)}(t,\mathbf{r}_1)\right] \Leftrightarrow \mathbf{Occ}\left[\mathbf{S}^{\mathbf{col}}_{2(1)}(t,\mathbf{r}_2)\right], \qquad (4.1.8)$$

(see Chapt. III, Definition 3.2.8(2)).

(iv) Let $\mathbf{S}^{\mathbf{m\text{-}col}}_{1(2)}(t,\mathbf{r}_1)$ and $\mathbf{S}^{\mathbf{col}}_{2(1)}(t,\mathbf{r}_2)$ be a physical events defined by formulae (4.1.6) and (4.1.7) correspondingly, then

$$\mathbf{Occ}\left[\mathbf{S}^{\mathbf{m\text{-}col}}_{1(2)}(t,\mathbf{r}_1)\right] \Leftrightarrow \mathbf{Occ}\left[\mathbf{S}^{\mathbf{col}}_{2(1)}(t,\mathbf{r}_2)\right], \qquad (4.1.9)$$

3.3. No any additional information about the system S_1 upon measurement at instant t can be found out upon measurement on the system S_2 upon measurement at instant t except the canonical information which can be predicted by using correlation relations which follows from concrete type of entanglement.

Remark 4.1.6. Note that conditions (1.1.8) - (1.1.9) are very similarly to condition (4.1.3) and give only an additional properties of the algebra $\mathcal{F}^{\#}_{M_4}$.

Remark 4.1.7. Note that from (4.1.8) it follows that

$$\left\{\mathbf{S}^{\mathbf{s\text{-}col}}_{1(2)}(t,\mathbf{r}_1),\mathbf{S}^{\mathbf{col}}_{2(1)}(t,\mathbf{r}_2)\right\} \in [\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]^{\Leftrightarrow}_{\mathbf{s.l.s.}} \qquad (4.1.10)$$

from (4.1.9) it follows that

$$\left\{\mathbf{S}^{\mathbf{m\text{-}col}}_{1(2)}(t,\mathbf{r}_1),\mathbf{S}^{\mathbf{col}}_{2(1)}(t,\mathbf{r}_2)\right\} \in [\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]^{\Leftrightarrow}_{\mathbf{s.l.s.}} \qquad (4.1.11)$$

Remark 4.1.8. Note that:

(i) collapse of a state vector $|\mathbf{S}_{2(1)}(t,\mathbf{r}_2)\rangle$ given by (4.1.5) occurs without any interaction between quantum systems $S_{1(2)}$ and $S_{2(1)}$ but only by property given by formulae (4.1.8);

(ii) collapse of a state vector $|\mathbf{S}_{2(1)}(t,\mathbf{r}_2)\rangle$ given by (4.1.7) occurs without any interaction between quantum systems $S_{1(2)}$ and $S_{2(1)}$ but only by property given by formulae (4.1.9).

Remark 4.1.9. We find that the EPR-paradox can be resolved by nonprincipal and convenient relaxing of the Einstein's locality principle. However it follows also, that the nonlocalities which are introduced above cannot be explained within the conventional quantum theory.

IV.2. Generalized EPR argument and Postulate of Nonlocality

Entanglement is one of the most interesting properties of quantum mechanics, and is an important ingredient of quantum information protocols such as quantum dense coding and quantum computation. In the Schrödinger picture, a necessary and sufficient criterion for the emergence of entanglement is that the state describing the entire system is inseparable, i.e. the wave function of the total system cannot be factored into a product of separate contributions from each sub-system. Using the Heisenberg approach, a sufficient criterion for the presence of entanglement is that correlations between conjugate observables of two subsystems allow the statistical inference of either observable in one sub-system, upon a measurement in the other, to be smaller than the standard quantum limit, i.e. the presence of non-classical correlations. The latter approach was originally proposed in the paper of Einstein, Podolsky and Rosen [1]. These two different pictures result in two distinct methods of characterizing entanglement. One is to identify an observable signature of the mathematical criterion for wave function entanglement, i.e. inseparability of the state. The second looks directly for the onset of non-classical correlations. For pure states these two approaches return the same result suggesting consistency of the two methods. However, when decoherence is present, causing the state to be mixed, difference scan occur.

IV.2.1. The EPR-Reid criterion

We remind now EPR-Reid criterion [2]-[5]. EPR originally argued as follows. Consider two spatially separated subsystems at A and B. EPR considered two observables $\hat{x}$ (the position) and $\hat{p}$ (momentum) for subsystem A, where $\hat{x}$ and $\hat{p}$ do not commute, so that (C is nonzero)

$$[\hat{x},\hat{p}] = 2C. \tag{4.2.1}$$

Suppose now that one may predict with certainty the result of measurement $\hat{x}$ based on the result of a measurement performed at B. Also, for a different choice of measurement at B, suppose one may predict the result of measurement $\hat{p}$ at A. Such correlated systems are predicted by quantum theory. Assuming local realism EPR deduce the existence of an element of reality, $\tilde{x}$, for the physical quantity $\hat{x}$ and also an element of reality, $\tilde{p}$, for $\hat{p}$. Local realism implies the existence of two hidden variables $\tilde{x}$ and $\tilde{p}$ that simultaneously predetermine, with no uncertainty, the values for the result of an $\hat{x}$ or $\hat{p}$ measurement on subsystem A, should it be performed. This hidden variable state for the subsystem A alone is not describable within quantum mechanics, since simultaneous eigenstates of $\hat{x}$ and $\hat{p}$ do not exist. Hence, EPR argued, if quantum mechanics is to be compatible with local realism, we must regard quantum mechanics to be incomplete.

We remind that in original publication [1] Einstein, Podolsky and Rosen describe two particles A and B with correlated position

$$x_B = x_A + x_0 \tag{4.2.2}$$

and anti-correlated momentum

$$p_B = -p_A, \tag{4.2.3}$$

(see Fig. 4.2.1).

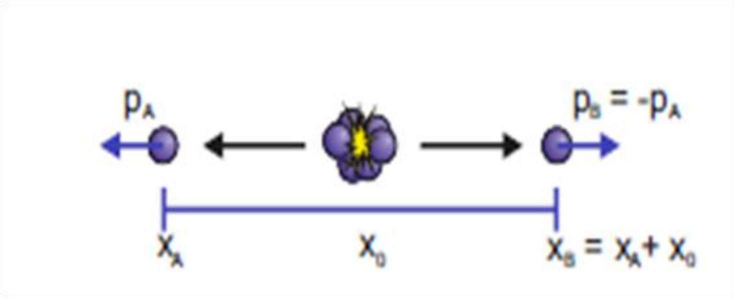

Fig. 4.2.1. Particles A and B with correlated position and anti-correlated momentum

In the idealized entangled state proposed by EPR,

$$|EPR\rangle = \int_{-\infty}^{\infty} |x, x\rangle dx = \int_{-\infty}^{\infty} |p, p\rangle dp$$

the positions and momenta of the two particles are perfectly correlated. Note that: this state is non-normalizable and cannot be realized in the laboratory. When coordinates x^A and p^A are measured in independent realizations of the same state, the correlations allow for an exact prediction of x^B and p^B. EPR assumed that such exact predictions necessitate an element of reality which predetermines the outcome of the measurement. Quantum mechanics however prohibits the exact knowledge of two noncommuting variables like x^B and p^B, since their measurement uncertainties are subject to the Heisenberg relation

$$\Delta x^B \Delta p^B \geq \hbar/2. \qquad (4.2.4)$$

Classical notion of EPR correlations was generalized to a more realistic scenario, yielding a Reid criterion [6] for the uncertainties $\Delta x^B_{\mathbf{inf}}$ and $\Delta p^B_{\mathbf{inf}}$ of the inferred predictions for x^B and p^B. The EPR criterion is met if these uncertainties violate the Heisenberg inequality for the inferred uncertainties $\Delta x^B_{\mathbf{inf}} \Delta p^B_{\mathbf{inf}} \geq \hbar/2$.
Reid extended classical EPR argument to situations where the result of measurement $\hat{x}$ at A cannot be predicted with absolute certainty [2]-[5]. The assumption of local realism allows us to deduce the existence of an element of reality of some type for $\hat{x}$ at A, since we can make a prediction of the result at A, without disturbing the subsystem at A, under the locality assumption. Let $\Psi\left(\hat{x}^A, \hat{x}^B\right)$ be a wave function of composite system $A \cup B$. Let x_i^B be the result of a measurement, $\hat{x}^B$ say, performed at B, where i is used to label the possible results, discrete or otherwise, of the measurement $\hat{x}^B$. As a result of the measurement of the coordinate, we have a new wave function of composite system $A \cup B$ which is given by Eq. (4.2.3) (see Remark 4.2.1)

$$\Psi_{x_i^B}\left(x, \hat{x}^B\right) = \Psi_{x_i^B}\left(\hat{x}^A, \hat{x}^B\right) = R(\hat{x}^B - x_i^B)\Psi\left(\hat{x}^A, \hat{x}^B\right) = R(\hat{x}^B - x_i^B)\Psi\left(x, \hat{x}^B\right) \quad (4.2.5)$$

and therefore adjoint probability density $p_{x_i^B}\left(x, \hat{x}^B\right) = p\left(x, \hat{x}^B \middle| x_i^B\right)$ at instant at once after measurement is given by

$$p_{x_i^B}\left(x, \hat{x}^B\right) = p\left(x, \hat{x}^B \middle| x_i^B\right) = \left\| R(\hat{x}^B - x_i^B)\Psi\left(x, \hat{x}^B\right) \right\|^2 \qquad (4.2.6)$$

Then the conditional probability density $p_{x_i^B}(x) = p(x|x_i^B)$ conditional on a result x_i^B for QM measurement at B is given by

$$p_{x_i^B}(x) = p(x|x_i^B) = \int_{-\infty}^{\infty} p_{x_i^B}\left(x,\hat{x}^B\right)d\hat{x}^B = \int_{-\infty}^{\infty} d\hat{x}^B \left\| \Psi_{x_i^B}\left(x,\hat{x}^B\right) \right\|^2 = \int_{-\infty}^{\infty} d\hat{x}^B \left\| R(\hat{x}^B - x_i^B)\Psi\left(x,\hat{x}^B\right) \right\|^2. \tag{4.2.7}$$

The predicted results for the measurement at A, based on the measurement at B, are however no longer a set of definite numbers with zero uncertainty, but become fuzzy, being described by a set of distributions $P(x|x_i^B)$ giving the probability of a result for the measurement at $A,$ conditional on a result x_i^B for measurement at B. We define $\Delta_i^2 x$ to be the variance of the conditional distribution $P(x|x_i^B)$. Similarly we may infer the result of measurement $\hat{p}$ at A, based on a (different) measurement, $\hat{p}^B$ say, at B. Denoting the results of the measurement $\hat{p}^B$ at B by p_j^B, we then define the probability distribution, $P\left(p|p_j^B\right)$ which is the predicted result of the measurement for $\hat{p}$ at A conditional on the result p_j^B for the measurement $\hat{p}^B$ at B. The variance of the conditional distribution $P\left(p|p_j^B\right)$ is denoted by $\Delta_j^2 p$.

Remark 4.2.1. We remind now that the QM-measurement is represented by the canonical scheme

$$|\psi\rangle \overset{a'}{\to} |\psi_{a'}\rangle = \Re_{a'}|\psi\rangle, \int da' \Re_{a'}^{\dagger}\Re_{a'} = \mathbf{1}, p_{a'} = \|\psi_{a'}\|^2 = \langle\psi|\Re_{a'}^{\dagger}\Re_{a'}|\psi\rangle, \tag{4.2.8}$$

where $p_{a'}$ is a corresponding probability density. To obtain the probability that the parameter a' turns out to belong to the set Δ one has to integrate over this set:

$$\mathbf{P}[a' \in \Delta] = \int_{\Delta} da' p_{a'}. \tag{4.2.9}$$

If the state $|\psi\rangle$ is represented by the wave function $\psi(a)$ the operator $\Re_{a'}$ describing the measurement giving the result a' will be taken in the following form

$$\Re_{a'}\psi(a) = R(a-a')\psi(a), \qquad (4.2.10)$$

where $R(a)$ is a function with a support concentrated in some vicinity of zero and representing the 'fuzziness' of the measurement. It is a characteristic function of the measurement and may, for example, be (and typically is) a Gaussian function. The width of this function corresponds to the resolution of the measurement.

Normalization $\int da'\Re^{\dagger}_{a'}\Re_{a'} = \mathbf{1}$ of the operators $\Re_{a'}$ is provided by the corresponding normalization of the function $R(a)$ as follows:

$$\int da|R^2(a)| = 1. \qquad (4.2.11)$$

If the measurement is described by the Gaussian function

$$R(a) = \exp\left[-\frac{(a-a')^2}{4\Delta^2}\right] \qquad (4.2.12)$$

it is a minimally disturbing measurement of the coordinate a' with resolution Δ [7].

Remark 4.2.2. Consider the momentum representation $\tilde{\psi}(p)$ of the initial wave function $\psi(q)$

$$\tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}}\int dq\psi(q)\exp\left[-\frac{i}{\hbar}pq\right]. \qquad (4.2.13)$$

As a result of the measurement of the coordinate,

$$\Re_{q'}\psi(q) = R(q-q')\psi(q) = \psi_{q'}(q), \tag{4.2.14}$$

we have a new wave function and its momentum representation has the form

$$\widetilde{\psi}_{q'}(p) = \int dp'\widetilde{R}_{q'}(p-p')\widetilde{\psi}(p'), \tag{4.2.15}$$

where $\widetilde{R}_{q'}(p)$ is a momentum representation of the function $R_{q'}(q)$. Note that

$$\begin{aligned} \widetilde{R}_{q'}(p) &= \widetilde{R}(p)\exp\left[-\frac{i}{\hbar}pq'\right], \\ \widetilde{R}(p) &= \frac{1}{2\pi\hbar}\int dqR(q)\exp\left[-\frac{i}{\hbar}pq\right]. \end{aligned} \tag{4.2.16}$$

Remark 4.2.3. Consider now a coordinate measurement having a Gaussian characteristic function of width of the order of Δ

$$R(q) \simeq \exp\left[-\frac{q^2}{4\Delta^2}\right]. \tag{4.2.17}$$

Then the momentum representation of this function (characterizing the structure of the momentum uncertainty $\hbar/\Delta$ acquired in the measurement) is also Gaussian with width of the order of $\hbar/\Delta$:

$$\widetilde{R}(p) \simeq \exp\left[\frac{-p^2}{\left(\frac{\hbar}{\Delta}\right)^2}\right]. \tag{4.2.18}$$

For a given experiment one could in principle measure the individual variances $\Delta_i^2 x$ of the conditional distributions $P(x|x_i^B)$ (and also $\Delta_j^2 p$ for the $P(p|p_j^B)$). Obviously if each of the variances $\Delta_i^2 x$ and $\Delta_j^2 p$ satisfy $\Delta_i^2 x = 0$ and $\Delta_j^2 p = 0$ this would imply the demonstration of the original

EPR paradox. This situation however is not practical for continuous variable measurements. Instead of considering the problem of simultaneous eigenstates as originally proposed by EPR, one can suggest a different and experimentally realizable criterion based on the Heisenberg Uncertainty Principle: $\Delta\hat{x}\Delta\hat{p} \geq C$. For the sake of notational convenience we now consider in the remainder of this subsection that appropriate scaling enables $\hat{x}$ and $\hat{p}$ to be dimensionless and $C = 1$.
EPR correlations however would be demonstrated in a convincing manner if the experimentalist could measure each of the conditional distributions $P(x|x_i^B)$ and establish that each of the distributions is very narrow, in fact constrained such that [2]-[5]:

$$\begin{aligned} P(x|x_i^B) &= 0 \quad \text{iff} \quad |x - \mu_i| > \delta, \\ P(p|p_j^B) &= 0 \quad \text{iff} \quad |p - \nu_j| > \delta. \end{aligned} \tag{4.2.19}$$

Here μ_i is the mean value of the conditional distribution $P(x|x_i^B)$ and ν_j is the mean value of the conditional distribution $P(p|p_j^B)$. In this case the assumption of local realism would imply, since the measurement $\hat{x}^B$ at B will always imply the result of $\hat{x}$ at A to be within the range $\mu_i \pm \delta_x$, that the result of the measurement at A is predetermined to be within a bounded range of width 2δ. In a straightforward extension of EPR's argument, we replace the words predict with certainty with predict with certainty that the result is constrained to be within the range $\mu_i \pm \delta$, and then define an element of reality with this intrinsic bounded by fuzziness δ. We now consider the situation where an experimenter has demonstrated that for *every outcome* x_i^B (and p_j^B) for the measurement $\hat{x}^B$ (and $\hat{p}^B$) performed at B, the variance $\Delta_i x$ (and $\Delta_j p$) of the appropriate conditional distribution satisfies

$$\Delta_i x < 1, \Delta_j p < 1 \tag{4.2.20}$$

for any $i,j \in \mathbb{N}$. The measurement at B always allows an inference of the result at A to a precision better than given by the uncertainty bound 1.

Remark 4.2.4. In this case we do not predict a result at A with certainty, as in EPR's original paradox. The measurement $\hat{x}^B$ at B however does predict by Eq. (4.2.3) [or by Eq. (4.2.9) in general case] with a certain probability constraints on the result for $\hat{x}$ at A.

Remark 4.2.5. Following the EPR argument, which assumes **no action-at-a-distance**, so that the measurement at B does not cause any instantaneous influence to the system at A, one can attribute a probabilistic predetermined element of reality to the system at A.

Remark 4.2.6. There is a similar predicted result for the measurement $\hat{p}$ at A based on a result of measurement at B, and a corresponding predetermined description based again on the **no-action-at-a-distance** assumption.

Remark 4.2.7. The important point in establishing the EPR paradox for this more general yet practical situation is that under the EPR premises the predetermined statistics (or generalised elements of reality) for the physical quantities $\hat{x}$ **and** $\hat{p}$ **are attributed simultaneously to the subsystem at** A.

Assuming no action-at-a-distance, the choice of the experimenter (Bob) at B to infer information about either $\hat{x}$ or $\hat{p}$ cannot actually induce the result of the measurement at A.

As there is no disturbance created by Bob's measurement, the (appropriately extended) EPR definition of realism is that the prediction for x is something (a probabilistic element of reality) that can be attributed to the subsystem at A, whether or not Bob makes his measurement.

Remark 4.2.8. This is also true of the prediction for $\hat{p}$, and therefore the two elements of reality representing the physical quantities $\hat{x}$ and $\hat{p}$ exist to describe the predictions for $\hat{x}$ and $\hat{p}$ simultaneously.

The paradox can then be established by proving the impossibility of such a simultaneous level of prediction for both $\hat{x}$ and $\hat{p}$ for any quantum description of the subsystem A alone. By this we mean explicitly that there can be no procedure allowed, within the predictions of quantum mechanics, to make *simultaneous inferences* by measurements performed at B or any other location, of both the result $\hat{x}$ and $\hat{p}$ at A , to the precision indicated by $\Delta_i x < 1,\ \Delta_j p < 1$.

Remark 4.2.9. Recall that the inference of the result at A by measurement at B is actually a measurement of $\hat{x}$ performed with the accuracy determined by the $\Delta_i x$. However simultaneous measurements of $\hat{x}$ and $\hat{p}$ to the accuracy (4.2.9) are not possible (predicted by quantum mechanics). The reduced density matrix describing the state at A after such measurements would violate the H.U.P. (Heisenberg Uncertainty Principle).

A simpler quantitative, experimentally testable criterion for EPR was proposed by Reid in 1989 see, for example, [2]-[5]. The 1989 inferred H.U.P. criterion is based on the average variance of the conditional distributions for inferring the result of measurement $\hat{x}$ (and also for $\hat{p}$). The EPR paradox is demonstrated when the product of the average errors in the inferred results for $\hat{x}$ and $\hat{p}$ violates the corresponding H.U.P. The spirit of the original EPR paradox is present, in that one can perform a measurement on B to enable an estimate of the result x at A (and similarly for $\hat{p}$).

Abbreviation 4.2.1. For the sake of notational convenience we now abbreviate in the remainder of the book: $\Delta_{\mathbf{loc}.i} x$ and $\Delta_{\mathbf{loc}.i} p$ instead $\Delta_i x$ and $\Delta_j p$ for the variance $\Delta_i x$ and $\Delta_j p$ which were calculated under assumption no action-at-a-distance, see Remarks 4.2.5 - 4.2.6.

We define now [2]:

$$\Delta^2_{\mathbf{loc.inf.}}\, x = \sum_i P(x_i^B)\Delta^2_{\mathbf{loc}.i} x,$$
$$\Delta^2_{\mathbf{loc.inf.}}\, p = \sum_j P\left(p_j^B\right)\Delta^2_{\mathbf{loc}.j} p. \qquad (4.2.21)$$

Here $\Delta^2_{\mathbf{loc.inf.}}\hat{x}$ is the average variance for the prediction (inference) under assumption no action-at-a-distance of the result x for $\hat{x}$ at A, conditional on a measurement $\hat{x}^B$ at B. Here $i \in \mathbb{N}$ labels all outcomes of the measurement $\hat{x}$ at A, and μ_i and $\Delta_i x$ are the mean and standard deviation, respectively, of the conditional distribution

$P(x|x_i^B)$, where x_i^B is the result of the measurement $\hat{x}^B$ at B. We define a $\Delta^2_{\mathbf{loc.inf.}}\hat{p}$ similarly to represent the weighted variance for the prediction (inference) under assumption no action-at-a-distance of the result $\hat{p}$ at A, based on the result of the measurement at B. Here $P(x_i^B)$ is the probability for a result x_i^B upon measurement of $\hat{x}^B$, and $P(p_j)$ is defined similarly.

The Reid's criterion to demonstrate the EPR "paradox", the Reid's local signature of the EPR paradox, is

$$\left(\Delta^2_{\mathbf{loc.inf.}}\, x\right)\left(\Delta^2_{\mathbf{loc.inf.}}\, p\right) < 1. \qquad (4.2.22)$$

This criterion is a clear criterion for the demonstration of the EPR "paradox", by way of the argument presented above. Such a prediction (4.2.21) for $\hat{x}$ and $\hat{p}$ with the average inference variances given, cannot be achieved by any quantum description of the subsystem alone. This EPR criterion has been achieved experimentally.

IV.2.2. The Postulate of Nonlocality and signature of the EPR paradox

Remark 4.2.10. A most critical component of the EPR argument was the principle of locality. Indeed, one may regard the EPR paradox as a statement of the mutual incompatibility of locality, entanglement, and completeness. Experimental tests of Bell's inequalities have indicated that quantum mechanics is complete by ruling out the possibility of hidden variables. Therefore, it is generally agreed that the assumption of locality is invalid for entangled states: measurement of either particle of an entangled system projects both particles onto a state consistent with the result of measurement, regardless of how far apart the particles are. In the situation proposed by EPR, the position or momentum of the unmeasured particle becomes a reality if and only if, the corresponding quantity of the other particle is measured.

Remark 4.2.11. The assumption of nonlocality allows us to deduce the existence of a fuzzy element of reality of some type for $\hat{x}$ at A, since we can make a prediction of the result at A, but with some disturbing of the subsystem at A, under the measurement, $\hat{x}^B$ say, performed at B. This prediction is subject to the result x_i^B of a measurement, $\hat{x}^B$ say, performed at B, where i is used to label the possible results, discrete or otherwise, of the measurement $\hat{x}^B$.
We accept now the following postulate:

Postulate of Nonlocality

(**i**) Let A and B two entangled particles. Let $\Psi\left(\hat{x}^A, \hat{x}^B\right)$ be a wave function of composite system $A \cup B$. Let x_i^B be the result of a measurement, $\hat{x}^B$ say, performed at B, where i is used to label the possible results, discrete or otherwise, of the measurement $\hat{x}^B$. As a result of the measurement of the coordinate, we have a new wave function of composite system $A \cup B$ which [in contrast with Eq.(4.2.5)] is given by

$$\begin{aligned} &\Psi_{x_i^B}\left(\hat{x}^A, \hat{x}^B\right) = R_2(\hat{x}^A - x_i^A(x_i^B))R_1(\hat{x}^B - x_i^B)\Psi\left(\hat{x}^A, \hat{x}^B\right), \\ &x_i^A(x_i^B) + x_0 \simeq x_i^B. \end{aligned} \tag{4.2.23}$$

(**ii**) Let A and B are two entangled particles. Let $\widetilde{\Psi}\left(\hat{p}^A, \hat{p}^B\right)$ be a wave function of composite system $A \cup B$. Let p_j^B be the result of a measurement, $\hat{p}^B$ say, performed at B, where j is used to label the possible results, discrete or otherwise, of the measurement $\hat{p}^B$. As a result of the measurement of the coordinate, we have a new wave function of composite system $A \cup B$ which is given by

$$\widetilde{\Psi}_{p_j^B}\left(\widehat{p}^A,\widehat{p}^B\right) = \widehat{R}_2(\widehat{p}^A - p_j^A\left(p_j^B\right))\widehat{R}_1(\widehat{p}^B - p_j^B)\widetilde{\Psi}\left(\widehat{p}^A,\widehat{p}^B\right),$$
$$p_j^A\left(p_j^B\right) \simeq -p_j^B. \tag{4.2.24}$$

Remark 4.2.12. The spirit of the original EPR paradox now is present, in that the canonical EPR correlations (4.2.2) and (4.2.3) well preserved.

Remark 4.2.13. Note that EPR correlations $x_i^A(x_i^B) + x_0 \simeq x_i^B$ and $p_j^A\left(p_j^B\right) \simeq -p_j^B$ however would be demonstrated in a convincing manner if the experimentalist could measure each of the conditional distributions $P(x|x_i^B)$ and establish that each of the distributions is very narrow, in fact constrained so that [2]-[5]

$$\begin{aligned} p(x|x_i^B) &\simeq 0 \quad \text{iff} \quad |x-\mu_i| > \delta, \\ p(p|p_j^B) &\simeq 0 \quad \text{iff} \quad |p-\nu_j| > \delta, \\ P(x|x_i^B) &\simeq 0 \quad \text{iff} \quad |x-\mu_i| > \delta, \\ P(p|p_j^B) &\simeq 0 \quad \text{iff} \quad |p-\nu_j| > \delta. \end{aligned} \tag{4.2.25}$$

Here μ_i is the mean of the conditional distribution $P(x|x_i^B)$ and ν_j is the mean of the conditional distribution $P(p|p_j^B)$.

Remark 4.2.14. We assume now that a coordinate and momentum measurements have a Gaussian characteristic function of width of the order of 2δ

$$\begin{aligned} R_1(x) = R_2(x) = R(x) &\simeq \exp\left[-\frac{x^2}{4\delta^2}\right] \\ \widehat{R}_1(p) = \widehat{R}_2(p) = \widehat{R}(p) &\simeq \exp\left[-\frac{p^2}{4\delta^2}\right] \end{aligned} \tag{4.2.26}$$

In this case the Postulate of Nonlocality would imply, since the measurement $\hat{x}^B$ at B will always imply the result of $\hat{x}$ at A to be within the range $\mu_i \pm \delta_x$, that the result of the measurement at A is predetermined to be within a bounded range of width 2δ. In a straightforward extension of EPR's argument, we replace the words ‘predict with certainty’ with ‘predict with certainty that the result is constrained to be within the range $\mu_i \pm \delta$'. We now consider the situation where an experimenter has demonstrated that for every outcome x_i^B (and p_j^B) for the measurement $\hat{x}^B$ (and $\hat{p}^B$) performed at B, the variance $\Delta_i x$ (and $\Delta_j p$) of the appropriate conditional distribution satisfies

$$\Delta_i x < 1, \Delta_j p < 1 \tag{4.2.27}$$

for all i,j.

The measurement at B always allows an inference of the result at A to a precision better than given by the uncertainty bound 1.

In this case we do not predict a result at A with certainty, as in EPR original paradox. The measurement $\hat{x}_B$ at B however does predict with a certain probability constraints on the result for $\hat{x}$ at A.

Remark 4.2.15. Note that adjoint probability density $p\left(\hat{x}^A, \hat{x}^B \middle| x_i^B\right)$ at instant at once after measurement [in contrast with Eq. (4.1.6)] is given by

$$\begin{aligned}
&p\left(\hat{x}^A, \hat{x}^B \middle| x_i^B\right) = \left\| \Psi_{x_i^B}\left(\hat{x}^A, \hat{x}^B\right) \right\|^2 = \\
&\left\| R(\hat{x}^A - x_i^A(x_i^B)) R(\hat{x}^B - x_i^B) \Psi\left(\hat{x}^A, \hat{x}^B\right) \right\|^2, \\
&x_i^A(x_i^B) + x_0 \simeq x^B \pm \mu_i.
\end{aligned} \tag{4.2.27a}$$

Then the conditional probability density $p_{x_i^B}(x) = p(x|x_i^B)$ depending on a result x_i^B for QM measurement at B is given by

$$\begin{aligned}
&p_{x_i^B}(x) = p(x|x_i^B) = \int_{-\infty}^{\infty} p_{x_i^B}\left(x, \hat{x}^B\right) d\hat{x}^B = \int_{-\infty}^{\infty} d\hat{x}^B \left\| \Psi_{x_i^B}\left(x, \hat{x}^B\right) \right\|^2 = \\
&\int_{-\infty}^{\infty} d\hat{x}^B \left\| R(\hat{x}^A - x_i^A(x_i^B)) R(\hat{x}^B - x_i^B) \Psi\left(x, \hat{x}^B\right) \right\|^2.
\end{aligned} \tag{4.2.28}$$

There is a similar predicted result for the measurement $\hat{p}$ at A based on a result of measurement at B, and a corresponding predetermined description based on the QM constraints

$$\begin{aligned}&\widetilde{\Psi}_{p_j^B}\left(\hat{p}^A,\hat{p}^B\right) = \hat{R}(\hat{p}^A - p_j^A\left(p_j^B\right))\hat{R}(\hat{p}^B - p_j^B)\widetilde{\Psi}\left(\hat{p}^A,\hat{p}^B\right),\\&p_j^A\left(p_j^B\right) \simeq -p^B.\end{aligned} \tag{4.2.29}$$

The spirit of the original EPR "paradox" is present, in that one can perform a measurement on B to enable an estimate of the result x at A (and similarly for $\hat{p}$).

Abbreviation 4.2.2. For the sake of notational convenience we now abbreviate in the remainder of the book: $\Delta_{\mathbf{nonloc}.i}x$ and $\Delta_{\mathbf{nonloc}.j}p$ instead $\Delta_i x$ and $\Delta_j p$ for the variance $\Delta_i x$ and $\Delta_j p$ which were calculated under nonlocality assumption (postulate) by conditional probability density given by Eq. (4.1.28).

We define now

$$\begin{aligned}&\Delta^2_{\mathbf{nonloc.inf.}}\,x = \sum\nolimits_i P(x_i^B)\Delta^2_{\mathbf{nonloc}.i}x,\\&\Delta^2_{\mathbf{nonloc.inf.}}\,p = \sum\nolimits_j P\left(p_j^B\right)\Delta^2_{\mathbf{nonloc}.j}p.\end{aligned} \tag{4.2.30}$$

Here $\Delta^2_{\mathbf{nonloc.inf.}}\,x$ is the average variance for the prediction (inference) of the result x for $\hat{x}$ at $A,$ conditional on a measurement $\hat{x}^B$ at $B.$ Here i labels all outcomes of the measurement $\hat{x}$ at $A,$ and μ_i and $\Delta_i x$ are the mean and standard deviation, respectively, of the conditional distribution $P(x|x_i^B),$ where x_i^B is the result of the measurement $\hat{x}^B$ at $B.$ We define a $\Delta^2_{\mathbf{nonloc.inf.}}\,p$ similarly to represent the weighted variance for the prediction (inference) of the result $\hat{p}$ at $A,$ based on the result of the measurement at B. Here $P(x_i^B)$ is the probability for a result x_i^B upon measurement of $\hat{x}^B$, and $P(p_j^B)$ is defined similarly. The criterion to demonstrate the EPR paradox, the signature of the EPR paradox, is the criterion to demonstrate the EPR paradox, the nonlocal signature of the EPR paradox, is given by

$$\left(\Delta_{\mathbf{nonloc.inf.}}\, x\right)\left(\Delta_{\mathbf{nonloc.inf.}}\, p\right) < 1. \tag{4.2.31}$$

This criterion is a clear criterion for the demonstration of the EPR paradox, by way of the argument presented above. Such a prediction for $\hat{x}$ and $\hat{p}$ with the average inference variances given, cannot be achieved by any quantum description of the subsystem alone.

IV.2.3. The EPR-nonlocality criteria

Remark 4.2.16. The principle of locality was a critical component of the EPR argument. Actually the EPR paradox is regarded as a statement of the mutual incompatibility of locality, entanglement, and completeness. Experimental studies of Bell's inequalities have shown that quantum mechanics is complete by ruling out the possibility of hidden variables. Consequently it is usually accepted that the assumption of locality is invalid for entangled states: measurement of either particle of an entangled system projects both particles onto a state corresponding with the measurement result, irrespective of how far apart these particles are. In the situation proposed by EPR, the position or momentum of the unmeasured particle becomes a reality if and only if, the corresponding quantity of the other particle is measured.
Since only one quantity or the other is measured, the position and the momentum of the unmeasured particle need not be simultaneous realities. In this way the EPR paradox also is resolved.
From Eq. (4.2.21) and Eq. (4.2.30) we obtain the EPR-nonlocality criteria

$$\begin{aligned} \left|\Delta^2_{\mathbf{loc.inf.}}\, x - \Delta^2_{\mathbf{nonloc.inf.}}\, x\right| &= \left|\sum_i P(x_i^B)[\Delta^2_{\mathbf{loc.}i} x - \Delta^2_{\mathbf{nonloc.}i} x]\right| > 0, \\ \left|\Delta^2_{\mathbf{loc.inf.}}\, p - \Delta^2_{\mathbf{nonloc.inf.}}\, p\right| &= \left|\sum_j P\left(p_j^B\right)\left[\Delta^2_{\mathbf{loc.}j} p - \Delta^2_{\mathbf{nonloc.}j} p\right]\right| > 0, \end{aligned} \tag{4.2.32}$$

and

$$\left|\left(\Delta_{\mathbf{nonloc.inf.}}\, x\right)\left(\Delta_{\mathbf{nonloc.inf.}}\, p\right) - \left(\Delta_{\mathbf{loc.inf.}}\, x\right)\left(\Delta_{\mathbf{loc.inf.}}\, p\right)\right| > 0. \qquad (4.2.33)$$

These EPR-nonlocality criteria has been achieved experimentally [8], [9], (see subsection IV.5, Remark 4.5.3 - Remark 4.5.4).

IV.3. Nonlocal Schrödinger equation implies the Postulate of Nonlocality

In this subsection we obtain nonlocal Schrödinger equation (NSE) which corresponding to position-momentum entangled pairs A and B (see Fig. 4.2.1) with well correlated position

$$\langle x_B \rangle \simeq \langle x_A \rangle + x_0 \qquad (4.3.1)$$

and anti-correlated momentum

$$\left\langle p_B \right\rangle \simeq -\langle p_A \rangle. \qquad (4.3.2)$$

Remark 4.3.1. As pointed out in subsection IV.2 it is generally agreed that the assumption of locality is invalid for entangled states: measurement of either particle of an entangled system projects both particles onto a state consistent with the result of measurement, regardless of how far apart the particles are. It allow us to use special nonlocal generalization of the canonical Schrödinger equation.

Remark 4.3.2. As pointed out in [10], [11] from nonlocal Schrödinger equation one obtains collapsed wave function corresponding to GRW collapse model.

It allow us to use similar nonlocal Schrödinger equation also for entangled states.

Remark 4.3.3. The spirit of the original EPR paradox is present, in that the canonical EPR correlations (4.3.1) and (4.3.2) give a boundary conditions for the solutions of the nonlocal Schrödinger equation.

Remark 4.3.4. In this subsection we denote (i) $x_A = x_1, x_B = x_2$,
(ii) $x^A = \widetilde{x}_1,\ \ x^B = \widetilde{x}_2 = \widetilde{x}_1 + x_0$.

Definition 4.3.1. Let us consider the time-dependent canonical Schrödinger equation

$$i\hbar\frac{\partial\Psi(x_1,x_2,t)}{\partial t} = H\Psi(x_1,x_2,t), \qquad (4.3.3)$$
$$t \in [0,T], (x_1,x_2) \in \mathbb{R}^2.$$

Let $\Psi(x_1,x_2,t)$ be a classical solution of the time-dependent Schrödinger equation (4.3.3). The time-dependent Schrödinger equation (4.3.3) is a weakly well preserved (in sense of Colombeau generalized functions) by corresponding to $\Psi(x_1,x_2,t)$ collapsed Colombeau generalized wave function $(\Psi^{\#}_{\varepsilon}(x_1,x_2,t))_{\varepsilon}, \varepsilon \in (0,1]$, where

$$(\Psi^{\#}_{\varepsilon}(x_1,x_2,t))_{\varepsilon} = (\Psi_{\varepsilon}(x_1,x_2,t;\widetilde{x}_1(t),\widetilde{x}_2(t)))_{\varepsilon} =$$

$$= \left(\frac{\Re_{1,2}(x_1,\widetilde{x}_1(t),x_2,\widetilde{x}_2(t);\delta,\varepsilon)\Psi(x_1,x_2,t)}{\|\Re_{1,2}(\widetilde{x}_1(t),\widetilde{x}_2(t);\delta,\varepsilon)\Psi(x_1,x_2,t)\|_2}\right)_{\varepsilon},$$

$$(\Re_{1,2}(x_1,\widetilde{x}_1,x_2,\widetilde{x}_2;\delta,\varepsilon))_{\varepsilon} = \prod_{i=1}^{2}(\Re_i(x_i,\widetilde{x}_i;\delta,\varepsilon))_{\varepsilon}, \qquad (4.3.4)$$

$$\Re_i(x,\widetilde{x}_i(t);\delta,\varepsilon) = \begin{cases} (\pi_\delta\delta^2)^{-1/4}\exp\left[-\frac{(x_i-\widetilde{x}_i(t))^2}{2\delta^2}\right] \text{ iff } \|x_i-\widetilde{x}_i\| \leq \varepsilon, \\ 0 \text{ iff } \|x_i-\widetilde{x}_i\| > \varepsilon. \end{cases}$$
$$i = 1,2.$$

in region $\Gamma \subseteq \mathbb{R}^2$ if there exists a solution $\Psi(x_1,x_2,t)$ of Schrödinger equation (4.2.1) such that the estimate

$$\left(\int_D \left\{ i\hbar \frac{\partial \Psi_\varepsilon^{\#}(x_1,x_2,t)}{\partial t} - \widehat{H}\Psi_\varepsilon^{\#}(x_1,x_2,t) \right\} dx_1 dx_2 \right)_\varepsilon = (O(\hbar^\alpha))_\varepsilon, \quad (4.3.5)$$

$$t \in [0,T], x_1, x_2 \in D,$$

with $1/2 \leq \alpha,$ is satisfied.

Definition 4.3.2. Equation (4.3.5) with a following boundary conditions

$$\begin{aligned} \langle x_B^t \rangle &\simeq \langle x_A^t \rangle + x_0, \\ \langle x_A^t \rangle &= \left(\int x_A \left| \Psi_\varepsilon^{\#}(x_A, x_B, t) \right|^2 dx_A dx_B \right)_\varepsilon, \\ \langle x_B^t \rangle &= \left(\int x_B \left| \Psi_\varepsilon^{\#}(x_A, x_B, t) \right|^2 dx_A dx_B \right)_\varepsilon, \end{aligned} \quad (4.3.6)$$

that is time-dependent nonlocal Schrödinger equation corresponding to EPR entangled state.

Definition 4.3.3. (i) The time-dependent integral equation (4.3.5) with a boundary conditions (4.3.6) is called the time-dependent nonlocal Schrödinger equation of the order $\hbar^\alpha$ corresponding to EPR entangled state.

(ii) Such collapsed wave function $\Psi^{\#}(x_1,x_2,t,)$ as mentioned in Definition 4.3.2 is called the $\hbar^\alpha$ - solution of the nonlocal Schrödinger equation (4.3.5)-(4.3.6) of the order α.

Definition 4.3.4. Let us consider the time-independent canonical Schrödinger equation

$$\widehat{H}\Psi(x_1,x_2) = 0, (x_1,x_2) \in \mathbb{R}^2. \quad (4.3.7)$$

Let $\Psi(x_1,x_2)$ be a classical solution of the time-independent Schrödinger equation (4.3.7). The time-independent Schrödinger equation (4.3.7) is a weakly well preserved (in sense of Colombeau generalized functions) by corresponding to $\Psi(x_1,x_2)$ Colombeau generalized collapsed wave function $(\Psi_\varepsilon^{\#}(x_1,x_2))_\varepsilon, \varepsilon \in (0,1],$ where

$$(\Psi_\varepsilon^\#(x_1,x_2,\delta))_\varepsilon = (\Psi_\varepsilon(x_1,x_2;\widetilde{x}_1,\widetilde{x}_2,\delta))_\varepsilon =$$

$$= \left(\frac{\Re_{1,2}(x_1,\widetilde{x}_1,x_2,\widetilde{x}_2;\delta,\varepsilon)\Psi(x_1,x_2)}{\|\Re_{1,2}(\widetilde{x}_1,\widetilde{x}_2;\delta,\varepsilon)\Psi(x_1,x_2)\|_2}\right)_\varepsilon,$$

$$(\Re_{1,2}(x_1,\widetilde{x}_1,x_2,\widetilde{x}_2;\delta,\varepsilon))_\varepsilon = \prod_{i=1}^{2}(\Re_i(x_i,\widetilde{x}_i;\delta,\varepsilon))_\varepsilon, \tag{4.3.8}$$

$$\Re_i(x,\widetilde{x}_i;\delta,\varepsilon) = \begin{cases} (\pi_\delta\delta^2)^{-1/4}\exp\left[-\dfrac{(x_i-\widetilde{x}_i)^2}{2\delta^2}\right] \text{ iff } \|x_i-\widetilde{x}_i\| \leq \varepsilon, \\ 0 \text{ iff } \|x_i-\widetilde{x}_i\| > \varepsilon. \end{cases}$$

in region $\Gamma \subseteq \mathbb{R}^2$ if there exists a solution $\Psi(x_1,x_2)$ of Schrödinger equation (4.3.7) such that the estimate

$$\left(\int_D \widehat{H}\Psi_\varepsilon^\#(x_1,x_2)dx_1dx_2\right)_\varepsilon = (O(\hbar^\alpha))_\varepsilon, \tag{4.3.9}$$

$$(x_1,x_2)\in D,$$

with $1/2 \leq \alpha$, is satisfied.

Definition 4.3.5. Equation (4.3.9) with boundary conditions

$$\begin{aligned} \langle x_B\rangle &\simeq \langle x_A\rangle + x_0, \\ \langle x_A\rangle &= \left(\int x_A|\Psi_\varepsilon^\#(x_A,x_B)|^2dx_Adx_B\right)_\varepsilon, \\ \langle x_B\rangle &= \left(\int x_B|\Psi_\varepsilon^\#(x_A,x_B)|^2dx_Adx_B\right)_\varepsilon, \end{aligned} \tag{4.3.10}$$

that is time-independent nonlocal Schrödinger equation corresponding to EPR entangled state.

Definition 4.3.6. (i) The time-independent integral equation (4.3.9) with a boundary conditions (4.3.10) is called the time-independent nonlocal Schrödinger equation of the order $\hbar^\alpha$ corresponding to EPR entangled state.

(ii) Such collapsed wave function $\Psi^{\#}(x_1,x_2)$ as mentioned in Definition 4.3.5 is called the $\hbar^{\alpha}$- solution of the nonlocal Schrödinger equation (4.3.9) - (4.3.10) of the order α.

Lemma 4.3.1. Let $\Phi(\lambda)$ be a function

$$\Phi(\lambda) = \int_0^a x^{\beta-1}\exp(-\lambda x^{\alpha})f(x)dx, \tag{4.3.11}$$

where $\lambda \gg 1,\ 0 < a < \infty, 0 < \beta, 0 < \alpha.$ Assume that $f(x)$ is continuous on $[0,a]$. Then

$$\Phi(\lambda) = \alpha^{-1}\Gamma\left(\frac{\beta}{\alpha}\right)[f(0) + o(1)]\lambda^{-\beta/\alpha} \tag{4.3.12}$$

Lemma 4.3.2. Let $f(x)$ be a function such that $f \in C^2(x < x_0)$ and $f \in C^2(x > x_0)$. Then

$$\begin{gathered} f'(x) = \left\{f'(x)\right\}_{x\neq x_0} + [f]_{x_0}\delta(x - x_0), \\ f''(x) = \left\{f''(x)\right\}_{x\neq x_0} + \left[f'\right]_{x_0}\delta(x - x_0) + [f]_{x_0}\delta'(x - x_0), \\ [f]_{x_0} = f(x_0 + 0) - f(x_0 - 0), \\ \left[f'\right]_{x_0} = f'(x_0 + 0) - f'(x_0 - 0). \end{gathered} \tag{4.3.13}$$

Theorem 4.3.1. Assume that there exists a classical solution $\Psi(x_1,x_2)$ of the Schrödinger equation (4.3.7) such that

$$\begin{gathered} \sup_{(x_1,x_2)\in D} |\Psi(x_1,x_2)| = O(\hbar^{-1/2}), \\ \sup_{(x_1,x_2)\in D} |\partial\Psi(x_1,x_2)/\partial x_1| = O(\hbar^{-3/2}), \quad \sup_{(x_1,x_2)\in D} |\partial\Psi(x_1,x_2)/\partial x_2| = O(\hbar^{-3/2}). \end{gathered} \tag{4.3.14}$$

Then any collapsed wave function $\Psi^{\#}(x)$ is given by Eq. (4.3.8) with $\sqrt{\hbar/\delta} = \hbar^{\alpha}, 1/4 < \alpha < 1/2$ that is $\hbar^{\alpha}$-solution of the time-independent

nonlocal Schrödinger equation (4.3.9)-(4.3.10) of the order α.
Proof. The Schrödinger equation (4.3.7) has the following form

$$H\Psi(x_1,x_2) = \hbar^2 \frac{\partial^2\Psi(x_1,x_2)}{\partial x_1^2} + \hbar^2 \frac{\partial^2\Psi(x_1,x_2)}{\partial x_2^2} + V(x_1,x_2)\Psi(x_1,x_2) = 0. \qquad (4.3.15)$$

Let $\Psi^{\#}_{\delta}(x_1,x_2)$ be a function

$$\Psi^{\#}_{\delta}(x_1,x_2) = R_{\delta}(x_1,\widetilde{x}_1)R_{\delta}(x_2,\widetilde{x}_2)\Psi(x_1,x_2), \qquad (4.3.16)$$

where

$$R_{\delta}(x_i,\widetilde{x}_i) = \begin{cases} (\pi_{\delta}\delta^2)^{-1/4}\exp\left[-\frac{(x_i-\widetilde{x}_i)^2}{2\delta^2}\right] \text{ iff } \|x_i-\widetilde{x}_i\| \leq \varepsilon, \\ 0 \text{ iff } \|x_i-\widetilde{x}_i\| > \varepsilon. \end{cases} \qquad (4.3.17)$$

From Eq. (4.3.17) by using Eq. (4.3.13) we obtain

$$\begin{gathered} \frac{\partial R_{\delta}(x_1,\widetilde{x}_1)}{\partial x_1} = -(\pi_{\delta}\delta)^{-1/4}\delta^{-1}(x_1-\widetilde{x}_1)\exp\left[-\frac{(x_1-\widetilde{x}_1)^2}{2\delta}\right] + \\ +\left([R_{\delta}(x_1,\widetilde{x}_1)]_{\widetilde{x}_1-\varepsilon}\right)\delta(x_1-\widetilde{x}_1+\varepsilon) + ([R_{\delta}(x_1,\widetilde{x}_1)]_{\widetilde{x}+\varepsilon})\delta(x_1-\widetilde{x}_1-\varepsilon), \\ \frac{\partial^2 R_{\delta}(x_1,\widetilde{x}_1)}{\partial x_1^2} = -(\pi_{\delta}\delta)^{-1/4}\delta^{-1}\exp\left[-\frac{(x_1-\widetilde{x}_1)^2}{2\delta}\right] + \\ (\pi_{\delta}\delta)^{-1/4}\delta^{-2}(x_1-\widetilde{x}_1)^2\exp\left[-\frac{(x_1-\widetilde{x}_1)^2}{2\delta}\right] + \\ \left(\left[\frac{\partial R_{\delta}(x_1,\widetilde{x}_1}{\partial x_1}\right]_{\widetilde{x}-\varepsilon}\right)\delta(x_1-\widetilde{x}_1+\varepsilon) + \left(\left[\frac{R_{\delta}(x_1,\widetilde{x}_1}{\partial x_1}\right]_{\widetilde{x}+\varepsilon}\right)\delta(x_1-\widetilde{x}_1-\varepsilon) + \\ ([R_{\delta}(x_1,\widetilde{x}_1]_{\widetilde{x}-\varepsilon})\delta'(x_1-\widetilde{x}_1+\varepsilon) + ([R_{\delta}(x_1,\widetilde{x}_1]_{\widetilde{x}+\varepsilon})\delta'(x_1-\widetilde{x}_1-\varepsilon) \end{gathered} \qquad (4.3.18)$$

and

$$\frac{\partial R_\delta(x_2,\widetilde{x}_2)}{\partial x_2} = -(\pi_\delta\delta)^{-1/4}\delta^{-1}(x_2-\widetilde{x}_2)\exp\left[-\frac{(x_2-\widetilde{x}_2)^2}{2\delta}\right]+$$
$$+\left([R_\delta(x_2,\widetilde{x}_2)]_{\widetilde{x}_1-\varepsilon}\right)\delta(x_2-\widetilde{x}_2+\varepsilon)+([R_\delta(x_2,\widetilde{x}_2)]_{\widetilde{x}+\varepsilon})\delta(x_2-\widetilde{x}_2-\varepsilon),$$
$$\frac{\partial^2 R_\delta(x_2,\widetilde{x}_2)}{\partial x_2^2} = -(\pi_\delta\delta)^{-1/4}\delta^{-1}\exp\left[-\frac{(x_2-\widetilde{x}_2)^2}{2\delta}\right]+$$
$$(\pi_\delta\delta)^{-1/4}\delta^{-2}(x_2-\widetilde{x}_2)^2\exp\left[-\frac{(x_2-\widetilde{x}_2)^2}{2\delta}\right]+ \tag{4.3.19}$$
$$\left(\left[\frac{\partial R_\delta(x_2,\widetilde{x}_2)}{\partial x_2}\right]_{\widetilde{x}-\varepsilon}\right)\delta(x_2-\widetilde{x}_2+\varepsilon)+\left(\left[\frac{R_\delta(x_2,\widetilde{x}_2)}{\partial x_2}\right]_{\widetilde{x}+\varepsilon}\right)\delta(x_2-\widetilde{x}_2-\varepsilon)+$$
$$+([R_\delta(x_2,\widetilde{x}_2)]_{\widetilde{x}-\varepsilon})\delta'(x_2-\widetilde{x}_2+\varepsilon)+([R_\delta(x_2,\widetilde{x}_2)]_{\widetilde{x}+\varepsilon})\delta'(x_2-\widetilde{x}_2-\varepsilon).$$

From Eq. (4.3.16) by differentiation we obtain

$$\frac{\partial^2\Psi^\#_\delta(x_1,x_2)}{\partial x_1^2} = \frac{\partial^2[R_\delta(x_1,\widetilde{x}_1)R_\delta(x_2,\widetilde{x}_2)\Psi(x_1,x_2)]}{\partial x_1^2} =$$
$$\frac{\partial}{\partial x_1}\left[\Psi(x_1,x_2)R_\delta(x_2,\widetilde{x}_2)\frac{\partial R_\delta(x_1,\widetilde{x}_1)}{\partial x_1}+R_\delta(x_1,\widetilde{x}_1)R_\delta(x_2,\widetilde{x}_2)\frac{\partial\Psi(x_1,x_2)}{\partial x_1}\right] = \tag{4.3.20}$$
$$2\frac{\partial\Psi(x_1,x_2)}{\partial x_1}R_\delta(x_2,\widetilde{x}_2)\frac{\partial R_\delta(x_1,\widetilde{x}_1)}{\partial x_1}+$$
$$+\Psi(x_1,x_2)R_\delta(x_2,\widetilde{x}_2)\frac{\partial^2 R_\delta(x_1,\widetilde{x}_1)}{\partial x_1^2}+R_\delta(x_1,\widetilde{x}_1)R_\delta(x_2,\widetilde{x}_2)\frac{\partial^2\Psi(x_1,x_2)}{\partial x_1^2}$$

and

$$\frac{\partial^2\Psi^\#_\delta(x_1,x_2)}{\partial x_2^2} = \frac{\partial^2[R_\delta(x_1)R_\delta(x_2,\widetilde{x}_2)\Psi(x_1,x_2)]}{\partial x_2^2} =$$
$$\frac{\partial}{\partial x_2}\left[\Psi(x_1,x_2)R_\delta(x_1,\widetilde{x}_1)\frac{\partial R_\delta(x_2,\widetilde{x}_2)}{\partial x_2}+R_\delta(x_1,\widetilde{x}_1)R_\delta(x_2,\widetilde{x}_2)\frac{\partial\Psi(x_1,x_2)}{\partial x_2}\right] = \tag{4.3.21}$$
$$2\frac{\partial\Psi(x_1,x_2)}{\partial x_2}R_\delta(x_1,\widetilde{x}_1)\frac{\partial R_\delta(x_2,\widetilde{x}_2)}{\partial x_2}+$$
$$+\Psi(x_1,x_2)R_\delta(x_1,\widetilde{x}_1)\frac{\partial^2 R_\delta(x_2,\widetilde{x}_2)}{\partial x_2^2}+R_\delta(x_1,\widetilde{x}_1)R_\delta(x_2,\widetilde{x}_2)\frac{\partial^2\Psi(x_1,x_2)}{\partial x_2^2}.$$

By substitution Eq. (4.3.15) and Eq. (4.3.20) - Eq. (4.3.21) into LHS of the Eq. (4.3.9) we obtain

$$\begin{gathered}
\int_{\Gamma} \widehat{H}\Psi_{\delta}^{\#}(x_1,x_2)dx_1dx_2 = \\
\int_{\Gamma} dx_1dx_2 R_{\delta}(x_1,\widetilde{x}_1)R_{\delta}(x_2,\widetilde{x}_2) \times \\
\left[\hbar^2\frac{\partial^2\Psi(x_1,x_2)}{\partial x_1^2} + \hbar^2\frac{\partial^2\Psi(x_1,x_2)}{\partial x_2^2} + V(x_1,x_2)\Psi(x_1,x_2)\right] + \\
+\hbar^2\int_{\Gamma} dx_1dx_2 \times \\
\left[2\frac{\partial\Psi(x_1,x_2)}{\partial x_1}R_{\delta}(x_2,\widetilde{x}_2)\frac{\partial R_{\delta}(x_1,\widetilde{x}_1)}{\partial x_1} + \Psi(x_1,x_2)R_{\delta}(x_2,\widetilde{x}_2)\frac{\partial^2 R_{\delta}(x_1,\widetilde{x}_1)}{\partial x_1^2}\right] + \\
+\hbar^2\int_{\Gamma} dx_1dx_2 \times \\
\left[2\frac{\partial\Psi(x_1,x_2)}{\partial x_2}R_{\delta}(x_1,\widetilde{x}_1)\frac{\partial R_{\delta}(x_2,\widetilde{x}_2)}{\partial x_2} + \Psi(x_1,x_2)R_{\delta}(x_1,\widetilde{x}_1)\frac{\partial^2 R_{\delta}(x_2,\widetilde{x}_2)}{\partial x_2^2}\right] = \\
= \Sigma_1(\hbar,\delta) + \Sigma_2(\hbar,\delta).
\end{gathered} \tag{4.3.22}$$

Now we go to estimate the quantities

$$\begin{gathered}
\Sigma_1(\hbar,\delta) = \hbar^2\int_{\Gamma} dx_1dx_2 \times \\
\left[2\frac{\partial\Psi(x_1,x_2)}{\partial x_1}R_{\delta}(x_2,\widetilde{x}_2)\frac{\partial R_{\delta}(x_1,\widetilde{x}_1)}{\partial x_1} + \Psi(x_1,x_2)R_{\delta}(x_2,\widetilde{x}_2)\frac{\partial^2 R_{\delta}(x_1,\widetilde{x}_1)}{\partial x_1^2}\right]
\end{gathered} \tag{4.3.23}$$

and

$$\begin{gathered}
\Sigma_2(\hbar,\delta) = \hbar^2\int_{\Gamma} dx_1dx_2 \times \\
\left[2\frac{\partial\Psi(x_1,x_2)}{\partial x_2}R_{\delta}(x_1,\widetilde{x}_1)\frac{\partial R_{\delta}(x_2,\widetilde{x}_2)}{\partial x_2} + \Psi(x_1,x_2)R_{\delta}(x_1,\widetilde{x}_1)\frac{\partial^2 R_{\delta}(x_2,\widetilde{x}_2)}{\partial x_2^2}\right]
\end{gathered} \tag{4.3.24}$$

From Eq. (4.3.23) using Eq. (4.3.14) we obtain

$$
\begin{gathered}
|\Sigma_1(\hbar,\delta)| \leq \hbar^2 \int_\Gamma dx_1 dx_2 \times \\
\left[2\left|\frac{\partial\Psi(x_1,x_2)}{\partial x_1}\right| R_\delta(x_1,\widetilde{x}_1) \left|\frac{\partial R_\delta(x_2,\widetilde{x}_2)}{\partial x_1}\right| + |\Psi(x_1,x_2)| R_\delta(x_1,\widetilde{x}_1) \left|\frac{\partial^2 R_\delta(x_2,\widetilde{x}_2)}{\partial x_1^2}\right| \right] \\
\leq 2O(\hbar^{1/2}) \int_\Gamma R_\delta(x_2,\widetilde{x}_2) \left|\frac{\partial R_\delta(x_1,\widetilde{x}_1)}{\partial x_1}\right| dx_1 dx_2 + \\
+ O(\hbar^{3/2}) \int_\Gamma R_\delta(x_2,\widetilde{x}_2) \frac{\partial^2 R_\delta(x_1,\widetilde{x}_1)}{\partial x_1^2} dx_1 dx_2 = \\
= 2O(\hbar^{1/2}) \int_{\mathbb{R}} R_\delta(x_2,\widetilde{x}_2) dx_2 \int_{\mathbb{R}} \left|\frac{\partial R_\delta(x_1,\widetilde{x}_1)}{\partial x_1}\right| dx_1 + \\
O(\hbar^{3/2}) \int_{\mathbb{R}} R_\delta(x_2,\widetilde{x}_2) dx_2 \int_{\mathbb{R}} \left|\frac{\partial^2 R_\delta(x_1,\widetilde{x}_1)}{\partial x_1^2}\right| dx_1 = \\
= \int_{\mathbb{R}} R_\delta(x_2,\widetilde{x}_2) dx_2 \left[2O(\hbar^{1/2}) \int_{\mathbb{R}} \left|\frac{\partial R_\delta(x_1,\widetilde{x}_1)}{\partial x_1}\right| dx_1 + O(\hbar^{3/2}) \int_{\mathbb{R}} \left|\frac{\partial^2 R_\delta(x_1,\widetilde{x}_1)}{\partial x_1^2}\right| dx_1 \right].
\end{gathered} \tag{4.3.25}
$$

From Eq. (4.3.24) using Eq. (4.3.14) we obtain

$$
\begin{gathered}
|\Sigma_2(\hbar,\delta)| \leq \hbar^2 \int_\Gamma dx_1 dx_2 \times \\
\left[2\left|\frac{\partial\Psi(x_1,x_2)}{\partial x_2}\right| R_\delta(x_1,\widetilde{x}_1) \left|\frac{\partial R_\delta(x_2,\widetilde{x}_2)}{\partial x_2}\right| + |\Psi(x_1,x_2)| R_\delta(x_1,\widetilde{x}_1) \left|\frac{\partial^2 R_\delta(x_2,\widetilde{x}_2)}{\partial x_2^2}\right| \right] \leq \\
2O(\hbar^{1/2}) \int_\Gamma R_\delta(x_1,\widetilde{x}_1) \left|\frac{\partial R_\delta(x_2,\widetilde{x}_2)}{\partial x_2}\right| dx_1 dx_2 + \\
O(\hbar^{3/2}) \int_\Gamma R_\delta(x_1,\widetilde{x}_1) \left|\frac{\partial^2 R_\delta(x_2,\widetilde{x}_2)}{\partial x_2^2}\right| dx_1 dx_2 \\
2O(\hbar^{1/2}) \int_{\mathbb{R}} R_\delta(x_1,\widetilde{x}_1) dx_1 \int_{\mathbb{R}} \left|\frac{\partial R_\delta(x_2,\widetilde{x}_2)}{\partial x_2}\right| dx_2 + \\
O(\hbar^{3/2}) \int_{\mathbb{R}} R_\delta(x_1,\widetilde{x}_1) dx_1 \int_{\mathbb{R}} \left|\frac{\partial^2 R_\delta(x_2,\widetilde{x}_2)}{\partial x_2^2}\right| dx_2 = \\
\int_{\mathbb{R}} R_\delta(x_1,\widetilde{x}_1) dx_1 \left[2O(\hbar^{1/2}) \int_{\mathbb{R}} \left|\frac{\partial R_\delta(x_2,\widetilde{x}_2)}{\partial x_2}\right| dx_2 + O(\hbar^{3/2}) \int_{\mathbb{R}} \left|\frac{\partial^2 R_\delta(x_2,\widetilde{x}_2)}{\partial x_2^2}\right| dx_2 \right].
\end{gathered} \tag{4.3.26}
$$

Having substituted Eq. (4.3.18) into Eq. (4.3.25) and Eq. (4.3.19) into Eq. (4.3.26) and having applied Lemma 4.3.1 we have finalized the proof of the Eq. (4.3.9).
We assume now that

$$\int_{\mathbb{R}}\int_{\mathbb{R}}|\Psi(x_1,x_2)|^2 dx_1 dx_2 = 1. \qquad (4.3.27)$$

From Eq. (4.3.27) and Eq. (4.3.17) by Lemma 4.3.1 we obtain

$$\int_{\mathbb{R}}\int_{\mathbb{R}} R_{\delta}^2(x_1,\widetilde{x}_1)R_{\delta}^2(x_2,\widetilde{x}_2)|\Psi(x_1,x_2)|^2 dx_1 dx_2 \simeq 1 \qquad (4.3.28)$$

From Eq. (4.3.16) - Eq. (4.3.17) by Lemma 4.3.1 we obtain

$$\begin{gathered}\langle x_A\rangle = \int\int x_1 \Psi_{\delta}^{\#}(x_1,x_2)dx_1dx_2 = \\ \int\int x_1 R_{\delta}^2(x_1,\widetilde{x}_1)R_{\delta}^2(x_2,\widetilde{x}_2)|\Psi(x_1,x_2)|^2 dx_1 dx_2 \simeq \widetilde{x}_1, \\ \langle x_B\rangle = \int\int x_2 \Psi_{\delta}^{\#}(x_1,x_2)dx_1dx_2 = \\ \int\int x_2 R_{\delta}^2(x_1,\widetilde{x}_1)R_{\delta}^2(x_2,\widetilde{x}_2)|\Psi(x_1,x_2)|^2 dx_1 dx_2 \simeq \widetilde{x}_2.\end{gathered} \qquad (4.3.29)$$

We choose now: $\widetilde{x}_1 = x_A, \widetilde{x}_2 = x_B = x_A + x_0,$ then a boundary condition $\langle x_B \rangle \simeq \langle x_A\rangle + x_0$ given by Eq. (4.3.10) is satisfied.

IV.4. Position-momentum entangled photon pairs in non-linear wave-guide

The physical system where we expect the entangled photon states to appear include: (A) a Kerr-type nonlinear single-mode wave-guide characterized by strong photon-photon coupling [12] or (B) a chain of coupled non-linear resonators. For two photons with momenta $\hbar k_1 = \hbar(k_0 - \delta k) and \hbar k^2 = \hbar(k_0 + \delta_k)$ and dispersion

$$\omega k_0 + \delta k \approx \omega k_0 + v\delta k + \beta \delta k^2/2, \qquad (4.4.1)$$

where v is the photon group velocity, the variation of the energy of a photon pair is

$$\Delta^{(2)}\omega = \omega k_0 - \delta k + \omega k_0 + \delta k - 2\omega k_0 \approx \beta \delta k^2. \tag{4.4.2}$$

As the photon-photon interaction conserves both energy and longitudinal momentum, the two-photon states propagating along the non-linear transmission line can be described by the Fock function

$$|\psi\rangle_{2k_0} = \int dk_1 dk_2 \delta(k_1 + k_2 - 2k_0) f(k_1 - k_2)|k_1, k_2\rangle \tag{4.4.3}$$

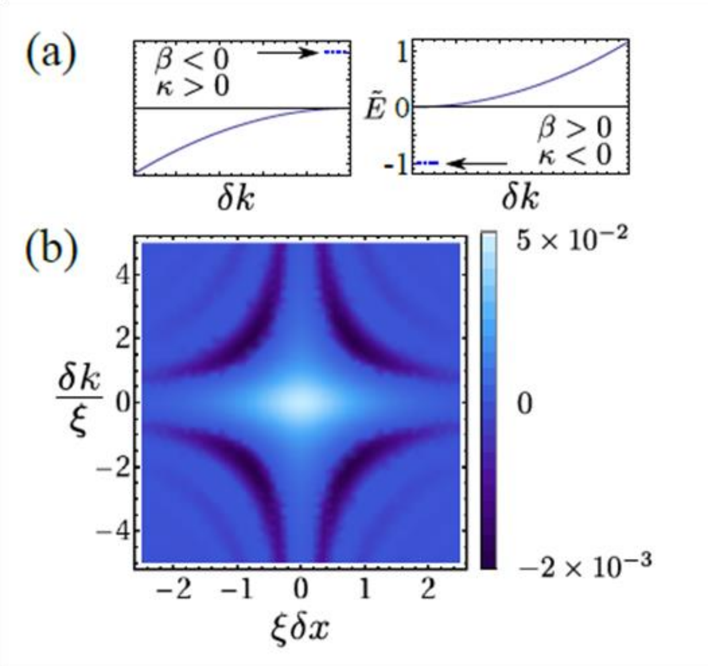

Fig. 4.4.1. Entangled two-photon states in non-linear wave guides. Adapted from [12]

(a) Spectrum of a two-photon state, $\tilde{E} = (E - 2\omega k_0)|\beta|/\kappa^2$, with total momentum $2k^0$ in a wave-guide with quadratic dispersion (4.3.1) for $\beta < 0, \kappa > 0$ (left) and $\beta > 0, \kappa < 0$ (right). Solid line corresponds to the continuous spectrum, while the single eigenvalue corresponding to the entangled state is shown by dashed line.
(b) Wigner function of the two-photon entangled state. It takes negative values, which is a hallmark of non-Gaussian entangled states.

To demonstrate the principle of position-momentum entanglement of photons in Kerr-nonlinear systems, we, first, consider the entangled photon pairs in non-linear optical wave-guides. Classically, Kerr nonlinearity in an isotopic medium manifests itself in the third-order polarisation $\mathbf{P}^{(3)(+)} = \chi^{(3)}[(\mathbf{E}^{(-)} \ \mathbf{E}^{(+)})\mathbf{E}^{(+)} + \alpha(\mathbf{E}^{(+)} \ \mathbf{E}^{(+)})\mathbf{E}^{(-)}]$, where + and - correspond to positive and negative frequency parts, E is the electric field, $\chi^{(3)}$ is the susceptibility of the medium $\chi^{(3)} = \chi^{(3)}_{xyxy}$, $\alpha = \chi^{(3)}_{xxyy}/(2\chi^{(3)})$. Quantizing electromagnetic field, integrating over transverse degrees of freedom, and neglecting magneto-optical effects ($\alpha = 0$) leading to entanglement over polarization degrees of freedom, one obtains the following Hamiltonian ($\hbar = c = 1$):

$$H = H_0 + H_{int}, H_0 = \sum_k k\omega k a_k^\dagger a_k,$$
$$H_{int} = \frac{k}{L}\sum_{k_1,k_2,k_3,k_4} \delta(k_1 + k_2, k_3 + k_4) a_{k_4}^\dagger a_{k_3}^\dagger a_{k_1} a_{k_2}, \quad (4.4.4)$$

where $a_k(a_k^\dagger)$ is the annihilation (creation) operator of a photon with longitudinal momentum k and energy ω_k, L is the length of the system. The non-linear term H_{int} in Eq. (4.4.3) describes photon-photon interaction with coupling $\kappa = \pi\omega^2\chi^{(3)}/2n_r^4 A\epsilon_0$, where n_r is the refraction index, A is the area occupied by the wave-guide mode and ϵ_0 is the vacuum permittivity. Hamiltonian (4.4.3) can be diagonalized exactly in the case of $\Delta^{(2)}\omega \propto \delta k^2$. We consider a sector of the Hilbert space, which consists of all the two-photon states with the total pair momentum $2k_0$ and assume the effective mass approximation for the wave-guide dispersion given by Eq. (4.4.1). In the coordinate domain, $a_x = 1/\sqrt{L}\sum_k \ a_k \exp[i(k - k_0)x]$, the Hamilton Eq.(4.4.3) takes the form

$$H = \int dx\left(\omega_{k_0} a_x^\dagger a_x - iv a_x^\dagger \partial_x a_x - \frac{1}{2}\beta a_x^\dagger \partial^2 a_x\right) + \frac{1}{2}\int dx_1 dx_2 a_{x_1}^\dagger a_{x_2}^\dagger U(x_1 - x_2) a_{x_1} a_{x_2}, \quad (4.4.5)$$

where $U(x_1 - x_2) = 2\kappa\delta(x_1 - x_2)$. For a two-photon state, described by the wave-function

$$\psi_t^{\mathbf{A/B}}(x_1,x_2) = |\,\psi\rangle = \int dx_1 dx_2 f(x_1,x_2) a_{x_1}^{\dagger} a_{x_2}^{\dagger}|0\rangle, \qquad (4.4.6.a)$$

one obtains the following Schrödinger equation:

$$[2\omega_{k_0} - iv(\partial_{x_1} + \partial_{x_2}) - \frac{1}{2}\beta(\partial_{x_1}^2 + \partial_{x_2}^2) + 2\kappa\delta(x_1 - x_2)]f(x_1,x_2) = Ef(x_1,x_2), \qquad (4.4.6.b)$$

where E is the energy of a two-photon state. Equation (4.4.6b) has scattering state solutions, which correspond to the continuous spectrum of non-interacting photons with energies given by Eq.(4.4.2) (See Fig. 4.4.1(a)). When the curvature of the wave-guide dispersion β and the photon coupling constant κ are of opposite signs, $\beta\kappa < 0$, there exists a bound state solution with

$$f(x_1,x_2) = \sqrt{\frac{\xi}{2L}}\, exp[-|x_1 - x_2|\xi], \xi = |\kappa/\beta| \qquad (4.4.7)$$

The energy of this state is split from the continuum of weakly correlated scattering states, as we show in Fig. 4.4.1(a), and it is given by

$$E_b = 2\omega_{k_0} - \kappa^2/\beta, \qquad (4.4.8)$$

as expected from binding of a one-dimensional massive particle to an attractive δ-functional potential well [13]. In the momentum domain, the two-photon bound state wave-function is given by Eq.(4.4.3) with

$$f(k_1 - k_2) = \frac{8\xi^{3/2}}{\sqrt{2L}\,[(k_1 - k_2)^2 + 4\xi^2]} \qquad (4.4.9)$$

The state (4.4.9) can be characterised by the Wigner function defined as the expectation value

$$\mathbf{W}(x_1,k_1;x_2,k_2) = \pi^{-2}\langle\psi|\Pi(x_1,k_1)\otimes\Pi(x_2,k_2)|\psi\rangle$$

of the parity operator

$$\Pi(x,k) = \int d\zeta e^{-2ix\zeta} a^{\dagger}_{k+\zeta}|0\rangle\langle 0|a_{k-\zeta}.$$

After straightforward calculations, one obtains

$$\mathbf{W}(x_1,k_1;x_2,k_2) = \frac{\xi^2 e^{-2\xi|\delta x|}}{2\pi^2(\delta k^2+\xi^2)}\cos(2\delta k|\delta x|) + \frac{\xi}{\delta k}\sin(2\delta k|\delta x|)\delta(k_1+k_2;2k_0), \quad (4.4.10)$$

where $\delta x = x_1 - x_2$. This function is negative for $\cos(2\delta k|\delta x|) + (\xi/\delta k)\sin(2\delta k|\delta x|) < 0,$ as shown in Fig. 4.4.1(b), which implies that the state (4.4.9) is entangled in position-momentum degrees of freedom. Moreover, for $\xi \to \infty,$ the two-photon wave-function approaches the ideal Einstein-Podolsky-Rosen state in which position and momenta are perfectly (anti-) correlated:

$$|\psi\rangle = \int d(\delta k)|k_0+\delta k, k_0-\delta k\rangle = \int dx e^{2ik_0x}|x,x\rangle.$$

Alternatively, to demonstrate that the state (4.4.9) is entangled in position-momentum degrees of freedom, one can find the uncertainties $\Delta(x_1-x_2)$ and $\Delta(k_1+k_2)$ calculated over the joint probability distributions $\mathbf{P}(x_1,x_2)$ and $\mathbf{P}(k_1,k_2)$ respectively, for which, the separability criterion:

$$[\Delta(x_2-x_1)]^2[\Delta(k_2+k)]^2 \geq 1, \quad (4.4.11)$$

can be applied. Although, the states for which the inequality (4.4.11) is violated are inseparable, they do not necessarily lead to EPR paradox. In order for an EPR "paradox" to arise, correlations must violate a more strict inequality:

$$[\Delta(x_2 - x_1)]^2[\Delta(k_1 + k_2)]^2 \geq 1/4, \qquad (4.4.12)$$

which can be accessible experimentally.
Nonlocal Schrödinger equation (4.3.9) corresponding to Schrödinger equation (4.4.6) (see subsection IV.3) is

$$\int\int dx_1 dx_2 \Big\{[2\omega_{k_0} - iv(\partial_{x_1} + \partial_{x_2}) - \frac{1}{2}\beta(\partial^2_{x_1} + \partial^2_{x_2}) + 2\kappa\delta(x_1 - x_2)]f^{\#}(x_1, x_2) - \\ -Ef(x_1, x_2)\Big\} = 0, \qquad (4.4.13)$$

where $f^{\#}(x_1, x_2)$ is given by Eq. (4.3.8).
Remark 4.4.1. Note that. We assume now the canonical postulate of locality. Then:
(a) Whenever a measurement of the coordinate x_2 of a particle **B** is performed at instant t with the result $\bar{x}_2^{\mathbf{B}} \in [x_2 - \varepsilon, x_2 + \varepsilon], \varepsilon \ll 1$ according to quantum mechanics a state vector $|\psi_t^{x_2}\rangle_{\mathbf{B}}$ collapses at instant t to the state vector

$$\left|\psi^{x_2}_{t,\delta,\varepsilon,\bar{x}^{\mathbf{B}}}\right\rangle_{\mathbf{B}} \sim \widehat{L}^{\mathbf{B}}_{\bar{x}^{\mathbf{B}}}(\delta, \varepsilon)|\psi_t^{x_2}\rangle_{\mathbf{B}}, \qquad (4.4.14)$$

see Chapter I.
(b) Under conditions given by Eq. (4.4.14) two-particle wave function $\psi_t^{\mathbf{A/B}}(x_1, x_2)$ given by Eq. (4.4.6b), collapses at instant t by the law

$$\psi_t^{\mathbf{A/B}}(x_1, x_2) \overset{\text{collapse}}{\longrightarrow} \widehat{L}^{\mathbf{B}}_{\bar{x}^{\mathbf{B}}}(\delta, \varepsilon)\psi_t^{\mathbf{A/B}}(x_1, x_2). \qquad (4.4.15)$$

Remark 4.4.2. Note that. We assume now the postulate of nonlocality. Then:
(i) Whenever a measurement of the coordinate x_1 of a particle **B** is performed at instant t with the result $\bar{x}^{\mathbf{B}} \in [x_1 - \varepsilon, x_1 + \varepsilon], \varepsilon \ll 1.$ Then:
(a) According to quantum mechanics a state vector $|\psi_t^{x}\rangle_{\mathbf{B}}$ collapses at instant t to the state vector

$$\left|\psi^{x_1}_{t,\delta,\varepsilon,\bar{x}^{\mathbf{B}}}\right\rangle_{\mathbf{B}} \sim \widehat{L}^{\mathbf{B}}_{\bar{x}^{\mathbf{B}}}(\delta, \varepsilon)|\psi_t^{x_1}\rangle_{\mathbf{B}}, \qquad (4.4.16)$$

where $\hat{L}^{\mathbf{B}}_{\bar{x}^{\mathbf{B}}}(\delta,\varepsilon)$ is a norm-reducing, positive, self-adjoint, linear operator in the 2-particle non projective Hilbert space $\mathbf{H}$, representing the localization of particle $\mathbf{B}$ around the point $x^{\mathbf{B}}$, see Chapter I.

(b) According postulate of nonlocality (see Chapter I) a state vector $|\psi_t^{x_2}\rangle_{\mathbf{A}}$ immediately collapses at instant to the state vector

$$\left|\psi^{x_1}_{t,\delta,\varepsilon,\bar{x}^{\mathbf{A}}}\right\rangle_{\mathbf{A}} \sim \hat{L}^{\mathbf{A}}_{\bar{x}^{\mathbf{B}}+x_0}(\delta,\varepsilon)|\psi_t^{x_1}\rangle_{\mathbf{A}} \tag{4.4.17}$$

and this is true independent of the distance in Minkovski spacetime $M_4 = \mathbb{R}^{1,3}$ that separates the particles. Thus

$$|\psi_t^x\rangle_{\mathbf{B}} \overset{\text{collapse}}{\rightarrow} \left|\psi^x_{t,\delta,\varepsilon,x^{\mathbf{B}}}\right\rangle_{\mathbf{B}} \Rightarrow |\psi_t^x\rangle_{\mathbf{A}} \overset{\text{collapse}}{\rightarrow} \left|\psi^x_{t,\delta,\varepsilon,x^{\mathbf{B}}+x_0}\right\rangle_{\mathbf{A}}. \tag{4.4.18}$$

(ii) Under conditions given by Eq. (4.4.16) - Eq. (4.4.18) two-particle wave function $\psi_t^{\mathbf{A/B}}(x_1,x_2)$ given by Eq. (4.4.6b) collapses at instant t by the law

$$\psi_t^{\mathbf{A/B}}(x_{\mathbf{1}},x_{\mathbf{2}}) \overset{\text{collapse}}{\rightarrow} \hat{L}^{\mathbf{A}}_{\bar{x}^{\mathbf{B}}+x_0}\hat{L}^{\mathbf{B}}_{\bar{x}^{\mathbf{B}}}(\delta,\varepsilon)\psi_t^{\mathbf{A/B}}(x_1,x_2). \tag{4.4.19}$$

IV.5. Position-momentum entangled photon pairs and the experimental verification of the postulate of nonlocality

In paper [14] it is reported on a demonstration of the EPR paradox using position-entangled and momentum-entangled photon pairs produced by spontaneous parametric down conversion. Transverse correlations from parametric down conversion have been studied both theoretically and experimentally. It was find experimentally that the position and momentum correlations are strong enough to allow the position or momentum of a photon to be inferred from that of its partner with a product of variances $\leq 0.01\hbar^2$, which violates the separability bound by 2 orders of magnitude. In the idealized entangled state proposed by EPR, the positions and momenta of the two particles are perfectly correlated. However such idealized entangled state is non-normalizable and cannot be realized in the laboratory. However, the state of the light produced in parametric down conversion can be made to approximate the EPR state under suitable conditions. In parametric down conversion, a pump photon is absorbed by a nonlinear medium and reemitted as two photons (conventionally called signal and idler photons), each with approximately half the energy of the pump photon. Considering only the transverse components, the momentum conservation of the down conversion process requires $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_p$, where $1,2,$ and p refers to the signal, idler, and pump photons, respectively. Provided the uncertainty in the pump transverse momentum is small, the transverse momenta of the signal and idler photons are highly anticorrelated. The exact degree of correlation depends on the structure of the signal idler state. In the regime of weak generation, this state has the form

$$|\psi\rangle_{1,2} = |\mathbf{vac}\rangle + \int d\mathbf{p}_1 d\mathbf{p}_2 A(\mathbf{p}_1, \mathbf{p}_2)|\mathbf{p}_1, \mathbf{p}_2\rangle, \tag{4.5.1}$$

where $|\text{vac}\rangle$ denotes the vacuum state and the two-photon amplitude $A(\mathbf{p}_1, \mathbf{p}_2)$ is

$$A(\mathbf{p}_1, \mathbf{p}_2) = \chi E_p(\mathbf{p}_1, \mathbf{p}_2)\frac{\exp(i\Delta \mathbf{k}_z L) - 1}{i\Delta \mathbf{k}_z}. \tag{4.5.2}$$

Here is the coefficient of the nonlinear interaction, E_p is the amplitude of the plane-wave component of the pump with transverse momentum $p_1 p_2, L$ is the length of the nonlinear medium, and

$\Delta\mathbf{k}_z = \mathbf{k}_{p,z} - \mathbf{k}_{1,z} - \mathbf{k}_{2,z}$ (where $\mathbf{k} = \mathbf{p}/\hbar$) is the longitudinal wave vector mismatch, which generally increases with transverse momentum and limits the angular spread of signal and idler photons. The vacuum component of the state makes no contribution to photon counting measurements and may be ignored. Also, there is no inherent difference between different transverse components; so without loss of generality, we consider the scalar position and momentum. The narrower the angular spectrum of the pump field and the wider the angular spectrum of the generated light, the more closely the integral (4.4.1) approximates

$$\int dp_1 dp_2 \delta(p_1 + p_2)|p_1, p_2\rangle = |EPR\rangle$$

and the stronger the correlations in the position and momentum become. The experimental setup used to determine position and momentum correlations is portrayed in Fig. 4.5.1(a)-(b). The idea is to measure the positions and momenta by measuring the down converted photons in the near and far fields, respectively [15]. The source of entangled photons is spontaneous parametric down conversion generated by pumping a 2 mm thick type-II-barium-borate (BBO) crystal with a 30 mW, cw, 390 nm laser beam. A prism separates the pump light from the down converted light. The signal and idler photons have orthogonal polarizations and are separated by a polarizing beam splitter. In each arm, the light passes through a narrow 40 m vertical slit, a 10 nm spectral filter, and a microscope objective. The objective focuses the transmitted light onto a multimode fiber which is coupled to an avalanche photodiode single-photon counting module. The spectral filter ensures that only photons with nearly equal energies are detected.

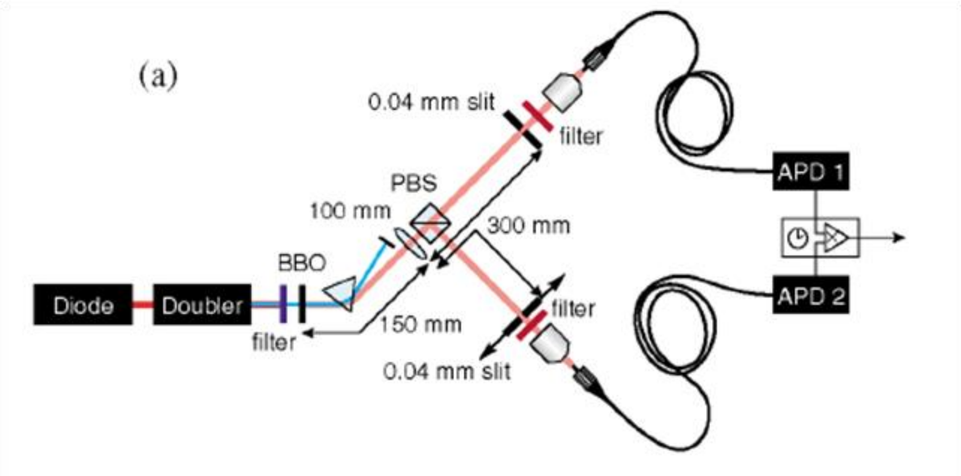

Fig. 4.5.1(a). Experimental setup for measuring position photon correlations. Position correlations are obtained by imaging the birth place of each photon of a pair onto a separate detector. Adapted from [14]

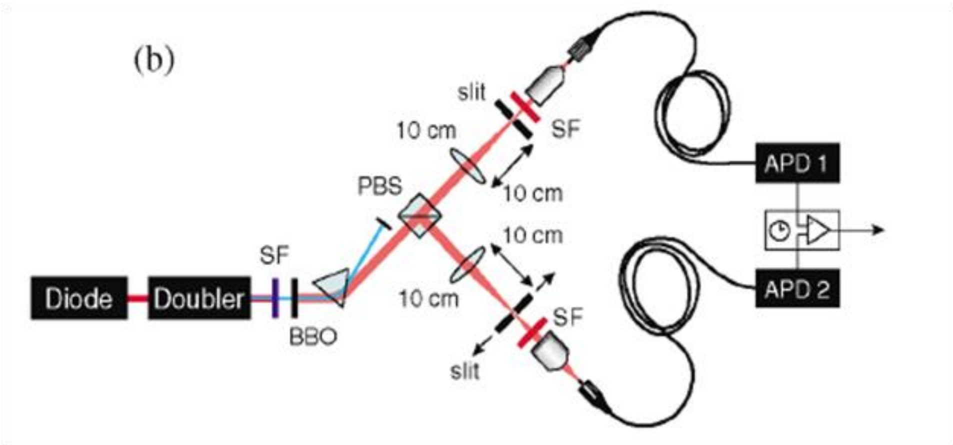

Fig. 4.5.1(b). Experimental setup for measuring correlations in transverse momentum. Correlations in transverse momentum are obtained by imaging the propagation direction of each photon of a pair onto a separate detector. Adapted from [14]

To measure correlations in the positions of the photons, a lens of focal length 100 mm (placed prior to the beam splitter) is used to image the exit face of the crystal onto the planes of the two slits [Fig.4.5.1(a)]. One slit is fixed at the location of peak signal intensity. The other slit is mounted on a translation stage. The photon coincidence rate is then recorded as a function of the displacement of the second slit.
To measure correlations in the transverse momenta of the photons, the imaging lens is replaced by two lenses of focal length 100 mm, one in each arm, at distance f from the planes of the two slits [Fig.4.5.1(b)]. These lenses map transverse momenta to transverse positions, such that a photon with transverse momentum $\hbar k_\perp$ comes to a focus at the point $x = fk_\perp/k$ in the plane of the slit. Again, one slit is fixed at the location of the peak count rate while the other is translated to obtain the coincidence distribution.
By normalizing the coincidence distributions, the conditional probability density functions $p(x_2|x_1)$ and $p(p_2|p_1)$ were obtained (see Fig. 4.5.2-4.5.3).

These probability densities are then used to calculate the uncertainty in the inferred position or momentum of photon 2 given the position or momentum of photon 1:

$$\begin{aligned} \Delta x_2^2(x_1) &= \int x_2^2 p(x_2|x_1)dx_2 - \left(\int x_2 p(x_2|x_1)dx_2\right)^2, \\ \Delta p_2^2(p_1) &= \int p_2^2 p(p_2|p_1)dx_2 - \left(\int p_2 p(p_2|p_1)dx_2\right)^2. \end{aligned} \tag{4.5.3}$$

Because of the finite width of the slits, the raw data in Fig. 4.5.2-4.5.3 describe a slightly broader distribution than is associated with the down conversion process itself.
By adjusting the computed values of Δx_2 and Δp_2 to account for this broadening (an adjustment smaller than 10%), we obtain the correlation uncertainties $\Delta x_2 = 0.027$ mm and $\Delta p_2 = 3.7\hbar$ mm^{-1}.

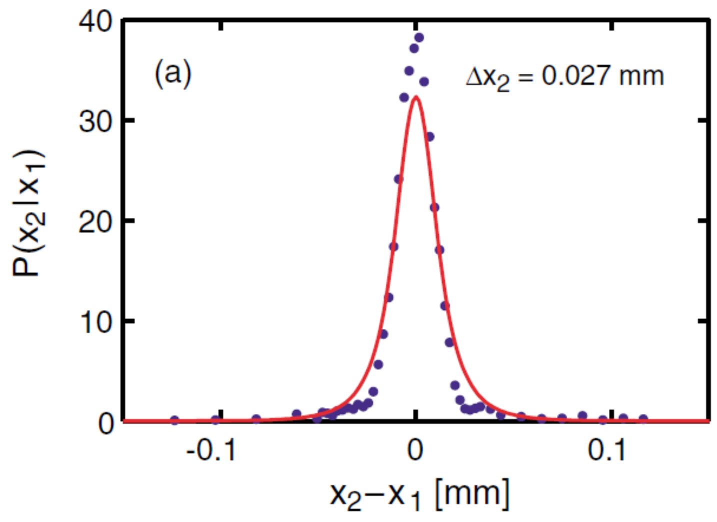

Fig. 4.5.2. The conditional probability distribution of the relative birthplace of the entangled photons. The solid line is the theoretical prediction and the dots are the experimented data. Adapted from [14]

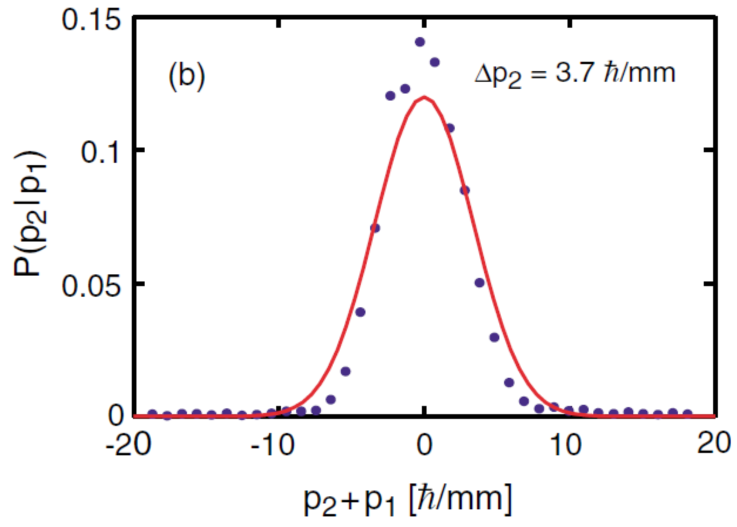

Fig. 4.5.3. The conditional probability distribution of the relative transverse momentum of the entangled photons. The solid line is the theoretical prediction and the dots are the experimented data. Adapted from [14]

The widths of the distributions determine the uncertainties in inferring the position or the momentum of one photon from that of the other. The experimentally measured variance product is then [14]

$$[\Delta^{\exp} x_2^2(x_1)][\Delta^{\exp} p_2^2(p_1)] = 0.01\hbar^2. \qquad (4.5.4)$$

Also shown in Fig. 4.5.2-4.5.3 are the predicted probability densities. These curves contain no free parameters and are obtained directly from the two-photon amplitude $A(p_1, p_2)$ [14], which is determined by the optical properties of BBO and the measured profile of the pump beam. Figure 4.5.2 indicates that the correlation widths obtained are intrinsic to the down conversion process and are limited only by the degree to which it deviates from the idealized EPR state (4.5.1). The value of $\Delta(p_2 + p_1)$ is limited by the finite width of the pump beam. The pump photons in a Gaussian beam of width w have an uncertainty $\hbar/2w$ in transverse momentum which, due to conservation of momentum, is imparted to the total momentum $p_1 + p_2$ of the signal and idler photons. The value of $\Delta(x_2 - x_1)$ is limited by the range of angles over which the crystal generates signal and idler photons. If the angular width of emission is $\Delta\phi$, then the principle of diffraction indicates that the photons cannot have a smaller transverse dimension than $\sim(k_{s,i}\Delta\phi)^{-1}$. Careful analysis based on the angular distribution of emission yields $\Delta(x_2 - x_1) = 1.88(k_{s,i}\Delta\phi)^{-1}$ [14]. With the measured beam width of $w = 0.17$ mm and predicted angular width 0.012 rad, the theory predicts [14]:

$$[\Delta^{\mathbf{th}} x_2^2(x_1)][\Delta^{\mathbf{th}} p_2^2(p_1)] = 0.0036\hbar^2. \qquad (4.5.5)$$

Remark 4.5.1. This is somewhat smaller than the experimentally calculated value of $0.01\hbar^2$, even though the data appear to closely match the theoretical curves

$$\Delta^{\exp} x_2^2(x_1)\Delta^{\exp} p_2^2(p_1) - \Delta^{\mathbf{th}} x_2^2(x_1)\Delta^{\mathbf{th}} p_2^2(p_1) = 0.01\hbar^2 - 0.0036\hbar^2 = 0.0064\hbar^2. \qquad (4.5.6)$$

Remark 4.5.2. The reason for this discrepancy is that the experimental distributions have small (1% of the peak) but very broad wings.

Remark 4.5.3. The origin of these uncoincidence counts is unknown [14].

Remark 4.5.4. In paper [14] it was assumed that these counts are perhaps due to scattering from optical components. If these counts are treated as a noise background and subtracted, the experimentally obtained uncertainties come into somewhat better agreement with the theoretically predicted values, yielding an uncertainty product of $0.004\hbar^2$:

$$\delta_{\mathbf{EPR}}^{\mathbf{nonloc.}}(x_2 - x_1, p_2 + p_1) = \Delta^{\exp} x_2^2(x_1)\Delta^{\exp} p_2^2(p_1) - \Delta^{\mathbf{th}} x_2^2(x_1)\Delta^{\mathbf{th}} p_2^2(p_1) = 0.006\hbar^2. \quad (4.5.7)$$

Thus final value of uncoincidence counts is

$$\delta_{\mathbf{EPR}}^{\mathbf{nonloc.}}(x_2 - x_1, p_2 + p_1) = 0.006\hbar^2. \quad (4.5.8)$$

Remark 4.5.5. Note that the separability criterion derived by Mancini et al. [16] is more useful here. We remind that it states that separable systems satisfy the joint uncertainty product

$$\Delta(x_2 - x_1)\Delta(p_2 + p_1) \geq \hbar^2, \quad (4.5.9)$$

where the uncertainties are calculated over the joint probability distributions $P(x_1, x_2)$ and $P(p_1, p_2)$, respectively. In these experiments the widths of the conditional probability distributions are P.

Therefore the results of [14] constitute a 2-order-of-magnitude violation of Mancini's separability criterion as well as a strong violation of EPR criterion.

IV.6. The EPR Paradox Resolution by using quantum mechanical formalism based on the probability representation of quantum states

IV.6.1. Preleminaries

We remind that any given n-dimensional quantum system is identified by a set $\mathbf{Q}$:

$$\mathbf{Q} \triangleq \langle \mathbf{H}, \mathfrak{I}, \mathfrak{R}, \mathscr{L}_{2,1}, \mathfrak{I}^*(\mathbf{H}), \mathbf{G}, |\psi_t\rangle\rangle \tag{4.6.1}$$

where:

(i) $\mathbf{H}$ that is some infinite-dimensional complex Hilbert space,

(ii) $\mathfrak{I} = (\Omega, \mathcal{F}, \mathbf{P})$ that is complete probability space,

(iii) $\mathfrak{R} = (\mathbb{R}^n, \Sigma)$ that is measurable space,

(iv) $\mathscr{L}_{2,1}(\Omega)$ that is complete space of complex valued random variables $X : \Omega \to \mathbb{C}^n$ such that

$$\int_\Omega \|X(\omega)\| d\mathbf{P} < \infty, \int_\Omega \|X(\omega)\|^2 d\mathbf{P} < \infty, \tag{4.6.2}$$

see Chapter II subsection II.1 postulate Q.I.1.

Remark 4.6.1. Let $\mathbf{B}_{M_4}$ be a Boolean algebra of physical events in Minkowski spacetime and let $\mathcal{F}_{M_4}^{ph} = (\mathbf{B}_{M_4}, \mathbf{P})$ be a measure algebra of physical events in Minkowski spacetime, i.e., $\mathcal{F}_{M_4}^{ph}$ that is a Boolean algebra $\mathbf{B}_{M_4}$ with a probability measure $\mathbf{P}$, see Chapter III subsection III.2, Definition 3.2.3.

We remind that we denote such physical events by $A(\mathbf{x}), B(\mathbf{x}), \ldots$ etc., where $\mathbf{x} = (t, x_1, x_2, x_3) \in M_4$ or $A, B, \ldots$ etc., and we write for a short $A^{Oc}(\mathbf{x}), B^{Oc}(\mathbf{x}), \ldots$ iff there physical events $A(\mathbf{x}), B(\mathbf{x})\ldots$ were occured.

Remark 4.6.2. We assume that particle A is initially in the state $|\psi_{\mathbf{A}}\rangle \in \mathbf{H}$. *Let* $A(q,t) \triangleq A\left(|\psi_{\mathbf{A}}\rangle, \widehat{Q}, q, \delta q, t\right) \in \mathbf{B}_{M_4}$ *be a physical event* which consists of the performing a measurement of the observable $\widehat{Q} = \int_{q_1}^{q_2} q|q\rangle\langle q| dq$ with the accuracy δq, and the result is obtained in the range $(q - \delta q, q + \delta q)$ at the instant t.

We assume that $A\left(|\psi_{\mathbf{A}}\rangle, \widehat{Q}, q, \delta q, t\right) \in \mathcal{F}_{M_4}^{ph}$.

Remark 4.6.3. Note that: if the physical event $A\left(|\psi_{\mathbf{A}}\rangle, \widehat{Q}, q, \delta q, t\right)$ was occurred then immediately after the measurement at the instant t

unconditional measure $\mathbf{P}$ collapses to conditional measure $\mathbf{P}\left(X \mid A\left(|\psi_{\mathbf{A}}\rangle, \widehat{Q}, q, \delta q, t\right)\right)$, where $X \in \mathcal{F}_{M_4}^{ph}$:

$$\mathbf{P}\left(X \mid A\left(|\psi_{\mathbf{A}}\rangle, \widehat{Q}, q, \delta q, t\right)\right) = \frac{\mathbf{P}\left(X \wedge A\left(|\psi_{\mathbf{A}}\rangle, \widehat{Q}, q, \delta q, t\right)\right)}{\mathbf{P}\left(A\left(|\psi_{\mathbf{A}}\rangle, \widehat{Q}, q, \delta q, t\right)\right)}. \quad (4.6.3)$$

Remark 4.6.4. Remind if we are to suppose that a particle at a definite position x is to be assigned a state vector $|x\rangle \in \mathbf{H}$, and if further we are to suppose that the possible positions are continuous over the range $(-\infty, \infty)$ and that the associated states are complete, then we are lead to requiring that any state $|\psi_{\mathbf{A}}\rangle$ of the particle must be expressible as

$$|\psi_{\mathbf{A}}\rangle = \int_{-\infty}^{\infty} |x\rangle\langle x|\psi_{\mathbf{A}}\rangle dx \quad (4.6.4)$$

with the states $|x\rangle$ by δ-function normalised, i.e. $\langle x|x'\rangle = \delta(x - x')$.

Definition 4.6.1. Let $B^{\infty} = \cup_{[a,b]\subset\mathbb{R}} \Sigma_{a,b}$ where $\Sigma_{a,b} = B([a,b])$ is the Borel algebra on a set $[a,b]$. Let $|\psi\rangle \in \mathbf{H}$. We define now a signed measure $\mathbf{P}_{|\psi_{\mathbf{A}}\rangle} : B^{\infty} \to \mathbb{R}$ by formula

$$\mathbf{P}_{|\psi_{\mathbf{A}}\rangle}(A) = \int_A x p_{|\psi_{\mathbf{A}}\rangle}(x) d\mu(x), \quad (4.6.5)$$

where $p_{|\psi_{\mathbf{A}}\rangle}(x) = |\langle x|\psi_{\mathbf{A}}\rangle|^2$.

Remark 4.6.5. We assume now that $(\Omega, \mathcal{F}, \mathbf{P}) = (\mathbb{R}, B^{\infty}, \mathbf{P}_{B^{\infty}})$ and $\mathbf{P}_{|\psi\rangle} \ll \mathbf{P}_{\mathbf{B}^{\infty}}$, i.e. $\mathbf{P}_{|\psi\rangle}$ is absolutely continuous with respect to $\mathbf{P}$ .By Radon-Nicodym theorem we obtain for any $A \in \Sigma_{a,b}$:

$$\mathbf{P}_{|\psi\rangle}(A) = \int_A X_{|\psi\rangle}(\omega) d\mathbf{P}(\omega), \quad (4.6.6)$$

i.e.

$$X_{|\psi\rangle}(\omega) = \frac{d\mathbf{P}_{|\psi\rangle}}{d\mathbf{P}}. \tag{4.6.7}$$

Remark 4.6.6. We assume now that: (i) a measure algebra $\mathcal{F}_{M_4}^{ph} = (\mathbf{B}_{M_4}, \mathbf{P})$ admits a representation $\Re[\bullet] : \mathcal{F}_{M_4}^{ph} \to (\mathbb{R}, B^{\infty}, \mathbf{P}_{B^{\infty}})$ of the measure algebra $\mathcal{F}_{M_4}^{ph} = (\mathbf{B}_{M_4}, \mathbf{P})$ in the measure algebra $\mathbf{B}^{\infty} \triangleq (\mathbb{R}, B^{\infty}, \mathbf{P}_{B^{\infty}})$, such that (ii) $\mathbf{P}_{B^{\infty}}(X) = \mathbf{P}(\Re^{-1}[X])$ for any $X \in B^{\infty}$ and (iii) for any physical event such that $A\left(|\psi_A\rangle, \widehat{Q}, q, \delta q, t\right) \in \mathcal{F}_{M_4}^{ph}$ (see Remark 4.6.2) the following condition holds

$$\Re\left[A\left(|\psi_{\mathbf{A}}\rangle, \widehat{Q}, q, \delta q, t\right)\right] = \{\omega | q - \delta q \leq X_{|\psi_{\mathbf{A}}\rangle}(\omega) \leq q + \delta q\}, \tag{4.6.8}$$

where $\{\omega | q - \delta q < X_{|\psi_{\mathbf{A}}\rangle}(\omega) < q + \delta q\} \in \mathbf{B}^{\infty}$.

IV.6.2.The EPR Paradox Resolution

The classical weak EPR argument

We briefly remind now the EPR argument [1]. Suppose that a system of two identical particles is prepared in a state such that their relative distance is large and constant $|r_1 - r_2| = L = x_0$, i.e., they are space-like separated, and the total momentum is zero $\vec{p}_1 + \vec{p}_2 = 0$ (see Fig. 4.6.1).

This preparation is, in principle, possible because the two observables, say $x_1 - x_2$ and $\vec{p}_1 + \vec{p}_2$, are compatible, i.e., both of them can be set to certain values with certainty on the same state. Correspondingly according to quantum mechanics they are in fact represented by commuting operators [1].

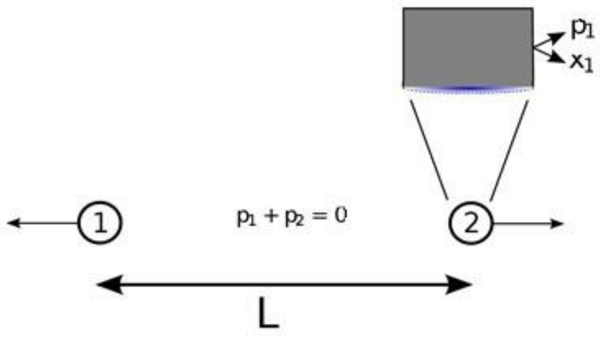

Fig. 4.6.1. Schematic representation of EPR thought experiment

Remark 4.6.7. Then one can measure the value of either of the two incompatible single particle observables, say x_1 or p_1 and correspondingly deduce the value of either $x_2 = x_0 - x_1$ or $p_2 = -p_1$ without interacting with particle 2. Because of this they correspond, according to the EPR argument, to elements of reality of the state of particle 2 that are independent of measurements and should be predictable by the theory [1]. On the other hand, quantum mechanics cannot predict the value of both x_2 and p_2 on the same state, because they are incompatible observables and this would be in contrast to Heisenberg uncertainty principle.

Remark 4.6.8. Thus, to conclude EPR, there are elements of reality of a state that cannot be predicted by the theory and therefore the theory is incomplete [1].

The strong EPR argument

Remark 4.6.9. Note then in additional to canonical EPR thought experiment: (i) one can measure at instant t the value of single particle 1 observable, say x_1^t and deduce the value $x_2^t = x_0 - x_1^t$ of particle 2 at instant t without interacting with particle 2 which at instant t is in a state, say ψ_2^t. Such a measurement however is not disturbed by the particle 2 and thus is not altered by its state ψ_2^t and therefore the value of single particle 2 observable, say p_2 is the same as before the measurement on particle 1. Therefore one can measure the value p_2^t in the state ψ_2^t exactly without any

uncertainty. On the other hand, Heisenberg uncertainty principle predicts that the position x_2^t and the momentum p_2^t of any particle cannot both be measured or predicted exactly, at the same time t, even in the theory.
Let $\mathbf{A}$ and $\mathbf{B}$ be two particles A and B with a state vector $|\psi_{\mathbf{A}}\rangle$

$$|\psi_{\mathbf{A}}\rangle = \int_{-\infty}^{\infty} |x\rangle\langle x|\psi_{\mathbf{A}}\rangle dx \tag{4.6.9}$$

and with a state vector $|\psi_{\mathbf{B}}\rangle$

$$|\psi_{\mathbf{B}}\rangle = \int_{-\infty}^{\infty} |x\rangle\langle x|\psi_{\mathbf{B}}\rangle dx \tag{4.6.10}$$

respectively, and with perfectly correlated position

$$x_{\mathbf{B}} = x_{\mathbf{A}} + x_0 \tag{4.6.11}$$

and perfectly anti-correlated momentum

$$p_{\mathbf{B}} = -p_{\mathbf{A}}. \tag{4.6.12}$$

We define now a signed measures $\mathbf{P}_{|\psi_{\mathbf{A}}\rangle} : B^{\infty} \to \mathbb{R}$ and $\mathbf{P}_{|\psi_{\mathbf{B}}\rangle} : B^{\infty} \to \mathbb{R}$ by formulas

$$\mathbf{P}_{|\psi_{\mathbf{A}}\rangle}(A) = \int_A x p_{|\psi_{\mathbf{A}}\rangle}(x) d\mu(x), \tag{4.6.13}$$

and

$$\mathbf{P}_{|\psi_{\mathbf{B}}\rangle}(A) = \int_A x p_{|\psi_{\mathbf{B}}\rangle}(x) d\mu(x), \tag{4.6.14}$$

where $p_{|\psi_{\mathbf{A}}\rangle}(x) = |\langle x|\psi_{\mathbf{A}}\rangle|^2$ and $p_{|\psi_{\mathbf{B}}\rangle}(x) = |\langle x|\psi_{\mathbf{B}}\rangle|^2$ respectively.
Remark 4.6.10. We assume now that $(\Omega, \mathcal{F}, \mathbf{P}) = (\mathbb{R}, B^{\infty}, \mathbf{P})$ and
(i) $\mathbf{P}_{|\psi_{\mathbf{A}}\rangle} \ll \mathbf{P}$,
(ii) $\mathbf{P}_{|\psi_{\mathbf{B}}\rangle} \ll \mathbf{P}$.

We define now random variables $X_{|\psi_{\mathbf{A}}\rangle}(\omega)$ and $X_{|\psi_{\mathbf{B}}\rangle}(\omega)$ by formulas

$$X_{|\psi_{\mathbf{A}}\rangle}(\omega) = \frac{d\mathbf{P}_{|\psi_{\mathbf{A}}\rangle}}{d\mathbf{P}}, X_{|\psi_{\mathbf{B}}\rangle}(\omega) = \frac{d\mathbf{P}_{|\psi_{\mathbf{B}}\rangle}}{d\mathbf{P}} \quad (4.6.15)$$

respectively. Notice that from Eq. (4.6.11), Eq. (4.6.13)-(4.6.14) and Eqs.(4.6.15) it follows that

$$X_{|\psi_{\mathbf{B}}\rangle}(\omega) = X_{|\psi_{\mathbf{A}}\rangle}(\omega) + x_0, \text{a.s.} \quad (4.6.16)$$

Let $B\left(|\psi_{\mathbf{B}}\rangle, \widehat{X}, x_{\mathbf{B}}, \delta x, t\right) \in \mathbf{B}_{M_4}$ be a physical event which consists of performing a measurement of the observable $\widehat{X} = \int_{x_1}^{x_2} x|x\rangle\langle x|dx$ with an accuracy δx, and the result is obtained in the range $(x_{\mathbf{B}} - \delta x, x_{\mathbf{B}} + \delta x)$ at instant t.

Remark 4.6.11. Note that: if the physical event $B\left(|\psi_{\mathbf{B}}\rangle, \widehat{X}, x_{\mathbf{B}}, \delta x, t\right)$ was occurred then immediately after the measurement at the instant t unconditional measure $\mathbf{P}$ collapses to conditional measure $\mathbf{P}\left(X \mid B\left(|\psi_{\mathbf{B}}\rangle, \widehat{X}, x_{\mathbf{B}}, \delta x, t\right)\right)$, where $X \in \mathcal{F}_{M_4}^{ph}$:

$$\mathbf{P}\left(X \mid B\left(|\psi_{\mathbf{B}}\rangle, \widehat{X}, x_{\mathbf{B}}, \delta x, t\right)\right) = \frac{\mathbf{P}\left(X \wedge B\left(|\psi_{\mathbf{B}}\rangle, \widehat{X}, x_{\mathbf{B}}, \delta x, t\right)\right)}{\mathbf{P}\left(B\left(|\psi_{\mathbf{B}}\rangle, \widehat{X}, x_{\mathbf{B}}, \delta x, t\right)\right)}, \quad (4.6.17)$$

see Remark 4.6.3.
Notice that: (i) from Eq. (4.6.8) it follows that

$$\Re\left[B\left(|\psi_{\mathbf{B}}\rangle, \widehat{X}, x_{\mathbf{B}}, \delta x, t\right)\right] = \Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}}, \delta x), \quad (4.6.18)$$

where we write for short $\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}}, \delta x)$ instead $\{\omega | x_{\mathbf{B}} - \delta x \leq X_{|\psi_{\mathbf{B}}\rangle}(\omega) \leq x_{\mathbf{B}} + \delta x\}$, i.e.

$$\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x) \triangleq \{\omega | x_{\mathbf{B}} - \delta x \leq X_{|\psi_{\mathbf{B}}\rangle}(\omega) \leq x_{\mathbf{B}} + \delta x\}, \tag{4.6.19}$$

see Remark 4.6.2;

(ii) from Eq. (4.6.11), Eq. (4.6.16) and Eq. (4.6.19) it follows that

$$\begin{aligned} &\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x) \triangleq \{\omega | x_{\mathbf{B}} - \delta x \leq X_{|\psi_{\mathbf{B}}\rangle}(\omega) \leq x_{\mathbf{B}} + \delta x\} = \\ &\{\omega | (x_{\mathbf{B}} - x_0) - \delta x \leq X_{|\psi_{\mathbf{B}}\rangle}(\omega) - x_0 \leq (x_{\mathbf{B}} - x_0) + \delta x\} = \\ &\{\omega | x_{\mathbf{A}} - \delta x \leq X_{|\psi_{\mathbf{A}}\rangle}(\omega) \leq x_{\mathbf{A}} + \delta x\} \triangleq \Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{A}},\delta x), \end{aligned} \tag{4.6.20}$$

and thus

$$\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x) = \Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{A}},\delta x) \tag{4.6.21}$$

(iii) from Eq. (4.6.17) - (4.6.19) it follows that: (i) unconditional measure $\mathbf{P}_{B^{\infty}}$ immediately after the measurement at instant t collapses to conditional measure $\mathbf{P}_{B^{\infty}}(X|\Sigma_{|\psi_{\mathbf{B}}\rangle}(x_{\mathbf{B}},\delta x))$, where $X \in B^{\infty}$:

$$\mathbf{P}_{B^{\infty}}\left(X \,\middle|\Sigma_{|\psi_{\mathbf{B}}\rangle}(x_{\mathbf{B}},\delta x)\right) = \frac{\mathbf{P}_{B^{\infty}}\left(X \wedge \Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x)\right)}{\mathbf{P}_{B^{\infty}}\left(\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x)\right)}. \tag{4.6.22}$$

Remark 4.6.12. (i) From Eq. (4.6.22) it follows that the unconditional probability density function $p_{\mathbf{B}}(x) = |\langle x|\psi_{\mathbf{B}}\rangle|^2$ immediately after the measurement at instant t collapses to the following conditional probability density function as

$$p_{\mathbf{B}}\left(x|\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x)\right) = \begin{cases} \dfrac{p_{\mathbf{B}}(x)}{\mathbf{P}_{B^{\infty}}\left(\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x)\right)} & \Leftrightarrow \ x \in \Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x) \\ 0 & \Leftrightarrow \ x \notin \Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x) \end{cases} \tag{4.6.23}$$

see Appendix B.

(ii) From Eq. (4.6.21) and Eq. (4.6.22) it follows that the unconditional probability density function $p_{\mathbf{A}}(x) = |\langle x|\psi_{\mathbf{A}}\rangle|^2$ immediately after the measurement at instant t collapses to the following conditional probability density function as

$$p_{\mathbf{A}}\left(x|\Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x)\right) = \begin{cases} \dfrac{p_{\mathbf{A}}(x)}{\mathbf{P}_{B^{\infty}}\left(\Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x)\right)} & \Leftrightarrow \ x \in \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x) \\ 0 & \Leftrightarrow \ x \notin \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x) \end{cases} \quad (4.6.24)$$

From Eq. (4.6.23) it follows that a wave function $\psi_{\mathbf{B}}(x) = \langle x|\psi_{\mathbf{B}}\rangle$ immediately after the measurement at instant t collapses to the following wave function

$$\psi_{\mathbf{B}}^{\mathbf{coll}}(x) = \begin{cases} \dfrac{\psi_{\mathbf{B}}(x)}{\sqrt{\mathbf{P}_{B^{\infty}}\left(\Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x)\right)}} & \Leftrightarrow \ x \in \Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x) \\ 0 & \Leftrightarrow \ x \notin \Sigma_{X_{|\psi_{\mathbf{B}}\rangle}}(x_{\mathbf{B}},\delta x) \end{cases} \quad (4.6.25)$$

From Eq. (4.6.24) it follows that immediately after the measurement on particle $\mathbf{B}$ at instant t a wave function $\psi_{\mathbf{A}}(x) = \langle x|\psi_{\mathbf{A}}\rangle$ collapses to the following wave function

$$\psi_{\mathbf{A}}^{\mathbf{coll}}(x) = \begin{cases} \dfrac{\psi_{\mathbf{A}}(x)}{\sqrt{\mathbf{P}_{B^{\infty}}\left(\Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x)\right)}} & \Leftrightarrow \ x \in \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x) \\ 0 & \Leftrightarrow \ x \notin \Sigma_{X_{|\psi_{\mathbf{A}}\rangle}}(x_{\mathbf{A}},\delta x) \end{cases} \quad (4.6.26)$$

Thus the measurement on particle $\mathbf{B}$ alters a wave function $\psi_{\mathbf{A}}(x)$ even if particles $\mathbf{A}$ and $\mathbf{B}$ are space-like separated and therefore EPR paradox disappears.

Chapter V

ERP-B PARADOX RESOLUTION

V.1. EPR-B experiment

The EPR-B, the spin version of the Einstein-Podolsky-Rosen experiment proposed by Bohm, see [17], [18] Bohm: "We consider a molecule of total spin zero consisting of two atoms, each of spin one-half. The wave function of the system is therefore

$$\psi = 1/\sqrt{2}\,[\psi_+(1)\psi_-(2) - \psi_-(1)\psi_+(2)]$$

where $\psi_+(1)$ refers to the wave function of the atomic state in which one particle (A) has spin $+\hbar/2,$ etc. The two atoms are then separated by a method that does not influence the total spin. After they have separated enough so that they cease to interact, any desired component of the spin of the first particle **A** is measured. Then, because the total spin is still zero, it can immediately be concluded that the same component of the spin of the other particle **B** is opposite to that of **A.** If this were a classical system, there would be no difficulty in interpreting the above results, because all components of the spin of each particle are well defined at each instant of time. Thus, in the molecule, each component of the spin of particle A has, from the very beginning, a value opposite to that of the same component of **B**; and this relationship does not change when the atom disintegrates. In other words, the two spin vectors are correlated. Hence, the measurement of any component of the spin of **A** permits us to conclude also that the same component of **B** is opposite in value. **The possibility of obtaining knowledge of the spin of particle B in this way evidently does not imply any interaction of the apparatus with particle B or any interaction between A and B.**

In quantum theory, a difficulty arises, in the interpretation of the above experiment, because only one component of the spin of each particle can have a definite value at a given time. Thus, if the x component is definite, then the y and z components are indeterminate and we may regard them more or less as in a kind of random fluctuation.

In spite of the effective fluctuation described above, however, the quantum theory still implies that no matter which component of the spin of **A** may be measured the same component of the spin of **B** will have a definite and opposite value when the measurement is over. Of course, the wave function then reduces to $\psi_{+}(1)\psi_{-}(2)$ or $\psi_{-}(1)\psi_{+}(2)$, in accordance with the result of the measurement. Hence, there will then be no correlations between the remaining components of the spins of the two atoms. Nevertheless, before the measurement has taken place (even while the atoms are still in flight) we are free to choose any direction as the one in which the spin of particle **A** (and therefore of particle **B**) will become definite.

In order to bring out the difficulty of interpreting the result, let us recall that originally, the indeterminacy principle was regarded as representing the effects of the disturbance of the observed system by the indivisible quanta connecting it with the measuring apparatus. This interpretation leads to no serious difficulties for the case of a single particle. For example, we could say that on measuring the z component of the spin of particle **A**, we disturb the x and y components and make them fluctuate. This point of view more generally implies that the definiteness of any desired component of the spin is (along with the indefiniteness of the other two components) a potentiality which can be realized with the aid of a suitably oriented spinmeasuring apparatus.

In the case of complementary pairs of continuous variables, such as position and momentum, one obtains from this point of view the well known wave-particle duality. In other words, the electron, for example, has potentialities for mutually incompatible wave-like and particle-like behavior, which are realized under suitable external conditions. In the laboratory those conditions are generally determined by the measuring apparatus although, more generally, they may be determined by any arrangement of matter with which the electron interacts. But in any case, it is essential that there must be an external interaction, which disturbs the observed system in such a way as to bring about the realization of one of its various mutually incompatible potentialities. As a result of this disturbance, when any one variable is made definite, other (noncommuting) variables must necessarily become indefinite and undergo fluctuation.

Evidently, the foregoing interpretation is not satisfactory when applied to the experiment of ERP. It is of course acceptable for particle **A** alone (the particle whose spin is measured directly). But it does not explain why particle **B** (which does not interact with **A** or with the measuring apparatus) realizes its potentiality for a definite spin in precisely the same direction as that of **A**. Moreover, it cannot explain the fluctuations of the other two components of the spin of particle **B** as the result of disturbances due to the measuring apparatus.

In this subsection we explain EPR-B experiment using reduction to a sort of generic EPR correlations for two particles **A** and **B** with maximally correlated position z_A and z_B. This explanation avoids the EPR-Bohm paradox.

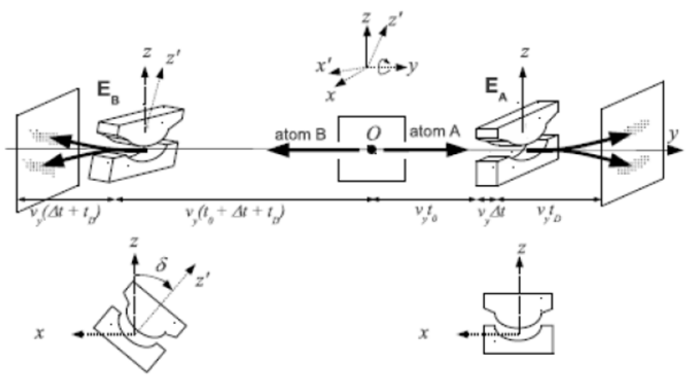

Fig. 5.1.1. Einstein-Podolsky-Rosen-Bohm experiment

Fig. 5.1.1 presents the Einstein-Podolsky-Rosen-Bohm experiment. A source S created in O pairs of identical atoms **A** and **B**, but with opposite spins. The atoms **A** and **B** split following the y-axis in opposite directions, and head towards two identical Stern-Gerlach apparatus $\mathbf{E_A}$ and $\mathbf{E_B}$. The electromagnet $\mathbf{E_A}$ "measures" the spin of **A** along the z-axis and the electromagnet $\mathbf{E_B}$ "measures" the spin of **B** along the z'-axis, which is obtained after a rotation of an angle δ around the y-axis.

Remark 5.1.1. So far we have consistently made use of the idea that if we know something definite about the state of such a physical system, say that we know z component of the spin of a particle is $S_z = \pm\frac{1}{2}\hbar,$ then we assign to the system the state $|S_z\rangle = \left|\pm\frac{1}{2}\hbar\right\rangle$, or, more simply, $|\pm\rangle.$

Remark 5.1.2. We can also note that these two states $|+\rangle$ and $|-\rangle$ are mutually exclusive, i.e. if an atom is in the state $|+\rangle$, then the result $S_z = -\frac{1}{2}\hbar$ is never observed, and furthermore, we note that the two states $|+\rangle$ and $|-\rangle$ cover all possible values for S_z.

Remark 5.1.3. When we say that we `know' the value of some physical observable of a quantum system, we are presumably implying that some kind of measurement has been made that provided us with this knowledge. Furthermore, it is assumed that in the process of acquiring this knowledge, the system, after the measurement has been performed, survives the measurement, and moreover if we immediately remeasured the same quantity, we would get the same result.

This is certainly the situation with the measurement of spin in the Stern-Gerlach experiment. If an atom emerges from one such a set of apparatus in a beam which indicates $S_z = \frac{1}{2}\hbar$ for that atom, and we passed the atom through a second apparatus, also with its magnetic field oriented in the z direction, we would find the atom emerging in the $S_z = \frac{1}{2}\hbar$ beam once again. Under such circumstances, we would be justified in saying that the *atom has been prepared* in the state

$\left|S_z = \frac{1}{2}\hbar\right\rangle,$ etc.

Definition 5.1.1. Assume that *atom* **A** *has been prepared in the state* $\left|S_z = \frac{1}{2}\hbar\right\rangle,\ \left|S_z = -\frac{1}{2}\hbar\right\rangle,$ etc. Then we will say that these events $\left|S_z = \frac{1}{2}\hbar\right\rangle, \left|S_z = -\frac{1}{2}\hbar\right\rangle,$ etc. occur. We will denote these events by symbols $\left|S_z = \frac{1}{2}\hbar\right\rangle^{\mathbf{A}}, \left|S_z = -\frac{1}{2}\hbar\right\rangle^{\mathbf{A}},$ etc., or $\left|\frac{1}{2}\hbar\right\rangle^{\mathbf{A}}, \left|-\frac{1}{2}\hbar\right\rangle^{\mathbf{A}},$ etc.

Definition 5.1.2. Assume that we know exactly that *atom* **A** *is in the state* $\left|\frac{1}{2}\hbar\right\rangle, \left|-\frac{1}{2}\hbar\right\rangle,$ etc.

Then we will say that these events $\left|\frac{1}{2}\hbar\right\rangle, \left|-\frac{1}{2}\hbar\right\rangle,$ etc. occur and we will denote these events again by symbols $\left|\frac{1}{2}\hbar\right\rangle^{\mathbf{A}}, \left|-\frac{1}{2}\hbar\right\rangle^{\mathbf{A}},$ etc.

Definition 5.1.3. Assume that these events $\left|\frac{1}{2}\hbar\right\rangle^{\mathbf{A}}, \left|-\frac{1}{2}\hbar\right\rangle^{\mathbf{A}},$ etc. occur in the point $\mathbf{x} = (t, x_1, x_2, x_3) = (t, \mathbf{r}) \in M_4$ of Minkowski spacetime M_4. Then we will denote these events by symbols $\left|\frac{1}{2}\hbar\right\rangle^{\mathbf{A}}_{\mathbf{x}}, \left|-\frac{1}{2}\hbar\right\rangle^{\mathbf{A}}_{\mathbf{x}},$ etc. or $\left|\frac{1}{2}\hbar\right\rangle^{\mathbf{A}}_{(t_A, z_A)}, \left|-\frac{1}{2}\hbar\right\rangle^{\mathbf{A}}_{(t_A, z_A)},$ etc.

Assumption 5.1.1. We claim for any $\mathbf{x} \in M_4$ that:

$$\left|\tfrac{1}{2}\hbar\right\rangle^{\mathbf{A}}_{\mathbf{x}} \in \mathcal{F}_{M_4}, \left|-\tfrac{1}{2}\hbar\right\rangle^{\mathbf{A}}_{\mathbf{x}} \in \mathcal{F}_{M_4}, etc. \tag{5.1.1}$$

Here $\mathcal{F}_{M_4}$ is a measure algebra $\mathcal{F}_{M_4} = (\mathbf{B}_{M_4}, \mathbf{P})$ with a probability measure $\mathbf{P},$ see Chapt. III, subsection III.2, Definition 3.2.3.

Remark 5.1.4. Note that for any $\mathbf{x} \in M_4$ and for any atom **A** these events $\left|\frac{1}{2}\hbar\right\rangle^{\mathbf{A}}_{\mathbf{x}}, \left|-\frac{1}{2}\hbar\right\rangle^{\mathbf{A}}_{\mathbf{x}}$ are mutually exclusive, see Remark 5.1.2, and therefore for any $\mathbf{x} \in M_4$

$$\mathbf{P}\left(\left|\tfrac{1}{2}\hbar\right\rangle^{\mathbf{A}}_{\mathbf{x}} \bigwedge \left|-\tfrac{1}{2}\hbar\right\rangle^{\mathbf{A}}_{\mathbf{x}}\right) = 0. \tag{5.1.2}$$

Remark 5.1.5. We remind that if an atom is prepared in an arbitrary initial state $|S\rangle,$ then the probability amplitude of finding it in some other state $|S'\rangle$ is given by

$$\langle S'|S\rangle = \langle S'|+\rangle\langle +|S\rangle + \langle S'|-\rangle\langle -|S\rangle \tag{5.1.3}$$

which leads, by the cancellation trick to

$$|S\rangle = |+\rangle\langle+|S\rangle + |-\rangle\langle-|S\rangle \tag{5.1.4}$$

and therefore the states $|\pm\rangle$ form a complete set of orthonormal basis states for the state space of the system.

Suppose we have an n-dimensional quantum system which contains only a quantum observable with discrete values such as $S_z,$ etc.
Then we claim the following:

Q $_{\mathbf{d}}$ .V.1. Any given n -dimensional quantum system which contains only a quantum observable with discrete values such that mentioned above is identified by a set $\mathbf{Q_d}$:

$$\mathbf{Q_d} \triangleq \left\langle \mathbf{H_d}, \mathfrak{I}_\mathbf{d}, \mathfrak{R}_\mathbf{d}, \mathcal{L}_{2,1}^{\mathbf{d}}, \mathbf{G_d}, |\psi_t\rangle \right\rangle, \tag{5.1.5}$$

where:

(i) $\mathbf{H_d}$ that is some finite-dimensional complex Hilbert space,

(ii) $\mathfrak{I}_\mathbf{d} = (\Omega_\mathbf{d}, \mathcal{F}_\mathbf{d}, \mathbf{P_d})$ that is complete probability space,

(iii) $\mathfrak{R}_\mathbf{d} = (\mathbb{R}^n, \Sigma_\mathbf{d})$ that is measurable space,

(iv) $\mathcal{L}_{2,1}^{\mathbf{d}}(\Omega_\mathbf{d})$ that is complete space of discrete complex valued random variables $X_\mathbf{d} : \Omega_\mathbf{d} \to \mathbb{C}^n$ such that

$$\int_{\Omega_\mathbf{d}} \|X_\mathbf{d}(\omega)\| d\mathbf{P_d} < \infty, \int_{\Omega_\mathbf{d}} \|X_\mathbf{d}(\omega)\|^2 d\mathbf{P_d} < \infty \tag{5.1.6}$$

(v) $\mathbf{G_d} : \mathbf{H_d} \to \mathcal{L}_{2,1}(\Omega_\mathbf{d})$ that is one to one correspondence such that

$$\langle\psi|\widehat{Q}_\mathbf{d}|\psi\rangle = \int_{\Omega_\mathbf{d}} \left(\mathbf{G_d}\left[\widehat{Q}_\mathbf{d}|\psi\rangle\right](\omega)\right) d\mathbf{P_d} = \mathbf{E}_{\Omega_\mathbf{d}}\left(\mathbf{G_d}\left[\widehat{Q}_\mathbf{d}|\psi\rangle\right](\omega)\right) \tag{5.1.7}$$

for any $|\psi\rangle \in \mathbf{H_d}$ and for any Hermitian operator with discrete spectrum $\widehat{Q}_\mathbf{d} : \mathbf{H_d} \to \mathbf{H_d}$, where $\widehat{Q}_\mathbf{d} \in \mathfrak{I}^*(\mathbf{H_d}) \subsetneqq C^*(\mathbf{H_d}), C^*(\mathbf{H_d})$ is C^*-algebra of the Hermitian adjoint operators in $\mathbf{H_d}$ and $\mathfrak{I}^*(\mathbf{H_d})$ is commutative subalgebra of $C^*(\mathbf{H_d})$.

(vi) $|\psi_t\rangle$ is a continuous vector function $|\psi_t\rangle : \mathbb{R}_+ \to \mathbf{H_d}$ which represented the evolution of the quantum system $\mathbf{Q_d}$.

Q $_{\mathbf{d}}$ .V.2. For any $|\psi_1\rangle, |\psi_2\rangle \in \mathbf{H_d}$ and for any Hermitian operator

$\widehat{Q}_{\mathbf{d}} : \mathbf{H_d} \to \mathbf{H_d}$ such that

$$\left\langle \psi_1 \middle| \widehat{Q}_{\mathbf{d}} \middle| \psi_2 \right\rangle = \left\langle \psi_2 \middle| \widehat{Q}_{\mathbf{d}} \middle| \psi_1 \right\rangle = 0 \tag{5.1.8}$$

the equality is valid

$$\mathbf{G_d}\left[\widehat{Q}_{\mathbf{d}}(|\psi_1\rangle + |\psi_2\rangle)\right](\omega) = \mathbf{G_d}\left[\widehat{Q}_{\mathbf{d}}|\psi_1\rangle\right](\omega) + \mathbf{G_d}\left[\widehat{Q}_{\mathbf{d}}|\psi_2\rangle\right](\omega). \tag{5.1.9}$$

Remark 5.1.6. Let $S_z^+(\omega)$ and $S_z^-(\omega)$ be discrete random variables $S_z^+ : \Omega_{\mathbf{d}} \to \{1,-1\}$, $S_z^- : \Omega_{\mathbf{d}} \to \{-1,1\}$ correspondingly such that:

$$\begin{gathered}
\text{(i) } S_z^+(\omega) = \mathbf{G}[|+\rangle],\ \text{(ii) } \mathbf{P_d}(\Delta_+^{=+1}) = 1, \text{where } \Delta_+^{=+1} \triangleq \left\{\omega | S_z^+(\omega) = 1\right\},\\
\text{(iii) } \mathbf{P_d}(\Delta_+^{=-1}) = 0, \text{where } \Delta_+^{=-1} \triangleq \left\{\omega | S_z^+(\omega) = -1\right\}\\
\text{and}\\
\text{(i) } S_z^-(\omega) = \mathbf{G}[|-\rangle],\ \text{(ii) } \mathbf{P_d}(\Delta_-^{=-1}) = 1, \text{where } \Delta_-^{=-1} \triangleq \left\{\omega | S_z^-(\omega) = -1\right\},\\
\text{(iii) } \mathbf{P_d}(\Delta_-^{=+1}) = 0, \text{where } \Delta_-^{=+1} \triangleq \left\{\omega | S_z^-(\omega) = 1\right\}.
\end{gathered} \tag{5.1.10}$$

Let $\mathbf{Q_c}$ be any n-dimensional quantum system which contains only a quantum observable with continuous values. We remind that such a quantum system is identified with a set $\mathbf{Q}$

$$\mathbf{Q} \triangleq \langle \mathbf{H}, \Im, \Re, \mathcal{L}_{2,1}, \mathbf{G}, |\psi_t\rangle\rangle. \tag{5.1.11}$$

Definition 5.1.4. We define now a composite quantum system $\mathbf{Q_{c,d}}$ which contains both sort of quantum observables by a set $\mathbf{Q_{c,d}}$

$$\mathbf{Q_{c,d}} \triangleq \left\langle \mathbf{H_{c,d}}, \Im_{\mathbf{c,d}}, \Re_{\mathbf{c,d}}, \mathcal{L}_{2,1}^{\mathbf{c,d}}, \mathbf{G_{c,d}}, |\psi_t\rangle\right\rangle \tag{5.1.12}$$

where:

(i) $\mathbf{H_{c,d}} = \mathbf{H_c} \times \mathbf{H_d}$ that is composite complex Hilbert space,

(ii) $\Im_{\mathbf{c,d}} = (\Omega_{\mathbf{c,d}}, \mathcal{F}_{\mathbf{c,d}}, \mathbf{P_d})$ that is complete probability space, with $\Omega_{\mathbf{c,d}} = \Omega_{\mathbf{c}} \times \Omega_{\mathbf{d}}, \mathcal{F}_{\mathbf{c,d}} = \mathcal{F}_{\mathbf{c}} \times \mathcal{F}_{\mathbf{d}}, \Re_{\mathbf{c,d}} = \Re_{\mathbf{c}} \times \Re_{\mathbf{d}}, \mathcal{L}_{2,1}^{\mathbf{c,d}} = \mathcal{L}_{2,1}^{\mathbf{c}} \times \mathcal{L}_{2,1}^{\mathbf{d}}, \mathbf{G_{c,d}} = \mathbf{G_c} \times \mathbf{G_d}$,

(iii) $\Re_{\mathbf{c,d}} = (\mathbb{R}^n, \Sigma_{\mathbf{c,d}})$ that is measurable space with $\Sigma_{\mathbf{c,d}} = \Sigma_{\mathbf{c}} \times \Sigma_{\mathbf{d}}$,

(iv) $\mathcal{L}_{2,1}^{\mathbf{c},\mathbf{d}}(\Omega_{\mathbf{d}})$ that is complete space of random variables $X_{\mathbf{c},\mathbf{d}} : \Omega_{\mathbf{c},\mathbf{d}} \to \mathbb{R}^n$ such that

$$\int_{\Omega_{\mathbf{c},\mathbf{d}}} \|X_{\mathbf{c},\mathbf{d}}(\omega)\| d\mathbf{P}_{\mathbf{c}} \times d\mathbf{P}_{\mathbf{d}} < \infty, \int_{\Omega_{\mathbf{c},\mathbf{d}}} \|X_{\mathbf{c},\mathbf{d}}(\omega)\|^2 d\mathbf{P}_{\mathbf{c}} \times d\mathbf{P}_{\mathbf{d}} < \infty, \omega \in \Omega_{\mathbf{c},\mathbf{d}} \tag{5.1.13}$$

(v) $\mathbf{G}_{\mathbf{c},\mathbf{d}} : \mathbf{H}_{\mathbf{c},\mathbf{d}} \to \mathcal{L}_{2,1}^{c,\mathbf{d}}(\Omega_{\mathbf{d}})$ that is one to one correspondence such that

$$\langle \psi | \widehat{Q}_{\mathbf{c},\mathbf{d}} | \psi \rangle = \int_{\Omega_{\mathbf{c},\mathbf{d}}} \left(\mathbf{G}_{\mathbf{c},\mathbf{d}} \left[\widehat{Q}_{\mathbf{c},\mathbf{d}} | \psi \rangle \right](\omega) \right) d\mathbf{P}_{\mathbf{c}} \times d\mathbf{P}_{\mathbf{d}} = \mathbf{E}_{\Omega_{\mathbf{c},\mathbf{d}}} \left(\mathbf{G}_{\mathbf{c},\mathbf{d}} \left[\widehat{Q}_{\mathbf{c},\mathbf{d}} | \psi \rangle \right](\omega) \right) \tag{5.1.14}$$

for any $|\psi\rangle \in \mathbf{H}_{\mathbf{c},\mathbf{d}}$ and for any Hermitian operator $\widehat{Q}_{\mathbf{c},\mathbf{d}} : \mathbf{H}_{\mathbf{c},\mathbf{d}} \to \mathbf{H}_{\mathbf{c},\mathbf{d}}$, where $\widehat{Q}_{\mathbf{c},\mathbf{d}} \in \mathfrak{I}^*(\mathbf{H}_{\mathbf{c},\mathbf{d}}) \subsetneqq C^*(\mathbf{H}_{\mathbf{c},\mathbf{d}}), C^*(\mathbf{H}_{\mathbf{c},\mathbf{d}})$ is C^*-algebra of the Hermitian adjoint operators in $\mathbf{H}_{\mathbf{c},\mathbf{d}}$ and $\mathfrak{I}^*(\mathbf{H}_{\mathbf{c},\mathbf{d}})$ is commutative subalgebra of $C^*(\mathbf{H}_{\mathbf{c},\mathbf{d}})$.

(vi) $|\psi_t\rangle$ is a continuous vector function $|\psi_t\rangle : \mathbb{R}_+ \to \mathbf{H}_{\mathbf{d}}$ which represented the evolution of the quantum system $\mathbf{Q}_{\mathbf{c},\mathbf{d}}$.

V.2. EPR-B paradox resolution

The usual conclusion of EPR-B experiment is to reject the non-local realism for two reasons: the impossibility of decomposing a pair of entangled atoms into two states, one for each atom, and the impossibility of interaction faster than the speed of light.

Remark 5.2.1. We find that the EPRB-paradox can be resolved by nonprincipal and convenient relaxing of the Einstein's locality principle, that is the "relaxed locality principle" introduced in Chapter IV.1.

Remark 5.2.2. The solution to the entangled state is obtained by resolving the Pauli equation from an initial singlet wave function with a spatial extension as:

$$\Psi_0(\mathbf{r}_A,\mathbf{r}_B) = \frac{1}{\sqrt{2}} f(\mathbf{r}_A) f(\mathbf{r}_B)(|+_A\rangle \otimes |-_B\rangle - |-_A\rangle \otimes |+_B\rangle), \qquad (5.2.1)$$

The initial wave function of the entangled state is the singlet state (5.2.1) with

$$f(\mathbf{r}) \asymp \begin{cases} (2\pi\sigma_0^2)^{-\frac{1}{2}} e^{-\dfrac{x^2+y^2+z^2}{4\sigma_0^2}} \text{ iff } \|\mathbf{r}\| \leq \epsilon, \\ 0 \text{ iff } \|\mathbf{r}\| > \epsilon \end{cases} \qquad (5.2.2)$$
$$\mathbf{r} = (x,y,z), \sigma_0 \ll 1, \epsilon \ll 1$$

and where $|\pm_A\rangle$ and $|\pm_B\rangle$ are the eigenvectors of the operators σ_{z_A} and σ_{z_B} :

$$\sigma_{z_A}|\pm_A\rangle = \pm|\pm_A\rangle, \sigma_{z_B}|\pm_B\rangle = \pm|\pm_B\rangle. \qquad (5.2.3)$$

Remark 5.2.3. We treat the dependence with y strictly quasiclassically, i.e., with speed $-\mathbf{v}_{y\pm}^A(\mathbf{v}_0,\theta_0^A)$ for **A** and $\mathbf{v}_{y\pm}^B(\mathbf{v}_0,\theta_0^B)$ for **B** such that

$$\begin{gathered} \mathbf{P}\left\{\left| y + \mathbf{v}_{y\pm}^A(\mathbf{v}_0,\theta_0^A)t \right| \leq \epsilon\right\} = 1, \\ \mathbf{P}\left\{\left| y + \mathbf{v}_{y\pm}^A(\mathbf{v}_0,\theta_0^A)t \right| > \epsilon\right\} = 0, \\ \mathbf{P}\left\{\left| y - \mathbf{v}_{y\pm}^B(\mathbf{v}_0,\theta_0^B)t \right| \leq \epsilon\right\} = 1, \\ \mathbf{P}\left\{\left| y - \mathbf{v}_{y\pm}^B(\mathbf{v}_0,\theta_0^B)t \right| > \epsilon\right\} = 0, \\ \epsilon \ll 1, \end{gathered} \qquad (5.2.4)$$

where

$$\begin{gathered} \mathbf{v}_{y\pm}^A(\mathbf{v}_0,\theta_0^A) = \eta_{\theta_0^A}^{\pm}\mathbf{v}_0, \mathbf{v}_{y\pm}^B(\mathbf{v}_0,\theta_0^B) = \eta_{\theta_0^B}^{\pm}\mathbf{v}_0, \\ \eta_{\theta_0^A}^{+} = \cos^2\frac{\theta_0^A}{2}, \eta_{\theta_0^A}^{-} = \sin^2\frac{\theta_0^A}{2}, \\ \eta_{\theta_0^B}^{+} = \cos^2\frac{\theta_0^B}{2}, \eta_{\theta_0^B}^{-} = \sin^2\frac{\theta_0^B}{2}. \end{gathered} \qquad (5.2.5)$$

The wave function $\Psi(\mathbf{r}_A,\mathbf{r}_B,t)$ of the two identical particles **A** and **B**, electrically neutral and with magnetic moments μ_0, subject to magnetic fields $\mathbf{E_A}$ and $\mathbf{E_B}$, admits on the basis of $|\pm_A\rangle$ and $|\pm_B\rangle$ four components $\Psi^{a,b}(\mathbf{r}_A,\mathbf{r}_B,t)$ and satisfies the two-body Pauli equation

$$i\hbar\frac{\partial\Psi^{a,b}(t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\Delta_A - \frac{\hbar^2}{2m}\Delta_B\right)\Psi^{a,b}(t) + \mu B_j^{\mathbf{E_A}}(\sigma_j)_c^a\Psi^{c,b}(t) + \mu B_j^{\mathbf{E_B}}(\sigma_j)_d^b\Psi^{a,d}(t) \quad (5.2.6)$$

with the initial conditions:

$$\Psi^{a,b}(0,\mathbf{r}_A,\mathbf{r}_B) = \Psi_0^{a,b}(\mathbf{r}_A,\mathbf{r}_B), \quad (5.2.7)$$

where $\Psi_0^{a,b}(\mathbf{r}_A,\mathbf{r}_B)$ corresponds to the singlet state (5.2.1).

Below we explain the EPR-B experiment by using nonlocal two-body Pauli equation

$$\begin{aligned}&\int d\mathbf{r}_A d\mathbf{r}_B\int dt\Bigg[-i\hbar\frac{\partial\Psi^{\#a,b}(t,t',\mathbf{r}_A,\mathbf{r}_B)}{\partial t} + \left(-\frac{\hbar^2}{2m}\Delta_A - \frac{\hbar^2}{2m}\Delta_B\right)\Psi^{\#a,b}(t,t',\mathbf{r}_A,\mathbf{r}_B)\\ &+\mu B_j^{\mathbf{E_A}}(\sigma_j)_c^a\Psi^{\#c,b}(t,t',\mathbf{r}_A,\mathbf{r}_B) + \mu B_j^{\mathbf{E_B}}(\sigma_j)_d^b\Psi^{\#a,d}(t,t',\mathbf{r}_A,\mathbf{r}_B)\Bigg] = O(\hbar^\alpha),\\ &d\mathbf{r}_A = dx_A dy_A z_A, d\mathbf{r}_B = dx_B dy_B z_B\end{aligned} \quad (5.2.8)$$

with a boundary condition

$$\int d\mathbf{r}_A d\mathbf{r}_B z_A(t_1)\left|\Psi^{\#}(t_1,t',\mathbf{r}_A,\mathbf{r}_B)\right|^2 = -\int d\mathbf{r}_A d\mathbf{r}_B z_B(t_2)\left|\Psi^{\#}(t_2,t',\mathbf{r}_A,\mathbf{r}_B)\right|^2. \quad (5.2.9)$$

One of the difficulties of the canonical interpretation of the EPR-B experiment is the existence of two simultaneous measurements. By doing these measurements one after the other, the interpretation of the experiment will be facilitated. That is the purpose of the two-step version of the experiment EPR-B studied below.

V.2.1. First step EPR-B: Spin measurement of A

Consider that at time t_0 the particle **A** arrives at the entrance of electromagnet $\mathbf{E_A}$.

Remark 5.2.4. We assume that a particle **A** collapses in a magnetic field $\mathbf{E_A}$ at some instant t' into two particles $\mathbf{A}_+$ and $\mathbf{A}_-$, i.e. the spinor $\Psi(z,y,t)$ collapses in a magnetic field $\mathbf{E_A}$ at some instant t' into two spinors $\Psi_{+}(z,y,t,t',\delta)$ and $\Psi_{-}(z,y,t,t',\delta)$ given by Eq. (6.1.9a)-(6.1.9b), see Assumption 5.1.1.

Remark 5.2.5. The particles **A**$_+$ and **A** stay within the magnetic field for a time $\Delta t' \leq \Delta t = \frac{\Delta l}{\mathbf{v}_0}$.

Thus after exit of the magnetic field $\mathbf{E_A}$, at time $t_1 = t_0 + \triangle t + t,$ the wave functions $\Psi_{+}(z,y,t_0+\Delta t+t,\delta)$ and $\Psi_{-}(z,y,t_0+\Delta t+t,\delta)$ become

$$\Psi_{+}(\mathbf{r}_{\mathbf{A}_+},\mathbf{r}_{\mathbf{B}_-},t_0+\triangle t+t) = f(\mathbf{r}_{\mathbf{B}_-}) \times \Big(f^{+}(\mathbf{r}_{\mathbf{A}_+},t)|+_{\mathbf{A}}\rangle \otimes |-_{\mathbf{B}}\rangle\Big) \qquad (5.2.10.a)$$

and

$$\Psi_{-}(\mathbf{r}_{\mathbf{A}_-},\mathbf{r}_{\mathbf{B}_+},t_0+\triangle t+t) = f(\mathbf{r}_{\mathbf{B}_+}) \times \Big(f^{-}(\mathbf{r}_{\mathbf{A}_-},t)|-_{\mathbf{A}}\rangle \otimes |+_{\mathbf{B}}\rangle\Big) \qquad (5.2.10.b)$$

respectively, with

$$\begin{aligned} f^{+}(\mathbf{r},t) &= \cos\frac{\theta_0}{2} f(x,z-z_{\triangle}-ut)\exp\Big[i\Big(\frac{muz}{\hbar}+\varphi^{+}(t)\Big)\Big] \\ f^{-}(\mathbf{r},t) &= \sin\frac{\theta_0}{2} f(x,z+z_{\triangle}+ut)\exp\Big[i\Big(-\frac{muz}{\hbar}+\varphi^{-}(t)\Big)\Big] \end{aligned} \qquad (5.2.11)$$

where $z_{\triangle}$ and u are given by

$$z_{\Delta} = \frac{\mu_B B_0'(\Delta t)^2}{2m} = 10^{-5}m, \qquad u = \frac{\mu_B B_0'(\Delta t)}{m} = 1m/s. \qquad (5.2.12)$$

Remark 5.2.6. We deduce that:

the beam of particle **A** is divided into two **A**$_+$ and **A**$_-$, and the beam of particle **B** is divided into two **B**$_+$ and **B**$_-$ ·

Remark 5.2.7. Our first conclusion is: the position of **B**$_+$ and **B**$_-$ does not depend on the spin measurement of **A**$_+$ and **A**$_-$, only the spins are involved.

We conclude from equation (5.2.10) that the spins of **A**$_+$ and **B**$_-$

(**A**$_-$ and **B**$_+$) remain opposite throughout the experiment. These are the two properties used in the relaxed causal interpretation.

Remark 5.2.8. By "relaxed locality principle" and decoherence it follows that the interaction between **A**$_+$, **A**$_-$, **B**$_+$, and **B**$_-$ is absent, we assume the existence of wave functions

$$\Psi_0^{\mathbf{A}_+}(\mathbf{r}_{\mathbf{A}_+},\theta_0^{\mathbf{A}_+},\varphi_0^{\mathbf{A}_+}),\Psi_0^{\mathbf{A}_-}(\mathbf{r}_{\mathbf{A}_+},\theta_0^{\mathbf{A}_-},\varphi_0^{\mathbf{A}_-}),\Psi_0^{\mathbf{B}_+}(\mathbf{r}_{\mathbf{B}},\theta_0^{\mathbf{B}_+},\varphi_0^{\mathbf{B}_+}),\Psi_0^{\mathbf{B}_-}(\mathbf{r}_{\mathbf{B}},\theta_0^{\mathbf{B}_-},\varphi_0^{\mathbf{B}_-}). \quad (5.2.13)$$

V.2.2. Second step EPR-B: Spin measurement of B

The second step is a continuation of the first one and corresponds to the EPR-B experiment broken down into two steps. On a pairs of particles $\mathbf{A}_+$, $\mathbf{B}_-$ and $\mathbf{A}_-$, $\mathbf{B}_+$ in a singlet state, first we made the Stern and Gerlach measurement on the $\mathbf{A}_+$ and $\mathbf{A}_-$ atom at instant t_1 between t_0 and $t_0+\triangle t+t_D$:

$$t_0 < t_1 < t_0+\triangle t+t_D. \quad (5.2.14)$$

Secondly, we make the Stern and Gerlach measurement on the $\mathbf{B}_+$ and $\mathbf{B}_-$ atom with an electromagnet $\mathbf{E_B}$ forming an angle δ with $\mathbf{E_A}$ at instant t_2 between $t_0+\triangle t+t_D$ and $t_0+2(\triangle t+t_D)$:

$$t_0+\triangle t+t_D < t_2 \le t_0+2(\triangle t+t_D) \quad (5.2.15)$$

At the exit of magnetic field $\mathbf{E_A}$, at time $t_0+\triangle t+t_D,$ the pair of particles wave functions is given by Eq. (5.2.10a) and Eq. (5.2.10b) respectively. Immediately after the measurements of $\mathbf{A}_+$ and $\mathbf{A}_-$, still at time $t_0+\triangle t+t_D,$ the wave functions of $\mathbf{B}_+$ and $\mathbf{B}_-$ depend on the measurements $\pm$ of **A** respectively such that:

$$\Psi_{\mathbf{B}_-/+\mathbf{A}}(\mathbf{r}_{\mathbf{B}_-},t_0+\triangle t+t_1) = f(\mathbf{r}_{\mathbf{B}_-})|-_{\mathbf{B}}\rangle, \quad (5.2.16.a)$$

and

$$\Psi_{\mathbf{B}_+/-\mathbf{A}}(\mathbf{r}_{\mathbf{B}_+},t_0+\triangle t+t_1) = f(\mathbf{r}_{\mathbf{B}_+})|+_{\mathbf{B}}\rangle. \quad (5.2.16.b)$$

Then, the measurement of $\mathbf{B}_+$ and $\mathbf{B}_-$ at time $t_2 > t_0+2(\triangle t+t_D)$ yields, in this two-step version of the EPR-B experiment, the same results for spatial quantization and correlations of spins as in the EPR-B experiment.

V.2.3. Resolution of the EPR-B experiment in de Broglie-Bohm interpretation by the "relaxed locality principle"

We assume, at the creation of the two entangled particles **A** and **B**, that each of the two particles **A** and **B** has an initial wave function with opposite spins:

$$\Psi_0^A(\mathbf{r}_A,\theta_0^A,\varphi_0^A) = f(\mathbf{r}_A)\left(\cos\frac{\theta_0^A}{2}|+_A\rangle + \sin\frac{\theta_0^A}{2}e^{i\varphi_0^A}|-_A\rangle\right) \qquad (5.2.17)$$

and

$$\Psi_0^B(\mathbf{r}_B,\theta_0^B,\varphi_0^B) = f(\mathbf{r}_B)\left(\cos\frac{\theta_0^B}{2}|+_B\rangle + \sin\frac{\theta_0^B}{2}e^{i\varphi_0^B}|-_B\rangle\right) =$$

$$f(\mathbf{r}_B)\left[\cos\left(\frac{\pi}{2}-\frac{\theta_0^A}{2}\right)|+_B\rangle + \sin\left(\frac{\pi}{2}-\frac{\theta_0^A}{2}\right)e^{i(\varphi_0^A-\pi)}|-_B\rangle\right] = \qquad (5.2.18)$$

$$\Psi_0^B(\mathbf{r}_B,\theta_0^B,\varphi_0^B) = f(\mathbf{r}_B)\left(\sin\frac{\theta_0^A}{2}|+_B\rangle + \cos\frac{\theta_0^A}{2}e^{i\varphi_0^A}|-_B\rangle\right)$$

with $\theta_0^B = \pi - \theta_0^A$ and $\varphi_0^B = \varphi_0^A - \pi$. The two particles **A** and **B** are statistically prepared as in the Stern and Gerlach experiment. Then the Pauli principle tells us that the two-body wave function must be antisymmetric; after calculation we find the same singlet state (5.2.1):

$$\Psi_0(\mathbf{r_A},\theta^\mathbf{A},\varphi^\mathbf{A},\mathbf{r_B},\theta^\mathbf{B},\varphi^\mathbf{B}) = -e^{i\varphi^\mathbf{A}}f(\mathbf{r_A})f(\mathbf{r_B})\times(|+_\mathbf{A}\rangle\otimes|-_\mathbf{B}\rangle - |-_\mathbf{A}\rangle\otimes|+_\mathbf{B}\rangle). \qquad (5.2.19)$$

Thus, we can consider that the singlet wave function is the wave function of a family of two fermions **A** and **B** with opposite spins: the direction of initial spin **A** and **B** exists, but is not *known*. It is a local hidden variable which is therefore necessary to add in the initial conditions of the model.
Here, we assume that at the initial time we know the spin of each particle (given by each initial wave function) and the initial position of each particle.

V.2.3.1. Step 1: spin measurement of A in de Broglie-Bohm interpretation

In Eq. (5.2.19) particle **A** can be considered independent on **B**. We can therefore give it the wave function

$$\Psi^A(\mathbf{r}_A, t_0 + \triangle t + t) = \cos\frac{\theta_0^A}{2} f^+(\mathbf{r}_A, t)|+_A\rangle + \sin\frac{\theta_0^A}{2} e^{i\varphi_0^A} f^-(\mathbf{r}_A, t)|-_A\rangle \qquad (5.2.20)$$

which is the wave function of a free particle in a Stern Gerlach apparatus and whose initial spin is given by $(\theta_0^A, \varphi_0^A)$.

For an initial polarization $(\theta_0^A, \varphi_0^A)$ and an initial position z_0^A, we obtain, in the de Broglie-Bohm interpretation [17] of the Stern and Gerlach experiment, an evolution of the position $z_{A_\pm}(t)$ and of the spin orientation of **A** $_\pm, \theta^{A_\pm}(z_{A_\pm}(t), t)$, see [19].

The case of particles **B**$_\pm$ is different. **B**$_\pm$ follows a rectilinear trajectories with $y_{B_\pm}(t) = \mathbf{v}_y^\pm(\mathbf{v}_0, \theta_0)t$, $z_{B_\pm}(t) = z_0^B$ and $x_{B_\pm}(t) = x_0^B$. By contrast, the orientation of its spin moves with the orientation of the spin of **A**$_\pm$:

$$\theta^{\mathbf{B}_\mp}(t) = \pi - \theta^{\mathbf{A}_\pm}(z_{\mathbf{A}_\pm}(t), t) \qquad (5.2.21)$$

and

$$\varphi^{\mathbf{B}_\mp}(t) = \varphi^{\mathbf{A}_\pm}(z_{\mathbf{A}_\pm}(t), t) - \pi. \qquad (5.2.22)$$

Remark 5.2.9. Let $\mathbf{A}_\pm(t, \mathbf{r}_{\mathbf{A}_\pm}(t), \theta^{\mathbf{A}_\pm}(z_{\mathbf{A}_\pm}(t), t), \varphi^{\mathbf{A}_\pm}(z_{\mathbf{A}_\pm}(t), t))$ denote events such that: "at instant t particles **A**$_\pm$ obtain the position coordinates $\mathbf{r}_{\mathbf{A}_\pm}(t) = \{x_{\mathbf{A}_\pm}(t), y_{\mathbf{A}_\pm}(t), z_{\mathbf{A}_\pm}(t)\}$ and spin orientation $\theta^{\mathbf{A}_\pm}(t) = \theta^{\mathbf{A}_\pm}(z_{\mathbf{A}_\pm}(t), t)$ and $\varphi^{\mathbf{A}_\pm}(t) = \varphi^{\mathbf{A}_\pm}(z_{\mathbf{A}_\pm}(t), t)$. Let $\mathbf{B}_\mp(t, \mathbf{r}_{\mathbf{B}_\mp}(t), \theta^{\mathbf{B}_\mp}(z_{\mathbf{B}_\mp}(t), t))$ denote events such that: "at instant t particles $\mathbf{B}_\mp$ obtain the position coordinates $\mathbf{r}_{\mathbf{B}_\mp}(t) = \{x_{\mathbf{B}_\mp}(t), y_{\mathbf{B}_\mp}(t), z_{\mathbf{B}_\mp}(t)\}$ and spin orientation $\theta^{\mathbf{B}_\mp} = \theta^{\mathbf{B}_\mp}(z_{\mathbf{B}_\mp}(t), t)$ and $\varphi^{\mathbf{B}_\mp}(z_{\mathbf{B}_\mp}(t), t)$.

Then in accordance with the relaxed principle of locality (see subsection IV.1) we assume that

$$\{\mathbf{A}_{\pm}(t_1,\mathbf{r}_{\mathbf{A}_{\pm}}(t),\theta^{\mathbf{A}_{\pm}},\varphi^{\mathbf{A}_{\pm}}(t)),\mathbf{B}_{\mp}(t,\mathbf{r}_{\mathbf{B}_{\mp}}(t),\theta^{\mathbf{B}_{\mp}}(t)),\varphi^{\mathbf{B}_{\mp}}(t)\}_{\mathbf{s.l.s.}} \in$$
$$\in [\mathcal{F}^{\#}_{M_4},\{(t_1,\mathbf{r}_1),(t_2,\mathbf{r}_2)\}]^{\Leftrightarrow}_{\mathbf{s.l.s.}}, \qquad (5.2.23)$$

see subsection IV.1, Definition 4.1.2. We can then associate the wave functions:

$$\Psi^{B_+}(\mathbf{r}_{B_+},t_0+\triangle t+t) = f(\mathbf{r}_{B_+})\cos\frac{\theta^{B_+}(t)}{2}|+_B\rangle \qquad (5.2.24)$$

and

$$\Psi^{B_-}(\mathbf{r}_{B_-},t_0+\triangle t+t) = f(\mathbf{r}_{B_-})\sin\frac{\theta^{B_-}(t)}{2}e^{i\varphi^{B_-}(t)}|-_{B_-}\rangle \qquad (5.2.25)$$

These wave functions are specific, because they depend upon initial conditions of **A** (position and spin). The orientation of spin of the particles $\mathbf{B}_{\pm}$ is driven by the particles $\mathbf{A}_{\mp}$ respectively through the singlet wave functions.

V.2.3.2. Step 2: spin measurement of $\mathbf{B}_{\mp}$ in de Broglie-Bohm interpretation

V.2.3.2.1. The prediction of the result of the spin measurement of $\mathbf{B}_{\mp}$ under assumption of canonical postulate of locality

At the time $t_0+\Delta t+t_D$, immediately after the measurement of **A**, $\theta^{B_{\mp}}(t_0+\Delta t+t_D)=\pi$ or 0 in accordance with the value of $\theta^{A_{\pm}}(z_{A_{\pm}}(t),t)$ and the wave functions of $\mathbf{B}_{\mp}$ are given by Eq. (5.2.16a) and Eq.(5.2.16b) respectively.

The frame $(Ox'yz')$ corresponds to the frame $(Oxyz)$ after a rotation of an angle δ around the y-axis (see Fig. 5.1.1). $\theta^{B_{\pm}}$ corresponds to the

B$_{\mp}$-spin angle with the z-axis, and $\theta'^{B_{\pm}}$ to the **B**-spin angle with the z' -axis, then $\theta'^{B_{\pm}}(t_0 + \Delta t + t_D) = \pi + \delta$ or δ.

In this second step, we are exactly in the case of a particle in a simple Stern and Gerlach experiment (with magnet $\mathbf{E_B}$) with a specific initial polarization equal to $\pi + \delta$ or δ and not random like in step 1.

Then, the measurement of **B**, at time $t_0 + 2(\triangle t + t_D)$), gives, in this interpretation of the two-step version of the EPR-B experiment, the same results as in the EPR-B experiment above. Thus we obtain EPR-B paradox again in de Broglie-Bohm interpretation.

Remark 5.2.10. Note that the derivation EPR-B paradox in the de Broglie-Bohm interpretation completely based on the canonical postulate of locality.

V.2.3.2.2. The prediction of the result of the spin measurement of B$_{\mp}$ under assumption of postulate of nonlocality

We assume now a weak or strong postulate of nonlocality, see subsections I.4.1-I.4.2. At the time $t_1 = t_0 + \Delta t + t_D$, immediately after the spin measurement of **A**$_{\pm}$, $\theta^{B_{\mp}}(t_0 + \Delta t + t_D) = \pi$ or 0 in accordance with the value of $\theta^{A_{\pm}}(z_{A_{\pm}}(t), t)$ and the wave functions of **B**$_{\mp}$ are given by Eq. (5.2.16a) and Eq.(5.2.16b) respectively.

Remark 5.2.11. In accordance with the postulate of nonlocality it follows:

(**i**) Whenever a measurement of the spin of a particle **A**$_{+}$ is performed at instant t_1 and particle **A**$_{+}$ is found in the state $|\uparrow\rangle_z$, i.e., a state $|\psi_{t_1}\rangle_{\mathbf{A}_+}$ collapses at instant t_1 to the state $|\uparrow\rangle_{z,\mathbf{A}_+}$ with respect of the Heisenberg spin uncertainty relations, then a state $|\psi_{t_1}\rangle_{\mathbf{B}_-}$

immediately collapses at instant t_1 to the state $|\downarrow\rangle_{z,\mathbf{B}_-}$ with respect of the Heisenberg spin uncertainty relations, and this is true independent of the distance in Minkowski spacetime that separates the particles, e.g.,

$$|\psi_{t_1}\rangle_{\mathbf{A}_+} \overset{\text{collapse}}{\rightarrow} |\uparrow\rangle_{z,\mathbf{A}_+} \Rightarrow |\psi_{t_1}\rangle_{\mathbf{B}_-} \overset{\text{collapse}}{\rightarrow} |\downarrow\rangle_{z,\mathbf{B}_-} \quad (5.2.26)$$

In accordance with Heisenberg spin uncertainty relations (1.4.5) spin of a particle $\mathbf{B}_-$ obtains an uncertainty along direction Oz' (see Fig. 5.1.1) and therefore EPR-B paradox disappears.

(**ii**) Whenever a measurement of the spin of a particle **A** is performed at instant t_1 and particle $\mathbf{A}_-$ is found in the state $|\downarrow\rangle_z$, i.e., a state $|\psi_{t_1}\rangle_{\mathbf{A}_-}$ collapses at instant t_1 to the state $|\downarrow\rangle_{z,\mathbf{A}_-}$ with respect of the Heisenberg spin uncertainty relations (1.4.5), then a state $|\psi_{t_1}\rangle_{\mathbf{B}_+}$ immediately collapses at instant t_1 to the state $|\uparrow\rangle_{z,\mathbf{B}_+}$ with respect of the Heisenberg spin uncertainty relations (1.4.5), and this is true independent of the distance in Minkowski spacetime that separates the particles, e.g.,

$$|\psi_{t_1}\rangle_{\mathbf{A}_-} \overset{\text{collapse}}{\rightarrow} |\downarrow\rangle_{z,\mathbf{A}_-} \Rightarrow |\psi_{t_1}\rangle_{\mathbf{B}_+} \overset{\text{collapse}}{\rightarrow} |\uparrow\rangle_{z,\mathbf{B}_+}. \quad (5.2.27)$$

In accordance with Heisenberg spin uncertainty relations (1.4.5) spin of a particle $\mathbf{B}_-$ obtains an uncertainty along direction Oz' (see Fig. 5.1.1) and therefore EPR-B paradox disappears.

V.2.4. Physical explanation of non-local influences using the relaxed principle of locality

From the wave function of two entangled particles, we find spins, trajectories and also a wave function for each of the two particles. In this interpretation, the quantum particle has a local position like a classical particle, but it has also a non-local behavior through the wave function. So, it is the wave function that creates the non classical properties. We can keep a view of a local realist world for the particle, but we should add a non-local vision through the wave function. As we saw in step 1, the non-local influences in the EPR-B experiment only concern the spin orientation, not the motion of the particles themselves. Indeed only spins are entangled in the wave function but not positions and motions like in the initial EPR experiment. This is a key point in the search for a physical explanation of non-local influences.

V.3. EPR-B paradox resolution by using quantum mechanical formalism based on the probability representation of quantum states

V.3.1. Preleminaries

We remind now EPR-B argument [17] in original D. Bohm formulation: "The Hypothetical Experiment of Einstein, Rosen, and Podolsky. We shall now describe the hypothetical experiment of Einstein, Rosen, and Podolsky. We have modified the experiment somewhat, but the form is conceptually equivalent to that suggested by them, and considerably easier to treat mathematically.

Suppose that we have a molecule containing two atoms in a state in which the total spin is zero and that the spin of each atom is $\hbar/2$. Roughly speaking, this means that the spin of each particle points in a direction exactly opposite to that of the other, insofar as the spin may be said to have any definite direction at all. Now suppose that the molecule is disintegrated by some process that does not change the total angular momentum. The two atoms will begin to separate and will soon cease to interact appreciably. Their combined spin angular momentum, however, remains equal to zero, because by hypothesis, no torques have acted on the system.

Now, if the spin was a classical angular momentum variable, the interpretation of this process would be as follows: while the two atoms were together in the form of a molecule, each component of the angular momentum of each atom would have a definite value that was always opposite to that of the other, thus making the total angular momentum equal to zero. When the atoms separated, each atom would continue to have every component of its spin angular momentum opposite to that of the other. The two spin-angular-momentum vectors would therefore be correlated. These correlations were originally produced when the atoms interacted in such a way as to form a molecule of zero total spin, but after the atoms separate, the correlations are maintained by the deterministic equations of motion of each spin vector separately, which bring about conservation of each component of the separate spin-angular-momentum vectors.

Suppose now that one measures the spin angular momentum of any one of the particles, say No. 1. Because of the existence of correlations, one can immediately conclude that the angular-momentum vector of the other particle (No. 2) is equal and opposite to that of No. 1. In this way, one can measure the angular momentum of particle No. 2 indirectly by measuring the corresponding vector of particle No. 1.

Let us now consider how this experiment is to be described in the quantum theory. Here, the investigator can measure either the $x, y,$ or z component of the spin of particle No. 1, but not more than one of these components, in any one experiment. Nevertheless, it still turns out as we shall see that whichever component is measured, the results are correlated, so that if the same component of the spin of atom No. 2 is measured, it will always turn out to have the opposite value. This means that a measurement of any component of the spin of atom No. 1 provides, as in classical theory, an indirect measurement of the same component of the spin of atom No. 2. Since, by hypothesis, the two particles no longer interact, we have obtained a way of measuring an arbitrary component of the spin of particle No. 2 without in any way disturbing that particle. If we accept the definition of an element of reality suggested by ERP, it is clear that after we have measured σ_z for particle 1, then σ_z for particle 2 must be regarded as an element of reality; existing separately in particle No. 2 alone. If this is true, however, this element of reality must have existed in particle No. 2 even before the measurement of σ_z at for particle No. 1 took place. For since there is no interaction with particle No. 2, the process of measurement cannot have affected this particle in any way. But now let us remember that, in each case, the observer is always free to reorient the apparatus in an arbitrary direction while the atoms are still in flight, and thus to obtain a definite (but unpredictable) value of the spin component in any direction that he chooses. Since this can be accomplished without in any way disturbing the second atom, we conclude that if criterion of ERP is applicable, precisely defined elements of reality must exist in the second atom, corresponding to the simultaneous definition of all three components of its spin. Because the wave function can specify, at most, only one of these components at a time with complete precision, we are then led to the conclusion that the wave function does not provide a complete description of all elements of reality existing in the second atom."

Actually, most experiments have been performed using polarization of photons. The quantum state of the pair of entangled photons is not the singlet state.The polarization of a photon is measured in a pair of perpendicular directions. Relative to a given orientation, polarization is either vertical (denoted by V or by $+$) or horizontal (denoted by H or by $-$). The photon pairs are generated in the quantum state

$$|\psi_{\mathrm{EPRB}}\rangle = \frac{1}{\sqrt{2}}(|V\rangle_s \otimes |V\rangle_i + |H\rangle_s \otimes |H\rangle_i) \qquad (5.3.1)$$

where $|V\rangle$ and $|H\rangle$ denotes the state of a single vertically or horizontally polarized photon, respectively, relative to a fixed and common reference direction for both particles.This state cannot be factored into a simple product of signal and idler states:

$$|\psi_{\mathrm{EPRB}}\rangle \neq |A\rangle_s \otimes |B\rangle_i \qquad (5.3.2)$$

for any choice of $|A\rangle_s$ and $|B\rangle_i$. This means the state of one particle cannot be specified without making reference to the other particle. Such particles are said to be entangled and $|\psi_{\text{EPRB}}\rangle$ is an entangled state. If we measure the polarizations of signal and idler photons in the H, V basis there are two possible outcomes: both vertical or both horizontal. Each occurs half of the time. We could instead measure the polarizations with polarizers rotated by an angle α. We use the rotated polarization basis

$$\begin{aligned} |V_\alpha\rangle &= \cos\alpha|V\rangle - \sin\alpha|H\rangle, \\ |H_\alpha\rangle &= \sin\alpha|V\rangle + \cos\alpha|H\rangle, \end{aligned} \tag{5.3.3}$$

where $|V_\alpha\rangle$ describes a state with polarization rotated by α from the vertical, while $|H_\alpha\rangle$ is α from the horizontal. In this basis the state is

$$|\psi_{\text{EPRB}}\rangle = \frac{1}{\sqrt{2}}(|V_\alpha\rangle \otimes |V_\alpha\rangle + |H_\alpha\rangle \otimes |H_\alpha\rangle). \tag{5.3.4}$$

Remark 5.3.1. After the signal photon is measured the idler is equally likely to be V_α or H_α. A measurement of its polarization, at any angle β, finds V_β or H_β with probability

$$\begin{aligned} P_{V_\beta}(\alpha, \beta) &= \frac{1}{2}|\langle V_\beta|V_\alpha\rangle|^2 + \frac{1}{2}|\langle V_\beta|H_\alpha\rangle|^2 = \\ &= \frac{1}{2}[\cos^2(\beta-\alpha) + \sin^2(\beta-\alpha)] = \frac{1}{2}, \\ P_{H_\beta}(\alpha, \beta) &= \frac{1}{2}|\langle H_\beta|V_\alpha\rangle|^2 + \frac{1}{2}|\langle H_\beta|H_\alpha\rangle|^2 = \frac{1}{2}. \end{aligned} \tag{5.3.5}$$

Remark 5.3.2. Let $(\Omega, \mathcal{F}, \nu)$ be a measure space and f be a Borel function. Note that

$$\lambda(A) = \int_A f d\nu, A \in \mathcal{F} \tag{5.3.6}$$

is a signed measure satisfying $\nu(A) = 0 \Rightarrow \lambda(A)$. We say λ is absolutely continuous with respect to (w.r.t.) ν and write $\lambda \ll \nu$.

Theorem (Radon-Nikodym). Let X be a set, let $\mathcal{F}$ be a σ-algebra of subsets of X, let ν be a σ-nite measure defined on $\mathcal{F}$, and let λ be a signed defined on $\mathcal{F}$. Suppose that $\lambda << \nu$. Then there exists a function $f : X \to \mathbb{R}$ on X that is integrable w.r.t. the measure ν and that satisfies Eq. (5.3.6) for all $A \in \mathcal{F}$. Moreover any two functions with this property are equal almost everywhere (a.e.) on X.

Remark 5.3.3. If $\int_X f d\nu = 1$ for any $f \geq 0$ a. e. ν, then λ is a probability measure and f is called its probability density function (p.d.f.) w.r.t. ν.

Remark 5.3.4. Remind that: (i) A random variable $X(\omega)$ is a measurable function from a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ to the reals $\mathbb{R}$, i.e., it is a function $X : \Omega \to \mathbb{R}$ such that for every Borel set $B \in \mathbf{B}^\infty : X^{-1}(B) = \{X \in B\} \in \mathcal{F}$, where we use the shorthand notation $\{X \in B\} \triangleq \{\omega \in \Omega | X(\omega) \in B\}$ and where $\mathbf{B}^\infty$ is Borel algebra of the all Borel subset B of $\mathbb{R}$.

(ii) If $X(\omega)$ is a random variable, then for every Borel subset B of of $\mathbb{R}, X^{-1}(B) \in \mathcal{F}$ and we can define a function on Borel sets by

$$\mathbf{P}_X(B) = \mathbf{P}[X^{-1}(B)]. \tag{5.3.7}$$

(iii) This function $\mathbf{P}_X : \mathbf{B}^\infty \to \mathbb{R}$ is in fact a probability measure, and and $(\mathbb{R}, \mathbf{B}^\infty, X)$ is a probability space.

(iv) The measure $\mathbf{P}_X$ is called the distribution of the random variable $X(\omega)$. If $X(\omega)$ gives measure one to a countable set of reals, then $X(\omega)$ is called a discrete random variable.

(v) Let $X(\omega)$ be a discrete random variable.Then the probability mass function $f_X : A(A \subseteq \mathbb{R}) \to [0,1]$ for $X(\omega)$ is defined as

$$f_X(x) = \mathbf{P}\{\omega \in \Omega | X(\omega) = x\}. \tag{5.3.8}$$

The total probability for all hypothetical outcomes $x : \sum_{x \in A} f_X(x) = 1.$

(vi) Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space mentioned above, see Remark 5.3.4. The distribution $\mathbf{P}_X$ is often given in terms of the cumulative distribution function (c.d.f.) F_X defined by $F_X(x) = \mathbf{P}\{X \le x\}$.

Remark 5.3.5. For any probability measure $\mathbf{P}_X$ on $(\mathbb{R}, \mathbf{B}^\infty)$ corresponding to a random variable X (or to a c.d.f. F_X) if $\mathbf{P}_X$ has a p.d.f. p_X w.r.t. a measure $\mathbf{P}$, then p_X is also called the p.d.f. of F_X or X w.r.t. $\mathbf{P}$.

Remark 5.3.6. (Discrete c.d.f. and p.d.f.). Let $a_1 < a_2 <\ldots< a_n$ be a sequence of real numbers and let $p_i, i = 1,2,\ldots,n$ be a sequence of positive numbers such that $\sum_{i=1}^n p_i = 1$. Then

$$F(x) = \begin{cases} \sum_{i=1}^m p_i & a_i \le x \le a_{i+1}, i = 1,2,\ldots,n-1 \\ 0 & -\infty < x < a_1 \end{cases} \tag{5.3.9}$$

is a stepwise c.d.f. It has a jump of size p_i at each a_i and is flat between a_i and a_{i+1}. Such a c.d.f. is called a discrete c.d.f. The corresponding probability measure $\mathbf{P}_F$ is

$$\mathbf{P}_F(A) = \sum_{\{i|a_i \in A\}} p_i, A \in \mathcal{F}. \tag{5.3.10}$$

Remark 5.3.7. Remind that the counting measure ν_Σ on a measurable space (Ω, Σ) is the positive measure

$$\nu_\Sigma(A) = \begin{cases} |A| & \text{if } A \text{ is finite} \\ +\infty & \text{if } A \text{ is infinite} \end{cases} \tag{5.3.11}$$

for all $A \in \Sigma$, where $|A|$ denotes the cardinality of the set A.

Let $\nu_{\mathcal{F}}$ be the counting measure on $\mathcal{F}$. Then

$$\mathbf{P}_F(A) = \int_A f_F d\nu_{\mathcal{F}} = \sum_{a_i \in A} f_F(a_i), A \subset \Omega, \tag{5.3.12}$$

where $f_F(a_i) = p_i, i = 1,2,\ldots$. That is, f_F is the p.d.f. of $\mathbf{P}_F$ or F w.r.t. v. Hence, any discrete c.d.f. has a p.d.f. w.r.t. counting measure. A p.d.f. w.r.t. counting measure is called a discrete p.d.f.

Definition 5.3.1. Let $(\Omega,\mathcal{F},\mathbf{P})$ be a probability space and X is a discrete random variable $X:\Omega\to\mathbb{R}$. The conditional probability of event $A\in\mathcal{F}$ given (X,y) is defined as

$$\mathbf{P}(A|X=y)=\frac{\mathbf{P}[A\wedge(X=y)]}{\mathbf{P}(X=y)}=\frac{\mathbf{P}\left[A\cap\left(X^{-1}(y)\right)\right]}{\mathbf{P}\left(X^{-1}(y)\right)}, \qquad (5.3.13)$$

where $\mathbf{P}\left(X^{-1}(y)\right)>0.$

Definition 5.3.2. (**I**) Let $X_{|V_\alpha\rangle}(\omega)$ be a discrete random variable $X_{|V_\alpha\rangle}:\Omega\to\mathbb{R}$ with the probability mass function (see Remark 5.3.4.v) $f_{X_{|V_\alpha\rangle}}$ defined by

$$f_{X_{|V_\alpha\rangle}}(x)=\begin{cases}\frac{1}{2} & \text{if } x=1\\ \frac{1}{2} & \text{if } x=0\end{cases} \qquad (5.3.14)$$

Thus there exist: (i) $\Delta_{1,\alpha}\subsetneqq\Omega$ such that $\Delta_{1,\alpha}=X^{-1}_{X_{|V_\alpha\rangle}}(1)$ and $\mathbf{P}(\Delta_{1,\alpha})=1/2,$

(ii) $\Delta_{2,\alpha}\subsetneqq\Omega$ such that $\Delta_2=X^{-1}_{X_{|V_\alpha\rangle}}(0)$ and $\mathbf{P}(\Delta_{2,\alpha})=1/2,$

(iii) $\Delta_{1,\alpha}=\Omega\backslash\Delta_{2,\alpha}\,\mathrm{mod}(\Lambda),\mathbf{P}(\Lambda)=0.$

(**II**) Let $X_{|H_\alpha\rangle}(\omega)$ be a discrete random variable $X_{|H_\alpha\rangle}:\Omega\to\mathbb{R}$ such that

$$\forall\omega(\omega\in\Delta_{1,\alpha})[X_{|H_\alpha\rangle}(\omega)=1-X_{|V_\alpha\rangle}(\omega)]. \qquad (5.3.15)$$

Therefore $\Delta_{1,\alpha}=X^{-1}_{X_{|H_\alpha\rangle}}(0),\Delta_{2,\alpha}=X^{-1}_{X_{|H_\alpha\rangle}}(1)$ and the probability mass function $f_{X_{|H_\alpha\rangle}}$ is

$$f_{X_{|H_\alpha\rangle}}(x)=\begin{cases}\frac{1}{2} & \text{if } x=1\\ \frac{1}{2} & \text{if } x=0\end{cases} \qquad (5.3.16)$$

Remark 5.3.8. Let $\mathbf{B}_{M_4}$ be a Boolean algebra of physical events in Minkowski spacetime and let $\mathcal{F}^{ph}_{M_4}=(\mathbf{B}_{M_4},\mathbf{P})$ be measure algebra of physical events in Minkowski spacetime, i.e., $\mathcal{F}^{ph}_{M_4}$ that is a Boolean algebra $\mathbf{B}_{M_4}$ with a probability measure $\mathbf{P}$, see Chapter III subsection III.2, Definition 3.2.3.

We remind that we denote such physical events by $A(\mathbf{x}), B(\mathbf{x}), \ldots$ etc., where $\mathbf{x} = (t, x_1, x_2, x_3) \in M_4$ or $A, B, \ldots$ etc., and we write for a short $A^{Oc}(\mathbf{x}), B^{Oc}(\mathbf{x}), \ldots$ iff there physical events $A(\mathbf{x}), B(\mathbf{x}) \ldots$ were occured.

Definition 5.3.3. Let σ_α be the measurement operator corresponding to measurments the photon polarization (see Appendix A) in polarization basis $\{|V_\alpha\rangle, |H_\alpha\rangle\}$, see Eq. (5.3.3). Let $A(|\psi_A^\alpha\rangle, \sigma_\alpha, t) \in \mathbf{B}_{M_4}$ be a physical event which consists on performing a measurement with absolute certainty of the observable σ_α at instant t.

Remark 5.3.9. We assume that:

(i) particle A is initially in the state $|\psi_A^\alpha\rangle = \frac{1}{\sqrt{2}}(|V_\alpha\rangle + |H_\alpha\rangle)$,

(ii) $A(|\psi_A^\alpha\rangle, \sigma_\alpha, t) \in \mathcal{F}^{ph}_{M_4}$.

Remark 5.3.10. Note that: (i) if there physical event $A(|\psi_A^\alpha\rangle, \sigma_\alpha, t)$ was occurred then immediately after the measurement at the instant t a particle A will be in the state $|V_\alpha^A\rangle = |V_\alpha\rangle$ or in the state $|H_\alpha^A\rangle = |H_\alpha\rangle$,

(ii) immediately after the measurement at the instant t the unconditional measure $\mathbf{P}$ collapses to the conditional measure $\mathbf{P}\left(X \mid A^{Oc}(|\psi_A^\alpha\rangle, \sigma_\alpha, t)\right)$:

$$\mathbf{P}\left(X \mid A^{Oc}(|\psi_A^\alpha\rangle, \sigma_\alpha, t)\right) = \frac{\mathbf{P}(X \wedge A^{Oc}(|\psi_A^\alpha\rangle, \sigma_\alpha, t))}{\mathbf{P}(A^{Oc}(|\psi_A^\alpha\rangle, \sigma_\alpha, t))}, \qquad (5.3.17)$$

where $X \in \mathcal{F}^{ph}_{M_4}$.

Remark 5.3.11. Let $A(|\psi_A\rangle, t)$ be a physical event which consists that a particle A at the instant t is in the state $|\psi_A\rangle$. Note that:

(i) $A^{Oc}(|\psi_A^\alpha\rangle, \sigma_\alpha, t) \Rightarrow A^{Oc}(|V_\alpha^A\rangle, t) \vee A^{Oc}(|H_\alpha^A\rangle, t)$,

(ii) $A^{Oc}(|V_\alpha^A\rangle,t) \wedge A^{Oc}(|H_\alpha^A\rangle,t) \notin \mathcal{F}_{M_4}^{ph}$,

(iii) from (i), (ii) and (5.3.17) it follows that:

$$\mathbf{P}\big(X \,|A^{Oc}(|\psi_A^\alpha\rangle,\sigma_\alpha,t)\big) = \mathbf{P}\big(X \,|A(|V_\alpha^A\rangle,t)\big) = \frac{\mathbf{P}(X \wedge A(|V_\alpha^A\rangle,t))}{\mathbf{P}(A(|V_\alpha^A\rangle,t))} \quad (5.3.18)$$

or

$$\mathbf{P}\big(X \,|A^{Oc}(|\psi_A^\alpha\rangle,\sigma_\alpha,t)\big) = \mathbf{P}\big(X \,|A(|H_\alpha^A\rangle,t)\big) = \frac{\mathbf{P}(X \wedge A(|H_\alpha^A\rangle,t))}{\mathbf{P}(A(|H_\alpha^A\rangle,t))}, \quad (5.3.19)$$

where $X \in \mathcal{F}_{M_4}^{ph}$.

Remark 5.3.12. We assume now that: (i) a measure algebra $\mathcal{F}_{M_4}^{ph} = (\mathbf{B}_{M_4},\mathbf{P})$ admits a representation $\Re[\bullet] : \mathcal{F}_{M_4}^{ph} \to (\mathbb{R},\mathbf{B},\mathbf{P_B})$ of the measure algebra $\mathcal{F}_{M_4}^{ph} = (\mathbf{B}_{M_4},\mathbf{P})$ in the measure algebra $\mathbf{B} \triangleq (\mathbb{R},\mathbf{B},\mathbf{P_B})$, such that

(ii) $\mathbf{P}_{B^\infty}(X) = \mathbf{P}(\Re^{-1}[X])$ for any $X \in \mathbf{B}$ and

(iii) for physical events $A((|V_\alpha^A\rangle,t)) \in \mathcal{F}_{M_4}^{ph}$ and $A((|H_\alpha^A\rangle,t)) \in \mathcal{F}_{M_4}^{ph}$ (see Remark 5.3.10**)** the following conditions hold

$$\Re[A^{Oc}(|V_\alpha^A\rangle,t)] = \Delta_{1,\alpha} \quad (5.3.20)$$

and

$$\Re[A^{Oc}(|H_\alpha^A\rangle,t)] = \Delta_{2,\alpha}, \quad (5.3.21)$$

where $\Delta_{1,\alpha} = X^{-1}_{X_{|V_\alpha\rangle}}(1)$ and $\Delta_{2,\alpha} = X^{-1}_{X_{|H_\alpha\rangle}}(1)$, see Definition 5.3.2.

Remark 5.3.13. We note that the product of the noise in a polarization measurement in the polarization basis $\{|V_\alpha^A\rangle, |H_\alpha^A\rangle\}$ and the polarization disturbance in the polarization basis $\{|V_\beta^A\rangle, |H_\beta^A\rangle\}$ caused by that measurement should be no less than $\Gamma(\alpha,\beta)$:

$$\Gamma(\alpha,\beta) = \frac{1}{2}|\langle\psi_A^\alpha|[\sigma_\alpha,\sigma_\beta]|\psi_A^\alpha\rangle| > 0, \quad (5.3.22)$$

see Appendix A. Note that Heisenberg's noise-disturbance uncertainty relation for the case of the polarization measurement is: for any apparatus $\mathbf{A}$ to measure an observable σ_α, the relation

$$\epsilon(\sigma_\alpha, |\psi_A^\alpha\rangle, \mathbf{A})\eta(\sigma_\beta, |\psi_A^\alpha\rangle, \mathbf{A}) \geq \frac{1}{2}|\langle\psi_A^\alpha|[\sigma_\alpha, \sigma_\beta]|\psi_A^\alpha\rangle| = \Gamma(\alpha, \beta) \tag{5.3.23}$$

holds for any input state $|\psi_A^\alpha\rangle$ and any observable σ_β, where $\epsilon(\sigma_\alpha, \psi_A^\alpha, \mathbf{A})$ stands for the noise of the σ_α measurement in the state ψ_A^α using apparatus **A** and $\eta(\sigma_\beta, \psi_A^\alpha, \mathbf{A})$ stands for the disturbance of σ_β in the state ψ_A^α caused by apparatus **A.** We refer to the above relation as the Heisenberg noise-disturbance uncertainty relation in polarization measurements, see Appendix A.

Remark 5.3.14. We use the rotated polarization basis $\{|V_{\gamma\pm\delta_\gamma}\rangle, |H_{\gamma\pm\delta_\gamma}\rangle\}$ with uncertainty $\delta_\gamma = \epsilon(\sigma_\gamma, |\psi_A^\alpha\rangle, \mathbf{A})$

$$\begin{gathered}
|V_{\gamma\pm\delta_\gamma}\rangle \triangleq \{|V_{\widetilde{\gamma}}\rangle | \gamma - \delta_\gamma \leq \widetilde{\gamma} \leq \gamma + \delta_\gamma\}, \\
|V_{\widetilde{\gamma}}\rangle = (\cos\widetilde{\gamma})|V\rangle - (\sin\widetilde{\gamma})|H\rangle, \\
\\
|H_{\gamma\pm\delta_\gamma}\rangle \triangleq \{|H_{\widetilde{\gamma}}\rangle | \gamma - \delta_\gamma \leq \widetilde{\gamma} \leq \gamma + \delta_\gamma\}, \\
|H_{\widetilde{\gamma}}\rangle = (\sin\widetilde{\gamma})|V\rangle + (\cos\widetilde{\gamma})|H\rangle
\end{gathered} \tag{5.3.24}$$

where $|V_{\gamma\pm\delta_\gamma}\rangle$ describes states with polarization rotated by any $\widetilde{\gamma} \in [\gamma - \delta_\gamma, \gamma + \delta_\gamma]$ from the vertical, while $|H_{\gamma\pm\delta_\gamma}\rangle$ is $\widetilde{\gamma}$ from the horizontal.

We abbreviate for short

$$\begin{gathered}
|V_{\gamma\pm\delta_\gamma}\rangle = \cos(\gamma \pm \delta_\gamma)|V\rangle - \sin(\gamma \pm \delta_\gamma)|H\rangle, \\
|H_{\gamma\pm\delta_\gamma}\rangle = \sin(\gamma \pm \delta_\gamma)|V\rangle + \cos(\gamma \pm \delta_\gamma)|H\rangle,
\end{gathered}$$

Remark 5.3.15. After the signal photon is measured the idler is equally likely to be $V_{\alpha\pm\delta_\alpha}$ or $H_{\alpha\pm\delta_\alpha}$. A measurement of its polarization, at any angle $\widetilde{\beta} : \beta - \delta_\beta \leq \widetilde{\beta} \leq \beta + \delta_\beta$, finds $V_{\widetilde{\beta}}$ or $H_{\widetilde{\beta}}$ with probability

$$P_{V_{\widetilde{\beta}}}(\alpha,\widetilde{\beta}) = \frac{1}{2}|\langle V_{\beta\pm\delta_\beta}|V_{\alpha\pm\delta_\alpha}\rangle|^2 + \frac{1}{2}|\langle V_{\beta\pm\delta_\beta}|H_{\alpha\pm\delta_\alpha}\rangle|^2 =$$
$$= \frac{1}{2}\{\cos^2[(\beta\pm\delta_\beta) - (\alpha \pm \delta_\alpha)] + \sin^2[(\beta\pm\delta_\beta) - (\alpha \pm \delta_\alpha)]\} = \frac{1}{2}, \quad (5.3.25)$$
$$P_{H_{\widetilde{\beta}}}(\alpha,\widetilde{\beta}) = \frac{1}{2}|\langle H_{\beta\pm\delta_\beta}|V_{\alpha\pm\delta_\alpha}\rangle|^2 + \frac{1}{2}|\langle H_{\beta\pm\delta_\beta}|H_{\alpha\pm\delta_\alpha}\rangle|^2 = \frac{1}{2}.$$

where $\delta_\alpha = \epsilon(\sigma_\alpha, |\psi_A^\alpha\rangle, \mathbf{A}), \delta_\beta = \eta\left(\sigma_\beta, \left|\psi_A^{\alpha\pm\delta_\alpha}\right\rangle, \mathbf{A}\right)$ and $\left|\psi_A^{\alpha\pm\delta_\alpha}\right\rangle = \frac{1}{\sqrt{2}}(|V_{\alpha\pm\delta_\alpha}\rangle + |H_{\alpha\pm\delta_\alpha}\rangle)$.

Definition 5.3.4. (**I**) Let $X_{|V_{\alpha\pm\delta_\alpha}\rangle}(\omega)$ be a discrete random variable $X_{|V_{\alpha\pm\delta_\alpha}\rangle} : \Omega \to \mathbb{R}$ with the probability mass function (see Remark 5.3.4.v) $f_{X_{|V_{\alpha\pm\delta_\alpha}\rangle}}$ defined by

$$f_{X_{|V_{\alpha\pm\delta_\alpha}\rangle}}(x) = \begin{cases} \frac{1}{2} & \text{if } x = 1 \\ \frac{1}{2} & \text{if } x = 0 \end{cases} \quad (5.3.26)$$

Thus there exist: (i) $\Delta_{1,\alpha\pm\delta_\alpha} \subsetneqq \Omega$ such that $\Delta_{1,\alpha\pm\delta_\alpha} = X^{-1}_{X_{|V_{\alpha\pm\delta_\alpha}\rangle}}(1)$ and $\mathbf{P}(\Delta_{1,\alpha\pm\delta_\alpha}) = 1/2$,

(ii) $\Delta_{1,\alpha\pm\delta_\alpha} \subsetneqq \Omega$ such that $\Delta_{2,\alpha\pm\delta_\alpha} = X^{-1}_{X_{|V_{\alpha\pm\delta_\alpha}\rangle}}(0)$ and $\mathbf{P}(\Delta_{2,\alpha\pm\delta_\alpha}) = 1/2$,

(iii) $\Delta_{1,\alpha\pm\delta_\alpha} = \Omega\backslash\Delta_{2,\alpha\pm\delta_\alpha} \operatorname{mod}(\Lambda), \mathbf{P}(\Lambda) = 0$.

Thus there exist: (i) $\Delta_{1,\alpha\pm\delta_\alpha} \subsetneqq \Omega$ such that $\Delta_{1,\alpha\pm\delta_\alpha} = X^{-1}_{X_{|V_\alpha\rangle}}(1)$ and $\mathbf{P}(\Delta_{1,\alpha\pm\delta_\alpha}) = 1/2$,

(ii) $\Delta_{2,\alpha\pm\delta_\alpha} \subsetneqq \Omega$ such that $\Delta_{2,\alpha\pm\delta_\alpha} = X^{-1}_{X_{|V_\alpha\rangle}}(0)$ and $\mathbf{P}(\Delta_{2,\alpha\pm\delta_\alpha}) = 1/2$,

(iii) $\Delta_{1,\alpha\pm\delta_\alpha} = \Omega\backslash\Delta_{2,\alpha\pm\delta_\alpha} \operatorname{mod}(\Lambda), \mathbf{P}(\Lambda) = 0$.

(**III**) Let $X_{|H_{\alpha\pm\delta_\alpha}\rangle}(\omega)$ be a discrete random variable $X_{|H_{\alpha\pm\delta_\alpha}\rangle} : \Omega \to \mathbb{R}$ such that

$$\forall\omega(\omega \in \Delta_{1,\alpha\pm\delta_\alpha})[X_{|H_{\alpha\pm\delta_\alpha}\rangle}(\omega) = 1 - X_{|V_{\alpha\pm\delta_\alpha}\rangle}(\omega)]. \quad (5.3.27)$$

Therefore $\Delta_{1,\alpha\pm\delta_\alpha} = X^{-1}_{X_{|H_{\alpha\pm\delta_\alpha}\rangle}}(0), \Delta_{2,\alpha\pm\delta_\alpha} = X^{-1}_{X_{|H_{\alpha\pm\delta_\alpha}\rangle}}(1)$ and the probability mass function $f_{X_{|H_{\alpha\pm\delta_\alpha}\rangle}}$ is

$$f_{X_{|H_{\alpha\pm\delta_\alpha}\rangle}}(x) = \begin{cases} \frac{1}{2} & \text{if } x = 1 \\ \frac{1}{2} & \text{if } x = 0 \end{cases} \tag{5.3.28}$$

Remark 5.3.16. Let $\mathbf{B}_{M_4}$ be a Boolean algebra of physical events in Minkowski spacetime and let $\mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4}, \mathbf{P})$ be a measure algebra of physical events in Minkowski spacetime, i.e., $\mathcal{F}^{ph}_{M_4}$ that is a Boolean algebra $\mathbf{B}_{M_4}$ with a probability measure $\mathbf{P}$, see Chapter III subsection III.2, Definition 3.2.3. We remind that we denote such physical events by $A(\mathbf{x}), B(\mathbf{x}), \ldots$ etc., where $\mathbf{x} = (t, x_1, x_2, x_3) \in M_4$ or $A, B, \ldots$ etc., and we write for a short $A^{Oc}(\mathbf{x}), B^{Oc}(\mathbf{x}), \ldots$ iff there physical events $A(\mathbf{x}), B(\mathbf{x}) \ldots$ were occured.

Definition 5.3.5. Let σ_α be the measurement operator corresponding to measurements of the photon polarization (see Appendix A) in the polarization basis $\{|V_\alpha\rangle, |H_\alpha\rangle\}$, see Eq. (5.3.3). Let $A(|\psi^\alpha_A\rangle, \sigma_\alpha, \delta_\alpha, t) \in \mathbf{B}_{M_4}$ be a physical event which consists of the performing a measurement (on the particle A at the instant t) of the observable σ_α with accuracy $\delta_\alpha = \epsilon(\sigma_\alpha, |\psi^\alpha_A\rangle, \mathbf{A})$, where the particle A is initially in the state ψ^α_A at the instant t. Here $\delta_\alpha = \epsilon(\sigma_\alpha, \psi, \mathbf{A})$ stands for the noise of the A measurement in the state ψ using apparatus **A** and $\delta_\beta = \eta(\sigma_\beta, \psi, \mathbf{A})$ stands for the disturbance of σ_β in the state ψ caused by apparatus **A.**

Remark 5.3.17. We assume that: (i) the particle A is initially in the state $|\psi^\alpha_A\rangle = \frac{1}{\sqrt{2}}(|V_\alpha\rangle + |H_\alpha\rangle)$, (ii) $A(|\psi^\alpha_A\rangle, \sigma_\alpha, \delta_\alpha, t) \in \mathcal{F}^{ph}_{M_4}$.

Remark 5.3.18. Note that: (i) if the physical event $A(|\psi^\alpha_A\rangle, \sigma_\alpha, \delta_\alpha, t)$ was occurred then immediately after the measurement at the instant t the particle A would be in the state $|V^A_{\alpha\pm\delta_\alpha}\rangle = |V_{\alpha\pm\delta_\alpha}\rangle$ or in the state

$\left|H^A_{\alpha\pm\delta_\alpha}\right\rangle = |H_{\alpha\pm\delta_\alpha}\rangle,$

(ii) immediately after the measurement at the instant t unconditional measure $\mathbf{P}$ collapses to conditional measure $\mathbf{P}\left(X \,\middle|A^{Oc}(|\psi^\alpha_A\rangle,\sigma_\alpha,\delta_\alpha,t)\right)$:

$$\mathbf{P}\left(X \,\middle|A^{Oc}(|\psi^\alpha_A\rangle,\sigma_\alpha,\delta_\alpha,t)\right) = \frac{\mathbf{P}(X \wedge A^{Oc}(|\psi^\alpha_A\rangle,\sigma_\alpha,\delta_\alpha,t))}{\mathbf{P}(A^{Oc}(|\psi^\alpha_A\rangle,\sigma_\alpha,\delta_\alpha,t))}, \tag{5.3.29}$$

where $X \in \mathcal{F}^{ph}_{M_4}$.

Remark 5.3.19. Let $A(|\psi_A\rangle,t)$ be a physical event which consists of that a particle A at the instant t is in the state $|\psi_A\rangle$. Note that:

(i) $A^{Oc}(|\psi^\alpha_A\rangle,\sigma_\alpha,\delta_\alpha,t) \Rightarrow A^{Oc}(\left|V^A_{\alpha\pm\delta_\alpha}\right\rangle,t) \vee A^{Oc}(\left|H^A_{\alpha\pm\delta_\alpha}\right\rangle,t),$

(ii) $A^{Oc}(\left|V^A_{\alpha\pm\delta_\alpha}\right\rangle,t) \wedge A^{Oc}(\left|H^A_{\alpha\pm\delta_\alpha}\right\rangle,t) \notin \mathcal{F}^{ph}_{M_4},$

(iii) from (i), (ii) and (5.3.17) it follows that:

$$\mathbf{P}\left(X \,\middle|A^{Oc}(|\psi^\alpha_A\rangle,\sigma_\alpha,\delta_\alpha,t)\right) = \frac{\mathbf{P}(X \wedge A(\left|V^A_{\alpha\pm\delta_\alpha}\right\rangle,t))}{\mathbf{P}(A(\left|V^A_{\alpha\pm\delta_\alpha}\right\rangle,t))} \tag{5.3.30}$$

or

$$\mathbf{P}\left(X \,\middle|A^{Oc}(|\psi^\alpha_A\rangle,\sigma_\alpha,\delta_\alpha,t)\right) = \frac{\mathbf{P}(X \wedge A(\left|H^A_{\alpha\pm\delta_\alpha}\right\rangle,t))}{\mathbf{P}(A(\left|H^A_{\alpha\pm\delta_\alpha}\right\rangle,t))}, \tag{5.3.31}$$

where $X \in \mathcal{F}^{ph}_{M_4}$.

Remark 5.3.20. We assume now that: (i) a measure algebra $\mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4},\mathbf{P})$ admits a representation $\Re[\cdot] : \mathcal{F}^{ph}_{M_4} \to (\mathbb{R},\mathbf{B},\mathbf{P_B})$ of the measure algebra $\mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4},\mathbf{P})$ in the measure algebra $\mathbf{B} \triangleq (\mathbb{R},\mathbf{B},\mathbf{P_B})$, such that

(ii) $\mathbf{P}_{B^\infty}(X) = \mathbf{P}(\Re^{-1}[X])$ for any $X \in \mathbf{B}$ and

(iii) for physical events $A((\left|V^A_{\alpha\pm\delta_\alpha}\right\rangle,t)) \in \mathcal{F}^{ph}_{M_4}$ and $A((\left|H^A_{\alpha\pm\delta_\alpha}\right\rangle,t)) \in \mathcal{F}^{ph}_{M_4}$ (see Remark 5.3.18) the following conditions hold

$$\Re[A^{Oc}(|V^{A}_{\alpha\pm\delta_\alpha}\rangle, t)] = X^{-1}_{|V^{A}_{\alpha\pm\delta_\alpha}\rangle}(1) = \Delta_{1,\alpha\pm\delta_\alpha} \quad (5.3.32)$$

and

$$\Re[A^{Oc}(|H^{A}_{\alpha\pm\delta_\alpha}\rangle, t)] = X^{-1}_{|H^{A}_{\alpha\pm\delta_\alpha}\rangle}(1) = \Delta_{2,\alpha\pm\delta_\alpha}, \quad (5.3.33)$$

where $\Delta_{1,\alpha\pm\delta_\alpha} = X^{-1}_{X_{|V_{\alpha\pm\delta_\alpha}\rangle}}(1)$ and $\Delta_{2,\alpha\pm\delta_\alpha} = X^{-1}_{X_{|H_{\alpha\pm\delta_\alpha}\rangle}}(1)$.

V.3.2. The EPR-B Paradox Resolution

Remind that in the well-known Einstein-Podolsky-Rosen-Bohm gedanken experiment with photons (Fig. 5.3.1), a source emits pairs of photons in a nonfactorizing state:

$$|\psi_{\mathrm{EPRB}}\rangle = \frac{1}{\sqrt{2}}(|V_\alpha\rangle_1 \otimes |V_\alpha\rangle_2 + |H_\alpha\rangle_1 \otimes |H_\alpha\rangle_2). \quad (5.3.34)$$

After the particles have been space-like separated, one performs correlated measurements of their polarizations along arbitrary directions $\vec{a}$ and $\vec{b}$. Two photons in a singlet state are space-like separated. The linear polarizations of photon 1 and photon 2 are measured along $\vec{a}$ and $\vec{b}$. Quantum mechanics predicts strong correlations between these measurements.

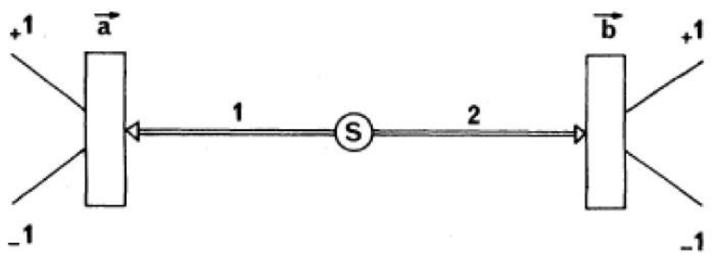

Fig. 5.3.1. Einstein-Podolsky-Rosen-Bohm gedanken experiment with photons

Quantum mechanics predicts that by a measurement of the linear polarization of photon 1 using a noiseless measuring apparatus **A** one obtais definite outcome exactly only $|V_\alpha\rangle_1$ or $|H_\alpha\rangle_1$. In the canonical Copenhagen interpretation QM predicts that the state $|\psi_{\mathrm{EPRB}}\rangle$ has collapsed, at the moment of measurement, from $|\psi_{\mathrm{EPRB}}\rangle$ to either $|V_\alpha\rangle_1 \otimes |V_\alpha\rangle_2$ or $|H_\alpha\rangle_1 \otimes |H_\alpha\rangle_2$.

Remark 5.3.21. The process described above seems to be *nonlocal*: the state changes instantly even though the particles could be space-like separated. We are accustomed to saying that this sort of instantaneous action at a distance is forbidden by relativity.

Assume that the state $|\psi_{\mathrm{EPRB}}\rangle$ has collapsed to $|V_\alpha\rangle_1 \otimes |V_\alpha\rangle_2$. Thus in the canonical Copenhagen interpretation result of the measurement of the polarizations of photon 1 predicts exactly the polarization of photon 2. This means that if we measure the linear polarization of photon 1 (using a noiseless measuring apparatus **A)** in any basis $\{|V_\beta\rangle_1, |H_\beta\rangle_2\}$ the result will be completely random ($|V_\beta\rangle_1$ or $|H_\beta\rangle_1$ with equal probability 1/2).

Remark 5.3.22. However, there is a *perfect correlation*: whenever we measure with certainty the linear polarization of photon 1 with outcome say $|V_\beta\rangle_1$ (using a noiseless measuring apparatus **A)** then we will measure with certainty (using a noiseless measuring apparatus **B)** the linear polarization of photon 2 exactly with outcome $|V_\beta\rangle_2$ **.**

Remark 5.3.23. We note that such a perfect correlation implies that the corresponding probability mass functions: $f_{X_{|V\alpha\rangle_1}}$, $f_{X_{|H\alpha\rangle_1}} f_{X_{|V\alpha\rangle_2}}$ and $f_{X_{|H\alpha\rangle_2}}$ (see Definition 5.3.2) are perfectly contracted for any α, by the following equations:

$$f_{X_{|V\alpha\rangle_1}}(x) = f_{X_{|V\alpha\rangle_2}}(x), \tag{5.3.35}$$

and

$$f_{X_{|H\alpha\rangle_1}}(x) = f_{X_{|H\alpha\rangle_2}}(x), \tag{5.3.36}$$

where $f_{X_{|H_\alpha\rangle_1}}(x) = 1 - f_{X_{|V_\alpha\rangle_1}}(x)$.

Now we go to prove that Eqs. (5.3.35) - (5.3.36) hold without any instantaneous action at a distance.

Remark 5.3.24. We assume now that: (i) photon 1 is initially in the state $|\psi_1^\alpha\rangle = \frac{1}{\sqrt{2}}(|V_\alpha\rangle_1 + |H_\alpha\rangle_1)$,

(ii) $A(|\psi_1^\alpha\rangle, \sigma_\alpha, t) \in \mathcal{F}_{M_4}^{ph}$, see Remark 5.3.9.

If the measurement of the linear polarization of photon 1 (using a noiseless measuring apparatus **A**) was performed at the instant t, i.e., $A^{Oc}(|\psi_A^\alpha\rangle, \sigma_\alpha, t)$ after the measurement at the instant t the unconditional measure $\mathbf{P}$ collapses to the conditional measure

$$\mathbf{P}\left(X \mid A^{Oc}(|\psi_1^\alpha\rangle, \sigma_\alpha, t)\right) = \mathbf{P}\left(X \mid A^{Oc}(|V_\alpha\rangle_1, t)\right) = \frac{\mathbf{P}(X \wedge A^{Oc}(|V_\alpha\rangle_1, t))}{\mathbf{P}(A(|V_\alpha\rangle_1, t))} \quad (5.3.37)$$

or collapses to the conditional measure

$$\mathbf{P}\left(X \mid A^{Oc}(|\psi_1^\alpha\rangle, \sigma_\alpha, t)\right) = \mathbf{P}\left(X \mid A^{Oc}(|H_\alpha\rangle_1, t)\right) = \frac{\mathbf{P}\left(X \wedge A^{Oc}\left(|H_\alpha^A\rangle_1, t\right)\right)}{\mathbf{P}(A(|H_\alpha\rangle_1, t))}, \quad (5.3.38)$$

where $X \in \mathcal{F}_{M_4}^{ph}$, see Remark 5.3.19.

Remark 5.3.25. We assume now that: (i) immediately after the measurement at the instant t the particle 1 is in the state $|V_\alpha\rangle_1$. In this case immediately after the measurement at the instant t the unconditional measure $\mathbf{P}$ collapses to the conditional measure $\mathbf{P}\left(X \mid A(|V_\alpha\rangle_1, t)\right)$ which is given by

$$\mathbf{P}\left(X \mid A^{Oc}(|V_\alpha\rangle_1, t)\right) = \frac{\mathbf{P}(X \wedge A^{Oc}(|V_\alpha\rangle_1, t))}{\mathbf{P}(A(|V_\alpha\rangle_1, t))} \quad (5.3.39)$$

where $X \in \mathcal{F}_{M_4}^{ph}$,

(ii) immediately after the measurement at the instant t the particle 1 is in the state $|H_\alpha\rangle_1$. In this case immediately after the measurement at the instant t the unconditional measure $\mathbf{P}$ collapses to the conditional measure $\mathbf{P}\big(X \mid A^{Oc}(|V_\alpha\rangle_1, t)\big)$ given by

$$\mathbf{P}\big(X \mid A^{Oc}(|H_\alpha\rangle_1, t)\big) = \frac{\mathbf{P}(X \wedge A^{Oc}(|H_\alpha\rangle_1, t))}{\mathbf{P}(A(|H_\alpha\rangle_1, t))}, \tag{5.3.40}$$

where $X \in \mathcal{F}^{ph}_{M_4}$.

(1) From Eq. (5.3.39) and Eq. (5.3.20) - Eq. (5.3.21) we obtain

$$\mathbf{P}\big(A(|H_\alpha\rangle_2, t) \mid A^{Oc}(|V_\alpha\rangle_1, t)\big) = \frac{\mathbf{P}(A(|H_\alpha\rangle_2, t) \wedge A^{Oc}(|V_\alpha\rangle_1, t))}{\mathbf{P}(A(|V_\alpha\rangle_1, t))} =$$
$$\frac{\mathbf{P_B}(\Re[A(|H_\alpha\rangle_2, t)] \wedge \Re[A^{Oc}(|V_\alpha\rangle_1, t)])}{\mathbf{P_B}(\Re[A(|V_\alpha\rangle_1, t)])} = \frac{\mathbf{P_B}(\Delta_{2,\alpha} \cap \Delta_{1,\alpha})}{\mathbf{P_B}(\Delta_{1,\alpha})} = \tag{5.3.41}$$
$$\frac{\mathbf{P_B}(\varnothing)}{\mathbf{P_B}(\Delta_{1,\alpha})} = 0.$$

Therefore $A^{Oc}(|H_\alpha\rangle_2, t)$ and $A^{Oc}(|V_\alpha\rangle_1, t)$ are mutually exclusive (disjoint) physical events, i.e., they cannot both occur simultaneously:

$$A^{Oc}(|H_\alpha\rangle_2, t) \wedge A^{Oc}(|V_\alpha\rangle_1, t) \notin \mathcal{F}^{ph}_{M_4}. \tag{5.3.42}$$

(2) From Eq. (5.3.39) and Eq. (5.3.20) - Eq. (5.3.21) we obtain

$$\mathbf{P}\big(A(|V_\alpha\rangle_2, t) \mid A^{Oc}(|V_\alpha\rangle_1, t)\big) = \frac{\mathbf{P}(A(|V_\alpha\rangle_2, t) \wedge A^{Oc}(|V_\alpha\rangle_1, t))}{\mathbf{P}(A(|V_\alpha\rangle_1, t))} =$$
$$\frac{\mathbf{P_B}(\Re[A(|V_\alpha\rangle_2, t)] \wedge \Re[A^{Oc}(|V_\alpha\rangle_1, t)])}{\mathbf{P_B}(\Re[A(|V_\alpha\rangle_1, t)])} = \frac{\mathbf{P_B}(\Delta_{1,\alpha} \cap \Delta_{1,\alpha})}{\mathbf{P_B}(\Delta_{1,\alpha})} = \tag{5.3.43}$$
$$\frac{\mathbf{P_B}(\Delta_{1,\alpha})}{\mathbf{P_B}(\Delta_{1,\alpha})} = 1.$$

Therefore physical events $A^{Oc}(|V_\alpha\rangle_2, t)$ and $A^{Oc}(|V_\alpha\rangle_1, t)$ always occur simultaneously even particle 1 and particle 2 are space-like separated:

$$A^{Oc}(|V_\alpha\rangle_2, t) \Leftrightarrow A^{Oc}(|V_\alpha\rangle_1, t). \tag{5.3.44}$$

(3) From Eq. (5.3.40) and Eq. (5.3.20) - Eq.(5.3.21) we obtain

$$\mathbf{P}\big(A(|H_\alpha\rangle_2,t)\ \big|A^{Oc}(|H_\alpha\rangle_1,t)\big) = \frac{\mathbf{P}(A(|H_\alpha\rangle_2,t)\wedge A^{Oc}(|H_\alpha\rangle_1,t))}{\mathbf{P}(A(|H_\alpha\rangle_1,t))} =$$
$$\frac{\mathbf{P_B}(\Re[A(|H_\alpha\rangle_2,t)]\wedge\Re[A^{Oc}(|H_\alpha\rangle_1,t)])}{\mathbf{P_B}(\Re[A(|H_\alpha\rangle_1,t)])} = \frac{\mathbf{P_B}(\Delta_{2,\alpha}\cap\Delta_{2,\alpha})}{\mathbf{P_B}(\Delta_{1,\alpha})} = \tag{5.3.45}$$
$$\frac{\mathbf{P_B}(\Delta_{2,\alpha})}{\mathbf{P_B}(\Delta_{2,\alpha})} = 1.$$

Therefore physical events $A^{Oc}(|H_\alpha\rangle_2,t)$ and $A^{Oc}(|H_\alpha\rangle_1,t)$ always occur simultaneously even particle 1 and particle 2 are space-like separated:

$$A^{Oc}(|H_\alpha\rangle_2,t) \Leftrightarrow A^{Oc}(|H_\alpha\rangle_1,t). \tag{5.3.46}$$

(4) From Eq. (5.3.40) and Eq. (5.3.20) - Eq.(5.3.21) we obtain

$$\mathbf{P}\big(A(|V_\alpha\rangle_2,t)\ \big|A^{Oc}(|H_\alpha\rangle_1,t)\big) = \frac{\mathbf{P}(A(|V_\alpha\rangle_2,t)\wedge A^{Oc}(|H_\alpha\rangle_1,t))}{\mathbf{P}(A(|H_\alpha\rangle_1,t))} =$$
$$\frac{\mathbf{P_B}(\Re[A(|V_\alpha\rangle_2,t)]\wedge\Re[A^{Oc}(|H_\alpha\rangle_1,t)])}{\mathbf{P_B}(\Re[A(|H_\alpha\rangle_1,t)])} = \frac{\mathbf{P_B}(\Delta_{1,\alpha}\cap\Delta_{2,\alpha})}{\mathbf{P_B}(\Delta_{2,\alpha})} = \tag{5.3.47}$$
$$\frac{\mathbf{P_B}(\varnothing)}{\mathbf{P_B}(\Delta_{2,\alpha})} = 0.$$

Therefore $A^{Oc}(|H_\alpha\rangle_2,t)$ and $A^{Oc}(|V_\alpha\rangle_1,t)$ are mutually exclusive (disjoint) physical events, i.e., they cannot both occur simultaneously:

$$A^{Oc}(|V_\alpha\rangle_2,t)\wedge A^{Oc}(|H_\alpha\rangle_1,t) \notin \mathcal{F}^{ph}_{M_4}. \tag{5.3.48}$$

Remark 5.3.26. Under rigorous consideration using Heisenberg noise-disturbance uncertainty relation (see Appendix A) quantum mechanics predicts that by a measurement of the linear polarization of photon 1 using measuring apparatus **A** one obtains definite outcome exactly only $|V_{\alpha\pm\delta_\alpha}\rangle_1$ or $|H_{\alpha\pm\delta_\alpha}\rangle_1$, where $\delta_\alpha = \epsilon(\sigma_\alpha,\psi^\alpha_1,\mathbf{A})$ stands for the noise of the σ_α measurement in state ψ^α_1 using apparatus **A,** see Appendix A, Eq. (A.13)**.** In the canonical Copenhagen interpretation QM predicts that the state $|\psi_{\mathrm{EPRB}}\rangle$ has collapsed, at the moment of the measurement, from $|\psi_{\mathrm{EPRB}}\rangle$ to either $|V_{\alpha\pm\delta_\alpha}\rangle_1\otimes|V_\alpha\rangle_2$ or $|H_{\alpha\pm\delta_\alpha}\rangle_1\otimes|H_\alpha\rangle_2$.

Remark 5.3.27. However, there is a *perfect correlation*: whenever we measure with uncertainty δ_α the linear polarization of photon 1 with outcome say $|V_{\alpha\pm\delta_\alpha}\rangle_1$ using measuring apparatus **A** then we will measure with uncertainty δ_α (using a similar measuring apparatus **B** with $\epsilon(\sigma_\alpha, \psi_1^\alpha, \mathbf{A}) \simeq \delta_\alpha$) the linear polarization of photon 2 exactly with outcome $|V_{\alpha\pm\delta_\alpha}\rangle_2$.

Remark 5.3.28. We note that such a perfect correlation implies that the corresponding probability mass functions: $f_{X_{|V_{\alpha\pm\delta_\alpha}\rangle_1}}$, $f_{X_{|H_{\alpha\pm\delta_\alpha}\rangle_1}} f_{X_{|V_{\alpha\pm\delta_\alpha}\rangle_2}}$ and $f_{X_{|H_{\alpha\pm\delta_\alpha}\rangle_2}}$ (see Definition 5.3.4) are perfectly contracted for any α, by the following equations:

$$f_{X_{|V_{\alpha\pm\delta_\alpha}\rangle_1}}(x) = f_{X_{|V_{\alpha\pm\delta_\alpha}\rangle_2}}(x), \tag{5.3.49}$$

and

$$f_{X_{|H_{\alpha\pm\delta_\alpha}\rangle_1}}(x) = f_{X_{|H_{\alpha\pm\delta_\alpha}\rangle_2}}(x), \tag{5.3.50}$$

where $f_{X_{|H_{\alpha\pm\delta_\alpha}\rangle_1}}(x) = 1 - f_{X_{|V_{\alpha\pm\delta_\alpha}\rangle_1}}(x)$.

Now we go to prove that Eqs. (5.3.49) - (5.3.50) hold without any instantaneous action at a distance.

Remark 5.3.29. We assume now that: (i) photon 1 is initially in the state $|\psi_1^\alpha\rangle = \frac{1}{\sqrt{2}}(|V_\alpha\rangle_1 + |H_\alpha\rangle_1)$, (ii) $A(|\psi_1^\alpha\rangle, \sigma_\alpha, \delta_\alpha, t) \in \mathcal{F}_{M_4}^{ph}$, where $\delta_\alpha = \epsilon(\sigma_\alpha, \psi_1^\alpha, \mathbf{A})$ stands for the noise of the σ_α measurement in state ψ_1^α using apparatus **A.**

If the measurement of the linear polarization of photon 1 using measuring apparatus **A** was performed at the instant t, i.e., $A^{Oc}(|\psi_A^\alpha\rangle, \sigma_\alpha, \delta_\alpha, t)$ after the measurement at the instant t the unconditional measure $\mathbf{P}$ collapses to the conditional measure

$$\mathbf{P}\left(X \,\middle|A^{Oc}(|\psi_1^\alpha\rangle, \sigma_\alpha, t)\right) = \mathbf{P}\left(X \,\middle|A^{Oc}(|V_{\alpha\pm\delta_\alpha}\rangle_1, t)\right) = \frac{\mathbf{P}(X \wedge A^{Oc}(|V_{\alpha\pm\delta_\alpha}\rangle_1, t))}{\mathbf{P}(A(|V_{\alpha\pm\delta_\alpha}\rangle_1, t))} \tag{5.3.51}$$

or collapses to the conditional measure

$$\mathbf{P}\left(X \left| A^{Oc}(|\psi_1^\alpha\rangle, \sigma_\alpha, t)\right.\right) = \mathbf{P}\left(X \left| A^{Oc}(|H_{\alpha\pm\delta_\alpha}\rangle_1, t)\right.\right) = \frac{\mathbf{P}\left(X \wedge A^{Oc}\left(|H^A_{\alpha\pm\delta_\alpha}\rangle_1, t\right)\right)}{\mathbf{P}(A(|H_{\alpha\pm\delta_\alpha}\rangle_1, t))}, \quad (5.3.52)$$

where $X \in \mathcal{F}^{ph}_{M_4}$, see Remark 5.3.19.

Remark 5.3.30. We assume now that: (i) immediately after the measurement at the instant t the particle 1 is in the state $|V_{\alpha\pm\delta_\alpha}\rangle_1$. In this case immediately after the measurement at the instant t the unconditional measure $\mathbf{P}$ collapses to the conditional measure $\mathbf{P}\left(X \left| A(|V_{\alpha\pm\delta_\alpha}\rangle_1, t)\right.\right)$ given by

$$\mathbf{P}\left(X \left| A^{Oc}(|V_{\alpha\pm\delta_\alpha}\rangle_1, t)\right.\right) = \frac{\mathbf{P}(X \wedge A^{Oc}(|V_{\alpha\pm\delta_\alpha}\rangle_1, t))}{\mathbf{P}(A(|V_{\alpha\pm\delta_\alpha}\rangle_1, t))} \quad (5.3.53)$$

where $X \in \mathcal{F}^{ph}_{M_4}$,

(ii) immediately after the measurement at the instant t the particle 1 is in the state $|H_{\alpha\pm\delta_\alpha}\rangle_1$. In this case immediately after the measurement at the instant t the unconditional measure $\mathbf{P}$ collapses to the conditional measure $\mathbf{P}\left(X \left| A^{Oc}(|H_{\alpha\pm\delta_\alpha}\rangle_1, t)\right.\right)$ given by

$$\mathbf{P}\left(X \left| A^{Oc}(|H_{\alpha\pm\delta_\alpha}\rangle_1, t)\right.\right) = \frac{\mathbf{P}(X \wedge A^{Oc}(|H_{\alpha\pm\delta_\alpha}\rangle_1, t))}{\mathbf{P}(A(|H_{\alpha\pm\delta_\alpha}\rangle_1, t))}, \quad (5.3.54)$$

where $X \in \mathcal{F}^{ph}_{M_4}$.

(1) From Eq. (5.3.53) and Eq. (5.3.32) - Eq.(5.3.33) we obtain

$$\begin{gathered}\mathbf{P}\left(A(|H_{\alpha\pm\delta_\alpha}\rangle_2, t) \left| A^{Oc}(|V_{\alpha\pm\delta_\alpha}\rangle_1, t)\right.\right) = \frac{\mathbf{P}(A(|H_{\alpha\pm\delta_\alpha}\rangle_2, t) \wedge A^{Oc}(|V_{\alpha\pm\delta_\alpha}\rangle_1, t))}{\mathbf{P}(A(|V_{\alpha\pm\delta_\alpha}\rangle_1, t))} = \\ \frac{\mathbf{P_B}(\Re[A(|H_{\alpha\pm\delta_\alpha}\rangle_2, t)] \wedge \Re[A^{Oc}(|V_{\alpha\pm\delta_\alpha}\rangle_1, t)])}{\mathbf{P_B}(\Re[A(|V_{\alpha\pm\delta_\alpha}\rangle_1, t)])} = \frac{\mathbf{P_B}(\Delta_{2,\alpha\pm\delta_\alpha} \cap \Delta_{1,\alpha\pm\delta_\alpha})}{\mathbf{P_B}(\Delta_{1,\alpha\pm\delta_\alpha})} = \\ \frac{\mathbf{P_B}(\varnothing)}{\mathbf{P_B}(\Delta_{1,\alpha\pm\delta_\alpha})} = 0.\end{gathered} \quad (5.3.55)$$

Therefore $A^{Oc}(|H_{\alpha\pm\delta_\alpha}\rangle_2, t)$ and $A^{Oc}(|V_{\alpha\pm\delta_\alpha}\rangle_1, t)$ are mutually exclusive (disjoint) physical events, i.e., they cannot both occur simultaneously:

$$A^{Oc}(|H_{\alpha\pm\delta_\alpha}\rangle_2,t)\wedge A^{Oc}(|V_{\alpha\pm\delta_\alpha}\rangle_1,t)\notin\mathcal{F}^{ph}_{M_4}. \quad (5.6.56)$$

(2) From Eq. (5.3.39) and Eq. (5.3.20) - Eq. (5.3.21) we obtain

$$\begin{gathered}\mathbf{P}\big(A(|V_{\alpha\pm\delta_\alpha}\rangle_2,t)\ \big|A^{Oc}(|V_{\alpha\pm\delta_\alpha}\rangle_1,t)\big)=\frac{\mathbf{P}(A(|V_{\alpha\pm\delta_\alpha}\rangle_2,t)\wedge A^{Oc}(|V_{\alpha\pm\delta_\alpha}\rangle_1,t))}{\mathbf{P}(A(|V_{\alpha\pm\delta_\alpha}\rangle_1,t))}=\\ \frac{\mathbf{P_B}(\Re[A(|V_{\alpha\pm\delta_\alpha}\rangle_2,t)]\wedge\Re[A^{Oc}(|V_{\alpha\pm\delta_\alpha}\rangle_1,t)])}{\mathbf{P_B}(\Re[A(|V_{\alpha\pm\delta_\alpha}\rangle_1,t)])}=\frac{\mathbf{P_B}(\Delta_{1,\alpha\pm\delta_\alpha}\cap\Delta_{1,\alpha\pm\delta_\alpha})}{\mathbf{P_B}(\Delta_{1,\alpha\pm\delta_\alpha})}=\\ =\frac{\mathbf{P_B}(\Delta_{1,\alpha\pm\delta_\alpha})}{\mathbf{P_B}(\Delta_{1,\alpha\pm\delta_\alpha})}=1.\end{gathered} \quad (5.3.57)$$

Therefore physical events $A^{Oc}(|V_{\alpha\pm\delta_\alpha}\rangle_2,t)$ and $A^{Oc}(|V_{\alpha\pm\delta_\alpha}\rangle_1,t)$ always occur simultaneously even particle 1 and particle 2 are space-like separated:

$$A^{Oc}(|V_{\alpha\pm\delta_\alpha}\rangle_2,t)\Leftrightarrow A^{Oc}(|V_{\alpha\pm\delta_\alpha}\rangle_1,t). \quad (5.3.58)$$

(3) From Eq. (5.3.40) and Eq. (5.3.20) - Eq.(5.3.21) we obtain

$$\begin{gathered}\mathbf{P}\big(A(|H_{\alpha\pm\delta_\alpha}\rangle_2,t)\ \big|A^{Oc}(|H_{\alpha\pm\delta_\alpha}\rangle_1,t)\big)=\frac{\mathbf{P}(A(|H_{\alpha\pm\delta_\alpha}\rangle_2,t)\wedge A^{Oc}(|H_{\alpha\pm\delta_\alpha}\rangle_1,t))}{\mathbf{P}(A(|H_{\alpha\pm\delta_\alpha}\rangle_1,t))}=\\ \frac{\mathbf{P_B}(\Re[A(|H_{\alpha\pm\delta_\alpha}\rangle_2,t)]\wedge\Re[A^{Oc}(|H_{\alpha\pm\delta_\alpha}\rangle_1,t)])}{\mathbf{P_B}(\Re[A(|H_{\alpha\pm\delta_\alpha}\rangle_1,t)])}=\frac{\mathbf{P_B}(\Delta_{2,\alpha\pm\delta_\alpha}\cap\Delta_{2,\alpha\pm\delta_\alpha})}{\mathbf{P_B}(\Delta_{1,\alpha\pm\delta_\alpha})}=\\ \frac{\mathbf{P_B}(\Delta_{2,\alpha\pm\delta_\alpha})}{\mathbf{P_B}(\Delta_{2,\alpha\pm\delta_\alpha})}=1.\end{gathered} \quad (5.3.59)$$

Therefore physical events $A^{Oc}(|H_{\alpha\pm\delta_\alpha}\rangle_2,t)$ and $A^{Oc}(|H_{\alpha\pm\delta_\alpha}\rangle_1,t)$ always occur simultaneously even particle 1 and particle 2 are space-like separated:

$$A^{Oc}(|H_{\alpha\pm\delta_\alpha}\rangle_2,t)\Leftrightarrow A^{Oc}(|H_{\alpha\pm\delta_\alpha}\rangle_1,t). \quad (5.3.60)$$

(4) From Eq. (5.3.40) and Eq. (5.3.20) - Eq. (5.3.21) we obtain

$$\mathbf{P}\big(A(|V_{\alpha\pm\delta_\alpha}\rangle_2,t)\ \big|A^{Oc}(|H_{\alpha\pm\delta_\alpha}\rangle_1,t)\big) = \frac{\mathbf{P}(A(|V_{\alpha\pm\delta_\alpha}\rangle_2,t)\wedge A^{Oc}(|H_{\alpha\pm\delta_\alpha}\rangle_1,t))}{\mathbf{P}(A(|H_{\alpha\pm\delta_\alpha}\rangle_1,t))} =$$
$$\frac{\mathbf{P_B}(\Re[A(|V_{\alpha\pm\delta_\alpha}\rangle_2,t)]\wedge\Re[A^{Oc}(|H_{\alpha\pm\delta_\alpha}\rangle_1,t)])}{\mathbf{P_B}(\Re[A(|H_{\alpha\pm\delta_\alpha}\rangle_1,t)])} = \frac{\mathbf{P_B}(\Delta_{1,\alpha\pm\delta_\alpha}\cap\Delta_{2,\alpha\pm\delta_\alpha})}{\mathbf{P_B}(\Delta_{2,\alpha\pm\delta_\alpha})} = \tag{5.3.61}$$
$$\frac{\mathbf{P_B}(\varnothing)}{\mathbf{P_B}(\Delta_{2,\alpha\pm\delta_\alpha})} = 0.$$

Therefore $A^{Oc}(|H_{\alpha\pm\delta_\alpha}\rangle_2,t)$ and $A^{Oc}(|V_{\alpha\pm\delta_\alpha}\rangle_1,t)$ are mutually exclusive (disjoint) physical events, i.e., they cannot both occur simultaneously:

$$A^{Oc}(|V_{\alpha\pm\delta_\alpha}\rangle_2,t)\wedge A^{Oc}(|H_{\alpha\pm\delta_\alpha}\rangle_1,t)\notin\mathcal{F}^{ph}_{M_4}. \tag{5.3.62}$$

Definition 5.3.6. Let σ_α be the measurement operator corresponding to measurements of the photon polarization (see Appendix A) in the polarization basis $\{|V_\alpha\rangle,|H_\alpha\rangle\}$ and let σ_β be the measurement operator corresponding to measurements of the photon polarization in the polarization basis $\{|V_\beta\rangle,|H_\beta\rangle\}$ see Eq.(1.3.3). We assume that particle 1 is initially in the state $|\psi_1^\alpha\rangle$. Let $A(|\psi_1^\alpha\rangle,\sigma_\alpha,\delta_\alpha;|\widetilde{\psi}_1^\alpha\rangle,\sigma_\beta,\delta_\beta,t)\in\mathbf{B}_{M_4}$ be a physical event which consists of:
(i) performing a measurement (on particle A at instant t) of the observable σ_α with accuracy $\delta_\alpha=\epsilon(\sigma_\alpha,|\psi_A^\alpha\rangle,\mathbf{A})$, where particle A is initially in the state ψ_A^α at the instant t,
(ii) immediately after the measurement on particle 1 at the instant t the particle 1 is in the state $|\widetilde{\psi}_1^\alpha\rangle$,
(iii) performing a measurement (on particle 1 at the instant t) of the observable σ_α with accuracy $\delta_\alpha=\epsilon(\sigma_\alpha,|\psi_A^\alpha\rangle,\mathbf{A})$ particle A obtains disturbance $\delta_\beta=\eta(\sigma_\beta,|\widetilde{\psi}_1^\alpha\rangle,\mathbf{A})$ of the observable σ_β in the state $|\widetilde{\psi}_1^\alpha\rangle$ caused by apparatus **A.**

Definition 5.3.7. (**I**) Let $X_{|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle}(\omega)$ be a discrete random variable $X_{|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle}:\Omega\to\mathbb{R}$ with the probability mass function (see Remark 5.3.4.v) $f_{X_{|V_{\alpha\pm\delta_\alpha},\beta\pm\delta_\beta\rangle}}$ defined by

$$f_{X|_{V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}}\rangle}(x) = \begin{cases} \frac{1}{2} & \text{if } x = 1 \\ \frac{1}{2} & \text{if } x = 0 \end{cases} \quad (5.3.63)$$

Thus there exist: (i) $\Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} \subsetneqq \Omega$ such that $\Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} = X^{-1}_{X|_{V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}}\rangle}(1)$ and $\mathbf{P}(\Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}) = 1/2$,

(ii) $\Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} \subsetneqq \Omega$ such that $\Delta_{2,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} = X^{-1}_{X|_{V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}}\rangle}(0)$ and $\mathbf{P}(\Delta_{2,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}) = 1/2$,

(iii) $\Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} = \Omega \backslash \Delta_{2,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} \bmod(\Lambda), \mathbf{P}(\Lambda) = 0.$

Thus there exist: (i) $\Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} \subsetneqq \Omega$ such that $\Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} = X^{-1}_{X|_{V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}}\rangle}(1)$ and $\mathbf{P}(\Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}) = 1/2$,

(ii) $\Delta_{2,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} \subsetneqq \Omega$ such that $\Delta_{2,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} = X^{-1}_{X|_{V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}}\rangle}(0)$ and $\mathbf{P}(\Delta_{2,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}) = 1/2$,

(iii) $\Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} = \Omega \backslash \Delta_{2,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} \bmod(\Lambda), \mathbf{P}(\Lambda) = 0.$

(**II**) Let $X|_{H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}}\rangle(\omega)$ be a discrete random variable $X|_{H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}}\rangle : \Omega \to \mathbb{R}$ such that

$$\forall\omega(\omega \in \Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta})\Big[X|_{H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}}\rangle(\omega) = 1 - X|_{V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}}\rangle(\omega)\Big]. \quad (5.3.64)$$

Therefore $\Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} = X^{-1}_{X|_{H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}}\rangle}(0), \Delta_{2,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} = X^{-1}_{X|_{H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}}\rangle}(1)$ and the probability mass function $f_{X|_{H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}}\rangle}$ is

$$f_{X|_{H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}}\rangle}(x) = \begin{cases} \frac{1}{2} & \text{if } x = 1 \\ \frac{1}{2} & \text{if } x = 0 \end{cases} \quad (5.3.65)$$

Remark 5.3.31. Let $\mathbf{B}_{M_4}$ be a Boolean algebra of physical events in Minkowski spacetime and let $\mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4}, \mathbf{P})$ be a measure algebra of physical events in Minkowski spacetime, i.e., $\mathcal{F}^{ph}_{M_4}$ that is a Boolean algebra $\mathbf{B}_{M_4}$ with a probability measure $\mathbf{P}$, see Chapter III subsection III.2, Definition 3.2.3. We remind that we denote such physical events by $A(\mathbf{x}), B(\mathbf{x}), \ldots$ etc., where $\mathbf{x} = (t, x_1, x_2, x_3) \in M_4$ or $A, B, \ldots$ etc., and we write for a short $A^{Oc}(\mathbf{x}), B^{Oc}(\mathbf{x}), \ldots$ iff the physical events $A(\mathbf{x}), B(\mathbf{x})\ldots$ were occurred.

Remark 5.3.32. We assume that: (i) particle 1 is initially in the state $|\psi_1^\alpha\rangle = \frac{1}{\sqrt{2}}(|V_\alpha\rangle + |H_\alpha\rangle)$, (ii) $A(|\psi_1^\alpha\rangle, \sigma_\alpha, \delta_\alpha; |\widetilde{\psi}_1^\alpha\rangle, \sigma_\beta, \delta_\beta, t) \in \mathcal{F}_{M_4}^{ph}$.

Remark 5.3.33. Note that: (i) if the physical event $A(|\psi_1^\alpha\rangle, \sigma_\alpha, \delta_\alpha; |\widetilde{\psi}_1^\alpha\rangle, \sigma_\beta, \delta_\beta, t)$ was occurred then immediately after the measurement at the instant t particle 1 will be in the state $\left|V^1_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\right\rangle = |V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle$ or in the state $\left|H^1_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\right\rangle = |H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle$,

(ii) immediately after the measurement at the instant t unconditional measure $\mathbf{P}$ collapses to conditional measure $\mathbf{P}\left(X \,\middle|\, A^{Oc}(|\psi_1^\alpha\rangle, \sigma_\alpha, \delta_\alpha; |\widetilde{\psi}_1^\alpha\rangle, \sigma_\beta, \delta_\beta, t)\right)$:

$$\begin{aligned} &\mathbf{P}\left(X \,\middle|\, A^{Oc}(|\psi_1^\alpha\rangle, \sigma_\alpha, \delta_\alpha; |\widetilde{\psi}_1^\alpha\rangle, \sigma_\beta, \delta_\beta, t)\right) = \\ &= \frac{\mathbf{P}(X \wedge A^{Oc}(|\psi_1^\alpha\rangle, \sigma_\alpha, \delta_\alpha; |\widetilde{\psi}_1^\alpha\rangle, \sigma_\beta, \delta_\beta, t))}{\mathbf{P}(A^{Oc}(|\psi_1^\alpha\rangle, \sigma_\alpha, \delta_\alpha; |\widetilde{\psi}_1^\alpha\rangle, \sigma_\beta, \delta_\beta, t))}, \end{aligned} \tag{5.3.66}$$

where $X \in \mathcal{F}_{M_4}^{ph}$.

Remark 5.3.34. Let $A(|\psi_1\rangle, t)$ be a physical event which consists of that at the instant t particle 1 is in the state $|\psi_1\rangle$. Note that:

(i) $A^{Oc}(|\psi_1^\alpha\rangle, \sigma_\alpha, \delta_\alpha; |\widetilde{\psi}_1^\alpha\rangle, \sigma_\beta, \delta_\beta, t) \Rightarrow A^{Oc}\left(\left|V^1_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\right\rangle, t\right) \vee A^{Oc}\left(\left|H^1_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\right\rangle, t\right)$,

(ii) $A^{Oc}\left(\left|V^1_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\right\rangle, t\right) \wedge A^{Oc}\left(\left|H^A_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\right\rangle, t\right) \notin \mathcal{F}_{M_4}^{ph}$,

(iii) from (i), (ii) and (5.3.66) it follows that:

$$\mathbf{P}\left(X \,\middle|\, A^{Oc}(|\psi_1^\alpha\rangle, \sigma_\alpha, \delta_\alpha; |\widetilde{\psi}_1^\alpha\rangle, \sigma_\beta, \delta_\beta, t)\right) = \frac{\mathbf{P}\left(X \wedge A\left(\left|V^1_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\right\rangle, t\right)\right)}{\mathbf{P}\left(A\left(\left|V^1_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\right\rangle, t\right)\right)} \tag{5.3.67}$$

or

$$\mathbf{P}\Big(X \,\Big|A^{Oc}(|\psi_1^{\alpha}\rangle,\sigma_{\alpha},\delta_{\alpha};|\widetilde{\psi}_1^{\alpha}\rangle,\sigma_{\beta},\delta_{\beta},t)\Big) = \frac{\mathbf{P}\Big(X \wedge A\Big(\big|H^1_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\big\rangle,t\Big)\Big)}{\mathbf{P}\Big(A\Big(\big|H^1_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\big\rangle,t\Big)\Big)}, \qquad (5.3.68)$$

where $X \in \mathcal{F}^{ph}_{M_4}$.

Remark 5.3.35. We assume now that: (i) a measure algebra $\mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4},\mathbf{P})$ admits a representation $\Re[\bullet] : \mathcal{F}^{ph}_{M_4} \to (\mathbb{R},\mathbf{B},\mathbf{P}_{\mathbf{B}})$ of the measure algebra $\mathcal{F}^{ph}_{M_4} = (\mathbf{B}_{M_4},\mathbf{P})$ in the measure algebra $\mathbf{B} \triangleq (\mathbb{R},\mathbf{B},\mathbf{P}_{\mathbf{B}})$, such that

(ii) $\mathbf{P}_{B^\infty}(X) = \mathbf{P}(\Re^{-1}[X])$ for any $X \in \mathbf{B}$ and

(iii) for physical events $A\Big(\Big(\big|V^1_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\big\rangle,t\Big)\Big) \in \mathcal{F}^{ph}_{M_4}$ and $A\Big(\Big(\big|H^1_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\big\rangle,t\Big)\Big) \in \mathcal{F}^{ph}_{M_4}$ (see Remark 5.3.33) the following conditions hold

$$\Re\Big[A^{Oc}\Big(\big|V^1_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\big\rangle,t\Big)\Big] = X^{-1}_{\big|V^A_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\big\rangle}(1) = \Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} \qquad (5.3.69)$$

and

$$\Re\Big[A^{Oc}\Big(\big|H^1_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\big\rangle,t\Big)\Big] = X^{-1}_{\big|H^1_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\big\rangle}(1) = \Delta_{2,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}, \qquad (5.3.70)$$

where

$\Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} = X^{-1}_{X_{\big|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\big\rangle}}(1)$ and $\Delta_{2,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} = X^{-1}_{X_{\big|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\big\rangle}}(1)$.

Remark 5.3.36. We assume now that: (i) immediately after the measurement at the instant t particle 1 is in the state $|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1$. In this case immediately after the measurement at the instant t unconditional measure $\mathbf{P}$ collapses to conditional measure $\mathbf{P}\Big(X \,\Big|A\Big(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1,t\Big)\Big)$ given by

$$\mathbf{P}\Big(X \,\Big|A^{Oc}\Big(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1,t\Big)\Big) = \frac{\mathbf{P}\Big(X \wedge A^{Oc}\Big(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1,t\Big)\Big)}{\mathbf{P}\Big(A\Big(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1,t\Big)\Big)} \qquad (5.3.71)$$

where $X \in \mathcal{F}^{ph}_{M_4}$,

(ii)immediately after the measurement at the instant t particle 1 is in the state $|H_{\alpha\pm\delta_\alpha}\rangle_1$. In this case immediately after the measurement at the instant t unconditional measure $\mathbf{P}$ collapses to conditional measure $\mathbf{P}\left(X \,\middle|\, A^{Oc}\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right)$ given by

$$\mathbf{P}\left(X \,\middle|\, A^{Oc}\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right) = \frac{\mathbf{P}\left(X \wedge A^{Oc}\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right)}{\mathbf{P}\left(A\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right)}, \tag{5.3.72}$$

where $X \in \mathcal{F}^{ph}_{M_4}$.

(1) From Eq. (5.3.71) and Eq. (5.3.69) - (5.3.70) we obtain

$$\mathbf{P}\left(A\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_2, t\right) \,\middle|\, A^{Oc}\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right) =$$

$$\frac{\mathbf{P}\left(A\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_2, t\right) \wedge A^{Oc}\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right)}{\mathbf{P}\left(A\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right)} =$$

$$\frac{\mathbf{P_B}\left(\Re\left[A\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_2, t\right)\right] \wedge \Re\left[A^{Oc}\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right]\right)}{\mathbf{P_B}\left(\Re\left[A\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right]\right)} = \tag{5.3.73}$$

$$\frac{\mathbf{P_B}\left(\Delta_{2,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} \cap \Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\right)}{\mathbf{P_B}\left(\Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\right)} = \frac{\mathbf{P_B}(\varnothing)}{\mathbf{P_B}\left(\Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\right)} = 0.$$

Therefore $A^{Oc}\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_2, t\right)$ and $A^{Oc}\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)$ are mutually exclusive (disjoint) physical events, i.e., they both cannot occur simultaneously:

$$A^{Oc}\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_2, t\right) \wedge A^{Oc}\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right) \notin \mathcal{F}^{ph}_{M_4}. \tag{5.3.74}$$

(2) From Eq. (5.3.71) and Eq. (5.3.69) - Eq.(5.3.70) we obtain

$$\mathbf{P}\left(A\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_2, t\right) \,\middle|\, A^{Oc}\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right) =$$

$$\frac{\mathbf{P}\left(A\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_2, t\right) \wedge A^{Oc}\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right)}{\mathbf{P}\left(A\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right)} =$$

$$\frac{\mathbf{P_B}\left(\Re\left[A\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_2, t\right)\right] \wedge \Re\left[A^{Oc}\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right]\right)}{\mathbf{P_B}\left(\Re\left[A\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right]\right)} = \tag{5.3.75}$$

$$\frac{\mathbf{P_B}\left(\Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} \cap \Delta_{1,\alpha\pm\delta_\alpha}\right)}{\mathbf{P_B}\left(\Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\right)} == \frac{\mathbf{P_B}\left(\Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\right)}{\mathbf{P_B}\left(\Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\right)} = 1.$$

Therefore physical events $A^{Oc}(|V_{\alpha\pm\delta_\alpha}\rangle_2, t)$ and $A^{Oc}(|V_{\alpha\pm\delta_\alpha}\rangle_1, t)$ always occur simultaneously even particle 1 and particle 2 are space-like separated:

$$A^{Oc}\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_2, t\right) \Leftrightarrow A^{Oc}\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right). \quad (5.3.76)$$

(3) From Eq. (5.3.72) and Eq. (5.3.69) - (5.3.70) we obtain

$$\begin{gathered}\mathbf{P}\left(A\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_2, t\right) \mid A^{Oc}\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right) = \\ \frac{\mathbf{P}\left(A\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_2, t\right) \wedge A^{Oc}\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right)}{\mathbf{P}\left(A\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right)} = \\ \frac{\mathbf{P_B}\left(\Re\left[A\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_2, t\right)\right] \wedge \Re\left[A^{Oc}\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right]\right)}{\mathbf{P_B}\left(\Re\left[A\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right]\right)} = \\ \frac{\mathbf{P_B}(\Delta_{2,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} \cap \Delta_{2,\alpha\pm\delta_\alpha})}{\mathbf{P_B}(\Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta})} = \frac{\mathbf{P_B}(\Delta_{2,\alpha\pm\delta_\alpha})}{\mathbf{P_B}(\Delta_{2,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta})} = 1.\end{gathered} \quad (5.3.77)$$

Therefore physical events $A^{Oc}(|H_{\alpha\pm\delta_\alpha}\rangle_2, t)$ and $A^{Oc}(|H_{\alpha\pm\delta_\alpha}\rangle_1, t)$ always occur simultaneously even particle 1 and particle 2 are space-like separated:

$$A^{Oc}\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_2, t\right) \Leftrightarrow A^{Oc}\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right). \quad (5.3.78)$$

(4) From Eq. (5.3.40) and Eq. (5.3.20) - Eq. (5.3.21) we obtain

$$\begin{gathered}\mathbf{P}\left(A\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_2, t\right) \mid A^{Oc}\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right) = \\ \frac{\mathbf{P}\left(A\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_2, t\right) \wedge A^{Oc}\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right)}{\mathbf{P}\left(A\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right)} = \\ \frac{\mathbf{P_B}\left(\Re\left[A\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_2, t\right)\right] \wedge \Re\left[A^{Oc}\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right]\right)}{\mathbf{P_B}\left(\Re\left[A\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1, t\right)\right]\right)} = \\ \frac{\mathbf{P_B}(\Delta_{1,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta} \cap \Delta_{2,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta})}{\mathbf{P_B}(\Delta_{2,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta})} = \frac{\mathbf{P_B}(\varnothing)}{\mathbf{P_B}(\Delta_{2,\alpha\pm\delta_\alpha,\beta\pm\delta_\beta})} = 0.\end{gathered} \quad (5.3.79)$$

Therefore $A^{Oc}\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_2,t\right)$ and $A^{Oc}\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1,t\right)$ are mutually exclusive (disjoint) physical events, i.e., they both cannot occur simultaneously:

$$A^{Oc}\left(|V_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_2,t\right) \wedge A^{Oc}\left(|H_{\alpha\pm\delta_\alpha,\beta\pm\delta_\beta}\rangle_1,t\right) \notin \mathcal{F}_{M_4}^{ph}. \qquad (5.3.80)$$

Chapter VI

SCHRÖDINGER'S CAT MEASURED SPIN. SCHRÖDINGER'S CAT PARADOX RESOLUTION

VI.1. Stern-Gerlach experiment revisited

In 1922, by studying the deflection of a beam of silver atoms in a strongly inhomogeneous magnetic field (Fig. 6.1.1) Otto Stern and Walter Gerlach obtained an experimental result that contradicts the common sense prediction: the beam, instead of expanding, splits into two separate beams giving two spots of equal intensity N^+ and N^- on a detector, at equal distances from the axis of the original beam. Historically, this is the experiment which helped establish spin quantization. Theoretically, it is the seminal experiment posing the problem of measurement in quantum mechanics.

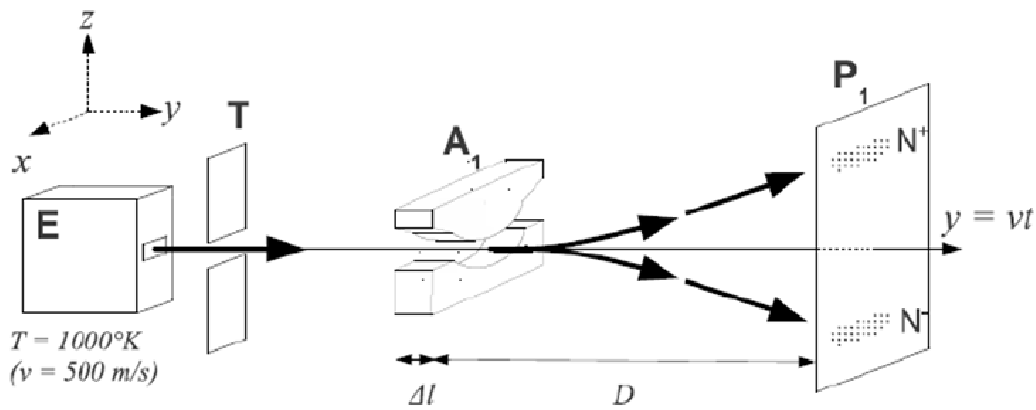

Fig. 6.1.1. Schematic configuration of the Stern-Gerlach experiment. Adapted from [19]

$$\Psi(z,y,t)|_{t=0} = \Psi^0(z,y) = \Psi_1^0(z)\Psi_2^0(y). \tag{6.1.1}$$

We assume now that both density $\Psi^0(z)$ and $\Psi^0(y)$ is very narrow, in fact constrained such that

$$\begin{aligned}\Psi_1^0(z) &= \Psi_1^0(z,\delta) = 0 \quad \text{iff} \quad |x| > \delta,\\ \Psi_2^0(y) &= \Psi_2^0(y,\delta) = 0 \quad \text{iff} \quad |y| > \delta,\end{aligned} \tag{6.1.2}$$

and

$$\begin{aligned}&\Psi_1^0(z) = \Psi_1^0(z,\delta) = (2\pi_{\sigma_0}\sigma_0^2)^{-\frac{1}{4}} e^{-\frac{z^2}{4\sigma_0^2}} \begin{pmatrix} \cos\dfrac{\theta_0}{2} e^{-i\frac{\varphi_0}{2}} \\ \sin\dfrac{\theta_0}{2} e^{i\frac{\varphi_0}{2}} \end{pmatrix} \text{ iff } |z| \leq \delta,\\ &\|\Psi_1^0(z,\delta)\|_2^2 = 1;\\ &\Psi_2^0(y) = \Psi_2^0(y,\delta) = (2\pi_{\sigma_0}\sigma_0^2)^{-\frac{1}{4}} e^{-\frac{y^2}{4\sigma_0^2}} \text{ iff } |y| \leq \delta,\\ &\|\Psi_2^0(y,\delta)\|_2^2 = 1\\ &\sigma_0 \ll 1.\end{aligned} \tag{6.1.3}$$

Silver atoms contained in the oven E (Fig. 6.1.1) are heated to a high temperature and escape through a narrow opening. A second aperture, T, selects those atoms whose velocity, $\mathbf{v}_0$, is parallel to the y-axis. The atomic beam crosses the gap of the electromagnet A_1 before condensing on the P_1 detector. Before crossing the electromagnet, the magnetic moment of each silver atom is oriented randomly (isotropically). In the beam, we represent each atom by its wave function; one can assume that at the entrance to the electromagnet, A_1, and at the initial time $t = 0$, each atom can be approximatively described by a quasi-Gaussian spinor in plain (z,y) given by Eqs. (6.1.1-6.1.3) corresponding to a pure state. As it will be proved later the variable y will be treated strictly quasiclassically, i.e. almost classically, with

$$\begin{aligned}\mathbf{P}\left\{\left| y - \mathbf{v}_y^{\pm}(\mathbf{v}_0,\theta_0)t \right| < \delta\right\} &= 1,\\ \mathbf{P}\left\{\left| y - \mathbf{v}_y^{\pm}(\mathbf{v}_0,\theta_0)t \right| \geq \delta\right\} &= 0\end{aligned} \tag{6.1.4}$$

and $\sigma_0 \leq \sigma_0' = 10^{-4}m,$ where σ_0' corresponds to the size of the slot T along the z-axis and where the expression of the functions $\mathbf{v}_y^{+}(\mathbf{v}_0,\theta_0)$

and $\mathbf{v}_y^-(\mathbf{v}_0,\theta_0)$ will be given later.

The approximation by a quasi-Gaussian initial spinor will allow explicit calculations. Because the slot is much wider along the x-axis, the variable z will be also treated strictly quasiclassically with

$$\mathbf{P}\left\{\left| z - z_\Delta^\pm - \mathbf{v}_z^\pm(u,\theta_0)t\right| \le \delta\right\} = 1,$$
$$\mathbf{P}\left\{\left| z - \mathbf{v}_z^\pm(u,\theta_0)t\right| > \delta\right\} = 0, \tag{6.1.5}$$

where the expression of the functions $\mathbf{v}_z^\pm(u,\theta_0), u = \frac{\mu_B B_0'(\Delta t)}{m}$ will be given later. In order to obtain an explicit solution of the Stern-Gerlach experiment, we take for the silver atom, we have $m = 1.8\times 10^{-25}kg, \mathbf{v}_0 = 500\ m/s$ (corresponding to the temperature $T = 1000°K$). In Eq. (6.1.3.) and in Fig. 6.1.2., θ_0 and φ_0 are the polar angles characterizing the initial orientation of the magnetic moment, θ_0 corresponds to the angle with the z-axis. The experiment is a statistical mixture of pure states where the θ_0 and the φ_0 are randomly chosen: θ_0 is drawn in a uniform way from $[0,\pi]$ and that φ_0 is drawn in a uniform way from $[0,2\pi]$.

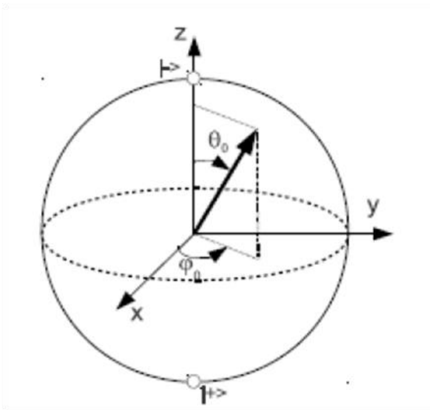

Fig. 6.1.2. Orientation of the magnetic moment θ_0 and φ_0 are the polar angles characterizing the spin vector in the de Broglie-Bohm interpretation. Adapted from [19]

Assumption 6.1.1. We assume that a particle collapses in a magnetic field $\mathbf{B}$ at some instant t' by two particles, i.e. the spinor $\Psi(z,y,t)$ collapses in a magnetic field $\mathbf{B}$ at some instant t' by two spinors $\Psi_{+}(z,y,t,t',\delta)$ and $\Psi_{-}(z,y,t,t',\delta)$ given by Eq. (6.1.9a) - Eq.(6.1.9b). Note that such a collapse obviously occurs except spinors such that: $\sqrt{2^{-1}}\left(\psi_{+}^{(z)}+\psi_{-}^{(z)}\right)=\psi_{+}^{(x)}$, etc.

Remark 6.1.1. Note that the standard assumption consists of that spinor collapses on detector P1 with respect to the Born rule. Thus the evolution of the spinor

$$\Psi(z,y,t,t') = \begin{pmatrix} \Psi_{+}(z,y,t,t') \\ \Psi_{-}(z,y,t,t') \end{pmatrix}$$

in a magnetic field $\mathbf{B}$ is then given by the nonlocal Pauli equation:

$$i\hbar\begin{pmatrix} \int dzdy \int dt \frac{\partial\Psi_{+}(z,y,t,t')}{\partial t} \\ \int dzdy \int dt \frac{\partial\Psi_{-}(z,y,t,t')}{\partial t} \end{pmatrix} = \\ = -\frac{\hbar^2}{2m}\int dt \int dzdy\Delta\begin{pmatrix} \Psi_{+}(z,y,t,t') \\ \Psi_{-}(z,y,t,t') \end{pmatrix} + \mu_B \int dt \int dzdy\mathbf{B}\sigma\begin{pmatrix} \Psi_{+}(z,y,t,t') \\ \Psi_{-}(z,y,t,t') \end{pmatrix} \qquad (6.1.6)$$

where $\mu_B = \frac{e\hbar}{2m_e}$ is the Bohr magneton and where $\sigma = (\sigma_x,\sigma_y,\sigma_z)$ corresponds to the three Pauli matrices.

Remark 6.1.2. First the particle enters an electromagnetic field $\mathbf{B}$ directed along the z-axis, $B_x = B_0'x, B_y = 0, B_z = B_0 - B_0'z,$ with $B_0 = 5\ Tesla$, $B_0' = \left|\frac{\partial B}{\partial z}\right| = 10^3\ Tesla/m$ over a length $\Delta l = 1\ cm$.

Remark 6.1.3. In exiting the magnetic field, the both particles are free until they reach the detector P_1 placed at distance $D = 20\ cm$.

The particles stay within the magnetic field for a time Δt with

$$\Delta t = \frac{\Delta l}{\mathbf{v}_0}. \qquad (6.1.7)$$

Assumption 6.1.2. We assume now for simplification that

$$t' \approx \Delta t. \tag{6.1.8}$$

Thus during this time $t \in [0, t') \approx [0, \Delta t)$, the spinor $\Psi(z, y, t, t', \delta)$ is:

$$\Psi(z,y,t,t',\delta) = \begin{pmatrix} \Psi_+(z,y,t,t',\delta) \\ \Psi_-(z,y,t,t',\delta) \end{pmatrix} = \begin{pmatrix} \Psi_+(z,t,t',\delta)\Psi(y,t,t',\delta) \\ \Psi_-(z,t,t',\delta)\Psi(y,t,t',\delta) \end{pmatrix}, \tag{6.1.9.a}$$

where

$$\Psi_+(z,t,t',\delta) = \cos\frac{\theta_0}{2} e^{-i\frac{\varphi_0}{2}} (2\pi\sigma_0^2)^{-\frac{1}{2}} \exp\left[-\frac{\left(z - \frac{\mu_B B_0'}{2m} t^2\right)^2}{4\sigma_0^2} \right] \times$$

$$\exp\left[i\frac{\mu_B B_0' t z - \frac{\mu_B^2 B_0'^2}{6m} t^3 + \mu_B B_0 t + 0.5\hbar\varphi_0}{\hbar} \right] \text{ iff } \left| z - \frac{\mu_B B_0'}{2m} t^2 \right| \leq \delta,$$

$$\Psi_+(z,t,\delta) = 0 \text{ iff } \left| z - \frac{\mu_B B_0'}{2m} t^2 \right| > \delta,$$

$$\Psi_-(z,t,t',\delta) = i\sin\frac{\theta_0}{2} e^{i\frac{\varphi_0}{2}} (2\pi\sigma_0^2)^{-\frac{1}{2}} \exp\left[-\frac{\left(z + \frac{\mu_B B_0'}{2m} t^2\right)^2}{4\sigma_0^2} \right] \times \tag{6.1.9.b}$$

$$\exp\left[i\frac{-\mu_B B_0' t z - \frac{\mu_B^2 B_0'^2}{6m} t^3 - \mu_B B_0 t - 0.5\hbar\varphi_0}{\hbar} \right] \text{ iff } \left| z + \frac{\mu_B B_0'}{2m} t^2 \right| \leq \delta,$$

$$\Psi_-(z,t,t',\delta) = 0 \text{ iff } \left| z + \frac{\mu_B B_0'}{2m} t^2 \right| > \delta;$$

$$\Psi(y,t,t',\delta) = (2\pi\sigma_0^2)^{-\frac{1}{2}} \exp\left[-\frac{(y - \mathbf{v}_0 t)^2}{4\sigma_0^2} \right] \text{ iff } |y - \mathbf{v}_0 t| \leq \delta,$$

$$\Psi(y,t,t',\delta) = 0 \text{ iff } |y - \mathbf{v}_0 t| > \delta.$$

After the magnetic field, at time $t + \Delta t$ $(t > 0)$ in the free space, the both spinors become:

$$\Psi_+(z,y,t+t',\delta) \simeq \Psi_+(z,y,t+\Delta t,\delta) = \Psi_+(z,t+\Delta t,\delta)\Psi(y,t+\Delta t,\delta) \tag{6.1.10}$$

and

$$\Psi_{-}(z,y,t+t',\delta) \simeq \Psi_{-}(z,y,t+\Delta t,\delta) = \Psi_{-}(z,t+\Delta t,\delta)\Psi(y,t+\Delta t,\delta). \qquad (6.1.11)$$

Here

$$\Psi_{+}(z,t+\Delta t,\delta) \simeq \begin{cases} \cos\frac{\theta_0}{2}(2\pi\sigma_0^2)^{-\frac{1}{2}}\exp\left[-\frac{(z-z_\Delta-ut)^2}{4\sigma_0^2}\right]e^{i\frac{muz+h\varphi_+}{h}} \text{ iff } |z-z_\Delta-ut| \leq \delta, \\ 0 \text{ iff } |z-z_\Delta-ut| > \delta \end{cases} \qquad (6.1.12)$$

and

$$\Psi_{-}(z,t+\Delta t,\delta) \simeq \begin{cases} \sin\frac{\theta_0}{2}(2\pi\sigma_0^2)^{-\frac{1}{4}}\exp\left[-\frac{(z+z_\Delta+ut)^2}{4\sigma_0^2}\right]e^{i\frac{-muz+h\varphi_-}{h}} \text{ iff } |z+z_\Delta+ut| \leq \delta, \\ 0 \text{ iff } |z+z_\Delta+ut| > \delta, \end{cases} \qquad (6.1.13)$$

and

$$\begin{gathered} \Psi(y,t,\delta) = (2\pi\sigma_0^2)^{-\frac{1}{4}}\exp\left[-\frac{(y-\mathbf{v}_0(t+\Delta t))^2}{4\sigma_0^2}\right] \text{ iff } |y-\mathbf{v}_0(t+\Delta_{\theta_0}t)| \leq \delta, \\ \Psi(y,t,\delta) = 0 \text{ iff } |y-\mathbf{v}_0(t+\Delta t)| > \delta. \end{gathered} \qquad (6.1.14)$$

where

$$z_\Delta = \frac{\mu_B B_0'([\Delta t]^2}{2m}, \qquad u = \frac{\mu_B B_0'(\Delta t)}{m}. \qquad (6.1.15)$$

From Eq. (6.1.10) - Eq. (6.1.12) and Eq. (6.1.14) we obtain

$$\Psi_{+}(z,y,t+\Delta t,\delta) = \begin{cases} \cos\frac{\theta_0}{2}(2\pi\sigma_0^2)^{-\frac{1}{2}}\exp\left[-\frac{(z-z_\Delta-ut)^2}{4\sigma_0^2}\right]e^{i\frac{muz+h\varphi_+}{h}} \times & \text{iff} \\ & |z-z_\Delta-ut| \leq \delta \\ \exp\left[-\frac{(y-\mathbf{v}_0(t+\Delta t))^2}{4\sigma_0^2}\right] & \text{and} \\ & |y-\mathbf{v}_0(t+\Delta t)| \leq \delta \\ 0 & \text{otherwise} \end{cases} \qquad (6.1.16)$$

From Eq. (6.1.16) by the postulate for the probability density with respect to observable z we obtain the expression

$$c_{|\Psi_+\rangle}(z,t) =$$

$$\begin{cases} (2\pi\sigma_0^2)^{-\frac{1}{2}} \exp\left[-\frac{\left(\frac{z}{\eta_{\theta_0}^+} - z_\Delta - ut \right)^2}{2\sigma_0^2} \right] & \text{iff } \left| \frac{z}{\eta_{\theta_0}^+} - z_\Delta - ut \right| \le \delta \\ 0 & \text{otherwise} \end{cases} \quad (6.1.17)$$

$$\eta_{\theta_0}^+ = \cos^2 \frac{\theta_0}{2}$$

and with respect to observable y we obtain the expression

$$c_{|\Psi_+\rangle}(y,t) =$$

$$\begin{cases} (2\pi\sigma_0^2)^{-\frac{1}{2}} \exp\left[-\frac{\left(\frac{y}{\eta_{\theta_0}^+} - \mathbf{v}_0(t+\Delta t) \right)^2}{4\sigma_0^2} \right] & \text{iff } \left| \frac{y}{\eta_{\theta_0}^+} - \mathbf{v}_0(t+\Delta t) \right| \le \delta \\ 0 & \text{otherwise} \end{cases} \quad (6.1.18)$$

$$\eta_{\theta_0}^+ = \cos^2 \frac{\theta_0}{2}$$

and therefore the corresponding particle moves by the strictly quasiclassical law

$$\begin{gathered} \mathbf{P}\left\{ \left| z^+(t) - z_\Delta^+ - \mathbf{v}_z^+(u,\theta_0)t \right| \le \delta \right\} = 1, \\ \mathbf{P}\left\{ \left| z^+(t) - z_\Delta^+ - \mathbf{v}_z^+(u,\theta_0)t \right| > \delta \right\} = 0, \\ \mathbf{P}\{ |y^+(t) - \mathbf{v}_y^+(\mathbf{v}_0,\theta_0)(t+\Delta t)| \le \delta \} = 1, \\ \mathbf{P}\{ |y^+(t) - \mathbf{v}_y^+(\mathbf{v}_0,\theta_0)(t+\Delta t)| > \delta \} = 0, \\ z_\Delta^+ = \eta_{\theta_0}^+ z_\Delta, \\ \mathbf{v}_z^+(\theta_0) = \eta_{\theta_0}^+ u, \mathbf{v}_y^+(\theta_0) = \eta_{\theta_0}^+ \mathbf{v}_0. \end{gathered} \quad (6.1.19)$$

From Eq. (6.1.11), Eq. (6.1.13) and Eq. (6.1.14) we obtain

$$\Psi_{-}(z,y,t+\Delta t,\delta) =$$

$$\begin{cases} \sin\frac{\theta_0}{2}(2\pi\sigma_0^2)^{-\frac{1}{4}}\exp\left[-\frac{(z+z_\Delta+ut)^2}{4\sigma_0^2}\right]e^{i\frac{-muz+\hbar\varphi_-}{\hbar}}\times & \text{iff} \\ & |z+z_\Delta+ut| \le \delta \\ \exp\left[-\frac{(y-\mathbf{v}_0(t+\Delta t))^2}{4\sigma_0^2}\right] & \text{and} \\ & |y-\mathbf{v}_0(t+\Delta_{\theta_0}t)| \le \delta \\ 0 & \text{otherwise} \end{cases} \tag{6.1.20}$$

From Eq. (6.1.20) by the postulate for the probability density with respect to observable z we obtain the expression

$$c_{|\Psi_-\rangle}(z,t) =$$

$$\begin{cases} (2\pi\sigma_0^2)^{-\frac{1}{2}}\exp\left[-\frac{\left(\frac{z}{\eta_{\theta_0}^-}+z_\Delta+ut\right)^2}{2\sigma_0^2}\right] & \text{iff } \left|\frac{z}{\eta_{\theta_0}^-}+z_\Delta+ut\right| \le \delta \\ 0 & \text{otherwise} \end{cases} \tag{6.1.21}$$

$$\eta_{\theta_0}^- = \sin^2\frac{\theta_0}{2}$$

and with respect to observable y we obtain the expression

$$c_{|\Psi_-\rangle}(y,t) =$$

$$\begin{cases} (2\pi\sigma_0^2)^{-\frac{1}{2}}\exp\left[-\frac{\left(\frac{y}{\eta_{\theta_0}^-}-\mathbf{v}_0(t+\Delta t)\right)^2}{4\sigma_0^2}\right] & \text{iff } \left|\frac{y}{\eta_{\theta_0}^-}-\mathbf{v}_0(t+\Delta_{\theta_0}t)\right| \le \delta \\ 0 & \text{otherwise} \end{cases} \tag{6.1.22}$$

$$\eta_{\theta_0}^- = \sin^2\frac{\theta_0}{2}$$

and therefore the corresponding particle moves by the strictly quasiclassical law

$$\begin{gathered} \mathbf{P}\left\{\left|z^-(t)+z_\Delta^-+\mathbf{v}_z^-(u,\theta_0)t\right| \le \delta\right\} = 1, \\ \mathbf{P}\left\{\left|z^-(t)+z_\Delta^-+\mathbf{v}_z^-(u,\theta_0)t\right| > \delta\right\} = 0, \\ \mathbf{P}\{|y^-(t)-\mathbf{v}_y^-(\mathbf{v}_0,\theta_0)(t+\Delta t)| \le \delta\} = 1, \\ \mathbf{P}\{|y^-(t)-\mathbf{v}_y^-(\mathbf{v}_0,\theta_0)(t+\Delta t)| > \delta\} = 0, \\ z_\Delta^- = \eta_{\theta_0}^- z_\Delta, \\ \mathbf{v}_z^-(u,\theta_0) = \eta_{\theta_0}^- u, \mathbf{v}_y^-(\mathbf{v}_0,\theta_0) = \eta_{\theta_0}^- \mathbf{v}_0. \end{gathered} \tag{6.1.23}$$

All interpretations are based on the Eq. (6.1.18) - (6.1.21). One deduce from Eq. (6.1.18) - (6.1.21) the probability density of a pure state in the free space after the electromagnet:

$$\rho_{\theta_0}(z,y,t+\Delta t) = (2\pi\sigma_0^2)^{-\frac{1}{2}}\rho_{\theta_0}(z,t+\Delta_{\theta_0}t)\sum_{\pm}\exp\left[-\frac{\left(\frac{y}{\eta_{\theta_0}^{\pm}}-\mathbf{v}_0(t+\Delta t)\right)^2}{4\sigma_0^2}\right];$$

$$\rho_{\theta_0}(z,t+\Delta t) = (2\pi\sigma_0^2)^{-\frac{1}{2}}\left\{\cos^{-2}\frac{\theta_0}{2}(2\pi\sigma_0^2)^{-\frac{1}{2}}\exp\left[-\frac{(\frac{z}{\eta_{\theta_0}^{+}}-z_\Delta-ut)^2}{2\sigma_0^2}\right]+\right. \tag{6.1.24}$$

$$\left.\sin^{-2}\frac{\theta_0}{2}(2\pi\sigma_0^2)^{-\frac{1}{2}}\exp\left[-\frac{(\frac{z}{\eta_{\theta_0}^{-}}+z_\Delta+ut)^2}{4\sigma_0^2}\right]\right\}.$$

The decoherence time $t_{\mathbf{dec}}$, where the two spots N^+ and N^- are separated, is then given by the equation:

$$t_{\mathbf{dec}} \asymp \frac{3\sigma_0 - z_\Delta}{u(\eta_{\theta_0}^{+}+\eta_{\theta_0}^{-})} = \frac{3\sigma_0 - z_\Delta}{u}. \tag{6.1.25}$$

This decoherence time is usually the time required to diagonalize the marginal density matrix $\rho_{\theta_0}^{S}(t,\delta)$ of spin variables associated with a pure state

$$\rho_{\theta_0}^{S}(t,\delta) = \begin{pmatrix} \int|\Psi_+(z,y,t+\Delta t,\delta)|^2dzdy & \int\Psi_-^*(z,y,t+\Delta t,\delta)\Psi_+(z,y,t+\Delta t,\delta)dzdy \\ \int\Psi_-(z,y,t+\Delta t,\delta)\Psi_+^*(z,y,t+\Delta t,\delta)dzdy & \int|\Psi_+(z,y,t+\Delta t,\delta)|^2dzdy \end{pmatrix} \tag{6.1.26}$$

For $t \geq t_{\mathbf{dec}}$, the product $\Psi_-(z,y,t+\Delta t,\delta)\Psi_+^*(z,y,t+\Delta t,\delta)$ is null and the density matrix $\rho_{\theta_0}^{S}(t,\delta)$ is diagonal. We then obtain atoms with a spin oriented only along the z -axis (positively or negatively). Let us consider the spinor $\Psi(z,y,t+\Delta t,\delta)$ given by Eq. (6.1.10) - (6.1.15).

Remark 6.1.4. Experimentally, we do not measure the spin directly, but the z position of the particle impact on the detector P1 (Fig.6.1.3.).

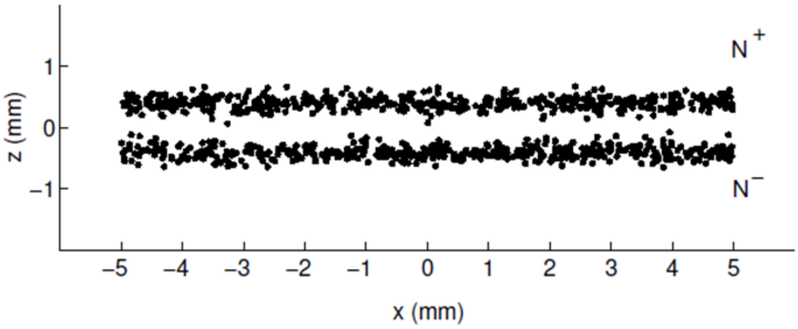

Fig. 6.1.3. Silver atom impacts on the detector P1.

Adapted from [19]

Remark 6.1.5. Note that if we measure the z -position of the particle at the instant $t,$ we also measure the y -position of the particle at the same instant $t.$

Remark 6.1.6. Let $\mathbf{P}_t(D-\delta, D, y_t^+)$ be the probability of obtaining the result y_t^+ at the instant $t,$ lying in the range $(D-\delta, D)$ on measuring observable y in respect to spinor $\Psi_+(z, y, t+\Delta t, \delta).$ From Eq. (6.1.19) we obtain

$$\begin{gathered}\mathbf{P}_t(D-\delta, D+\delta, y_t^+) = 1 \text{ iff} \\ y^+(t) = D \text{ and } \left|y^+(t) - \mathbf{v}_y^+(\mathbf{v}_0, \theta_0)(t+\Delta t)\right| \leq \delta.\end{gathered} \tag{6.1.27}$$

From Eq. (6.1.27) it follows that:

$$\mathbf{P}_t(D-\delta, D+\delta, y_t^+) = 1$$
if
$$t \triangleq t_+(D) \approx \frac{D}{\mathbf{v}_y^+(\mathbf{v}_0,\theta_0)} = \frac{D}{\mathbf{v}_0 \cos^2 \frac{\theta_0}{2}}. \qquad (6.1.28)$$

Remark 6.1.7. Let $\mathbf{P}_t(\widetilde{z}_+ - \delta, \widetilde{z}_+ + \delta, z_t^+)$ be the probability of obtaining the result z_t^+ at the instant $t,$ lying in the range $(\widetilde{z}_+ - \delta, \widetilde{z}_+ + \delta), \widetilde{z}_+ \in N^+$ on measuring observable z in respect to spinor $\Psi_+(z,y,t+\Delta t,\delta).$ From Eq.(6.1.19) we obtain

$$\mathbf{P}_t(\widetilde{z}_+ - \delta, \widetilde{z}_+ + \delta, z_t^+) = 1 \text{ iff}$$
$$z^+(t) = \widetilde{z}_+ \text{ and } |z^+(t) - z_\Delta^+ - \mathbf{v}_z^+(u,\theta_0)t| \leq \delta. \qquad (6.1.29)$$

From Eq. (6.1.29) it follows that:

$$\mathbf{P}_t(\widetilde{z}_+ - \delta, \widetilde{z}_+ + \delta, z_t^+) = 1$$
if
$$t \triangleq t(\widetilde{z}_+) \approx \frac{\widetilde{z}_+}{\mathbf{v}_z^+(u,\theta_0)} = \frac{\widetilde{z}_+}{u \cos^2 \frac{\theta_0}{2}}. \qquad (6.1.30)$$

Remark 6.1.8. Note that from Remark 6.1.6 it follows that $t(\widetilde{z}_+) \approx t(D)$ and therefore from Eq. (6.1.28) and Eq. (6.1.30) one obtains

$$\frac{\widetilde{z}_+}{u \cos^2 \frac{\theta_0}{2}} \approx \frac{D}{\mathbf{v}_0 \cos^2 \frac{\theta_0}{2}} \Rightarrow \frac{\widetilde{z}_+}{u} \approx \frac{D}{\mathbf{v}_0} \qquad (6.1.31)$$

as it should be, because the equality $\frac{\widetilde{z}_+}{u} \approx \frac{D}{\mathbf{v}_0}$ is required by the condition of the Stern-Gerlach experiment.

Remark 6.1.9. Let $\mathbf{P}_t(D-\delta, D, y_t^-)$ be the probability of obtaining the result y_t^- at the instant $t,$ lying in the range $(D-\delta, D)$ on measuring observable y in respect to spinor $\Psi_-(z,y,t+\Delta t,\delta).$ From Eq. (6.1.23) we obtain

$$\begin{gathered}\mathbf{P}_t(D-\delta, D+\delta, y_t^-) = 1 \text{ iff} \\ y^-(t) = D \text{ and } \left|y^-(t) - \mathbf{v}_y^-(\mathbf{v}_0, \theta_0)(t+\Delta t)\right| \leq \delta.\end{gathered} \tag{6.1.32}$$

From Eq. (6.1.32) it follows that:

$$\begin{gathered}\mathbf{P}_t(D-\delta, D+\delta, y_t^-) = 1 \\ \text{if} \\ t \triangleq t_-(D) \approx \frac{D}{\mathbf{v}_y^-(\mathbf{v}_0, \theta_0)} = \frac{D}{\mathbf{v}_0 \sin^2 \frac{\theta_0}{2}}.\end{gathered} \tag{6.1.33}$$

Remark 6.1.10. Let $\mathbf{P}_t(\widetilde{z}_- - \delta, \widetilde{z}_- + \delta, z_t^-)$ be the probability of obtaining the result z_t^- at the instant $t,$ lying in the range $(\widetilde{z}_- - \delta, \widetilde{z}_- + \delta), \widetilde{z}_- \in N^-$ on measuring observable z in respect to spinor $\Psi_-(z, y, t+\Delta t, \delta).$ From Eq.(6.1.32) we obtain

$$\begin{gathered}\mathbf{P}_t(\widetilde{z}_- - \delta, \widetilde{z}_- + \delta, z_t^-) = 1 \text{ iff} \\ z^-(t) = \widetilde{z}_- \text{ and } |z^-(t) - z_\Delta^- - \mathbf{v}_z^-(u, \theta_0)t| \leq \delta.\end{gathered} \tag{6.1.34}$$

From Eq. (6.1.29) it follows that:

$$\begin{gathered}\mathbf{P}_t(\widetilde{z}_- - \delta, \widetilde{z}_- + \delta, z_t^-) = 1 \\ \text{if} \\ t \triangleq t(\widetilde{z}_-) \approx \frac{|\widetilde{z}_-|}{\mathbf{v}_z^-(u, \theta_0)} = \frac{|\widetilde{z}_-|}{u \sin^2 \frac{\theta_0}{2}}.\end{gathered} \tag{6.1.35}$$

Remark 6.1.11. Note that from Remark 6.1.5 it follows that $t(\widetilde{z}_+) \approx t(D)$ and therefore from Eq. (6.1.33) and Eq. (6.1.35) one obtains

$$\frac{|\widetilde{z}_-|}{u \sin^2 \frac{\theta_0}{2}} \approx \frac{D}{\mathbf{v}_0 \sin^2 \frac{\theta_0}{2}} \Rightarrow \frac{\widetilde{z}_-}{u} \approx \frac{D}{\mathbf{v}_0} \tag{6.1.36}$$

as it should be, because the equality $\frac{|\widetilde{z}_-|}{u} \approx \frac{D}{\mathbf{v}_0}$ is required by the condition of the Stern-Gerlach experiment.

VI.2. Schrödinger's cat which measures spin. Schrödinger's cat paradox resolution

Another known in literature special sort of the Schrödinger cat paradox can be simply illustrated with the famous Stern-Gerlach experiment (Fig. 6.1.1). Silver atoms boiled off from a furnace are sent through a non-uniform magnetic field, and impinge on a photographic plate. Instead of a continuous distribution of spots, one sees two spots, corresponding to spin up and spin down relative to the magnetic field axis. Each atom goes up OR down, but one cannot predict which in any given run - the results of the experiment are probabilistic. There is a 50% chance of an atom going up, and a 50% chance that it will go down.

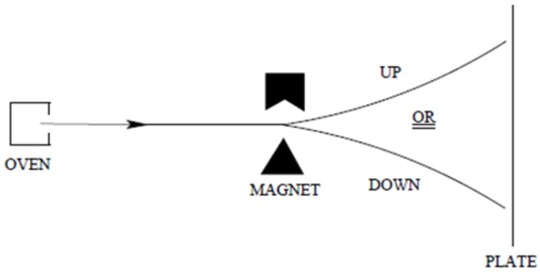

Fig. 6.1.4. Stern-Gerlach experiment. Adapted from [20]

Remark 6.2.1. We remind that from the point of view of the Schrödinger equation of the quantum theory, this result has no any rigorous explanation.

Remark 6.2.2. In the quantum theory, the state of the particle is described by its wave function, and the Schrödinger equation says that at a post-measurement final time $t_{\mathbf{f}}$, the wave function is related to that at a pre-measurement initial time $t_{\mathbf{i}}$, by known deterministic relation

$$\Psi(t_f) = U(t_f, t_i)\Psi(t_i),$$
$$U(t_f, t_i) = \exp\left[i\widehat{H}(t_f - t_i) \right]$$

with the transition unitary operator U completely specified by the Hamiltonian $\widehat{H}$. To explain what is observed, the Schrödinger equation must be supplemented by the reduction postulate and the Born rule.

This state that the wave function only gives a description of probabilities when a measurement is made, with the probabilities for an ‘up’ outcome and a ‘down’ outcome given by the squares of the coefficients of the corresponding components in the initial wave function $\Psi(t_i)$, with the sum of the ‘up’ and ‘down’ probabilities equal to one.

The reduction postulate and the Born rule are an add-on to the Schrödinger equation. According to the Copenhagen interpretation of quantum mechanics, the Schrödinger equation is applied when a microscopic system, the silver atom, is time-evolving in isolation. But when the atom interacts with a macroscopic measuring apparatus, as in the Stern-Gerlach setup, you have to use the reduction postulate and the Born rule.

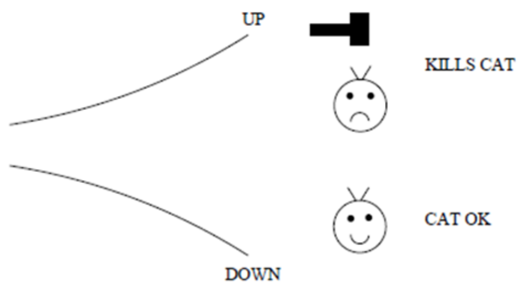

Fig. 6.1.5. The Stern-Gerlach apparatus with a Schrödinger cat as the outcome registration. Adapted from [20]

Remark 6.2.3. This situation leads to puzzles that have been debated for over eighty years. If quantum mechanics describes the whole universe, then why can't one use the Schrödinger equation to describe the system consisting of the silver atom plus the measuring apparatus? But we never see a superposition state of the atom plus apparatus. This is the Schrödinger's famous cat paradox. Arrange the experiment so that an up outcome triggers a mechanism that kills the cat, while a down outcome keeps the cat alive. Of course we don't do this, but if we were to do it, we would always see a live cat OR a dead one, never a superposition of the two (Fig. 6.1.5). So we have the problem of definite outcomes: where does the either - or dichotomy arises?
Let us consider again the Schrödinger's cat which measures spin by using the Stern-Gerlach apparatus, see (Fig. 6.1.4). When a measurement is made, with the up outcome Schrödinger's cat is dead. When a measurement is made, with the down outcome Schrödinger's cat is alive. As it is known many years that conventional QM with canonical explanation of the Stern-Gerlach experiment cannot give predicable and definite outcomes for Schrödinger's cat which measures spin.
Theorem 6.2.1. Any spinor

$$\begin{pmatrix} \Psi_{+}(z,y,t,t') \\ \Psi_{-}(z,y,t,t') \end{pmatrix} \tag{6.2.1}$$

given by Eq. (6.1.9a) - (6.1.9b) with θ_0 such that $\cos\frac{\theta_0}{2} \neq 0$ always kills the Schrödinger's cat at the instant t :

$$t \approx \frac{D}{\mathbf{v}_0 \cos^2\frac{\theta_0}{2}}. \tag{6.2.2}$$

Proof. Immediately from Eq. (6.1.31) and Eq. (6.1.36).

Chapter VII

THE BELL INEQUALITIES REVISITED

One of the Bell's assumptions in the original derivation of his inequalities was the hypothesis of locality, i.e., the absence of the influence of two remote measuring instruments on one another. That is why violations of these inequalities observed in experiments are often interpreted as a manifestation of the nonlocal nature of quantum mechanics, or a refutation of a local realism. In [1], [2] Bell's inequality was derived in its traditional form, without resorting to the hypothesis of locality and without the introduction of hidden variables, the only assumption being that the probability distributions are nonnegative. This can therefore be regarded as a rigorous proof that the hypothesis of locality and the hypothesis of existence of the hidden variables not relevant to violations of Bell's inequalities. The physical meaning of the obtained results is examined. Physical nature of the violation of the Bell inequalities is explained (see VII.2) under EPR-B nonlocality postulate.

VII.1. Bell theorem without the hypothesis of locality and without the introduction of hidden variables

VII.1.1. Clauser-Horne-Shimony-Holt (CHSH) inequality

In a typical Bell experiment, two systems which may have previously interacted - for instance they may have been produced by a common source - are now spatially separated and are each measured by one of two distant observers, Alice and Bob (see Fig.7.2.1). Alice may choose one out of several possible measurements to perform on her system and we let x denote her measurement choice. For instance, x may refer to the position of a knob on her measurement apparatus. Similarly, we let y denote

Bob's measurement choice. Once the measurements are performed, they yield outcomes a and b on the two systems.

Remark 7.1.1. The actual values assigned to the measurement choices x,y and outcomes a,b are purely conventional; they are mere macroscopic labels distinguishing the different possibilities.

Remark 7.1.2. From one run of the experiment to the other, the outcomes a and b that are obtained may vary, even when the same choices of measurements x and y are made.

Assumption 7.1.1. These outcomes a and b are thus in general governed by a Kolmogorovian probability distribution $p(ab|xy),$ which can of course depend on the particular experiment being performed. By repeating the experiment a sufficient number of times and collecting the observed data, one can get a fair estimate of such Kolmogorovian probabilities [3]-[4].

Assumption 7.1.2. When such an experiment is actually performed-say, by generating pairs of spin $-1/2$ particles and measuring the spin of each particle in different directions - it will in general be found that

$$p(ab|xy) \neq p(a|x)p(b|y), \qquad (7.1.1)$$

implying that the outcomes on both sides are not statistically independent of each other. Even though the two systems may be separated by a large distance – and may even be space-like separated - the existence of such correlations is nothing mysterious. In particular, it does not necessarily imply some kind of direct influence of one system on the other, for these correlations some dependence relation between the two systems which was established when they interacted in the past may simply reveal.
This is at least what one would expect in a local theory.

Let us formulate the idea of a local theory more precisely.

Assumption 7.1.3. The assumption of locality implies that we should be able to identify a set of past factors, described by some variables λ, having a joint causal influence on both outcomes, and which fully account for the dependence between a and b. Once all such factors have been taken into account, the residual indeterminacies about the outcomes must now be decoupled, that is, the Kolmogorovian probabilities for a and b should factorize:

$$p(ab|xy,\lambda) = p(a|x,\lambda)p(b|y,\lambda). \tag{7.1.2}$$

Remark 7.1.3. This factorability condition simply expresses that we have found an explanation according to which the probability for a only depends on the past variables λ and on the local measurement x, but not on the distant measurement and outcome, and analogously for the probability to obtain b.

The variable λ will not necessarily be constant for all runs of the experiment, even if the procedure which prepares the particles to be measured is held fixed, because λ may involve physical quantities that are not fully controllable. The different values of λ across the runs should thus be characterized by a probability distribution $q(\lambda)$. Combined with the above factorability condition, we can thus write

$$p(ab|xy) = \int_{\Lambda} d\lambda q(\lambda)p(a|x,\lambda)p(b|y,\lambda), \tag{7.1.3}$$

where we also implicitly assumed that the measurements x and y can be freely chosen in a way that is independent on λ, i.e., that $q(\lambda|x,y) = q(\lambda)$. This decomposition now represents a precise condition for locality in the context of Bell experiments.

Remark 7.1.4. Note that no assumptions of determinism or of a classical behaviour are being involved in the condition (7.1.3): we assumed that a (and similarly b) is only probabilistically determined by the measurement x and the variable λ, with no restrictions on the physical laws governing this causal relation. Locality is the crucial assumption behind (7.1.3). In relativistic terms, it is the requirement that events in one region of space-time should not influence events in space-like separated regions.

Let us consider for simplicity an experiment where there are only two measurement choices per observer $x,y \in \{0,1\}$ and where the possible outcomes take also two values labelled $a,b \in \{-1,+1\}$. Let $\langle a_x b_y \rangle$ be the expectation value of the product ab for given measurement choices (x,y) :

$$\langle a_x b_y \rangle = \sum_{a,b} abp(ab|xy). \tag{7.1.4}$$

Consider the following expression

$$S = \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle, \tag{7.1.5}$$

which is a function of the probabilities $p(ab|xy)$. If these probabilities satisfy the locality decomposition (7.1.3), we necessarily have that

$$S = \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle \leq 2, \tag{7.1.6}$$

which is known as the Clauser-Horne-Shimony-Holt (CHSH) inequality [5].

To derive this inequality, we can use (7.1.3) in the Definition (7.1.4) of $\langle a_x b_y \rangle$, which allows us to express this expectation value as an average

$$\langle a_x b_y \rangle = \int_\Lambda d\lambda q(\lambda) d\lambda q(\lambda) \langle a_x \rangle_\lambda \langle b_y \rangle_\lambda \tag{7.1.7}$$

of a product of local expectations

$$\langle a_x \rangle_\lambda = \sum_a ap(a|x,\lambda) \qquad (7.1.8)$$

and

$$\langle b_y \rangle_\lambda = \sum_b bp(b|y,\lambda) \qquad (7.1.9)$$

taking values in $[-1,1]$. Inserting these expressions (7.1.7) - (7.1.9) in Eq. (7.1.5), we can write

$$S = \int_\Lambda d\lambda q(\lambda) S_\lambda, \qquad (7.1.10)$$

where

$$S_\lambda = \langle a_0 \rangle_\lambda \langle b_0 \rangle_\lambda + \langle a_0 \rangle_\lambda \langle b_1 \rangle_\lambda + \langle a_1 \rangle_\lambda \langle b_0 \rangle_\lambda - \langle a_1 \rangle_\lambda \langle b_1 \rangle_\lambda. \qquad (7.1.11)$$

Since $\langle a_0 \rangle_\lambda, \langle b_0 \rangle_\lambda \in [-1,1]$, this last expression is smaller than S'_λ

$$S_\lambda \le S'_\lambda = |\langle b_0 \rangle_\lambda + \langle b_1 \rangle_\lambda| + |\langle b_0 \rangle_\lambda - \langle b_1 \rangle_\lambda|. \qquad (7.1.12)$$

Without loss of generality, we can assume that $\langle b_0 \rangle_\lambda \ge \langle b_1 \rangle_\lambda \ge 0$ which yields $S_\lambda \le 2\langle b_0 \rangle_\lambda \le 2$ and thus $S \le 2$.

Consider now the quantum predictions for an experiment in which the two systems measured by Alice and Bob are two qubits in the singlet state $\Psi^- = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$, where we have used the shortcut notation $|ab\rangle = |a\rangle \otimes |b\rangle$, and where $|0\rangle$ and $|1\rangle$ are conventionally the eigenstates of σ_z for the eigenvalues $+1$ and -1 respectively.

Let the measurement choices x and y be associated with vectors $\vec{x}$ and $\vec{y}$ corresponding to measurements of $\vec{x} \cdot \vec{\sigma}$ on the first qubit and of $\vec{y} \cdot \vec{\sigma}$ on the second qubit, where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ denotes the Pauli vector. According to the quantum theory we (then) have the expectations $\langle a_x b_y \rangle = -\vec{x}\cdot\vec{y}$. Let the two settings $x \in \{0,1\}$ correspond to measurements in the orthogonal directions $\hat{e}_1$ and $\hat{e}_2$ respectively and the settings $y \in \{0,1\}$ to measurements in the directions $-\frac{1}{\sqrt{2}}(\hat{e}_1 + \hat{e}_2)$ and $\frac{1}{\sqrt{2}}(-\hat{e}_1 + \hat{e}_2)$. We then have $\langle a_0 b_0 \rangle = \langle a_0 b_1 \rangle = \langle a_1 b_0 \rangle = \frac{1}{\sqrt{2}}$ and $-\langle a_1 b_1 \rangle = -\frac{1}{\sqrt{2}}$ whence

$$S = 2\sqrt{2} \tag{7.1.13}$$

in contradiction with CHSH inequality (7.1.6).

VII.1.2. Clauser Horne Inequality

Suppose that some observable of the two particles is registered as a count in a detector.

If the composite state consists of two photons, the detector registers a hit if the polarization is along some direction. The inequality will be determined by counting. There will be a total of N events, with $N_1(a)$ counts in detector 1 when it is set to select a and $N_2(b)$ counts in detector 2 when it is set to select b. The number of coincidences of the two detectors with settings a and b respectively is $N_{12}(a,b)$. The probabilities are

$$p_1(a) = \frac{N_1(a)}{N}, p_2(\hat{b}) = \frac{N_2(b)}{N}, p_{12}(\hat{a}) = \frac{N_{12}(a,b)}{N}. \tag{7.1.14}$$

Remember that in the Bell formulation, the hidden variable determined absolutely the value of the polarization for a particular measurement. Then

$$p_1(a) = \int d\lambda w(\lambda) p_1(a,\lambda), p_2(b) = \int d\lambda w(\lambda) p_2(b,\lambda),$$
$$p_{12}(a,b,\lambda) = p_1(a,\lambda) p_2(b,\lambda), \tag{7.1.15}$$
$$p_{12}(a,b) = \int d\lambda w(\lambda) p_1(a,\lambda) p_2(b,\lambda).$$

Remind that for any four real numbers $x, x', y, y' \in [0,1]$ the inequality holds

$$xy - xy' + x'y + x'y' \leq x' + y. \tag{7.1.16}$$

We denote now

$$x = p_1(a,\lambda), y = p_2(b,\lambda), x' = p_1(a',\lambda), y' = p_2(b',\lambda), \tag{7.1.17}$$

and substitute into inequality (7.1.16) we get

$$p_1(a,\lambda)p_2(b,\lambda) - p_1(a,\lambda)p_2(b',\lambda) + p_1(a',\lambda)p_2(b,\lambda) + p_1(a',\lambda)p_2(b',\lambda) \leq \\ \leq p_1(a',\lambda) + p_2(b,\lambda). \tag{7.1.18}$$

Next multiplying by $w(\lambda)$ and integrating over all λ we get

$$p_{12}(a,b) - p_{12}(a,b') + p_{12}(a',b) + p_{12}(a',b') \leq p_1(a') + p_2(b). \tag{7.1.19}$$

Consider now the quantum predictions for an experiment with 2 photons.

An atomic s-state with zero total angular momentum and even parity decays in two steps.

Photon γ_1 is emitted in the $E1$ transition from the S-state to a P-state with $m = \pm 1, 0.$

Photon γ_2 is emitted in the second $E1$ transition to the ground state. The initial state of the atom also has zero angular momentum and even parity. Therefore the photons which are emitted back to back have the same helicity, so that their total angular momentum is zero.

The two photons have different energies, ω_1 and ω_2. The helicity of each of the photons is determined by the intermediate state. If the intermediate state is $m = +1$ then the helicity of both photons is odd and if $m = -1$ then the helicities are even. The energy of the intermediate state is degenerate. There is no magnetic field that might split the energies of the $m = \pm 1, 0$ levels. The final pure photon state is therefore the linear combination

$$|\alpha\rangle = \frac{1}{\sqrt{2}}(|+1\rangle \otimes |+1\rangle + |-1\rangle \otimes |-1\rangle). \tag{7.1.20}$$

It will be more interesting if the measurements of the photon polarizations are in the linear basis. Then we can look for correlations of the measurement of linear polarization by detector 1 along a and by 2 along b. So let us write $|\alpha\rangle$ in the linear polarization basis. The linear and circular polarization bases are related according to

$$|x, y, k\rangle = \frac{1}{\sqrt{2}}(|1\rangle \pm i|-1\rangle) \tag{7.1.21}$$

and

$$|x, y, -k\rangle = \frac{1}{\sqrt{2}}(|1\rangle \mp i|-1\rangle). \tag{7.1.22}$$

Thus we can rewrite

$$|\alpha\rangle = \frac{1}{\sqrt{2}}(|x\rangle_1 \otimes |x\rangle_2 + |y\rangle_1 \otimes |y\rangle_2). \tag{7.1.23}$$

Evidently if detector 1 measures horizontal polarization then so will detector 2, etc. In general we want to measure the correlation $p_{12}(\theta_1, \theta_2)$, that is the probability that we get a count on detector 1 with polarization axis θ_1 coincident with a count in detector 2 with polarization axis θ_2. The observable is the operator

$$\Gamma(\theta_1,\theta_2) = |\theta_1\rangle_1|\theta_2\rangle_2\langle\theta_1|_1\langle\theta_2|_2. \tag{7.1.24}$$

Assume now that

$$\Gamma(\theta_1,\theta_2) = \Gamma(\theta_1-\theta_2), \tag{7.1.25}$$

since there is zero angular momentum in the final state, there is rotation symmetry so the observable can only depend on the difference of the polarization angles. The expectation value of Γ is

$$\begin{gathered}\langle\Gamma(\theta_1-\theta_2)\rangle = \langle\alpha|\Gamma(\theta_1-\theta_2)|\alpha\rangle = \\ \tfrac{1}{2}(\langle x|_1\langle x|_2 + \langle y|_1\langle y|_2)|\theta_1\rangle_1|\theta_2\rangle_2\langle\theta_1|_1\langle\theta_2|_2(|x\rangle_1|x\rangle_2 + |y\rangle_1|y\rangle_2).\end{gathered} \tag{7.1.26}$$

Note that $\langle x|\theta\rangle = \cos\theta, \langle y|\theta\rangle = \sin\theta.$ Finally one obtains

$$\langle\Gamma(\theta_1-\theta_2)\rangle = \tfrac{1}{4}(1+\cos 2(\theta_1-\theta_2)). \tag{7.1.27}$$

The Clauser Horne inequality is

$$\frac{N_{12}(a,b)+N_{12}(b,a')+N_{12}(a',b')-N_{12}(a,b')}{N_1(a')+N_2(b)} \leq 1. \tag{7.1.28}$$

Assume now that a,b,a',b' are all separated by the angle φ then

$$\frac{N_{12}(\varphi)+N_{12}(\varphi)+N_{12}(\varphi)-N_{12}(3\varphi)}{N_1(a')+N_2(b)} = \frac{3N_{12}(\varphi)-N_{12}(3\varphi)}{N_1(a')+N_2(b)} \leq 1. \tag{7.1.29}$$

Next relate coincidences to expectation values. Note that $N_{12}(\varphi) = N\langle\alpha|\Gamma(\varphi)|\alpha\rangle.$ As regards the singles counts $N_1(a')$ and $N_2(b)$, we know that the number of counts must be independent of the direction of a' or b and that for any direction $N_1 = \frac{1}{2}N$, since half the photons will be polarized along and direction. Therefore the inequality (7.1.29) becomes

$$\Delta(\varphi) = \frac{\frac{3}{4}(1+\cos 2\varphi) - \frac{1}{4}(1+\cos 6\varphi)}{\frac{1}{2}+\frac{1}{2}} = \frac{1}{2} + \frac{3}{4}\cos 2\varphi - \frac{1}{4}\cos 6\varphi \leq 1. \quad (7.1.30)$$

The inequality (7.1.30) is maximally violated if $\varphi = \frac{\pi}{8}$:

$$\Delta\left(\frac{\pi}{8}\right) = \frac{1}{2} + \frac{3\sqrt{2}}{4} + \frac{1}{2\sqrt{2}} \simeq 1.2 \quad (7.1.31)$$

which is not less than 1.

VII.1.3. Violation of Bell's inequality under strict Einstein locality conditions

The assumption of locality in the derivation of Bell's theorem requires that the measurement processes of the two observers are space-like separated (Fig. 7.1.1). This means that it is necessary to freely choose a direction for analysis, to set the analyzer and finally to register the particle such that it is impossible for any information about these processes to travel via any (possibly unknown) channel to the other observer before he, in turn, finishes his measurement.

Selection of an analyzer direction has to be completely unpredictable which necessitates a physical random number generator. A numerical pseudo-random number generator can not be used, since its state at any time is predetermined. Furthermore, to achieve complete independence of both observers, one should avoid any common context as would be conventional registration of coincidences as in all previous experiments. Rather the individual events should be registered on both sides completely independently and compared only after the measurements are finished. This requires independent and highly accurate time bases on both sides. In our experiment for the first time any mutual influence between the two observations is excluded within the realm of Einstein locality.

To achieve this condition the observers Alice and Bob were spatially separated by 400 m across the Innsbruck university science campus. In [5] polarization entangled photon pairs which were sent to the observers through optical fibers were used. About 250 m of each 500 m long cable was laid out and the rest was left coiled at the source. This, we remark, has no influence on the timing argument because the optical elements of the source and the locally coiled fibers can be seen as jointly forming the effective source of the experiment (Fig. 7.1.1).

Remark 7.1.5. The difference in fiber length was less than $1m$ which means that the photons were registered simultaneously within interval $5ns$.

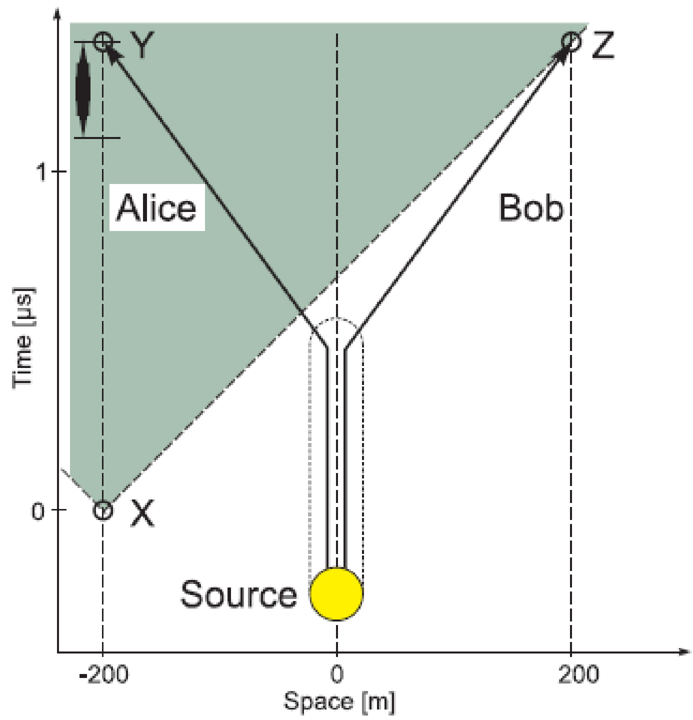

Fig. 7.1.1. Spacetime diagram of Bell experiment [5]

Selecting a random analyzer direction, setting the analyzer and finally detecting a photon constitute the measurement process. This

process on Alice's side must fully lie inside the shaded region which is, during Bob's own measurement, invisible to him as a matter of principle. For setup this means that the decision about the setting has to be made after point “X” if the corresponding photons are detected at spacetime points “Y” and “Z” respectively. In this experiment the measurement process (indicated by a short black bar) including the choice of a random number only took less than a tenth of the maximum allowed time. The vertical parts of the kinked photon world lines emerging from the source represent the fiber coils at the source location.
The source of polarization entangled photon pairs is degenerate type-II parametric down-conversion where a BBO-crystal was pumped with 400 mW of 351nm light from an Argon-ion-laser. A telescope was used to narrow the UV-pump beam, in order to enhance the coupling of the 702 nm photons into the two single mode glass fibers. On the way to the fibers, the photons passed a half-wave plate and the compensator crystals necessary to compensate for in-crystal birefringence and to adjust the internal phase of the entangled state $|\Psi\rangle = 1/\sqrt{2}\,(|H\rangle_1|V\rangle_2 + e^{i\varphi}|V\rangle_1|H\rangle_2),$ which was chosen $\varphi = \pi.$

Remark 7.1.6. The horizontal and the vertical polarization jointly define a basis denoted by $\hat{z},$ which can take on the values $|H\rangle$ or $|V\rangle$.
The modulation systems (high-voltage amplifier and electro-optic modulator) had a frequency range from DC to 30 MHz. In operating the systems at high frequencies a reduced polarization contrast of 97% (Bob) and 98% (Alice) was observed.
This, however, is no real depolarization but merely reflects the fact that we are averaging over the polarization rotation induced by an electrical signal from the high-voltage amplifier, which is not of perfectly rectangular shape.

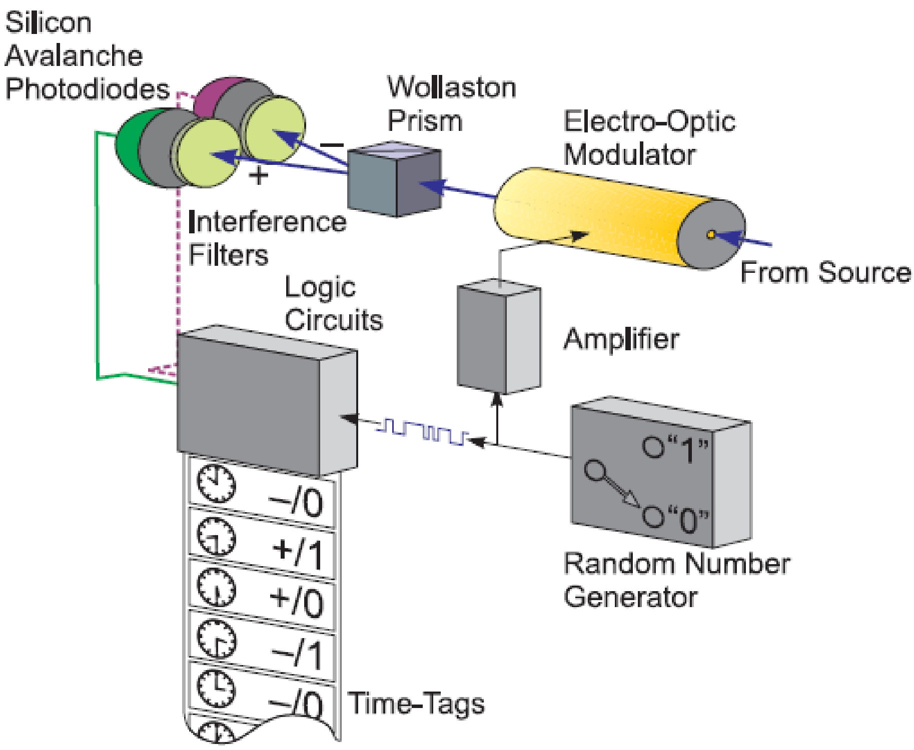

Fig. 7.1.2. One of the two observer stations [5]

A random number generator is driving the electro-optic modulator. Silicon avalanche photodiodes are used as detectors. A time tag is stored for each detected photon together with the corresponding random number 0 or 1 and the code for the detector + or - corresponding to the two outputs of the Wollaston prism polarizer. All alignments and adjustments were pure local operations that did not rely on a common source or on communication between the observers.

The actual orientation for local polarization analysis was determined independently by a physical random number generator. This generator has a light-emitting diode (coherence time $t_c \approx$ 10 fs) illuminating a beam splitter whose outputs are monitored by photomultipliers. The subsequent electronic circuit sets its output to 0(1) upon receiving a pulse from photomultiplier 0(1).

Remark 7.1.7. Events where both photomultipliers register a photon within $\triangle t \leq 2$ ns are ignored [5].

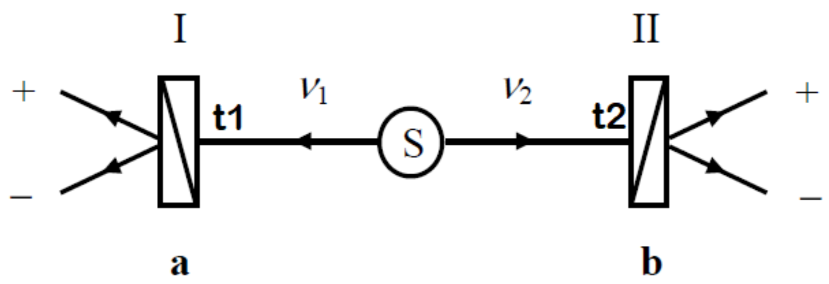

Fig. 7.1.3. (i) $t_1 - t_2 = 0$, (ii) $t_1 - t_2 = \delta > 0$, (iii) $t_1 - t_2 = -\delta < 0$

A down-converter (one way to produce an entangled pair) throws two entangled photons ν_1 and ν_2 in opposite directions. Polarization of the photons ν_1 and ν_2 is measured by polarizers **I** and **II** respectively.

Remark 7.1.8. Assume that photon ν_1 collapses in polarizer **I** at instant t_1 and photon ν_2 collapses in polarizer **II** at instant t_2 respectively. Note that in general case $t_1 \neq t_2$ even if photons ν_1 and ν_2 were registered simultaneously (within $5ns$ interval, see Remark 7.1.5).

Notice that obviously there exist only three possibilities:

(i) $t_1 - t_2 = 0$, (ii) $t_1 - t_2 = \tau_{\min} = \delta > 0$, (iii) $t_1 - t_2 = -\tau_{\min} = -\delta < 0$.

We have chosen here $\tau_{\min} = const = \delta$.

The resulting binary random number generator has a maximum toggle frequency of 500 MHz. By changing the source intensity the mean interval was adjusted to about 10 ns in order to have a high primary random bit rate. Certainly this kind of random-number generator is not necessarily evenly distributed. For a test of Bell's inequality it is, however, not necessary to have perfectly even distribution, because all correlation functions are normalized to the total number of events for a certain combination of the analyzers' settings. Still, we kept the distribution even to within 2% in order to obtain an approximately equal number of samples for each setting. The distribution was adjusted by equalizing the number of counts of the two photomultipliers through changing their internal photoelectron amplification. Due to the limited speed of the

subsequent modulation system it was sufficient to sample this random number generator periodically at a rate of 10 MHz.
There are many variants of Bell's inequalities. In G. Weihs, T. Jennewein experiment [5] a version first derived by Clauser et al. [4] (CHSH) was used since it applies directly to Zeilinger's experimental configuration. The number of coincidences between Alice's detector *i* and Bob's detector *j* is denoted by $C_{ij}(\alpha,\beta)$ with $i,j \in \{+,-\}$ where α and β are the directions of the two polarization analyzers and $+$ and $-$ denote the two outputs of a two-channel polarizer respectively. If we assume that the detected pairs are a fair sample of all pairs emitted, then the normalized expectation value $E(\alpha,\beta)$ of the correlation between Alice's and Bob's local results is

$$E(\alpha,\beta) = [C_{++}(\alpha,\beta) + C_{--}(\alpha,\beta) - C_{+-}(\alpha,\beta) - C_{-+}(\alpha,\beta)]/N, \qquad (7.1.32)$$

where N is the sum of all coincidence rates.
Remark 7.1.9. We define now:
(i) $C_{ij}^{=}(\alpha,\beta) \triangleq C_{ij}(\alpha,\beta,t_1,t_2),$ where $t_1 - t_2 = 0;$
(ii) $C_{ij}^{>}(\alpha,\beta) \triangleq C_{ij}(\alpha,\beta,t_1,t_2),$ where $t_1 - t_2 = \delta;$
(iii) $C_{ij}^{<}(\alpha,\beta) \triangleq C_{ij}(\alpha,\beta,t_1,t_2),$ where $t_1 - t_2 = -\delta.$
Remark 7.1.10. Note that

$$C_{ij}(\alpha,\beta) \simeq C_{ij}^{=}(\alpha,\beta) + C_{ij}^{>}(\alpha,\beta) + C_{ij}^{+}(\alpha,\beta). \qquad (7.1.33)$$

In a rather general form the CHSH inequality reads

$$S(\alpha,\alpha',\beta,\beta') = |E(\alpha,\beta) - E(\alpha',\beta)| + |E(\alpha,\beta') + E(\alpha',\beta')| \leq 2. \qquad (7.1.34)$$

Quantum theory predicts a sinusoidal dependence for the coincidence rate

$$C_{++}^{QM}(\alpha,\beta) \asymp \sin^2(\beta - \alpha) \qquad (7.1.35)$$

on the difference angle of the analyzer directions in Alice's and Bob's experiments. The same behavior can also be seen in the correlation function

$$E^{QM}(\alpha,\beta) = -\cos[2(\beta-\alpha)]. \tag{7.1.36}$$

Thus, for various combinations of analyzer directions $\alpha,\beta,\alpha',\beta'$ these functions violate CHSH inequality. Maximum violation is obtained using the following set of angles

$$S_{\max}^{QM} = S^{QM}(0^\circ,45^\circ,22.5^\circ,67.5^\circ) = 2\sqrt{2} = 2.82 > 2. \tag{7.1.37}$$

If, however, the perfect correlations ($\alpha-\beta=0^\circ$ or 90°) have a reduced visibility V ≤ 1 then the quantum theoretical predictions for E and S are reduced as well by the same factor independent of the angle. Thus, because the visibility of the perfect correlations in this experiment was about 97% and was expected S to be not higher than 2.74 if alignment of all angles is perfect and all detectors are equally efficient. Various measurements with the described setup were performed [5]. The data presented in Fig. 7.1.4 are the result of a scan of the DC bias voltage in Alice's modulation system over a 200 V range in 5 V steps. At each point a synchronization pulse triggered a measurement period of 5 s on each side. From the time-tag series we extracted coincidences after all measurements had been finished. Fig. 7.1.4 shows four of the 16 resulting coincidence rates as functions of the bias voltage. Each curve corresponds to a certain detector and a certain modulator state on each side. A nonlinear χ^2 fit showed perfect agreement with the sine curve predicted by quantum theory. Visibility was 97% as one could have expected from the previously measured polarization contrast. No oscillations in the singles count rates were found. We want to stress again that the accidental coincidences have not been subtracted from the plotted data.

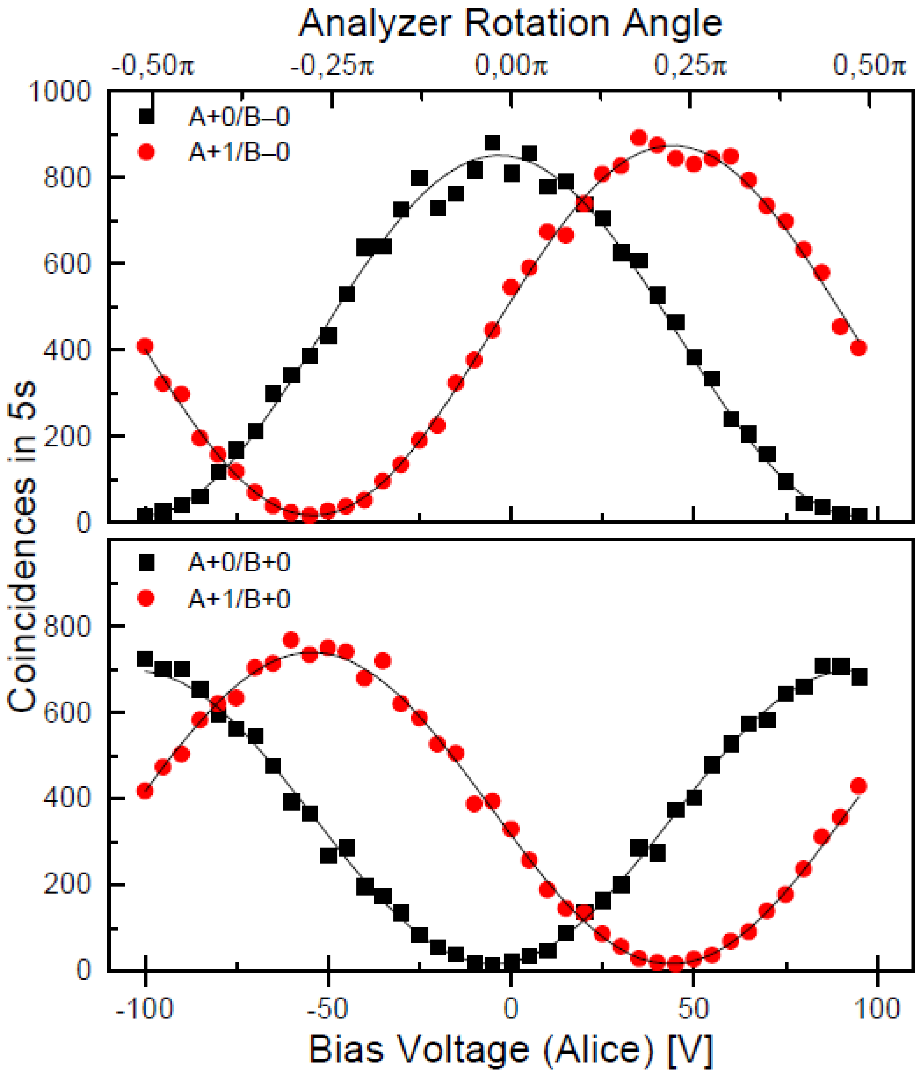

Fig. 7.1.4. Four out of sixteen coincidence rates between various detection channels as functions of bias voltage (analyzer rotation angle) on Alice's modulator. A+1/B−0, for example, are the coincidences between Alice's "+" detector with switch having been in position "1" and Bob's "−" detector with switch position "0". The difference in height is explained by different efficiencies of the detectors

In order to give quantitative results for the violation of Bell's inequality with better statistics, experimental runs were performed with the settings $0°, 45°$ for Alice's and $22.5°, 67.5°$ for Bob's polarization analyzer. A typical observed value of the function S in such a measurement was $S = 2.73 \pm 0.02$ for 14700 coincidence events collected in $10s$. This corresponds to a violation of the CHSH inequality of 30 standard deviations assuming only statistical errors. If we allow for asymmetries between the detectors and minor errors of the modulator voltages this result agrees very well with the quantum theoretical prediction.

VII.1.4. CHSH theorem without the hypothesis of locality

One of the Bell's assumptions in the original derivation of his inequalities was the hypothesis of locality, i.e., of the absence of the influence of two remote measuring instruments on one another. That is why violations of these inequalities observed in experiments are often interpreted as a manifestation of the nonlocal nature of quantum mechanics, or a refutation of local realism. In [1, 2], CHSH inequality was derived in its traditional form, without resorting to the hypothesis of locality, the only assumption being that the probability distributions are nonnegative. This can therefore be regarded as a rigorous proof that the hypothesis of locality is not relevant to violations of CHSH inequalities.

Let A, A', B, B' be random variables with values in the set $\{-1, +1\}$, i.e.,

$$A = \pm 1, A' = \pm 1, B = \pm 1, B' = \pm 1. \qquad (7.1.38)$$

Assume that there exist joint probability distribution functions $W(A, A', B, B')$, of A, A', B, B' defining probabilities for each possible set of outcomes such that:

(i)

$$P(A, A', B, B') \geq 0, P(A, B, B') \geq 0, P(A', B, B') \geq 0, \text{etc.}, \qquad (7.1.39)$$

(ii)

$$\sum_{A,A',B,B'} P(A,A',B,B') = 1, \sum_{A,A',B,B'} P(A,B,B') = 1, \sum_{A,A',B,B'} P(A',B,B') = 1, \text{etc.,} \quad (7.1.40)$$

(iii)

$$\begin{gathered} P(A,A',B,B') + P(-A,A',B,B') = P(A',B,B') \geq P(A,A',B,B'), \\ P(A,A',B,B') + P(A,-A',B,B') = P(A',B,B') \geq P(A,A',B,B'), \\ \text{etc.} \end{gathered} \quad (7.1.41)$$

From (7.1.41) one obtains

$$\begin{gathered} 0 \leq P(A,B,B') = P(A,A',B,B') + P(A,-A',B,B') \leq P(A',B') + P(-A',B) = \\ P(A',B') + P(B) - P(A',B). \end{gathered} \quad (7.1.42)$$

Similarly one obtains

$$\begin{gathered} 0 \leq P(A,-B,-B') = P(A,-B) + P(A,-B,B') = \\ P(A) - P(A,B) - P(A,B') + P(A,B,B') \end{gathered} \quad (7.1.43)$$

and therefore

$$-P(A,B,B') \leq P(A) - P(A,B) - P(A,B'). \quad (7.1.44)$$

From (7.1.42) and (7.1.44) we obtain

$$\begin{gathered} 0 \leq P(A,B,B') - P(A,B,B') = \\ P(A',B') + P(B) - P(A',B) + P(A) - P(A,B) - P(A,B') \end{gathered} \quad (7.1.45)$$

and therefore

$$\begin{gathered} 0 \leq P(A',B') + P(B) - P(A',B) + P(A) - P(A,B) - P(A,B') = \\ = P(A) + P(B) - P(A,B) - P(A',B) - P(A,B') + P(A',B'). \end{gathered} \quad (7.1.46)$$

From (7.1.46) one obtains

$$\Lambda(A,A',B,B') \equiv P(A,B) + P(A',B) + P(A,B') - P(A',B') - P(A) - P(B) \leq 0. \quad (7.1.47)$$

Note that

$$P(B,B') = P(B) - P(B,-B') \quad (7.1.48)$$

and

$$P(-B,-B') = P(-B') - P(B,-B') = 1 - P(B') - P(B,-B'). \quad (7.1.49)$$

From (7.1.49) and (7.1.48) we obtain

$$P(-B,-B') = 1 - P(B) - P(B') + P(B,B'). \quad (7.1.50)$$

Note that

$$0 \leq P(-A,-B,-B') = P(-B,-B') - P(A,-B,-B'). \quad (7.1.51)$$

Inserting (7.1.43) and (7.1.50) into (7.1.51) we obtain

$$\begin{gathered} 0 \leq 1 - P(A) - P(B) - P(B') + P(A,B) + P(A',B) + P(A,B') + \\ +P(B,B') - P(A,B,B') = 1 - P(A) - P(B) - P(B') + \\ +P(A,B) + P(A',B) + P(-A,B,B'). \end{gathered} \quad (7.1.52)$$

Note that

$$\begin{gathered} P(-A,B,B') = P(-A,A',B,B') + P(-A,-A',B,B') \leq P(A',B) + P(-A',B') = \\ = P(A',B) + P(B') - P(A',B'). \end{gathered} \quad (7.1.53)$$

From (7.1.53) we obtain

$$0 \le 1 - P(A) - P(B) - P(B') + P(A,B) + P(A',B) + P(A,B') - P(A',B'). \qquad (7.1.54)$$

From (7.1.54) and (7.1.47) we obtain

$$-1 \le \Lambda(A,A',B,B') \le 0. \qquad (7.1.55)$$

Note that the following representation of the quantities $\langle AB\rangle, \langle A'B\rangle, \langle AB'\rangle, \langle A'B'\rangle$ holds

$$\langle AB\rangle = P_{AB}(++) + P_{AB}(--) - P_{AB}(+-) - P_{AB}(-+), \text{ etc.,} \qquad (7.1.56)$$

where

$$P_{AB}(++) = P(A=1,B=1), P_{AB}(--) = P(A=-1,B=-1), \text{etc.} \qquad (7.1.57)$$

From (7.1.56) and (7.1.57) we obtain

$$\langle AB\rangle + \langle A'B\rangle + \langle AB'\rangle - \langle A'B'\rangle = \Lambda(++++) + \Lambda(----) - \\ -\Lambda(+-+-) - \Lambda(-++-+). \qquad (7.1.58)$$

From (7.1.55) we obtain

$$-2 \le \Lambda(++++) + \Lambda(----) \le 0 \qquad (7.1.59)$$

and

$$0 \le -\Lambda(+-+-) - \Lambda(-+-+) \le 2. \qquad (7.1.60)$$

From (7.1.59) and (7.1.60) we obtain

$$-2 \le \Lambda(++++) + \Lambda(----) - \Lambda(+-+-) - \Lambda(-+-+) \le 2. \qquad (7.1.61)$$

From (7.1.58) and (7.1.61) finally we obtain

$$|\langle AB\rangle + \langle A'B\rangle + \langle AB'\rangle - \langle A'B'\rangle| \leq 2. \qquad (7.1.62)$$

VII.1.5. CHSH theorem without the introduction of hidden variables

A hidden-variable theory is the traditional, but not unique, basis for constructing various types of Bell's theorem. The starting point may also be a recognition of the existence of a positive-definite probability distribution function. This assumption alone is used to formulate and prove Bell's paradoxes of different types [6, 7].

Let A, A', B, B' be random variables with values in the set $\{-1, +1\}$, i.e.,

$$A = \pm 1, A' = \pm 1, B = \pm 1, B' = \pm 1. \qquad (7.1.63)$$

Assume that there exists joint probability distribution function $P(A, A', B, B')$ of A, A', B, B' defining probabilities for each possible set of outcomes such that:

$$P(A, A', B, B') \geq 0, \qquad (7.1.64)$$

and

$$\sum_{A, A', B, B'} P(A, A', B, B') = 1, \qquad (7.1.65)$$

and

$$P(A, A', B, B') - P(-A, A', B, B') = P(A', B, B'). \qquad (7.1.66)$$

Let us consider Bell inequality of the form

$$\Pi = \frac{1}{2}|\langle AB\rangle + \langle A'B\rangle + \langle AB'\rangle - \langle A'B'\rangle| \leq 1. \qquad (7.1.67)$$

It is well known [1], [6] that under assumptions (7.1.64)-(7.1.66) Bell inequality is directly provable without any reference to Kolmogorov probability space $(\Omega, \Sigma, \mathbf{P})$.

We abbreviate now for short [1, 6]:

$$\begin{gathered} P_1 = P_{A,A',B,B'}(++++) = P(A=1, A'=1, B=1, B'=1), \\ P_2 = P_{A,A',B,B'}(+++-) = P(A=1, A'=1, B=1, B'=-1), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ P_{16} = P_{A,A',B,B'}(----) = P(A=-1, A'=-1, B=-1, B'=-1). \end{gathered} \qquad (7.1.68)$$

For the quantities $\langle AB\rangle, \langle A'B\rangle, \langle AB'\rangle$ and $\langle A'B'\rangle$ using Eq. (7.1.68) one obtains the representatives

$$\begin{gathered} \langle AB\rangle = P_{AB}(++) + P_{AB}(--) - P_{AB}(+-) - P_{AB}(-+), \\ \text{etc.}, \end{gathered} \qquad (7.1.69)$$

where

$$\begin{gathered} P_{AB}(++) = P_{A,A',B,B'}(++++) + P_{A,A',B,B'}(+++-) + \\ P_{A,A',B,B'}(+-++) + P_{A,A',B,B'}(+-+-), \\ \text{etc.} \end{gathered} \qquad (7.1.70)$$

Substituting Eq. (7.1.69) into the expression $\langle AB\rangle + \langle A'B\rangle + \langle AB'\rangle - \langle A'B'\rangle$ one obtains

$$\begin{gathered}\Pi = \frac{1}{2}(\langle AB\rangle + \langle A'B\rangle + \langle AB'\rangle - \langle A'B'\rangle) = \\ P_{A,A',B,B'}(++++) + P_{A,A',B,B'}(+++-) - P_{A,A',B,B'}(++-+) - \\ -P_{A,A',B,B'}(++--) - P_{A,A',B,B'}(+-++) - P_{A,A',B,B'}(+-+-) + \\ +P_{A,A',B,B'}(+--+) - P_{A,A',B,B'}(+---) - P_{A,A',B,B'}(-+++) + \\ +P_{A,A',B,B'}(-++-) - P_{A,A',B,B'}(-+-+) + P_{A,A',B,B'}(-+--) - \\ -P_{A,A',B,B'}(--++) - P_{A,A',B,B'}(--+-) + P_{A,A',B,B'}(---+) + \\ +P_{A,A',B,B'}(----).\end{gathered} \tag{7.1.71}$$

From (7.1.64) - (7.1.65) it obviously follows that $-1 \leq \Pi \leq 1$, and therefore Bell inequality (7.1.67) holds.

VII.2. Physical nature of the violation of the Bell inequalities

VII.2.1. Physical interpretation of the Bell test experiment under EPR-B nonlocality postulate

Actually, most experiments have been performed using polarization of photons. The quantum state of the pair of entangled photons is not the singlet state. The polarization of a photon is measured in a pair of perpendicular directions. Relative to a given orientation, polarization is either vertical (denoted by V or by $+$) or horizontal (denoted by H or by $-$). The photon pairs are generated in the quantum state

$$|\psi_{\mathrm{EPR}}\rangle = \frac{1}{\sqrt{2}}(|V\rangle_s \otimes |V\rangle_i + |H\rangle_s \otimes |H_i\rangle), \tag{7.2.1}$$

where $|V\rangle$ and $|H\rangle$ denote the state of a single vertically or horizontally polarized photon, respectively (relative to a fixed and common reference direction for both particles) and subscripts s and i indicate signal or idler photon respectively.

The source S produces pairs of "photons" sent in opposite directions. Each photon encounters a two-channel polariser whose orientation ($\mathbf{a}$ or $\mathbf{b}$) can be set by the experimenter. Emerging signals from each channel are detected and coincidences of four types ($++,--,+-$ and $-+$) are counted by the coincidence monitor.

This state cannot be factored into a simple product of signal and idler states: $|\psi_{\mathrm{EPR}}\rangle \neq |A\rangle_{\mathrm{s}} \otimes |B\rangle_{\mathrm{i}}$ for any choice of $|A\rangle_{\mathrm{s}}$ and $|B\rangle_{\mathrm{i}}$. This means the state of one particle cannot be specified without making reference to the other particle. Such particles are said to be "entangled" and $|\psi_{\mathrm{EPR}}\rangle$ is an entangled state. If we measure the polarizations of signal and idler photons in the H,V basis there are two possible outcomes: both vertical or both horizontal. Each occurs half of the time. We could instead measure the polarizations with polarizers rotated by an angle α. We use the rotated polarization basis

$$|V_\alpha\rangle = \cos\alpha|V\rangle - \sin\alpha|H\rangle, |H_\alpha\rangle = \sin\alpha|V\rangle + \cos\alpha|H\rangle. \quad (7.2.2)$$

Here $|V_\alpha\rangle$ describes a state with polarization rotated by α from the vertical, while $|H_\alpha\rangle$ is α from the horizontal. In this basis the state is

$$|\psi_{\mathrm{EPR}}\rangle = \frac{1}{\sqrt{2}}\left(|V_\alpha\rangle_{\mathrm{s}}|V_\alpha\rangle_{\mathrm{i}} + |H_\alpha\rangle_{\mathrm{s}}|H_\alpha\rangle_{\mathrm{i}}\right). \quad (7.2.3)$$

Remark 7.2.1. We will denote the events corresponding to coincidences (at the instant t) of four types $(++),(--),(+-),(-+)$ on the coincidence monitor by symbols

$$\{+,+;t\}, \{-,-;t\}, \{+,-;t\}, \{-,+;t\}$$

or by symbols $\{\mathbf{a}_+,\mathbf{b}_+;t\}, \{\mathbf{a}_-,\mathbf{b}_-;t\}, \{\mathbf{a}_+,\mathbf{b}_-;t\}, \{\mathbf{a}_-,\mathbf{b}_+;t\}$ respectively or simply $\{\mathbf{a}_+,\mathbf{b}_+\}, \{\mathbf{a}_-,\mathbf{b}_-\}, \{\mathbf{a}_+,\mathbf{b}_-\}, \{\mathbf{a}_-,\mathbf{b}_+\}$.

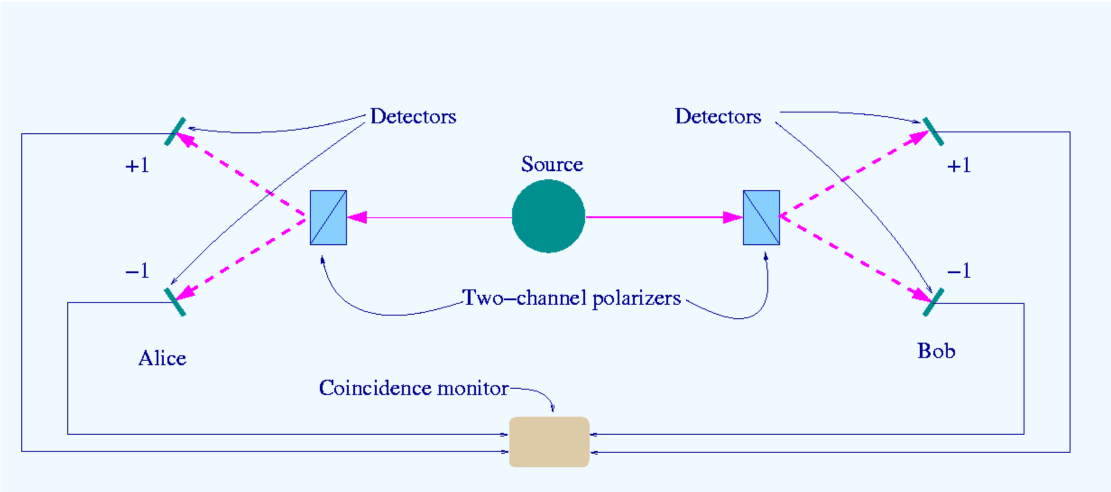

Fig. 7.2.1. Scheme of a “two-channel” Bell test

Remark 7.2.2. Clearly, if we measure in this rotated basis we get the same results: half the time both are $|V_\alpha\rangle$ and half of the time both are $|H_\alpha\rangle$. Knowing this, we can measure the signal polarization and infer with certainty the idler polarization. This is the situation EPR described, but we have used polarizations instead of position and momentum.

Remark 7.2.3. Note that there is an uncertainty relationship between polarizations in different bases. Knowledge of a photon polarization after the measurement such a polarization in the V_{0°, H_{0° basis implies complete uncertainty of its polarization in the $V_{45^\circ}, H_{45^\circ}$ basis, for example.

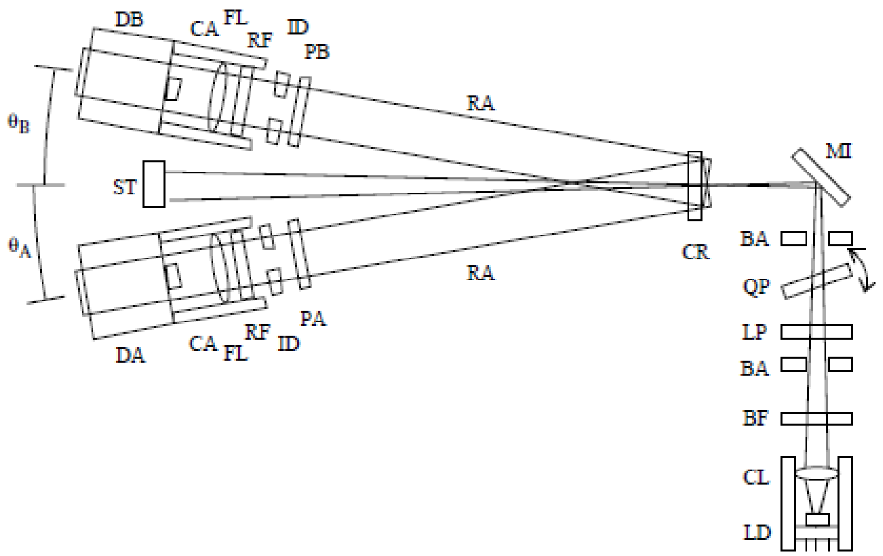

Fig. 7.2.2. Schematic of experimental setup [8]

Symbols: LD Laser Diode, CL Collimating Lens, BF Blue Filter, BA Beam Aperture, LP Laser Polarizer, QP Quartz Plate, MI Mirror, CR Downconversion Crystals, RA Rail, PA Polarizer **A**, PB Polarizer **B**, ID Iris Diaphragm, RF Red Filter, FL Focusing Lens, CA Cage Assembly, DA Detector **A**, DB Detector **B**, ST Beam Stop.

Figure 7.2.2 shows a schematic of an experimental setup to produce polarization - entangled photons [8]. A 5 mW free-running InGaN diode laser produces a beam of violet (405 nm) photons which passes through a blue filter, a linear polarizer, and a birefringent plate before reaching a pair of beta barium borate (BBO) crystals. In the crystals, a small fraction of the laser photons spontaneously decays into pairs of photons by the process of spontaneous parametric downconversion (SPD). In a given decay the downconverted photons emerge at the same time and on opposite sides of the laser beam. The detectors, two single-photon counting modules (SPCMs), are preceded by linear polarizers and red filters to block any scattered laser light. Even so, it is necessary to use coincidence detection to separate the downconverted photons from the background of other photons reaching the detectors. Because the photons of a downconverted pair are produced at the same time they cause coincident, i.e., nearly simultaneous, firings of the SPCMs. Coincidences are detected by a fast logic circuit and recorded by a personal computer. The detection components (SPCMs, irises, lenses and filters) are mounted on rails which pivot about a vertical axis passing through the crystals. This allows the detection of SPD photons at different angles with minimal realignment. The rails were positioned at $\theta_A = \theta_B = 2.5$ and the focusing lenses adjusted for maximum singles rates. With the irises fully open and polarizers both set to vertical, more than 300 counts per second were observed [8].

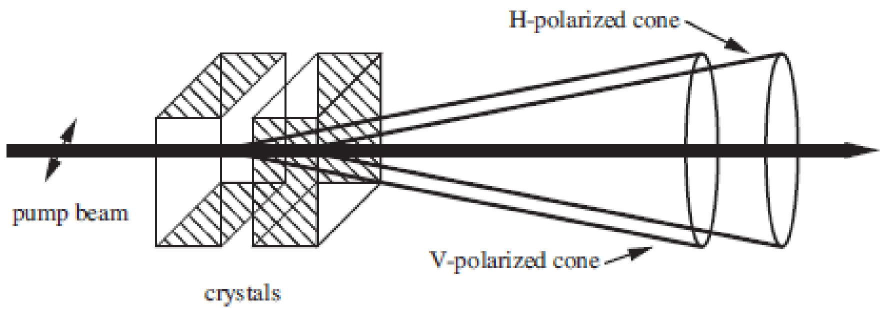

Fig. 7.2.3. Two-crystal down conversion source

The crystals are 0.1 mm thick and in contact face-to-face, while the pump beam is approximately 1 mm in diameter. Thus the cones of downconverted light from the two crystals overlap almost completely.

These BBO crystals are cut for Type I phase matching, which means that the signal and idler photons emerge with the same polarization, which is orthogonal to that of the pump photon. Each crystal can only support downconversion of one pump polarization. The other polarization passes through the crystal unchanged. We use two crystals, one rotated 90° from the other, so that either pump polarization can downconvert according to the rules

$$|V\rangle_p \to |H\rangle_s|H\rangle_i, |H\rangle_p \to \exp[i\Delta]|V\rangle_s|V\rangle_i. \tag{7.2.4}$$

where Δ is a phase due to dispersion and birefringence in the crystals. The geometry is shown schematically in Figure 7.2.3.

To create an entangled state, we first linearly polarize the laser beam at an angle θ_l from the vertical and then shift the phase of one polarization component by ϕ_l with the birefringent quartz plate. The laser photons (pump photons) are then in the state

$$|\psi_{\text{pump}}\rangle = \cos\theta_l|V\rangle_p + \exp[i\phi_l]\sin\theta_l|H\rangle_p \tag{7.2.5}$$

when they reach the crystals. The downconverted photons emerge in the state

$$|\psi_{\text{DC}}\rangle = \cos\theta_l|H\rangle_s|H\rangle_i + \exp[i\phi]\sin\theta_l|V\rangle_s|V\rangle_i \tag{7.2.6}$$

where $\phi \equiv \phi_l + \Delta$ is the total phase difference of the two polarization components [8].

Remark 7.2.4. This state is an entangled state and is already quite adjustable.

Further modifications can be made with ordinary optical components. For example, if $\theta_l = \pi/4$, $\phi = \pi$ then a half-wave plate in the signal beam could be used to switch the signal polarization $|H\rangle_s \leftrightarrow |V\rangle_s$ to produce $|\psi'_{\text{DC}}\rangle = (|V\rangle_s|H\rangle_i - |H\rangle_s|V\rangle_i)/\sqrt{2}$.

By placing polarizers rotated to angles α and β in the signal and idler paths, respectively, we measure the polarization of the downconverted photons. For a pair produced in the state $|\psi_{DC}\rangle$, the probability of coincidence detection is

$$P_{VV}(\alpha,\beta) = |\langle V_\alpha|_s \langle V_\beta|_i |\psi_{DC}\rangle|^2. \quad (7.2.7)$$

The VV subscripts on P indicate the measurement outcome $V_\alpha V_\beta$, both photons vertical in the bases of their respective polarizers. More generally, for any pair of polarizer angles α,β, there are four possible outcomes, $V_\alpha V_\beta, V_\alpha H_\beta, H_\alpha V_\beta$ and $H_\alpha H_\beta$ indicated by VV, VH, HV and HH, respectively. Using the basis of equation (7.2.2), we find

$$P_{VV}(\alpha,\beta) = |\sin\alpha\sin\beta\cos\theta_l + \exp[i\phi]\cos\alpha\cos\beta\sin\theta_l|^2 \quad (7.2.8)$$

or

$$P_{VV}(\alpha,\beta) = \sin^2\alpha\sin^2\beta\cos^2\theta_l + \cos^2\alpha\cos^2\beta\sin^2\theta_l + \tfrac{1}{4}\sin 2\alpha\sin 2\beta\sin 2\theta_l\cos\phi. \quad (7.2.9)$$

A special case occurs when $|\psi_{DC}\rangle = |\psi_{\mathrm{EPR}}\rangle$, i.e., when $\theta_l = \pi/4$ and $\phi = 0$. In this case

$$P_{VV}(\alpha,\beta) = \tfrac{1}{2}\cos^2(\beta-\alpha), \quad (7.2.10)$$

which depends only on the relative angle $\beta-\alpha$.

The last term in Eq. (7.2.9) is a cross term which accounts for the interference between the H,H and V,V parts of the state. The ϕ in this term is, through its dependence on Δ, a complicated function of pump photon wavelength, signal photon wavelength and angle as well as crystal characteristics. Because the laser has a finite line width and we collect photons over a finite solid angle and wavelength range, we collect a range of ϕ. To account for this, we replace $\cos\phi$ by its average $\langle\cos\phi\rangle \equiv \cos\phi_m$ to get

$$P_{VV}(\alpha,\beta) = \sin^2\alpha\sin^2\beta\cos^2\theta_l + \cos^2\alpha\cos^2\beta\sin^2\theta_l + \tfrac{1}{4}\sin 2\alpha\sin 2\beta\sin 2\theta_l\cos\phi_m \quad (7.2.11)$$

In the experiment, a fixed interval T of data acquisition (typically in the range 0.5 seconds to 15 seconds) was chosen and the number of coincidences $N(\alpha,\beta)$ during that interval was recorded [8].

Assuming a constant flux of photon pairs, the number collected will be

$$N(\alpha,\beta) = A\left(\sin^2\alpha\sin^2\beta\cos^2\theta_l + \cos^2\alpha\cos^2\beta\sin^2\theta_l + \tfrac{1}{4}\sin 2\alpha\sin 2\beta\sin 2\theta_l\cos\phi_m\right) + C \qquad (7.2.12)$$

where A is the total number of entangled pairs produced and C is an offset to account for imperfections in the polarizers and alignment of the crystals. This is necessary to account for the fact that some coincidences are observed even when the polarizers are set to $\alpha = 0^\circ, \beta = 90^\circ$.

Remark 7.2.5. We emphasize that (7.2.9) and (7.2.10) hold iff the measurements on photon v_1 and photon v_2 occur simultaneously, i.e., photon v_1 and photon v_2 collapse in polarizers **I** and **II** respectively at an instant $t = t_1 = t_2, t \in [0,T]$, (see Remark 7.1.8) and we will denote such events by

$$\left\{v_1^{t_1}, v_2^{t_2}\right\}^{=}$$

or simply $\{v_1, v_2\}^{=}$.

Remark 7.2.6. We will denote such entangled pairs of photons also by $\left\{v_1^{t_1}, v_2^{t_2}\right\}^{=}$ or simply $\{v_1, v_2\}^{=}$ and we will denote the total number of entangled pairs $\left\{v_1^{t_1}, v_2^{t_2}\right\}^{=}$ produced during interval T by $A_T^{=}$. The number of coincidences during interval T corresponding exactly to entangled pairs $\left\{v_1^{t_1}, v_2^{t_2}\right\}^{=}$ we will denote by

$$N_T^{=}(\alpha_+, \beta_+).$$

We rewrite now Eq. (7.2.11) in the following form:

$$P_{VV}^{=}(\alpha,\beta|\left\{v_1^{t_1}, v_2^{t_2}\right\}^{=}) \triangleq P_{VV}^{=}(\alpha,t_1;\beta,t_2|\left\{v_1^{t_1}, v_2^{t_2}\right\}^{=}) = \sin^2\alpha\sin^2\beta\cos^2\theta_l + \cos^2\alpha\cos^2\beta\sin^2\theta_l + \tfrac{1}{4}\sin 2\alpha\sin 2\beta\sin 2\theta_l\cos\phi_m \qquad (7.2.13)$$

Remark 7.2.7. Note that $P^{=}_{VV}(\alpha,t_1;\beta,t_2|\{v_1^{t_1},v_2^{t_2}\}^{=})$ is the conditional probability of the event $\{\alpha_+,\beta_+;t\}$ (see Remark 7.2.1) under condition that the event $\{v_1^{t_1},v_2^{t_2}\}^{=}$ was occured at instant $t = t_1 = t_2$.
We rewrite now Eq. (7.2.12) in the following form:

$$N^{=}_T(\alpha_+,\beta_+) = A^{=}_T\Big(\sin^2\alpha\sin^2\beta\cos^2\theta_l + \cos^2\alpha\cos^2\beta\sin^2\theta_l + \tfrac{1}{4}\sin2\alpha\sin2\beta\sin2\theta_l\cos\phi_m\Big) + C_1 \quad (7.2.14)$$

Remark 7.2.8. Let us now consider the complete set of probabilities $p_{\pm\pm}(\mathbf{a},t_1;\mathbf{b},t_2)$ of joint detections of v_1 and v_2 in the channels $+$ or $-$ of polarisers **I** or **II**, in orientations $\mathbf{a}$ and $\mathbf{b}$, see Fig. 7.2.4. The canonical Quantum Mechanical predictions for the the joint detection probabilities are

$$\begin{gathered} p_{++}(\mathbf{a},t_1;\mathbf{b},t_2) = p_{--}(\mathbf{a},t_1;\mathbf{b},t_2) = \tfrac{1}{2}\cos^2(\mathbf{a},\mathbf{b}), \\ p_{-+}(\mathbf{a},t_1;\mathbf{b},t_2) = p_{+-}(\mathbf{a},t_1;\mathbf{b},t_2) = \tfrac{1}{2}\sin^2(\mathbf{a},\mathbf{b}), \\ t_1 \simeq t_2. \end{gathered} \quad (7.2.15)$$

We emphasize that (7.2.15) holds iff the measurements on photon v_1 and photon v_2 occur simultaneously, i.e., iff photon v_1 and photon v_2 collapse in polarizers **I** and **II** respectively at instant $t = t_1 = t_2$, see Remark 7.1.8.
Remark 7.2.9. Suppose now that the measurement on photon v_1 occurs first, at instant t_1, and gives the result $+$, with the polarizer **I** in the orientation $\mathbf{a}$ and therefore the measurement on photon v_2 occurs at instant t_2, where $t_1 - t_2 = -\delta < 0$, (see Remark 7.1.8) and we will denote such events (i.e., if photon v_1 collapses in polarizer **I** at instant t_1 and photon v_2 collapses in polarizer **II** at instant t_2, where $t_1,t_2 \in [0,T]$ respectively) by

$$\{v_1^{t_1},v_2^{t_2}\}^{<},$$

or simply $\{v_1,v_2\}^{<}$.

Remark 7.2.10. We will denote such entangled pairs of photons also by $\{v_1^{t_1}, v_2^{t_2}\}^<, \quad t_1, t_2 \in [0, T],$ or simply $\{v_1, v_2\}^<$.

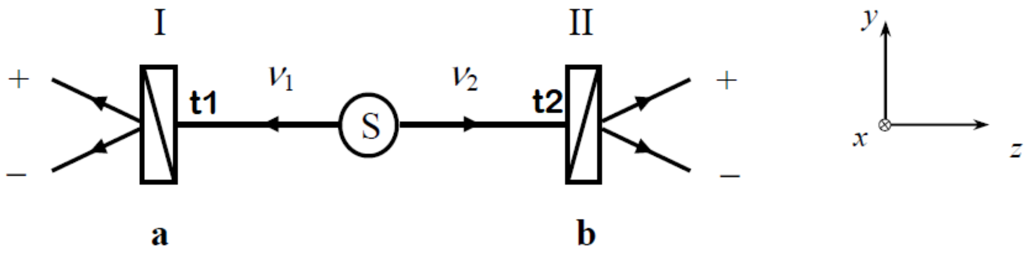

Fig. 7.2.4. Schematic of the experiment testing EPR non-locality.
(i) $t_1 - t_2 = 0$, (ii) $t_1 - t_2 = \delta > 0$, (iii) $t_1 - t_2 = -\delta < 0$

Entangled photons from the source are sent to two fast switches, that direct them to polarizing detectors. The switches change settings very rapidly, effectively changing the detector settings for the experiment while the photons are in flight.
$|\widetilde{\psi}(v_1, v_2)\rangle$ describing the pair is obtained by projection of the initial state vector $|\psi(v_1, v_2)\rangle = \frac{1}{\sqrt{2}}\{|x, x\rangle + |y, y\rangle\},$ where $|x\rangle$ and $|y\rangle$ are linear polarizations states, onto the eigenspace associated to the result $+$: this two dimensional eigenspace has a basis $\{|\mathbf{a}, x\rangle, |\mathbf{a}, y\rangle\}$. Using the corresponding projector, one finds after a little algebra

$$|\widetilde{\psi}(v_1, v_2)\rangle = |\mathbf{a}, \mathbf{a}\rangle. \tag{7.2.16}$$

This means that (i) immediately after the first measurement, photon v_1 takes the polarization $|\mathbf{a}\rangle$: this is obvious because it has been measured with a polarizer oriented along $\mathbf{a}$, and the result $+$ has been found, (ii) the distant photon v_2, which has not yet interacted with any polarizer, at instant t_1 has also been projected exactly into the state $|\mathbf{a}\rangle$ with a well defined polarization, parallel to the one found for photon ν_1.

Remark 7.2.11. Note that the standard Heisenberg's uncertainty principle predicts that if the polarization of the photon ν_2 along the direction $\mathbf{a}$ becomes certainty, i.e., known exactly, all information about the polarization of the photon ν_2 along the direction $\mathbf{b}$ becomes uncertainty, i.e., it will be completely lost. In order to overcome this problem we apply Heisenberg's noise - disturbance uncertainty relation, see Appendix A.

This relation is generally formulated as follows: for any apparatus $\mathbf{A}$ to measure an observable $A,$ the relation

$$\epsilon(A,\psi,\mathbf{A})\eta(B,\psi,\mathbf{A}) \geq \tfrac{1}{2}|\langle\psi|[A,B]|\psi\rangle| \tag{7.2.17}$$

holds for any input state ψ and any observable B, where $\epsilon(A,\psi,\mathbf{A})$ stands for the noise of the A measurement in state ψ using apparatus **A** and $\eta(B,\psi,\mathbf{A})$ stands for the disturbance of B in state ψ caused by apparatus **A.**

From (7.2.17) one obtains

$$\epsilon(\sigma_{\mathbf{a}},\psi_{12},\mathbf{P_I})\eta(\sigma_{\mathbf{b}},\psi_{12},\mathbf{P_I}) \geq \tfrac{1}{2}|\langle\psi_{12}|[\sigma_{\mathbf{a}},\sigma_{\mathbf{b}}]|\psi_{12}\rangle|, \tag{7.2.18}$$

where the measurement operators $\sigma_{\mathbf{a}}$ and $\sigma_{\mathbf{b}}$ measure the polarization in the $\mathbf{a}$-direction and $\mathbf{b}$-direction respectively and where $|\psi_{12}\rangle = |\psi(\nu_1,\nu_2)\rangle$.

Remark 7.2.12. Note that after the measurement on the photon ν_1 along the direction $\mathbf{a}$ the polarization of the photon ν_1 along the direction $\mathbf{b}$ obtains finite uncertainty $\eta_{\mathbf{b}} = \pm\eta(\sigma_{\mathbf{b}},\psi_{12},\mathbf{P_I})$.

Thus for the joint detection probabilities $p_{\pm\pm}(\mathbf{a},t_1;\mathbf{b},t_2)$ instead classical Eq. (7.2.15) by using the weak postulate of nonlocality, (see subsection I.4.1) we obtain

$$\begin{aligned} p^{<}_{++}(\mathbf{a},t_1;\mathbf{b},t_2) &= p^{<}_{--}(\mathbf{a},t_1;\mathbf{b},t_2) = \tfrac{1}{2}\cos^2(\mathbf{a}\pm\epsilon_{\mathbf{a}},\mathbf{b}\pm\eta_{\mathbf{b}}),\\ p^{<}_{-+}(\mathbf{a},t_1;\mathbf{b},t_2) &= p^{<}_{+-}(\mathbf{a},t_1;\mathbf{b},t_2) = \tfrac{1}{2}\sin^2(\mathbf{a}\pm\epsilon_{\mathbf{a}},\mathbf{b}\pm\eta_{\mathbf{b}}).\end{aligned} \tag{7.2.19}$$

Where $\epsilon_{\mathbf{a}} = \pm\epsilon(\sigma_{\mathbf{a}},\psi_{12},\mathbf{P}_{\mathbf{I}}), t_1 - t_2 = -\delta < 0.$

Remark 7.2.13. Suppose now that the measurement on photon ν_2 occurs first, at instant t_2, and gives the result $+$, with the polarizer **I** in the orientation $\mathbf{a}$ and therefore the measurement on photon ν_1 occurs at instant t_1, where $t_1 - t_2 = \delta > 0$, (see Remark 7.1.8) and we will denote such events (i.e., if photon ν_1 collapses in polarizer **I** at instant t_1 and photon ν_2 collapses in polarizer **II** at instant t_2 respectively, where $t_1 - t_2 = \delta > 0, t_1,t_2 \in [0,T]$) by

$$\left\{\nu_1^{t_1},\nu_2^{t_2}\right\}^{>}$$

or simply $\{\nu_1,\nu_2\}^{>}$.

Remark 7.2.14. We will denote such entangled pairs of photons by $\left\{\nu_1^{t_1},\nu_2^{t_2}\right\}^{>}$, $t_1,t_2 \in [0,T]$ or simply $\{\nu_1,\nu_2\}^{>}$.

Remark 7.2.15. Note that after the measurement on the photon ν_2 along the direction $\mathbf{b}$ the polarization of the photon ν_1 along the direction $\mathbf{a}$ obtains finite uncertainty $\eta_{\mathbf{a}} = \pm\eta(\sigma_{\mathbf{a}},\psi_{12},\mathbf{P}_{\mathbf{II}})$. Thus for the joint detection probabilities $p^{>}_{\pm\pm}(\mathbf{a},t_1;\mathbf{b},t_2)$ instead classical Eq.(7.2.15) by using the weak postulate of nonlocality, (see subsection I.4.1) we obtain

$$\begin{aligned} p^{>}_{++}(\mathbf{a},t_1;\mathbf{b},t_2) &= p^{>}_{--}(\mathbf{a},t_1;\mathbf{b},t_2) = \tfrac{1}{2}\cos^2(\mathbf{a}\pm\eta_{\mathbf{a}},\mathbf{b}\pm\epsilon_{\mathbf{b}}),\\ p^{>}_{-+}(\mathbf{a},t_1;\mathbf{b},t_2) &= p^{>}_{+-}(\mathbf{a},t_1;\mathbf{b},t_2) = \tfrac{1}{2}\sin^2(\mathbf{a}\pm\eta_{\mathbf{a}},\mathbf{b}\pm\epsilon_{\mathbf{b}}),\end{aligned} \tag{7.2.20}$$

where $\epsilon_{\mathbf{b}} = \pm\epsilon(\sigma_{\mathbf{b}}, \psi_{12}, \mathbf{P_{II}}), t_1 - t_2 = \delta > 0.$

Remark 7.2.16. Note that similarly as above, instead Eq. (7.2.11) and Eq. (7.2.12) we obtain:

(I)

$$P^{<}_{VV}(\alpha,\beta) \triangleq P^{<}_{VV}(\alpha,t_1;\beta,t_2) = |\langle V_{\alpha\pm\epsilon_\alpha}|_s\langle V_{\beta\pm\eta_\beta}|_i|\psi_{DC}\rangle|^2, \quad (7.2.21)$$

$$\begin{gathered} P^{<}_{VV}(\alpha,\beta) \triangleq P^{<}_{VV}(\alpha,t_1;\beta,t_2) = \sin^2(\alpha\pm\epsilon_\alpha)\sin^2(\beta\pm\eta_\beta)\cos^2\theta_l + \\ +\cos^2(\alpha\pm\epsilon_\alpha)\cos^2(\beta\pm\eta_\beta)\sin^2\theta_l + \tfrac{1}{4}\sin 2(\alpha\pm\epsilon_\alpha)\sin 2(\beta\pm\eta_\beta)\sin 2\theta_l\cos\phi_m \end{gathered} \quad (7.2.22)$$

where $t_1 - t_2 = -\delta < 0$ and

$$\begin{gathered} N^{<}_T(\alpha_+,\beta_+) = \\ A^{<}_T[\sin^2(\alpha\pm\epsilon_\alpha)\sin^2(\beta\pm\eta_\beta)\cos^2\theta_l + \cos^2\alpha\cos^2(\beta\pm\eta_\beta)\sin^2\theta_l + \\ +\tfrac{1}{4}\sin 2(\alpha\pm\epsilon_\alpha)\sin 2(\beta\pm\eta_\beta)\sin 2\theta_l\cos\phi_m\,\Big] + C_2 = \\ A^{<}_T[\sin^2\alpha\sin^2\beta\cos^2\theta_l + \cos^2\alpha\cos^2\beta\sin^2\theta_l + \\ +\tfrac{1}{4}\sin 2\alpha\sin 2\beta\sin 2\theta_l\cos\phi_m\,\Big] + A^{<}_T\widetilde{C}(\alpha,\beta,\theta_l,\phi_m,\epsilon_\alpha,\eta_\beta) + C_2, \end{gathered} \quad (7.2.23)$$

where $A^{<}_T$ is the total number of entangled pairs $\left\{v_1^{t_1}, v_2^{t_2}\right\}^{<}$ produced during interval T and

$$N^{<}_T(\alpha_+,\beta_+)$$

is the number of coincidences corresponding to entangled pairs $\left\{v_1^{t_1}, v_2^{t_2}\right\}^{<}$ produced during interval T.

Remark 7.2.17. Note that $P^{<}_{VV}(\alpha,t_1;\beta,t_2|\left\{v_1^{t_1}, v_2^{t_2}\right\}^{<})$ is the conditional probability of the event $\{\alpha_+,\beta_+;t_2\}$ (see Remark 7.2.1) under condition that the event $\left\{v_1^{t_1}, v_2^{t_2}\right\}^{<}$ was occurred at instant $t = t_2$.

(II)

$$P^{>}_{VV}(\alpha,\beta) \triangleq P^{>}_{VV}(\alpha,t_1;\beta,t_2) = |\langle V_{\alpha\pm\eta_\alpha}|_s\langle V_{\beta\pm\epsilon_\beta}|_i|\psi_{DC}\rangle|^2, \quad (7.2.24)$$

$$P_{VV}^{>}(\alpha,\beta) \triangleq P_{VV}^{>}(\alpha,t_1;\beta,t_2) = \sin^2(\alpha\pm\eta_\alpha)\sin^2(\beta\pm\epsilon_\beta)\cos^2\theta_l + \\ +\cos^2(\alpha\pm\eta_\alpha)\cos^2(\beta\pm\epsilon_\beta)\sin^2\theta_l + \frac{1}{4}\sin 2(\alpha\pm\eta_\alpha)\sin 2(\beta\pm\epsilon_\beta)\sin 2\theta_l\cos\phi_m \quad (7.2.25)$$

where $t_1 - t_2 = \delta > 0$ and

$$N_T^{>}(\alpha_+,\beta_+) = \\ A_T^{>}[\sin^2(\alpha\pm\eta_\alpha)\sin^2(\beta\pm\eta_\beta)\cos^2\theta_l + \cos^2\alpha\cos^2(\beta\pm\epsilon_\beta)\sin^2\theta_l + \\ +\frac{1}{4}\sin 2(\alpha\pm\eta_\alpha)\sin 2(\beta\pm\epsilon_\beta)\sin 2\theta_l\cos\phi_m\,] + C_3 = \\ A_T^{>}[\sin^2\alpha\sin^2\beta\cos^2\theta_l + \cos^2\alpha\cos^2\beta\sin^2\theta_l + \\ +\frac{1}{4}\sin 2\alpha\sin 2\beta\sin 2\theta_l\cos\phi_m\,] + A_T^{>}\widetilde{C}(\alpha,\beta,\theta_l,\phi_m,\epsilon_\beta,\eta_\alpha) + C_3, \quad (7.2.26)$$

where $A_T^{>}$ is the total number of entangled pairs $\left\{v_1^{t_1},v_2^{t_2}\right\}^{>}$ produced during interval T and

$$N_T^{>}(\alpha_+,\beta_+)$$

is the number of coincidences corresponding to entangled pairs $\left\{v_1^{t_1},v_2^{t_2}\right\}^{>}$ produced during interval T.

Remark 7.2.18. Note that $P_{VV}^{>}(\alpha,t_1;\beta,t_2|\left\{v_1^{t_1},v_2^{t_2}\right\}^{>})$ is the conditional probability of the event $\{\alpha_+,\beta_+;t_1\}$ (see Remark 7.2.1) under condition that the event $\left\{v_1^{t_1},v_2^{t_2}\right\}^{>}$ was occurred at instant $t = t_1$.

From Eq. (7.2.14), (7.2.23) and (7.2.26) we obtain
(I)

$$\widetilde{P}_{VV}^{=} = \widetilde{P}_{VV}^{=}(\alpha_+,\beta_+) \triangleq P_{VV}^{=}(\{\alpha_+,\beta_+\}\wedge\{v_1,v_2\}^{=}) = \frac{N_T^{=}(\alpha_+,\beta_+)}{A_T^{=}+A_T^{<}+A_T^{>}} = \\ \frac{A_T^{=}}{A_T^{=}+A_T^{<}+A_T^{>}}[\sin^2\alpha\sin^2\beta\cos^2\theta_l + \cos^2\alpha\cos^2\beta\sin^2\theta_l + \\ \frac{1}{4}\sin 2\alpha\sin 2\beta\sin 2\theta_l\cos\phi_m\,] + C, \quad (7.2.27)$$

where $\widetilde{P}^{=}_{VV}(\alpha_+,\beta_+)$ is the joint probability of both events $\{\alpha_+,\beta_+\}$ and $\{v_1,v_2\}^{=}$ being true.

(II)

$$\begin{gathered}\widetilde{P}^{<}_{VV}(\alpha_+,\beta_+) \triangleq P^{<}_{VV}(\{\alpha_+,\beta_+\} \wedge \{v_1,v_2\}^{<}) = \frac{N^{<}_{T}(\alpha_+,\beta_+)}{A^{=}_{T}+A^{<}_{T}+A^{>}_{T}} = \\ \frac{A^{<}_{T}}{A^{=}_{T}+A^{<}_{T}+A^{>}_{T}}[\sin^2(\alpha \pm \epsilon_\alpha)\sin^2(\beta \pm \eta_\beta)\cos^2\theta_l + \cos^2\alpha\cos^2(\beta \pm \eta_\beta)\sin^2\theta_l + \\ +\tfrac{1}{4}\sin 2(\alpha \pm \epsilon_\alpha)\sin 2(\beta \pm \eta_\beta)\sin 2\theta_l\cos\phi_m\Big] + C = \\ A^{<}_{T}[\sin^2\alpha\sin^2\beta\cos^2\theta_l + \cos^2\alpha\cos^2\beta\sin^2\theta_l + \\ +\tfrac{1}{4}\sin 2\alpha\sin 2\beta\sin 2\theta_l\cos\phi_m\Big] + \widetilde{C}(\alpha,\beta,\theta_l,\phi_m,\epsilon_\alpha,\eta_\beta),\end{gathered} \tag{7.2.28}$$

where $\widetilde{P}^{<}_{VV}(\alpha_+,\beta_+)$ is the joint probability of both events $\{\alpha_+,\beta_+\}$ and $\{v_1,v_2\}^{<}$ being true.

(III)

$$\begin{gathered}\widetilde{P}^{>}_{VV}(\alpha_+,\beta_+) \triangleq P^{>}_{VV}(\{\alpha_+,\beta_+\} \wedge \{v_1,v_2\}^{>}) = \frac{N^{>}_{T}(\alpha_+,\beta_+)}{A^{=}_{T}+A^{<}_{T}+A^{>}_{T}} = \\ \frac{A^{>}_{T}}{A^{=}_{T}+A^{<}_{T}+A^{>}_{T}}[\sin^2(\alpha \pm \eta_\alpha)\sin^2(\beta \pm \eta_\beta)\cos^2\theta_l + \cos^2\alpha\cos^2(\beta \pm \epsilon_\beta)\sin^2\theta_l + \\ +\tfrac{1}{4}\sin 2(\alpha \pm \eta_\alpha)\sin 2(\beta \pm \epsilon_\beta)\sin 2\theta_l\cos\phi_m\Big] + C = \\ A[\sin^2\alpha\sin^2\beta\cos^2\theta_l + \cos^2\alpha\cos^2\beta\sin^2\theta_l + \\ +\tfrac{1}{4}\sin 2\alpha\sin 2\beta\sin 2\theta_l\cos\phi_m\Big] + \widetilde{C}(\alpha,\beta,\theta_l,\phi_m,\epsilon_\beta,\eta_\alpha),\end{gathered} \tag{7.2.29}$$

where $\widetilde{P}^{>}_{VV}(\alpha_+,\beta_+)$ is the joint probability of both events $\{\alpha_+,\beta_+\}$ and $\{v_1,v_2\}^{>}$ being true.

From Eq. (7.2.27), (7.2.28) and (7.2.29) we obtain

$$P_{VV} = P_{VV}(\alpha_+,\beta_+) = \widetilde{P}^{=}_{VV}(\alpha_+,\beta_+) + \widetilde{P}^{<}_{VV}(\alpha_+,\beta_+) + \widetilde{P}^{>}_{VV}(\alpha_+,\beta_+), \tag{7.2.30}$$

where $P_{VV}(\alpha_+,\beta_+)$ is the unconditional probability of the event $\{\alpha_+,\beta_+\}$.

Remark 7.2.19. Note that

$$\begin{gathered}P^{=}_{VH}(\alpha_+,\beta_-) = |\langle V_\alpha|_s\langle H_\beta|_i|\psi_{DC}\rangle|^2, P^{=}_{HV}(\alpha_-,\beta_+) = |\langle V_\alpha|_s\langle H_\beta|_i|\psi_{DC}\rangle|^2,\\ P^{=}_{HH}(\alpha_-,\beta_-) = |\langle H_\alpha|_s\langle H_\beta|_i|\psi_{DC}\rangle|^2,\\ P^{<}_{VV}(\alpha,\beta) = |\langle V_{\alpha\pm\epsilon_\alpha}|_s\langle V_{\beta\pm\eta_\beta}|_i|\psi_{DC}\rangle|^2,\\ P^{>}_{VV}(\alpha,\beta) = |\langle V_{\alpha\pm\eta_\alpha}|_s\langle V_{\beta\pm\epsilon_\beta}|_i|\psi_{DC}\rangle|^2\end{gathered} \quad (7.2.31)$$

From Eqs. (7.2.31) similarly as above we obtain

(**I**)

$$P_{VH} = P_{VH}(\alpha_+,\beta_-) = \widetilde{P}^{=}_{VH}(\alpha_+,\beta_-) + \widetilde{P}^{<}_{VH}(\alpha_+,\beta_-) + \widetilde{P}^{>}_{VH}(\alpha_+,\beta_-), \quad (7.2.32)$$

where

(i) $P_{VH} = P_{VV}(\alpha_+,\beta_-)$ is the unconditional probability of the event $\{\alpha_+,\beta_-\}$,

(ii) $\widetilde{P}^{=}_{VH}(\alpha_+,\beta_-)$ is the joint probability of both events $\{\alpha_+,\beta_-\}$ and $\{v_1,v_2\}^{=}$ being true,

(iii) $\widetilde{P}^{<}_{VH}(\alpha_+,\beta_-)$ is the joint probability of both events $\{\alpha_+,\beta_-\}$ and $\{v_1,v_2\}^{<}$ being true,

(iv) $\widetilde{P}^{>}_{VH}(\alpha_+,\beta_-)$ is the joint probability of both events $\{\alpha_+,\beta_-\}$ and $\{v_1,v_2\}^{>}$ being true,

(**II**)

$$P_{HV} = P_{HV}(\alpha_-,\beta_+) = \widetilde{P}^{=}_{HV}(\alpha_-,\beta_+) + \widetilde{P}^{<}_{HV}(\alpha_-,\beta_+) + \widetilde{P}^{>}_{HV}(\alpha_-,\beta_+), \quad (7.2.33)$$

where

(i) $P_{HV} = P_{HV}(\alpha_-,\beta_+)$ is the unconditional probability of the event $\{\alpha_-,\beta_+\}$,

(ii) $\widetilde{P}^{=}_{HV}(\alpha_-,\beta_+)$ is the joint probability of both events $\{\alpha_-,\beta_+\}$ and $\{v_1,v_2\}^{=}$ being true,

(iii) $\widetilde{P}^{<}_{HV}(\alpha_-,\beta_+)$ is the joint probability of both events $\{\alpha_-,\beta_+\}$ and $\{v_1,v_2\}^{<}$ being true,

(iv) $\widetilde{P}^{>}_{HV}(\alpha_-,\beta_+)$ is the joint probability of both events $\{\alpha_-,\beta_+\}$ and $\{v_1,v_2\}^>$ being true,

(III)

$$P_{HH}=P_{HH}(\alpha_-,\beta_-)=\widetilde{P}^{=}_{HH}(\alpha_-,\beta_-)+\widetilde{P}^{<}_{HH}(\alpha_-,\beta_-)+\widetilde{P}^{>}_{HH}(\alpha_-,\beta_-), \qquad (7.2.34)$$

where

(i) $P_{HV}=P_{HV}(\alpha_-,\beta_-)$ is the unconditional probability of the event $\{\alpha_-,\beta_-\}$,

(ii) $\widetilde{P}^{=}_{HH}(\alpha_-,\beta_-)$ is the joint probability of both events $\{\alpha_-,\beta_-\}$ and $\{v_1,v_2\}^=$ being true,

(iii) $\widetilde{P}^{<}_{HH}(\alpha_-,\beta_-)$ is the joint probability of both events $\{\alpha_-,\beta_-\}$ and $\{v_1,v_2\}^<$ being true,

(iv) $\widetilde{P}^{>}_{HH}(\alpha_-,\beta_-)$ is the joint probability of both events $\{\alpha_-,\beta_-\}$ and $\{v_1,v_2\}^>$ being true.

From Eq. (7.2.14), (7.2.23) and (7.2.26) we obtain

$$\begin{gathered}
N_T^{tot}(\alpha_+,\beta_+)=N_T^{=}(\alpha_+,\beta_+)+N_T^{<}(\alpha_+,\beta_+)+N_T^{>}(\alpha_+,\beta_+)=\\
+A_T^{=}\left(\sin^2\alpha\sin^2\beta\cos^2\theta_l+\cos^2\alpha\cos^2\beta\sin^2\theta_l+\tfrac{1}{4}\sin2\alpha\sin2\beta\sin2\theta_l\cos\phi_m\right)+C_1+\\
+A_T^{<}[\sin^2(\alpha\pm\epsilon_\alpha)\sin^2(\beta\pm\eta_\beta)\cos^2\theta_l+\cos^2\alpha\cos^2(\beta\pm\eta_\beta)\sin^2\theta_l+\\
+\tfrac{1}{4}\sin2(\alpha\pm\epsilon_\alpha)\sin2(\beta\pm\eta_\beta)\sin2\theta_l\cos\phi_m\,]+C_2+\\
+A_T^{>}[\sin^2(\alpha\pm\eta_\alpha)\sin^2(\beta\pm\eta_\beta)\cos^2\theta_l+\cos^2\alpha\cos^2(\beta\pm\epsilon_\beta)\sin^2\theta_l+\\
+\tfrac{1}{4}\sin2(\alpha\pm\eta_\alpha)\sin2(\beta\pm\epsilon_\beta)\sin2\theta_l\cos\phi_m\,]+C_3=\\
(A_T^{=}+A_T^{<}+A_T^{>})\times\\
\times\left(\sin^2\alpha\sin^2\beta\cos^2\theta_l+\cos^2\alpha\cos^2\beta\sin^2\theta_l+\tfrac{1}{4}\sin2\alpha\sin2\beta\sin2\theta_l\cos\phi_m\right)+\\
\left(A_T^{<}\widetilde{C}_2(\alpha,\beta,\theta_l,\phi_m,\epsilon_\alpha,\eta_\beta)+A_T^{>}\widetilde{C}_3(\alpha,\beta,\theta_l,\phi_m,\epsilon_\beta,\eta_\alpha)\right)+C=\\
A_T\left(\sin^2\alpha\sin^2\beta\cos^2\theta_l+\cos^2\alpha\cos^2\beta\sin^2\theta_l+\tfrac{1}{4}\sin2\alpha\sin2\beta\sin2\theta_l\cos\phi_m\right)+\\
\left(A_T^{<}\widetilde{C}_2(\alpha,\beta,\theta_l,\phi_m,\epsilon_\alpha,\eta_\beta)+A_T^{>}\widetilde{C}_3(\alpha,\beta,\theta_l,\phi_m,\epsilon_\beta,\eta_\alpha)\right)+C_1+C_2+C_3=\\
A_T\left(\sin^2\alpha\sin^2\beta\cos^2\theta_l+\cos^2\alpha\cos^2\beta\sin^2\theta_l+\tfrac{1}{4}\sin2\alpha\sin2\beta\sin2\theta_l\cos\phi_m\right)+C,
\end{gathered} \qquad (7.2.35)$$

where $A_T=A_T^{=}+A_T^{<}+A_T^{>}$ is the total number of entangled pairs produced and

$$C = \left(A_T^{<}\widetilde{C}_2(\alpha,\beta,\theta_l,\phi_m,\epsilon_\alpha,\eta_\beta) + A_T^{>}\widetilde{C}_3(\alpha,\beta,\theta_l,\phi_m,\epsilon_\beta,\eta_\alpha)\right) + C_1 + C_2 + C_3. \qquad (7.2.36)$$

Remark 7.2.20. To create the state $|\psi_{\mathrm{EPR}}\rangle$ or something close to it, it is necessarily to adjust the parameters which determine the laser polarization. First one adjusts θ_l to equalize the coincidence counts $N(0^\circ,0^\circ)$ and $N(90^\circ,90^\circ)$.

Next one set ϕ_l by rotating the quartz plate about a vertical axis to maximize $N(45^\circ,45^\circ)$. When performing these optimizations, one typically collects a few hundred photons per point which requires an acquisition window of a few seconds.

Remark 7.2.21. A rough idea of the purity of the entangled state can be found by measuring $N(0^\circ,0^\circ)$, $N(90^\circ,90^\circ), N(45^\circ,45^\circ)$ and $N(0^\circ,90^\circ)$. Using the model of Eq. (7.2.19), one obtains

$$C = N(0^\circ,90^\circ), \qquad (7.2.37)$$

$$A = N(0^\circ,0^\circ) + N(90^\circ,90^\circ) - 2C, \qquad (7.2.38)$$

$$\tan^2\theta_l = \frac{N(90^\circ,90^\circ) - C}{N(0^\circ,0^\circ) - C}, \qquad (7.2.39)$$

$$\cos\phi_m = \frac{1}{\sin 2\theta_l}\left(4\frac{N(45^\circ,45^\circ) - C}{A} - 1\right). \qquad (7.2.40)$$

In a typical acquisition, after optimizing θ_l and ϕ_l we find, with $T = 10$ seconds, $N(0,0) = 293$, $N(90,90) = 307$, $N(0,90) = 22$,and $N(45,45) = 286$.

These give $C = 22, A = 556, \theta_l = 46,$ and $\phi_m = 26$. More extensive data are shown in Fig. 7.2.5 along with a fit to Eq. (7.2.19). The best fit parameters, $C = 31, A = 539, \theta_l = 46$ and $\phi_m = 26$ are in good agreement with the rough estimates made with just four points.

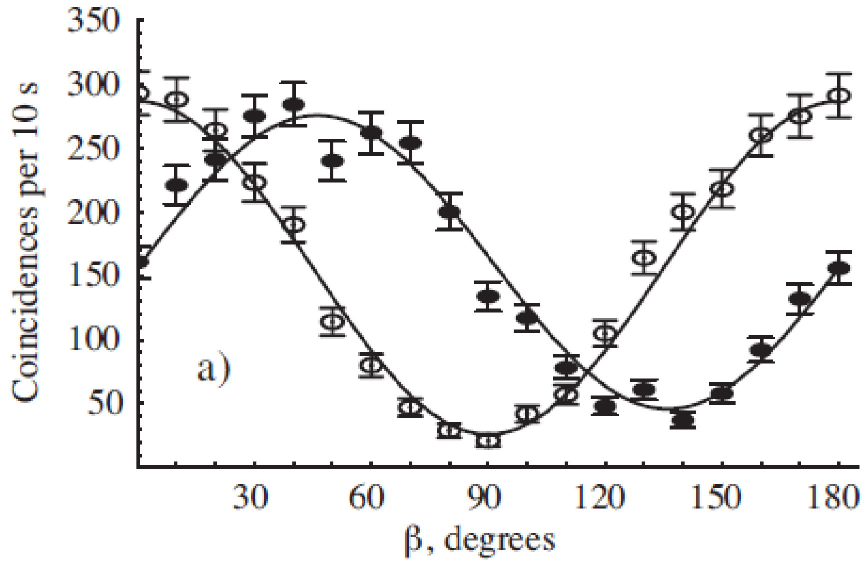

Fig. 7.2.5a Experimental polarization correlations.

$\alpha = 0^\circ$ (open circles) and $\alpha = 45^\circ$ (filled circles)

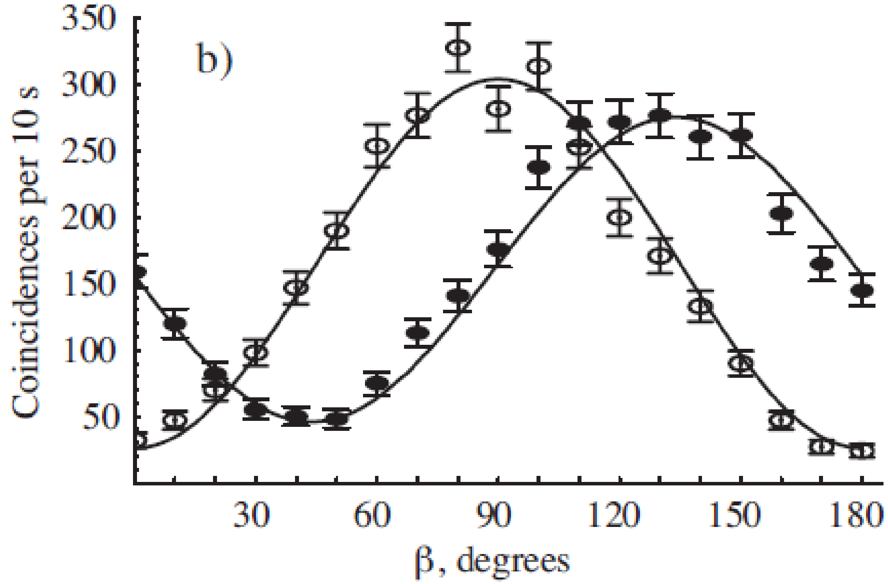

Fig. 7.2.5b. Experimental polarization correlations.
$\alpha = 90^\circ$ (open circles) and $\alpha = 135^\circ$ (filled circles)

Fig. 7.2.5a shows $\alpha = 0^\circ$ (open circles) and $\alpha = 45^\circ$ (filled circles). Fig.7.2.5b shows $\alpha = 90^\circ$ (open circles) and $\alpha = 135^\circ$ (filled circles). Error bars indicate plus/minus one standard deviation statistical uncertainty. Curves are a fit to Eq. (7.2.19).

Remark 7.2.22. Remind that in his comment on Bohr's lecture, Einstein noted that quantum mechanics allows a measurement of one particle to influence the state of another. To illustrate this for polarizations, we consider again the state $|\psi_{\text{EPR}}\rangle$ of Eq. (7.2.3). If the signal photon is measured with a polarizer set to α, the result will be H_α or V_α, each occurring half the time.
In the usual Copenhagen interpretation the state has collapsed, at the moment of measurement, from $|\psi_{\text{EPR}}\rangle$ to either $|V_\alpha\rangle_{\text{s}}|V_\alpha\rangle_{\text{i}}$ or $|H_\alpha\rangle_{\text{s}}|H_\alpha\rangle_{\text{i}}$. But the mere choice of α does not determine the state of the idler photon; it is the (random) outcome of the measurement on the signal photon that decides whether the idler ends up as $|V_\alpha\rangle_i$ or $|H_\alpha\rangle_i$.
Despite the randomness, the choice of α clearly has an effect on the state of the idler photon: it gives it a definite polarization in the $|V_\alpha\rangle_i, |H_\alpha\rangle_i$ basis, which it did not have before the measurement.

Remark 7.2.23. After the signal photon is measured the idler is equally likely to be V_α or H_α. A measurement of its polarization, at any angle β, finds V_β with the probability

$$\begin{aligned} P_V(\beta) &= \tfrac{1}{2}|\langle V_{\beta\pm\delta_{\mathbf{b}}}|V_\alpha\rangle|^2 + \tfrac{1}{2}|\langle V_{\beta\pm\delta_{\mathbf{b}}}|H_\alpha\rangle|^2 = \\ &= \tfrac{1}{2}[\cos^2(\beta\pm\delta_{\mathbf{b}} - \alpha) + \sin^2(\beta\pm\delta_{\mathbf{b}} - \alpha)] = \tfrac{1}{2}. \end{aligned} \tag{7.2.41}$$

This gives no information about the choice of α. It is also the probability we would find if the signal photon had not been measured.

VII.2.2. EPR-B nonlocality is the physical nature of the violation of the Bell inequalities

Remind that classical CHSH constrains the degree of polarization correlation under measurements at different polarizer angles.
The proof involves two measures of correlation, see subsection VII.2.3. The first measure is

$$E(\alpha,\beta) \equiv P_{VV}(\alpha,\beta) + P_{HH}(\alpha,\beta) - P_{VH}(\alpha,\beta) - P_{HV}(\alpha,\beta). \qquad (7.2.42)$$

This incorporates all possible measurement outcomes and varies from $+1$ when the polarizations always agree to -1 when they always disagree.

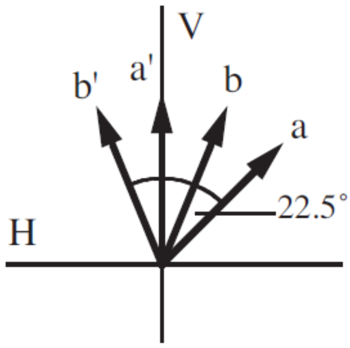

Fig. 7.2.6. Polarizer angles for maximal S^{QM}

The second measure is

$$S \equiv E(a,b) - E(a,b') + E(a',b) + E(a',b'), \qquad (7.2.43)$$

where a, a', b, b' are four different polarizer angles. S does not have a clear physical meaning. Its importance comes from the fact that Clauser, Horne, Shimony and Holt proved

$$|S^{HVT}| \leq 2 \qquad (7.2.44)$$

(see subsection VII.2.3) for any classical Hidden Variable Theory (HVT) and arbitrary a, a', b, b'.

Remark 7.2.24. Quantum mechanics under (i) canonical Copenhagen interpretation, and (ii) canonical SRT (Special Relativity Theory) locality for certain settings, can violate this inequality. If we choose the polarizer angles, $a = -45,\ a' = 0,\ b = 22.5$ and $b' = 22.5$, as shown in Fig. 7.2.6, then, using Eq. (7.2.15)

$$S^{(QM)} = 2\sqrt{2}. \qquad (7.2.45)$$

This result is specific to the state $|\psi_{\mathrm{EPR}}\rangle$. Other states give lower S values. It is interesting to note that for these angles our simple HVT (see subsection VII.2.3) gives

$$S^{(HVT)} = 2. \qquad (7.2.46)$$

The CHSH (Clauser, Horne, Shimony and Holt) inequality shows that no theory which is both local and realistic (or `complete' in the EPR sense) will ever agree with quantum mechanics.

Remark 7.2.25. Note that derivation of the CHSH inequality essentially depends on SR locality condition, see Remark 7.2.27. In subsection VII.2.4 we introduced generalized HVT (GHVT) based on EPR-B nonlocality condition. For such GHVT we derive the revised CHSH inequality

$$|S^{GHVT}| \leq 4. \qquad (7.2.47)$$

To find the probabilities P that make up E, we need four values of N, specificially

$$P_{VV}(\alpha,\beta) = N_T^{tot}(\alpha,\beta)/N_T^{tot}, P_{VH}(\alpha,\beta) = N_T^{tot}(\alpha,\beta_\perp)/N_T^{tot},$$
$$P_{HV}(\alpha,\beta) = N_T^{tot}(\alpha_\perp,\beta)/N_T^{tot}, P_{HH}(\alpha,\beta) = N_T^{tot}(\alpha_\perp,\beta_\perp)/N_T^{tot}, \quad (7.2.48)$$
$$N_T^{tot} = N_T^{tot}(\alpha,\beta) + N_T^{tot}(\alpha_\perp,\beta) + N_T^{tot}(\alpha,\beta_\perp) + N_T^{tot}(\alpha_\perp,\beta_\perp),$$

where N_{tot} is the total number of pairs detected during interval of time $[0,T]$ and $\alpha_\perp,\beta_\perp$ are the polarizer settings $\alpha+90,\ \ \beta+90$. This requires counting coincidences for equal intervals with the polarizer set four different ways.

The quantity $E(\alpha,\beta)$ requires four N measurements

$$E(\alpha,\beta) = \frac{N_T^{tot}(\alpha,\beta) + N_T^{tot}(\alpha_\perp,\beta_\perp) - N_T^{tot}(\alpha,\beta_\perp) - N_T^{tot}(\alpha_\perp,\beta)}{N_T^{tot}(\alpha,\beta) + N_T^{tot}(\alpha_\perp,\beta_\perp) + N_T^{tot}(\alpha,\beta_\perp) + N_T^{tot}(\alpha_\perp,\beta)} \quad (7.2.49)$$

and $S \equiv E(a,b) - E(a,b') + E(a',b) + E(a',b')$ requires sixteen.

Remark 7.2.26. By consideration above (see Eq. (7.2.35)) based on EPR-B nonlocality condition we find that the quantities: $N_T^{tot}(\alpha,\beta), N_T^{tot}(\alpha_\perp,\beta_\perp), N_T^{tot}(\alpha,\beta_\perp), N_T^{tot}(\alpha_\perp,\beta)$ have the representations

$$N_T^{tot}(\alpha,\beta) = N_T^{=}(\alpha,\beta) + N_T^{<}(\alpha,\beta) + N_T^{>}(\alpha,\beta),$$
$$N_T^{tot}(\alpha_\perp,\beta_\perp) = N_T^{=}(\alpha_\perp,\beta_\perp) + N_T^{<}(\alpha_\perp,\beta_\perp) + N_T^{>}(\alpha_\perp,\beta_\perp),$$
$$N_T^{tot}(\alpha,\beta_\perp) = N_T^{=}(\alpha,\beta_\perp) + N_T^{<}(\alpha,\beta_\perp) + N_T^{>}(\alpha,\beta_\perp), \quad (7.2.50)$$
$$N_T^{tot}(\alpha_\perp,\beta) = N_T^{=}(\alpha_\perp,\beta) + N_T^{<}(\alpha_\perp,\beta) + N_T^{>}(\alpha_\perp,\beta).$$

These representations have the rigorous physical meaning and they are essentially important for derivation of the revised CHSH inequality (7.2.47), see subsection VII.2.4.

Remark 7.2.27. Remind that any classical Bell's type inequality was derived under the condition of SR locality and the canonical Copenhagen interpretation of QM.

We consider now again the state $|\psi_{\text{EPR}}\rangle$ of Eq. (7.2.3). If the signal photon ν_1 is measured at instant t_1 with a polarizer **I** set to α, the result will be H_α or V_α, each occurring half the time. In the usual Copenhagen interpretation the state $|\psi_{\mathrm{EPR}}\rangle$ has immediately collapsed, at the moment of measurement, from $|\psi_{\mathrm{EPR}}\rangle$ to either $|V_\alpha\rangle_{\mathrm{s}}|V_\alpha\rangle_{\mathrm{i}}$ or $|H_\alpha\rangle_{\mathrm{s}}|H_\alpha\rangle_{\mathrm{i}}$. We assume now for definiteness that the state $|\psi_{\mathrm{EPR}}\rangle$ collapsed to $|V_\alpha\rangle_{\mathrm{s}}|V_\alpha\rangle_{\mathrm{i}}$ and thus the state of the idler photon v_2 is $|V_\alpha\rangle_{\mathrm{i}}$. In agreement with a strong SR locality the idler photon v_2 is not disturbed and its polarization can be measured with a polarizer **II** at instant t_2 such that $t_2 > t_1$. Thus under the canonical physical interpretation of the Bell test experiment given in physical literature [8] it is not important which event was occurred: $\{v_1^{t_1}, v_2^{t_2}\}^{=}, \{v_1^{t_1}, v_2^{t_2}\}^{<}$ or $\{v_1^{t_1}, v_2^{t_2}\}^{>}$, before the corresponding coincidence is revealed with detectors A and B.

Remark 7.2.28. We emphasize that the violation of the classical CHSH inequality (7.2.44) confirms EPR-B nonlocality condition.

VII.2.3. Canonical Local Realistic Hidden Variable Theory

Einstein believed that a theory could be found to replace quantum mechanics, one which was complete and contained only local interactions. Here we describe such a theory, a local realistic hidden variable theory (HVT) [8].

In a such HVT, each photon has a polarization angle λ, but this polarization does not behave like polarization in quantum mechanics. When a photon meets a polarizer set to an angle γ, it will always register as V_γ if λ is closer to γ than to $\gamma + \pi/2$, i.e.,

$$P_V^{(HVT)}(\gamma,\lambda) = \begin{cases} 1 & |\gamma-\lambda| \leq \pi/4 \\ 1 & |\gamma-\lambda| > 3\pi/4 \\ 0 & \text{otherwise.} \end{cases} \quad (7.2.51)$$

In each pair, the signal and idler photon have the same polarization

$\lambda_s = \lambda_i = \lambda$. As successive pairs are produced λ changes in an unpredictable manner that uniformly covers the whole range of possible polarizations.

The quantity λ is the hidden variable, a piece of information that is absent from quantum mechanics. HVTs do not have the spooky features of quantum mechanics. The theory is *local*: measurement outcomes are determined by features of objects present at the site of measurement. Any measurement on the signal (idler) photon is determined by λ_s and α (λ_i and β). The theory is also *realistic*: All measurable quantities have definite values, independent of our knowledge of them. Furthermore, the theory specifies all of these values (for a given λ), so it is *complete* in Einstein's sense of the word. Finally, there is no requirement that λ be random; it could be that λ is changing in a deterministic way that remains to be discovered.

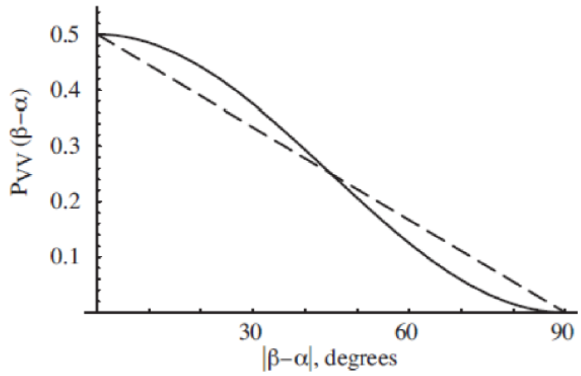

Fig. 7.2.7. Predicted polarization correlations for a quantum mechanical entangled state (solid curve) and a hidden-variable theory (dashed line)

To compare this theory to quantum mechanics, we need a prediction for the coincidence probability $P_{VV}^{(HVT)}(\alpha,\beta)$. A coincidence occurs when λ is in a range such that both α and β are close to λ. The probability of this is

$$P_{VV}^{(HVT)}(\alpha,\beta) = \frac{1}{\pi}\int_0^\pi P_V^{(HVT)}(\alpha,\lambda)P_V^{(HVT)}(\beta,\lambda)d\lambda = \frac{1}{2} - \frac{|\beta-\alpha|}{\pi}. \qquad (7.2.52)$$

This function and the corresponding quantum mechanical probability from Eq. (7.2.10) are plotted in Fig. 7.2.7. The predictions are fairly similar. Where they disagree quantum mechanics predicts stronger correlations (or stronger anti-correlations) than the HVT.

Remind that for any canonical HVT, the distribution of the hidden variable λ is described by a function $\rho(\lambda)$, where $\rho(\lambda) \geq 0$ and $\int \rho(\lambda)d\lambda = 1$.

The assumptions of locality and realism are embodied in the following: It is assumed that for the signal photon the outcome of a measurement is determined completely by λ and the measurement angle α. These outcomes are specified by the function $A(\lambda,\alpha)$, which can take on the values $+1$ for detection as V_α and -1 for detection as H_α. Similarly, a function $B(\lambda,\beta)$ describes the outcomes for the idler photon as $+1$ for V_β and -1 for H_β. A HVT would specify the functions ρ, A and B.

The probability of a particular outcome, averaged over an ensemble of photon pairs, is given by an integral. In particular

$$\begin{aligned} P_{VV}(\alpha,\beta) &= \int \tfrac{1+A(\lambda,\alpha)}{2}\tfrac{1+B(\lambda,\beta)}{2}\rho(\lambda)d\lambda, P_{VH}(\alpha,\beta) = \int \tfrac{1+A(\lambda,\alpha)}{2}\tfrac{1-B(\lambda,\beta)}{2}\rho(\lambda)d\lambda, \\ P_{HV}(\alpha,\beta) &= \int \tfrac{1-A(\lambda,\alpha)}{2}\tfrac{1+B(\lambda,\beta)}{2}\rho(\lambda)d\lambda, P_{HH}(\alpha,\beta) = \int \tfrac{1-A(\lambda,\alpha)}{2}\tfrac{1-B(\lambda,\beta)}{2}\rho(\lambda)d\lambda. \end{aligned} \qquad (7.3.53)$$

Let $E(\alpha,\beta)$ be

$$E(\alpha,\beta) \equiv P_{VV}(\alpha,\beta) + P_{HH}(\alpha,\beta) - P_{VH}(\alpha,\beta) - P_{HV}(\alpha,\beta). \qquad (7.2.54)$$

It is easy to see that $E(\alpha,\beta)$, given in Eq. (7.2.54), is

$$E(\alpha,\beta) = \int A(\lambda,\alpha)B(\lambda,\beta)\rho(\lambda)d\lambda. \qquad (7.2.55)$$

We define now the quantity s, which describes the polarization correlation in a single pair of particles:

$$\begin{aligned} s(\lambda,a,a',b,b') &= A(\lambda,a)B(\lambda,b) - A(\lambda,a)B(\lambda,b') + A(\lambda,a')B(\lambda,b) + A(\lambda,a')B(\lambda,b') \\ &= A(\lambda,a)[B(\lambda,b) - B(\lambda,b')] + A(\lambda,a')[B(\lambda,b) + B(\lambda,b')], \end{aligned} \quad (7.2.56)$$

where a,a',b,b' are four angles. Note that s can only take on the values ± 2. The average of s over an ensemble of pairs is

$$\begin{gathered} \langle s(\lambda,a,a',b,b')\rangle = \int s(\lambda,a,a',b,b')\rho(\lambda)d\lambda = \\ E(a,b) - E(a,b') + E(a',b) + E(a',b') = S(a,a',b,b'). \end{gathered} \quad (7.2.57)$$

Because s can only take on the values ± 2, its average S must satisfy $-2 \leq S \leq +2$, which is the Clauser, Horne, Shimony and Holt inequality

$$|S| \leq 2. \quad (7.2.58)$$

VII.2.4. Local Hidden Variable Theory revisited. Generalized Local Hidden Variable Theory Validity of CHSH-inequality for correlations taking into account EPR-B nonlocality

For any GHVT, the distribution of the hidden variable λ is described by a function $\rho(\lambda,t_1,t_2)$, where

$$\rho(\lambda,t_1,t_2) \geq 0 \quad (7.2.59)$$

and

$$\int \rho(\lambda,t_1,t_2)d\lambda = 1. \quad (7.2.60)$$

The assumptions of EPR-B nonlocality and realism are embodied in the following: It is assumed that for the signal photon the outcome of a measurement is determined completely by λ and the measurement angle α.

These outcomes are specified by the function $A(\lambda, \alpha, t_1)$, which can take on the values $+1$ for detection as V_α and -1 for detection as H_α. Similarly, a function $B(\lambda, \beta, t_2)$ describes the outcomes for the idler photon as $+1$ for V_β and -1 for H_β. A GHVT would specify the functions ρ, A and B.

The probability of a particular outcome, averaged over an ensemble of photon pairs, is given by an integral. In particular

$$\begin{aligned}
&P_{VV}(\alpha, t_1; \beta, t_2) = \\
&\int \frac{1+[A(\lambda,\alpha,t_1)+A(\lambda,\alpha,t_2)]}{2}\frac{1+[B(\lambda,\beta,t_1)+B(\lambda,\alpha,t_2)]}{2}\rho(\lambda,t_1,t_2)d\lambda, \\
&P_{VH}(\alpha, t_1; \beta, t_2) = \\
&\int \frac{1+[A(\lambda,\alpha,t_1)+A(\lambda,\alpha,t_2)]}{2}\frac{1-[B(\lambda,\beta,t_1)+B(\lambda,\alpha,t_2)]}{2}\rho(\lambda,t_1,t_2)d\lambda, \\
&P_{HV}(\alpha, t_1; \beta, t_2) = \\
&\int \frac{1-[A(\lambda,\alpha,t_1)+A(\lambda,\alpha,t_2)]}{2}\frac{1+[B(\lambda,\beta,t_1)+B(\lambda,\alpha,t_2)]}{2}\rho(\lambda,t_1,t_2)d\lambda, \\
&P_{HH}(\alpha, t_1; \beta, t_2) = \\
&\int \frac{1-[A(\lambda,\alpha,t_1)+A(\lambda,\alpha,t_2)]}{2}\frac{1-[B(\lambda,\beta,t_1)+B(\lambda,\alpha,t_2)]}{2}\rho(\lambda,t_1,t_2)d\lambda.
\end{aligned} \tag{7.2.61}$$

Let $E(\alpha, t_1; \beta, t_2)$ be

$$E(\alpha,t_1;\beta,t_2) = P_{VV}(\alpha,t_1;\beta,t_2) + P_{HH}(\alpha,t_1;\beta,t_2) - P_{VH}(\alpha,t_1;\beta,t_2) - P_{HV}(\alpha,t_1;\beta,t_2). \tag{7.2.62}$$

It is easy to see that $E(\alpha, t_1; \beta, t_2)$, given in Eq. (7.2.62), is

$$
\begin{aligned}
&E(\alpha,t_1;\beta,t_2) = \\
&\int[A(\lambda,\alpha,t_1)+A(\lambda,\alpha,t_2)][B(\lambda,\beta,t_1)+B(\lambda,\alpha,t_2)]\rho(\lambda,t_1,t_2)d\lambda = \\
&\int A(\lambda,\alpha,t_1)B(\lambda,\beta,t_1)\rho(\lambda,t_1,t_2)d\lambda+\int A(\lambda,\alpha,t_1)B(\lambda,\alpha,t_2)\rho(\lambda,t_1,t_2)d\lambda+ \\
&+\int A(\lambda,\alpha,t_2)B(\lambda,\beta,t_1)\rho(\lambda,t_1,t_2)d\lambda+\int A(\lambda,\alpha,t_2)B(\lambda,\alpha,t_2)\rho(\lambda,t_1,t_2)d\lambda = \\
&\int A(\lambda,\alpha,t_1)B(\lambda,\alpha,t_2)\rho(\lambda,t_1,t_2)d\lambda+\int A(\lambda,\alpha,t_2)B(\lambda,\beta,t_1)\rho(\lambda,t_1,t_2)d\lambda+ \\
&\int A(\lambda,\alpha,t_1)B(\lambda,\beta,t_1)\rho(\lambda,t_1,t_2)d\lambda+\int A(\lambda,\alpha,t_2)B(\lambda,\alpha,t_2)\rho(\lambda,t_1,t_2)d\lambda
\end{aligned}
\qquad (7.2.63)
$$

We assume now for simplicity that:
(i) $|t_1-t_2|=\delta>0,$ i.e. $(t_1>t_2)\vee(t_1<t_2)\wedge(t_1\neq t_2)$ and
(ii)

$$
\int A(\lambda,\alpha,t_1)B(\lambda,\beta,t_1)\rho(\lambda,t_1,t_2)d\lambda=0,\int A(\lambda,\alpha,t_2)B(\lambda,\alpha,t_2)\rho(\lambda,t_1,t_2)d\lambda=0. \qquad (7.2.64)
$$

From Eq. (7.2.63) and Eq. (7.2.64) we obtain

$$
\begin{aligned}
&E(\alpha,t_1;\beta,t_2) = \\
&\int A(\lambda,\alpha,t_1)B(\lambda,\alpha,t_2)\rho(\lambda,t_1,t_2)d\lambda+\int A(\lambda,\alpha,t_2)B(\lambda,\beta,t_1)\rho(\lambda,t_1,t_2)d\lambda
\end{aligned}
\qquad (7.2.65)
$$

We assume now for definiteness that: $t_1>t_2$ and rewrite Eq. (7.2.65) in the following form

$$
\begin{aligned}
E(\alpha,t_1;\beta,t_2) &= E_1(\alpha,t_1;\beta,t_2)+E_2(\alpha,t_1;\beta,t_2) \\
E_1(\alpha,t_1;\beta,t_2) &= \int A(\lambda,\alpha,t_1)B(\lambda,\alpha,t_2)\rho(\lambda,t_1,t_2)d\lambda, \\
E_2(\alpha,t_1;\beta,t_2) &= \int A(\lambda,\alpha,t_2)B(\lambda,\beta,t_1)\rho(\lambda,t_1,t_2)d\lambda.
\end{aligned}
\qquad (7.2.66)
$$

The second measure $S(a,a',b,b';t_1,t_2)$ now is

$$\begin{aligned}
&S(a,a',b,b';t_1,t_2) = S_1(a,a',b,b';t_1,t_2) + S_2(a,a',b,b';t_1,t_2) = \\
&E(a,t_1;b,t_2) - E(a,t_1;b',t_2) + E(a',t_1;b,t_2) + E(a',t_1;b',t_2) = \\
&E_1(\alpha,t_1;\beta,t_2) - E_1(a,t_1;b',t_2) + E_1(a',t_1;b,t_2) + E_1(a',t_1;b',t_2) + \\
&+E_2(a,t_1;b,t_2) - E_2(a,t_1;b',t_2) + E_2(a',t_1;b,t_2) + E_2(a',t_1;b',t_2),
\end{aligned} \tag{7.2.67}$$

where

$$\begin{aligned}
&S_1(a,a',b,b';t_1,t_2) = \\
&E_1(\alpha,t_1;\beta,t_2) - E_1(a,t_1;b',t_2) + E_1(a',t_1;b,t_2) + E_1(a',t_1;b',t_2), \\
&S_2(a,a',b,b';t_1,t_2) = \\
&E_2(a,t_1;b,t_2) - E_2(a,t_1;b',t_2) + E_2(a',t_1;b,t_2) + E_2(a',t_1;b',t_2).
\end{aligned} \tag{7.2.68}$$

Note that

$$\begin{aligned}
S_1(a,a',b,b';t_1,t_2) &= \int s_1(\lambda,a,a',b,b';t_1,t_2)\rho(\lambda,t_1,t_2)d\lambda, \\
S_2(a,a',b,b';t_1,t_2) &= \int s_2(\lambda,a,a',b,b';t_1,t_2)\rho(\lambda,t_1,t_2)d\lambda,
\end{aligned} \tag{7.2.69}$$

where

$$\begin{aligned}
&s_1(\lambda,a,a',b,b';t_1,t_2) = \\
&A(\lambda,a,t_1)B(\lambda,b,t_2) - A(\lambda,a,t_1)B(\lambda,b',t_2) + \\
&A(\lambda,a',t_1)B(\lambda,b,t_2) + A(\lambda,a',t_1)B(\lambda,b',t_2) = \\
&A(\lambda,a,t_1)[B(\lambda,b,t_2) - B(\lambda,b',t_2)] + \\
&A(\lambda,a',t_1)[B(\lambda,b,t_2) + B(\lambda,b',t_2)]
\end{aligned} \tag{7.2.70}$$

and

$$\begin{aligned}
&s_2(\lambda,a,a',b,b';t_1,t_2) = \\
&A(\lambda,a,t_2)B(\lambda,b,t_1) - A(\lambda,a,t_2)B(\lambda,b',t_1) + \\
&A(\lambda,a',t_2)B(\lambda,b,t_1) + A(\lambda,a',t_2)B(\lambda,b',t_1) = \\
&A(\lambda,a,t_2)[B(\lambda,b,t_1) - B(\lambda,b',t_1)] + \\
&A(\lambda,a',t_2)[B(\lambda,b,t_1) + B(\lambda,b',t_1)].
\end{aligned} \tag{7.2.71}$$

Remark 7.2.29. Note that $s_1(\lambda,a,a',b,b';t_1,t_2)$ and $s_2(\lambda,a,a',b,b';t_1,t_2)$ can only take on the values ± 2 and therefore its averages

$S_1(a,a',b,b';t_1,t_2)$ and $S_2(a,a',b,b';t_1,t_2)$ must satisfy

$$-2 \leq S_1 \leq 2, -2 \leq S_2 \leq 2. \tag{7.2.72}$$

From Eq. (7.2.67) we obtain

$$\begin{gathered}\left|S^{GHVT}(a,a',b,b';t_1,t_2)\right| = |S_1(a,a',b,b';t_1,t_2) + S_2(a,a',b,b';t_1,t_2)| \leq \\ |S_1(a,a',b,b';t_1,t_2)| + |S_2(a,a',b,b';t_1,t_2)|\end{gathered} \tag{7.2.73}$$

Thus from Eq. (7.2.72) and (7.2.73) finally we obtain

$$\left|S^{GHVT}\right| \leq 4. \tag{7.2.74}$$

VII.3. Bell inequalities revisited

VII.3.1. Clauser-Horne-Shimony-Holt (CHSH) inequality revisited. Validity of revised CHSH inequality

In a typical Bell experiment, two systems which may have previously interacted - for instance they may have been produced by a common source - are now spatially separated and are each measured by one of two distant observers, Alice and Bob (see Fig.7.2.1). Alice may choose one out of several possible measurements to perform on her system and we let x_{t_1} denote her measurement choice at instant t_1. For instance, x_{t_1} may refer to the position of a knob on her measurement apparatus at instant t_1. Similarly, we let y_{t_2} denote Bob's measurement choice. Once the measurements are performed, they yield outcomes a_{t_1} and b_{t_2} on the two systems.

Remark 7.3.1. The actual values assigned to the measurement choices x_{t_1}, y_{t_2} and outcomes a_{t_1}, b_{t_2} are purely conventional; they are mere macroscopic labels distinguishing the different possibilities.

Remark 7.3.2. From one run of the experiment to the other, the outcomes a_{t_1} and b_{t_2} that are obtained may vary, even when the same choices of measurements x_{t_1} and y_{t_2} are made.

Assumption 7.3.1. These outcomes a_{t_1} and b_{t_2} are thus in general governed by a Kolmogorovian probability distribution $p(a,t_1;b,t_2|x_{t_1}y_{t_2})$, which can of course depend on the particular experiment being performed. By repeating the experiment a sufficient number of times and collecting the observed data, one can get a fair estimate of such Kolmogorovian probabilities.

Assumption 7.3.2. The assumption of locality implies that we should be able to identify a set of past factors, described by some variables λ, having a joint causal influence on both outcomes, and which fully account for the dependence between a_{t_1} and b_{t_2}. Once all such factors have been taken into account, the residual indeterminacies about the outcomes must now be decoupled, that is, the Kolmogorovian joint probabilities for a_{t_2} and b_{t_2} should factorize:

$$p(a,t_1;b,t_2|xy,\lambda) = p(a,t_1|x,\lambda)p(b,t_2|y,\lambda). \quad (7.3.1)$$

Remark 7.3.3. This factorability condition simply expresses that we have found an explanation according to which the probability for a_{t_1} only depends on the past variables λ and on the local measurement x_{t_1}, but not on the distant measurement and outcome, and analogously for the probability to obtain b_{t_2}.

The variable λ will not necessarily be constant for all runs of the experiment, even if the procedure which prepares the particles to be measured is held fixed, because λ may involve physical quantities that are not fully controllable. The different values of λ across the runs should thus be characterized by a probability distribution $q(\lambda,t_1,t_2)$. Combined with the above factorability condition, we can thus write

$$p(a,t_1;b,t_2|xy) = \int_\Lambda d\lambda q(\lambda,t_1,t_2)p(a,t_1|x,\lambda)p(b,t_2|y,\lambda), \qquad (7.3.2)$$

where we also implicitly assumed that the measurements x_{t_1} and y_{t_2} can be freely chosen in a way that is independent of λ, i.e., that $q(\lambda,t_1,t_2|x_{t_1},y_{t_2}) = q(\lambda,t_1,t_2)$. This decomposition now represents a precise condition for locality in the context of Bell experiments.
Let us consider for simplicity an experiment where there are only two measurement choices per observer $x_{t_1},y_{t_2} \in \{0,1\}$ and where the possible outcomes take also two values labelled $a_{t_1},b_{t_2} \in \{-1,+1\}$. Let $\left\langle a_{t_1x_{t_1}} b_{t_2y_{t_2}} \right\rangle$ be the expectation value of the product $a_{t_1}b_{t_2}$ for given measurement choices (x_{t_1},y_{t_2}) :

$$\left\langle a_{t_1x_{t_1}} b_{t_2y_{t_2}} \right\rangle = \sum_{a,b} abp(ab,t_1,t_2|x_{t_1}y_{t_2}). \qquad (7.3.3)$$

Assumption 7.3.3. We assume now that $(t_1,t_2) \in \mathbb{Z}^2_{+\delta}, \mathbb{Z}_{+\delta} = \mathbb{Z}_+ \times \delta, 0 < \delta \ll 1,$

$$\begin{gathered} p(ab,t_1,t_2|x_{t_1}y_{t_2}) = p(ab,t_1-t_2|x_{t_1}y_{t_2}) \Leftrightarrow (|t_1-t_2| = \delta) \wedge (|t_1-t_2| = 0), \\ q(\lambda,t_1,t_2) = q(\lambda,t_1-t_2) \Leftrightarrow (|t_1-t_2| = \delta) \wedge (|t_1-t_2| = 0) \end{gathered} . \qquad (7.3.4)$$

Thus

$$\left\langle a_{t_1x_{t_1}} b_{t_2y_{t_2}} \right\rangle = \sum_{a,b} abp(ab,t_1-t_2|x_{t_1}y_{t_2}). \qquad (7.3.5)$$

Remark 7.3.4. We denote

$$\left\langle a_{t_1x_{t_1}} b_{t_2y_{t_2}} \right\rangle \triangleq \langle a_x b_y \rangle_= \qquad (7.3.6)$$

iff $|t_1-t_2| = 0.$ We denote

$$\left\langle a_{t_1 x_{t_1}} b_{t_2 y_{t_2}} \right\rangle \triangleq \langle a_x b_y \rangle_> \qquad (7.3.7)$$

iff $|t_1 - t_2| = \delta$ and $t_1 > t_2$. We denote

$$\left\langle a_{t_1 x_{t_1}} b_{t_2 y_{t_2}} \right\rangle \triangleq \langle a_x b_y \rangle_< \qquad (7.3.8)$$

iff $|t_1 - t_2| = \delta$ and $t_1 < t_2$. We denote

$$\langle a_x b_y \rangle \triangleq \langle a_x b_y \rangle_= + \langle a_x b_y \rangle_> + \langle a_x b_y \rangle_<. \qquad (7.3.9)$$

Consider the following expression

$$S = \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle, \qquad (7.3.10)$$

which is a function of the probabilities $p(ab|xy)$. If these probabilities satisfy the locality decomposition (7.3.2) and Eq. (7.3.4), we necessarily have that

$$S = S_= + S_> + S_< = \langle a_0 b_0 \rangle + \langle a_0 b_1 \rangle + \langle a_1 b_0 \rangle - \langle a_1 b_1 \rangle \leq 6, \qquad (7.3.11)$$

where

$$\begin{aligned} S_= &= \langle a_0 b_0 \rangle_= + \langle a_0 b_1 \rangle_= + \langle a_1 b_0 \rangle_= - \langle a_1 b_1 \rangle_=, \\ S_> &= \langle a_0 b_0 \rangle_> + \langle a_0 b_1 \rangle_> + \langle a_1 b_0 \rangle_> - \langle a_1 b_1 \rangle_>, \\ S_> &= \langle a_0 b_0 \rangle_< + \langle a_0 b_1 \rangle_< + \langle a_1 b_0 \rangle_< - \langle a_1 b_1 \rangle_<. \end{aligned} \qquad (7.3.12)$$

To derive this inequality, we can use (7.3.2) and Eq. (7.3.4) in the definitions (7.3.6) - (7.3.8) of $\langle a_0 b_0 \rangle_=, \langle a_x b_y \rangle_>$ and $\langle a_x b_y \rangle_<$ which allows us to express these expectation values as averages:
(i)

$$\langle a_x b_y \rangle_= = \int\limits_\Lambda d\lambda q(\lambda, t_1 - t_2)(\lambda) \langle a_x \rangle_\lambda^= \langle b_y \rangle_\lambda^=, \tag{7.3.13}$$

where $t_1 - t_2 = 0,$ and where we denote $\left\langle a_{t_1 x_{t_1}} \right\rangle_\lambda \triangleq \langle a_x \rangle_\lambda^=, \left\langle b_{t_2 y_{t_2}} \right\rangle_\lambda \triangleq \langle b_y \rangle_\lambda^=,$

(ii)

$$\langle a_x b_y \rangle_> = \int\limits_\Lambda d\lambda q(\lambda, t_1 - t_2)(\lambda) \langle a_x \rangle_\lambda^> \langle b_y \rangle_\lambda^>, \tag{7.3.14}$$

where $t_1 > t_2$ and $t_1 - t_2 = \delta,$ and where we denote $\left\langle a_{t_1 x_{t_1}} \right\rangle_\lambda \triangleq \langle a_x \rangle_\lambda^>, \left\langle b_{t_2 y_{t_2}} \right\rangle_\lambda \triangleq \langle b_y \rangle_{\lambda\lambda}^>,$

(iii)

$$\langle a_x b_y \rangle_< = \int\limits_\Lambda d\lambda q(\lambda, t_1 - t_2)(\lambda) \langle a_x \rangle_\lambda^< \langle b_y \rangle_\lambda^<, \tag{7.3.15}$$

where $t_1 < t_2,$ $t_1 - t_2 = -\delta,$ and where we denote $\left\langle a_{t_1 x_{t_1}} \right\rangle_\lambda \triangleq \langle a_x \rangle_\lambda^<, \left\langle b_{t_2 y_{t_2}} \right\rangle_\lambda \triangleq \langle b_y \rangle_\lambda^<,$ of a product of corresponding local expectations:

$$\langle a_x \rangle_\lambda^= = \sum\nolimits_a ap(a, t_1 | x, \lambda), \langle b_y \rangle_\lambda^= = \sum\nolimits_b bp(b, t_2 | y, \lambda), \tag{7.3.16}$$

and

$$\langle a_x \rangle_\lambda^> = \sum\nolimits_a ap(a, t_1 | x, \lambda), \langle b_y \rangle_\lambda^< = \sum\nolimits_b bp(b, t_2 | y, \lambda), \tag{7.3.17}$$

and

$$\langle a_x \rangle_\lambda^< = \sum\nolimits_a ap(a, t_1 | x, \lambda), \langle b_y \rangle_\lambda^< = \sum\nolimits_b bp(b, t_2 | y, \lambda), \tag{7.3.18}$$

taking values in $[-1,1]$. Inserting this expressions (7.3.16)-(7.3.18) in Eqs. (7.3.12), one obtains

$$S_{=} = \int_{\Lambda} d\lambda q(\lambda,0) S_{\lambda}^{=},\ S_{>} = \int_{\Lambda} d\lambda q(\lambda,\delta) S_{\lambda}^{>},\ S_{<} = \int_{\Lambda} d\lambda q(\lambda,-\delta) S_{\lambda}^{<}, \tag{7.3.19}$$

where

$$\begin{aligned} S_{\lambda}^{=} &= \langle a_0\rangle_{\lambda}^{=}\langle b_0\rangle_{\lambda}^{=} + \langle a_0\rangle_{\lambda}^{=}\langle b_1\rangle_{\lambda}^{=} + \langle a_1\rangle_{\lambda}^{=}\langle b_0\rangle_{\lambda}^{=} - \langle a_1\rangle_{\lambda}^{=}\langle b_1\rangle_{\lambda}^{=}, \\ S_{\lambda}^{>} &= \langle a_0\rangle_{\lambda}^{>}\langle b_0\rangle_{\lambda}^{>} + \langle a_0\rangle_{\lambda}^{>}\langle b_1\rangle_{\lambda}^{>} + \langle a_1\rangle_{\lambda}^{>}\langle b_0\rangle_{\lambda}^{>} - \langle a_1\rangle_{\lambda}^{>}\langle b_1\rangle_{\lambda}^{>}, \\ S_{\lambda}^{<} &= \langle a_0\rangle_{\lambda}^{<}\langle b_0\rangle_{\lambda}^{<} + \langle a_0\rangle_{\lambda}^{<}\langle b_1\rangle_{\lambda}^{<} + \langle a_1\rangle_{\lambda}^{<}\langle b_0\rangle_{\lambda}^{<} - \langle a_1\rangle_{\lambda}^{<}\langle b_1\rangle_{\lambda}^{<}, \end{aligned} \tag{7.3.20}$$

Since $\langle a_0\rangle_{\lambda}, \langle b_0\rangle_{\lambda} \in [-1,1]$, these last expressions are smaller than $\widetilde{S}_{\lambda}^{=}, \widetilde{S}_{\lambda}^{>}$ and $\widetilde{S}_{\lambda}^{<}$ correspondingly, where

$$\begin{aligned} S_{\lambda}^{=} \le \widetilde{S}_{\lambda}^{=} &= |\langle b_0\rangle_{\lambda}^{=} + \langle b_1\rangle_{\lambda}^{=}| + |\langle b_0\rangle_{\lambda}^{=} - \langle b_1\rangle_{\lambda}^{=}|, \\ S_{\lambda}^{>} \le \widetilde{S}_{\lambda}^{>} &= |\langle b_0\rangle_{\lambda}^{>} + \langle b_1\rangle_{\lambda}^{>}| + |\langle b_0\rangle_{\lambda}^{>} - \langle b_1\rangle_{\lambda}^{>}|, \\ S_{\lambda}^{<} \le \widetilde{S}_{\lambda}^{<} &= |\langle b_0\rangle_{\lambda}^{<} + \langle b_1\rangle_{\lambda}^{<}| + |\langle b_0\rangle_{\lambda}^{<} - \langle b_1\rangle_{\lambda}^{<}|. \end{aligned} \tag{7.3.21}$$

Without loss of generality, we can assume that

$$\langle b_0\rangle_{\lambda}^{=} \ge \langle b_1\rangle_{\lambda}^{=} \ge 0,\ \langle b_0\rangle_{\lambda}^{>} \ge \langle b_1\rangle_{\lambda}^{>} \ge 0,\ \langle b_0\rangle_{\lambda}^{<} \ge \langle b_1\rangle_{\lambda}^{<} \ge 0, \tag{7.3.22}$$

which yields

$$S_{\lambda}^{=} \le 2\langle b_0\rangle_{\lambda}^{=} \le 2, S_{\lambda}^{>} \le 2\langle b_0\rangle_{\lambda}^{>} \le 2, S_{\lambda}^{<} \le 2\langle b_0\rangle_{\lambda}^{<} \le 2 \tag{7.3.23}$$

and thus

$$S_{=} \le 2, S_{>} \le 2, S_{<} \le 2. \tag{7.3.24}$$

From (7.3.24) we obtain

$$S = S_{=} + S_{>} + S_{<} \leq 6. \tag{7.3.25}$$

The inequality (7.3.25) finalized the proof.

VII.3.2. Clauser-Horne inequality revisited. Validity of revised Clauser-Horne inequality

The inequality will be determined by counting during the large time t. There will be a total of N_t events, with $N_1(a,t)$ counts in detector 1 by the time t when it is set to select a and $N_2(b,t)$ counts in detector 2 by the time t when it is set to select b. The number of coincidences of the two detectors with settings a and b respectively is $N_{12}(a,b,t)$. The probabilities are

$$\begin{gathered} p_1(a,t) = \frac{N_1(a,t)}{N_t}, p_2(b,t) = \frac{N_2(b,t)}{N_t}, p_{12}(a,b,t) = \frac{N_{12}(a,b,t)}{N_t}, \\ p_{12}(a',b,t) = \frac{N_{12}(a',b,t)}{N_t}, p_{12}(a,b',t) = \frac{N_{12}(a,b',t)}{N_t}, p_{12}(a',b',t) = \frac{N_{12}(a',b',t)}{N_t}. \end{gathered} \tag{7.3.26}$$

Remember that in the Bell formulation, the hidden variable determined absolutely the value of the polarization for a particular measurement. Then

$$\begin{gathered} p_1(a,t_1) = \int d\lambda w(\lambda) p_1(a,t_1,\lambda), p_2(b,t_2) = \int d\lambda w(\lambda) p_2(b,t_2,\lambda), \\ p_{12}(a,b,t_1,t_2,\lambda) = p_1(a,t_1,\lambda) p_2(b,t_2,\lambda), \\ p_{12}(a,b,t_1,t_2) = \int d\lambda w(\lambda) p_1(a,t_1,\lambda) p_2(b,t_2,\lambda), \end{gathered} \tag{7.3.27}$$

where $t_1, t_2 \in [0,t]$. Remind that for any four real numbers $x, x', y, y' \in [0,1]$ the inequality holds

$$xy - xy' + x'y + x'y' \leq x' + y. \tag{7.3.28}$$

We denote now

$$x = p_1(a,t_1,\lambda), y = p_2(b,t_2,\lambda), x' = p_1(a',t_1,\lambda), y' = p_2(b',t_2,\lambda), \tag{7.3.29}$$

and substitute into inequality (7.3.28), we get

$$\begin{gathered} p_1(a,t_1,\lambda)p_2(b,t_2,\lambda) - p_1(a,t_1,\lambda)p_2(b',t_2,\lambda) + \\ +p_1(a',t_1,\lambda)p_2(b,t_2,\lambda) + p_1(a',t_1,\lambda)p_2(b',t_2,\lambda) \leq p_1(a',t_1,\lambda) + p_2(b,t_2,\lambda). \end{gathered} \tag{7.3.30}$$

Next, multiply by $w(\lambda)$ and integrate over all λ we get

$$\begin{gathered} p_{12}(a,b,t_1,t_2) - p_{12}(a,b',t_1,t_2) + p_{12}(a',b,t_1,t_2) + p_{12}(a',b',t_1,t_2) \leq \\ \leq p_1(a',t_1) + p_2(b,t_2). \end{gathered} \tag{7.3.31}$$

Remark 7.3.5. We assume now that $(t_1,t_2) \in \mathbb{Z}^2_{+\delta}, \mathbb{Z}_{+\delta} = \mathbb{Z}_+ \times \delta, 0 < \delta \ll 1,$

$$\begin{gathered} p_{12}(a,b,t_1,t_2) = p_{12}(a,b,t_1-t_2) \Leftrightarrow (|t_1-t_2| = \delta) \wedge (|t_1-t_2| = 0), \\ p_{12}(a,b',t_1,t_2) = p_{12}(a,b',t_1-t_2) \Leftrightarrow (|t_1-t_2| = \delta) \wedge (|t_1-t_2| = 0), \\ p_{12}(a',b,t_1,t_2) = p_{12}(a',b,t_1-t_2) \Leftrightarrow (|t_1-t_2| = \delta) \wedge (|t_1-t_2| = 0), \\ p_{12}(a',b',t_1,t_2) = p_{12}(a',b',t_1-t_2) \Leftrightarrow (|t_1-t_2| = \delta) \wedge (|t_1-t_2| = 0). \end{gathered} . \tag{7.3.32}$$

Where $t_1, t_2 \in [0,t]$.
Remark 7.3.6. We denote

$$p_{12}(a,b,t_1-t_2) \triangleq p^{=}_{12}(a,b) \tag{7.3.33}$$

iff $t_1 - t_2 = 0$.
We denote

$$p_{12}(a,b,t_1-t_2) \triangleq p^{>}_{12}(a,b) \tag{7.3.34}$$

iff $t_1 - t_2 = \delta$ and $t_1 > t_2$.
We denote

$$p_{12}(a,b,t_1-t_2) \triangleq p_{12}^{<}(a,b) \quad (7.3.35)$$

iff $t_1 - t_2 = \delta$ and $t_1 < t_2$.

Remark 7.3.7. We denote

$$p_{12}(a,b',t_1-t_2) \triangleq p_{12}^{=}(a,b') \quad (7.3.36)$$

iff $t_1 - t_2 = 0$.

We denote

$$p_{12}(a,b',t_1-t_2) \triangleq p_{12}^{>}(a,b') \quad (7.3.37)$$

iff $t_1 - t_2 = \delta$ and $t_1 > t_2$.

We denote

$$p_{12}(a,b',t-t_2) \triangleq p_{12}^{<}(a,b') \quad (7.3.38)$$

iff $t_1 - t_2 = -\delta$ and $t_1 < t_2$.

Remark 7.3.8. We denote

$$p_{12}(a',b,t_1-t_2) \triangleq p_{12}^{=}(a',b) \quad (7.3.39)$$

iff $t_1 - t_2 = 0$.

We denote

$$p_{12}(a',b,t_1-t_2) \triangleq p_{12}^{>}(a',b) \quad (7.3.40)$$

iff $t_1 - t_2 = \delta$ and $t_1 > t_2$.

We denote

$$p_{12}(a',b,t_1-t_2) \triangleq p_{12}^{<}(a',b) \quad (7.3.41)$$

iff $t_1 - t_2 = -\delta$ and $t_1 < t_2$.

Remark 7.3.9. We denote

$$p_{12}(a',b',t_1-t_2) \triangleq p_{12}^{=}(a',b') \quad (7.3.42)$$

iff $t_1 - t_2 = 0$.

We denote

$$p_{12}(a',b',t_1,t_2) \triangleq p_{12}^{>}(a',b') \quad (7.3.43)$$

iff $t_1 - t_2 = \delta$ and $t_1 > t_2$.

We denote

$$p_{12}(a',b',t_1-t_2) \triangleq p_{12}^{<}(a',b') \quad (7.3.44)$$

iff $t_1 - t_2 = -\delta$ and $t_1 < t_2$.

Remark 7.3.10. We denote

$$\begin{aligned} p_{12}(a,b) &= p^{=}_{12}(a,b) + p^{>}_{12}(a,b) + p^{>}_{12}(a,b), \\ p_{12}(a',b) &= p^{=}_{12}(a',b) + p^{>}_{12}(a',b) + p^{>}_{12}(a',b), \\ p_{12}(a,b') &= p^{=}_{12}(a,b') + p^{>}_{12}(a,b') + p^{>}_{12}(a,b'), \\ p_{12}(a',b') &= p^{=}_{12}(a',b') + p^{>}_{12}(a',b') + p^{>}_{12}(a',b'). \end{aligned} \tag{7.3.45}$$

From (7.3.31) - (7.3.32) we obtain

$$\begin{gathered} p_{12}(a,b,t_1-t_2) - p_{12}(a,b',t_1-t_2) + p_{12}(a',b,t_1-t_2) + p_{12}(a',b',t_1-t_2) \leq \\ \leq p_1(a',t_1) + p_2(b,t_2), \end{gathered} \tag{7.3.46}$$

where $(|t_1-t_2| = \delta) \wedge (|t_1-t_2| = 0)$.

From (7.3.46) we obtain:

(i)

$$\begin{gathered} p^{=}_{12}(a,b) - p^{=}_{12}(a,b') + p^{=}_{12}(a',b) + p^{=}_{12}(a',b') \leq \\ \leq p_1(a',t_1) + p_2(b,t_2), \end{gathered} \tag{7.3.47}$$

where $t_1 - t_2 = 0$;

(ii)

$$\begin{gathered} p^{>}_{12}(a,b) - p^{>}_{12}(a,b') + p^{>}_{12}(a',b) + p^{>}_{12}(a',b') \leq \\ \leq p_1(a',t_1) + p_2(b,t_2), \end{gathered} \tag{7.3.48}$$

where $t_1 - t_2 = \delta$ and $t_1 > t_2$;

(iii)

$$\begin{gathered} p^{<}_{12}(a,b) - p^{<}_{12}(a,b') + p^{<}_{12}(a',b) + p^{<}_{12}(a',b') \leq \\ \leq p_1(a',t_1) + p_2(b,t_2), \end{gathered} \tag{7.3.49}$$

where $t_1 - t_2 = -\delta$ and $t_1 < t_2$.

From (7.3.47) we obtain

(i)

$$p_{12}^{=}(a,b) - p_{12}^{=}(a,b') + p_{12}^{=}(a',b) + p_{12}^{=}(a',b') \leq$$
$$\leq \lim_{t_1\to\infty} p_1(a',t_1) + \lim_{t_2\to\infty} p_2(b,t_2) = p_1(a') + p_2(b), \quad (7.3.50)$$

where we denote $p_1(a') = \lim_{t_1\to\infty} p_1(a',t_1), p_2(b) = \lim_{t_2\to\infty} p_2(b,t_2);$

(ii)

$$p_{12}^{>}(a,b) - p_{12}^{>}(a,b') + p_{12}^{>}(a',b) + p_{12}^{>}(a',b') \leq$$
$$\leq \lim_{t_1\to\infty} p_1(a',t_1) + \lim_{t_2\to\infty} p_2(b,t_2) = p_1(a') + p_2(b), \quad (7.3.51)$$

where we denote $p_1(a') = \lim_{t_1\to\infty} p_1(a',t_1), p_2(b) = \lim_{t_2\to\infty} p_2(b,t_2);$

(iii)

$$p_{12}^{<}(a,b) - p_{12}^{<}(a,b') + p_{12}^{<}(a',b) + p_{12}^{<}(a',b') \leq$$
$$\leq \lim_{t_1\to\infty} p_1(a',t_1) + \lim_{t_2\to\infty} p_2(b,t_2) = p_1(a') + p_2(b), \quad (7.3.52)$$

where we denote $p_1(a') = \lim_{t_1\to\infty} p_1(a',t_1), p_2(b) = \lim_{t_2\to\infty} p_2(b,t_2);$

From (7.3.50) - (7.3.52) we obtain

(i)

$$p_{12}^{=}(a,b) - p_{12}^{=}(a,b') + p_{12}^{=}(a',b) + p_{12}^{=}(a',b') \leq p_1(a') + p_2(b), \quad (7.3.53)$$

(ii)

$$p_{12}^{>}(a,b) - p_{12}^{>}(a,b') + p_{12}^{>}(a',b) + p_{12}^{>}(a',b') \leq p_1(a') + p_2(b), \quad (7.3.54)$$

(iii)

$$p_{12}^{<}(a,b) - p_{12}^{<}(a,b') + p_{12}^{<}(a',b) + p_{12}^{<}(a',b') \leq p_1(a') + p_2(b). \quad (7.3.55)$$

From (7.3.53) - (7.3.55) we obtain

$$[p_{12}^{=}(a,b)+p_{12}^{>}(a,b)+p_{12}^{<}(a,b)]-[p_{12}^{=}(a,b')+p_{12}^{>}(a,b')+p_{12}^{<}(a,b')]+$$
$$+[p_{12}^{=}(a',b)+p_{12}^{>}(a',b)+p_{12}^{<}(a',b)]+[p_{12}^{=}(a',b')+p_{12}^{<}(a',b')+p_{12}^{<}(a',b')]\leq \qquad (7.3.56)$$
$$\leq 3p_1(a')+3p_2(b),$$

From (7.3.53) by Eqs. (7.3.45) we obtain

$$p_{12}(a,b)-p_{12}(a,b')+p_{12}(a',b)+p_{12}(a',b')\leq 3p_1(a')+3p_2(b). \qquad (7.3.57)$$

The revised Clauser Horne inequality is

$$\frac{N_{12}(a,b)+N_{12}(b,a')+N_{12}(a',b')-N_{12}(a,b')}{N_1(a')+N_2(b)}\leq 3. \qquad (7.3.58)$$

Assume now that a,b,a',b' are separated by the angle φ then

$$\frac{N_{12}(\varphi)+N_{12}(\varphi)+N_{12}(\varphi)-N_{12}(3\varphi)}{N_1(a')+N_2(b)}=\frac{3N_{12}(\varphi)-N_{12}(3\varphi)}{N_1(a')+N_2(b)}\leq 3. \qquad (7.3.59)$$

Next relate coincidences to expectation values. Note that $N_{12}(\varphi)=N\langle\alpha|\Gamma(\varphi)|\alpha\rangle$. As regards the singles counts $N_1(a')$ and $N_2(b)$, we know that the number of counts must be independent of the direction of a' or b and that for any direction $N_1=\frac{1}{2}N$, since half the photons will be polarized along the direction. Therefore, the inequality (7.3.59) becomes

$$\Delta(\varphi)=\frac{\frac{3}{4}(1+\cos 2\varphi)-\frac{1}{4}(1+\cos 6\varphi)}{\frac{1}{2}+\frac{1}{2}}=\frac{1}{2}+\frac{3}{4}\cos 2\varphi-\frac{1}{4}\cos 6\varphi\leq 3. \qquad (7.3.60)$$

The inequality (7.3.60) is not violated even if $\varphi=\frac{\pi}{8}$:

$$\max_{\varphi}\Delta(\varphi)=\Delta\left(\frac{\pi}{8}\right)=\frac{1}{2}+\frac{3\sqrt{2}}{4}+\frac{1}{2\sqrt{2}}\simeq 1.2, \qquad (7.3.61)$$

which of course is less than 3.

VII.3.3. Revised CHSH inequality without the hypothesis of locality

Let $A_t, A'_t, B_t, B'_t, t \in \mathbb{Z}_{+\delta} = \mathbb{Z}_+ \times \delta, 0 < \delta \ll 1,$ be stochastic processes defined on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with values in the set $\{-1, +1\}$, i.e.,

$$A_t = \pm 1, A'_t = \pm 1, B_t = \pm 1, B'_t = \pm 1, t \in [0, T]. \tag{7.3.62}$$

Assume that there exist joint probability distribution functions $W(A_{t_1}, A'_{t_2}, B_{t_3}, B'_{t_4}),$ of $A_{t_1}, A'_{t_2}, B_{t_3}, B'_{t_4}$ (where $(|t_i - t_j| = 0) \wedge (|t_i - t_j| = \delta), i,j = 1,2,3,4$) defining probabilities for each possible set of outcomes such that:

(i)

$$P(A_{t_1}, A'_{t_2}, B_{t_3}, B'_{t_4}) \geq 0, P(A_{t_1}, B_{t_3}, B'_{t_4}) \geq 0, P(A'_{t_2}, B_{t_3}, B'_{t_4}) \geq 0, \text{etc.}, \tag{7.3.63}$$

(ii)

$$\sum_{A_{t_1}, A'_{t_2}, B_{t_3}, B'_{t_4}} P(A_{t_1}, A'_{t_2}, B_{t_3}, B'_{t_4}) = 1,$$
$$\sum_{A_{t_1}, B_{t_3}, B'_{t_4}} P(A_{t_1}, B_{t_3}, B'_{t_4}) = 1, \sum_{A'_{t_2}, B_{t_3}, B'_{t_4}} P(A'_{t_2}, B_{t_3}, B'_{t_4}) = 1, \tag{7.3.64}$$
$$\text{etc.},$$

(iii)

$$P(A_{t_1}, A'_{t_2}, B_{t_3}, B'_{t_4}) + P(-A_{t_1}, A'_{t_2}, B_{t_3}, B'_{t_4}) = P(A'_{t_2}, B_{t_3}, B'_{t_4}) \geq P(A_{t_1}, A'_{t_2}, B_{t_3}, B'_{t_4}),$$
$$P(A_{t_1}, A'_{t_2}, B_{t_3}, B'_{t_4}) + P(A_{t_1}, -A'_{t_2}, B_{t_3}, B'_{t_4}) = P(A'_{t_2}, B_{t_3}, B'_{t_4}) \geq P(A_{t_1}, A'_{t_2}, B_{t_3}, B'_{t_4}), \tag{7.3.65}$$
$$\text{etc.}$$

From (7.3.65) one obtains

$$0 \le P(A_{t_1}, B_{t_3}, B'_{t_4}) = P(A_{t_1}, A'_{t_2}, B_{t_3}, B'_{t_4}) + P(A_{t_1}, -A'_{t_2}, B_{t_3}, B'_{t_4}) \le$$
$$\le P(A'_{t_2}, B'_{t_4}) + P(-A'_{t_2}, B_{t_3}) = \tag{7.3.66}$$
$$P(A'_{t_2}, B'_{t_4}) + P(B_{t_3}) - P(A'_{t_2}, B_{t_3}).$$

Similarly one obtains

$$0 \le P(A_{t_1}, -B_{t_3}, -B'_{t_4}) = P(A_{t_1}, -B_{t_3}) + P(A_{t_1}, -B_{t_3}, B'_{t_4}) =$$
$$P(A_{t_1}) - P(A_{t_1}, B_{t_3}) - P(A_{t_1}, B'_{t_4}) + P(A_{t_1}, B_{t_3}, B'_{t_4}) \tag{7.3.67}$$

and therefore

$$-P(A_{t_1}, B_{t_3}, B'_{t_4}) \le P(A_{t_1}) - P(A_{t_1}, B_{t_3}) - P(A_{t_1}, B'_{t_4}). \tag{7.3.68}$$

From (7.3.66) and (7.3.68) we obtain

$$0 \le P(A_{t_1}, B_{t_3}, B'_{t_4}) - P(A_{t_1}, B_{t_3}, B'_{t_4}) =$$
$$P(A'_{t_2}, B'_{t_4}) + P(B_{t_3}) - P(A'_{t_2}, B_{t_3}) + P(A_{t_1}) - P(A_{t_1}, B_{t_3}) - P(A_{t_1}, B'_{t_4}) \tag{7.3.69}$$

and therefore

$$0 \le P(A'_{t_2}, B'_{t_4}) + P(B_{t_3}) - P(A'_{t_2}, B_{t_3}) + P(A_{t_1}) - P(A_{t_1}, B_{t_3}) - P(A_{t_1}, B'_{t_4}) =$$
$$= P(A_{t_1}) + P(B_{t_3}) - P(A_{t_1}, B_{t_3}) - P(A'_{t_2}, B_{t_3}) - P(A_{t_1}, B'_{t_4}) + P(A'_{t_2}, B'_{t_4}). \tag{7.3.70}$$

From (7.3.70) one obtains

$$\Lambda(A_{t_1}, A'_{t_2}, B_{t_3}, B'_{t_4}) \equiv$$
$$P(A_{t_1}, B_{t_3}) + P(A'_{t_2}, B_{t_3}) + P(A_{t_1}, B'_{t_4}) - P(A'_{t_2}, B'_{t_4}) - P(A_{t_1}) - P(B_{t_3}) \le 0. \tag{7.3.71}$$

Note that

$$P(B_{t_3}, B'_{t_4}) = P(B_{t_3}) - P(B_{t_3}, -B'_{t_4}) \tag{7.3.72}$$

and

$$P(-B_{t_3},-B'_{t_4}) = P(-B'_{t_4}) - P(B_{t_3},-B'_{t_4}) = 1 - P(B'_{t_4}) - P(B_{t_3},-B'_{t_4}). \qquad (7.3.73)$$

From (7.3.73) and (7.3.72) we obtain

$$P(-B_{t_3},-B'_{t_4}) = 1 - P(B_{t_3}) - P(B'_{t_4}) + P(B_{t_3},B'_{t_4}). \qquad (7.3.73a)$$

Note that

$$0 \le P(-A_{t_1},-B_{t_3},-B'_{t_4}) = P(-B_{t_3},-B'_{t_4}) - P(A_{t_1},-B_{t_3},-B'_{t_4}). \qquad (7.3.74)$$

Inserting (7.3.67) and (7.3.73a) into (7.3.74) we obtain

$$\begin{aligned} 0 \le 1 - P(A_{t_1}) - P(B_{t_3}) - P(B'_{t_4}) + P(A_{t_1},B_{t_3}) + P(A'_{t_2},B_{t_3}) + P(A_{t_1},B'_{t_4}) + \\ +P(B_{t_3},B'_{t_4}) - P(A_{t_1},B_{t_3},B'_{t_4}) = 1 - P(A_{t_1}) - P(B_{t_3}) - P(B'_{t_4}) + \\ +P(A_{t_1},B_{t_3}) + P(A'_{t_2},B_{t_3}) + P(-A_{t_1},B_{t_3},B'_{t_4}). \end{aligned} \qquad (7.3.75)$$

Note that

$$\begin{aligned} P(-A_{t_1},B_{t_3},B'_{t_4}) = P(-A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}) + P(-A,-A',B,B') \le \\ \le P(A'_{t_2},B_{t_3}) + P(-A'_{t_2},B'_{t_4}) = P(A'_{t_2},B_{t_3}) + P(B'_{t_4}) - P(A'_{t_2},B'_{t_4}). \end{aligned} \qquad (7.3.76)$$

From (7.3.76) we obtain

$$\begin{aligned} 0 \le 1 - P(A_{t_1}) - P(B_{t_3}) - P(B'_{t_4}) + P(A_{t_1},B_{t_3}) + \\ +P(A'_{t_2},B_{t_3}) + P(A_{t_1},B'_{t_4}) - P(A'_{t_2},B'_{t_4}). \end{aligned} \qquad (7.3.77)$$

From (7.3.77) and (7.3.71) we obtain

$$-1 \le \Lambda(A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}) \le 0. \qquad (7.3.78)$$

(**I**)Let us define the following quantities: $\langle AB\rangle^{=}, \langle A'B\rangle^{=}, \langle AB'\rangle^{=}, \langle A'B'\rangle^{=}$

$$\begin{aligned}
\langle AB\rangle^{=} &= P^{=}_{AB}(++) + P^{=}_{AB}(--) - P^{=}_{AB}(+-) - P^{=}_{AB}(-+),\\
\langle A'B\rangle^{=} &= P^{=}_{A'B}(++) + P^{=}_{A'B}(--) - P^{=}_{A'B}(+-) - P^{=}_{A'B}(-+),\\
\langle AB'\rangle^{=} &= P^{=}_{AB'}(++) + P^{=}_{AB'}(--) - P^{=}_{AB'}(+-) - P^{=}_{AB'}(-+),\\
\langle A'B'\rangle^{=} &= P^{=}_{A'B'}(++) + P^{=}_{A'B'}(--) - P^{=}_{A'B'}(+-) - P^{=}_{A'B'}(-+),
\end{aligned} \tag{7.3.79}$$

where

$$\begin{aligned}
P^{=}_{AB}(++) &= P(A_{t_1} = 1, B_{t_3} = 1) \Leftrightarrow t_1 - t_3 = 0,\\
P^{=}_{AB}(--) &= P(A_{t_1} = -1, B_{t_3} = -1) \Leftrightarrow t_1 - t_3 = 0,\\
P^{=}_{AB}(+-) &= P(A_{t_1} = 1, B_{t_3} = -1) \Leftrightarrow t_1 - t_3 = 0,\\
P^{=}_{AB}(-+) &= P(A_{t_1} = -1, B_{t_3} = 1) \Leftrightarrow t_1 - t_3 = 0,\\
P^{=}_{A'B}(++) &= P(A'_{t_2} = 1, B_{t_3} = 1) \Leftrightarrow t_2 - t_3 = 0,\\
&\text{etc.}
\end{aligned} \tag{7.3.80}$$

From Eqs. (7.3.71), (7.3.79) and (7.3.80) we obtain

$$\begin{aligned}
\langle AB\rangle^{=} + \langle A'B\rangle^{=} + \langle AB'\rangle^{=} - \langle A'B'\rangle^{=} &= \Lambda^{=}(++++) + \Lambda^{=}(----) -\\
&-\Lambda^{=}(+-+-) - \Lambda^{=}(-+-+).
\end{aligned} \tag{7.3.81}$$

From (7.3.78) we obtain

$$-2 \le \Lambda^{=}(++++) + \Lambda^{=}(----) \le 0 \tag{7.3.82}$$

and

$$0 \le -\Lambda^{=}(+-+-) - \Lambda^{=}(-+-+) \le 2 \tag{7.3.83}$$

From (7.3.82) and (7.3.83) we obtain

$$-2 \le \Lambda^{=}(++++) + \Lambda^{=}(----) - \Lambda^{=}(+-+-) - \Lambda^{=}(-+-+) \le 2. \tag{7.3.84}$$

(**II**)Let us define the following quantities: $\langle AB\rangle^{>}, \langle A'B\rangle^{>}, \langle AB'\rangle^{>}, \langle A'B'\rangle^{>}$

$$\begin{aligned}
\langle AB\rangle^{>} &= P^{>}_{AB}(++) + P^{>}_{AB}(--) - P^{>}_{AB}(+-) - P^{>}_{AB}(-+),\\
\langle A'B\rangle^{>} &= P^{>}_{A'B}(++) + P^{>}_{A'B}(--) - P^{>}_{A'B}(+-) - P^{>}_{A'B}(-+),\\
\langle AB'\rangle^{>} &= P^{>}_{AB'}(++) + P^{>}_{AB'}(--) - P^{>}_{AB'}(+-) - P^{>}_{AB'}(-+),\\
\langle A'B'\rangle^{>} &= P^{>}_{A'B'}(++) + P^{>}_{A'B'}(--) - P^{>}_{A'B'}(+-) - P^{>}_{A'B'}(-+),
\end{aligned} \tag{7.3.85}$$

where

$$\begin{aligned}
P^{>}_{AB}(++) &= P(A_{t_1} = 1, B_{t_3} = 1) \Leftrightarrow t_1 - t_3 = \delta,\\
P^{>}_{AB}(--) &= P(A_{t_1} = -1, B_{t_3} = -1) \Leftrightarrow t_1 - t_3 = \delta,\\
&\text{etc.}
\end{aligned} \tag{7.3.86}$$

From Eqs. (7.3.71), (7.3.85) and (7.3.86) we obtain

$$\begin{aligned}
\langle AB\rangle^{>} + \langle A'B\rangle^{>} + \langle AB'\rangle^{>} - \langle A'B'\rangle^{>} &= \Lambda^{>}(++++) + \Lambda^{>}(----) -\\
&-\Lambda^{>}(+-+-) - \Lambda^{>}(-++-+).
\end{aligned} \tag{7.3.87}$$

From (7.3.78) we obtain

$$-2 \leq \Lambda^{>}(++++) + \Lambda^{>}(----) \leq 0 \tag{7.3.88}$$

and

$$0 \leq -\Lambda^{>}(+-+-) - \Lambda^{>}(-+-+) \leq 2 \tag{7.3.89}$$

From (7.3.88) and (7.3.89) we obtain

$$-2 \leq \Lambda^{>}(++++) + \Lambda^{>}(----) - \Lambda^{>}(+-+-) - \Lambda^{>}(-+-+) \leq 2. \tag{7.3.90}$$

(**III**) Let us define the following quantities: $\langle AB\rangle^{<}, \langle A'B\rangle^{<}, \langle AB'\rangle^{<}, \langle A'B'\rangle^{<}$:

$$\begin{aligned}\langle AB\rangle^{<} &= P^{>}_{AB}(++) + P^{>}_{AB}(--) - P^{>}_{AB}(+-) - P^{>}_{AB}(-+),\\ \langle A'B\rangle^{<} &= P^{>}_{A'B}(++) + P^{>}_{A'B}(--) - P^{>}_{A'B}(+-) - P^{>}_{A'B}(-+),\\ \langle AB'\rangle^{<} &= P^{>}_{AB'}(++) + P^{>}_{AB'}(--) - P^{>}_{AB'}(+-) - P^{>}_{AB'}(-+),\\ \langle A'B'\rangle^{<} &= P^{>}_{A'B'}(++) + P^{>}_{A'B'}(--) - P^{>}_{A'B'}(+-) - P^{>}_{A'B'}(-+),\end{aligned} \tag{7.3.91}$$

where

$$\begin{aligned}P^{<}_{AB}(++) &= P(A_{t_1} = 1, B_{t_3} = 1) \Leftrightarrow t_1 - t_3 = -\delta,\\ P^{<}_{AB}(--) &= P(A_{t_1} = -1, B_{t_3} = -1) \Leftrightarrow t_1 - t_3 = -\delta,\\ &\text{etc.}\end{aligned} \tag{7.3.92}$$

From Eqs. (7.3.71), (7.3.91) and (7.3.92) we obtain

$$\begin{aligned}\langle AB\rangle^{<} + \langle A'B\rangle^{<} + \langle AB'\rangle^{<} - \langle A'B'\rangle^{<} = \Lambda^{<}(++++) + \Lambda^{<}(----) -\\ -\Lambda^{<}(+-+-) - \Lambda^{<}(-+-+).\end{aligned} \tag{7.3.93}$$

From (7.3.78) we obtain

$$-2 \leq \Lambda^{<}(++++) + \Lambda^{<}(----) \leq 0 \tag{7.3.94}$$

and

$$0 \leq -\Lambda^{<}(+-+-) - \Lambda^{<}(-+-+) \leq 2. \tag{7.3.95}$$

From (7.3.94) and (7.3.95) we obtain

$$-2 \leq \Lambda^{<}(++++) + \Lambda^{<}(----) - \Lambda^{<}(+-+-) - \Lambda^{<}(-+-+) \leq 2. \tag{7.3.96}$$

(IV) Let us define the following quantities: $\Lambda(++++), \Lambda(----), \Lambda(+-+-)$ and $\Lambda(-+-+)$

$$
\begin{aligned}
\Lambda(++++) &= \Lambda^{=}(++++) + \Lambda^{>}(++++) + \Lambda^{<}(++++),\\
\Lambda(----) &= \Lambda^{=}(----) + \Lambda^{>}(----) + \Lambda^{<}(----),\\
\Lambda(+-+-) &= \Lambda^{=}(+-+-) + \Lambda^{>}(+-+-) + \Lambda^{<}(+-+-),\\
\Lambda(-+-+) &= \Lambda^{=}(-+-+) + +\Lambda^{>}(-+-+)\Lambda^{<}(-+-+).
\end{aligned} \tag{7.3.97}
$$

From Eq. (7.3.97) and the inequalities (7.3.96), (7.3.90) and (7.3.84) we obtain

$$
-6 \leq \Lambda(++++) + \Lambda(----) - \Lambda(+-+-) - \Lambda(-+-+) \leq 6. \tag{7.3.98}
$$

(**V**) Let us define the following quantities: $\langle AB\rangle, \langle A'B\rangle$

$$
\begin{aligned}
\langle AB\rangle &= \langle AB\rangle^{=} + \langle AB\rangle^{>} + \langle AB\rangle^{<},\\
\langle A'B\rangle &= \langle A'B\rangle^{=} + \langle A'B\rangle^{>} + \langle A'B\rangle^{<},\\
\langle AB'\rangle &= \langle AB'\rangle^{=} + \langle AB'\rangle^{>} + \langle AB'\rangle^{<},\\
\langle A'B'\rangle &= \langle A'B'\rangle^{=} + \langle A'B'\rangle^{>} + \langle A'B'\rangle^{<}.
\end{aligned} \tag{7.3.99}
$$

From Eqs. (7.3.93), (7.3.87), (7.3.84), (7.3.97) and (7.3.99) we obtain

$$
\begin{aligned}
\langle AB\rangle + \langle A'B\rangle + \langle AB'\rangle - \langle A'B'\rangle = \Lambda(++++) + \Lambda(----) -\\
-\Lambda(+-+-) - \Lambda(-+-+).
\end{aligned} \tag{7.3.100}
$$

From inequality (7.3.98) and Eq. (7.3.100) finally we obtain

$$
|\langle AB\rangle + \langle A'B\rangle + \langle AB'\rangle - \langle A'B'\rangle| \leq 6. \tag{7.3.101}
$$

VII.3.4. Revised CHSH inequality without the introduction of hidden variables

Let $A_t, A'_t, B_t, B'_t, t \in \mathbb{Z}_{+\delta} = \mathbb{Z}_+ \times \delta, 0 < \delta \ll 1,$ be stochastic processes defined on a common probability space $(\Omega, \mathcal{F}, \mathbf{P})$ with values in the set $\{-1,+1\}$, i.e.,

$$A_t = \pm 1, A'_t = \pm 1, B_t = \pm 1, B'_t = \pm 1, t \in [0,T]. \quad (7.3.102)$$

Assume that there exist joint probability distribution functions $P(A_{t_1}, A'_{t_2}, B_{t_3}, B'_{t_4})$, of $A_{t_1}, A'_{t_2}, B_{t_3}, B'_{t_4}$ (where $(|t_i - t_j| = 0) \vee (|t_i - t_j| = \delta > 0), i,j = 1,2,3,4$) defining probabilities for each possible set of outcomes such that:

(i)

$$P(A_{t_1}, A'_{t_2}, B_{t_3}, B'_{t_4}) \geq 0, P(A_{t_1}, B_{t_3}, B'_{t_4}) \geq 0, P(A'_{t_2}, B_{t_3}, B'_{t_4}) \geq 0, \text{etc.}, \quad (7.3.103)$$

and

$$\sum_{A_{t_1}, A'_{t_2}, B_{t_3}, B'_{t_4}} P(A_{t_1}, A'_{t_2}, B_{t_3}, B'_{t_4}) = 1, \quad (7.3.104)$$

and

$$P(A_{t_1}, A'_{t_2}, B_{t_3}, B'_{t_4}) - P(-A_{t_1}, A'_{t_2}, B_{t_3}, B'_{t_4}) = P(A'_{t_2}, B_{t_3}, B'_{t_4}). \quad (7.3.105)$$

We abbreviate now for short:

$$
\begin{gathered}
P^{=}_{A,A',B,B'}(++++) \triangleq \\
P^{=}_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(++++) = P(A_{t_1} = 1, A'_{t_2} = 1, B_{t_3} = 1, B'_{t_4} = 1) \Leftrightarrow t_1 = t_3, \\
P^{=}_{A,A',B,B'}(+++-) \triangleq \\
P^{=}_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(+++-) = P(A_{t_1} = 1, A'_{t_2} = 1, B_{t_3} = 1, B'_{t_4} = -1) \Leftrightarrow t_1 = t_3, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
P^{=}_{A,A',B,B'}(----) \triangleq \\
P^{=}_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(----) = P(A_{t_1} = -1, A'_{t_2} = -1, B_{t_3} = -1, B'_{t_4} = -1) \Leftrightarrow t_1 = t_3; \\
P^{>}_{A,A',B,B'}(++++) \triangleq \\
P^{>}_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(++++) = P(A_{t_1} = 1, A'_{t_2} = 1, B_{t_3} = 1, B'_{t_4} = 1) \Leftrightarrow t_1 - t_3 = \delta, \\
P^{>}_{A,A',B,B'}(+++-) \triangleq \\
P^{>}_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(+++-) = P(A_{t_1} = 1, A'_{t_2} = 1, B_{t_3} = 1, B'_{t_4} = -1) \Leftrightarrow t_1 - t_3 = \delta, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
P^{>}_{A,A',B,B'}(----) \triangleq \\
P^{>}_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(----) = P(A_{t_1} = -1, A'_{t_2} = -1, B_{t_3} = -1, B'_{t_4} = -1) \Leftrightarrow t_1 - t_3 = \delta; \\
P^{<}_{A,A',B,B'}(++++) \triangleq \\
P^{<}_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(++++) = P(A_{t_1} = 1, A'_{t_2} = 1, B_{t_3} = 1, B'_{t_4} = 1) \Leftrightarrow t_1 - t_3 = -\delta, \\
P^{<}_{A,A',B,B'}(+++-) \triangleq \\
P^{<}_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(+++-) = P(A_{t_1} = 1, A'_{t_2} = 1, B_{t_3} = 1, B'_{t_4} = -1) \Leftrightarrow t_1 - t_3 = -\delta, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
P^{<}_{A,A',B,B'}(----) \triangleq \\
P^{<}_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(----) = P(A_{t_1} = -1, A'_{t_2} = -1, B_{t_3} = -1, B'_{t_4} = -1) \Leftrightarrow t_1 - t_3 = -\delta;
\end{gathered}
\tag{7.3.106}
$$

and

$$
\begin{gathered}
P^{=}_{A_{t_1}B_{t_3}}(++) = P_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(++++) + P_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(+++-) + \\
P_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(+-++) + P_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(+-+-) \Leftrightarrow t_1 = t_3,
\end{gathered}
\tag{7.3.107}
$$

etc.,

$$
\begin{gathered}
P^{>}_{A_{t_1}B_{t_3}}(++) = P_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(++++) + P_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(+++-) + \\
P_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(+-++) + P_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(+-+-) \Leftrightarrow t_1 - t_3 = \delta,
\end{gathered}
\tag{7.3.108}
$$

etc.,

$$P^{<}_{A_{t_1}B_{t_3}}(++) = P_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(++++) + P_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(+++-) + P_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(+-++) + P_{A_{t_1},A'_{t_2},B_{t_3},B'_{t_4}}(+-+-) \Leftrightarrow t_1 - t_3 = -\delta, \quad (7.3.109)$$

etc.

For the quantities (i) $\langle AB\rangle^{=}, \langle A'B\rangle^{=}, \langle AB'\rangle^{=}, \langle A'B'\rangle^{=}$;
(ii) $\langle AB\rangle^{>}, \langle A'B\rangle^{>}, \langle AB'\rangle^{>}, \langle A'B'\rangle^{>}$; and
(iii) $\langle AB\rangle^{<}, \langle A'B\rangle^{<}, \langle AB'\rangle^{<}, \langle A'B'\rangle^{<}$, using Eqs. (7.3.107) - (7.3.109) one obtains the representatives

$$\langle AB\rangle^{=} = P^{=}_{AB}(++) + P^{=}_{AB}(--) - P^{=}_{AB}(+-) - P^{=}_{AB}(-+),$$

etc.,

$$\langle AB\rangle^{>} = P^{>}_{AB}(++) + P^{>}_{AB}(--) - P^{>}_{AB}(+-) - P^{>}_{AB}(-+), \quad (7.3.110)$$

etc.,

$$\langle AB\rangle^{<} = P^{<}_{AB}(++) + P^{<}_{AB}(--) - P^{<}_{AB}(+-) - P^{<}_{AB}(-+).$$

Substituting Eqs. (7.3.110) into expressions

$$\begin{aligned} \Pi^{=} &= \frac{1}{2}[\langle AB\rangle^{=} + \langle A'B\rangle^{=} + \langle AB'\rangle^{=} - \langle A'B'\rangle^{=}], \\ \Pi^{>} &= \frac{1}{2}[\langle AB\rangle^{>} + \langle A'B\rangle^{>} + \langle AB'\rangle^{>} - \langle A'B'\rangle^{>}], \quad (7.3.111) \\ \Pi^{<} &= \frac{1}{2}[\langle AB\rangle^{<} + \langle A'B\rangle^{<} + \langle AB'\rangle^{<} - \langle A'B'\rangle^{<}], \end{aligned}$$

one obtains

$$\begin{aligned} \Pi^{=} = \frac{1}{2}[\langle AB\rangle^{=} + \langle A'B\rangle^{=} + \langle AB'\rangle^{=} - \langle A'B'\rangle^{=}] = \\ P^{=}_{A,A',B,B'}(++++) + P^{=}_{A,A',B,B'}(+++-) - P^{=}_{A,A',B,B'}(++-+) - \\ -P^{=}_{A,A',B,B'}(++--) - P^{=}_{A,A',B,B'}(+-++) - P^{=}_{A,A',B,B'}(+-+-) + \\ +P^{=}_{A,A',B,B'}(+--+) - P^{=}_{A,A',B,B'}(+---) - P^{=}_{A,A',B,B'}(-+++) + \quad (7.3.112) \\ +P^{=}_{A,A',B,B'}(-++-) - P^{=}_{A,A',B,B'}(-+-+) + P^{=}_{A,A',B,B'}(-+--) - \\ -P^{=}_{A,A',B,B'}(--++) - P^{=}_{A,A',B,B'}(--+-) + P^{=}_{A,A',B,B'}(---+) + \\ +P^{=}_{A,A',B,B'}(----); \end{aligned}$$

$$
\begin{aligned}
\Pi^{>} = \frac{1}{2}[\langle AB\rangle^{>} + \langle A'B\rangle^{>} + \langle AB'\rangle^{>} - \langle A'B'\rangle^{>}] = \\
P^{>}_{A,A',B,B'}(++++) + P^{>}_{A,A',B,B'}(+++-) - P^{>}_{A,A',B,B'}(++-+) - \\
-P^{>}_{A,A',B,B'}(++--) - P^{>}_{A,A',B,B'}(+-++) - P^{>}_{A,A',B,B'}(+-+-) + \\
+P^{>}_{A,A',B,B'}(+--+) - P^{>}_{A,A',B,B'}(+---) - P^{>}_{A,A',B,B'}(-+++) + \\
+P^{>}_{A,A',B,B'}(-++-) - P^{>}_{A,A',B,B'}(-+-+) + P^{>}_{A,A',B,B'}(-+--) - \\
-P^{>}_{A,A',B,B'}(--++) - P^{>}_{A,A',B,B'}(--+-) + P^{>}_{A,A',B,B'}(---+) + \\
+P^{>}_{A,A',B,B'}(----);
\end{aligned}
\qquad (7.3.113)
$$

$$
\begin{aligned}
\Pi^{<} = \frac{1}{2}[\langle AB\rangle^{<} + \langle A'B\rangle^{<} + \langle AB'\rangle^{<} - \langle A'B'\rangle^{<}] = \\
P^{<}_{A,A',B,B'}(++++) + P^{<}_{A,A',B,B'}(+++-) - P^{<}_{A,A',B,B'}(++-+) - \\
-P^{<}_{A,A',B,B'}(++--) - P^{<}_{A,A',B,B'}(+-++) - P^{<}_{A,A',B,B'}(+-+-) + \\
+P^{<}_{A,A',B,B'}(+--+) - P^{<}_{A,A',B,B'}(+---) - P^{<}_{A,A',B,B'}(-+++) + \\
+P^{<}_{A,A',B,B'}(-++-) - P^{<}_{A,A',B,B'}(-+-+) + P^{<}_{A,A',B,B'}(-+--) - \\
-P^{<}_{A,A',B,B'}(--++) - P^{<}_{A,A',B,B'}(--+-) + P^{<}_{A,A',B,B'}(---+) + \\
+P^{<}_{A,A',B,B'}(----).
\end{aligned}
\qquad (7.3.114)
$$

From (7.3.103)-(7.3.105) it obviously follows that

$$
-2 \le \Pi^{=} \le 2, -2 \le \Pi^{>} \le 2, -2 \le \Pi^{<} \le 2. \qquad (7.3.115)
$$

Thus we obtain

$$
|\Pi^{=}| \le 2, |\Pi^{>}| \le 2, |\Pi^{<}| \le 2. \qquad (7.3.116)
$$

We define now the quantityes

$$
\Pi = \Pi^{=} + \Pi^{>} + \Pi^{<} \qquad (7.3.117)
$$

and

$$\begin{aligned}\langle AB\rangle &= \langle AB\rangle^{=} + \langle AB\rangle^{>} + \langle AB\rangle^{<},\\ \langle A'B\rangle &= \langle A'B\rangle^{=} + \langle A'B\rangle^{>} + \langle A'B\rangle^{<},\\ \langle AB'\rangle &= \langle AB'\rangle^{=} + \langle AB'\rangle^{>} + \langle AB'\rangle^{<},\\ \langle A'B'\rangle &= \langle A'B'\rangle^{=} + \langle A'B'\rangle^{>} + \langle A'B'\rangle^{<}.\end{aligned} \tag{7.3.118}$$

From (7.3.116) and (7.3.117) we obtain

$$|\Pi| = |\Pi^{=} + \Pi^{>} + \Pi^{<}| \le |\Pi^{=}| + |\Pi^{>}| + |\Pi^{<}| \le 6. \tag{7.3.119}$$

From (7.3.117) and (7.3.118) we obtain

$$\Pi = \Pi^{=} + \Pi^{>} + \Pi^{<} = \langle AB\rangle + \langle A'B\rangle + \langle AB'\rangle - \langle A'B'\rangle. \tag{7.3.120}$$

From (7.3.119) and (7.3.120) finally we obtain

$$|\langle AB\rangle + \langle A'B\rangle + \langle AB'\rangle - \langle A'B'\rangle| \le 6. \tag{7.3.121}$$

VII.4. Leggett inequality revisited

VII.4.1. Classical Leggett inequality

Leggett have introduced the class of non-local models and formulated an incompatibility theorem [8]. Such models were extended so as to make them applicable to real experimental situations and also to allow simultaneous tests of all local hidden-variable models. Finally, an experiment was performed that violates the new inequality and hence excludes a broad class of non-local hidden-variable theories [9].

These theories are based on the following assumptions: (1) all measurement outcomes are determined by pre-existing properties of particles independent of the measurement (realism); (2) physical states are statistical mixtures of subensembles with definite polarization, where (3) polarization is defined such that expectation values taken for each subensemble obey Malus' law (that is, the well-known cosine dependence of the intensity of a polarized beam after an ideal polarizer).

These assumptions are in a way appealing, because they provide a natural explanation of quantum mechanically separable states (polarization states indeed obey Malus' law). In addition, they do not explicitly demand locality; that is, measurement outcomes may very well depend on parameters in space-like separated regions.

As a consequence, such theories can explain important features of quantum mechanically entangled (non-separable) states of two particles: first, they do not allow information to be transmitted faster than the speed of light; second, they reproduce perfect correlations for all measurements in the same bases, which is a fundamental feature of the Bell singlet state; and third, they provide a model for all thus far performed experiments in which the Clauser, Horne, Shimony and Holt (CHSH) inequality was violated.

A general framework of such models is the following: assumption (1) requires that an individual binary measurement outcome A for a polarization measurement along direction $\vec{a}$ (that is, whether a single photon is transmitted or absorbed by a polarizer set at a specific angle) is predetermined by some set of hidden-variables λ, and a three-dimensional vector $\vec{u}$, as well as by some set of other possibly non-local parameters η (for example, measurement settings in space-like separated regions) - that is, $A = A(\lambda, \vec{u}, \vec{a}, \eta)$.

According to assumption (3), particles with the same $\vec{u}$ but with different λ build up subensembles of 'definite polarization' described by a probability distribution $\rho_{\vec{u}}(\lambda)$. The expectation value $\bar{A}(\vec{u})$, obtained by averaging over λ, fulfills Malus' law, that is,

$\bar{A}(\vec{u}) = \int d\lambda \rho_{\vec{u}}(\lambda) A(\lambda, \vec{u}, \vec{a}, \eta) = \vec{u} \cdot \vec{a}$.

Finally, with assumption (2), the measured expectation value for a general physical state is given by averaging over the distribution $F(\vec{u})$ of subensembles, that is, $\langle A \rangle = \int d\vec{u} F(\vec{u}) \bar{A}(\vec{u})$.

Let us consider a specific source, which emits pairs of photons with well-defined polarizations $\vec{u}$ and $\vec{v}$ to laboratories of Alice and Bob, respectively. The local polarization measurement outcomes A and B are fully determined by the polarization vector, by an additional set of hidden variables λ specific to the source and by any set of parameters η outside the source. For reasons of clarity, we choose an explicit non-local dependence of the outcomes on the settings $\vec{a}$ and $\vec{b}$ of the measurement devices. Note, however, that this is just an example of a possible non-local dependence, and that one can choose any other set out of η. Each emitted pair is fully defined by the subensemble distribution $\rho_{\vec{u},\vec{v}}(\lambda)$. In agreement with assumption (3) we impose the following conditions on the predictions for local averages of such measurements (all polarizations and measurement directions are represented as vectors on the Poincaré sphere [10]):

$$\bar{A}(\vec{u}) = \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda) A(\vec{a}, \vec{b}, \lambda) = \vec{u} \cdot \vec{a}, \tag{7.4.1}$$

$$\bar{B}(\vec{v}) = \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda) B(\vec{b}, \vec{a}, \lambda) = \vec{v} \cdot \vec{b}. \tag{7.4.2}$$

It is important to note that the validity of Malus' law imposes the non-signalling condition on the investigated non-local models, as the local expectation values do only depend on local parameters. The correlation function of measurement results for a source emitting well-polarized photons is defined as the average of the products of the individual measurement outcomes:

$$\overline{AB}(\vec{u},\vec{v}) = \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda) A(\vec{a}, \vec{b}, \lambda) B(\vec{b}, \vec{a}, \lambda) \tag{7.4.3}$$

For a general source producing mixtures of polarized photons the observable correlations are averaged over a distribution of the polarizations $F(\vec{u},\vec{v})$, and the general correlation function E is given by:

$$E = \langle AB \rangle = \int d\vec{u} d\vec{v} F(\vec{u},\vec{v}) \overline{AB}(\vec{u},\vec{v}) \qquad (7.4.4)$$

It is a very important trait of this model that there exist subensembles of definite polarizations (independent of measurements) and that the predictions for the subensembles agree with Malus' law. It is clear that other classes of non-local theories, possibly even fully compliant with all quantum mechanical predictions, might exist that do not have this property when reproducing entangled states. There the non-local correlations are a consequence of the non-local quantum potential, which exerts suitable torque on the particles leading to experimental results compliant with quantum mechanics. In that theory, neither of the two particles in a maximally entangled state carries any angular momentum at all when emerging from the source [11].

In contrast, in the Leggett model, it is the total ensemble emitted by the source that carries no angular momentum, which is a consequence of averaging over the individual particles' well defined angular momenta (polarization).

The theories described here are incompatible with quantum theory. Remind the basic idea of the incompatibility theorem [8] uses the following identity, which holds for any numbers $A = \pm 1$ and $B = \pm 1$:

$$-1 + |A + B| = AB = 1 - |A - B|. \qquad (7.4.5)$$

One can apply this identity to the dichotomic measurement results $A = A(\vec{a}, \vec{b}, \lambda) = \pm 1$ and $B = B(\vec{b}, \vec{a}, \lambda) = \pm 1$. The identity holds even if the values of A and B mutually depend on each other. For example, the value of a specific outcome A can depend on the value of an actually obtained result B. In contrast, in the derivation of the CHSH inequality it is necessary to assume that A and B do not depend on each other.

Therefore, any kind of non-local dependencies used in the present class of theories are allowed. Taking the average over the subensembles with definite polarizations we obtain:

$$-1 + \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda)|A + B| = \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda)AB = 1 - \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda)|A - B| \qquad (7.4.6)$$

Denoting these averages by bars, one arrives at the shorter expression:

$$-1 + \langle|A + B|\rangle = \langle AB\rangle = 1 - \langle|A - B|\rangle. \qquad (7.4.7)$$

As the average of the modulus is greater than or equal to the modulus of the averages, one gets the set of inequalities:

$$-1 + |\langle A\rangle + \langle B\rangle| \le \langle AB\rangle \le 1 - |\langle A\rangle - \langle B\rangle|. \qquad (7.4.8)$$

By inserting Malus' law, equations (7.4.1) and (7.4.2), in equation (7.4.8), and by using expression (7.4.4), one arrives at a set of inequalities for experimentally accessible correlation functions (for a detailed derivation see Appendix B). In particular, if we let Alice choose her observable from the set of two settings $\vec{a}_1$ and $\vec{a}_2$, and Bob from the set of three settings $\vec{b}_1$, $\vec{b}_2$ and $\vec{b}_3 = \vec{a}_2$, the following generalized Leggett-type inequality is obtained:

$$S_{NLHV} = |E_{11}(\varphi) + E_{23}(0)| + |E_{22}(\varphi) + E_{23}(0)| \le 4 - \tfrac{4}{\pi}|\sin\tfrac{\varphi}{2}|, \qquad (7.4.9)$$

where $E_{kl}(\varphi)$ is a uniform average of all correlation functions, defined in the plane of $\vec{a}_k$ and $\vec{b}_l$, with the same relative angle φ; the subscript NLHV stands for 'non-local hidden- variables'. For the inequality to be applied, vectors $\vec{a}_1$ and $\vec{b}_1$ necessarily have to lie in a plane orthogonal to the one defined by $\vec{a}_2$ and $\vec{b}_2$. This contrasts with the standard experimental configuration used to test the CHSH inequality, which is maximally violated for settings in one plane.

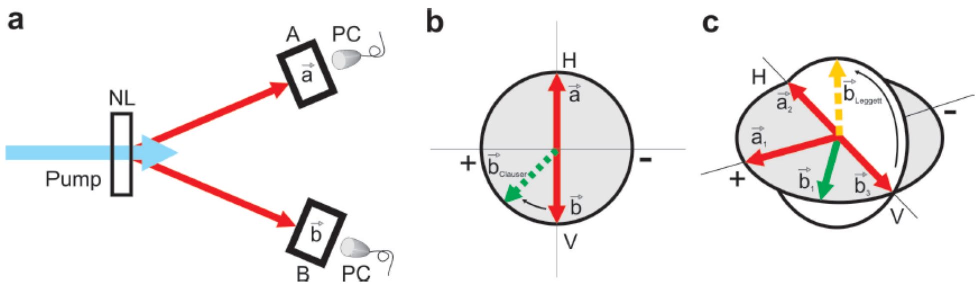

Fig. 7.4.1. Testing non-local hidden-variable theories [9]

(a) Diagram of a standard two-photon experiment to test for hidden variable theories. When pumping a nonlinear crystal (NL) with a strong pump field, photon pairs are created via spontaneous parametric down-conversion (SPDC) and their polarization is detected with single-photon counters (PC). Local measurements at A and B are performed along directions $\vec{a}$ and $\vec{b}$ on the Poincaré sphere, respectively. Depending on the measurement directions, the obtained correlations can be used to test Bell inequalities (b) or Leggett-type inequalities (c).

(b) Correlations in one plane. Shown are measurements along directions in the linear plane of the Poincaré sphere (H (V) denotes horizontal (vertical) polarization). The original experiments by Wu and Shaknov [12] and Kocher and Commins [13], designed to test quantum predictions for correlated photon pairs, measured perfect correlations (solid lines). Measurements along the dashed line allow a Bell test, as was first performed by Freedman and Clauser [14].

(c) Correlations in orthogonal planes. All current experimental tests to violate Bell's inequality (CHSH) are performed within the shaded plane. Out-of-plane measurements are required for a direct test of the class of non-local hidden-variable theories, as was first suggested by Leggett [15].

The situation resembles in a way the status of the Einstein, Podolsky and Rosen (EPR) paradox before the advent of Bell's theorem and its first experimental tests. The experiments of Wu and Shaknov [12] and of Kocher and Commins [13] were designed to demonstrate the validity of a quantum description of photon-pair correlations.

As this task only required the testing of correlations along the same polarization direction, their results could not provide experimental data for the newly derived Bell inequalities (Fig. 7.4.1a, b). Curiously, as was shown by Clauser, Horne, Shimony and Holt, only a small modification of the measurement directions, such that non-perfect correlations of an entangled state are probed, was sufficient to test Bell's inequalities.
The seminal experiment by Freedman and Clauser [14] was the first direct and successful test. Today, all Bell tests - that is, tests of local realism - are performed by testing correlations of measurements along directions that lie in the same plane of the Poincaré sphere. Similar to the previous case, violation of the Leggett-type inequality requires only small modifications to that arrangement: To test the inequality, correlations of measurements along two orthogonal planes have to be probed (Fig. 7.4.1c). Therefore, the existing data of all Bell tests cannot be used to test the class of nonlocal theories considered in [14].

Quantum theory violates inequality (7.4.9). Consider the quantum predictions for the polarization singlet state of two photons, $|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}[|H\rangle_A|V\rangle_B - |V\rangle_A|H\rangle_B]$, where, for example, $|H\rangle_A$ denotes a horizontally polarized photon propagating to Alice. The quantum correlation function for the measurements $\vec{a}_k$ and $\vec{b}_l$ performed on photons depends only on the relative angles between these vectors, and therefore $E_{kl} = -\vec{a}_k \cdot \vec{b}_l = -\cos\varphi$. Thus the left hand side of inequality (7.4.9), for quantum predictions, reads $|2(\cos\varphi + 1)|$. The maximal violation of inequality (7.4.9) is for $\varphi_{max} = 18.8^\circ$. For this difference angle, the bound given by inequality (7.4.9) equals 3.792 and the quantum value is 3.893. Although this excludes the non-local models, it might still be possible that the obtained correlations could be explained by a local realistic model. In order to avoid that, we have to exclude both local realistic and non-local realistic hidden-variable theories. Note however that such local realistic theories need not be constrained by assumptions (1) - (3). The violation of the CHSH inequality invalidates all local realistic models. If one takes

$$S_{CHSH} = |E_{11} + E_{12} - E_{21} + E_{22}| \le 2 \qquad (7.4.10)$$

the quantum value of the left hand side for the settings used to maximally violate inequality (7.4.9) is 2.2156.

The correlation function determined in an actual experiment is typically reduced by a visibility factor V to $E^{exp} = -V\cos\varphi$ owing to noise and imperfections. Thus to observe violations of inequality (7.4.9) (and inequality (7.4.10)) in the experiment, one must have a sufficiently high experimental visibility of the observed interference. For the optimal difference angle $\varphi_{max} = 18.8^\circ$, the minimum required visibility is given by the ratio of the bound (3.792) and the quantum value (3.893) of inequality (7.4.9), or $\sim 97.4\%$. We note that in standard Bell-type experiments, a minimum visibility of only $\sim 71\%$ is sufficient to violate the CHSH inequality, inequality (7.4.10), at the optimal settings. For the settings used here, the critical visibility reads $2/2.2156 \approx 90.3\%$, which is much lower than 97.4%.

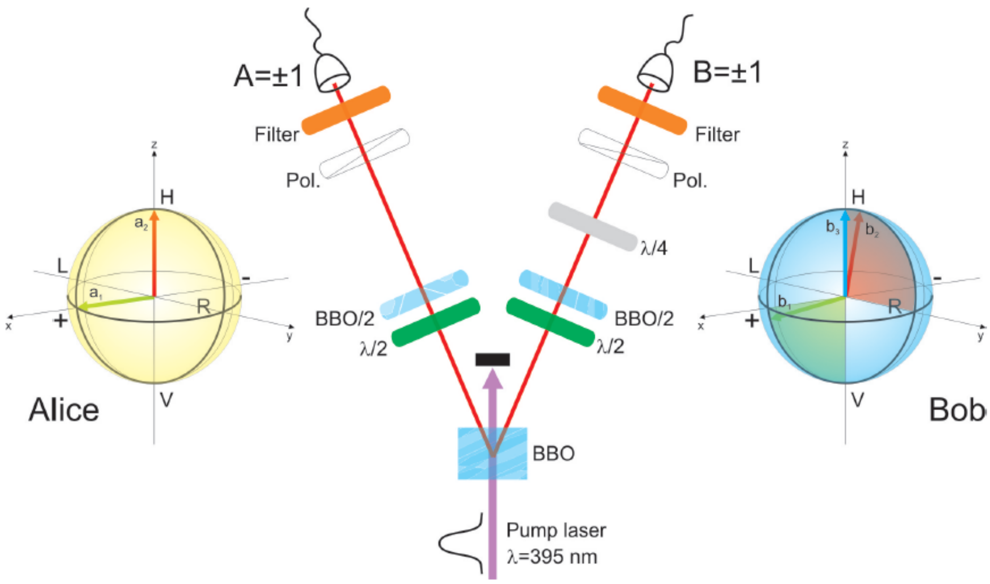

Fig. 7.4.2. Experimental set-up [9]

A 2-mm-thick type-II - barium-borate (BBO) crystal is pumped with a pulsed frequency- doubled Ti: sapphire laser (180 fs) at λ = 395 nm wavelength and ~ 150 mW optical c.w. power. The crystal is aligned to produce the polarization-entangled singlet state $|\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}[|H\rangle_A|V\rangle_B - |V\rangle_A|H\rangle_B]$. Spatial and temporal distinguishability of the produced photons (induced by birefringence in the BBO) are compensated by a combination of half-wave plates $(\lambda/2)$ and additional BBO crystals (BBO/2), while spectral distinguishability (due to the broad spectrum of the pulsed pump) is eliminated by narrow spectral filtering of 1 nm bandwidth in front of each detector. In addition, the reduced pump power diminishes higher-order SPDC emissions of multiple photon pairs. This allows us to achieve a two-photon visibility of about 99%, which is well beyond the required threshold of 97:4%. The arrows in the Poincaré spheres indicate the measurement settings of Alice's and Bob's polarizers for the maximal violation of inequality (7.4.9). Note that setting $\vec{b}_2$ lies in the $y-z$ plane and therefore a quarter-wave plate has to be introduced on Bob's side. The coloured planes indicate the measurement directions for various difference angles φ for both inequalities.

In the experiment [9] (see Fig. 7.4.2), a pairs of polarization entangled photons was generated via SPDC. The photon source is aligned to produce pairs in the polarization singlet state. We observed maximal coincidence count rates (per 10 s), in the H/V basis, of around 3,500 with single count rates of 95,000 (Alice) and 105,000 (Bob), 3,300 coincidences in the $\pm 45^\circ$ basis (75,000 singles at Alice and 90,000 at Bob), and 2,400 coincidences in the R/L basis (70,000 singles at Alice and 70,000 at Bob). The reduced count rates in the R/L basis are due to additional retarding elements in the beam path. The two-photon visibilities are approximately $99.0 \pm 1.2\%$ in the H/V basis, $99.2 \pm 1.6\%$ in the $\pm 45^\circ$ basis and $98.9 \pm 1.7\%$ in the R/L basis, which - to our knowledge - is the highest reported visibility for a pulsed SPDC scheme. So far, no experimental evidence against the rotational invariance of the singlet state exists. We therefore replace the rotation averaged correlation functions in inequality (7.4.9) with their values measured for one pair of settings (in the given plane).

In terms of experimental count rates, the correlation function $E(\vec{a},\vec{b})$ for a given pair of general measurement settings is defined by

$$E(\vec{a},\vec{b}) = \frac{N_{++} + N_{--} - N_{+-} - N_{-+}}{N_{++} + N_{--} + N_{+-} + N_{-+}} \qquad (7.4.11)$$

where N_{AB} denotes the number of coincident detection events between Alice's and Bob's measurements within the integration time. We ascribe the number $+1$, if Alice (Bob) detects a photon polarized along $\vec{a}$ $\left(\vec{b}\right)$, and -1 for the orthogonal direction $\vec{a}^{\perp}$ $\left(\vec{b}^{\perp}\right)$. For example, N_{+-} denotes the number of coincidences in which Alice obtains $\vec{a}$ and Bob $\vec{b}^{\perp}$. Note that $E(\vec{a}_k,\vec{b}_l) = E_{kl}(\varphi)$, where φ is the difference angle between the vectors $\vec{a}$ and $\vec{b}$ on the Poincaré sphere.

To test inequality (7.4.9), three correlation functions ($E(\vec{a}_1,\vec{b}_1,)$, $E(\vec{a}_2,\vec{b}_2)$, $E(\vec{a}_2,\vec{b}_3)$) have to be extracted from the measured data. We choose observables $\vec{a}_1$ and $\vec{b}_1$ as linear polarization measurements (in the x - z plane on the Poincaré sphere; see Fig. 7.4.2) and $\vec{a}_2$ and $\vec{b}_2$ as elliptical polarization measurements in the y - z plane. Two further correlation functions ($E(\vec{a}_2,\vec{b}_1)$ and $E(\vec{a}_1,\vec{b}_2)$) are extracted to test the CHSH inequality (7.4.10).

The first set of correlations, in the x - z plane, is obtained by using linear polarizers set to α_1 and β_1 (relative to the z - axis) at Alice's and Bob's location, respectively. In particular, $\alpha_1 = \pm 45^\circ$, while β_1 is chosen to lie between 45° and 160° (green arrows in Fig. 7.4.2). The second set of correlations (necessary for CHSH) is obtained in the same plane for $\alpha_2 = 0^\circ/90^\circ$ and β_1 between 45° and 160°. The set of correlations for measurements in the y - z plane is obtained by introducing a quarter-wave plate with the fast axis aligned along the (horizontal) 0°- direction at Bob's site, which effectively rotates the polarization state by 90° around the z-axis on the Poincaré sphere (red arrows in Fig. 7.4.2). The polarizer angles are then set to $\alpha_2 = 0^\circ/90^\circ$ and β_2 is scanned between 0° and 115°. With the same β_2 and $\alpha_1 = \pm 45^\circ$, the expectation values specific only for the CHSH case are measured. The remaining measurement for inequality (7.4.9) is the check of perfect correlations, for which we choose $\alpha_2 = \beta_3 = 0^\circ$, that is, the intersection of the two orthogonal planes.

Fig. 7.4.3 shows the experimental violation of inequalities (7.4.9) and (7.4.10) for various difference angles. Maximum violation of inequality (7.4.9) is achieved, for example, for the settings $\{\alpha_1,\alpha_2,\beta_1,\beta_2,\beta_3\} = \{45^\circ,0^\circ,55^\circ,10^\circ,0^\circ\}$.

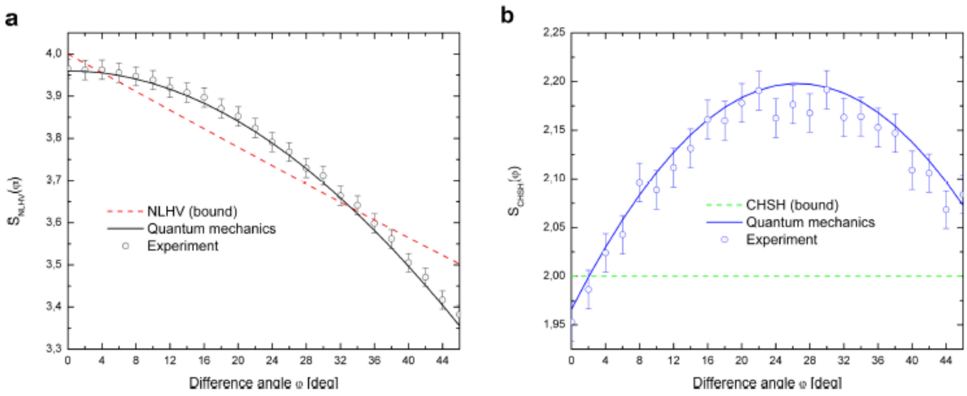

Fig. 7.4.3. Experimental violation of the inequalities for non-local hidden-variable theories (NLHV) and for local realistic theories (CHSH)

(a) Dashed line indicates the bound of inequality (7.4.9) for the investigated class of nonlocal hidden variable theories. The solid line is the quantum theoretical prediction reduced by the experimental visibility. The shown experimental data were taken for various difference angles φ (on the Poincaré sphere) of local measurement settings. The bound is clearly violated for $4^{\circ} < \varphi < 36^{\circ}$. Maximum violation is observed for $\varphi_{max} = 20^{\circ}$.

(b) At the same time, no local realistic theory can model the correlations for the investigated settings as the same set of data also violates the CHSH inequality (7.4.10). The bound (dashed line) is overcome for all values φ around φ_{max}, and hence excludes any local realistic explanation of the observed correlations in **a**. Again, the solid line is the quantum prediction for the observed experimental visibility. Error bars indicate s.d.

The following expectation values for a difference angle $\varphi = 20^{\circ}$ (the errors are calculated assuming that the counts follow a poissonian distribution) were obtained [9]: $E(\vec{a}_1, \vec{b}_1) = -0.9298 \pm 0.0105$, $E(\vec{a}_2, \vec{b}_2) = -0.942 \pm 0.0112$, $E(\vec{a}_2, \vec{b}_3) = -0.9902 \pm 0.0118$. This results in

$S_{NLHV} = 3.8521 \pm 0.0227$, which violates inequality (7.4.9) by 3.2 standard deviations (see Fig. 7.4.3). At the same time, we can extract the additional correlation functions $E(\vec{a}_2, \vec{b}_1) = 0.3436 \pm 0.0088$, $E(\vec{a}_1, \vec{b}_2) = 0.0374 \pm 0.0091$ required for the CHSH inequality. We obtain $S_{CHSH} = 2.178 \pm 0.0199$, which is a violation by ~ 9 standard deviations. The stronger violation of inequality (7.4.10) is due to the relaxed visibility requirements on the probed entangled state.

VII.4.2. Derivation of the Canonical Leggett inequality

With the assumption that photons with well defined polarization obey Malus' law:

$$\langle A \rangle = \vec{u} \cdot \vec{a}, \qquad \langle B \rangle = \vec{v} \cdot \vec{b}, \tag{7.4.12}$$

the upper bound of Eq. (7.4.8) becomes:

$$\langle AB \rangle \leq 1 - |\vec{u} \cdot \vec{a}_k - \vec{v} \cdot \vec{b}_l|, \tag{7.4.13}$$

where $\vec{a}_k$ and $\vec{b}_l$ are unit vectors associated with the k-th measurement setting of Alice and the l -th of Bob, respectively. Taking the average over arbitrary polarizations we obtain

$$E_{kl}^{HVT} \leq 1 - \int d\vec{u} d\vec{v} F(\vec{u}, \vec{v}) |\vec{u} \cdot \vec{a}_k - \vec{v} \cdot \vec{b}_l|, \tag{7.4.14}$$

where E_{kl}^{HVT} is the correlation function which can be experimentally measured when Alice chooses to measure $\vec{a}_k$ and Bob chooses $\vec{b}_l$. Let us denote by u_{kl} and v_{kl} the length of projections of vectors $\vec{u}$ and $\vec{v}$ onto the plane spanned by $\vec{a}_k$ and $\vec{b}_l$. Since one can decompose vectors $\vec{u}$ and $\vec{v}$ into a vector orthogonal to the plane of the settings and a vector within the plane the scalar products read

$$\vec{u}\cdot\vec{a}_k = u_{kl}\cos(\phi_{a_k}-\phi_u), \vec{v}\cdot\vec{b}_l = v_{kl}\cos(\phi_{b_l}-\phi_v), \qquad (7.4.15)$$

where all the ϕ angles are relative to some axis within the plane of the settings; angles ϕ_u and ϕ_v describe the position of the projections of vectors $\vec{u}$ and $\vec{v}$, respectively, whereas angles ϕ_{a_k} and ϕ_{b_l} describe the position of the setting vectors. With this notation the inequality (7.4.14) becomes:

$$E_{kl}^{HVT} \leq 1 - \int d\vec{u}d\vec{v}F(\vec{u},\vec{v})|u_{kl}\cos(\phi_{a_k}-\phi_u) - v_{kl}\cos(\phi_{b_l}-\phi_v)|. \qquad (7.4.16)$$

The magnitudes of the projections can always be decomposed into the sum and the difference of two real numbers $u_{kl} = n_1 + n_2$ and $v_{kl} = n_1 - n_2$.

We insert this decomposition into the last inequality, and hence the terms multiplied by n_1 and n_2 are

$$\cos(\phi_{a_k}-\phi_u) - \cos(\phi_{b_l}-\phi_v) = 2\sin\frac{\phi_{a_k}+\phi_{b_l}-(\phi_u+\phi_v)}{2}\sin\frac{-(\phi_{a_k}-\phi_{b_l})+\phi_u-\phi_v}{2} \qquad (7.4.17)$$

and

$$\cos(\phi_{a_k}-\phi_u) - \cos(\phi_{b_l}-\phi_v) = 2\cos\frac{\phi_{a_k}+\phi_{b_l}-(\phi_u+\phi_v)}{2}\cos\frac{\phi_{a_k}-\phi_{b_l}-(\phi_u-\phi_v)}{2}, \qquad (7.4.18)$$

respectively. We make the following substitution for the measurement angles

$$\xi_{kl} = \frac{\phi_{a_k}+\phi_{b_l}}{2}, \varphi_{kl} = \phi_{a_k}-\phi_{b_l}, \qquad (7.4.19)$$

and parameterize the position of the projections within their plane by

$$\psi_{uv} = \frac{\phi_u + \phi_v}{2}, \qquad \chi_{uv} = \phi_u - \phi_v. \tag{7.4.20}$$

Using these new angles one obtains that

$$\begin{gathered} E_{kl}^{HVT}(\xi_{kl}, \varphi_{kl}) \leq \\ 1 - 2\int d\vec{u} d\vec{v} F(\vec{u}, \vec{v}) |n_2 \cos\frac{\varphi_{kl} - \chi_{uv}}{2} \cos(\xi_{kl} - \psi_{uv}) - n_1 \sin\frac{\varphi_{kl} - \chi_{uv}}{2} \sin(\xi_{kl} - \psi_{uv})|, \end{gathered} \tag{7.4.21}$$

where in the correlation function $E_{kl}(\xi_{kl}, \varphi_{kl})$ we explicitly state the angles it is dependent on. The expression within the modulus is a linear combination of two harmonic functions of $\xi_{kl} - \psi_{uv}$, and therefore is a harmonic function itself. Its amplitude reads

$$\sqrt{n_2^2 \cos^2\left(\frac{\varphi_{kl} - \chi_{uv}}{2}\right) + n_1^2 \sin^2\left(\frac{\varphi_{kl} - \chi_{uv}}{2}\right)}, \tag{7.4.22}$$

and the phase is some fixed real number α

$$\begin{gathered} E_{kl}^{HVT}(\xi_{kl}, \varphi_{kl}) \leq \\ 1 - 2\int d\vec{u} d\vec{v} F(\vec{u}, \vec{v}) \sqrt{n_2^2 \cos^2\left(\frac{\varphi_{kl} - \chi_{uv}}{2}\right) + n_1^2 \sin^2\left(\frac{\varphi_{kl} - \chi_{uv}}{2}\right)} \times \\ \times |\cos(\xi_{kl} - \psi_{uv} + \alpha)|. \end{gathered} \tag{7.4.23}$$

In the next step we average both sides of this inequality over the *measurement angle* $\xi_{kl} = \frac{\phi_{a_k} + \phi_{b_l}}{2}$. This means an integration over $\xi_{kl} \in [0, 2\pi)$ and a multiplication by $\frac{1}{2\pi}$. The integral of the ξ_{kl} dependent part of the right-hand side of (7.4.23) reads:

$$\frac{1}{2\pi}\int_0^{2\pi} d\xi_{kl} |\cos(\xi_{kl} - \psi_{uv} + \alpha)| = \frac{2}{\pi}. \tag{7.4.24}$$

By denoting the average of the correlation function over the angle ξ_{kl} as:

$$E_{kl}^{HVT}(\varphi_{kl}) \equiv \frac{1}{2\pi}\int_0^{2\pi} d\xi_{kl} E_{kl}(\xi_{kl},\varphi_{kl}), \tag{7.4.25}$$

one can write (7.4.23) as

$$E_{kl}^{HVT}(\varphi_{kl}) \leq 1 - \frac{4}{\pi}\int d\vec{u}d\vec{v}F(\vec{u},\vec{v})\sqrt{n_2^2\cos^2\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right) + n_1^2\sin^2\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right)} \tag{7.4.26}$$

This inequality is valid for any choice of observables in the plane defined by $\vec{a}_k$ and $\vec{b}_l$. One can introduce two new observable vectors in this plane and write the inequality for the averaged correlation function $E_{k'l'}^{HVT}(\varphi'_{k'l'})$ of these new observables. The sum of these two inequalities is

$$\begin{aligned} E_{kl}^{HVT}(\varphi_{kl}) + E_{k'l'}^{HVT}(\varphi'_{k'l'}) &\leq 2 - \frac{4}{\pi}\int d\vec{u}d\vec{v}F(\vec{u},\vec{v}) \\ \times\Bigg(\sqrt{n_2^2\cos^2\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right) + n_1^2\sin^2\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right)} &+ \\ +\sqrt{n_2^2\cos^2\left(\frac{\varphi'_{k'l'}-\chi_{uv}}{2}\right) + n_1^2\sin^2\left(\frac{\varphi'_{k'l'}-\chi_{uv}}{2}\right)} \end{aligned} \tag{7.4.27}$$

One can use the triangle inequality

$$\begin{gathered} \|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|, \\ \sqrt{(x_1+y_1)^2 + (x_2+y_2)^2} \leq \sqrt{x_1^2+x_2^2} + \sqrt{y_1^2+y_2^2}, \end{gathered} \tag{7.4.28}$$

for the two-dimensional vectors $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$, with components defined by

$$x_1 = \left|n_2\cos\frac{\varphi_{kl}-\chi_{uv}}{2}\right|, \quad y_1 = \left|n_2\cos\frac{\varphi'_{k'l'}-\chi_{uv}}{2}\right|, \tag{7.4.29}$$

and

$$x_2 = \left| n_1 \sin\frac{\varphi_{kl} - \chi_{uv}}{2} \right|, \quad y_2 = \left| n_1 \sin\frac{\varphi'_{k'l'} - \chi_{uv}}{2} \right|. \tag{7.4.30}$$

One can further estimate this bound by using the following relations

$$\left|\cos\left(\frac{\varphi_{kl} - \chi_{uv}}{2}\right)\right| + \left|\cos\left(\frac{\varphi'_{k'l'} - \chi_{uv}}{2}\right)\right| \geq \left|\sin\left(\frac{\varphi_{kl} - \varphi'_{k'l'}}{2}\right)\right| \tag{7.4.31}$$

and

$$\left|\sin\left(\frac{\varphi_{kl} - \chi_{uv}}{2}\right)\right| + \left|\sin\left(\frac{\varphi'_{k'l'} - \chi_{uv}}{2}\right)\right| \geq \left|\sin\left(\frac{\varphi_{kl} - \varphi'_{k'l'}}{2}\right)\right|. \tag{7.4.32}$$

This estimate follows if one uses the formula for the sine of the difference angle to the right-hand side argument $\frac{\varphi_{kl} - \varphi'_{k'l'}}{2} = \frac{\varphi_{kl} - \chi_{uv}}{2} - \frac{\varphi'_{k'l'} - \chi_{uv}}{2}$. Namely,

$$\begin{aligned} \left|\sin\frac{\varphi_{kl} - \varphi'_{k'l'}}{2}\right| &= \left|\sin\frac{\varphi_{kl} - \chi_{uv}}{2}\cos\frac{\varphi'_{k'l'} - \chi_{uv}}{2} - \cos\frac{\varphi_{kl} - \chi_{uv}}{2}\sin\frac{\varphi'_{k'l'} - \chi_{uv}}{2}\right| \\ &\leq \left|\sin\frac{\varphi_{kl} - \chi_{uv}}{2}\right|\left|\cos\frac{\varphi'_{k'l'} - \chi_{uv}}{2}\right| + \left|\cos\frac{\varphi_{kl} - \chi_{uv}}{2}\right|\left|\sin\frac{\varphi'_{k'l'} - \chi_{uv}}{2}\right|. \end{aligned} \tag{7.4.33}$$

After these estimates, the lower bound of $E_{kl} + E_{k'l'}$ (following form the left-hand side inequality in (7.4.8)) is equal to minus the upper bound, and thus one can apply the upper bound to the modulus of the left hand side of (7.4.27). This is because the only formal difference between expressions in the estimates seeking the lower bound of the averaged Eq. (7.4.8) compared to those seeking the upper bound boils down to the interchange between n_1 and n_2. After applying (7.4.31) and (7.4.32), this makes no difference anymore. One can shortly write

$$|E_{kl}^{HVT}(\varphi_{kl}) + E_{k'l'}^{HVT}(\varphi'_{k'l'})| \le 2 - \frac{4}{\pi}\left|\sin\frac{\varphi_{kl} - \varphi'_{k'l'}}{2}\right| \int d\vec{u}d\vec{v}F(\vec{u},\vec{v})\sqrt{n_2^2 + n_1^2}\,. \quad (7.4.34)$$

Going back to the magnitudes:

$$|E_{kl}^{HVT}(\varphi_{kl}) + E_{k'l'}^{HVT}(\varphi'_{k'l'})| \le 2 - \frac{2\sqrt{2}}{\pi}\left|\sin\frac{\varphi_{kl} - \varphi'_{k'l'}}{2}\right| \int d\vec{u}d\vec{v}F(\vec{u},\vec{v})\sqrt{u_{kl}^2 + v_{kl}^2}\,. \quad (7.4.35)$$

This inequality is valid for *any* choice of the plane of observables. The bound involves only the projections of vectors $\vec{u}$ and $\vec{v}$ onto the plane of the settings. The integrations in the bound can be thought of as a mean value of expression $\sqrt{u_{kl}^2 + v_{kl}^2}$ averaged over the distribution of the vectors. For the plane orthogonal to the initial one the inequality is

$$|E_{pq}^{\perp HVT}(\varphi_{pq}) + E_{p'q'}^{\perp HVT}(\varphi'_{p'q'})| \le 2 - \frac{2\sqrt{2}}{\pi}\left|\sin\frac{\varphi_{pq} - \varphi'_{p'q'}}{2}\right| \int d\vec{u}d\vec{v}F(\vec{u},\vec{v})\sqrt{u_{pq}^2 + v_{pq}^2}\,, \quad (7.4.36)$$

where u_{pq} and v_{pq} denote the projections of vectors $\vec{u}$ and $\vec{v}$, respectively, onto the plane spanned by the settings $\vec{a}_p$ and $\vec{b}_q$ (which is by construction orthogonal to the plane spanned by $\vec{a}_k$ and $\vec{b}_l$). We add the inequalities for orthogonal observation planes, (7.4.35) and (7.4.36), and choose $\varphi'_{k'l'} = \varphi'_{p'q'} = 0$ and $\varphi_{kl} = \varphi_{pq} = \varphi$. This gives

$$\begin{aligned} &|E_{kl}^{HVT}(\varphi) + E_{k'l'}^{HVT}(0)| + |E_{pq}^{\perp HVT}(\varphi) + E_{p'q'}^{\perp HVT}(0)| \le \\ &4 - \frac{2\sqrt{2}}{\pi}\left|\sin\frac{\varphi}{2}\right| \int d\vec{u}d\vec{v}F(\vec{u},\vec{v})\left(\sqrt{u_{kl}^2 + v_{kl}^2} + \sqrt{u_{pq}^2 + v_{pq}^2}\right) \end{aligned} \quad (7.4.37)$$

We apply the triangle inequality (7.4.28) to the expression within the bracket. These time vectors $\vec{x}$ and $\vec{y}$ have the following components:

$$\vec{x} = (u_{kl}, u_{pq}), \vec{y} = (v_{kl}, v_{pq}). \quad (7.4.38)$$

The integrand is bounded by:

$$\sqrt{u_{kl}^2 + v_{kl}^2} + \sqrt{u_{pq}^2 + v_{pq}^2} \geq \sqrt{(u_{kl} + u_{pq})^2 + (v_{kl} + v_{pq})^2}. \quad (7.4.39)$$

Let us consider the term involving vector $\vec{u}$ only. Since the lengths are positive

$$(u_{kl} + u_{pq})^2 \geq u_{kl}^2 + u_{pq}^2. \quad (7.4.40)$$

Recall that u_{kl} and u_{pq} are projections onto orthogonal planes. One can introduce normal vectors to these planes, $\vec{n}_{kl}$ and $\vec{n}_{pq}$, respectively, and write

$$(\vec{n}_{kl} \cdot \vec{u})^2 + u_{kl}^2 = 1, (\vec{n}_{pq} \cdot \vec{u})^2 + u_{pq}^2 = 1. \quad (7.4.41)$$

Note that the scalar products are two components of vector $\vec{u}$ in the Cartesian frame build out of vectors $\vec{n}_{kl}$, $\vec{n}_{pq}$, and the one which is orthogonal to these two. Since vector $\vec{u}$ is normalized one has:

$$(\vec{n}_{kl} \cdot \vec{u})^2 + (\vec{n}_{pq} \cdot \vec{u})^2 \leq 1, \quad (7.4.42)$$

which implies for the sum of equations (7.4.41)

$$u_{kl}^2 + u_{pq}^2 \geq 1. \quad (7.4.43)$$

The same applies to vector $\vec{v}$ and one can conclude that

$$\sqrt{u_{kl}^2 + v_{kl}^2} + \sqrt{u_{pq}^2 + v_{pq}^2} \geq \sqrt{2}. \quad (7.4.44)$$

Since the weight function $F(\vec{u},\vec{v})$ is normalized, the final Laggett type inequality is

$$\left|E_{kl}^{HVT}(\varphi)+E_{k'l'}^{HVT}(0)|+|E_{pq}^{\perp HVT}(\varphi)+E_{p'q'}^{\perp HVT}(0)\right| \le 4-\frac{4}{\pi}\left|\sin\frac{\varphi}{2}\right|. \qquad (7.4.45)$$

VII.4.3. Leggett inequality revisited. Validity of revised Leggett inequality

A general framework of such models is the following: assumption (1) requires that an individual binary measurement outcome A_{t_1} for a polarization measurement at instant t_1 along direction $\vec{a}$ (that is, whether a single photon is transmitted or absorbed at instant t_1 by a polarizer set at a specific angle) is predetermined by some set of hidden-variables λ, and a three-dimensional vector $\vec{u}$, as well as by some set of other possibly non-local parameters η (for example, measurement settings in space-like separated regions) - that is, $A_{t_1}(\lambda,\vec{u},\vec{a},\eta)=A(\lambda,\vec{u},\vec{a},\eta,t_1)$. According to assumption (3), particles with the same $\vec{u}$ but with different λ build up subensembles of 'definite polarization' described by a probability distribution $\rho_{\vec{u}}(\lambda,t_1)$. The expectation value $\langle A_{t_1}(\vec{u})\rangle=\langle A(\vec{u},t_1)\rangle,$ obtained by averaging over λ, fulfils Malus' law, that is, $\langle A_{t_1}(\vec{u})\rangle=\int d\lambda\rho_{\vec{u}}(\lambda,t_1)A(\lambda,\vec{u},\vec{a},\eta,t_1)=\vec{u}\bullet\vec{a}$.
Finally, with assumption (2), the measured expectation value for a general physical state is given by averaging over the distribution $F(\vec{u},t_1)$ of subensembles, that is, $\langle A_{t_1}\rangle=\int d\vec{u}F(\vec{u},t_1)\langle A_{t_1}(\vec{u})\rangle$.
Let us consider a specific source, which emits pairs of photons with well-defined polarizations $\vec{u}$ and $\vec{v}$ to laboratories of Alice and Bob, respectively. The local polarization measurement outcomes A_{t_1} and B_{t_2} are fully determined by the polarization vector, by an additional set of hidden variables λ specific to the source and by any set of parameters η outside the source. For reasons of clarity, we choose an explicit non-local dependence of the outcomes on the settings $\vec{a}$

and $\vec{b}$ of the measurement devices. Note, however, that this is just an example of a possible non-local dependence, and that one can choose any other set out of η. Each emitted pair is fully defined by the subensemble distribution $\rho_{\vec{u},\vec{v}}(\lambda, t_1, t_2)$. In agreement with assumption (3) we impose the following conditions on the predictions for local averages of such measurements (all polarizations and measurement directions are represented as vectors on the Poincaré sphere [10]):

$$\langle A_{t_1}(\vec{u})\rangle = \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda, t_1, t_2) A(\vec{a}, \vec{b}, \lambda, t_1) = \vec{u} \cdot \vec{a}, \tag{7.4.46}$$

$$\langle B_{t_2}(\vec{v})\rangle = \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda, t_1, t_2) B(\vec{b}, \vec{a}, \lambda, t_2) = \vec{v} \cdot \vec{b}. \tag{7.4.47}$$

It is important to note that the validity of Malus' law imposes the non-signalling condition on the investigated non-local models, as the local expectation values do only depend on local parameters. The correlation function of measurement results for a source emitting well-polarized photons is defined as the average of the products of the individual measurement outcomes:

$$\langle A_{t_1} B_{t_2}(\vec{u}, \vec{v})\rangle = \langle AB(\vec{u}, \vec{v}, t_1, t_2)\rangle = \int d\lambda \rho_{\vec{u},\vec{v}}(\lambda, t_1, t_2) A(\vec{a}, \vec{b}, \lambda, t_1) B(\vec{b}, \vec{a}, \lambda, t_2) \tag{7.4.48}$$

For a general source producing mixtures of polarized photons the observable correlations are averaged over a distribution of the polarizations $F(\vec{u}, \vec{v}, t_1, t_2)$, and the general correlation function $E(t_1, t_2)$ is given by:

$$E(t_1, t_2) = \langle A_{t_1} B_{t_2}\rangle_F = \int d\vec{u} d\vec{v} F(\vec{u}, \vec{v}, t_1, t_2) \langle AB(\vec{u}, \vec{v}, t_1, t_2)\rangle \tag{7.4.49}$$

It is a very important trait of this model that there exist subensembles of definite polarizations (independent of measurements) and that the predictions for the subensembles agree with Malus' law. There the non-local correlations are a consequence of the non-local quantum potential, which exerts suitable torque on the particles leading to experimental results compliant with quantum mechanics. In that theory, neither of the two particles in a maximally entangled state carries any angular momentum at all when emerging from the source [11]. In contrast, in the Leggett model, it is the total ensemble emitted by the source that carries no angular momentum, which is a consequence of averaging over the individual particles' well defined angular momenta (polarization).

Assumption 7.4.1. We assume now that $|t_1 - t_2| = \delta > 0,$ and

$$\begin{aligned} \langle AB(\vec{u},\vec{v},t_1,t_2)\rangle &= \langle AB(\vec{u},\vec{v},t_1-t_2)\rangle, \\ F(\vec{u},\vec{v},t_1,t_2) &= F(\vec{u},\vec{v},t_1-t_2), \\ E(t_1,t_2) &= E(t_1-t_2). \end{aligned} \tag{7.4.50}$$

Remark 7.4.1. We abbreviate now for short

$$\begin{aligned} \langle A_{t_1}B_{t_2}(\vec{u},\vec{v})\rangle &\triangleq \langle AB(\vec{u},\vec{v})\rangle^{>} \text{ iff } t_1 - t_2 = \delta > 0, \\ \langle A_{t_1}B_{t_2}(\vec{u},\vec{v})\rangle &\triangleq \langle AB(\vec{u},\vec{v})\rangle^{<} \text{ iff } t_1 - t_2 = -\delta < 0, \\ E(t_1,t_2) &\triangleq E^{>} \text{ iff } t_1 - t_2 = \delta > 0, \\ E(t_1,t_2) &\triangleq E^{<} \text{ iff } t_1 - t_2 = -\delta < 0. \end{aligned} \tag{7.4.51}$$

We take a source which distributes pairs of well-polarized photons. Different pairs can have different polarizations. The size of a subensemble in which photons have polarizations $\vec{u}$ and $\vec{v}$ is described by the weight function $F(\vec{u},\vec{v},t_1,t_2) = F(\vec{u},\vec{v},t_1-t_2)$. All polarizations and measurement directions are represented as vectors on the Poincaré sphere. In every such subensemble individual measurement outcomes are determined by hidden variables λ. The hidden variables are allocated according to the distribution $\rho_{\vec{u},\vec{v}}(\lambda,t_1,t_2) = \rho_{\vec{u},\vec{v}}(\lambda,t_1-t_2)$.

For any dichotomic measurement results, $A_{t_1} = \pm 1$ and $B_{t_2} = \pm 1$, the following identity holds:

$$-1+|A_{t_1}+B_{t_2}|=A_{t_1}B_{t_2}=1-|A_{t_1}-B_{t_2}|. \tag{7.4.52}$$

If the signs of A_{t_1} and B_{t_2} are the same $|A_{t_1}+B_{t_2}|=2$ and $|A_{t_1}-B_{t_2}|=0$, and if $A_{t_1}=-B_{t_2}$ then $|A_{t_1}+B_{t_2}|=0$ and $|A_{t_1}-B_{t_2}|=2$. Any kind of non-local dependencies is allowed, i.e. $A_{t_1}=A(\vec{a},\vec{b},\vec{u},\vec{v},\lambda,t_1\ldots.)$ and $B_{t_2}=B(\vec{a},\vec{b},\vec{u},\vec{v},\lambda,t_2\ldots.)$. Taking the average over the subensemble with definite polarizations gives

$$\begin{aligned}&-1+\int d\lambda\rho_{\vec{u},\vec{v}}(\lambda,t_1,t_2)|A_{t_1}+B_{t_2}|=\\&=\int d\lambda\rho_{\vec{u},\vec{v}}(\lambda,t_1,t_2)A_{t_1}B_{t_2}=1-\int d\lambda\rho_{\vec{u},\vec{v}}(\lambda,t_1,t_2)|A_{t_1}-B_{t_2}|,\end{aligned} \tag{7.4.53}$$

which in an abbreviated notation, where the averages are denoted by $\langle\cdot\rangle$, is

$$-1+\langle|A_{t_1}+B_{t_2}|\rangle=\langle A_{t_1}B_{t_2}\rangle=1-\langle|A_{t_1}-B_{t_2}|\rangle. \tag{7.4.54}$$

As the average of the modulus is greater than or equal to the modulus of the averages, one gets the set of inequalities:

$$-1+|\langle A_{t_1}\rangle+\langle B_{t_2}\rangle|\ \leq\langle A_{t_1}B_{t_2}\rangle\leq 1-|\langle A_{t_1}\rangle-\langle B_{t_2}\rangle|. \tag{7.4.55}$$

From now on only the upper bound will be considered, however all steps apply to the lower bound as well. With the assumption that photons with well defined polarization obey Malus' law

$$\langle A_{t_1}\rangle=\vec{u}\cdot\vec{a},\ \langle B_{t_2}\rangle=\vec{v}\cdot\vec{b}, \tag{7.4.56}$$

the upper bound of the inequality (7.4.55) becomes

$$\langle A_{t_1}B_{t_2}\rangle\leq 1-|\vec{u}\cdot\vec{a}_k-\vec{v}\cdot\vec{b}_l|, \tag{7.4.57}$$

where $\vec{a}_k$ and $\vec{b}_l$ are unit vectors associated with the k-th measurement setting of Alice and the l-th of Bob, respectively. Taking the average over arbitrary polarizations we obtain

$$E_{kl}(t_1,t_2) \leq 1 - \int d\vec{u}d\vec{v}F(\vec{u},\vec{v},t_1,t_2)|\vec{u}\bullet\vec{a}_k - \vec{v}\bullet\vec{b}_l|. \quad (7.4.58)$$

Remark 7.4.2. We assume now that $t_1 - t_2 = \delta > 0$ and by using (7.4.50) – (7.4.51) we rewrite now the inequalities (7.4.57) and (7.4.58) as

$$\langle AB\rangle^{>} \leq 1 - |\vec{u}\bullet\vec{a}_k - \vec{v}\bullet\vec{b}_l| \quad (7.4.59)$$

and

$$E_{kl}^{>} \leq 1 - \int d\vec{u}d\vec{v}F(\vec{u},\vec{v},\delta)|\vec{u}\bullet\vec{a}_k - \vec{v}\bullet\vec{b}_l|. \quad (7.4.60)$$

respectively.

Remark 7.4.3. We assume now that $t_1 - t_2 = -\delta < 0$ and by using (7.4.50)-(7.4.51) we rewrite now the inequalities (7.4.57) and (7.4.58) as

$$\langle AB\rangle^{<} \leq 1 - |\vec{u}\bullet\vec{a}_k - \vec{v}\bullet\vec{b}_l| \quad (7.4.61)$$

and

$$E_{kl}^{<} \leq 1 - \int d\vec{u}d\vec{v}F(\vec{u},\vec{v},-\delta)|\vec{u}\bullet\vec{a}_k - \vec{v}\bullet\vec{b}_l|. \quad (7.4.62)$$

respectively.

Remark 7.4.4. We define now the full avereges E_{kl}^{GHVT} as

$$E_{kl}^{GHVT} = E_{kl}^{>} + E_{kl}^{<}, \quad (7.4.63)$$

where E_{kl}^{GHVT} is the correlation function which can be experimentally measured when Alice chooses to measure $\vec{a}_k$ and Bob chooses $\vec{b}_l$. Let us denote by u_{kl} and v_{kl} the length of projections of vectors $\vec{u}$ and $\vec{v}$ onto the plane spanned by $\vec{a}_k$ and $\vec{b}_l$.
Thus from the inequalities (7.4.60) and (7.4.62) by using Eq. (7.4.63) we obtain

$$\begin{aligned} E_{kl}^{GHVT} &\leq 2 - \int d\vec{u}d\vec{v}\widetilde{F}(\vec{u},\vec{v})|\vec{u} \bullet \vec{a}_k - \vec{v} \bullet \vec{b}_l|, \\ \widetilde{F}(\vec{u},\vec{v},\delta) &= F(\vec{u},\vec{v},\delta) + F(\vec{u},\vec{v},-\delta). \end{aligned} \quad (7.4.64)$$

Compare the inequality (7.4.64) with the inequality (7.4.14). Let us denote by u_{kl} and v_{kl} the length of projections of vectors $\vec{u}$ and $\vec{v}$ onto the plane spanned by $\vec{a}_k$ and $\vec{b}_l$. Since one can decompose vectors $\vec{u}$ and $\vec{v}$ into a vector orthogonal to the plane of the settings and a vector within the plane the scalar products read

$$\vec{u} \bullet \vec{a}_k = u_{kl}\cos(\phi_{a_k} - \phi_u), \vec{v} \bullet \vec{b}_l = v_{kl}\cos(\phi_{b_l} - \phi_v), \quad (7.4.65)$$

where all the ϕ angles are relative to some axis within the plane of the settings; angles ϕ_u and ϕ_v describe the position of the projections of vectors $\vec{u}$ and $\vec{v}$, respectively, whereas angles ϕ_{a_k} and ϕ_{b_l} describe the position of the setting vectors. With this notation the inequality (7.4.64) becomes

$$E_{kl}^{GHVT} \leq 2 - \int d\vec{u}d\vec{v}\widetilde{F}(\vec{u},\vec{v},\delta)|u_{kl}\cos(\phi_{a_k} - \phi_u) - v_{kl}\cos(\phi_{b_l} - \phi_v)|. \quad (7.4.66)$$

Compare the inequality (7.4.66) with the inequality (7.4.16). The magnitudes of the projections can always be decomposed into the sum and the difference of two real numbers $u_{kl} = n_1 + n_2$ and $v_{kl} = n_1 - n_2$. We insert this decomposition into the last inequality, and hence the terms multiplied by n_1 and n_2 are

$$\cos(\phi_{a_k}-\phi_u)-\cos(\phi_{b_l}-\phi_v) = \\ = 2\sin\frac{\phi_{a_k}+\phi_{b_l}-(\phi_u+\phi_v)}{2}\sin\frac{-(\phi_{a_k}-\phi_{b_l})+\phi_u-\phi_v}{2} \tag{7.4.67}$$

and

$$\cos(\phi_{a_k}-\phi_u)-\cos(\phi_{b_l}-\phi_v) = \\ = 2\cos\frac{\phi_{a_k}+\phi_{b_l}-(\phi_u+\phi_v)}{2}\cos\frac{\phi_{a_k}-\phi_{b_l}-(\phi_u-\phi_v)}{2}, \tag{7.4.68}$$

respectively. We make the following substitution for the measurement angles

$$\xi_{kl} = \frac{\phi_{a_k}+\phi_{b_l}}{2}, \varphi_{kl} = \phi_{a_k}-\phi_{b_l}, \tag{7.4.69}$$

and parameterize the position of the projections within their plane by

$$\psi_{uv} = \frac{\phi_u+\phi_v}{2}, \chi_{uv} = \phi_u-\phi_v. \tag{7.4.70}$$

Using these new angles one obtains that

$$E_{kl}^{GHVT}(\xi_{kl},\varphi_{kl}) \leq \\ 2-2\int d\vec{u}d\vec{v}\widetilde{F}(\vec{u},\vec{v},\delta)|n_2\cos\frac{\varphi_{kl}-\chi_{uv}}{2}\cos(\xi_{kl}-\psi_{uv})-n_1\sin\frac{\varphi_{kl}-\chi_{uv}}{2}\sin(\xi_{kl}-\psi_{uv})|, \tag{7.4.71}$$

where in the correlation function $E_{kl}(\xi_{kl},\varphi_{kl})$ we explicitly state the angles it is dependent on. Compare the inequality (7.4.71) with the inequality (7.4.21). The expression within the modulus is a linear combination of two harmonic functions of $\xi_{kl}-\psi_{uv}$, and therefore is a harmonic function itself. Its amplitude reads

$$\sqrt{n_2^2\cos^2\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right)+n_1^2\sin^2\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right)}, \tag{7.4.72}$$

and the phase is some fixed real number α

$$E_{kl}^{GHVT}(\xi_{kl},\varphi_{kl}) \leq 2-2\int d\vec{u}d\vec{v}\widetilde{F}(\vec{u},\vec{v},\delta)\sqrt{n_2^2\cos^2\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right)+n_1^2\sin^2\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right)}\times |\cos(\xi_{kl}-\psi_{uv}+\alpha)|. \tag{7.4.73}$$

In the next step we average both sides of this inequality over the *measurement angle* $\xi_{kl} = \dfrac{\phi_{a_k}+\phi_{b_l}}{2}$. This means an integration over $\xi_{kl} \in [0, 2\pi)$ and a multiplication by $\frac{1}{2\pi}$. The integral of the ξ_{kl} dependent part of the right-hand side of (7.4.73) reads:

$$\frac{1}{2\pi}\int_0^{2\pi} d\xi_{kl}|\cos(\xi_{kl}-\psi_{uv}+\alpha)| = \frac{2}{\pi}. \tag{7.4.74}$$

By denoting the average of the correlation function over the angle ξ_{kl} as:

$$E_{kl}^{GHVT}(\varphi_{kl}) \equiv \frac{1}{2\pi}\int_0^{2\pi} d\xi_{kl}E_{kl}^{GHVT}(\xi_{kl},\varphi_{kl}), \tag{7.4.75}$$

one can write (7.4.73) as

$$E_{kl}^{GHVT}(\varphi_{kl}) \leq 2-\frac{4}{\pi}\int d\vec{u}d\vec{v}\widetilde{F}(\vec{u},\vec{v},\delta)\sqrt{n_2^2\cos^2\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right)+n_1^2\sin^2\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right)} \tag{7.4.76}$$

This inequality is valid for any choice of observables in the plane defined by $\vec{a}_k$ and $\vec{b}_l$. One can introduce two new observable vectors in this plane and write the inequality for the averaged correlation function $E_{k'l'}^{GHVT}(\varphi'_{k'l'})$ of these new observables. The sum of these two inequalities is

$$E_{kl}^{GHVT}(\varphi_{kl}) + E_{k'l'}^{HVT}(\varphi'_{k'l'}) \leq 4 - \frac{4}{\pi}\int d\vec{u}d\vec{v}\widetilde{F}(\vec{u},\vec{v},\delta)$$
$$\times\left(\sqrt{n_2^2\cos^2\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right) + n_1^2\sin^2\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right)} + \right. \qquad (7.4.77)$$
$$\left. + \sqrt{n_2^2\cos^2\left(\frac{\varphi'_{k'l'}-\chi_{uv}}{2}\right) + n_1^2\sin^2\left(\frac{\varphi'_{k'l'}-\chi_{uv}}{2}\right)}\right).$$

One can use the triangle inequality

$$\|\vec{x}+\vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|,$$
$$\sqrt{(x_1+y_1)^2 + (x_2+y_2)^2} \leq \sqrt{x_1^2+x_2^2} + \sqrt{y_1^2+y_2^2}, \qquad (7.4.78)$$

for the two-dimensional vectors $\vec{x} = (x_1, x_2)$ and $\vec{y} = (y_1, y_2)$, with components defined by

$$x_1 = \left|n_2\cos\frac{\varphi_{kl}-\chi_{uv}}{2}\right|, \quad y_1 = \left|n_2\cos\frac{\varphi'_{k'l'}-\chi_{uv}}{2}\right|, \qquad (7.4.79)$$

and

$$x_2 = \left|n_1\sin\frac{\varphi_{kl}-\chi_{uv}}{2}\right|, \quad y_2 = \left|n_1\sin\frac{\varphi'_{k'l'}-\chi_{uv}}{2}\right|. \qquad (7.4.80)$$

One can further estimate this bound by using the following relations

$$\left|\cos\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right)\right| + \left|\cos\left(\frac{\varphi'_{k'l'}-\chi_{uv}}{2}\right)\right| \geq \left|\sin\left(\frac{\varphi_{kl}-\varphi'_{k'l'}}{2}\right)\right| \qquad (7.4.81)$$

and

$$\left|\sin\left(\frac{\varphi_{kl}-\chi_{uv}}{2}\right)\right| + \left|\sin\left(\frac{\varphi'_{k'l'}-\chi_{uv}}{2}\right)\right| \geq \left|\sin\left(\frac{\varphi_{kl}-\varphi'_{k'l'}}{2}\right)\right|. \qquad (7.4.82)$$

This estimate follows if one uses the formula for the sine of the difference angle to the right-hand side argument $\frac{\varphi_{kl} - \varphi'_{k'l'}}{2} = \frac{\varphi_{kl} - \chi_{uv}}{2} - \frac{\varphi'_{k'l'} - \chi_{uv}}{2}$. Namely,

$$\begin{aligned} \left|\sin\frac{\varphi_{kl} - \varphi'_{k'l'}}{2}\right| &= \left|\sin\frac{\varphi_{kl} - \chi_{uv}}{2}\cos\frac{\varphi'_{k'l'} - \chi_{uv}}{2} - \cos\frac{\varphi_{kl} - \chi_{uv}}{2}\sin\frac{\varphi'_{k'l'} - \chi_{uv}}{2}\right| \\ &\leq \left|\sin\frac{\varphi_{kl} - \chi_{uv}}{2}\right|\left|\cos\frac{\varphi'_{k'l'} - \chi_{uv}}{2}\right| + \left|\cos\frac{\varphi_{kl} - \chi_{uv}}{2}\right|\left|\sin\frac{\varphi'_{k'l'} - \chi_{uv}}{2}\right|. \end{aligned} \quad (7.4.83)$$

After these estimates, the lower bound of $E_{kl}^{GHVT} + E_{k'l'}^{GHVT}$ (following form the left-hand side inequality in (7.4.55)) is equal to minus the upper bound, and thus one can apply the upper bound to the modulus of the left hand side of (7.4.27). This is because the only formal difference between expressions in the estimates seeking the lower bound of the averaged Eq. (7.4.55) compared to those seeking the upper bound boils down to the interchange between n_1 and n_2. After applying (7.4.81) and (7.4.82), this makes no difference anymore. One can shortly write

$$\left|E_{kl}^{GHVT}(\varphi_{kl}) + E_{k'l'}^{GHVT}(\varphi'_{k'l'})\right| \leq 4 - \frac{4}{\pi}\left|\sin\frac{\varphi_{kl} - \varphi'_{k'l'}}{2}\right| \int d\vec{u}d\vec{v}\widetilde{F}(\vec{u},\vec{v},\delta)\sqrt{n_2^2 + n_1^2}. \quad (7.4.84)$$

Going back to the magnitudes:

$$|E_{kl}^{GHVT}(\varphi_{kl}) + E_{k'l'}^{GHVT}(\varphi'_{k'l'})| \leq 4 - \frac{2\sqrt{2}}{\pi}\left|\sin\frac{\varphi_{kl} - \varphi'_{k'l'}}{2}\right| \int d\vec{u}d\vec{v}\widetilde{F}(\vec{u},\vec{v},\delta)\sqrt{u_{kl}^2 + v_{kl}^2}. \quad (7.4.85)$$

The inequality (7.4.85) is valid for *any* choice of the plane of observables. The bound involves only the projections of vectors $\vec{u}$ and $\vec{v}$ onto the plane of the settings. The integrations in the bound can be thought of as a mean value of expression $\sqrt{u_{kl}^2 + v_{kl}^2}$ averaged over the distribution of the vectors. For the plane orthogonal to the initial one the inequality is

$$|E^{\perp GHVT}_{pq}(\varphi_{pq}) + E^{\perp GHVT}_{p'q'}(\varphi'_{p'q'})| \leq 2 - \frac{2\sqrt{2}}{\pi}\left|\sin\frac{\varphi_{pq} - \varphi'_{p'q'}}{2}\right| \int d\vec{u}d\vec{v}\widetilde{F}(\vec{u},\vec{v},\delta)\sqrt{u^2_{pq} + v^2_{pq}}, \quad (7.4.86)$$

where u_{pq} and v_{pq} denote the projections of vectors $\vec{u}$ and $\vec{v}$, respectively, onto the plane spanned by the settings $\vec{a}_p$ and $\vec{b}_q$ (which is by construction orthogonal to the plane spanned by $\vec{a}_k$ and $\vec{b}_l$). We add the inequalities for orthogonal observatin planes, (7.4.85) and (7.4.86), and choose $\varphi'_{k'l'} = \varphi'_{p'q'} = 0$ and $\varphi_{kl} = \varphi_{pq} = \varphi$. This gives

$$\begin{aligned} &\left|E^{GHVT}_{kl}(\varphi) + E^{GHVT}_{k'l'}(0)| + |E^{\perp GHVT}_{pq}(\varphi) + E^{\perp GHVT}_{p'q'}(0)\right|| \leq \\ &8 - \frac{2\sqrt{2}}{\pi}\left|\sin\frac{\varphi}{2}\right| \int d\vec{u}d\vec{v}\widetilde{F}(\vec{u},\vec{v},\delta)\left(\sqrt{u^2_{kl} + v^2_{kl}} + \sqrt{u^2_{pq} + v^2_{pq}}\right) \end{aligned} \quad (7.4.87)$$

We apply the triangle inequality (7.4.78) to the expression within the bracket. These time vectors $\vec{x}$ and $\vec{y}$ have the following components:

$$\vec{x} = (u_{kl}, u_{pq}), \quad \vec{y} = (v_{kl}, v_{pq}). \quad (7.4.88)$$

The integrand is bounded by:

$$\sqrt{u^2_{kl} + v^2_{kl}} + \sqrt{u^2_{pq} + v^2_{pq}} \geq \sqrt{(u_{kl} + u_{pq})^2 + (v_{kl} + v_{pq})^2}. \quad (7.4.89)$$

Let us consider the term involving vector $\vec{u}$ only. Since the lengths are positive

$$(u_{kl} + u_{pq})^2 \geq u^2_{kl} + u^2_{pq}. \quad (7.4.90)$$

Recall that u_{kl} and u_{pq} are projections onto orthogonal planes. One can introduce normal vectors to these planes, $\vec{n}_{kl}$ and $\vec{n}_{pq}$, respectively, and write

$$(\vec{n}_{kl} \cdot \vec{u})^2 + u_{kl}^2 = 1, (\vec{n}_{pq} \cdot \vec{u})^2 + u_{pq}^2 = 1. \qquad (7.4.91)$$

Note that the scalar products are two components of vector $\vec{u}$ in the Cartesian frame build out of vectors $\vec{n}_{kl}$, $\vec{n}_{pq}$, and the one which is orthogonal to these two. Since vector $\vec{u}$ is normalized one has:

$$(\vec{n}_{kl} \cdot \vec{u})^2 + (\vec{n}_{pq} \cdot \vec{u})^2 \leq 1, \qquad (7.4.92)$$

which implies for the sum of equations (7.4.90)

$$u_{kl}^2 + u_{pq}^2 \geq 1. \qquad (7.4.93)$$

The same applies to vector $\vec{v}$ and one can conclude that

$$\sqrt{u_{kl}^2 + v_{kl}^2} + \sqrt{u_{pq}^2 + v_{pq}^2} \geq \sqrt{2}. \qquad (7.4.94)$$

Since the weight function $F(\vec{u},\vec{v})$ is normalized, the final Laggett type inequality is

$$\left|E_{kl}^{GHVT}(\varphi) + E_{k'l'}^{GHVT}(0)\right| + \left|E_{pq}^{\perp GHVT}(\varphi) + E_{p'q'}^{\perp GHVT}(0)\right| \leq 8 - \frac{4}{\pi}\left|\sin\frac{\varphi}{2}\right|. \qquad (7.4.95)$$

APPENDICES

Appendix A

HEISENBERG'S NOISE-DISTURBANCE UCERTAINTY PRINCIPLE

In the wave description of a photon, polarization can be visualized as the way the wave is rotated. A photon possesses horizontal $|H\rangle$ or vertical $|V\rangle$ polarization, but until its polarization is measured, these two states are said to be in a superposition, described by

$$|\Psi\rangle = \alpha|H\rangle + \beta|V\rangle \qquad (A.1)$$

where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2$ is the probability of finding the photon in state $|H\rangle$ and $|\beta|^2$ in state $|V\rangle$. The horizontal and the vertical polarization jointly define a basis denoted by $\hat{z}$, which can take on the values $|H\rangle$ or $|V\rangle$. However, the polarization can be described in an additional basis as well, the $\hat{x}$ basis, which is shifted 45° in positive direction, see Fig. A.1.

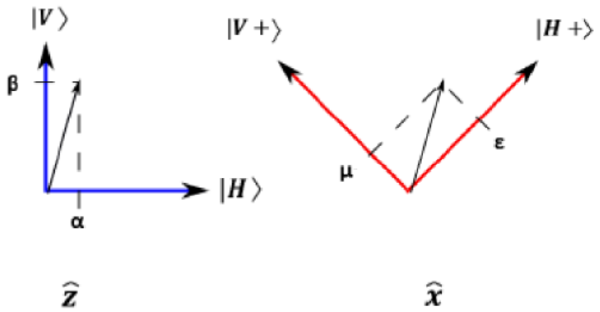

Fig. A.1. A graphical representation of the bases $\hat{z}$ and $\hat{x}$

Just like the particle superposition consists of $|H\rangle$ and $|V\rangle$ in $\hat{z}$ basis, the two new states $|H+\rangle$ and $|V+\rangle$ will describe the particle superposition in the $\hat{x}$ basis as

$$|\Psi\rangle = \epsilon|H+\rangle + \mu|V+\rangle \qquad (A.2)$$

where $|\epsilon|^2$ describes the probability of finding the photon in state $|H+\rangle$ and $|\mu|^2$ is state $|V+\rangle$.

An analogy to the intrinsic property spin is the polarization of photons. Since the polarization is just another example of an intrinsic property a particle could exhibit, one can repeat the Stern-Gerlach experiment with the use of photons. Like the spin of the electron, one can write a polarization state of one basis as a superposition constituting the eigenstates of the other basis.

The eigenstates obtained in the $\hat{z}$ basis are given by

$$\begin{aligned} |H\rangle &= \tfrac{1}{\sqrt{2}}(|H+\rangle + |V+\rangle), \\ |V\rangle &= \tfrac{1}{\sqrt{2}}(|H+\rangle - |V+\rangle). \end{aligned} \qquad (A.3)$$

while polarization states in the $\hat{x}$ basis are described by

$$\begin{aligned} |H+\rangle &= \tfrac{1}{\sqrt{2}}(|H\rangle + |V\rangle), \\ |V+\rangle &= \tfrac{1}{\sqrt{2}}(|H\rangle - |V\rangle). \end{aligned} \qquad (A.4)$$

Every SG (Stern-Gerlach) - apparatus is replaced by three half wave plates (HWP) and a polarizing beam splitter (PBS) in between, see Fig. A.2. This setup will have the same effect on a photon as the SG-apparatus had on the electrons since it measures the polarization of the photon which thereby collapses into one of the eigenstates. A mathematical description of the setup, in order to explain the expectation value which will make us able to predict the paths of the photons throughout the measurements, follows below.

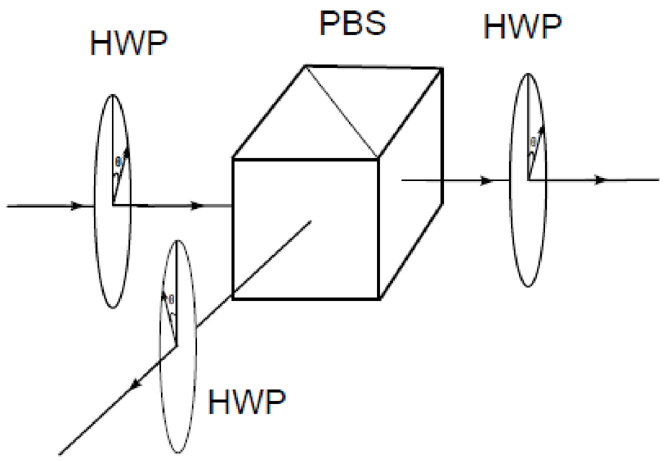

Fig. A.2. A representation of a setup corresponding to a SG-apparatus

The HWP rotates the polarization of the transmitted light and thereby shifts between the two bases. The HWP is described mathematically by the operator

$$\hat{R}_\theta = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix} \qquad (A.5)$$

and when set to basis $\hat{z} : \theta = 0^\circ$, and when set to $\hat{x} : \theta = 22.5^\circ$. The PBS divides the incident beam into two output beams and can mathematically be described as the operator

$$\hat{P} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \qquad (A.6)$$

In this instance the matrix representing the measurement operator σ_z in the $\hat{z}$ basis is derived by

$$\sigma_z = \widehat{R}_{0^\circ}\widehat{P}\widehat{R}_{0^\circ}. \qquad (A.7)$$

Similarly, the matrix representing the measurement operator σ_x in the $\hat{x}$ basis is derived by

$$\sigma_x = \widehat{R}_{22.5^\circ}\widehat{P}\widehat{R}_{22.5^\circ}. \qquad (A.8)$$

Though the expressions above look alike, note that does not take on the same values when set to measure $\hat{z}$ and $\hat{x}$. The measurement operator σ_z measures the polarization in the z -direction. Hence the photon will always be thrown into one of the eigenstates $|H\rangle$ or $|V\rangle$. When measured in σ_x, it will always be thrown into $|H+\rangle$ or $|V+\rangle$. Thus, one can say that the states are eigenstates to the respective operator. The two operators σ_z and σ_x are represented by

$$\sigma_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \qquad (A.9)$$

Notice that

$$[\sigma_z, \sigma_x] = 2\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = 2C. \qquad (A.10)$$

The Robertson uncertainty relation is given by

$$\langle\sigma_z\rangle\langle\sigma_x\rangle \geq \tfrac{1}{2}|\langle[\sigma_z, \sigma_x]\rangle| = \langle C\rangle. \qquad (A.11)$$

Or in general case

$$\sigma(A,\psi)\sigma(B,\psi) \geq \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle| \qquad (A.12)$$

for any observables A, B and any state $\psi,$ where the standard deviation $\sigma(X,\psi)$ of an observable X in state ψ is defined by

$$\sigma^2(X,\psi) = \langle\psi|X^2|\psi\rangle - \langle\psi|X|\psi\rangle^2 .$$

This relation was proven mathematically from fundamental postulates of quantum mechanics. Nevertheless, this relation describes the limitation on preparing microscopic objects but has no direct relevance to the limitation of accuracy of measuring devices. It is a common understanding that the uncertainty principle implies or is implied by a limitation on measuring a system without disturbing it as a position measurement typically disturbs the momentum. However, the limitation has eluded a correct quantitative expression on the trade-off between noise and disturbance.

Heisenberg noise-disturbance uncertainty relation

By the γ -ray thought experiment, Heisenberg [10, 16] argued that the product of the noise in a position measurement and the momentum disturbance caused by that measurement should be no less than $\hbar/2.$ This relation is generally formulated as follows: for any apparatus $\mathbf{A}$ to measure an observable $A,$ the relation

$$\epsilon(A,\psi,\mathbf{A})\eta(B,\psi,\mathbf{A}) \geq \frac{1}{2}|\langle\psi|[A,B]|\psi\rangle| \qquad (A.13)$$

holds for any input state ψ and any observable B, where $\epsilon(A,\psi,\mathbf{A})$ stands for the noise of the A measurement in state ψ using apparatus **A** and $\eta(B,\psi,\mathbf{A})$ stands for the disturbance of B in state ψ caused by apparatus **A.** We refer to the above relation as the Heisenberg noise-disturbance uncertainty relation.

Heisenberg uncertainty relation for joint measurements

Very similarly to the above relation (A.13), the Heisenberg uncertainty relation for joint measurements is generally formulated as follows: for any apparatus **A** with two outputs for the joint measurement of A and $B,$ the relation

$$\epsilon(A,\psi,\mathbf{A})\epsilon(B,\psi,\mathbf{A}) \geq \tfrac{1}{2}|\langle\psi|[A,B]|\psi\rangle| \qquad (A.14)$$

holds for any input state, where $\epsilon(X,\psi,\mathbf{A})$ stands for the noise of the X measurement in state ψ using apparatus **A** for $X=(A,B)$. This relation was proven under the joint unbiasedness condition requiring that the (experimental) mean values of the outcome x_A of the A measurement and the outcome y_B of the B measurement should coincide with the (theoretical) mean values of observables A and $B,$ respectively, on any input state ψ. It is a common opinion that currently available measuring devices satisfy this relation [12]-[14].

Appendix B

CONDITIONAL PROBABILITY DENSITY FUNCTIONS

If X is given to be a continuous random variable with a defined density function say $f(x)$ and E is an event which has positive probability then we define the conditional density function as

$$f(x|E) = \begin{cases} \dfrac{f(x)}{\mathbf{P}(E)} & \Leftrightarrow \; x \in E \\ 0 & \Leftrightarrow \; x \notin E \end{cases} \qquad (B.1)$$

Appendix C

FOURIER TRANSFORM AND HEISENBERG UNCERTAINTY PRINCIPLE

Definition C.1. Given any $\mathcal{L}_{1,2}(\mathbb{R})$ function f, where $\mathcal{L}_{1,2}(\mathbb{R}) = \mathcal{L}_1(\mathbb{R}) \cap \mathcal{L}_2(\mathbb{R})$, we define two operators as follows:

(1) $f(x) \mapsto \hat{f}(p) = \mathcal{F}[f](p) = \int_{\mathbb{R}} f(x)\exp(-2\pi ixp)dx,$

(2) $f(p) \mapsto \mathcal{F}^{-1}[f](x) = \int_{\mathbb{R}} f(p)\exp(2\pi ixp)dp.$

Theorem C.1. (Heisenberg's Uncertainty Principle). Let $\psi \in \mathcal{L}_{1,2}(\mathbb{R})$ with the condition $\int_{\mathbb{R}} |\psi(x)|^2 dx = 1.$ Then

$$\left(\int_{\mathbb{R}} x^2|\psi(x)|^2dx\right)\left(\int_{\mathbb{R}} p^2\left|\hat{f}(p)\right|^2\right) \geq \frac{1}{16\pi^2}, \qquad (C.1)$$

and

$$\left(\int_{\mathbb{R}} (x-x_0)^2|\psi(x)|^2dx\right)\left(\int_{\mathbb{R}} (p-p_0)^2\left|\hat{f}(p)\right|^2\right) \geq \frac{1}{16\pi^2}. \qquad (C.2)$$

Thinking of $\|(x-x_0)\psi(x)\|_2$ as the standard deviation of the displacement written Δx and similarly for $\|(p-p_0)\psi(p)\|_2$ written Δp we may write

$$\Delta x\Delta p \geq const. \qquad (C.3)$$

which is the usual quantum-mechanical way of Heisenberg's uncertainty principle.

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