Foundation of paralogical nonstandard analysis and its application to some famous problems of trigonometrical and orthogonal series. Part II.

Jaykov Foukzon

I. Introduction.

L. Carleson's celebrated theorem of 1965 [1] asserts the pointwise convergence of the partial Fourier sums of square integrable functions. The Fourier transform has a formulation on each of the Euclidean groups R, Z and T. Carleson's original proof worked on T. Fefferman's proof translates very easily to R. M´at´e [2] extended Carleson's proof to Z. Each of the statements of the theorem can be stated in terms of a maximal Fourier multiplier theorem [5]. Inequalities for such operators can be transferred between these three Euclidean groups, and was done P. Auscher and M.J. Carro [3]. But L. Carleson's original proof and another proofs very long and very complicated. We give a very short and very "simple" proof of this fact. Our proof uses PNSA technique only, developed in part I, and does not uses complicated technical formations unavoidable by the using of purely standard approach to the present problems. In contradiction to Carleson's method, which is based on profound properties of trigonometric series, the proposed approach is quite general and allows to research a wide class of analogous problems for the general orthogonal series. Lets suppose that there are general orthogonal series in space $L_2(\Omega), \Omega \subseteq R^d, d = 1,2...$

$$\sum_{n=0}^{\infty} c_n f_n(x), (c_n) \in l_2, f_n \in L_2(\Omega). \qquad (1.1)$$

We shall say that sequence $\{f_n(x)\}_{n=1}^{\infty}$ or series (1.1) possesses by LC-property if series (1.1) converges a.e. It is good known that a general orthogonal row does not possesses by LC-property. A problem of possession by LC-property is still open for many orthogonal series, as example for the series by Jakoby's polynomial. In the present work we shall obtain a general sufficient condition guaranteeing the presence of LC-property in series (1.1). We shall say that the general orthogonal series (1.1) in space L_2 is strong orthogonal series, if the following condition is executed:

$$ {}^{\#}\left(\int^{\#} f_i(x)^{\#} f_j(x) d^{\#}\mu(x)\right) =^{\bullet} \delta_{ij} . \tag{1.2} $$

Where $\delta_{ij} =^{\bullet} 1$, if $i = j$, and $\delta_{ij} =^{\bullet} 0$, if $i \neq^{\bullet} j$ $i, j \in {}^{\#}N$. In other words it is said here that orthogonality persists in strong (non-paralogical) sense after using of #-mapping. The main result is that strong orthogonality plus condition ${}^{\#}(c_n) \in^{\bullet} {}^{\#}l_2$ pulls the LC-property.

During last time L. Carleson's result [1] was generalized in various directions [4], [5]. For non-orthogonal series a special analogue of L. Carleson's celebrated theorem was obtained in work [4]. A Hilbert space of Dirichlet series is obtained by considering the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ that satisfy. These series converge in the half plane $\operatorname{Re} s > 1/2$ and define a functions that are locally L_2 on the boundary $\operatorname{Re} s = 1/2$. An analog of L.Carleson's celebrated convergence theorem is obtained [2]: each such Dirichlet series converges almost everywhere on the critical line $\operatorname{Re} s = 1/2$. To each Dirichlet series of the above type corresponds a "trigonometric" series $\sum_{n=1}^{\infty} a_n \chi(n)$, where χ is a multiplicative character from the positive integers to the unit circle. The space of characters is naturally identified with the infinite-dimensional torus T^{∞}, where each dimension comes from a prime number. The secondanalog of Carleson's theorem reads: The above "trigonometric" series converges for almost all characters χ. At the same time classical technique was found not enough effective for the study of a problem of LC-property presence in a case of multidimensional Fourier series. Carleson's results are trivially transferred on N-multiple Fourier series, for the case of convergence by cubes, but in the case of arbitral convergence Carleson methods does not works and, in general, the problem for N-multiple Fourier series is still open. Particularly, this problem is open for the case of orbicular amounts $E_M[f(x)]$

$$ E_M[f(x)] = (2\pi)^N \sum_{|n|^2 \leq M} f_n \exp(inx), n \in Z^N \tag{1.3} $$

of Fourier series of function $f \in L_2(T^N)$. We shall demonstrate that in the case of orbicular amounts $E_M[f(x)]$ LC-property is right.

In 1971 R. Cooke proved Cantor-Lebesque theorem in two dimentions [6]: if

$$\sum_{|n|^2=k} c_n \exp(inx) \to 0, k \to \infty \tag{1.4}$$

a.e. on T^2, then

$$\sum_{|n|^2=k} |c_n|^2 \to 0, k \to \infty . \tag{1.5}$$

As it good known, this Cooke‘s result took the part of the last lacking element in solution of the old problem of representation of two variable quantities function by trigonometric series. Unfortunately, Cooke’s proof is essentially based on specific particular qualities of two-dimensional case and for $N > 2$ it could not be principally adapted. We shall demonstrate that if for $N > 2$ the following condition is executed:

$$\sum_{|n|^2=k} {}^{\#}c_n \exp(inx) \approx^{\bullet} 0, k \in {}^{\#}N \setminus N, n \in {}^{\#}Z^N , \tag{1.6}$$

#-a.e. on ${}^{\#}\mathrm{T}^N$ then

$$\sum_{|n|^2=k} \left|{}^{\#}c_n\right|^2 \approx 0 ,$$

and by that (1.5) is true.

II. Lebesgue #-measure.

Definition 2.1. A collection of subsets $\Re$ of a set ${}^{\#}R$ is a #-σ-algebra if

(i) $\varnothing, {}^{\#}R \in \Re$.

(ii) $X \in \Re \Rightarrow {}^{\#}R \setminus X \in \Re$.

(iii) $\{X_i\}_{i=1}^{{}^{\#}\infty} \subset \Re \Rightarrow \bigcup_{i=1}^{{}^{\#}\infty} X_i \in \Re$.

(iv) Let $\Re$ denote the smallest collection of subsets of ${}^{\#}R$ that includes all the open sets and is closed under #-countable (#-countable$\Leftrightarrow{}^{\#}$countable) unions, #-countable intersections and complements. These sets are called the #-Borel sets. In fact the #-Borel sets form a #–σ-algebra: ${}^{\#}R, \varnothing \in \Re$, and $\Re$ is closed under #-countable unions and intersections. We will call #-Borel sets measurable. Many subsets of ${}^{\#}R$ are not #-Borel measurable, but all the ones you can write down or that might arise in a practical setting are #-measurable.

The Lebesgue #-measure on ${}^{\#}R$ is a map $\mu(\circ) : \Re \to {}^{\#}R \cup \{{}^{\#}\infty\}$ with the properties that:

(i) $\mu(\{\varnothing\}) = 0$.
(ii) $\mu([a,b]) = \mu((a,b)) = b - a$.
(iii) $\mu\left(\bigcup_{i=1}^{^{\#}\infty} A_i\right) = \sum_{i=1}^{^{\#}\infty} \mu(A_i), A_k \cap A_l = \varnothing, k \neq^{\bullet} l$.

Notice that the Lebesgue #-measure attaches a #-measure to all #-measurable sets. Sets of #-measure zero are called null sets, and something that happens everywhere except on a set of #-measure zero is said to happen #-almost everywhere, often written simply #-a.e. For technical reasons, allow any subset of a null set to also be regarded as #-measurable, with measure zero. Evidently that $\mu({}^{\#}Q) = 0$, $\mu(R) = 0$, $\mu({}^{*}R) = 0$.

Definition 2.2. A function $f : {}^{\#}R \to {}^{\#}R \cup \{{}^{\#}\pm\infty\}$ is a #-Borel measurable function if $f^{-1}(A) \in \Re$ for every $A \in \Re$.
Definition 2.3.The characteristic function $\chi_E(x)$, defined by $\chi_E(x) = 1$ if $x \in^{\bullet} E$, $\chi_E(x) = 0$ if $x \notin^{\bullet} E$,and $\chi_E(x) = 1$ if$(x \in E) \wedge (x \notin E)$.
Definition 2.4.A simple function is a map $f : {}^{\#}R \to {}^{\#}R$ of the form

$$f(x) = \sum_{i=1}^{n} c_i \chi_{E_i}, n \in {}^{\#}N, \tag{2.1}$$

where the c_i are non-zero constants and the E_i are disjoint #-measurable sets with $\mu(E_i) = 1$.
The integral of the simple function (2.1) is defined to be

$$\int_E f(x) d^{\#}\mu(x) = \sum_{i=1}^{n} c_i \mu(E \cap E_i)$$

for any #-measurable set E.
The basic approximation fact in the Lebesgue integral is the following: $f : {}^{\#}R \to {}^{\#}R \cup \{{}^{\#}\pm\infty\}$ is #-measurable and weak non-negative, then there is an increasing sequence (f_n) of simple functions with the property that $f_n \to^{\#} f$ #-a.e. We write this as $f_n \uparrow^{\#} f$ #-a.e., and define the integral of f to be

$$\int_E f(x) d^{\#}\mu(x) = \lim_{n \to^{\#}\infty} \int_E f_n(x) d^{\#}\mu(x).$$

Notice that (once we allow the value ${}^{\#}\infty$), the limit is guaranteed to exist since the sequence is weak increasing. Let there is a strong limit $s-\lim_{n\to^{\#}\infty}\int_E f_n(x)d^{\#}\mu(x)$, then we shall say that function $f(x)$ integrable in strong sense and also we shall define strong Lebesgue integral $\int_E f(x)d^{\#}_{\bullet}\mu(x)$ by corellation

$$\int_E f(x)d^{\#}_{\bullet}\mu(x) = s-\lim_{n\to^{\#}\infty}\int_E f_n(x)d^{\#}\mu(x).$$

Notice that (once we allow the value ${}^{\#}\infty$), the limit is guaranteed to exist since the sequence is weak increasing. For a general measurable function f, write $f = f^{+} - f^{-}$ where both f^{+} and f^{-} are weak non-negative and #-measurable, then define

$$\int_E f(x)d^{\#}\mu(x) = \int_E f^{+}(x)d^{\#}\mu(x) - \int_E f^{-}(x)d^{\#}\mu(x).$$

If $f(x)={}^{\#}g(x), E\subseteq R$

$$\int_{{}^{\#}E} f(x)d^{\#}\mu(x) = {}^{\#}\left(\int_E g(x)d\mu(x)\right)$$

Example . Let $\Im(x) = \chi_{{}^{*}R\cap^{\#}[0,1]}$. Then $f(x)$ is itself a simple function, so

$$\int_0^1 \Im(x)d^{\#}\mu(x) = \mu\left({}^{*}R\cap^{\#}[0,1]\right) = 0.$$

A #-measurable function f on $[a,b]$ is essentially hyper-bounded if there is a hyper-finite constant $\eta\in{}^{\#}R$ such that $f(x)\leq\eta$ #-a.e. on $[a,b]$. The essential supremum of such a function f is the infimum of all such essential hyper-bounds η, written

$$\|f\|_{\infty}^{\#} \equiv \|f\|_{{}^{\#}\infty} = ess.\sup_{x\in[a,b]}|f(x)|$$

Definition 2.5. Define $\Im_p^{\#}[a,b], p\in{}^{\#}R$ to be the non-standard linear space of #-measurable functions f on $[a,b]$ for which

$$\|f\|_p^{\#} = \left(\int_a^b |f|^p d^{\#}\mu \right)^{\frac{1}{p}} <^{\bullet} {}^{\#}\infty$$

for $p \in [1, {}^{\#}\infty]$ and $\Im_{{}^{\#}\infty}[a,b]$ to be the non-linear space of essentially hyper-bounded functions. Notice that $\|\circ\|_p^{\#}$ on $\Im_p^{\#}$ is only a semi-norm, since many functions will for example have $\|f\|_p^{\#} = 0$. Define an equivalence relation on $\Im_p^{\#}$ by $f \sim g$ if $\{x \in {}^{\#}R \mid f(x) \neq g(x)\}$ is a null set. Then define $L_p^{\#} = \Im_p^{\#} / \sim$ the space of $L_p^{\#}$ functions.
In practice we will not think of elements of $L_p^{\#}$ as equivalence classes of functions, but as functions defined #-a.e. A similar definition may be made of p-integrable functions on ${}^{\#}R$, giving the linear space $L_p^{\#}({}^{\#}R)$.
Theorem 2.1.The normed spaces $L_p^{\#}[a,b]$ and $L_p^{\#}({}^{\#}R)$ are (#-separable) non-standard Banach spaces under the norm $\|\circ\|_p^{\#}$.
Theorem 2.2. If $r^{-1} = q^{-1} + p^{-1}$, then

$$\|f \cdot g\|_r^{\#} \leq \|f\|_p^{\#} \cdot \|g\|_q^{\#}$$

for any $f \in L_p^{\#}, g \in L_q^{\#}$. It follows that for any #-measurable f on $[a,b]$,

$$\|f\|_1^{\#} \leq \|f\|_2^{\#} \leq \ldots \leq \|f\|_k^{\#} \leq \ldots \leq \|f\|_{{}^{\#}\infty}.$$

Hence

$$L_1^{\#}[a,b] \supset L_2^{\#}[a,b] \supset \ldots \supset L_k^{\#}[a,b] \supset \ldots \supset L_{{}^{\#}\infty}[a,b].$$

In the theorem we allow p and q to be anything in $[1, {}^{\#}\infty]$ with the obvious interpretation of $1/{}^{\#}\infty$.
Note the “opposite" behaviour to the non-standard sequence spaces $l_p^{\#} \supset {}^{\#}l_p$, $p \in {}^{\#}N$, where we evidently saw that $l_1^{\#} \subset l_2^{\#} \subset \ldots \subset l_{{}^{\#}\infty}$.
Two easy consequences of Holder's inequality are the Cauchy-Schwartz inequality,

$$\|f \cdot g\|_1^\# \le \|f\|_2^\# \cdot \|g\|_2^\#,$$

and Minkowski's inequality,

$$\|f + g\|_p^\# \le \|f\|_p^\# + \|g\|_p^\#.$$

Theorem 2.3. Let $(f_n(x))$ be a sequence of #-measurable functions on a #-measurable set E such that $f_n(x) \to f(x)$ #-a.e. and there exists an integrable function $g(x)$ such that $|f(x)| \le g(x)$ #-a.e. Then

$$\int_E f(x) d^\# \mu(x) = \lim_{n \to^\# \infty} \int_E f_n(x) d^\# \mu(x)$$

Let X and Y be two subsets of ${}^\# R$. Let A, B denote the $\#\text{–}\sigma$-algebra of #-Borel sets in X and Y respectively. Subsets of $X \times Y$ of the form $A_1 \times B_1 = \{(x, y) \mid x \in A_1, y \in B_2\}$ with $A_1 \in A, B_1 \in B$ are called rectangles. Let $A \times B$ denote the smallest $\#\text{–}\sigma$-algebra on $X \times Y$ containing all the #-measurable rectangles. Notice that, despite the notation, this is much larger than the set of all #-measurable rectangles.
The #-measure space $(X \times Y, A \times B)$ is the Cartesian product of (X, A) and (Y, B).
Let μ_X and μ_Y denote Lebesgue #-measure on X and Y. Then there is a unique measure $\lambda_{X \times Y}$ on $X \times Y$ with the property that $\lambda_{X \times Y}(A \times B) = \mu_X(A) \cdot \mu_Y(B)$ for all #-measurable rectangles $A \times B$. This #-measure is called the product #-measure of μ_X and μ_Y and we write $\lambda_{X \times Y} = \mu_X \times \mu_Y$. The most important result on product #-measures is Fubini's theorem.
Theorem 2.4. If $h(x, y)$ is an integrable function on $X \times Y$, then $x \to h(x, y)$ is an integrable function of X for #-a.e. y, $y \to h(x, y)$ is an integrable function of y for #-a.e. y, and

$$\int_{X \times Y} h(x, y) d^\# \lambda_{X \times Y}(x, y) = \int_X \int_Y h(x, y) d^\# \mu_X(x) d^\# \mu_Y(y).$$

III. Short proof for L. Carleson's celebrated theorem and general criteria for LC-property.

Definition 3.1. Let φ - is some statement about sets which is true in ZFC. We shall say that characteristic φ is kept by #-mapping in strong sense, if statement ${}^\# \varphi$ is not paralogical. Let φ is some statement about sets, which is true in ZFC. We shall say

that characteristic φ is kept by #-mapping in strong sense, if statement ${}^{\#}\varphi$ is not paralogical.
Most characteristics of classical objects is not kept by #-mapping in strong point, i.e. becomes paralogical characteristics. Particularly, the orthogonality characteristic for the sequence of functions $\{f_n\}_{n=1}^{\infty} \in L_2$ is kept by #-mapping in strong sense only for the special functions sequences, for instance, for $\{\exp(inx\}_{n=1}^{\infty}$ and is not kept in a general case.

Definition 3.2.Lets suppose that there are general orthogonal series in space $L_2(\Omega), \Omega \subseteq R^d, d = 1,2...$

$$\sum_{n=0}^{\infty} c_n f_n(x), (c_n) \in l_2, f_n \in L_2(\Omega). \tag{3.1}$$

We shall say that the general orthogonal series (3.1) (or sequence $\{f(x)_i\}_{i=1}^{\infty}$) in space $L_2(\Omega)$ is strong orthogonal series (sequence), if the following condition is executed:

$$^{\#}\left(\int {}^{\#} f_i(x) {}^{\#} f_j(x) d\, {}^{\#}\mu(x)\right) =^{\bullet} \delta_{ij}. \tag{3.2}$$

Where $\delta_{ij} =^{\bullet} 1$, if $i = j$, and $\delta_{ij} =^{\bullet} 0$, if $i \neq^{\bullet} j$ $i, j \in {}^{\#}N$. In other words it is said here that orthogonality persists in strong (non-paralogical) sense after using of #-mapping.
Otherwise we shall say that sequence (orthogonal series 3.1) $\{f_i(x)\}_{i=1}^{\infty}$ is #-paraorthogonal. For example, Haar's and Redemacher's are paraorthogonal.

Lemma 3.1. Sequence $\{\exp(inx\}_{n=1}^{\infty}$ is strong orthogonal.

Proof. $\int\limits_{-^{\#}\pi}^{^{\#}\pi} {}^{\#}\exp(imx) {}^{\#}\exp(inx) d\, {}^{\#}x = \int\limits_{-^{\#}\pi}^{^{\#}\pi} {}^{\#}\exp(imx) {}^{\#}\exp(inx) d_{\bullet}^{\#} x = \begin{cases} 0, m \neq^{\bullet} n, \\ 2^{\#}\pi, m =^{\bullet} n. \end{cases}$

Lemma 3.2. Sequence $\{\exp(inx\}, n \in Z^d, x \in \mathrm{T}^d$ is strong orthogonal.
Theorem 3.1. Let $\sum_{n=1}^{\infty} c_n^2 < \infty$ and $\sum_{n=1}^{^{\#}\infty} {}^{\#}c_n^2 <^{\bullet} \infty$. Then trigonometric series $\sum_{n=1}^{\infty} c_n \exp(inx)$ could not be non-converted on the set of positive measure. Then trigonometric series $\sum_{n=1}^{\infty} c_n \exp(inx)$ could not be non-converted on the set of positive measure

Proof. Let function $\psi(x)=\sum_{n=1}^{\infty} c_n \exp(inx)$ is equal ∞ on set $E, mes(E)>0$. Then such a hyper-ultimate $M \in {}^{\#}N$ will be found that function $\Theta(x)=\Psi(x)\overline{\Psi}(x)$ $\Psi(x)=\sum_{n=1}^{M} {}^{\#}c_n {}^{\#}\exp(inx)$ receive infinitely large positive meanings on set ${}^{\#}E \subset [-{}^{\#}\pi, \pi], {}^{\#}mes({}^{\#}E)>0$. But then $\eta=\sum_{n=1}^{M} {}^{\#}c_n^2 = \int_{-{}^{\#}\pi}^{{}^{\#}\pi} \Theta(x) d_{\bullet}^{\#}x$ is infinitely large positive number that contradicts to the theorem conditions.

Theorem 3.2. Let $\sum_{n=1}^{\infty} c_n^2 < \infty$ and $\sum_{n=1}^{{}^{\#}\infty} {}^{\#}c_n^2 <^{\bullet} \infty$. Then trigonometric series $\sum_{n=1}^{\infty} c_n \exp(inx)$ converts a.e. The proof of this theorem is executed analogous to theorem 3.1.

Theorem 3.3. Let $\sum_{n=1}^{\infty} c_n^2 < \infty$ and $\sum_{n=1}^{{}^{\#}\infty} {}^{\#}c_n^2 <^{\bullet} \infty$. Then paraorthogonal of orbicular amounts $E_M[f(x)]=(2\pi)^N \sum_{|n|^2 \le M} f_n \exp(inx), n \in Z^N$ converts a.e.

By the proof of theorem 3.1-3.2 only a characteristic of the strong orthogonality of trigonometric system was used. Like that the following theorem is true.

Theorem 3.4. (General result) Let $\sum_{n=1}^{\infty} c_n^2 < \infty$ and $\sum_{n=1}^{{}^{\#}\infty} {}^{\#}c_n^2 <^{\bullet} \infty$. Then the general strong orthogonal series

$$\sum_{n=0}^{\infty} c_n f_n(x), f_n \in L_2(\Omega), \Omega \subseteq R^d$$

converts a.e.

Lets remind a definition of some special orthogonal systems of functions. Jacoby's polynomes $p_n(x;\alpha,\beta)$ are determined by the following formula [7]:

$$p_n^{(\alpha,\beta)}(x)=\sqrt{\frac{2n+\alpha+\beta+1}{2^{\alpha+\beta+1}}}\sqrt{\frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}} J_n^{(\alpha,\beta)}(x), n=1,2\ldots, \tag{3.3}$$

where function $J_n^{(\alpha,\beta)}(x)$ is determined from equation

$$(1-x)^{\alpha}(1+x)^{\beta} J_n^{(\alpha,\beta)}(x)=\frac{(-1)^n}{n!2^n}\frac{d^n}{dx^n}[(1-x)^{\alpha+n}(1+x)^{\beta+n}]. \tag{3.4}$$

Sequence $\{p_n^{(\alpha,\beta)}(x)\}_{n=1}^{\infty}$ is orthogonal on $[-1,1]$ with weight $(1-x)^{\alpha}(1+x)^{\beta}$. Particular cases $\alpha=\beta=0$ and $\alpha=\beta=-1/2$ correspond to Legandre and Chebyshev's polynomes.

Lemma 3.3. Sequence $\{p_n^{(\alpha,\beta)}(x)\}_{n=1}^{\infty}$ is strong orthogonal on $[-1,1]$ с весом $(1-x)^{\alpha}(1+x)^{\beta}$.

Proof. Using #-mapping to equality (3.3) we shall obtain

$$ {}^{\#}p_n^{(\alpha,\beta)}(x) = {}^{\#}\sqrt{\frac{2n+\alpha+\beta+1}{2^{\alpha+\beta+1}}}\ {}^{\#}\sqrt{\frac{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}}\ {}^{\#}J_n^{(\alpha,\beta)}(x), n\in{}^{\#}N, \tag{3.5}$$

where function ${}^{\#}J_n^{(\alpha,\beta)}$ responds to equation

$$ {}^{\#}(1-x)^{\alpha}\,{}^{\#}(1+x)^{\beta}\,{}^{\#}J_n^{(\alpha,\beta)}(x) =_{\bullet} \frac{(-1)^n}{n!2^n}\frac{d_{\bullet}^{\#n}}{d_{\bullet}^{\#}x^n}[{}^{\#}(1-x)^{\alpha+n}\,{}^{\#}(1+x)^{\beta+n}], \tag{3.6}$$

and occupies a place of the polynome of degree $n\in{}^{\#}N$. We have to demonstrate that function ${}^{\#}(1-x)^{\alpha}\,{}^{\#}(1+x)^{\beta}\,{}^{\#}J_n^{(\alpha,\beta)}(x)$ is strong orthogonal to arbitral polynome $P(x)$ of degree $r\leq_{\bullet} n-1$. Let $R_{\bullet}^{\#(k)}(x)=\left(\frac{d_{\bullet}^{\#}}{d_{\bullet}^{\#}x}\right)^k R(x)$, where $R(x)={}^{\#}(1-x)^{\alpha+n}\,{}^{\#}(1+x)^{\beta+n}$ - polynome of degree $n\in{}^{\#}N$. We have to demonstrate that function ${}^{\#}(1-x)^{\alpha}\,{}^{\#}(1+x)^{\beta}\,{}^{\#}J_n^{(\alpha,\beta)}(x)$ is strong orthogonal to arbitral polynome $P(x)$ of degree $r\leq_{\bullet} n-1$. Let $R_{\bullet}^{\#(k)}(x)=\left(\frac{d_{\bullet}^{\#}}{d_{\bullet}^{\#}x}\right)^k R(x)$, where $R(x)={}^{\#}(1-x)^{\alpha+n}\,{}^{\#}(1+x)^{\beta+n}$.

Elementary equality:

$$\begin{aligned}(-1)^n n!2^n \int_{-1}^{1} {}^{\#}(1-x)^{\alpha}\,{}^{\#}(1+x)^{\beta}\,{}^{\#}J_n^{(\alpha,\beta)}(x)P(x)d_{\bullet}^{\#}x &=_{\bullet} \int_{-1}^{1} R_{\bullet}^{\#(n)}(x)P(x)d_{\bullet}^{\#}x =_{\bullet} \\ =_{\bullet} \left[R_{\bullet}^{\#(n-1)}(x)P(x) - R_{\bullet}^{\#(n-2)}(x)P_{\bullet}^{\#/}(x)+\ldots+(-1)^{n-1}R(x)P_{\bullet}^{\#(n-1)}(x)\right]_{-1}^{+1} &+ \\ +(-1)^n\int_{-1}^{1} R(x)P_{\bullet}^{\#(n)}(x)d_{\bullet}^{\#}x,\end{aligned} \tag{3.7}$$

is true for $0\leq k\leq_{\bullet} n-1, R_{\bullet}^{\#(k)}(-1)=_{\bullet} R_{\bullet}^{\#(k)}(1)=_{\bullet} 0$, from where, taking into account the (3.7), we obtain the required statement.

Let

$$ T_0(x) = p_0^{(-0.5,-0.5)} \equiv \sqrt{\pi^{-1}},\ p_n^{(-0.5,-0.5)}(x) = \sqrt{\frac{2}{\pi}}\cos(n\arccos x), n=1,2,\ldots \tag{3.8}$$

Sequence (3.8) is called Chebyshev's polynomes system. As it known, Chebyshev's polynomes are orthogonal on $[-1,1]$ with weight $(1-x^2)^{-\frac{1}{2}}$. By virtue of lemma (3.2) Chebyshev's polynomes are strong orthogonal on $[-1,1]$ with weight $(1-x^2)^{-\frac{1}{2}}$. This fact is easy to prove directly too. For $m \neq^{\bullet} n$ we obviously have

$$\int_{-1}^{1} {}^{\#}(1-x^2)^{-\frac{1}{2}} [{}^{\#}\cos(m({}^{\#}\arccos x))][{}^{\#}\cos(n({}^{\#}\arccos x))] d_{\bullet}^{\#} x =^{\bullet} \int_{0}^{{}^{\#}\pi} ({}^{\#}\cos m\theta)({}^{\#}\cos n\theta) d_{\bullet}^{\#}\theta =^{\bullet} 0,$$

if $m =^{\bullet} n$, then $\int_{-1}^{1} {}^{\#}(1-x^2)^{-\frac{1}{2}} [{}^{\#}\cos(n({}^{\#}\arccos x))]^2 d_{\bullet}^{\#} x =^{\bullet} \int_{0}^{{}^{\#}\pi} ({}^{\#}\cos n\theta)^2 d_{\bullet}^{\#}\theta =^{\bullet} \begin{cases} {}^{\#}\pi/2, n \geq^{\bullet} 1, \\ {}^{\#}\pi, n =^{\bullet} 0. \end{cases}$

Theorem 3.5. Let $\sum_{n=1}^{\infty} c_n^2 < \infty$ and $\sum_{n=1}^{{}^{\#}\infty} {}^{\#}c_n^2 <^{\bullet} \infty$. Then series $\sum_{n=1}^{\infty} c_n p_n^{(\alpha,\beta)}$ converts a.e.

Strong orthogonality is not a sufficient condition for the presence of LC-property in orthogonal series (3.1) because #-paraorthogonal functions systems exist which, nevertheless, possess by LC-property. The simplest example of such a system is Haar's system. Haar's system is determined on interval $[0,1]$ by the following way [7]: $\chi_0^{(0)}(x) \equiv 1$,

$$\chi_0^{(1)}(x) = \begin{cases} 1, x \in [0,1/2), \\ 0, x = 1/2, \\ -1, x \in (1/2,1]. \end{cases} \qquad \chi_m^{(k)}(x) = \begin{cases} \sqrt{2^m}, x \in \left((k-1)/2^m, (k-1/2)/2^m\right), \\ -\sqrt{2^m}, x \in \left((k-1/2)/2^m, k/2^m\right), \\ 0, x \in \left((l-1)/2^m, l/2^m\right), l \neq k, 1 \leq l \leq 2^m, \end{cases} \tag{3.9}$$

in the point of fracture the function $\chi_m^{(k)}(x)$ is defined as arithmetical mean of its meanings which the function receives in intervals close to this point; in point 0 function $\chi_m^{(k)}(x)$ receives the same meaning that it received in interval $(0,1/2^{m+1})$; in point 1 function $\chi_m^{(k)}(x)$ receives the same meaning that it received in interval $(1-1/2^{m+1},1)$. Haar's system, as it known, is orthogonal

$$\int_{0}^{1} \chi_m^{(j)}(x)\chi_n^{(i)}(x)dx = 0, m \neq n, \tag{3.10}$$

and Haar's series of arbitral L- integrated function possesses by LC-property. At the same time non-standard function ${}^{\#}\chi_m^{(j)}(x), m, j \in {}^{\#}N, (m>0) \wedge (j>0)$, as it easily seen, is not L-#-integrated on $[0,1]$ in strong sense and, moreover,

$$\int_{0}^{1} {}^{\#}\chi_m^{(j)}(x) {}^{\#}\chi_n^{(i)}(x) d^{\#}x \neq^{\bullet} 0, m \neq^{\bullet} n,$$

and instead the condition of strong orthogonality (3.2) considerably more weak condition is executed

$$\int\limits_{E_1(m,k)} {}^{\#}\chi_m^{(j)}(x)\,{}^{\#}\chi_n^{(i)}(x)d^{\#}x + \int\limits_{E_2(m,k)} {}^{\#}\chi_m^{(j)}(x)\,{}^{\#}\chi_n^{(i)}(x)d^{\#}x + \sum_{l=1}^{2^m} \int\limits_{E_3(m,k,l)} {}^{\#}\chi_m^{(j)}(x)\,{}^{\#}\chi_n^{(i)}(x)d^{\#}x =^{\bullet} 0,$$

$$E_1(m,k) = \left(\frac{k-1}{2^m}, \frac{k-1/2}{2^m}\right)_{\bullet}, E_2(m,k) = \left(\frac{k-1/2}{2^m}, \frac{k}{2^m}\right)_{\bullet}, E_3(m,k,l) = \left(\frac{l-1}{2^m}, \frac{l}{2^m}\right)_{\bullet}.$$

A cause of this fact is obvious and concerned with the circumstance, that logical conditions of the type (3.9) after using of #-mapping become paralogical statements in $ZFC^{\#}$. For example, formula $(x \in [0,1/2)) \vee (x = 1/2) \vee (x \in (1/2,1])$ of ZFC theory, after using of #-mapping, transfers in formula ${}^{\#}(x \in [0,1/2)) \vee^{\#} (x = 1/2) \vee^{\#} (x \in (1/2,1])$ of $ZFC^{\#}$ theory. But such $\xi \in^{\#} R$ exist, for which statement ${}^{\#}(\xi \in [0,1/2)) \wedge^{\#} (\xi = 1/2) \wedge^{\#} (\xi \in (1/2,1])$ is true in $ZFC^{\#}$; as a result function ${}^{\#}\chi_0^{(1)}(x)$ is $3!$- dimensional in a standard sense on the set of the positive #-measure.

By analogous way function ${}^{\#}\chi_m^{(j)}(x)$ is $2^{m+1}!$-dimensional in a standard sense on the set of the positive #-measure.

According to the fundamental D. Menshov's theorem [8], for arbitral sequence $\{c_n\}_{n=1}^{\infty}$ responding condition $\sum_{n=1}^{\infty} c_n < \infty$ there is orthogonal series $\mathrm{M}(x) \sim \sum_{n=1}^{\infty} c_n \mathrm{M}_n(x)$ which disperses everywhere on $(0,1)$. In virtue of the main theorem, Menshov's series $\mathrm{M}(x)$ could not be orthogonal in strong sense, but it is paraorthogonal. This fact is possible to prove directly too, after considering of the concrete structure of the sequence of non-standard functions $\{{}^{\#}\mathrm{M}_n\}_{n=1}^{K}, K \in^{\#} N \setminus N$.

Theorem 3.6. Let

$$\sum_{|n|^2 = m} f_n \exp(inx) \to 0, n \in Z^N, m \to \infty, \text{ a.e.,} \tag{3.11}$$

let also characteristic (3.11) is kept by #-mapping, i.e.

$$\sum_{|n|^2 = M} {}^{\#}f_n \, {}^{\#}\exp(inx) \to^{\bullet} 0, n \in^{\#} Z^N, M \to^{\#} \infty, \text{ \#-a.e.,} \tag{3.12}$$

then $\sum_{|n|^2 = m} | f_n |^2 \to 0, m \to \infty$.

Proof. From condition (3.11) it follows that $\sum_{|n|^2=M}{}^{\#} f_n \exp(inx) \approx 0 \ \forall M, M \in {}^{\#}N \setminus N$, #-a.e. , from where, taking into account the condition (3.12), we have

$$\sum_{|n|^2=M} f_n \exp(inx) \approx^{\bullet} 0 \ \forall M, M \in {}^{\#}N \setminus N, \text{ \#-a.e.} \qquad (3.13)$$

Taking into account the condition (3.13), as a result of elementary calculations, we obtain

$$\sum_{|n|^2=M} | f_n |^2 \approx^{\bullet} 0, \forall M, M \in {}^{\#}N \setminus N,$$

from where the theorem confirmation follows.

References

[1] Lennart Carleson, On convergence and growth of partial sumas of Fourier series, Acta Math. 116 (1966), 135–157. 33 #7774

[2] Attila M´at´e, Convergence of Fourier series of square integrable functions, Mat. Lapok 18 (1967), 195–242. MR 39 #701 (Hungarian, with English summary)

[3] P. Auscher and M. J. Carro, On relations between operators on RN, TN and ZN, Studia Math. 101 (1992), 165–182. MR 94b:42007.

[4] Hakan Hedenmalm and Eero Saksman, **CARLESON'S CONVERGENCE THEOREM FOR DIRICHLET SERIES,** Pacific Journal of Mathematics Vol. 208, No. 1, 2003.

[5] Michael T. Lacey, Georgia Institute of Technology, Carleson's Theorem: Proof, Complements, Variations, arXiv:math.CA/0307008 v2 22 Oct 2003.

[6] Cooke R.A. Cantor-Lebesque theorem in two dimensions, Proc. Amer. Math. Soc.-1971.Vol.30,p.547-550.

[7] Alexits G. Convergence problems of orthogonal series, Budapest, 1961.

[8] Menshov D.E. Sur les series de function ortogonales. Fund. Math,

1923,vol.4,p.82-105