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Generalized Löb's Theorem. Strong Reflection Principles and Large Cardinal Axioms

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ABSTRACT

In this article, a possible generalization of the Löb's theorem is considered. Main result is: let κ be an inaccessible cardinal, then $\neg\mathrm{Con}\left(ZFC+\left(V=H_\kappa\right)\right)$.

Keywords: Löb's Theorem; Second Godel Theorem; Consistency; Formal System; Uniform Reflection Principles; ω-Model of ZFC; Standard Model of ZFC; Inaccessible Cardinal

1. Introduction

Let Th be some fixed, but unspecified, consistent formal theory.

Theorem 1 [1]. (Löb's Theorem**).**

If $Th\vdash\exists x\,\mathrm{Prov}_{Th}\left(x,\breve{n}\right)\to\phi_n$ where x is the Gödel number of the proof of the formula with Gödel number n, and $\breve{n}$ is the numeral of the Gödel number of the formula φ_n, then $Th\vdash\phi_n$. Taking into account the second Gödel theorem it is easy to be able to prove $\exists x\,\mathrm{Prov}_{Th}\left(x,\breve{n}\right)\to\varphi_n$, for disprovable (refutable) and undecidable formulas φ_n. Thus summarized, Löb's theorem says that for refutable or undecidable formula φ, the intuition "if exists proof of φ then φ" is fails.

Definition 1. Let M_ω^{Th} be an ω-model of the *Th*. We said that, $Th^\#$ is a nice theory over *Th* or a nice extension of the *Th* iff:

1) $Th^\#$ contains *Th*;

2) Let Φ be any closed formula, then

$$\left[Th\vdash\mathrm{Pr}_{Th}\left(\left[\Phi\right]^c\right)\right]\&\left[M_\omega^{Th}\vDash\Phi\right]$$

implies $Th^\#\vdash\Phi$.

Definition 2. We said that, $Th^\#$ is a maximally nice theory over *Th* or a maximally nice extension of the *Th* iff $Th^\#$ is consistent and for any consistent nice extension Th' of the *Th*: $\mathrm{Ded}\left(Th^\#\right)\subseteq\mathrm{Ded}\left(Th'\right)$ implies $\mathrm{Ded}\left(Th^\#\right)=\mathrm{Ded}\left(Th'\right)$.

Theorem 2. (Generalized Löb's Theorem). Assume that 1) Con(*Th*) and 2) *Th* has an ω-model M_ω^{Th}. Then theory *Th* can be extended to a maximally consis- tent nice theory $Th^\#$.

2. Preliminaries

Let *Th* be some fixed, but unspecified, consistent formal theory. For later convenience, we assume that the encoding is done in some fixed formal theory *S* and that *Th* contains *S*. We do not specify *S*—it is usually taken to be a formal system of arithmetic, although a weak set theory is often more convenient. The sense in which *S* is contained in *Th* is better exemplified than explained: If *S* is a formal system of arithmetic and *Th* is, say, *ZFC*, then *Th* contains *S* in the sense that there is a well-known embedding, or interpretation, of *S* in *Th*. Since encoding is to take place in *S*, it will have to have a large supply of constants and closed terms to be used as codes. (e.g. in formal arithmetic, one has $\overline{0},\overline{1},\cdots$) *S* will also have certain function symbols to be described shortly. To each formula, Φ, of the language of *Th* is assigned a closed term, $\left[\Phi\right]^c$, called the code of Φ. [N. B. If $\Phi(x)$ is a formula with a free variable *x*, then $\left[\Phi(x)\right]^c$ is a closed term encoding the formula $\Phi(x)$ with *x* viewed as a syntactic object and not as a parameter.] Corresponding to the logical connectives and quantifiers are function symbols, $\mathrm{neg}(\cdot),\mathrm{imp}(\cdot)$, etc., such that, for all formulae $\Phi,\Psi: S|-\mathrm{neg}\left(\left[\Phi\right]^c\right)=\left[\neg\Phi\right]^c, S|-\mathrm{imp}\left(\left[\Phi\right]^c,\left[\Psi\right]^c\right)=\left[\Phi\to\Psi\right]^c$ etc.

Of particular importance is the substitution operator, represented by the function symbol $\mathrm{sub}(\cdot,\cdot)$. For formulae $\Phi(x)$, terms *t* with codes $[t]^c$:

$$S|-\mathrm{sub}\left(\left[\Phi(x)\right]^c,[t]^c\right)=\left[\Phi(t)\right]^c. \quad (2.1)$$

Iteration of the substitution operator *sub* allows one to define function symbols $\mathrm{sub}_3,\mathrm{sub}_4,\cdots,\mathrm{sub}_n$ such that

$$S|-\mathrm{sub}_n\left(\left[\Phi(x_1,x_2,\cdots,x_n)\right]^c,[t_1]^c,[t_2]^c,\cdots,[t_n]^c\right) \\ =\left[\Phi(t_1,t_2,\cdots,t_n)\right]^c \quad (2.2)$$

It well known [2,3] that one can also encode derivations and have a binary relation $\mathrm{Prov}_{Th}(x,y)$ (read "*x* proves *y*" or "*x* is a proof of *y*") such that for closed $t_1,t_2: S|-\mathrm{Prov}_{Th}(t_1,t_2)$ iff t_1 is the code of a derivation in *Th* of the formula with code t_2. It follows that

$$Th\vdash\Phi\leftrightarrow S\vdash\mathrm{Prov}_{Th}\left(t,[\Phi]^c\right) \quad (2.3)$$

for some closed term *t*. Thus one can define predicate $\mathrm{Pr}_{Th}(y)$:

$$\mathrm{Pr}_{Th}(y)\leftrightarrow\exists x\,\mathrm{Prov}_{Th}(x,y), \quad (2.4)$$

and therefore one obtain a predicate asserting provability.

Remark 2.1. We note that is not always the case that [2,3]:

$$Th\vdash\Phi i\leftrightarrow S\vdash\mathrm{Pr}_{Th}\left([\Phi]^c\right). \quad (2.5)$$

It well known [3] that the above encoding can be carried out in such a way that the following important conditions $D1, D2$ and $D3$ are met for all sentences [2,3]:

$$D1.\ Th\vdash\Phi \text{ implies } S\vdash\mathrm{Pr}_{Th}\left([\Phi]^c\right),$$
$$D2.\ S\vdash\mathrm{Pr}_{Th}\left([\Phi]^c\right)\to\mathrm{Pr}_{Th}\left(\left[\mathrm{Pr}_{Th}\left([\Phi]^c\right)\right]^c\right), \quad (2.6)$$
$$D3.\ S\vdash\mathrm{Pr}_{Th}\left([\Phi]^c\right)\wedge\mathrm{Pr}_{Th}\left([\Phi\to\Psi]^c\right) \\ \to\mathrm{Pr}_{Th}\left([\Psi]^c\right).$$

Conditions $D1, D2$ and $D3$ are called the Derivability Conditions.

Assumption 2.1. We assume now that:

1) the language of *Th* consists of:
numerals $\overline{0},\overline{1},\cdots$
countable set of the numerical variables: $\{v_0,v_1,\cdots\}$
countable set F of the set variables:
$F=\{x,y,z,X,Y,Z,\Re,\cdots\}$
countable set of the *n*-ary function symbols: $f_0^n,f_1^n,\cdots$
countable set of the *n*-ary relation symbols: $R_0^n,R_1^n,\cdots$
connectives: $\neg,\to$
quantifier: $\forall$.

2) *Th* contains

$$Th^*\triangleq ZFC+\exists(\omega-\text{model of } ZFC)$$

3) *Th* has an ω-model M_ω^{Th}.

Theorem 2.1. (Löb's Theorem). Let be 1) $\mathrm{Con}(Th)$ and 2) ϕ be closed. Then

$$Th\vdash\mathrm{Pr}_{Th}\left([\phi]^c\right)\to\phi \text{ iff } Th\vdash\phi. \quad (2.7)$$

It well known that replacing the induction scheme in Peano arithmetic ***PA*** by the ω-rule with the meaning "if the formula $A(n)$ is provable for all *n*, then the formula $A(x)$ is provable":

$$\frac{A(0),A(1),\cdots,A(n),\cdots}{\forall xA(x)}, \quad (2.8)$$

leads to complete and sound system PA_∞ where each true arithmetical statement is provable. S. Feferman showed that an equivalent formal system $Th^\#$ can be obtained by erecting on $Th=PA$ a transfinite progression of formal systems PA_∞ according to the following scheme

$$PA_0=PA$$
$$PA_{\alpha+1}=PA_\alpha+\left\{\forall x\,\mathrm{Pr}_{PA_\alpha}\left(\left[A(\dot{x})\right]^c\right)\to\forall xA(x)\right\}, \quad (2.9)$$
$$PA_\lambda=\bigcup_{\alpha<\lambda}PA_\alpha$$

where $A(x)$ is a formula with one free variable and λ is a limit ordinal. Then $Th=\bigcup_{\alpha\in O}PA_\alpha, O$ being Kleene's system of ordinal notations, is equivalent to $Th^\#=PA_\infty$. It is easy to see that $Th^\#=PA^\#$, *i.e.* $Th^\#$ is a maximally nice extension of the ***PA***.

3. Generalized Löb's Theorem

Definition 3.1. An $Th-\mathrm{wff}\,\Phi$ (well-formed formula Φ) is closed *i.e.*, Φ is a *Th*-sentence iff it has no free variables; a wff Ψ is open if it has free variables. We'll use the slang "*k*-place open wff" to mean a wff with *k* distinct free variables. Given a model M^{Th} of the ***Th*** and a ***Th***-sentence Φ, we assume known the meaning of $M\vDash\Phi$—*i.e.* Φ is true in M^{Th}, (see for example [4-6]).

Definition 3.2. Let M_ω^{Th} be an ω-model of the *Th*. We said that, $Th^\#$ is a nice theory over *Th* or a nice extension of the *Th* iff:

1) $Th^\#$ contains *Th*;

2) Let Φ be any closed formula, then

$$\left[Th\vdash\mathrm{Pr}_{Th}\left([\Phi]^c\right)\right]\&\left[M_\omega^{Th}\vDash\Phi\right]$$

implies $Th^\#\vdash\Phi$.

Definition 3.3. We said that $Th^\#$ is a maximally nice theory over *Th* or a maximally nice extension of the *Th* iff $Th^\#$ is consistent and for any consistent nice extension Th' of the Th: $\mathrm{Ded}(Th^{\#}) \subseteq \mathrm{Ded}(Th')$ implies $\mathrm{Ded}(Th^{\#}) = \mathrm{Ded}(Th')$.

Lemma 3.1. Assume that: 1) $\mathrm{Con}(Th)$; and 2) $Th \vdash \mathrm{Pr}_{Th}\left([\Phi]^c\right)$, where Φ is a closed formula. Then $Th \nvdash \mathrm{Pr}_{Th}\left([\neg\Phi]^c\right)$.

Proof. Let $\mathrm{Con}_{Th}(\Phi)$ be the formula

$$\begin{aligned}&\mathrm{Con}_{Th}(\Phi)\\&\triangleq \forall t_1 \forall t_2 \neg\left[\mathrm{Prov}_{Th}\left(t_1,[\Phi]^c\right) \wedge \mathrm{Prov}_{Th}\left(t_2,\mathrm{neg}\left([\Phi]^c\right)\right)\right]\\&\leftrightarrow \neg\exists t_1 \neg\exists t_2\left[\mathrm{Prov}_{Th}\left(t_1,[\Phi]^c\right) \wedge \mathrm{Prov}_{Th}\left(t_2,\mathrm{neg}\left([\Phi]^c\right)\right)\right]\end{aligned} \quad (3.1)$$

where t_1, t_2 is a closed term. We note that under canonical observation, one obtain $Th + \mathrm{Con}(Th) \vdash \mathrm{Con}_{Th}(\Phi)$ for any closed wff Φ. Suppose that $Th \vdash \mathrm{Pr}_{Th}\left([\neg\Phi]^c\right)$, then assumption (*ii*) gives

$$Th \vdash \mathrm{Pr}_{Th}\left([\Phi]^c\right) \wedge \mathrm{Pr}_{Th}\left([\neg\Phi]^c\right). \quad (3.2)$$

From (3.1) and (3.2) one obtain

$$\exists t_1 \exists t_2\left[\mathrm{Prov}_{Th}\left(t_1,[\Phi]^c\right) \wedge \mathrm{Prov}_{Th}\left(t_2,\mathrm{neg}\left([\Phi]^c\right)\right)\right]. \quad (3.3)$$

But the Formula (3.3) contradicts the Formula (3.1). Therefore: $Th \nvdash \mathrm{Pr}_{Th}\left([\neg\Phi]^c\right)$.

Lemma 3.2. Assume that: 1) $\mathrm{Con}(Th)$; and 2) $Th \vdash \mathrm{Pr}_{Th}\left([\neg\Phi]^c\right)$, where Φ is a closed formula. Then $Th \nvdash \mathrm{Pr}_{Th}\left([\Phi]^c\right)$.

Theorem 3.1. [7,8]. (Generalized Löb's Theorem). Assume that: $\mathrm{Con}(Th)$. Then theory *Th* can be extended to a maximally consistent nice theory $Th^{\#}$ over *Th*.

Proof. Let $\Phi_1 \cdots \Phi_i \cdots$ be an enumeration of all wff's of the theory ***Th*** (this can be achieved if the set of propositional variables can be enumerated). Define a chain $\wp = \{Th_i \mid i \in \mathbb{N}\}, Th_1 = Th$ of consistent theories inductively as follows: assume that theory Th_i is defined.

1) Suppose that a statement (3.4) is satisfied

$$\begin{gathered}Th \vdash \mathrm{Pr}_{Th}\left([\Phi_i]^c\right) \text{ and}\\ [Th_i \nvdash \Phi_i] \& \left[M_\omega^{Th} \vDash \Phi_i\right]\end{gathered}. \quad (3.4)$$

Then we define theory Th_{i+1} as follows

$$Th_{i+1} \triangleq Th_i \cup \{\Phi_i\}.$$

2) Suppose that a statement (3.5) is satisfied

$$\begin{gathered}Th \vdash \mathrm{Pr}_{Th}\left([\neg\Phi_i]^c\right) \text{ and}\\ [Th_i \nvdash \neg\Phi_i] \& \left[M_\omega^{Th} \vDash \neg\Phi_i\right]\end{gathered}. \quad (3.5)$$

Then we define theory Th_{i+1} as follows:

$$Th_{i+1} \triangleq Th_i \cup \{\neg\Phi_i\}.$$

3) Suppose that a statement (3.6) is satisfied

$$Th \vdash \mathrm{Pr}_{Th}\left([\Phi_i]^c\right) \text{ and } Th_i \vdash \Phi_i. \quad (3.6)$$

Then we define theory Th_{i+1} as follows:

$$Th_{i+1} \triangleq Th_i \cup \{\Phi_i\}.$$

4) Suppose that a statement (3.7) is satisfied

$$Th \vdash \mathrm{Pr}_{Th}\left([\neg\Phi_i]^c\right) \text{ and } Th \vdash \neg\Phi_i. \quad (3.7)$$

Then we define theory Th_{i+1} as follows:

$$Th_{i+1} \triangleq Th_i.$$

We define now theory $Th^{\#}$ as follows:

$$Th^{\#} \triangleq \bigcup_{i \in \mathbb{N}} Th_i. \quad (3.8)$$

First, notice that each Th_i is consistent. This is done by induction on *i* and by Lemmas 3.1-3.2. By assumption, the case is true when $i = 1$. Now, suppose Th_i is consistent. Then its deductive closure $\mathrm{Ded}(Th_i)$ is also consistent. If a statement (3.6) is satisfied *i.e.*, $Th \vdash \mathrm{Pr}_{Th}\left([\Phi_i]^c\right)$ and $Th \vdash \Phi_i$, then clearly $Th_{i+1} \triangleq Th_i \cup \{\Phi_i\}$ is consistent since it is a subset of closure $\mathrm{Ded}(Th_i)$. If a statement (3.7) is satisfied, *i.e.*, $Th \vdash \mathrm{Pr}_{Th}\left([\neg\Phi_i]^c\right)$ and $Th_i \vdash \neg\Phi_i$, then clearly $Th_{i+1} \triangleq Th_i \cup \{\neg\Phi_i\}$ is consistent since it is a subset of closure $\mathrm{Ded}(Th_i)$.

Otherwise:

1) if a statement (3.4) is satisfied, *i.e.* $Th_i \vdash \mathrm{Pr}_{\mathrm{Th}_i}\left([\Phi_i]^c\right)$ and $Th_i \nvdash \Phi_i$, then clearly $Th_{i+1} \triangleq Th_i \cup \{\Phi_i\}$ is consistent by Lemma 3.1 and by one of the standard properties of consistency: $\Delta \cup \{A\}$ is consistent iff $\Delta \nvdash \neg A$;

2) if a statement (3.5) is satisfied, *i.e.* $Th \vdash \mathrm{Pr}_{Th}\left([\neg\Phi_i]^c\right)$ and $Th_i \nvdash \neg\Phi_i$, then clearly $Th_{i+1} \triangleq Th_i \cup \{\neg\Phi_i\}$ is consistent by Lemma 3.2 and by one of the standard properties of consistency: $\Delta \cup \{\neg A\}$ is consistent iff $\Delta \nvdash A$.

Next, notice $\mathrm{Ded}(Th^{\#})$ is a maximally consistent nice extension of the set $\mathrm{Ded}(Th)$. A set $\mathrm{Ded}(Th^{\#})$ is consistent because, by the standard Lemma 3.3 below, it is the union of a chain of consistent sets. To see that $\mathrm{Ded}(Th^{\#})$ is maximal, pick any wff Φ. Then Φ is some Φ_i in the enumerated list of all wff's. Therefore for any Φ such that $Th \vdash \mathrm{Pr}_{Th}\left([\Phi]^c\right)$ or $Th \vdash \mathrm{Pr}_{Th}\left([\neg\Phi]^c\right)$, either $\Phi \in Th^{\#}$ or $\neg\Phi \in Th^{\#}$.

Since $\mathrm{Ded}(Th_{i+1}) \subseteq \mathrm{Ded}(Th^{\#})$, we have $\Phi \in \mathrm{Ded}(Th^{\#})$ or $\neg\Phi \in \mathrm{Ded}(Th^{\#})$, which implies that $\mathrm{Ded}(Th^{\#})$ is maximally consistent nice extension of the $\mathrm{Ded}(Th)$.

Lemma 3.3. The union of a chain $\wp = \{\Gamma_i | i \in \mathbb{N}\}$ of the consistent sets Γ_i, ordered by $\subseteq$, is consistent.

Definition 3.4. (a) Assume that a theory *Th* has an ω-model M_ω^{Th} and Φ is a *Th*-sentence. Let Φ_ω be a *Th*-sentence Φ with all quantifiers relativized to ω-model M_ω^{Th} [9];

(b) Assume that a theory *Th* has a standard model SM^{Th}

And Φ is a *Th*-sentence. Let Φ_{SM} be a Th-sentence Φ with all quantifiers relativized to the model SM^{Th} [9].

Definition 3.5. (a) Assume that *Th* has an ω-model M_ω^{Th}. Let Th_ω be a theory *Th* relativized to a model M_ω^{Th}—*i.e.*, any Th_ω-sentence has a form Φ_ω for some *Th*-sentence Φ [9];

(b) Assume that *Th* has an standard model SM^{Th}. Let Th_{SM} be a theory *Th* relativized to a model SM^{Th}—*i.e.*, any Th_{SM}-sentence has a form Φ_{SM} for some *Th*-sentence Φ [9].

Definition 3.6. (a) For a given ω-model M_ω^{Th} of the *Th* and for any Th_ω-sentence Φ_ω, we define $M_\omega^{Th} \vDash *\Phi_\omega$ such that the equivalence:

$$M_\omega^{Th} \vDash *\Phi_\omega \text{ iff } Th^\dagger \vdash \Phi_\omega \wedge \left(Th_\omega \vdash \mathrm{Pr}_{Th_\omega}\left([\Phi_\omega]^c\right)\right) \leftrightarrow Th^\dagger \vdash \Phi_\omega, \tag{3.9a}$$

where $Th^\dagger \triangleq Th + \exists M_\omega^{Th}$ is satisfied;

(b) For a given standard model SM^{Th} of the *Th* and for any Th_{SM}-sentence Φ_{SM}, we define $SM^{Th} \vDash *\Phi_{SM}$ such that the equivalence:

$$SM^{Th} \vDash *\Phi_{SM} \text{ iff } Th^\dagger \vdash \Phi_{SM} \wedge \left(Th_{SM} \vdash \mathrm{Pr}_{Th_{SM}}\left([\Phi_{SM}]^c\right)\right) \leftrightarrow Th^\dagger \vdash \Phi_{SM}, \tag{3.9b}$$

where $Th^\dagger \triangleq Th + \exists SM^{Th}$ is satisfied.

Theorem 3.2. (***Strong Reflection Principle***). Assume that: 1) $\mathrm{Con}(Th)$, 2) *Th* has an ω-model M_ω^{Th} and 3) $M_\omega^{Th} \models *Th_\omega$. Then

$$Th_\omega \vdash \mathrm{Pr}_{Th_\omega}\left([\Phi_\omega]^c\right) \Leftrightarrow Th_\omega \vdash \Phi_\omega. \tag{3.10}$$

Proof. The one direction is obvious. For the other, assume that

$$Th_\omega \vdash \mathrm{Pr}_{Th_\omega}\left([\Phi_\omega]^c\right), \tag{3.11}$$

$Th_\omega \nvdash \Phi_\omega$ and $Th_\omega \vdash \neg\Phi_\omega$. Then

$$Th_\omega \vdash \mathrm{Pr}_{Th_\omega}\left([\neg\Phi_\omega]^c\right). \tag{3.12}$$

Note that 1) + 2) implies $\mathrm{Con}(Th_\omega)$.

Let Con_{Th_ω} be the formula

$$\begin{aligned}\mathrm{Con}_{Th_\omega} &\triangleq \forall t_1 \forall t_2 \forall t_3 \left(t_3 = [\Phi_\omega]^c\right) \\ &\neg\left[\mathrm{Prov}_{Th_\omega}\left(t_1,[\Phi_\omega]^c\right) \wedge \mathrm{Prov}_{Th_\omega}\left(t_2,\mathrm{neg}\left([\Phi_\omega]^c\right)\right)\right] \\ &\leftrightarrow \neg\exists t_1 \neg\exists t_2 \neg\exists t_3 \left(t_3 = [\Phi_\omega]^c\right) \\ &\times\left[\mathrm{Prov}_{Th_\omega}\left(t_1,[\Phi_\omega]^c\right) \wedge \mathrm{Prov}_{Th_\omega}\left(t_2,\mathrm{neg}\left([\Phi_\omega]^c\right)\right)\right].\end{aligned} \tag{3.13}$$

where t_1, t_2, t_3 is a closed term. Note that in any ω-model M_ω^{Th} by the canonical observation one obtain the equivalence: $\mathrm{Con}(Th_\omega) \leftrightarrow \mathrm{Con}_{Th_\omega}$ But the Formulae (3.11)-(3.12) contradicts the Formula (3.13). Therefore $Th_\omega \nvdash \Phi_\omega$ and $Th_\omega \nvdash \mathrm{Pr}_{Th_\omega}\left([\neg\Phi_\omega]^c\right)$.

Then theory $Th'_\omega = Th_\omega + \neg\Phi_\omega$ is consistent and from the above observation one obtain that: $\mathrm{Con}(Th'_\omega) \leftrightarrow \mathrm{Con}_{Th'_\omega}$, where

$$\begin{aligned}\mathrm{Con}_{Th'_\omega} &\leftrightarrow \neg\exists t_1 \neg\exists t_2 \neg\exists t_3 \left(t_3 = [\Phi_\omega]^c\right) \\ &\times\left[\mathrm{Prov}_{Th'_\omega}\left(t_1,[\Phi_\omega]^c\right) \wedge \mathrm{Prov}_{Th'_\omega}\left(t_2,\mathrm{neg}\left([\Phi_\omega]^c\right)\right)\right].\end{aligned} \tag{3.14}$$

On the other hand one obtain

$$Th'_\omega \vdash \mathrm{Pr}_{Th'_\omega}\left([\Phi_\omega]^c\right), Th'_\omega \vdash \mathrm{Pr}_{Th'_\omega}\left([\neg\Phi_\omega]^c\right). \tag{3.15}$$

But the Formula (3.15), contradicts the Formula (3.14). This contradiction completed the proof.

Definition 3.7. (a) Assume that: (i) *Th* has an ω-model M_ω^{Th} and (ii) $M_\omega^{Th} \vDash *Th_\omega$. Then we said that M_ω^{Th} is a strong ω-model of the *Th* and denote such ω-model of the *Th* as $M_{\omega,\vDash *}^{Th}$.

(b) Assume that: (i) *Th* has an standard model SM^{Th} and (ii) $SM^{Th} \vDash *Th_{SM}$. Then we said that SM^{Th} is a strong standard model of the *Th* and denote such standard model of the *Th* as $SM_{\vDash *}^{Th}$.

Definition 3.8. (a) Assume that *Th* has a strong ω-model $M_{\omega,\vDash *}^{Th}$. Then we said that *Th* is a *strongly consistent*.

(b) Assume that *Th* has a strong standard model $SM_{\vDash *}^{Th}$ Then we said that *Th* is a *strongly SM-consistent*

Definition 3.9. (a) Assume that *Th* has a strong ω-model $M_{\omega,\vDash *}^{Th}$ and Φ is a *Th*-sentence. Let $\Phi_{\omega,\vDash *}$ be a *Th*-sentence Φ with all quantifiers relativized to a strong ω-model $M_{\omega,\vDash *}^{Th}$.

(b) Assume that Th has a strong standard model $SM_{\models *}^{Th}$ and Φ is a *Th*-sentence. Let $\Phi_{SM,\models *}$ be a *Th*-sentence Φ with all quantifiers relativized to $SM_{\models *}^{Th}$.

Definition 3.10. Assume that *Th* has a strong ω-model $M_{\omega,\models *}^{Th}$. Let $Th_{\omega,\models *}$ be a theory *Th* relativised to $M_{\omega,\models *}^{Th}$ *i.e.*, any $Th_{\omega,\models *}$-sentence has the form $\Phi_{\omega,\models *}$ for some *Th*-sentence Φ.

Let *Th* be a theory such that Assumption 1.1 is satisfied. Let $\operatorname{Con}\left(Th; M_{\omega,\models *}^{Th}\right)$ be a sentence in *Th* asserting that *Th* has a strong ω-model $M_{\omega,\models *}^{Th}$. Let *Th** be a theory: $Th^* = Th + \operatorname{Con}\left(Th; M_{\omega,\models *}^{Th}\right)$.

Let $\operatorname{Con}\left(Th^*; M_{\omega,\models *}^{Th^*}\right)$ be a sentence in *Th** asserting that *Th** has a strong ω-model $M_{\omega,\models *}^{Th^*}$. We assume throughout that *Th* is a strongly consistent, *i.e.* a sentence $\operatorname{Con}\left(Th; M_{\omega,\models *}^{Th}\right)$ is true in any ω-model M_{ω}^{Th} of the *Th*. Note that:

$$\begin{aligned} &\operatorname{Con}\left(Th; M_{\omega,\models *}^{Th}\right) \leftrightarrow \operatorname{Con}_{Th_{\omega,\models *}} \\ &\operatorname{Con}_{Th_{\omega,\models *}} \leftrightarrow \neg \operatorname{Pr}_{Th_{\omega,\models *}}\left(\left[\Phi_{\omega,\models *}\right]^c\right) \end{aligned}, \tag{3.16}$$

where a sentence $\Phi_{\omega,\models *}$ is refutable in $Th_{\omega,\models *}$ and

$$\begin{aligned} &\operatorname{Con}\left(Th^*; M_{\omega,\models *}^{Th^*}\right) \leftrightarrow \operatorname{Con}_{Th^*_{\omega,\models *}} \\ &\operatorname{Con}_{Th^*_{\omega,\models *}} \leftrightarrow \neg \operatorname{Pr}_{Th^*_{\omega,\models *}}\left(\left[\Phi^*_{\omega,\models *}\right]^c\right) \end{aligned}, \tag{3.17}$$

where a sentence $\Phi^*_{\omega,\models *}$ is refutable in $Th^*_{\omega,\models *}$.

Lemma 3.4. *Th** is a strongly consistent.

Proof. Assume that *Th** is no strongly consistent, that is, has no any strong ω-model $M_{\omega,\models *}^{Th^*}$. This means that there is no any ω-model M_{ω}^{Th} of the *Th* in which $\operatorname{Con}\left(Th; M_{\omega,\models *}^{Th}\right)$ is true and therefore from Formula (3.16) one obtain, that a formula $\neg \operatorname{Con}_{Th_{\omega,\models *}}$ is true in any ω-model M_{ω}^{Th} of the *Th*. So from Formula (3.16) by using a Strong Reflection Principle (Theorem 3.2) one obtain that a sentence $\neg \operatorname{Con}\left(Th; M_{\omega,\models *}^{Th}\right)$ is provable in Th_{ω}, *i.e.* $Th_{\omega} \vdash \neg \operatorname{Con}\left(Th; M_{\omega,\models *}^{Th}\right)$. But a sentence $\neg \operatorname{Con}\left(Th; M_{\omega,\models *}^{Th}\right)$ contrary to the assumption that *Th* is a strongly consistent. This contradiction completed the proof.

Theorem 3.3. *Th* has no any strong ω-model $M_{\omega,\models *}^{Th}$.
Proof. By Lemma 3.4 and Formula (3.17) one obtain that $Th^*_{\omega,\models *} \vdash \operatorname{Con}_{Th^*_{\omega,\models *}}$. But Godel's Second Incompleteness Theorem applied to $Th^*_{\omega,\models *}$ asserts that $\operatorname{Con}_{Th^*_{\omega,\models *}}$ is unprovable in $Th^*_{\omega,\models *}$. This contradiction completed the proof.

Theorem 3.4. *ZFC* has no any strong ω-model $M_{\omega,\models *}^{ZFC}$.

Proof. Immediately follows from Theorem 3.3 and definitions.

Theorem 3.5. *ZFC* has no any strong standard model. $SM_{\models *}^{ZFC}$.

Proof. Immediately follows from Theorem 3.4 and definitions.

Theorem 3.6. $ZFC + \operatorname{Con}(ZFC)$ is incompatible with all the usual large cardinal axioms [10,11] which imply the existence of a strong standard model of *ZFC*.

Proof. Theorem 3.6 immediately follows from Theorem 3.5.

Theorem 3.7. Let κ be an inaccessible cardinal. Then $\neg \operatorname{Con}(ZFC + \exists \kappa)$.

Proof. Let H_κ be a set of all sets having hereditary size less then κ. It easy to see that H_κ forms a strong standard model of *ZFC*. Therefore Theorem 3.7 immediately follows from Theorem 3.6.

4. Conclusion

In this paper we proved so-called strong reflection principles corresponding to formal theories *Th* which has ω-models M_{ω}^{Th} and in particular to formal theories *Th*, which has a standard models SM^{Th}. The assumption that there exists a standard model of *Th* is stronger than the assumption that there exists a model of *Th*. This paper examined some specified classes of the standard models of *ZFC* so-called strong standard models of *ZFC*. Such models correspond to large cardinals axioms. In particular we proved that theory $ZFC + \operatorname{Con}(ZFC)$ is incompatible with existence of any inaccessible cardinal κ. Note that the statement: Con ($ZFC + \exists$ some inaccessible cardinal κ) is Π_1^0. Thus Theorem 3.6 asserts there exist numerical counterexample which would imply that a specific polynomial equation has at least one integer root.

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