# Model $\mathrm{P}(\varphi)_4$ Quantum Field Theory: A Nonstandard Approach Based on Nonstandard Pointwise-Defined Quantum Fields

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**Abstract.** A new non-Archimedean approach to interacted quantum fields is presented. In proposed approach, a field operator $\varphi(x,t)$ no longer a standard tempered operator-valued distribution, but a non-classical operator-valued function. We prove using this novel approach that the quantum field theory with Hamiltonian $P(\varphi)_4$ exists and that the corresponding $C^*$- algebra of bounded observables satisfies all the Haag-Kastler axioms except Lorentz covariance. We prove that the $\lambda(\varphi^4)_4$ quantum field theory model is Lorentz covariant.

## INTRODUCTION

Extending the real numbers $\mathbb{R}$ to include infinite and infinitesimal quantities originally enabled Laugwitz [1] to view the delta distribution $\delta(x)$ as a nonstandard point function. Independently Robinson [2] demonstrated that distributions could be viewed as generalized polynomials. Luxemburg [3] and Sloan [4] presented an alternate re-presentative of distributions as internal functions within the context of canonical Robinson's theory of nonstandard analysis. For further information on nonstandard real analysis, we refer to [5, 6].

**Abbreviation 1.1.**1In this paper we adopt the following notations. For a standard set $E$ we often write $E_{\mathrm{st}}$. For a set $E_{\mathrm{st}}$ let ${}^{\sigma}E_{\mathrm{st}}$ be a set ${}^{\sigma}E_{\mathrm{st}} = \{{}^*x | x \in E_{\mathrm{st}}\}$. We identify $z$ with ${}^{\sigma}z$ i.e., $z \equiv {}^{\sigma}z$ for all $z \in \mathbb{C}$. Hence, ${}^{\sigma}E_{\mathbf{st}} = E_{\mathrm{st}}$ if $E \subseteq \mathbb{C}$, e.g., ${}^{\sigma}\mathbb{C} = \mathbb{C}$, ${}^{\sigma}\mathbb{R} = \mathbb{R}$, ${}^{\sigma}P = P$, ${}^{\sigma}L_+^{\uparrow} = L_+^{\uparrow}$, etc. Let ${}^*\mathbb{R}_{\approx}$, ${}^*\mathbb{R}_{\approx +}$, ${}^*\mathbb{R}_{\mathrm{fin}}$, ${}^*\mathbb{R}_{\infty}$, and ${}^*\mathbb{N}_{\infty}$ denote the sets of infinitesimal hyper-real numbers, positive infinitesimal hyper-real numbers, finite hyper-real numbers, infinite hyper-real numbers and infinite hyper natural numbers, respectively. Note that ${}^*\mathbb{R}_{\mathrm{fin}} = {}^*\mathbb{R} \backslash {}^*\mathbb{R}_{\infty}$, ${}^*\mathbb{C} = {}^*\mathbb{R} + \mathrm{i}\,{}^*\mathbb{R}$, ${}^*\mathbb{C}_{\mathrm{fin}} = {}^*\mathbb{R}_{\mathrm{fin}} + \mathrm{i}\,{}^*\mathbb{R}_{\mathrm{fin}}$.

**Definition 1.1**Let $\{X, O\}$ be a standard topological space and let ${}^*X$ be the nonstandard extension of $X$. Let $O_x$ de-note the set of open neighbourhoods of point $x \in X$. The monad $mon_O(x)$ of $x$ is the subset of ${}^*X$ defined by $mon_O(x) = \cap \{{}^*O | O \subset O_x\}$.The set of near standard points of ${}^*X$ is the subset of ${}^*X$ defined by $nst\,({}^*X) = \cup \{mon_O(x) | x \in X\}$. It is shown that $\{X, O\}$ is Hausdorff space if and only if $x \neq y$ implies $mon_O(x) \cap mon_O(y) = \emptyset$. Thus for any Hausdorff space $\{X, O\}$, we can define the equivalence relation $\approx_O$ on $nst\,({}^*X)$ so that $x \approx_O y$ if and only if $x \in mon_O(z)$ and $y \in mon_O(z)$ for some $z \in X$.

**Definition 1.2** The standard Schwartz space of rapidly decreasing test functions on $\mathbb{R}^n, n \in \mathbb{N}$ is the standard function space is defined by $S(\mathbb{R}^n, \mathbb{C}) = \{f \in C^{\infty}(\mathbb{R}^n, \mathbb{C}) | \forall \alpha, \beta \in \mathbb{N}^n [\|f\|_{\alpha,\beta} < \infty]\}$, where

$$\|f\|_{\alpha,\beta} = \sup_{x \in \mathbb{R}^n} \left| x^{\alpha} \left( D^{\beta} f(x) \right) \right|.$$

**Remark 1.1** If $f$ is a rapidly decreasing function, then for all $\alpha \in \mathbb{N}^n$ the integral of $\left|x^{\alpha} D^{\beta} f(x)\right|$ exists

$$\int_{\mathbb{R}^n} \left| x^{\alpha} D^{\beta} f(x) \right| d^n x < \infty$$

**Definition 1.3** The internal Schwartz space of rapidly decreasing test functions on ${}^*\mathbb{R}^n, n \in {}^*\mathbb{N}$ is the function space defined by ${}^*S({}^*\mathbb{R}^n, {}^*\mathbb{C}) = \{{}^*f \in {}^*C^{{}^*\infty}({}^*\mathbb{R}^n, {}^*\mathbb{C}) | \forall \alpha, \beta \in {}^*\mathbb{N}^n [{}^*\|{}^*f\|_{\alpha,\beta} < {}^*\infty]\}$, where

$$ {}^*\|{}^*f\|_{\alpha,\beta} = \sup{}^*\left\{x^\alpha\left(D^\beta f(x)\right)|x \in \mathbb{R}^n\right\}. $$

**Remark 1.2** If $f$ is a rapidly decreasing function, $f \in S(\mathbb{R}^n, \mathbb{C})$, then for all $\alpha, \beta \in {}^*\mathbb{N}^n$ the internal integral of $\left|{}^*x^\alpha D^\beta\, {}^*f(x)\right|$ exists

$$ {}^*\!\!\int_{{}^*\mathbb{R}^n}\left|{}^*x^\alpha D^\beta\, {}^*f(x)\right| d^n x < {}^*\infty. $$

Here $D^\beta\, {}^*f(x) = {}^*\!\left(D^\beta f(x)\right)$.

**Definition 1.4** The Schwartz space of essentially rapidly decreasing test functions on ${}^*\mathbb{R}^n, n \in {}^*\mathbb{N}$ is the function space defined by

$$ {}^*S_{\text{fin}}({}^*\mathbb{R}^n, {}^*\mathbb{C}) = $$

$$ \left\{{}^*f \in {}^*C^{{}^*\infty}({}^*\mathbb{R}^n, {}^*\mathbb{C}) | \forall(\alpha,\beta)(\alpha,\beta \in {}^*\mathbb{N}^n)\exists c_{\alpha\beta}\left(c_{\alpha\beta} \in {}^*\mathbb{R}_{\text{fin}}\right)\forall x(x \in {}^*\mathbb{R}^n)\left[\left|x^\alpha\left({}^*D^\beta\, {}^*f(x)\right)\right| < c_{\alpha\beta}\right]\right\} $$

**Remark 1.3** If ${}^*f \in {}^*S_{\text{fin}}({}^*\mathbb{R}^n, {}^*\mathbb{C})$, then for all $\alpha \in {}^*\mathbb{N}^n$ the internal integral of $\left|{}^*x^\alpha D^\beta\, {}^*f(x)\right|$ exists and finitely bounded above

$$ {}^*\!\!\int_{{}^*\mathbb{R}^n}\left|{}^*x^\alpha D^\beta\, {}^*f(x)\right| d^n x < d_{\alpha\beta}, d_{\alpha\beta} \in {}^*\mathbb{R}_{\text{fin}}. $$

**Abbreviation 1.2** The standard Schwartz space of rapidly decreasing test functions on $\mathbb{R}^n$ we will be denote by $S(\mathbb{R}^n)$. Let ${}^*S({}^*\mathbb{R}^n), n \in {}^*\mathbb{N}$ denote the space of ${}^*\mathbb{C}$-valued rapidly decreasing internal test functions on ${}^*\mathbb{R}^n, n \in {}^*\mathbb{N}$ and let ${}^*S_{\text{fin}}({}^*\mathbb{R}^n), n \in {}^*\mathbb{N}$ denote the set of ${}^*\mathbb{C}_{\text{fin}}$-valued essentially rapidly decreasing test functions on ${}^*\mathbb{R}^n, n \in {}^*\mathbb{N}$. If $h(\omega, x): \mathbb{R} \times \mathbb{R}^n$ and $f: \mathbb{R}^n \to \mathbb{C}$ are Lebesgue measurable on $\mathbb{R}^{4n}$ we shall write $\langle {}^*h, {}^*f\rangle$ for internal Lebesgue integral ${}^*\!\int_{{}^*\mathbb{R}^n} {}^*h\, {}^*f\, d^n x$ with ${}^*f \in {}^*S_{\text{fin}}({}^*\mathbb{R}^n)$. Certain internal functions ${}^*h(\omega, x): {}^*\mathbb{R} \times {}^*\mathbb{R}^n \to {}^*\mathbb{C}$ define classical distribution $\tau(f)$ by the rule [3, 4]:

$$ \tau(f) = \text{st}(\langle {}^*h, {}^*f\rangle). \tag{1} $$

Here $\text{st}(a)$ is the standard part of $a$ and $\text{st}(\langle {}^*h, {}^*f\rangle)$ exists [5].

**Definition 1.5** We shall say that ${}^*h(\omega, x)$ with $\omega = \varpi \in {}^*\mathbb{R}_\infty$ is an internal representative to distribution $\tau(f)$ and we will write symbolically $\tau(x_1, \ldots, x_n) \approx {}^*h(\omega, x_1, \ldots, x_n)$ if the equation (1) holds.

**Definition 1.6** [6] We shall say that certain internal functions ${}^*h(\omega, x): {}^*\mathbb{R} \times {}^*\mathbb{R}^n \to {}^*\mathbb{C}$ is a finite tempered distribution if ${}^*f \in {}^*S_{\text{fin}}({}^*\mathbb{R}^n)$ implies $|{}^*h, {}^*f| \in {}^\sigma\mathbb{R} = \mathbb{R}$. A functions ${}^*h(\omega, x): {}^*\mathbb{R} \times {}^*\mathbb{R}^n \to {}^*\mathbb{C}$ is called infinitesimal tempered distribution if ${}^*f \in {}^*S_{\text{fin}}({}^*\mathbb{R}^n)$ implies $|{}^*h, {}^*f| \in {}^*\mathbb{R}_\approx$. The space of infinitesimal tempered distribution is denoted by ${}^*S_\approx({}^*\mathbb{R}^n)$.

**Definition 1.7** We shall say that certain internal functions ${}^*h(\omega, x): {}^*\mathbb{R} \times {}^*\mathbb{R}^{4n} \to {}^*\mathbb{C}$ is a Lorentz $\approx$-invariant tempered distribution if ${}^*f \in {}^*S_{\text{fin}}({}^*\mathbb{R}^n)$ and $\Lambda \in {}^\sigma L_+^\uparrow$ implies $\langle {}^*h, {}^*f(\Lambda x_1, \ldots, \Lambda x_n)\rangle \approx \langle {}^*h, {}^*f(x_1, \ldots, x_n)\rangle$.

**Example 1.1** Let us consider Lorentz invariant distribution

$$ D(x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{i\boldsymbol{kr}}\frac{\sin\omega t}{\omega}d^3k = \frac{1}{2\pi}\delta(r^2 - t^2)\text{sign}(t). \tag{2} $$

Here $\omega = |\boldsymbol{k}| = \sqrt{k_1^2 + k_2^2 + k_3^2}$ and $\boldsymbol{r} = (x_1, x_2, x_3)$, $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$. It easily verify that distribution $D(x)$ has the following internal representative

$$ D(x, \varpi) = \frac{1}{(2\pi)^3}{}^*\!\!\int_{|\boldsymbol{k}| \le \varpi} e^{i\boldsymbol{kr}}\frac{\sin\omega t}{\omega}d^3k. \tag{3} $$

Here $\varpi \in {}^*\mathbb{R}_\infty$. By integrating in (3) over angle variables we get

$$ D(x, \varpi) = \frac{1}{8\pi^2 r}{}^*\!\!\int_0^\varpi\left\{e^{i\omega(r-t)} + e^{-i\omega(r-t)} - e^{i\omega(r+t)} - e^{-i\omega(r+t)}\right\}d\omega. \tag{4} $$

From (4) by canonical calculation finally we get

$$ D(x, \varpi) \approx \frac{1}{4\pi^2 r}\left[\frac{\sin\varpi(r-t)}{r-t} - \frac{\sin\varpi(r+t)}{r+t}\right] \approx \frac{\delta(r-t) - \delta(r+t)}{4\pi^2 r} = \frac{1}{2\pi}\delta(r^2 - t^2)\text{sign}(t). \tag{5} $$

**Example 1.2** We consider now the following Lorentz invariant distribution:

$$ D_1(x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}^3} e^{i\boldsymbol{kr}}\frac{\cos\omega t}{\omega}d^3k = \frac{1}{2\pi^2}\frac{1}{x^2}. \tag{6} $$

It easily verify that distribution $D(x)$ has the following internal representative

$$ D_1(x, \varpi) = \frac{1}{(2\pi)^3}{}^*\!\!\int_{|\boldsymbol{k}| \le \varpi} e^{i\boldsymbol{kr}}\frac{\cos\omega t}{\omega}d^3k. \tag{7} $$

Here $\varpi \in {}^*\mathbb{R}_\infty$. By integrating in (7) over angle variables we get

$$ D_1(x, \varpi) \approx -\frac{i}{8\pi^2 r}{}^*\!\!\int_0^\varpi\left\{e^{i\omega(r-t)} - e^{-i\omega(r-t)} + e^{i\omega(r+t)} - e^{-i\omega(r+t)}\right\}d\omega. \tag{8} $$

From (8) finally we get

$$D_1(x,\varpi) \approx -\frac{i}{8\pi^2 r}\left[\frac{-2}{i(r-t)} + \frac{-2}{i(r+t)} + \frac{2\cos\varpi(r-t)}{i(r-t)} + \frac{2\cos\varpi(r+t)}{i(r+t)}\right] \approx \frac{1}{2\pi^2}\frac{1}{x^2}. \quad (9)$$

Example 3.We consider now the following Lorentz invariant distribution

$$\Delta_c(x) = \frac{1}{2(2\pi)^3}\int_{\mathbb{R}^3} e^{i(\boldsymbol{kr}-\varepsilon(\boldsymbol{k})|t|)}\frac{d^3k}{\varepsilon(\boldsymbol{k})} = -\frac{m}{8\pi}\frac{H_1^{(2)}\left(-im\sqrt{|x^2|}\right)}{m\sqrt{|x^2|}}. \quad (10)$$

Here $-x^2 < 0$, $\varepsilon(\boldsymbol{k}) = \sqrt{|\boldsymbol{k}^2| + m^2}$ and $H_1^{(2)}$ is a Hankel function of the second kind. It easily verify that distribution $\Delta_c(x)$ has the following internal representative

$$\Delta_c(x,\varpi) = \frac{1}{2(2\pi)^3}{}^*\int_{|\boldsymbol{k}|\le\varpi} e^{i(\boldsymbol{kr}-\varepsilon(\boldsymbol{k})|t|)}\frac{d^3k}{\varepsilon(\boldsymbol{k})} \quad (11)$$

From (10)-(11) it follows ${}^*\Delta_c(x) = \Delta_c(x,\varpi) + \breve{\Delta}_c(x)$ where

$$\breve{\Delta}_c(x) = \frac{1}{2(2\pi)^3}{}^*\int_{|\boldsymbol{k}|>\varpi} e^{i(\boldsymbol{kr}-\varepsilon(\boldsymbol{k})|t|)}\frac{d^3k}{\varepsilon(\boldsymbol{k})}. \quad (12)$$

Note that for all $\Lambda \in {}^\sigma L_+^\uparrow$, $\breve{\Delta}_c(\Lambda x) \in {}^*S_\approx({}^*\mathbb{R}^n)$ and therefore for all $\Lambda \in {}^\sigma L_+^\uparrow$, $\Delta_c(\Lambda x,\varpi) \approx \Delta_c(x,\varpi)$, i.e., $\Delta_c(x,\varpi)$ is a Lorentz $\approx$ -invariant tempered distribution, see definition 4. Thus we can set $t = 0$ in (11). By integrating in (11) over angle variables and using substitution of variables $|\boldsymbol{k}| = m\sinh(u)$ we get

$$\Delta_c(x,\varpi) \approx \frac{m}{8\pi^2 ir}{}^*\int_{-\ln\varpi}^{\ln\varpi}\exp\left(imr\sinh(u)\right)du. \quad (13)$$

Note that

$${}^*H_1^{(2)}(x) = \frac{\pi}{i}\int_{-{}^*\infty}^{{}^*\infty}\exp\left(imr\sinh(u)\right)du = \Delta_c(x,\varpi) + \Xi(x,\varpi), \quad (14)$$

$$\Xi(x,\varpi) = \frac{\pi}{i}\int_{-{}^*\mathbb{R}}^{-\ln\varpi}\exp\left(imr\sinh(u)\right)du + \int_{\ln\varpi}^{{}^*\mathbb{R}}\exp\left(imr\sinh(u)\right)du. \quad (15)$$

From (13)-(15) finally we obtain $\Delta_c(x,\varpi) \approx H_1^{(2)}(x)$ , since $\Xi(x,\varpi) \in {}^*S_\approx({}^*\mathbb{R}^n)$.

**Example 1.4** Let us consider Lorentz invariant distribution

$$\Delta(x-y) = \int\{\exp[-ip(x-y)] - \exp[ip(x-y)]\}\,\delta(p^2-m^2)\vartheta(p^0)d^4p. \quad (16)$$

From (16) one obtains $\Delta(x-y) = \Xi_1(x-y) - \Xi_2(x-y)$, where

$$\Xi_1(x-y) = \int\left\{\exp\{[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})] - i\omega(\boldsymbol{p})(x^0-y^0)\}\right\}\frac{d^3p}{\sqrt{\boldsymbol{p}^2+m^2}}, \quad (17)$$

$$\Xi_2(x-y) = \int\left\{\exp\{[-i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})] + i\omega(\boldsymbol{p})(x^0-y^0)\}\right\}\frac{d^3p}{\sqrt{\boldsymbol{p}^2+m^2}}, \quad (18)$$

$\omega(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2+m^2}$. It easily verify that distribution (17) and (18) has the following internal representatives

$$\Xi_1(x-y,\varpi) = {}^*\int_{|\boldsymbol{k}|\le\varpi}\left\{\exp\{[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})] - i\omega(\boldsymbol{p})(x^0-y^0)\}\right\}\frac{d^3p}{\sqrt{\boldsymbol{p}^2+m^2}}. \quad (19)$$

$$\Xi_2(x-y,\varpi) = {}^*\int_{|\boldsymbol{k}|\le\varpi}\left\{-\exp\left[[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})] + i\omega(\boldsymbol{p})(x^0-y^0)\right]\right\}\frac{d^3p}{\sqrt{\boldsymbol{p}^2+m^2}}. \quad (20)$$

Note that ${}^*\Delta(x-y) = [\Xi_1(x-y,\varpi) + \Xi_2(x-y,\varpi)] + \left[\breve{\Xi}_1(x-y,\varpi) + \breve{\Xi}_2(x-y,\varpi)\right]$, where

$$\breve{\Xi}_1(x-y,\varpi) = {}^*\int_{|\boldsymbol{k}|>\varpi}\left\{\exp\{[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})] - i\omega(\boldsymbol{p})(x^0-y^0)\}\right\}\frac{d^3p}{\sqrt{\boldsymbol{p}^2+m^2}}, \quad (21)$$

$$\breve{\Xi}_2(x-y,\varpi) = {}^*\int_{|\boldsymbol{k}|>\varpi}\left\{-\exp\left[[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})] + i\omega(\boldsymbol{p})(x^0-y^0)\right]\right\}\frac{d^3p}{\sqrt{\boldsymbol{p}^2+m^2}}. \quad (22)$$

Note that for all $\Lambda \in {}^\sigma L_+^\uparrow$, $\breve{\Xi}_1(\Lambda(x-y),\varpi) + \breve{\Xi}_1(\Lambda(x-y),\varpi) \in {}^*S_\approx({}^*\mathbb{R}^n)$ and therefore for all $\Lambda \in {}^\sigma L_+^\uparrow$, ${}^*\Delta\left(\Lambda(x-y)\right) \approx \Delta(\Lambda(x-y),\varpi) = \Xi_1(\Lambda(x-y),\varpi) + \Xi_2(\Lambda(x-y),\varpi)$, i.e., $\Delta(x-y,\varpi)$ is a Lorentz $\approx$-invariant tempered distribution, see definition 4. From (20) by replacement $\boldsymbol{p} \to -\boldsymbol{p}$ we obtain

$$\Xi_1(x-y,\varpi) = -{}^*\int_{|\boldsymbol{k}|\le\varpi}\left\{\exp\{[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})] + i\omega(\boldsymbol{p})(x^0-y^0)\}\right\}\frac{d^3p}{\sqrt{\boldsymbol{p}^2+m^2}}. \quad (23)$$

From (19) and (23) we get

$$\Delta(x-y,\varpi) = \Xi_1(x-y,\varpi) + \Xi_2(x-y,\varpi) = {}^*\int_{|\boldsymbol{k}|\le\varpi}\sin[\omega(\boldsymbol{p})(x^0-y^0)]\exp[i\boldsymbol{p}(\boldsymbol{x}-\boldsymbol{y})]\frac{d^3p}{\sqrt{\boldsymbol{p}^2+m^2}}. \quad (24)$$

Thus for any points $x$ and $y$ separated by space-like interval from (24) we obtain that

$$\Delta(x-y,\varpi) \approx 0, \quad (25)$$

since $\Delta(x-y,\varpi)$ is a Lorentz $\approx$-invariant tempered distribution. From (25) for any points $x$ and $y$ separated by spacelike interval we obtain that: $\mathrm{st}\left(\Delta(x-y,\varpi)\right) \equiv 0$.

**Definition 1.8** [7] Let for each $m > 0$: $H_m = \{p \in \mathbb{R}^4 | p\cdot\tilde{p} = m^2, m >, p_0 > 0\}$, where $\tilde{p} = (p^0,-p^1,-p^2,-p^3)$. Here the sets $H_m$ which are standard mass hyperboloids, are invariant under ${}^\sigma L_+^\uparrow$. Let $j_m$ be the homeomorphism of $H_m$ onto $\mathbb{R}^3$ given by $j_m\colon (p_0,p_1,p_2,p_3) \to (p_1,p_2,p_3) = \boldsymbol{p}.$ Define a measure $\Omega_m(E)$ on $H_m$ by

$$\Omega_m(E) = \int_{j_m(E)} \frac{d^3\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}} .$$

The measure $\Omega_m(E)$ is ${}^{\sigma}L_+^{\uparrow}$-invariant [7].

**Theorem 1.1** [7] Let $\mu$ is a polynomially bounded measure with support in $\bar{V}_+$. If $\mu$ is ${}^{\sigma}L_+^{\uparrow} = L_+^{\uparrow}$- invariant, there exists a polynomially bounded measure $\rho$ on $[0,\infty)$ and a constant $c$ so that for any $f \in S(\mathbb{R}^4)$

$$\int_{\mathbb{R}^4} f\, d\,\mu = cf(0) + \int_0^{\infty} d\,\rho\,(m)\left(\int_{\mathbb{R}^3} \frac{f\left(\sqrt{|\boldsymbol{p}|^2+m^2}, p_1, p_2, p_3\right) d^3\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}\right). \tag{26}$$

**Theorem 1.2** Let $\mu$ is a polynomially bounded $L_+^{\uparrow}$- invariant measure with support in $\bar{V}_+$. Let $\mathcal{F}(f)$ be a linear $*$-continuous functional $\mathcal{F}: {}^*S_{\text{fin}}\,({}^*\mathbb{R}^4) \to {}^*\mathbb{R}_{\text{fin}}$ defined by ${}^*\!\int_{{}^*\mathbb{R}^4} {}^*f\, d\,\mu$ and there exists a polynomially bounded measure $\rho$ on $[0,\infty)$ such that $\int_0^{{}^*\infty} d\,{}^*\rho\,(m) \in {}^*\mathbb{R}_{\text{fin}}$ and a constant $c \in {}^*\mathbb{R}_{\text{fin}}$. Then for any $f \in {}^*S_{\text{fin}}({}^*\mathbb{R}^4)$ and for any $\varkappa \in {}^*\mathbb{R}_\infty$ the following property holds

$$\mathcal{F}({}^*f) \approx c\,{}^*f(0) + {}^*\!\int_0^{{}^*\infty} d\,{}^*\rho\,(m)\left({}^*\!\int_{|p|\le\varkappa} \frac{{}^*f\left(\sqrt{|\boldsymbol{p}|^2+m^2}, p_1, p_2, p_3\right) d^{\#3}\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}\right) \tag{27}$$

**Definition 1.9** Let $\chi(\varkappa,\boldsymbol{p})$ be a function such that: $\chi(\varkappa,\boldsymbol{p}) \equiv 1$ if $|\boldsymbol{p}| \le \varkappa, \chi(\varkappa,\boldsymbol{p}) \equiv 0$ if $|\boldsymbol{p}| > \varkappa$, $\varkappa \in {}^*\mathbb{R}_\infty$.Define internal measure $\Omega_{m,\varkappa}$ on ${}^*H_m$ by

$$\Omega_{m,\varkappa}(E) = {}^*\!\int_{{}^*H_m} \frac{\chi(\varkappa,\boldsymbol{p})\, d^3\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}. \tag{28}$$

**Theorem 1.3** [7] Let $W_2(x_1,x_2)$ be the two-point function of a field theory satisfying the Wightman axioms and the additional condition that $(\psi_0, \varphi(f)\psi_0) = 0$ for all $f \in S(\mathbb{R}^4)$. Then there exists a polynomially bounded positive measure $\rho(m)$ on $[\![0,\infty)$ so that for all for all $f \in S(\mathbb{R}^4)$

$$W_2(f) = \left(\psi_0, \varphi(\bar{f})\varphi(f)\psi_0\right) = \int \bar{f}(x_1) f(x_2)\, W_2(x_1 - x_2) d^4x d^4y = \int_0^{\infty}\left(\int_{H_m} \hat{f}\, d\Omega_m\right) d\rho(m). \tag{29}$$

**Theorem 1.4** Let $W_2(x_1,x_2)$ be the two-point function of a field theory mentioned in Theorem 1.3. Then for all $f \in S_{\text{fin}}({}^*\mathbb{R}^4)$ and for any $\varkappa \in {}^*\mathbb{R}_\infty$ the following property holds

$${}^*W_2(f) \approx {}^*\!\int_0^{{}^*\infty}\left({}^*\!\int_{{}^*H_m} \hat{f}\, d\Omega_{m,\varkappa}\right) d\,{}^*\rho(m). \tag{30}$$

**Definition 1.10** (1) Let $L(H)$ be algebra of the all densely defined linear operators in standard Hilbert space $H$. Operator-valued distribution on $\mathbb{R}^n$, that is a map $\varphi: S(\mathbb{R}^n) \to L(H)$ such that there exists a dense subspace $D \subset H$ satisfying:

1. for each $f \in S(\mathbb{R}^n)$ the domain of $\varphi$ contains $D$,
2. the induced map: $S \to End(D)$, $f \to \varphi\,(f)$, is linear,
3. for each $h_1 \in D$ and $h_2 \in H$ the assignment $f \to \langle h_2, \varphi(f)h_1\rangle$ is a tempered distribution.

(2) Certain operator-valued internal function $\varphi\,({}^*f, \varpi): {}^*S\left({}^*\mathbb{R}^n\right) \to {}^*L({}^*H)$ is an internal representative for standard operator valued distribution $\varphi\,(f)$ if for each near standard vectors $\tilde{h}_1 \in {}^*D$ and $\tilde{h}_2 \in {}^*H$ the equality holds

$$\langle h_2, \varphi(f)h_1\rangle = \operatorname{st}\left({}^*\langle\tilde{h}_2, \varphi({}^*f,\varpi)\tilde{h}_1\rangle\right), \tag{31}$$

where $h_1 \approx \tilde{h}_1$ and $h_2 \approx \tilde{h}_2$.

**Definition 1.11** [8] Let $H$ be a Hilbert space and denote by $H^n$ the $n$-fold tensor product $H^n = H \otimes H \otimes \cdots \otimes H$. Set $H^0 = \mathbb{C}$ and define $\mathcal{F}(H) = H^n$. $\mathcal{F}(H)$ is called the Fock space over Hilbert space $H$. Notice $\mathcal{F}(H)$ will be separable if $H$ is. We set now $H = L_2(\mathbb{R}^3)$ then an element $\psi \in \mathcal{F}(H)$ is a sequence of $\mathbb{C}$-valued functions $\psi = \{\psi_0, \psi_1(x_1), \psi_2(x_1,x_2), \psi_2(x_1,x_2,x_3), \dots, \psi_n(x_1,\dots,x_n)\}$, $n \in \mathbb{N}$ and such that the following condition holds

$$|\psi_0|^2 + \textstyle\sum_{n\in\mathbb{N}}\left(\int |\psi_n(x_1,\dots,x_n)|^2 d^{3n}x\right) < \infty.$$

**Definition 1.12** [7] Let us define now external operator $a(p)$ on $\mathcal{F}_s$ with domain $D_S$ by

$$(a(p)\psi)^{(n)} = \sqrt{n+1}\,\psi^{(n+1)}(p, k_1, \dots k_n). \tag{32}$$

The formal adjoint of the operator $a(p)$ reads

$$(a^{\dagger}(p)\psi)^{(n)} = \frac{1}{\sqrt{n}}\sum_{l=1}^{n} \delta^{(3)}(p - k_l)\psi^{(n-1)}(k_1,\dots,k_{l-1},k_{l+1},\dots,k_n) \tag{33}$$

**Definition 1.13** [7] Let $\psi^{\text{fin}}$ be a vector $\psi^{\text{fin}} = \left\{\psi^{(n)}\right\}_{n=1}^{\infty}$ for which $\psi^{(n)} = 0$ for all except finitely many $n$ is called a finite particle vector. We will denote the set of finite particle vectors by $F_0$. The vector $\Omega_0 = \langle 1,0,0,\dots\rangle$ is called the vacuum.

**Definition 1.14** We let now ${}^*D_{{}^*S} = \left\{{}^*\psi \,|\, {}^*\psi \in {}^*F_0, {}^*\psi^{(n)} \in {}^*S\,({}^*\mathbb{R}^{3n}), n \in {}^*\mathbb{N}\right\}$ and for each $p \in {}^*\mathbb{R}^{3n}$ we define an internal operator ${}^*a(p)$ on ${}^*\mathcal{F}_s$ with domain ${}^*D_{{}^*S}$ by

$$({}^*a(p)\psi)^{(n)} = \sqrt{n+1}\,{}^*\psi^{(n+1)}(p,k_1,\dots k_n). \tag{34}$$

The formal $*$-adjoint of the operator ${}^*a$ reads

$$({}^*a^\dagger(p)\psi)^{(n)} = \frac{1}{\sqrt{n}}\sum_{l=1}^{n}{}^*\delta^{(3)}(p-k_l){}^*\psi^{(n-1)}(k_1,\dots,k_{l-1},k_{l+1},\dots,k_n). \tag{35}$$

We express the free internal scalar field and the time zero fields with hyperfinite momentum cut-off $\varkappa \in {}^*\mathbb{R}_\infty$ in terms of ${}^*a^\dagger(p)$ and ${}^*a(p)$ as quadratic forms on ${}^*D_{{}^*S}$ by

$$\begin{gathered}{}^*\Phi_{m,\varkappa}(x,t) = \\ (2\pi)^{-3/2}\,{}^*\!\int_{|p|\leq\varkappa}\left\{\left(\exp(\mu(p)t-ipx)\right){}^*a^\dagger(p) + \left(\exp(\mu(p)t+ipx)\right){}^*a\,(p)\right\}\frac{d^3p}{\sqrt{2\mu(p)}},\end{gathered} \tag{36}$$

$${}^*\varphi_{m,\varkappa}(x,t) = (2\pi)^{-3/2}\,{}^*\!\int_{|p|\leq\varkappa}\left\{\left(\exp(-ipx)\right){}^*a^\dagger(p) + \left(\exp(ipx)\right){}^*a\,(p)\right\}\frac{d^3p}{\sqrt{2\mu(p)}}, \tag{37}$$

$${}^*\pi_{m,\varkappa}(x,t) = (2\pi)^{-3/2}\,{}^*\!\int_{|p|\leq\varkappa}\left\{\left(\exp(-ipx)\right){}^*a^\dagger(p) + \left(\exp(ipx)\right){}^*a\,\,(p)\right\}\frac{d^3p}{\sqrt{\mu(p)/2}}. \tag{38}$$

**Theorem 1.5** Let $\Phi_m(x,t)$ and $\varphi_m(x,t)$, $\pi_m(x,t)$ be the free standard scalar field and the time zero fields respectively. Then for any $\varkappa \in {}^*\mathbb{R}_\infty$ the operator valued internal functions (35)-(37) gives internal representatives for standard operator valued distributions $\Phi_m(x,t)$ and $\varphi_m(x,t)$, $\pi_m(x,t)$ respectively.

**Definition 1.15** Let $\{X,\|\cdot\|\}$ be a standard Banach space. For $x \in {}^*X$ and $\varepsilon > 0, \varepsilon \approx 0$ we define the open $\approx$-ball about $x$ of radius $\varepsilon$ to be the set $B_\varepsilon(x) = \{y \in {}^*X|{}^*\|x-y\| < \varepsilon\}$.

**Definition 1.16** Let $\{\{X,\|\cdot\|\}$ be a standard Banach space, $Y \subset X$, thus ${}^*Y \subset {}^*X$ and let $x \in {}^*X$.Then $x$ is an $*$-accumu-lotion point of ${}^*Y$ if for any $\varepsilon \in {}^*\mathbb{R}_{\approx+}$ there is a hyper infinite sequence $\{x_n\}_{n=1}^{^*\infty}$in ${}^*Y$ such that $\{x_n\}_{n=1}^{^*\infty} \cap (B_\varepsilon(x)\backslash\{x\} \neq \emptyset)$.

**Definition 1.17** Let $\{\{X,\|\cdot\|\}$ be a standard Banach space, let${}^*Y \subseteq {}^*X$,${}^*Y$ is $*$-closed if any $*$-accumulation point of ${}^*Y$ is an element of ${}^*Y$.

**Definition 1.18** Let $\{\{X,\|\cdot\|\}$ be a standard Banach space. We shall say that internal hyper infinite sequence $\{x_n\}_{n=1}^{^*\infty}$in ${}^*X$ is $*$-converges to $x \in {}^*X$ as $n \to {}^*\infty$ if for any $\varepsilon \in {}^*\mathbb{R}_{\approx+}$ there is $N \in {}^*\mathbb{N}$ such that for any $n > N$: ${}^*\|x-y\| < \varepsilon$.

**Definition 1.19** Let $\{\{X,\|\cdot\|_X\}$, $\{\{Y,\|\cdot\|_Y\}$ be a standard Banach spaces. A linear internal operator $A: D(A) \subseteq {}^*X \to {}^*Y$ is $*$-closed if for every internal hyper infinite sequence $\{x_n\}_{n=1}^{^*\infty}$ in $D(A)$ $*$-converging to $x \in {}^*X$ such that $Ax_n \to y \in {}^*Y$ as $n \to {}^*\infty$ one has $x \in D(A)$ and $Ax = y$. Equivalently , $A$ is $*$-closed if its graph is $*$-closed in the direct sum ${}^*X \oplus {}^*Y$.

**Definition 1.20** Let $H$ be a standard external Hilbert space. The graph of the internal linear transformation $T: {}^*H \to {}^*H$ is the set of pairs $\{\langle\varphi, T\varphi\rangle|\varphi \in D(T)\}$. The graph of $T$, denoted by $\Gamma(\mathrm{T})$, is thus a subset of ${}^*H \times {}^*H$ which is internal Hilbert space with inner product $(\langle\varphi_1,\psi_1\rangle,\langle\varphi_2,\psi_2\rangle) = (\varphi_1,\varphi_2) + (\psi_1,\psi_2)$.The operator $T$ is called a $*$-closed operator if $\Gamma(\mathrm{T})$ is a $*$-closed subset of Cartesian product ${}^*H \times {}^*H$.

**Definition 1.21** Let $H$ be a standard Hilbert space. Let $T_1$ and $T$ be internal operators on internal Hilbert space ${}^*H$. Note that if $\Gamma(T_1) \supset \Gamma(T)$, then $T_1$ is said to be an extension of $T$ and we write $T_1 \supset T$. Equivalently, $T_1 \supset T$ if and only if $D(T_1) \supset D(T)$ and $T_1\varphi\ = T\varphi$ for all $\varphi \in D(T)$.

**Definition 1.22** Any internal operator $T$ on ${}^*H$ is $*$-closable if it has a $*$-closed extension. Every $*$-closable internal operator $T$ has a smallest $*$-closed extension, called its $*$-closure, which we denote by $*$-$\bar{T}$.

**Definition 1.23** Let $H$ be a standard Hilbert space. Let $T$ be a $*$-densely defined internal linear operator on internal Hilbert space ${}^*H$. Let $D(T^*)$ be the set of $\varphi \in {}^*H$ for which there is a vector $\xi \in {}^*H$ with $(T\psi,\varphi) = (\varphi,\xi)$ for all $\psi \in D(T)$, then for each $\varphi \in D(T^*)$, we define $T^*\varphi = \xi$. $T^*$ is called the $*$-adjoint of $T$. Note that $S \subset T$ implies $T^* \subset S^*$.

**Definition 1.24** Let $H$ is a standard Hilbert space. A $*$-densely defined internal linear operator $T$ on internal Hilbert space ${}^*H$ is called symmetric (or Hermitian) if $T \subset T^*$. Equivalently, T is symmetric if and only if $(T\varphi,\psi) = (\varphi,T\psi)$ for all $\varphi,\psi \in D(T)$.

**Definition 1.25** Let $H$ be a standard Hilbert space. A symmetric internal linear operator $T$ on internal Hilbert space ${}^*H$ is called essentially self- $*$-adjoint if its $*$-closure $*$-$\bar{T}$ is self- $*$-adjoint. If $T$ is $*$-closed, a subset $D \subset D(T)$ is called a $*$-core for $T$ if $*$- $\left(\overline{T \restriction D}\right) = T$. If $T$ is essentially self- $*$-adjoint, then it has one and only one self -$*$-adjoint extension.

**Theorem 1.6** Let $n_1,n_2 \in \mathbb{N}$ and suppose that $W(k_1,\dots k_{n_1},p_1,\dots,p_{n_2}) \in {}^*L_2({}^*\mathbb{R}^{3(n_1+n_2)})$ where $W(k_1,\dots k_{n_1},p_1,\dots,p_{n_2})$ is a ${}^*\mathbb{C}$-valued internal function on ${}^*\mathbb{R}^{3(n_1+n_2)}$. Then there is a unique operator $T_W$ on ${}^*\mathcal{F}({}^*L_2({}^*\mathbb{R}^3))$ so that ${}^*D_{{}^*S} \subset D(T_W)$ is a $*$ - core for $T_W$ and

(1) as ${}^*\mathbb{C}$-valued quadratic forms on ${}^*D_{{}^*S} \times {}^*D_{{}^*S}$

$$T_W = {}^*\int_{{}^*\mathbb{R}^{3(n_1+n_2)}} W(k_1, \dots k_{n_1}, p_1, \dots, p_{n_2}) \left(\prod_{i=1}^{n_1} {}^*a^\dagger(k_i)\right)\left(\prod_{i=1}^{n_2} {}^*a(p_i)\right) d^{n_1}k d^{n_2}p$$

(2) As ${}^*\mathbb{C}$-valued quadratic forms on $D_{{}^*S} \times D_{{}^*S}$

$$T_W^* = {}^*\int_{{}^*\mathbb{R}^{3(n_1+n_2)}} W(k_1, \dots k_{n_1}, p_1, \dots, p_{n_2}) \left(\prod_{i=1}^{n_1} {}^*a^\dagger(k_i)\right)\left(\prod_{i=1}^{n_2} {}^*a(p_i)\right) d^{n_1}k d^{n_2}p$$

(3) On vectors in ${}^*F_0$ the operators $T_W$ and $T_W^*$ are given by the explicit formulas

$$\left(T_W({}^*\psi)\right)^{(l-n_2+n_1)} =$$

$$K(l, n_1, n_2)\,{}^*\mathbf{S}\left[{}^*\int_{|p_1|\leq\varpi} \dots {}^*\int_{|p_{n_2}|\leq\varpi} W(k_1, \dots k_{n_1}, p_1, \dots, p_{n_2})\,{}^*\psi^{(l)}(p_1, \dots, p_{n_2}, k_1, \dots k_{n_1}) d^{3n_2}p\right], \tag{39}$$

$$\left(T_W^*({}^*\psi)\right)^n = 0 \text{ if } n < n_1 - n_2,$$

$$\left(T_W^*({}^*\psi)\right)^{(l-n_1+n_2)} =$$

$$K(l, n_2, n_1)\,{}^*\mathbf{S}\left[{}^*\int_{|p_1|\leq\varpi} \dots {}^*\int_{|p_{n_2}|\leq\varpi} W(k_1, \dots k_{n_1}, p_1, \dots, p_{n_2})\,{}^*\psi^{(l)}(p_1, \dots, p_{n_2}, k_1, \dots k_{n_1}) d^{3n_1}k\right] \tag{40}$$

$$\left(T_W^*({}^*\psi)\right)^n = 0, \text{ if } n < n_2 - n_1.$$

Here $\mathbf{S}$ is the symmetrization operator defined in [8] and $K(l, n_2, n_1) = \left[\frac{l!(l+n_1-n_2)!}{(l-n_2)^2}\right]^{1/2}$, $n_1, n_2 \in \mathbb{N}, l \in {}^*\mathbb{N}$.

**Proof.** For vectors ${}^*\psi \in D_{{}^*S}$ we define $T_W({}^*\psi)$ by the formula (39). By the Schwarz inequality and the fact that ${}^*\mathbf{S}$ is a projection we get

$$\left({}^*\left\|\left(T_W({}^*\psi)\right)^{(l-n_2+n_1)}\right\|\right)^2 \leq K(l, n_1, n_2)\,{}^*\left\|\left({}^*\psi^{(l)}\right)\right\|^2 {}^*\|W\|^2. \tag{41}$$

Let us now define the operator $T_W^*({}^*\psi)$ on $D_{{}^*S}$ by the formula (39), then for all ${}^*\varphi, {}^*\psi \in D_{{}^*S}$, then one obtains directly ${}^*({}^*\varphi, T_W {}^*\psi) = {}^*(T_W^* {}^*\varphi, {}^*\psi)$. Thus, $T_W$ is $*$-closable and $T_W^*$ is the restriction of the $*$-adjoint of $T_W$ on $D_{{}^*S}$. We will use $T_W$ to denote $*$-$\overline{T}_W$ and $T_W^*$ to denote the $*$-adjoint of $T_W$. By the definition of $T_W$, $D_{{}^*S}$ is a $*$-core and further, since $T_W$ is bounded on the $l$-particle vectors in $D_{{}^*S}$ we get ${}^*F_0 \subset D(T_W)$. Since the right-hand side of (39) is also bounded on the $l$-particle vectors, equation (38) represents $T_W$ on all $l$-particle vectors. The proof of the statement (2) about $T_W^*$ is the same.

**Definition 1.26** [7] Define standard $Q$ -space by $Q = \times_{k=1}^\infty \mathbb{R}$. Let $\sigma$ be the $\sigma$-algebra generated by infinite products of measurable sets in $\mathbb{R}$ and set $\mu = \otimes_{k=1}^\infty \mu_k$ with $d\mu_k = \pi^{-1/2}\exp(-x_k^2/2)$. Denote the points of $Q$ by $q = \langle q_1, q_2, \dots \rangle$. Then $\langle Q, \mu\rangle$ is a measure space and the set of the all functions of the form $P_n(q) = P(q_1, q_2, \dots, q_n)$, where $P_n(q)$ is a polynomial and $n \in \mathbb{N}$ is arbitrary, is dense in $L_2(Q, d\mu)$. Remind that there exists a unitary map $S\colon \mathcal{F}_s(H) \to L_2(Q, d\mu)$ of Fock space $\mathcal{F}_s(H)$ onto $L_2(Q, d\mu)$ so that $\mathrm{S}\varphi(f_k)\mathrm{S}^{-1} = q_k$ and $\mathrm{S}\Omega_0 = 1$. Here $\{f_k\}_{k=1}^\infty$ is an orthonormal basis for $H$. Then by transfer one obtains internal measure space ${}^*\langle Q, \mu\rangle = \langle {}^*Q, {}^*\mu\rangle$ and internal unitary map ${}^*\mathrm{S}\colon \mathcal{F}_s(H) \to {}^*L_2({}^*Q, d{}^*\mu)$ so that ${}^*\mathrm{S}\varphi(f_r){}^*\mathrm{S}^{-1} = q_r$, $r \in {}^*\mathbb{N}$ and ${}^*\mathrm{S}\Omega_0 = 1$. Here $\{f_r\}_{r=1}^{{}^*\infty}$ is an orthonormal basis for ${}^*H$.

**Theorem 1.7** Let ${}^*\varphi_\varkappa(x, t)$ be internal free scalar boson field of mass $m$ at time $t = 0$ with hyperfinite momentum cutoff $\varkappa$ in four-dimensional space-time. Let $g(x)$ be a real-valued internal function in ${}^*L_2({}^*\mathbb{R}^3) \cap {}^*L_1({}^*\mathbb{R}^3)$. Then the operator

$${}^*H_{I,\varkappa}(g) = \lambda(\varkappa)\,{}^*\int_{{}^*\mathbb{R}^3} g(x) : {}^*\varphi_\varkappa^4(x) : d^3x \tag{42}$$

is a well-defined internal symmetric operator on ${}^*D_{{}^*S_{\mathrm{fin}}}$. Here $: {}^*\varphi_\varkappa^4(x) := {}^*\varphi_\varkappa^4(x) + d_2(\varkappa)\left({}^*\varphi_\varkappa^2(x)\right) + d_1(\varkappa)$. where the coefficients $d_2(\varkappa)$ and $d_1(\varkappa)$ are independent of $x$. Let S denote the unitary map of $\mathcal{F}_s(H)$ onto $L_2(Q, d\mu)$ considered in [7]. Then $V = {}^*\mathrm{S}\,{}^*H_{I,\varkappa}(g){}^*\mathrm{S}^{-1}$ is multiplication by internal function $V_{I,\varkappa}(q)$ which satisfies:

(a) $V_{I,\varkappa}(q) \in {}^*L_p({}^*Q, d{}^*\mu)$ for all $p \in {}^*\mathbb{N}$, (b) $\exp\left(-tV_{I,\varkappa}(q)\right) \in {}^*L_1({}^*Q, d{}^*\mu)$ for all $t \in [0, {}^*\infty)$.

Proof: Note that for each $x \in {}^*\mathbb{R}^3$, the operator ${}^*\mathrm{S}({}^*\varphi_\varkappa(x)){}^*\mathrm{S}^{-1}$ is just the operator on internal measurable space ${}^*L_2({}^*\mathrm{Q}, d{}^*\mu)$ on which this operator acts by multiplying by the function $\sum_{k=1}^{{}^*\infty} c_k(x, \varkappa) q_k$, where $c_k(x, \varkappa) = (2\pi)^{3/2}\left(f_k, (\mu(p))^{1/2}\exp(ipx)\right)$. Furthermore, $\sum_{k=1}^{{}^*\infty} |c_k(x, \varkappa)|^2 = (2\pi)^{3/2}\,{}^*\left\|\mu(p)^{1/2}\right\|_2^2$ so ${}^*\mathrm{S}\left({}^*\varphi_\varkappa^4(x)\right){}^*\mathrm{S}^{-1}$and ${}^*\mathrm{S}\left({}^*\varphi_\varkappa^2(x)\right){}^*\mathrm{S}^{-1}$ are in ${}^*L_2({}^*\mathrm{Q}, d{}^*\mu)$ and the corresponding ${}^*L_2({}^*\mathrm{Q}, d{}^*\mu)$-norms are uniformly bounded in $x$.

Therefore, since $g \in {}^*L_1({}^*\mathbb{R}^3)$ the operator ${}^*\mathrm{S}\left({}^*H_{I,\varkappa}(g)\right){}^*\mathrm{S}^{-1}$ is just the operator on internal measurable space ${}^*L_2({}^*\Omega, d^*\mu)$ on which this operator acts by multiplying by the ${}^*L_2({}^*\mathrm{Q}, d^*\mu)$-function which we denote by $V_{\varkappa,\lambda}(q)$. Let us consider now the expression for ${}^*H_{I,\varkappa}(g){}^*\Omega$, obviously this is a vector $(0,0,0,0,\psi^4,0,\dots)$ with

$$\psi^4(p_1,p_2,p_3,p_4) = {}^*\!\!\int_{{}^*\mathbb{R}^3} \frac{\lambda(\varkappa)g(x)\prod_{i=1}^{4}[\chi(\varkappa,p_i)]\exp\left(-ix\sum_{i=1}^{i=4}p_i\right)d^3x}{(2\pi)^{3/2}\prod_{i=1}^{4}[2\mu(p_i)]^{1/2}}. \quad (43)$$

Here $\chi(\varkappa,p) \equiv 1$ if $|p| \le \varkappa, \chi(\varkappa,p) \equiv 0$ if $|p| > \varkappa$, $\varkappa \in {}^*\mathbb{R}_\infty$. We choose now the parameter $\lambda = \lambda(\varkappa) \approx 0$ such that ${}^*\|\psi^4\|_2^2 \in \mathbb{R}$ and therefore we obtain ${}^*\left\| {}^*H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0\right\|_2^2 \in \mathbb{R}$, since ${}^*\left\| {}^*H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0\right\|_2^2 = {}^*\|\psi^4\|_2^2$. But, since ${}^*\mathrm{S}{}^*\Omega_0 = 1$, we get the equalities

$${}^*\left\| {}^*H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0\right\|_2 = \left\|{}^*\mathrm{S}\, H_{I,\varkappa,\lambda(\varkappa)}(g){}^*\mathrm{S}^{-1}\right\|_{{}^*L_2({}^*Q,d^*\mu)} = {}^*\left\|V_{I,\varkappa,\lambda(\varkappa)}(q)\right\|_{{}^*L_2({}^*Q,d^*\mu)}. \quad (44)$$

From (43) we get that ${}^*\left\|V_{I,\varkappa,\lambda(\varkappa)}(q)\right\|_{{}^*L_2({}^*Q,d^*\mu)} \in \mathbb{R}$ and it is easily verify, that each polynomial $P(q_1,q_2,\dots,q_n)$, is $n \in {}^*\mathbb{N}$ in the domain of the operator $V_{I,\varkappa,\lambda(\varkappa)}(q)$ and ${}^*\mathrm{S}\,{}^*H_{I,\varkappa,\lambda(\varkappa)}(g){}^*\mathrm{S}^{-1} \equiv V_{I,\varkappa,\lambda(\varkappa)}(q)$ on that domain. Since ${}^*\Omega_0$ is in the domain of ${}^*H^p{}_{I,\varkappa,\lambda(\varkappa)}(g), p \in {}^*\mathbb{N}$, 1 is in the domain of the operator $V^p{}_{I,\varkappa,\lambda(\varkappa)}(q)$ for all $p \in {}^*\mathbb{N}$. Thus, for all $p \in {}^*\mathbb{N}$ $V_{I,\varkappa,\lambda(\varkappa)}(q) \in {}^*L_{2p}({}^*Q, d^*\mu)$, since ${}^*\mu\,({}^*Q)$ is finite, we conclude that $V_{I,\varkappa,\lambda(\varkappa)}(q) \in {}^*L_p({}^*Q, d^*\mu)$ for all $p \in {}^*\mathbb{N}$.

(b) Remind Wick's theorem asserts that $:{}^*\varphi_{m,\varkappa}^j(x) := \sum_{i=0}^{[j/2]}(-1)^i \frac{j!}{(j-2i)!i!} c_\varkappa^i\, {}^*\varphi_{m,\varkappa}^{(j-2i)}(x)$ with $c_\varkappa = {}^*\left\|{}^*\varphi_{m,\varkappa}(x){}^*\Omega_0\right\|_2^2$. For $j = 4$ we get $-O(c_\varkappa^2) \le :{}^*\varphi_{m,\varkappa}^4(x):$ and therefore $-\left({}^*\!\int_{{}^*\mathbb{R}^3} g(x)\, d^3x\right) O(c_\varkappa^2) \le {}^*H_{I,\varkappa,\lambda(\varkappa)}(g)$. Finally we obtain ${}^*\!\int_{{}^*Q} \exp\left(-t(:{}^*\varphi_{m,\varkappa}^4(x):)\right) d\,{}^*\mu \le \exp\left(O(c_\varkappa^2)\right)$ and this inequality finalized the proof.

**Theorem 1.8** [7] Let $\langle M,\mu\rangle$ be a $\sigma$-measure standard space with $\mu(M) = 1$and let $H_0$ be the generator of a hyper- contractive semigroup on $L_2(M, d\mu)$. Let $V$ be a $\mathbb{R}$-valued measurable function on $\langle M,\mu\rangle$ such that $V \in L_p(M, d\mu)$ for all $p \in [\![1,\infty)$ and $\exp(-tV) \in L_1(M, d\mu)$ for all $t > 0$. Then $H_0 + V$ is essentially self-adjoint on $C^\infty(H_0) \cap D(V)$ and is bounded below. Here $C^\infty(H_0) = \bigcap_{p\in\mathbb{N}} D\left(H_0^p\right)$.

**Theorem 1.9** Let $\langle M,\mu\rangle$ be a $\sigma$-measure space with $\mu(M) = 1$and let $H_0$ be the generator of a hypercontractive semi-group on $L_2(M, d\mu)$. Let$V$ be a ${}^*\mathbb{R}$-valued internal measurable function on $\langle {}^*M, {}^*\mu\rangle$ such that $V \in {}^*L_p({}^*M, d^*\mu)$ for all $p \in [1, {}^*\infty)$ and ${}^*\exp(-tV) \in {}^*L_1({}^*M, d^*\mu)$ for all $t > 0$. Assume that a set $C^{{}^*\infty}({}^*H_0) \cap D(V)$ is internal. Then operator ${}^*H_0 + V$ is essentially self-$*$-adjoint internal operator on $C^{{}^*\infty}({}^*H_0) \cap D(V)$ and it is hyper finitely bounded below. Here $C^{{}^*\infty}({}^*H_0) = \bigcap_{p\in{}^*\mathbb{N}} D\left({}^*H_0^p\right)$.

**Proof.** It follows immediately by transfer from theorem 8.

**Remark 1.4** Let $V_{I,\varkappa,\lambda}$ be operator on internal measurable space ${}^*L_2({}^*\Omega, d^*\mu)$ on which this operator acts by multiplying by the ${}^*L_2({}^*\mathrm{Q}, d^*\mu)$-function$V_{I,\varkappa,\lambda}$, see proof to Theorem 1.7. Note that for this operator a set $C^{{}^*\infty}({}^*H_0) \cap D\left(V_{I,\varkappa,\lambda}\right)$ is not internal and therefore Theorem9 no longer holds. But without this theorem we cannot conclude that operator ${}^*H_0 + V_{I,\varkappa,\lambda}$ is essentially self-$*$-adjoint internal operator on $C^{{}^*\infty}({}^*H_0) \cap D\left(V_{I,\varkappa,\lambda}\right)$. Thus Robinson's transfer is of no help in the case corresponding to operator $V_{I,\varkappa,\lambda}$ considered above. In order to resolve this issue, we will use non conservative extension of the model theoretical nonstandard analysis, see [9-13].

## NON CONSERVATIVE EXTENSION OF THE MODEL THEORETICAL NONSTANDARD ANALYSIS

Remind that Robinson nonstandard analysis (RNA) many developed using set theoretical objects called super-structures [2-6, 14]. A superstructure V(S) over a set S is defined in the following way: $\mathrm{V}_0(\mathrm{S}) = \mathrm{S}$, $\mathrm{V}_{n+1}(\mathrm{S}) = \mathrm{V}_n(\mathrm{S}) \cup \mathrm{P}\left(\mathrm{V}_n(\mathrm{S})\right)$, $\mathrm{V}(\mathrm{S}) = \bigcup_{n\in\mathbb{N}} \mathrm{V}_{n+1}(\mathrm{S})$. Making $\mathrm{S} = \mathbb{R}$ will suffice for virtually any construction necessary in analysis. Bounded formulas are formulas where all quantifiers occur in the form: $\forall x\,(x \in y \to \cdots)$, $\exists x\,(x \in y \to \cdots)$. A nonstandard embedding is a mapping $*: \mathrm{V}(X) \to \mathrm{V}(Y)$ from a superstructure $\mathrm{V}(X)$ called the standard universe, into another superstructure $\mathrm{V}(Y)$ called nonstandard universe, satisfying the following postulates:

1. $Y = {}^*X$

2. **Transfer Principle** For every bounded formula $\Phi(x_1,\dots,x_n)$ and elements $a_1,\dots,a_n \in V(X)$ the property $\Phi(a_1,\dots,a_n)$ is true for $a_1,\dots,a_n$ in the standard universe if and only if it is true for ${}^*a_1,\dots,{}^*a_n$ in the nonstandard universe $V(X)\quad \Phi(x_1,\dots,x_n) \leftrightarrow V(Y)\quad \Phi({}^*a_1,\dots,{}^*a_n)$.

3. **Non-triviality** For every infinite set $A$ in the standard universe, the set $\{{}^*a|a\in A\}$ is a proper subset of ${}^*A$.

**Definition 2.1** A set $x$ is internal if and only if $x$ is an element of ${}^*A$ for some $A \in \mathrm{V}(\mathbb{R})$. Let $X$ be a set and $A=\{A_i\}_{i\in I}$ a family of subsets of $X$ .Then the collection $A$ has the infinite intersection property, if any infinite sub collection $J \subset I$ has non-empty intersection. Nonstandard universe is $\sigma$ -saturated if whenever $\{A_i\}_{i\in I}$ is a collection of internal sets with the infinite intersection property and the cardinality of $I$ is less than or equal to $\sigma$.

**Remark 2.1** For each standard universe $U=V(X)$ there exists canonical language $L_U$ and for each nonstandard universe $W=V(Y)$ there exists corresponding canonical nonstandard language ${}^*L=L_W$ [5, 14]

4.***The restricted rules of conclusion*** If Let $A$ and $B$ well formed, closed formulas so that $A,B\in{}^*L$. If $W\vDash A$, then $\neg A \nvdash_{RMP} B$. Thus, if a statement $A$ holds in nonstandard universe, we cannot obtain from formula $\neg A$ any formula $B$ whatsoever.

**Definition 2.2** [9-13] A set $S\subset{}^*\mathbb{N}$ is a hyper inductive if the following statement holds in $V(Y)$:

$$\bigwedge\nolimits_{\alpha\in{}^*\mathbb{N}}(\alpha\in S\to\alpha^+\in S).$$

Here $\alpha^+=\alpha+1$.Obviously a set ${}^*\mathbb{N}$ is a hyper inductive.

5. Axiom of hyper infinite induction

$$\forall S(S\subset{}^*\mathbb{N})\{\forall\beta(\beta\subset{}^*\mathbb{N})\left[\wedge_{1\le\alpha<\beta}(\alpha\in S\to\alpha^+\in S)\right]\to S={}^*\mathbb{N}\}.$$

**Example 2.1** Remind the proof of the following statement: structure $(\mathbb{N},<,=)$ is a well-ordered set.

**Proof.** Let $X$ be a nonempty subset of $\mathbb{N}$. Suppose X does not have a <-least element. Then consider the set $\mathbb{N}\backslash X$.

Case1. $\mathbb{N}\backslash X=\emptyset$. Then $X=\mathbb{N}$ and so 0 is a $<$ -least element but this is a contradiction.

Case2. $\mathbb{N}\backslash X\neq\emptyset$. Then $1\in\mathbb{N}\backslash X$ otherwise 1 is a $<$ -least element but this is a contradiction. Assume now that there exists some $n\in\mathbb{N}\backslash X$ such that $n\neq 1$, but since we have supposed that $X$ does not have a $<$ -least element, thus $n+1\notin X$. Thus we see that for all $n$ the statement $n\in\mathbb{N}\backslash X$ implies that $n+1\in\mathbb{N}\backslash X$. We can conclude by axiom of induction that $n\in\mathbb{N}\backslash X$ for all $n\in\mathbb{N}$. Thus $\mathbb{N}\backslash X=\mathbb{N}$ implies $X=\emptyset$. This is a contradiction to $X$ being a non-empty subset of $\mathbb{N}$. Remind that structure $({}^*\mathbb{N},<,=)$ is not a well-ordered set [5, 6, 14]. We set now $X_1={}^*\mathbb{N}\backslash\mathbb{N}$ and thus ${}^*\mathbb{N}\backslash X_1=\mathbb{N}$. In contrast with a set $X$ mentioned above the assumption $n\in{}^*\mathbb{N}\backslash X_1$ implies that $n+1\in{}^*\mathbb{N}\backslash X_1$ if and only if $n$ is finite, since for any infinite $n\in{}^*\mathbb{N}\backslash\mathbb{N}$ the assumption $n\in{}^*\mathbb{N}\backslash X_1$ contradicts with a true statement $V(Y)\vDash n\notin{}^*\mathbb{N}\backslash X_1$=$\mathbb{N}$ and therefore in accordance with postulate 4 we cannot obtain from $n\in{}^*\mathbb{N}\backslash X_1$ any closed formula $B$ whatsoever.

**Theorem 2.1** [13] (Generalized Recursion Theorem) Let $S$ be a set, $c\in S$ and $g\colon S\times{}^*\mathbb{N}\to S$ is any function with $\mathrm{dom}(g)=S\times{}^*\mathbb{N}$ and $\mathrm{range}(g)\subseteq S$, then there exists a function $\mathcal{F}\colon{}^*\mathbb{N}\to S$ such that: 1) $\mathrm{dom}(\mathcal{F})={}^*\mathbb{N}$ and $\mathrm{range}(\mathcal{F})\subseteq S$; 2) $\mathcal{F}(1)=c$; 3) for all $x\in{}^*\mathbb{N},\mathcal{F}(n+1)=g(\mathcal{F}(n),n)$.

**Definition 2.3** [11-13] (1) Suppose that $S$ is a standard set on which a binary operations $(\cdot+\cdot)$ and $(\cdot\times\cdot)$ is defined and under which $S$ is closed. Let $\{x_k\}_{k\in{}^*\mathbb{N}}$ be any hyper infinite sequence of terms of ${}^*S$. For every hyper natural $n\in{}^*\mathbb{N}$ we denote by $Ext\text{-}\sum_{k=1}^{n}x_k$ the element of ${}^*S$ uniquely determined by the following canonical conditions: (a) $Ext\text{-}\sum_{k=1}^{1}x_k=x_1$; (b) $Ext\text{-}\sum_{k=1}^{n+1}x_k=Ext\text{-}\sum_{k=1}^{n}x_k+x_{n+1}$ for all $n\in{}^*\mathbb{N}$.

(2) For every hyper natural $n\in{}^*\mathbb{N}_\infty$ we denote by $Ext\text{-}\prod_{k=1}^{n}x_k$ the element of ${}^*S$ uniquely determined by the following canonical conditions: (a) $Ext\text{-}\prod_{k=1}^{1}x_k=x_1$; (b) $Ext\text{-}\prod_{k=1}^{n+1}x_k=(Ext\text{-}\prod_{k=1}^{n}x_k)\times x_{n+1}$ for all $n\in{}^*\mathbb{N}$.

**Theorem 2.2.** [13] (1) suppose that $S$ is a standard set on which a binary operation $(\cdot+\cdot)$ is defined and under which $S$ is closed and that $(\cdot+\cdot)$ is associative on S. Let $\{x_k\}_{k\in{}^*\mathbb{N}}$ be any hyper infinite sequence of terms of ${}^*S$. Then for any $n,m\in{}^*\mathbb{N}$ we have: $Ext\text{-}\sum_{k=1}^{n+m}x_k=Ext\text{-}\sum_{k=1}^{n}x_k+Ext\text{-}\sum_{k=1}^{m}x_k$ ;

(2) suppose that $S$ is a standard set on which a binary operation $(\cdot\times\cdot)$ is defined and under which $S$ is closed and that $(\cdot\times\cdot)$ is associative on S. Let $\{x_k\}_{k\in{}^*\mathbb{N}}$ be any hyper infinite sequence of terms of ${}^*S$. Then for any $n,m\in{}^*\mathbb{N}$ we have: $Ext\text{-}\prod_{k=1}^{n+m}x_k=(Ext\text{-}\prod_{k=1}^{n}x_k)\times(Ext\text{-}\prod_{k=1}^{m}x_k)$; (3) for any $z\in{}^*S$ and for any $n\in{}^*\mathbb{N}_\infty$ we have:

$$z\times(Ext\text{-}\textstyle\sum_{k=1}^{n}x_k)=Ext\text{-}\sum_{k=1}^{n}z\times x_k.$$

## External non-Archimedean Field ${}^*\mathbb{R}_c^\#$ by Cauchy Completion of the Internal Non - Archimedean Field ${}^*\mathbb{R}$.

**Definition 2.4** A hyper infinite sequence of hyperreal numbers from ${}^*\mathbb{R}$ is a function $a: {}^*\mathbb{N} \to {}^*\mathbb{R}$ from the hyper- natural numbers ${}^*\mathbb{N}$ into the hyperreal numbers ${}^*\mathbb{R}$.We usually denote such a function by $n \mapsto a_n$ , so the terms in the sequence are written as $\{a_1, a_2, \ldots, a_n, \ldots\}$.To refer to the whole hyper infinite sequence, we will write $\{a_n\}_{n=1}^{{}^*\infty}$ or $\{a_n\}_{n\in {}^*\mathbb{N}}$.

**Abbreviation 2.1** For a standard set $E$ we often write $E_{\mathrm{st}}$, let ${}^\sigma E_{\mathrm{st}} = \{{}^*x | x \in E_{\mathrm{st}}\}$.We identify $z$ with ${}^\sigma z$ i.e., $z \equiv {}^\sigma z$ for all $z \in \mathbb{C}$. Hence, ${}^\sigma E_{\mathrm{st}} = E_{\mathrm{st}}$ if $E \subseteq \mathbb{C}$, e.g., ${}^\sigma\mathbb{C} = \mathbb{C}$, ${}^\sigma\mathbb{R} = \mathbb{R}$, etc.Let ${}^*\mathbb{R}_c^\#$, ${}^*\mathbb{R}_{c,\approx}^\#$ , ${}^*\mathbb{R}_{c,\approx+}^\#$ , ${}^*\mathbb{R}_{c,\mathrm{fin}}^\#$ , ${}^*\mathbb{R}_{c,\infty}^\#$ , ${}^*\mathbb{N}_\infty$ de-note the sets of Cauchy hyper-real numbers, Cauchy infinitesimal hyper-real numbers, Cauchy positive infinitesimal hyperreal numbers, Cauchy finite hyper-real numbers, Cauchy infinite hyper-real numbers and infinite hypernatural numbers, respectively. Note that ${}^*\mathbb{R}_{c,\mathrm{fin}}^\# = {}^*\mathbb{R}_c^\# \backslash {}^*\mathbb{R}_{c,\infty}^\#$.

**Definition 2.5** Let $\{a_n\}_{n=1}^{{}^*\infty}$ be a hyper infinite ${}^*\mathbb{R}$- valued sequence mentioned above. We shall say that $\{a_n\}_{n=1}^{{}^*\infty}$ #-tends to 0 if, given any $\varepsilon \in {}^*\mathbb{R}_{\approx+}$, there is a hyper natural number $N \in {}^*\mathbb{N}$ such that for all $n > N$, $|a_n| \leq \varepsilon$. We denote this symbolically by $a_n \to_\# 0$.

**Definition 2.6** Let $\{a_n\}_{n=1}^{{}^*\infty}$ be a hyper infinite ${}^*\mathbb{R}$-valued sequence mentioned above. We shall say that $\{a_n\}_{n=1}^{{}^*\infty}$ #-tends to $q \in {}^*\mathbb{R}$ if, given any $\varepsilon \in {}^*\mathbb{R}_{\approx+}$, there is a hyper natural number $N \in {}^*\mathbb{N}$ such that for all $n > N$, $|a_n - q| \leq \varepsilon$ and we denote this symbolically by $a_n \to_\# q$ or by $\#\text{-}\lim_{n\to{}^*\infty} a_n = q$.

**Definition 2.7** Let $\{a_n\}_{n=1}^{{}^*\infty}$ be a hyper infinite ${}^*\mathbb{R}$-valued sequence mentioned above. We shall say that sequence $\{a_n\}_{n=1}^{{}^*\infty}$ is bounded if there is a hyperreal $M \in {}^*\mathbb{R}$ such that for any $n \in {}^*\mathbb{N}$, $|a_n| \leq M$.

**Definition 2.8** Let $\{a_n\}_{n=1}^{{}^*\infty}$ be a hyper infinite ${}^*\mathbb{R}$-valued sequence mentioned above. We shall say that $\{a_n\}_{n=1}^{{}^*\infty}$ is a Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequence if , given any $\varepsilon \in {}^*\mathbb{R}_{\approx+}$ , there is a hyper natural number $N(\varepsilon) \in {}^*\mathbb{N}$ such that for any $m, n > N$, $|a_n - a_m| < \varepsilon$.

**Theorem 2.3** If $\{a_n\}_{n=1}^{{}^*\infty}$ is a #-convergent hyper infinite ${}^*\mathbb{R}$-valued sequence, i.e., that is, $a_n \to_\# q$ for some hyper-real number $q, q \in {}^*\mathbb{R}$ then $\{a_n\}_{n=1}^{{}^*\infty}$ is a Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequence.

**Theorem 2.4** If $\{a_n\}_{n=1}^{{}^*\infty}$ is a Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequence, then it is finitely bounded or hyper finitely bounded; that is, there is some finite or hyperfinite $M \in {}^*\mathbb{R}_+$ such that $|a_n| \leq M$ for all $n \in {}^*\mathbb{N}$.

**Definition 2.8** Let $S$ be a set, with an equivalence relation $(\cdot \sim \cdot)$ on pairs of elements. For$s \in S$, denote by $cl[s]$ the set of all elements in $S$ that are related to $s$. Then for any $s, t \in S$, either $cl[s] = cl[t]$ or $cl[s]$ and $cl[t]$ are dis-joint.

**Remark 2.2** The hyperreal numbers ${}^*\mathbb{R}_c^\#$ will be constructed as equivalence classes of Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences. Let $\mathcal{F}\{{}^*\mathbb{R}\}$ denote the set of all Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences of hyperreal numbers. We define the equivalence relation on a set $\mathcal{F}\{{}^*\mathbb{R}\}$.

**Definition 2.9** Let $\{a_n\}_{n=1}^{{}^*\infty}$ and $\{b_n\}_{n=1}^{{}^*\infty}$ be in $\mathcal{F}\{{}^*\mathbb{R}\}$. Say they are #-equivalent if $a_n - b_n \to_\# 0$ i.e., if and only if the hyper infinite ${}^*\mathbb{R}$-valued sequence $\{a_n - b_n\}_{n=1}^{{}^*\infty}$ #-tends to 0.

**Theorem 2.5** [13] Definition above yields an equivalence relation on a set $\mathcal{F}\{{}^*\mathbb{R}\}$.

**Definition 2.10** The external hyperreal numbers ${}^*\mathbb{R}_c^\#$ are the equivalence classes $cl[\{a_n\}]$ of Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences of hyperreal numbers, as per definition above. That is, each such equivalence class is an external hyperreal number.

**Definition 2.11** Given any hyperreal number $q \in {}^*\mathbb{R}$**,** define a hyperreal number $q^\#$to be the equivalence class of the hyper infinite ${}^*\mathbb{R}$-valued sequence $\{a_n = q\}_{n=1}^{{}^*\infty}$consisting entirely of $q \in {}^*\mathbb{R}$. So we view ${}^*\mathbb{R}$ as being inside ${}^*\mathbb{R}_c^\#$ by thinking of each hyperreal number $q \in {}^*\mathbb{R}$ as its associated equivalence class $q^\#$. It is standard to abuse this notation, and simply refer to the equivalence class as q as well.

**Definition 2.12** Let $s, t \in {}^*\mathbb{R}_c^\#$, so there are Cauchy hyper infinite ${}^*\mathbb{R}$-valued sequences $\{a_n\}_{n=1}^{{}^*\infty}$, $\{b_n\}_{n=1}^{{}^*\infty}$ of hyper-real numbers with $s = cl[\{a_n\}]$ and $t = cl[\{b_n\}]$.

(a) Define $s + t$ to be the equivalence class of the hyper infinite sequence $\{a_n + b_n\}_{n=1}^{{}^*\infty}$.

(b) Define $s \times t$ to be the equivalence class of the hyper infinite sequence $\{a_n + b_n\}_{n=1}^{{}^*\infty}$.

**Theorem 2.6** [13] The operations $+, \times$ in definition above by the requirements (a) and (b) are well-defined.

**Theorem 2.7** Given any hyperreal number $s \in {}^*\mathbb{R}_c^\#, s \neq 0$ there is a hyperreal number $t \in {}^*\mathbb{R}_c^\#$ such that $s \times t = 1$.

**Theorem 2.8** If $\{a_n\}_{n=1}^{*\infty}$ is a Cauchy hyper infinite sequence which does not #-tend to 0, then there is some $N \in {}^*\mathbb{N}$ such that, for all $n > N, a_n \neq 0$.

**Definition 2.13** Let $s \in {}^*\mathbb{R}_c^\#$. Say that $s$ is positive if $s \neq 0$, and if $s = cl[\{a_n\}]$ for some Cauchy hyper infinite sequence of hyperreal numbers such that for some $N \in {}^*\mathbb{N}, a_n > 0$ for all $n > N$. Then for a given two hyperreal numbers $s, t$, say that $s > t$ if $s - t$ is positive.

**Theorem 2.9** Let $s, t \in {}^*\mathbb{R}_c^\#$ be hyperreal numbers such that $s > t$, and let $r \in {}^*\mathbb{R}_c^\#$, then $s + r > t + r$.

**Theorem 2.10** Let $s, t \in {}^*\mathbb{R}_c^\#$ be hyperreal numbers such that $s, t > 0$. Then there is $m \in {}^*\mathbb{N}$ such that $m \times s > t$.

**Theorem 2.11** Given any hyperreal number $r \in {}^*\mathbb{R}_c^\#$, and any hyperreal number $\varepsilon > 0, \varepsilon \approx 0$, there is a hyperreal number $q \in {}^*\mathbb{R}_c^\#$ such that $|r - q| < \varepsilon$.

**Definition 2.14** Let $S \subsetneq {}^*\mathbb{R}_c^\#$ be a nonempty set of hyperreal numbers. A hyperreal number $x \in {}^*\mathbb{R}_c^\#$ is called an upper bound for $S$ if $x \geq s$ for all $s \in S$. A hyperreal number $x$ is the least upper bound (or supremum: sup$S$) for $S$ if $x$ is an upper bound for $S$ and $x \leq y$ for every upper bound $y$ of $S$.

**Remark 2.3** The order $\leq$ given by definition above obviously is $\leq$-incomplete.

**Definition 2.15** Let $S \subsetneq {}^*\mathbb{R}_c^\#$ be a nonempty set of hyperreal numbers. We will say that:

(1) $S$ is $\leq$ -admissible above if the following conditions are satisfied:

(a) $S$ is finitely bounded or hyper finitely bounded above;

(b) let $A(S)$ be a set such that $\forall x[x \in A(S) \Leftrightarrow x \geq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there are $\alpha \in S$ and $\beta \in A(S)$ such that $\beta - \alpha \leq \varepsilon \approx 0$.

(2) $S$ is $\leq$ -admissible belov if the following conditions are satisfied:

(a) $S$ is finitely bounded or hyper finitely bounded below;

(b) let $L(S)$ be a set such that $\forall x[x \in L(S) \Leftrightarrow x \leq S]$ then for any $\varepsilon > 0, \varepsilon \approx 0$ there are $\alpha \in S$ and $\beta \in L(S)$ such that $\alpha - \beta \leq \varepsilon \approx 0$.

**Theorem 2.12** [13] (a) Any $\leq$-admissible above subset $S \subset {}^*\mathbb{R}_c^\#$ has the least upper bound property.

(b) Any $\leq$-admissible above subset $S \subset {}^*\mathbb{R}_c^\#$ has the greatest lower bound property.

**Theorem 2.13** [13] (Generalized Nested Intervals Theorem) Let $\{I_n\}_{n=1}^{*\infty} = \{[a_n, b_n]\}_{n=1}^{*\infty}$, $[a_n, b_n] \subset {}^*\mathbb{R}_c^\#$ be a hyper infinite sequence of #-closed intervals satisfying each of the following conditions:

(a) $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \supseteq I_n \supseteq \cdots$

(b) $b_n - a_n \to_\# 0$ as $n \to {}^*\infty$, Then $\bigcap_{n=1}^{*\infty} I_n$consists of exactly one hyperreal number $x \in {}^*\mathbb{R}_c^\#$.

**Theorem 2.14** [13] (Generalized Squeeze Theorem) Let $\{a_n\}_{n=1}^{*\infty}$, $\{c_n\}_{n=1}^{*\infty}$ be two hyper infinite sequences #-converging to $L$, and $\{b_n\}_{n=1}^{*\infty}$ a hyper infinite sequence. If $\forall n > K, K \in {}^*\mathbb{N}$ we have $a_n \leq b_n \leq c_n$, then $b_n$ also #-converges to $L$.

**Theorem 2.15** [13] If #-$\lim_{n\to{}^*\infty,}| a_n| = 0$, then #-$\lim_{n\to{}^*\infty}, a_n = 0$.

**Theorem 2.16** [13] (Generalized Bolzano -Weierstrass Theorem) Any finitely or hyper finitely bounded hyper infinite ${}^*\mathbb{R}_c^\#$ -valued sequence has #-convergent hyper infinite subsequence.

**Definition 2.16** Let $\{a_n\}_{n=1}^{*\infty}$ be ${}^*\mathbb{R}_c^\#$-valued sequence. Say that a sequence $\{a_n\}_{n=1}^{*\infty}$ #-tends to 0 if, given any $\varepsilon > 0$, $\varepsilon \approx 0$, there is a hyper natural number $N \in {}^*\mathbb{N}_\infty, N = N(\varepsilon)$ such that, for all $n > N$, $|a_n| \leq \varepsilon$.

**Definition 2.17** Let $\{a_n\}_{n=1}^{*\infty}$ be ${}^*\mathbb{R}_c^\#$-valued hyper infinite sequence. We call $\{a_n\}_{n=1}^{*\infty}$ a Cauchy hyper infinite sequence if given any hyperreal number $\varepsilon \in {}^*\mathbb{R}_{c,\approx+}^\#$, there is a hypernatural number $N = N(\varepsilon)$ such that for any $m, n > N$, $|a_n - a_m| < \varepsilon$.

**Theorem 2.17** If $\{a_n\}_{n=1}^{*\infty}$is a #-convergent hyper infinite sequence i.e., $a_n \to_\# b$ for some hyperreal number $b \in {}^*\mathbb{R}_c^\#$, then $\{a_n\}_{n=1}^{*\infty}$ is a Cauchy hyper infinite sequence.

**Theorem 2.18** If $\{a_n\}_{n=1}^{*\infty}$ is a Cauchy hyper infinite sequence, then it is bounded; that is, there is some $M \in {}^*\mathbb{R}_c^\#$ such that $|a_n| \leq M$ for all $n \in {}^*\mathbb{N}$.

**Theorem 2.19** [13] Any Cauchy hyper infinite sequence $\{a_n\}_{n=1}^{*\infty}$ has a #-limit in ${}^*\mathbb{R}_c^\#$; that is, there exists $b \in {}^*\mathbb{R}_c^\#$ such that $a_n \to_\# b$.

**Remark 2.4** Note that there exists canonical natural embedding ${}^*\mathbb{R} \hookrightarrow {}^*\mathbb{R}_c^\#$.

**Remark 2.5** A nonempty set S of Cauchy hyperreal numbers ${}^*\mathbb{R}_c^\#$ is unbounded above if it has no hyperfinite upper bound, or unbounded below if it has no hyperfinite lower bound. It is convenient to adjoin to Cauchy hyperreal number system ${}^*\mathbb{R}_c^\#$ two points, $+\infty^\# = ({}^*{+}\infty)^\#$ (which we also write more simply as $\infty^\#$ ) and $-\infty^\#$, and to define the order relationships between them and any Cauchy hyperreal number $x \in {}^*\mathbb{R}_c^\#$ by $-\infty^\# < x < \infty^\#$.

**Definition 2.18** We will call $-\infty^\#$ and $\infty^\#$ are points at hyper infinity. If $S \subset {}^*\mathbb{R}_c^\#$ is a nonempty set of Cauchy hyperreals, we write $\sup(S) = \infty^\#$ to indicate that $S$ is unbounded above, and $\inf(S) = -\infty^\#$ to indicate that $S$ is unbounded below.

**Definition 2.19** That is $(\varepsilon, \delta)$ definition of the #-limit of a function $f: D \to {}^*\mathbb{R}_c^\#$ is as follows: let $f(x)$ is a ${}^*\mathbb{R}_c^\#$- valued function defined on a subset $D \subset {}^*\mathbb{R}_c^\#$ of the Cauchy hyperreal numbers. Let $c$ be a #-limit point of $D$ and let $L \in {}^*\mathbb{R}_c^\#$ be Cauchy hyperreal number. We say that $\#\text{-}\lim_{x \to_\# c} f(x) = L$ if for every $\varepsilon \approx 0, \varepsilon > 0$ there exists a $\delta \approx 0, \delta > 0$ such that, for all $x \in D$, if $0 < |x - c| < \delta$, then $|f(x) - L| < \varepsilon$.

**Definition 2.20** [12] The function $f: {}^*\mathbb{R}_c^\# \to {}^*\mathbb{R}_c^\#$ is a #-continuous (or micro continuous) at some point $c$ of its domain if the #-limit of $f(x)$, as $x$ #-approaches $c$ through the domain of $f$, exists and is equal to $f(c)$: $\#\text{-}\lim_{x \to_\# c} f(x) = f(c)$.

**Theorem 2.20** [13] Let $\{a_n\}_{n=1}^{{}^*\infty}$ and $\{b_n\}_{n=1}^{{}^*\infty}$ be ${}^*\mathbb{R}_c^\#$- valued hyper infinite sequences. Then the following equalities hold for any $n, k, l, j, m \in {}^*\mathbb{N}$ :

$$b \times (Ext\text{-}\textstyle\sum_{i=1}^n a_i) = Ext\text{-}\sum_{i=1}^n b \times a_i \tag{45}$$

$$Ext\text{-}\textstyle\sum_{i=1}^n a_i \pm Ext\text{-}\sum_{i=1}^n b_i = Ext\text{-}\sum_{i=1}^n (a_i \pm b_i) \tag{46}$$

$$Ext\text{-}\textstyle\sum_{i=k_0}^{k_1}\left(Ext\text{-}\sum_{j=l_0}^{l_1} a_{ij}\right) = Ext\text{-}\sum_{j=l_0}^{l_1}\left(Ext\text{-}\sum_{i=k_0}^{k_1} a_{ij}\right) \tag{47}$$

$$(Ext\text{-}\textstyle\sum_{i=1}^n a_i) \times \left(Ext\text{-}\sum_{j=1}^n b_j\right) = Ext\text{-}\sum_{i=1}^n\left(Ext\text{-}\sum_{j=1}^n a_i \times b_j\right) \tag{48}$$

$$(Ext\text{-}\textstyle\prod_{i=1}^n a_i) \times (Ext\text{-}\prod_{i=1}^n b_i) = Ext\text{-}\prod_{i=1}^n a_i \times b_i \tag{49}$$

$$(Ext\text{-}\textstyle\prod_{i=1}^n a_i)^m = Ext\text{-}\prod_{i=1}^n a_i^m. \tag{50}$$

**Theorem 2.21** [13] Let $\{a_n\}_{i=1}^n$ and $\{b_n\}_{i=1}^n$ be ${}^*\mathbb{R}_c^\#$- valued monotonically non-decreasing hyperfinite sequences. Suppose that $a_i \le b_i, 1 \le i \le n$, then the following equalities hold for any $n \in {}^*\mathbb{N}$ :

$$Ext\text{-}\textstyle\prod_{i=1}^n a_i \le Ext\text{-}\prod_{i=1}^n b_i. \tag{51}$$

**Theorem 2.22** [13] Let $\{a_n\}_{i=1}^n$ and $\{b_n\}_{i=1}^n$ be ${}^*\mathbb{R}_c^\#$- valued hyperfinite sequences. Then the following inequalities hold for any $n \in {}^*\mathbb{N}$ :

$$(Ext\text{-}\textstyle\prod_{i=1}^n a_i \times b_i)^2 \le (Ext\text{-}\prod_{i=1}^n a_i^2) \times (Ext\text{-}\prod_{i=1}^n b_i^2). \tag{52}$$

**Definition 2.21** [12] Assume that $\{a_n\}_{n=1}^{{}^*\infty}$ is a ${}^*\mathbb{R}_c^\#$- valued hyper infinite sequence, the symbol $Ext\text{-}\sum_{n=1}^{{}^*\infty} a_n$ is a hyper infinite series, and $a_n$ is the n-th term of the hyper infinite series.

**Definition 2.22** [12] We shall say that a series $Ext\text{-}\sum_{n=1}^{{}^*\infty} a_n$ #-converges to the sum $A \in {}^*\mathbb{R}_c^\#$, and write $Ext\text{-}\sum_{n=1}^{{}^*\infty} a_n = A$ if the hyper infinite sequence $\{A_n\}_{n=1}^{{}^*\infty}$ defined by $A_m = Ext\text{-}\sum_{n=1}^m a_n$ #-converges to the sum $A$. The hyperfinite sum $A_m$ is the $n$-th partial sum of $Ext\text{-}\sum_{n=1}^{{}^*\infty} a_n$. If $\#\text{-}\lim_{m \to {}^*\infty,} A_m = \infty^\#$ or $-\infty^\#$, we shall say that $Ext\text{-}\sum_{n=1}^{{}^*\infty} a_n$ #-diverges to $\infty^\#$ or to $-\infty^\#$.

**Theorem 2.23** [12] The hyper infinite sum $Ext\text{-}\sum_{n=1}^{{}^*\infty} a_n$ of a #-convergent hyper infinite series is unique.

## Hyper Infinite Sequences and Series of ${}^*\mathbb{R}_c^\#$- Valued Functions

**Definition 2.23** [12] If $f_1, f_2, \ldots, f_k, f_{k+1}, \ldots, f_n, \ldots n \in {}^*\mathbb{N}$ are ${}^*\mathbb{R}_c^\#$- valued functions on a subset $D \subset {}^*\mathbb{R}_c^\#$ we say that $\{f_n\}_{n=1}^{{}^*\infty}$ is a hyper infinite sequence of ${}^*\mathbb{R}_c^\#$- valued functions on $D$.

**Definition 2.24** [12] Suppose that $\{f_n\}_{n=1}^{{}^*\infty}$ is a hyper infinite sequence of ${}^*\mathbb{R}_c^\#$- valued functions on $D \subset {}^*\mathbb{R}_c^\#$ and the hyper infinite sequence of values $\{f_n(x)\}_{n=1}^{{}^*\infty}$ #-converges for each $x$ in some subset $S$ of $D$. Then we say that $\{f_n(x)\}_{n=1}^{{}^*\infty}$ #-converges pointwise on $S$ to the #-limit function $f$, defined by $f(x) = \lim_{n \to {}^*\infty} f_n(x)$.

**Definition 2.25** [12] If $\{f_n(x)\}_{n=1}^{{}^*\infty}$ is a hyper infinite sequence of ${}^*\mathbb{R}_c^\#$- valued functions on $D \subset {}^*\mathbb{R}_c^\#$, then

$$Ext\text{-}\textstyle\sum_{n=1}^{{}^*\infty} f_n(x) \tag{53}$$

is a hyper infinite series of functions on $D$. The partial sums of (1), are defined by $F_n(x) = Ext\text{-}\sum_{k=1}^n f_n(x)$. If hyper infinite sequence

$\{F_n(x)\}_{n=1}^{{}^*\infty}$#-

converges pointwise to the #-limit function $F(x)$ on a subset $S \subset D$, we say that $\{F_n(x)\}_{n=1}^{*\infty}$ #-converges pointwise to the sum $F(x)$ on $S$, and write $F(x) = Ext\text{-}\sum_{n=1}^{*\infty} f_n(x)$.

**Definition 2.26** [12] A hyper infinite series of the form $Ext\text{-}\sum_{n=1}^{*\infty}(x - x_0)^n, n \in {}^*\mathbb{N}$ is called a hyper infinite power series in $x - x_0$.

## The #-Derivatives and Riemann #-Integral of ${}^*\mathbb{R}_c^\#$-Valued Functions $f: D \to {}^*\mathbb{R}_c^{\#n}$

**Definition 2.27** [12] A function $f: D \to {}^*\mathbb{R}_c^\#$ #-differentiable at an #-interior point $x \in D$ of its domain $D \subset {}^*\mathbb{R}_c^\#$ if the difference quotient $f(x) - f(x_0)/x - x_0$ has a #-limit:

$$\#\text{-}\lim_{x \to_\# x_0}(f(x) - f(x_0)/x - x_0).$$

In this case the #-limit is called the #-derivative of $f$ at interior point $x_0$, and is denoted by $f^{\#'}(x_0)$ or by $d^\# f\,(x_0)/d^\# x$.

**Definition 2.28** If $f$ is defined on an #-open set $S \subset {}^*\mathbb{R}_c^\#$**,** we say that f is #-differentiable on $S$ if $f$ is #-differentiable at every point of $S$. If $f$ is #-differentiable on $S$, then $f^{\#'}(x)$ is a function on $S$.We say that $f$ is #-continuously #-differentiable on $S$ if $f^{\#'}(x)$ is #-continuous on $S$.

**Definition 2.29** If $f$ is #-differentiable on a #-neighbourhood of $x_0$, it is reasonable to ask if $f^{\#\prime}(x)$ is #-differentiable at $x_0$. If so, we denote the #-derivative of $f^{\#\prime}(x)$ at $x_0$ by $f^{\#\prime\prime}(x_0)$ or by $f^{\#(2)}(x_0)$ and this is the second #-derivative of $f$ at $x_0$. Continuing inductively by hyper infinite induction, if $f^{\#(n-1)}(x)$ is defined on a #-neighbourhood of $x_0$, then the $n$-th #-derivative of $f$ at $x_0$ denoted by $f^{\#(n)}(x_0)$ or by $d^{\#(n)} f\,(x_0)/d^\# x^n$, where $n \in {}^*\mathbb{N}$.

**Theorem 2.24** [12] If $f$ is #-differentiable at $x_0$ then $f$ is #-continuous at $x_0$.

**Theorem 2.25** [12] If $f$ and $g$ are #-differentiable at $x_0$, then so are $f \pm g$ and $f \times g$ with:

(a) $(f \pm g)^{\#\prime}(x_0) = f^{\#\prime}(x_0) \pm g^{\#\prime}(x_0)$, (b) $(f \times g)^{\#\prime}(x_0) = f^{\#\prime}(x_0)g(x_0) + g^{\#\prime}(x_0)f(x_0)$.

(c) The quotient $f/g$ is #-differentiable at $x_0$ if $g(x_0) \neq 0$ with $(f/g)^{\#\prime} = \frac{f^{\#\prime}(x_0)g(x_0) - g^{\#\prime}(x_0)f(x_0)}{g(x_0)^2}$.

(d) If $n \in {}^*\mathbb{N}$ and $f_i, 1 \leq i \leq n$ are #-differentiable at $x_0$, then so are $Ext\text{-}\sum_{i=1}^n f_i$ with:

$$(Ext\text{-}\textstyle\sum_{i=1}^n f_i)^{\#\prime}(x_0) = Ext\text{-}\sum_{i=1}^n f_i^{\#\prime}(x_0).$$

(e) If $n \in {}^*\mathbb{N}$ and $f^{\#(n)}(x_0), g^{\#(n)}(x_0)$ exist, then so does $(f \times g)^{\#(n)}(x_0)$ and

$$(f \times g)^{\#(n)}(x_0) = Ext\text{-}\sum_{i=0}^{n} \binom{n}{i} f^{\#(i)}(x_0) g^{\#(n-i)}(x_0)$$

**Theorem 2.26** [12] (The Chain Rule) Suppose that $g$ is #-differentiable at $x_0$ and $f$ is #-differentiable at $g(x_0)$. Then the composite function $h = f \circ g$ defined by $h(x) = f(g(x))$ is #-differentiable at $x_0$ with $h^{\#\prime}(x_0) = f^{\#\prime}(g(x_0))g^{\#\prime}(x_0)$.

**Theorem 2.27** [12] (Generalized Taylor's Theorem) Suppose that $f^{\#(n)}(x), n \in {}^*\mathbb{N}$ exists on an #-open interval $I$ about $x_0$, and let $x \in I$. Let $P_n(x, x_0)$ be the $n$-th Taylor hyper polynomial of $f$ about $x_0$, $P_n(x, x_0) = Ext\text{-}\sum_{r=0}^n \frac{f^{\#(r)}(x_0)(x - x_0)^r}{r!}$ Then the remainder $R(x, x_0) = f(x) - P_n(x, x_0)$ can be written as

$$R(x, x_0) = \frac{f^{\#(n+1)}(c)(x - x_0)^n}{(n+1)!}. \tag{54}$$

Here $c$ depends upon $x$ and is between $x$ and $x_0$.

**Definition 2.30** [12] Let $[a, b] \subset {}^*\mathbb{R}_c^\#$. A hyperfinite partition of $[a, b]$ is a hyperfinite set of subintervals $[x_0, x_1]$,…,$[x_{n-1}, x_n]$, with $n \in {}^*\mathbb{N}_\infty$, where $a = x_0 < x_1 \ldots < x_n = b$. A set of these points $x_0, x_1, \ldots, x_n$ defines a hyperfinite partition $P$ of $[a, b]$, which we denote by $P = \{x_i\}_{i=0}^n$. The points $x_0, x_1, \ldots, x_n$ are the partition points of $P$.The largest of the lengths of the subintervals $[x_{i-1}, x_i]$, $0 \leq i \leq n$ is the norm of $P = \{x_i\}_{i=0}^n$ denoted by $\|P\|$; thus, $\|P\| = \max_{1 \leq i \leq n}(x_i - x_{i-1})$.

**Definition 2.31** Let $P$ and $P'$ are hyperfinite partitions of $[a, b]$, then $P'$ is a refinement of $P$ if every partition point of $P$ is also a partition point of $P'$; that is, if $P'$ is obtained by inserting additional points between those of $P$.

**Definition 2.32** Let $f$ be ${}^*\mathbb{R}_c^\#$- valued function $f: [a, b] \to {}^*\mathbb{R}_c^\#$, then we say that external hyperfinite sum $\sigma^{Ext}$ defined by

$$\sigma^{Ext} = Ext\text{-}\textstyle\sum_{i=1}^n f(c_i)\,(x_i - x_{i-1}), x_{i-1} \leq c_i \leq x_i, \tag{55}$$

is a Riemann external hyperfinite sum of $f$ over the hyperfinite partition $P = \{x_i\}_{i=0}^{n}$.

**Definition 2.33** [12] Let $f$ be ${}^*\mathbb{R}_c^\#$- valued function $f: [a,b] \to {}^*\mathbb{R}_c^\#$, then we say that $f$ is Riemann #-integrable on $[a,b]$ if there is a number $L \in {}^*\mathbb{R}_c^\#$ with the following property: for every $\varepsilon \approx 0, \varepsilon > 0$, there is a $\delta \approx 0, \delta > 0$ such that $|L - \sigma^{Ext}| < \delta$ if $\sigma^{Ext}$ is any Riemann external hyperfinite sum of $f$ over a partition $P$ of $[a,b]$ such that $\|P\| < \delta$. In this case, we say that $L$ is the Riemann #-integral of $f$ over $[a,b]$, and we shall write

$$L = Ext\text{-}\int_a^b f(x) d^\# x. \tag{56}$$

Thus the Riemann #-integral of ${}^*\mathbb{R}_c^\#$- valued function $f: [a,b] \to {}^*\mathbb{R}_c^\#$ over $[a,b]$ is defined as #-limit of the external hyperfinite sums (55) with respect to partitions of the interval $[a,b]$:

$$Ext\text{-}\int_a^b f(x) d^\# x = \#\text{-}\lim_{n \to {}^*\infty} \left( Ext\text{-}\sum_{i=1}^{n} f(c_i)\,(x_i - x_{i-1}) \right). \tag{57}$$

**Definition 2.34** A coordinate rectangle $R$ in ${}^*\mathbb{R}_c^{\#\boldsymbol{n}}, n \in {}^*\mathbb{N}$ is the external finite or hyperfinite Cartesian product of $n$ #-closed intervals; that is, $R = Ext\text{-} \times_{i=1}^{n} [a_i, b_i]$. The content of $R$ is $V(R) = Ext\text{-}\prod_{i=1}^{n}(b_i - a_i)$. The hyperreal numbers $b_i - a_i$, $1 \le i \le n$ are the edge lengths of $R$. If they are equal, then $R$ is finite or hyperfinite coordinate cube. If $a_l = b_l$ for some $r$, then $V(R) = 0$ and we say that $R$ is degenerate; otherwise, $R$ is nondegenerate.

**Definition 2.35** If $R = Ext\text{-}\times_{i=1}^{n} [a_i, b_i]$ and $P_r = a_{r0} < a_{r1} < \cdots < a_{rm_r}$ is an external hyperfinite partition of $[a_r, b_r], 1 \le r \le n$, then the set of all rectangles in ${}^*\mathbb{R}_c^{\#\boldsymbol{n}}$ that can be written as $Ext\text{-} \times_{i=1}^{n} \left[a_{i,j_{i-1}}, a_{i,j_i}\right]$, $1 \le j_r \le m_r$, $1 \le r \le n$ is a partition of $R$. We denote this partition by $P = Ext\text{-} \times_{r=1}^{n} P_r$ and define its norm to be the maximum of the norms of $P_i, 1 \le i \le n$; thus, $\|P\| = \max_i\{P_i | 1 \le i \le n\}$.

**Definition 2.36** If $P = Ext\text{-} \times_{i=1}^{n} P_i$ and $P' = Ext\text{-} \times_{i=1}^{n} P_i'$ are partitions of the same rectangle, then $P'$ is a refinement of $P$ if $P_i'$ is a refinement of $P_i, 1 \le i \le n$ as defined above.

**Definition 2.37** Suppose that $f$ is a ${}^*\mathbb{R}_c^\#$- valued function defined on a rectangle $R$ in ${}^*\mathbb{R}_c^{\#\boldsymbol{n}}, n \in {}^*\mathbb{N}$, $P = \{P_i\}_{i=1}^{k}$is a partition of $R$, and $x_i$ is an arbitrary point in $R_i$, $1 \le j \le k$. Then a Riemann external hyperfinite sum $\sigma^{Ext}$ of $f$ over the partition $P$ is defined by

$$\sigma^{Ext} = Ext\text{-}\sum_{i=1}^{k} f(x_i)\, V(R_i) \tag{58}$$

**Definition 2.38** Let $f$ be a ${}^*\mathbb{R}_c^\#$- valued function defined on a rectangle $R$ in ${}^*\mathbb{R}_c^{\#\boldsymbol{n}}, n \in {}^*\mathbb{N}$. We say that $f$ is Riemann #-integrable on $R$ if there is a number L with the following property: for every $\varepsilon \approx 0, \varepsilon > 0$, there is a $\delta \approx 0, \delta > 0$ such that $|L - \sigma^{Ext}| < \delta$ if $\sigma^{Ext}$ is any Riemann external hyperfinite sum of $f$ over a partition $P$ of $R$ such that $\|P\| < \delta$. In this case, we say that $L$ is the Riemann #-integral of $f$ over $R$, and write

$$L = Ext\text{-}\int_R f(x) d^{\#n} x. \tag{59}$$

Thus the Riemann #-integral of ${}^*\mathbb{R}_c^\#$- valued function $f$ defined on a rectangle $R$ in ${}^*\mathbb{R}_c^{\#\boldsymbol{n}}$ is defined as #-limit of the external hyperfinite sums (58) with respect to partitions of the rectangle $R$:

$$Ext\text{-}\int_R f(x) d^{\#n} x = \#\text{-}\lim_{n \to {}^*\infty} \left( Ext\text{-}\sum_{i=1}^{k} f(x_i)\, V(R_i) \right). \tag{60}$$

## The ${}^*\mathbb{R}_c^\#$-Valued #-Exponential Function $\boldsymbol{Ext}$-$\exp(\boldsymbol{x})$ and ${}^*\mathbb{R}_c^\#$-Valued Trigonometric Functions $\boldsymbol{Ext}$-$\sin(\boldsymbol{x}), \boldsymbol{Ext}$-$\cos(\boldsymbol{x})$

We define the #-exponential function $Ext$-$\exp(x)$ as the solution of the #-differential equation

$$f^{\#\prime}(x) = f(x), f(0) = 1. \tag{61}$$

We solve it by setting $f(x) = Ext\text{-}\sum_{n=0}^{{}^*\infty} x^n$ , $f^{\#\prime}(x) = Ext\text{-}\sum_{n=0}^{{}^*\infty} n x^n$. Therefore

$$Ext\text{-}\exp(x) = Ext\text{-}\sum_{n=0}^{{}^*\infty} \frac{x^n}{n!}. \tag{62}$$

From (1) we get $\big(Ext\text{-}\exp(x)\big)\big(Ext\text{-}\exp(y)\big) = Ext\text{-}\exp(x+y)$ for any $x, y \in {}^*\mathbb{R}_c^\#$.

We define the #- trigonometric functions $Ext\text{-}\sin x$ and $Ext\text{-}\cos x$ by

$$Ext\text{-}\sin x = Ext\text{-}\sum_{n=0}^{{}^*\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!},\; Ext\text{-}\cos x = Ext\text{-}\sum_{n=0}^{{}^*\infty} (-1)^n \frac{x^{2n}}{(2n)!}. \tag{63}$$

It can be shown that the series (1) #-converges for all $x \in {}^*\mathbb{R}_c^\#$ #-differentiating yields

$$(Ext\text{-}\sin x)^{\#\prime} = Ext\text{-}\cos x,\; (Ext\text{-}\cos x)^{\#\prime} = -(Ext\text{-}\sin x). \tag{64}$$

## ${}^{*}\mathbb{R}_{\mathbf{c}}^{\#}$ -Valued Schwartz Distributions

**Definition 2.39** [12] Let $U$ be an #- open subset of ${}^{*}\mathbb{R}_{c}^{\#n}$ and $f: U \to {}^{*}\mathbb{R}_{c}^{\#}$. The partial derivative of $f$ at the point $x = (x_1, x_2, \dots, x_i, \dots, x_n)$ with respect to the $i$-th variable $x_i$ is defined as

$$\frac{\partial^{\#} f}{\partial^{\#} x_i} = \#\text{-}\lim_{h \to_{\#} 0} \frac{f(x_1, x_2, \dots, x_i + h, \dots, x_n) - f(x_1, x_2, \dots, x_i, \dots, x_n)}{h}.$$

**Definition 2.38** A multi-index of size $n \in {}^{*}\mathbb{N}$ is an element in ${}^{*}\mathbb{N}^{n}$, the length of a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in {}^{*}\mathbb{N}^{n}$ is defined as $Ext\text{-}\sum_{i=1}^{n} \alpha_i$ and denoted by $|\alpha|$. We introduce the following notations for a given multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in {}^{*}\mathbb{N}^{n}$: $x^{\alpha} = Ext\text{-}\prod_{i=1}^{n} x_i^{\alpha_i}$ ; $\partial^{\#\alpha} = Ext\text{-}\prod_{i=1}^{n} \frac{\partial^{\#\alpha_i}}{\partial^{\#} x_i^{\alpha_i}}$ or symbolically $\partial^{\#\alpha} = Ext\text{-}\frac{\partial^{\#\alpha}}{\partial^{\#} x_1^{\alpha_1} \dots \partial^{\#} x_n^{\alpha_n}}$..

**Definition 2.40** The Schwartz space of rapidly decreasing ${}^{*}\mathbb{C}_{c}^{\#}$- valued test functions on ${}^{*}\mathbb{R}_{c}^{\#n}$, $n \in {}^{*}\mathbb{N}$ is the function space defined by

$$\mathrm{S}^{\#}({}^{*}\mathbb{R}_{c}^{\#n}, {}^{*}\mathbb{C}_{c}^{\#}) = \left\{ f \in C^{{}^{*}\infty}({}^{*}\mathbb{R}_{c}^{\#n}, {}^{*}\mathbb{C}_{c}^{\#}) | \forall (\alpha, \beta)(\alpha, \beta \in {}^{*}\mathbb{N}^{n}) \forall x (x \in {}^{*}\mathbb{R}_{c}^{\#n}) \left[ \left| x^{\alpha} D^{\#\beta} f(x) \right| < \infty^{\#} \right] \right\}.$$

**Remark 2.6** Note that if $f \in S^{\#}({}^{*}\mathbb{R}_{c}^{\#n}, {}^{*}\mathbb{C}_{c}^{\#})$ the integral of $x^{\alpha} \left| D^{\#\beta} f(x) \right|$ exists

$$Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#n}} \left| x^{\alpha} D^{\#\beta} f(x) \right| d^{\#n} < \infty^{\#}.$$

**Definition 2.41** The Schwartz space of essentially rapidly decreasing ${}^{*}\mathbb{C}_{c}^{\#}$- valued test functions on ${}^{*}\mathbb{R}_{c}^{\#n}$, $n \in {}^{*}\mathbb{N}$ is the function space defined by

$$\mathrm{S}^{\#}({}^{*}\mathbb{R}_{c}^{\#n}, {}^{*}\mathbb{C}_{c}^{\#}) = \left\{ f \in C^{{}^{*}\infty}({}^{*}\mathbb{R}_{c}^{\#n}, {}^{*}\mathbb{C}_{c}^{\#}) | \forall \alpha (\alpha \in \mathbb{N}^{n}) \forall \beta (\beta \in {}^{*}\mathbb{N}^{n}) \forall x (x \in {}^{*}\mathbb{R}_{c}^{\#n}) \left[ \left| x^{\alpha} D^{\#\beta} f(x) \right| < \infty \right] \right\}.$$

**Remark 2.7** Note that if $f \in S^{\#}({}^{*}\mathbb{R}_{c}^{\#n}, {}^{*}\mathbb{C}_{c}^{\#})$ the integral of $x^{\alpha} \left| D^{\#\beta} f(x) \right|, \alpha \in \mathbb{N}^{n}, \beta \in {}^{*}\mathbb{N}^{n}$ exists and

$$Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#n}} \left| x^{\alpha} D^{\#\beta} f(x) \right| d^{\#n} < \infty .$$

**Definition 2.42** The Schwartz space of rapidly decreasing ${}^{*}\mathbb{C}_{c}^{\#}$- valued test functions on ${}^{*}\mathbb{R}_{c,\mathrm{fin}}^{\#n}$, $n \in {}^{*}\mathbb{N}$ is the function space defined by

$$\breve{\mathrm{S}}^{\#}\left({}^{*}\mathbb{R}_{c,\mathrm{fin}}^{\#n}, {}^{*}\mathbb{C}_{c}^{\#}\right) = \left\{ f \in C^{{}^{*}\infty}\left({}^{*}\mathbb{R}_{c,\mathrm{fin}}^{\#n}, {}^{*}\mathbb{C}_{c}^{\#}\right) | \forall (\alpha, \beta)(\alpha, \beta \in {}^{*}\mathbb{N}^{n}) \forall x \left(x \in {}^{*}\mathbb{R}_{c,\mathrm{fin}}^{\#n}\right) \left[ \left| x^{\alpha} D^{\#\beta} f(x) \right| < \infty^{\#} \right] \right\}.$$

**Remark 2.8** Note that if $f \in \breve{\mathrm{S}}^{\#}\left({}^{*}\mathbb{R}_{c,\mathrm{fin}}^{\#n}, {}^{*}\mathbb{C}_{c}^{\#}\right)$ the integral of $x^{\alpha} \left| D^{\#\beta} f(x) \right|, \alpha \in {}^{*}\mathbb{N}^{n}, \beta \in {}^{*}\mathbb{N}^{n}$ exists and

$$Ext\text{-}\int_{{}^{*}\mathbb{R}_{c,\mathrm{fin}}^{\#n}} \left| x^{\alpha} D^{\#\beta} f(x) \right| d^{\#n} < \infty^{\#} .$$

**Definition 28.43** The Schwartz space of essentially rapidly decreasing ${}^{*}\mathbb{C}_{c}^{\#}$- valued test functions on ${}^{*}\mathbb{R}_{c,\mathrm{fin}}^{\#n}$, $n \in {}^{*}\mathbb{N}$ is the function space defined by

$$\breve{\mathrm{S}}_{\mathrm{fin}}^{\#}\left({}^{*}\mathbb{R}_{c,\mathrm{fin}}^{\#n}, {}^{*}\mathbb{C}_{c}^{\#}\right) =$$

$$\left\{ f \in C^{{}^{*}\infty}\left({}^{*}\mathbb{R}_{c,\mathrm{fin}}^{\#n}, {}^{*}\mathbb{C}_{c}^{\#}\right) | \forall (\alpha, \beta)(\alpha \in \mathbb{N}^{n}, \beta \in \mathbb{N}^{n}) \exists c_{\alpha\beta} \left(c_{\alpha\beta} \in {}^{*}\mathbb{R}_{c,\mathrm{fin}}^{\#}\right) \forall x \left(x \in {}^{*}\mathbb{R}_{c,\mathrm{fin}}^{\#n}\right) \left[ \left| x^{\alpha} \left( D^{\#\beta} f(x) \right) \right| < c_{\alpha\beta} \right] \right\}.$$

**Remark 2.9** Note that if $f \in S_{\mathrm{fin}}^{\#}({}^{*}\mathbb{R}_{c}^{\#n}, {}^{*}\mathbb{C}_{c}^{\#})$ the integral of $\left| x^{\alpha} D^{\#\beta} f(x) \right|$ exists and finitely bounded above

$$Ext\text{-}\int_{{}^{*}\mathbb{R}_{c,\mathrm{fin}}^{\#n}} \left| x^{\alpha} D^{\#\beta} f(x) \right| d^{\#n} < d_{\alpha\beta}, d_{\alpha\beta} \in {}^{*}\mathbb{R}_{c,\mathrm{fin}}^{\#}.$$

**Abbreviation 2.2** 1) The Schwartz space of rapidly decreasing test functions on ${}^{*}\mathbb{R}_{c}^{\#n}$ we will be denoting by $\mathrm{S}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$ and let $\mathrm{S}_{\mathrm{fin}}^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$ denote the set of ${}^{*}\mathbb{C}_{c}^{\#}$-valued essentially rapidly decreasing test functions on ${}^{*}\mathbb{R}_{c}^{\#n}$. 2) The Schwartz space of rapidly decreasing ${}^{*}\mathbb{C}_{c}^{\#}$- valued test functions on ${}^{*}\mathbb{R}_{c,\mathrm{fin}}^{\#n}$ we will be denoting by $\breve{\mathrm{S}}^{\#}\left({}^{*}\mathbb{R}_{c,\mathrm{fin}}^{\#n}\right)$ and let $\breve{\mathrm{S}}_{\mathrm{fin}}^{\#}\left({}^{*}\mathbb{R}_{c,\mathrm{fin}}^{\#n}\right)$ denote the set of ${}^{*}\mathbb{C}_{c}^{\#}$-valued essentially rapidly decreasing test functions on ${}^{*}\mathbb{R}_{c,\mathrm{fin}}^{\#n}$.

**Definition 2.44** A linear functional $u: S^{\#}({}^{*}\mathbb{R}_{c}^{\#n}) \to {}^{*}\mathbb{C}_{c}^{\#}$ is a #-continuous if there exist $C, k \in {}^{*}\mathbb{N}$ and constants $c_{\alpha\beta}$ such that $|u(\varphi)| \leq C\left(Ext\text{-}\sum_{|\alpha| \leq k, |\beta| \leq k} c_{\alpha\beta}\right)$. Here $\forall x (x \in {}^{*}\mathbb{R}_{c}^{\#n}) \left[ \left| x^{\alpha} \left( D^{\#\beta} \varphi(x) \right) \right| < c_{\alpha\beta} \right]$.

**Definition 2.45** A linear functional $u: S^{\#}\left({}^{*}\mathbb{R}_{c,\mathrm{fin}}^{\#n}\right) \to {}^{*}\mathbb{C}_{c}^{\#}$ is a strongly #-continuous if there exist $C, k \in {}^{*}\mathbb{N}$ and constants $c_{\alpha\beta}$ such that $|u(\varphi)| \leq C\left(Ext\text{-}\sum_{|\alpha| \leq k, |\beta| \leq k} c_{\alpha\beta}\right) \in {}^{*}\mathbb{R}_{c,\mathrm{fin}}^{\#}$.

**Definition 2.46** A generalized function $u \in S^{\#\prime}({}^{*}\mathbb{R}_{c}^{\#n})$ is defined as a #-continuous linear functional on vector space $S^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$, symbolically it written as $u: \varphi \to (u, \varphi)$. Thus space $S^{\#\prime}({}^{*}\mathbb{R}_{c}^{\#n})$ of generalized functions is the space dual to $S^{\#}({}^{*}\mathbb{R}_{c}^{\#n})$.

**Definition 2.47** A generalized function $u \in S^{\#\prime}\left({}^*\mathbb{R}^{\#n}_{c,\mathrm{fin}}\right)$ is defined as a strongly #-continuous linear functional on vector space $S^{\#}\left({}^*\mathbb{R}^{\#n}_{c,\mathrm{fin}}\right)$, symbolically it written as $u\colon \varphi \to (u,\varphi)$. Thus space $S^{\#\prime}\left({}^*\mathbb{R}^{\#n}_{c,\mathrm{fin}}\right)$ of generalized functions is the space dual to $S^{\#}\left({}^*\mathbb{R}^{\#n}_{c,\mathrm{fin}}\right)$.

**Definition 2.48** Convergence of a hyper infinite sequence $\{u_n\}_{n=1}^{{}^*\infty}$ of generalized functions in $S^{\#\prime}({}^*\mathbb{R}^{\#n}_{c})$ is defined as weak #-convergence of the hyper infinite sequence of functionals in $S^{\#\prime}({}^*\mathbb{R}^{\#n}_{c})$ that is: $u_n \to_{\#} 0$, as $n \to {}^*\infty$, in $S^{\#\prime}({}^*\mathbb{R}^{\#n}_{c})$ means that $(u_n,\varphi) \to_{\#} 0$, as $n \to {}^*\infty$, for all $\varphi \in S^{\#}({}^*\mathbb{R}^{\#n}_{c})$.

**Definition 2.49** Convergence of a hyper infinite sequence $\{u_n\}_{n=1}^{{}^*\infty}$ of generalized functions in $S^{\#\prime}\left({}^*\mathbb{R}^{\#n}_{c,\mathrm{fin}}\right)$ is defined as weak #-convergence of functionals in $S^{\#\prime}\left({}^*\mathbb{R}^{\#n}_{c,\mathrm{fin}}\right)$ that is: $u_n \to_{\#} 0$, as $n \to {}^*\infty$, in $S^{\#\prime}\left({}^*\mathbb{R}^{\#n}_{c,\mathrm{fin}}\right)$ means that $(u_n,\varphi) \to_{\#} 0$, as $n \to {}^*\infty$, for all $\varphi \in S^{\#}\left({}^*\mathbb{R}^{\#n}_{c,\mathrm{fin}}\right)$.

**Definition 2.50** 1) Let $u \in S^{\#'}({}^*\mathbb{R}^{\#n}_{c})$ and let $x = Ay + b$ be a linear transformation of ${}^*\mathbb{R}^{\#n}_{c}$ onto ${}^*\mathbb{R}^{\#n}_{c}$. The generalized function $u(Ay+b) \in S^{\#'}({}^*\mathbb{R}^{\#n}_{c})$ is defined by

$$(u(Ay+b),\varphi) = \left(u, \frac{\varphi[A^{-1}(x-b)]}{|\det A|}\right). \tag{65}$$

Formula (1) enables one to define generalized functions that are translation invariant, spherically symmetric, centrally symmetric, homogeneous, periodic, Lorentz invariant, etc.

2) Let the function $\alpha(x) \in C^{\#1}({}^*\mathbb{R}^{\#}_{c})$ have only simple zeros $x_k \in {}^*\mathbb{R}^{\#}_{c}, k \in {}^*\mathbb{N}$, the function $\delta\big(\alpha(x)\big)$ is defined by

$$\delta\big(\alpha(x)\big) = Ext\text{-}\sum_{k=1}^{{}^*\infty} \frac{\delta(x-x_k)}{|\alpha^{\#\prime}(x_k)|}. \tag{66}$$

3) Let $u \in S^{\#'}({}^*\mathbb{R}^{\#n}_{c})$, the generalized (weak) #-derivative $\partial^{\#\alpha}u$ of $u$ of order $\alpha$ is defined as

$$(\partial^{\#\alpha}u,\varphi) = (-1)^{|\alpha|}(u,\partial^{\#\alpha}\varphi). \tag{67}$$

4) Let $u \in S^{\#'}({}^*\mathbb{R}^{\#n}_{c})$ and $g(x) \in C^{\#^*\infty}({}^*\mathbb{R}^{\#n}_{c})$, The product $gu = ug$ is defined by

$$(gu,\varphi) = (u,g\varphi). \tag{68}$$

5) Let $u_1 \in S^{\#'}({}^*\mathbb{R}^{\#n}_{c})$ and $u_2 \in S^{\#'}({}^*\mathbb{R}^{\#m}_{c})$ then their direct product is defined by the formula

$$(u_1 \times u_2,\varphi) = \big(u_1(x)(u_2(y),\varphi)\big),\ \varphi(x,y) \in S^{\#}({}^*\mathbb{R}^{\#n}_{c} \times {}^*\mathbb{R}^{\#m}_{c}). \tag{69}$$

6) The Fourier transform $\mathcal{F}[u]$ of a generalized function $u \in S^{\#'}({}^*\mathbb{R}^{\#n}_{c})$ is defined by the formula

$$(\mathcal{F}[u],\varphi) = (u,\mathcal{F}[\varphi]), \tag{70}$$

$$\mathcal{F}[\varphi] = Ext\text{-}\int_{{}^*\mathbb{R}^{\#n}_{c}} \varphi(x)(Ext\text{-}\exp[i(\xi,x)])d^{\#n}x. \tag{71}$$

Since the operation $\varphi(x) \to \mathcal{F}[\varphi](\xi)$ is an isomorphism of $\mathrm{S}^{\#}({}^*\mathbb{R}^{\#n}_{c})$ onto $\mathrm{S}^{\#}({}^*\mathbb{R}^{\#n}_{c})$, the operation $u \to \mathcal{F}[u]$ is an isomorphism of $S^{\#'}({}^*\mathbb{R}^{\#n}_{c})$ onto $S^{\#'}({}^*\mathbb{R}^{\#n}_{c})$ and the inverse of $\mathcal{F}[u]$ is given by: $\mathcal{F}^{-1}[u] = (2\pi)^{-n}\mathcal{F}[u(-\xi)]$. The following formulas hold for $u \in S^{\#'}({}^*\mathbb{R}^{\#n}_{c})$: (a) $\partial^{\#\alpha}\mathcal{F}[u] = \mathcal{F}[(ix)^{\alpha}u]$, (b) $\mathcal{F}[\partial^{\#\alpha}u] = (i\xi)^{\alpha}\mathcal{F}[u]$,(c) if the generalized function $u_1 \in S^{\#'}({}^*\mathbb{R}^{\#n}_{c})$ has #-com-pact support, then $\mathcal{F}[u_1 * u_2] = \mathcal{F}[u_1]\mathcal{F}[u_2]$.

7) If the generalized function $u$ is periodic with $n$-period $T = (T_1,\dots,T_n)$, then $u \in S^{\#'}({}^*\mathbb{R}^{\#n}_{c})$, and it can be expanded in a hyper infinite trigonometric series

$$u(x) = Ext\text{-}\sum_{|k|=0}^{{}^*\infty} c_k(u)(Ext\text{-}\exp[i(k\omega,x)]), |c_k(u)| \le A(1+|k|)^m. \tag{72}$$

The series (1) #-converges to $u(x)$ in $S^{\#'}({}^*\mathbb{R}^{\#n}_{c})$, here $\omega = \left(\frac{2\pi}{T_1},\dots,\frac{2\pi}{T_n}\right)$ and $k\omega = \left(\frac{2\pi k_1}{T_1},\dots,\frac{2\pi k_n}{T_n}\right)$.

## A NON-ARCHIMEDEAN METRIC SPACES ENDOWED WITH ${}^*\mathbb{R}^{\#}_{\mathbf{c}}$ -VALUED METRIC

**Definition 3.1** A non-Archimedean metric space is an ordered pair $(M,d^{\#})$ where $M$ a set and $d^{\#}$ is a #-metric on $M$ i.e., ${}^*\mathbb{R}^{\#}_{c+}$- valued function $d^{\#}\colon M \times M \to {}^*\mathbb{R}^{\#}_{c+}$such that for any triplet $x,y,z \in M$, the following holds:
1. $d^{\#}(x,y) = 0 \Longrightarrow x = y$. 2. $d^{\#}(x,y) = d^{\#}(y,x)$. 3. $d^{\#}(x,z) \le d^{\#}(x,y) + d^{\#}(y,z)$.

**Definition 3.2** A hyper infinite sequence $\{x_n\}_{n=1}^{{}^*\infty}$ of points in $M$ is called #-Cauchy in $(M,d^{\#})$ if for every hyperreal $\varepsilon \in {}^*\mathbb{R}^{\#}_{c+}$ there exists some $N \in {}^*\mathbb{N}$ such that $d^{\#}(x_n,x_m) < \varepsilon$ if $n,m > N$.

**Definition 3.3** A point $x$ of the non-Archimedean metric space $(M,d^{\#})$ is the #-limit of the hyper infinite sequence $\{x_n\}_{n=1}^{{}^*\infty}$ if for all $\varepsilon \in {}^*\mathbb{R}^{\#}_{c+}$, there exists some $N \in {}^*\mathbb{N}$ such that $d^{\#}(x_n,x) < \varepsilon$ if $n > N$.

**Definition 3.4** A non-Archimedean metric space is #-complete if any of the following equivalent conditions are satisfied:

1.Every hyper infinite #-Cauchy sequence $\{x_n\}_{n=1}^{*\infty}$ of points in $M$ has a #-limit that is also in $M$.

2.Every hyper infinite #-Cauchy sequence in $M$, #-converges in $M$ that is, to some point of $M$.

For any non-Archimedean metric space $(M, d^{\#})$ one can construct a #-complete non-Archimedean metric space $(M', d^{\#})$ which is also denoted as $(\#\text{-}\bar{M}, d^{\#})$ and which contains $M$ a #-dense subspace.

It has the following universal property: if $K$ is any #-complete non-Archimedean metric space and $f: M \to K$ is any uniformly #-continuous function from $M$ to $K$, then there exists a unique uniformly #-continuous function $f': M' \to K$ that extends $f$.The space $\#\text{-}\bar{M}$ is determined up to #-isometry by this property (among all #-complete metric spaces #- isometrically containing non-Archimedean metric space $(\#\text{-}\bar{M}, d^{\#})$, and is called the #-completion of $(M, d^{\#})$.

The #-completion of $M$ can be constructed as a set of equivalence classes of Cauchy hyper infinite sequences in $M$. For any two hyper infinite Cauchy sequences $\{x_n\}_{n=1}^{*\infty}$ and $\{y_n\}_{n=1}^{*\infty}$ in $M$, we may define their distance as $d^{\#\prime} = \#\text{-}\lim_{n \to \infty^{\#}} d^{\#}(x_n, y_n)$. This #-limit exists because the hyperreal numbers ${}^*\mathbb{R}_{\mathbf{c}}^{\#}$ are #-complete. This is only a pseudo metric, not yet a metric, since two different hyper infinite Cauchy sequences may have the distance 0. But having distance 0 is an equivalence relation on the set of all hyper infinite Cauchy sequences, and the set of equivalence classes is a metric space, the #-completion of M. The original space is embedded in this space via the identification of an element $x$ of $M'$ with the equivalence class of hyper infinite sequences in $M$ #-converging to $x$ i.e., the equivalence class containing a hyper infinite sequence with constant value $x$. This defines a #-isometry onto a #-dense subspace, as required.

**Example 3.1** Both ${}^*\mathbb{R}$ and ${}^*\mathbb{C}$ are internal metric spaces when endowed with the distance function $d(x, y) = |x - y|$.

**Definition 3.5** About any point $x \in M$ we define the #-open ball of radius $r \in {}^*\mathbb{R}_{c+}^{\#}$ about $x$ as the set $B_r(x) = \{y \in M | d^{\#}(x, y) < r\}$. These #-open balls form the base for a topology on $M$.

**Definition 3.6** A non-Archimedean metric space $(M, d^{\#})$ is called hyper finitely bounded if there exists some $r \in {}^*\mathbb{R}_{c,\mathrm{fin}+}$ such that $d^{\#}(x, y) < r$ for all $x, y \in M$.

**Definition 3.7** A non-Archimedean metric space $(M, d^{\#})$ is called finitely bounded if there exists some $r \in {}^*\mathbb{R}_{c,\infty+}$ such that $d^{\#}(x, y) < r$ for all $x, y \in M$.

**Definition 3.8** A non-Archimedean metric space $(M, d^{\#})$ is called hyper finitely bounded if there exists some $r \in {}^*\mathbb{R}_{c,\infty+}$ such that $d^{\#}(x, y) < r$ for all $x, y \in M$.

**Definition 3.9** Let $(M, d^{\#})$ be a non-Archimedean metric space. A set $A \subset X$ is called finitely bounded if there exists some $r \in {}^*\mathbb{R}_{c,\mathrm{fin}+}$ such that $A \subset B_r(a)$, $a \in X$.

**Definition 3.10** A non-Archimedean metric space $(M, d^{\#})$ is called #-compact if every hyper infinite sequence $\{x_n\}_{n=1}^{*\infty}$ in $M$ has a hyper infinite subsequence that #-converges to a point in $M$. This sort of compactness is known as hyper sequential compactness and, in a non-Archimedean metric spaces is equivalent to the topological notions of hyper countable #-compactness.

**Definition 3.11** A topological space $X$ is called hyper countably #-compact if it satisfies any of the following equivalent conditions: (a) every hyper countable open cover $U$ of $X$ (i.e., $card(U) = card({}^*\mathbb{N})$) has a finite or hyperfinite sub-cover.

For a function $f: M_1 \to M_2$ with a non-Archimedean metric spaces $(M_1, d_1^{\#})$ and $(M_2, d_2^{\#})$ the following definitions of uniform #-continuity and (ordinary) #-continuity hold.

**Definition 3.12** A function $f$ is called uniformly #-continuous if for every $\varepsilon \in {}^*\mathbb{R}_{c\approx+}^{\#}$there exists $\delta \in {}^*\mathbb{R}_{c\approx+}$ such that for every $x, y \in M_1$ with $d_1^{\#}(x, y) < \delta$ we get $d_2^{\#}(f(x), f(y)) < \varepsilon$.

**Definition 3.13** A function $f$ is called #-continuous at $x \in M_1$ if for every $\varepsilon \in {}^*\mathbb{R}_{c\approx+}^{\#}$there exists $\delta \in {}^*\mathbb{R}_{c\approx+}^{\#}$ such that for every $y \in M_1$ with $d_1^{\#}(x, y) < \delta$ we get $d_2^{\#}(f(x), f(y)) < \varepsilon$.

## LEBESGUE #-INTEGRATION OF ${}^*\mathbb{R}_{\mathbf{c}}^{\#}$ -VALUED FUNCTIONS

Let $C_0^{\#}({}^*\mathbb{R}_c^{\#n})$ be the space of all ${}^*\mathbb{R}_c^{\#}$-valued #-compactly supported #-continuous functions of ${}^*\mathbb{R}_c^{\#n}$. Define a #-norm on $C_0^{\#}$ by the Riemann #-integral [12]:

$$\|f\|_{\#} = Ext\text{-}\int |f(x)| d^{\#n} x, \tag{73}$$

Note that the Riemann #-integral exists for any #-continuous function $f: {}^*\mathbb{R}_c^{\#n} \to {}^*\mathbb{R}_c^{\#}$ , see [12]. Then $C_0^{\#}({}^*\mathbb{R}_c^{\#n})$ is a #-normed vector space and thus in particular, it is a non-Archimedean metric space. All non-Archimedean metric space, have a non-Archimedean #-completion $(\#\text{-}\bar{M}, d^{\#})$. Let $L_1^{\#}$ be this #-completion. This space $L_1^{\#}$ is isomorphic to the space of Lebesgue #-integrable functions modulo the subspace of functions with #-integral zero. Furthermore, the Riemann integral (1) is a uniformly #-continuous linear functional with respect to the #-norm on $C_0^{\#}({}^*\mathbb{R}_c^{\#n})$ which is #-dense in $L_1^{\#}$. Hence the Riemann #- integral $Ext\text{-}\int f(x)d^{\#n}x$ has a unique extension to all of $L_1^{\#}$. This integral is precisely the Lebesgue #-integral.

**Definition 4.1** Suppose that $1 \leq p < {}^*\infty$, and $[a,b]$ is an interval in ${}^*\mathbb{R}_c^{\#}$. We denote by $L_p^{\#}([a,b])$ the set of the all functions $f: [a,b] \to {}^*\mathbb{R}_c^{\#}$ such that $Ext\text{-}\int_a^b |f(x)|^p d^{\#}x < {}^*\infty$. We define the $L_p^{\#}$ -#-norm of $f$ by

$$\|f\|_{\#p} = \left(Ext\text{-}\int_a^b |f(x)|^p d^{\#}x\right)^{1/p}. \tag{74}$$

More generally, if $E$ is a subset of ${}^*\mathbb{R}_c^{\#n}$, which could be equal to ${}^*\mathbb{R}_c^{\#n}$ itself, then $L_p^{\#}(E)$ is the set of Lebesgue #-measurable functions $f : E \to {}^*\mathbb{R}_c^{\#}$ whose $p$-th power is Lebesgue #-integrable, with the #-norm

$$\|f\|_{\#p} = \left(Ext\text{-}\int_E |f(x)|^p d^{\#n}x\right)^{1/p}. \tag{75}$$

**Definition 4.2** A set $X \subset {}^*\mathbb{R}_c^{\#n}$ is #-measurable if there exists $Ext\text{-}\int 1_X \, d^{\#n}x$, where $1_X$ is the indicator function.

**Definition 4.3** A ${}^*\mathbb{R}_c^{\#}$ -valued function $f$ on ${}^*\mathbb{R}_c^{\#n}$ is a #-measurable if a set $\{x | f(x) > t\}$ is a #-measurable set for all $t \in {}^*\mathbb{R}_c^{\#n}$.

**Remark 4.1** To assign a value to the Lebesgue #-integral of the indicator function $1_X$ of a #-measurable set $X$ consistent with the given #-measure $\mu^{\#}$, the only reasonable choice is to set: $Ext\text{-}\int 1_X d\,\mu^{\#} = \mu^{\#}(X)$.

**Definition 4.4** A hyperfinite linear combination of indicator functions $f = Ext\text{-}\sum_{k=1}^{n} \alpha_k \, 1_{X_k}$ where the coefficients $\alpha_k \in {}^*\mathbb{R}_c^{\#}$ and $X_k$ are disjoint #-measurable sets, is called a #-measurable simple function.

**Definition 4,5** When the coefficients $\alpha_k$ are positive, we set $Ext\text{-}\int f d\,\mu^{\#} = Ext\text{-}\sum_{k=1}^{n} \alpha_k \, \mu^{\#}(X_k)$. For a non-negative #-measurable function $f$, let $\{f_n(x)\}_{n=1}^{^*\infty}$be a hyper infinite sequence of the simple functions $f_n(x)$ whose values is $\frac{k}{2^n}$ whenever $\frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n}$ for $k$ a non-negative hyperinteger less than $4^n$. Then we set

$$Ext\text{-}\int f d\,\mu^{\#} = \#\text{-}\lim_{n \to {}^*\infty}\left(Ext\text{-}\int f_n d\,\mu^{\#}\right).$$

**Definition 4.6** If $f$ is a #-measurable function of the set $E$ to the reals including $\pm\infty^{\#}$, then we can write $f = f^+ - f^-$, where: 1) $f^+(x) = f(x)$ if $f(x) > 0$ and $f^+(x) = 0$ if $f(x) \leq 0$; 2) $f^-(x) = f(x)$ if $f(x) < 0$ and $f^-(x) = 0$ if $f(x) \geq 0$. Note that both $f^+$ and $f^-$ are non-negative #-measurable functions and $|f| = f^+ + f^-$.

**Definition 4.7** We say that the Lebesgue #-integral of the #-measurable function $f$ exists, or is defined if at least one of $Ext\text{-}\int f^+ d\,\mu^{\#}$ and $Ext\text{-}\int f^- d\,\mu^{\#}$ is finite or hyperfinite. In this case we define

$$Ext\text{-}\int f d\,\mu^{\#} = \left(Ext\text{-}\int f^+ d\,\mu^{\#}\right) + \left(Ext\text{-}\int f^- d\,\mu^{\#}\right).$$

**Theorem 4.1** Assuming that $f$ is #-measurable and non-negative, the function $\check{f}(x) = \{x \in E | f(x) > t\}$ is monotonically non-increasing. The Lebesgue #-integral may then be defined as the improper Riemann #-integral of $\check{f}(x)$: $Ext\text{-}\int_E f d\,\mu^{\#} = Ext\text{-}\int_0^{^*\infty} \check{f}(x) d^{\#}x$.

**Definition 4.8** Let $X$ be any set. We denote by $2^X$ the set of all subsets of $X$.A family $\mathcal{F} \subset 2^X$ is called a #-$\sigma$-algebra on $X$ (or $\sigma^{\#}$-algebra on $X$) if: 1) $\emptyset \in \mathcal{F}$. 2) A family $\mathcal{F}$ is closed under complements, i.e. $A \in \mathcal{F}$ implies $X \backslash A \in \mathcal{F}$. 3) A family $\mathcal{F}$ is closed under hyper infinite unions, i.e. if $\{A_n\}_{n \in {}^*\mathbb{N}}$ is a hyper infinite sequence in $\mathcal{F}$ then $\bigcup_{n \in {}^*\mathbb{N}} A_n \in \mathcal{F}$.

**Theorem 4.2** If $\mathcal{F}$ is a #-$\sigma$-algebra on $X$ then: (1) $\mathcal{F}$ is closed under hyper infinite intersections, i.e., if $\{A_n\}_{n \in {}^*\mathbb{N}}$ is a hyper infinite sequence in $\mathcal{F}$ then $\bigcap_{n \in {}^*\mathbb{N}} A_n \in \mathcal{F}$. (2) $X \in \mathcal{F}$.3) $\mathcal{F}$ is closed under hyperfinite unions and hyperfinite intersections.(4) $\mathcal{F}$ is closed under set differences. (5) $\mathcal{F}$ is closed under symmetric differences.

**Theorem 4.3** If $\{A_\alpha\}_{\alpha \in I}$ is a collection of $\sigma^{\#}$-algebras on a set $X$, then $\bigcap_{\alpha \in I} A_\alpha$ , is also an $\sigma^{\#}$-algebras on a set $X$.

**Theorem 4.4** If $K \subset L$ then $\sigma^{\#}(K) \subset \sigma^{\#}(L)$.

**Definition 4.9** (Borel $\sigma^{\#}$-algebra) Given a topological space $X$, the Borel $\sigma^{\#}$-algebra is the $\sigma^{\#}$-algebra generated by the #-open sets. It is denoted by $\mathcal{B}^{\#}(X)$. We call sets in $\mathcal{B}^{\#}(X)$ a Borel set. Specifically in the case $X = {}^*\mathbb{R}_c^{\#n}$ we have that $\mathcal{B}^{\#}({}^*\mathbb{R}_c^{\#n}) = \{U | U \text{ is } \#\text{-open set}\}$. Note that the Borel $\sigma^{\#}$-algebra also contains all #-closed sets and is the smallest $\sigma^{\#}$-algebra with this property.

**Definition 4.10** (#- Measures) A pair $(X,\mathcal{F})$ where $\mathcal{F}$ is an $\sigma^{\#}$-algebra on $X$ is call a #- measurable space. Elements of $\mathcal{F}$ are called a #-measurable sets. Given a #-measurable space $(X,\mathcal{F})$, a function $\mu^{\#}:\mathcal{F}\to[0,{}^{*}\infty]$ is called a #-mea-sure on $(X,\mathcal{F})$ if: 1) $\mu^{\#}(\emptyset)=0$. 2) For all hyper infinite sequences $\{A_n\}_{n\in{}^{*}\mathbb{N}}$ of pairwise disjoint sets in $\mathcal{F}$

$$\mu^{\#}\left(\bigcup_{n=1}^{{}^{*}\infty}A_n\right)=Ext\text{-}\sum_{n=1}^{{}^{*}\infty}\mu^{\#}(A_n). \tag{76}$$

## A NON-ARCHIMEDEAN BANACH SPACES ENDOWED WITH ${}^{*}\mathbb{R}_{c}^{\#}$ -VALUED NORM

A non-Archimedean normed space with ${}^{*}\mathbb{R}_{c}^{\#}$ -valued norm (#-norm) is a pair $(X,\|\cdot\|_{\#})$ consisting of a vector space $X$ over a non-Archimedean scalar field ${}^{*}\mathbb{R}_{c}^{\#}$ or complex field ${}^{*}\mathbb{C}_{c}^{\#}={}^{*}\mathbb{R}_{c}^{\#}+\mathrm{i}{}^{*}\mathbb{R}_{c}^{\#}$ together with a norm $\|\cdot\|_{\#}:X\to{}^{*}\mathbb{R}_{c}^{\#}$. Like any norms, this norm induces a translation invariant distance function, called the norm induced non-Archimedean ${}^{*}\mathbb{R}_{c}^{\#}$ -valued metric $d^{\#}(x,y)$ for all vectors $x,y\in X$, defined by $d^{\#}(x,y)=\|x-y\|_{\#}=\|y-x\|_{\#}$. Thus $d^{\#}(x,y)$ makes $X$ into a non-Archimedean metric space $(X,d^{\#})$.

**Definition 5.1** A hyper infinite sequence $\{x_n\}_{n=1}^{{}^{*}\infty}$ in $X$ is called $d^{\#}$ - Cauchy or Cauchy in $(X,d^{\#})$ or $\|\cdot\|_{\#}$ -Cauchy if for every hyperreal $\varepsilon\in{}^{*}\mathbb{R}_{c+}^{\#}$ there exists some $N\in{}^{*}\mathbb{N}$ such that $d^{\#}(x_n,y_m)=\|x_n-y_n\|_{\#}<\varepsilon$ if $n,m>N$.

**Definition 5.2** The metric $d^{\#}$ is called a #-complete metric if the pair $(X,d^{\#})$ is a #-complete metric space, which by definition means for every $d^{\#}$- Cauchy sequence $\{x_n\}_{n=1}^{{}^{*}\infty}$ in $(X,d^{\#})$, there exists some $x\in X$ such that $\#\text{-}\lim_{n\to{}^{*}\infty}\|x_n-x\|_{\#}=0$.

### Semigroups on Non-Archimedean Banach Spaces and Their Generators

**Definition 5.3** A family of bounded operators $\{T(t)|0<t<{}^{*}\infty\}$ on external hyper infinite dimensional non-Archimedean Banach space $X$ endowed with ${}^{*}\mathbb{R}_{c}^{\#}$ -valued #-norm $\|\cdot\|_{\#}$ is called a strongly #-continuous semigroup if: (a) $T(0)=I$, (b) $T(s)T(t)=T(s+t)$ for all $s,t\in{}^{*}\mathbb{R}_{c,+}^{\#}$, (c) For each $\phi\in X, t\mapsto T(t)$ is #-continuous map-ping.

**Definition 5.4** A family $\{T(t)|0<t<{}^{*}\infty\}$ of bounded or hyper bounded operators on external hyper infinite dimensional Banach space $X$ is called a contraction semigroup if it is a strongly #-continuous semigroup and moreover $\|T(t)\|_{\#}<1$ for all $t\in[0,{}^{*}\infty)$.

**Theorem 5.1** Let $T(t)$ is a strongly #-continuous semigroup on a non-Archimedean Banach space $X$, let

$$A\varphi=\#\text{-}\lim_{r\to_{\#}0}A_r\varphi$$

where $A_r=r^{-1}\big(I-T(r)\big)$ and let $D(A)=\{\varphi|\exists\big(\#\text{-}\lim_{r\to_{\#}0}A_r\varphi\big)\}$, then the operator $A$ is #-closed and #-densely defined. Operator $A$ is called the infinitesimal generator of the semigroup $T(t)$.

**Definition 5.5** We will also say that $A$ generates the semigroup $T(t)$ and write $T(t)=Ext\text{-}\exp(-tA)$.

**Theorem 5.2** (Generalized Hille -Yosida theorem) A necessary and sufficient condition that #-closed linear operator $A$ on a non-Archimedean Banach space $X$ generate a contraction semigroup is that: (a) $(-{}^{*}\infty,0)\subset\rho(A)$, (b) $\|(\lambda+A)^{-1}\|_{\#}\leq\lambda^{-1}$ for all $\lambda>0$.

**Definition 5.6** Let $X$ be a non-Archimedean Banach space, $\varphi\in X$.An element $l\in X^{*}$ that satisfies $\|l\|_{\#}=\|\varphi\|_{\#}$, and $l(\varphi)=\|\varphi\|_{\#}^{2}$ is called a normalized tangent functional to $\varphi$. By the generalized Hahn-Banach theorem, each $\varphi\in X$ has at least one normalized tangent functional.

**Definition 5.7** A #-densely defined operator $A$ on a non-Archimedean Banach space $X$ is called accretive if for each $\varphi\in D(A)$, $\mathrm{Re}\big(l(A\varphi)\big)\geq 0$ for some normalized tangent functional to $\varphi$. Operator $A$ is called maximal accretive if $A$ is accretive and $A$ has no proper accretive extension.

**Remark 5.1** We remark that any accretive operator is #-closable. The #-closure of an accretive operator is again accretive, so every accretive operator has a smallest #-closed accretive extension.

**Theorem 5.3** A #-closed operator $A$ on a non-Archimedean Banach space $X$ is the generator of a contraction semigroup if and only if $A$ is accretive and $\mathrm{Ran}(\lambda_0+A)=X$ for some $\lambda_0>0$.

**Theorem 5.4** Let $A$ be a #-closed operator on a non-Archimedean Banach space $X$. Then, if both $A$ and it adjoint $A^{*}$ are accretive, $A$ generates a contraction semigroup.

**Theorem 5.5** Let A be the generator of a contraction semigroup on a non-Archimedean Banach space $X$. Let $D$ be a #-dense set, $D \subset D(A)$, so that $Ext\text{-}\exp(-tA) : D \to D$. Then $D$ is a #-core for $A$, i.e.,#-$\overline{A \restriction D} = A$.

## Hypercontractive Semigroups

In the previous section we discussed $L_{\#}^{p}$-contractive semigroups. In this section we give a self #- adjointness theorem for the operators of the form $A + V$, where $V$ is a multiplication operator and $A$ generates a $L_{\#}^{p}$-contractive semigroup that satisfies a strong additional property.

**Definition 5.8** Let $\langle M.\mu^{\#}\rangle$ be a #-measure space with $\mu^{\#}(M) = 1$ and suppose that $A$is a positive self-adjoint operator on $L_{\#}^{2}(M, d^{\#}\mu^{\#})$. We say that $Ext$-$\exp(-tA)$ is a hyper contractive semigroup if: (a) $Ext$-$\exp(-tA)$ is $L_{\#}^{p}$-contractive; (b) for some $b > 2$ and some constant $C_b$, there is a $T > 0$ so that $\|[Ext\text{-}\exp(-tA)]\varphi\|_{\#b} \leq \|\varphi\|_{\#2}$ for all $\varphi \in\ L_{\#}^{2}(M, d^{\#}\mu^{\#})$.

**Remark 5.2** Note that the condition (a) implies that $Ext$-$\exp(-tA)$ is a strongly #-continuous contraction semi-group for all $p < {}^{*}\infty$. Holder's inequality shows that $\|\cdot\|_{\#q} \leq \|\cdot\|_{\#p}$ if $p \geq q$. Thus the $L_{\#}^{p}$-spaces are a nested family of spaces which get smaller as $p$ gets larger; this suggests that (b) is a very strong condition. The following proposition shows that constant $b$ plays no special role.

**Theorem 5.6** Let $Ext$-$\exp(-tA)$ be a hypercontractive semigroup on $L_{\#}^{2}(M, d^{\#}\mu^{\#})$. Then for all $p, q \in (1, {}^{*}\infty)$ there is a constant $C_{p,q}$ and a $t_{p,q} > 0$ so that if $> t_{p,q}$ , then $\|Ext\text{-}\exp(-tA)\varphi\|_{\#p} < C_{p,q}\|\varphi\|_{\#q}$, for all $\varphi \in\ L_{q}^{\#}$.

**Theorem 5.7** Let $\langle M, \mu^{\#}\rangle$ be a $\sigma^{\#}$-measure space with $\mu^{\#}(M) = 1$and let $H_0$ be the generator of a hypercontractive semi-group on $L_2(M, d^{\#}\mu^{\#})$. Let $V$ be a ${}^{*}\mathbb{R}_{c}^{\#}$ -valued measurable function on $\langle M, \mu^{\#}\rangle$ such that $V \in L_{p}^{\#}(M, d^{\#}\mu^{\#})$ for all $p \in [\![1, {}^{*}\infty)$ and $Ext$-$\exp(-tV) \in L_{1}^{\#}(M, d^{\#}\mu^{\#})$ for all $t > 0$. Then $H_0 + V$ is essentially self#-adjoint on $C^{{}^{*}\infty}(H_0\ ) \cap D(V)$ and is bounded below. Here $C^{{}^{*}\infty}(H_0\ ) = \bigcap_{p \in {}^{*}\mathbb{N}} D\big(H_0^{p}\big)$.

## A NON-ARCHIMEDEAN HILBERT SPACES ENDOWED WITH ${}^{*}\mathbb{C}_{c}^{\#}$ -VALUED INNER PRODUCT

**Definition 6.1** Let $H$ be external hyper infinite dimensional vector space over complex field ${}^{*}\mathbb{C}_{c}^{\#} = {}^{*}\mathbb{R}_{c}^{\#} + \mathrm{i}\,{}^{*}\mathbb{R}_{c}^{\#}$. An inner product on $H$ is a${}^{*}\mathbb{C}_{c}^{\#}$-valued function, $\langle\cdot,\cdot\rangle: H \times H \to {}^{*}\mathbb{C}_{c}^{\#}$, such that (1) $\langle ax + by, z\rangle = \langle ax, z\rangle + \langle by, z\rangle$, (2) $\overline{\langle x, y\rangle} = \langle y, x\rangle$. (3) $\|x\|^2 \equiv \langle x, x\rangle \geq 0$ with equality $\langle x, x\rangle = 0$ if and only if $x = 0$.

**Theorem 6.1** (Generalized Schwarz Inequality) Let $\{H, \langle\cdot,\cdot\rangle\}$be an inner product space, then for all $x, y \in H$: $|\langle x, y\rangle| \leq \|x\|\|y\|$ and equality holds if and only if $x$ and $y$ are linearly dependent.

**Theorem 6.2** Let $\{H, \langle\cdot,\cdot\rangle\}$be an inner product space, and $\|x\|_{\#} = \sqrt{\langle x, x\rangle}$ . Then $\|\cdot\|_{\#}$ is a ${}^{*}\mathbb{R}_{c}^{\#}$ -valued #-norm on a space $H$. Moreover $\langle x, x\rangle$ is #-continuous on Cartesian product $H \times H$, where $H$ is viewed as the #-normed space $\{H, \|\cdot\|_{\#}\}$.

**Definition 6.2** A non-Archimedean Hilbert space is a #-complete inner product space.

**Example 6.1** The standard inner product on ${}^{*}\mathbb{C}_{c}^{\#n}, n \in {}^{*}\mathbb{N}_{\infty}$ is given by external hyperfinite sum

$$\langle x, y\rangle = Ext\text{-}\sum_{i=1}^{n} \overline{x_i}\, y_i. \tag{77}$$

Here $x = \{x_i\}_{i=1}^{n}, y = \{y_i\}_{i=1}^{n}$ , with $x_i, y_i \in {}^{*}\mathbb{C}_{c}^{\#}, 1 \leq i \leq n$ , see [13].

**Example 6.2** The sequence space $l_2^{\#}$ consists of all hyper infinite sequences $z = \{z_i\}_{i=1}^{{}^{*}\infty}$ of complex numbers in ${}^{*}\mathbb{C}_{c}^{\#}$ such that the hyper infinite series Ext-$\sum_{i=1}^{n}|z_i|^2$ #-converges. The inner product on $l_2^{\#}$ is defined by

$$\langle z, w\rangle = Ext\text{-}\sum_{i=1}^{{}^{*}\infty} \overline{z_i}\, w_i. \tag{78}$$

Here $z = \{z_i\}_{i=1}^{{}^{*}\infty}$, $w = \{w_i\}_{i=1}^{{}^{*}\infty}$ and the latter hyper infinite series #-converging as a consequence of the generalized Schwarz inequality and the #-convergence of the previous hyper infinite series.

**Example 6.3** Let $C^{\#}[a, b]$ be the space of the ${}^{*}\mathbb{C}_{c}^{\#}$- valued #-continuous functions defined on the interval $[a, b] \subset {}^{*}\mathbb{R}_{c}^{\#}$, see [13]. We define an inner product on the space $C^{\#}[a, b]$ by the formula

$$\langle f, g\rangle = Ext\text{-}\int_a^b \overline{f(x)} g(x)\, d^{\#}x. \tag{79}$$

This space is not #-complete, so it is not a non-Archimedean Hilbert space. The #-complettion of $C^{\#}[a, b]$ with respect to the #-norm

$$\|f\|_{\#} = \left(Ext\text{-}\int_a^b |f(x)|^2\, d^{\#}x\right)^{1/2}, \tag{80}$$

is denoted by $L_2^{\#}[a,b]$.

**Example 6.4** Let $C^{\#(k)}[a,b]$be the space of the ${}^*\mathbb{C}_c^{\#}$- valued functions with $k \in {}^*\mathbb{N}$ #-continuous #-derivatives on $[a,b] \subset {}^*\mathbb{R}_c^{\#}$, see [13].We define an inner product on the space $C^{\#(k)}[a,b]$ by the formula

$$\langle f, g\rangle = Ext\text{-}\sum_{i=0}^{k} \left(Ext\text{-}\int_a^b \overline{f^{\#(i)}(x)} g^{\#(i)}(x)\, d^{\#}x\right). \tag{81}$$

Here $f^{\#(i)}$ and $g^{\#(i)}$ denotes the $i$-th #-derivatives of $f$ and $g$ respectively.The corresponding #-norm is

$$\|f\|_{\#} = \left(Ext\text{-}\sum_{i=1}^{k} \left(Ext\text{-}\int_a^b \left|f^{\#(i)}(x)\right|^2 d^{\#}x\right)\right)^{1/2}. \tag{82}$$

This space is not #-complete, so it is not a non-Archimedean Hilbert space. The non-Archimedean Hilbert space obtained by #-completion of $C^{\#(k)}[a,b]$ with respect to the #-norm (1) is non-Archimedean Sobolev space, denoted by $H^{\#k}[a,b]$.

**Definition 6.3** The graph of the linear transformation $T: H \to H$ is the set of pairs $\{\langle \phi, T\phi\rangle | (\phi \in D(T))\}$. The graph of the operator $T$, denoted by $\Gamma(\mathrm{T})$, is thus a subset of $H \times H$ which is a non-Archimedean Hilbert space with the following inner product $(\langle \phi_1, \psi_1\rangle, \langle \phi_2, \psi_2\rangle)$. Operator $T$ is called a #-closed operator if $\Gamma(\mathrm{T})$ is a #-closed subset of $H \times H$.

**Definition 6.4** Let $T_1$ and $T$ be operators on H. If $\Gamma(T_1) \supset \Gamma(\mathrm{T})$, then $T_1$ is said to be an extension of $T$ and we write $T_1 \supset T$. Equivalently, $T_1 \supset T$ if and only if $D(T_1) \supset D(T)$ and $T_1\phi = T\phi$ for all $\phi \in D(T)$.

**Definition 6.5** An operator $T$ is #-closable if it has a #-closed extension. Every #-closable operator has a smallest #-closed extension, called its #-closure, which we denote by #-T.

**Theorem 6.3** If $T$ is #-closable, then $\Gamma(\#\text{-}\bar{T}) = \#\text{-}\overline{\Gamma(T)}$.

**Definition 6.6** Let $D(T^*)$ be the set of $\varphi \in H$ for which there is an $\xi \in H$ with $(T\psi, \varphi) = (\psi, \xi)$ for all $\psi \in D(T)$. For each $\varphi \in D(T^*)$, we define $T^*\varphi = \xi$.The operator $T^*$ is called the #-adjoint of $T$. Note that $\varphi \in D(T^*)$ if and only if $|(T\psi, \varphi)| \leq C\|\psi\|_{\#}$ for all $\psi \in D(T)$. Note that $S \subset T$ implies $T^* \subset S$.

**Remark 6.1** Note that for $\xi$ to be uniquely determined by the condition $(T\psi, \varphi) = (\psi, \xi)$ one need the fact that $D(T)$ is #-dense in $H$. If the domain $D(T^*)$ is #-dense in $H$, then we can define $T^{**} = (T^*)^*$.

**Theorem 6.4** Let $T$ be a #-densely defined operator on a non-Archimedean Hilbert space $H$. Then: (a) $T^*$ is #-closed. (b) The operator $T$ is #-closabie if and only if $D(T^*)$ is -dense in which case $T = T^{**}$. (c) If T is #-closable, then $(\#\text{-}\bar{T})^* = T^*$.

**Definition 6.7** Let $T$ be a #-closed operator on a non-Archimedean Hilbert space $H$. A complex number $\lambda \in {}^*\mathbb{C}_c^{\#}$ is in the resolvent set $\rho(T)$, if $\lambda I - T$ is a bijection of $D(T)$ onto $H$ with a finitely or hyper finitely bounded inverse. If complex number $\lambda \in \rho(T)$, $R_\lambda = (\lambda I - T)^{-1}$ is called the resolvent of $T$ at $\lambda$.

**Definition 6.8** A #-densely defined operator $T$ on a non-Archimedean Hilbert space is called symmetric or Hermitian if $T \subset T^*$, that is, $D(T) \subset D(T^*)$ and $T\varphi = T^*\varphi$ for all $\varphi \in D(T)$ and equivalently, $T$ is symmetric if and only if $(T\varphi, \psi) = (\varphi, T\psi)$ for all $\varphi, \psi \in D(T)$.

**Definition 6.9** A #-densely defined operator $T$ is called self-#-adjoint if $T = T^*$, that is, if and only if $T$ is symmetric and $D(T) = D(T^*)$.

**Remark 6.2** A symmetric operator $T$ is always #-closable, since $D(T)$ #-dense in $H$. If $T$ is symmetric, $T^*$ is a #-closed extension of $T$ so the smallest #-closed extension $T^{**}$ of $T$ must be contained in $T^*$. Thus for symmetric operators, we have $T \subset T^{**} \subset T^*$, for #-closed symmetric operators we have $T = T^{**} \subset T^*$ and, for self-#-adjoint operators we have $T = T^{**} = T^*$. Thus a #-closed symmetric operator $T$ is self-#-adjoint if and only if $T^*$ is symmetric.

**Definition 6.10** A symmetric operator $T$ is called essentially self-#-adjoint if its #-closure #-$\bar{T}$ is self-#-adjoint. If $T$ is #-closed, a subset $D \subset D(T)$ is called a core for $T$ if #-$\overline{T \restriction D} = T$.

**Remark 6.3** If $T$ is essentially self-#-adjoint, then it has one and only one self-#-adjoint extension.

**Definition 6.11** Let $A$ be an operator on a non-Archimedean Hilbert space $H^{\#}$.The set $C^{*\infty}(A) = \bigcap_{n=1}^{*\infty} D(A^n)$ is called the $C^{*\infty}$-vectors for $A$. A vector $\varphi \in C^{*\infty}(A)$ is called an #-analytic vector for $A$ if $Ext\text{-}\sum_{n=0}^{*\infty} \frac{\|A^n\|_{\#} t^n}{n!} < {}^*\infty$ for some $t > 0$. If $A$ is self-#-adjoint, then $C^{*\infty}(A)$ will be #-dense in $D(A)$.

**Theorem 6.5** (Generalized Nelson's analytic vector theorem) Let $A$ be a symmetric operator on a non-Archimedean Hilbert space H. If $D(A)$ contains a #-total set of #-analytic vectors, then $A$ is essentially self-#-adjoint.

**Definition 6.12** Operator $A$ is relatively bounded with respect to operator $T$ if $D(T) \subset D(A)$ and

$$\|Au\|_{\#} \leq a\|u\|_{\#} + b\|Tu\|_{\#}, u \in D(T).$$

**Theorem 6.6** Let $T$ be self-#-adjoint. If $A$ is symmetric and $T$-bounded with $T$-bound smaller than 1, then $T + A$ is also self-#-adjoint. In particular $T + A$ is self-#-adjoint if $A$ is bounded and symmetric with $D(T) \subset D(A)$.

**Theorem 6.7** Let $A$ be essentially self -#-adjoint on the domain $D(A)$ and let $B$ be a symmetric operator on $D(A)$. If there exists a constant $a \in {}^*\mathbb{R}_c^\#$ such that for all $\psi \in D(A)$ and for all $\beta \in {}^*\mathbb{R}_c^\#$ such that $0 \leq \beta \leq 1$ and the inequality holds $\|B\psi\|_\# \leq a\|(A + \beta B)\psi\|_\#$, then $A + B$ is essentíallv self -#-adjoint on $D(A)$ and its #-closure has domain $\boldsymbol{D}(\#\text{-}\bar{\boldsymbol{A}})$.

**Theorem 6.8** Let $A$ and $B$ be the same as in Theorem 6.7. Then $A$ and $A + B$ have the same #-cores. If $A$ is bounded from below, then $A + B$ is bounded from below.

## GENERALIZED TROTTER PRODUCT FORMULA

**Theorem 7.1** Let $A$ and $B$ be self-adjoint operators on non-Archimedean Hilbert space $H^\#$. Suppose that the opera-tor $A + B$ is self-#-adjoint on $D = D(A) \cap D(B)$, then the following equality holds

$$\text{s-\#-}\lim_{n\to{}^*\infty}\left[\left(Ext\text{-}\exp\left(\frac{itA}{n}\right)\right)\left(Ext\text{-}\exp\left(\frac{itB}{n}\right)\right)\right]^n = Ext\text{-}\exp[it(A+B)]. \quad (83)$$

**Theorem 7.2** Let $A$ and $B$ be self-adjoint operators on non-Archimedean Hilbert space $H^\#$. Suppose that the opera-tor $A + B$ is essentially self-#-adjoint on $D = D(A) \cap D(B)$, then the following equality holds

$$\text{s-\#-}\lim_{n\to{}^*\infty}\left[\left(Ext\text{-}\exp\left(\frac{itA}{n}\right)\right)\left(Ext\text{-}\exp\left(\frac{itB}{n}\right)\right)\right]^n = Ext\text{-}\exp[it(A+B)]. \quad (84)$$

**Theorem 7.3** Let $A$ and $B$ be the generators of contraction semigroups on non-Archimedean Banach space $B^\#$.Suppose that the #-closure of $(A + B) \upharpoonright D(A) \cap D(B)$ generates a contraction semigroup on $B^\#$. Then the following equality holds

$$\text{s-\#-}\lim_{n\to{}^*\infty}\left[\left(Ext\text{-}\exp\left(-\frac{tA}{n}\right)\right)\left(Ext\text{-}\exp\left(-\frac{tB}{n}\right)\right)\right]^n = Ext\text{-}\exp[-t(\#\text{-}\overline{A+B})]. \quad (85)$$

## FOCK SPACE OVER NONARCHIMEDEAN HILBERT SPACE

**Definition 8.1** Let $H^\#$ be a complex hyper infinite-dimensional non-Archimedean Hilbert space over field ${}^*\mathbb{C}_c^\#$ and denote by $H^{\#(n)}$ the $n$-fold tensor product: $H^{\#(n)} = Ext\text{-}\otimes_{k=1}^n H^\#$, $n \in {}^*\mathbb{N}$. Set $H^{\#(0)} = {}^*\mathbb{C}_c^\#$ and define $\mathcal{F}(H^\#) = Ext\text{-}\oplus_{n\in{}^*\mathbb{N}}\left(H^{\#(n)}\right)$. $\mathcal{F}(H^\#)$ is called the Fock space over non-Archimedean Hilbert space $H^\#$. Set $H^\# = L_2^\#({}^*\mathbb{R}_c^{\#3})$, then an element $\psi \in \mathcal{F}(H^\#)$ is a hyper infinite sequence of ${}^*\mathbb{C}_c^\#$-valued functions $\psi = \{\psi_0, \psi_1(x_1), \psi_2(x_1, x_2), \psi_2(x_1, x_2, x_3), \dots, \psi_n(x_1, \dots, x_n)\}$, $n \in {}^*\mathbb{N}$ and such that

$$\|\psi\|_\# = |\psi_0|^2 + Ext\text{-}\sum_{n\in{}^*\mathbb{N}}(Ext\text{-}\int|\psi_n(x_1, \dots, x_n)|^2 d^{\#3n}x) < {}^*\infty.$$

Actually, it is not $\mathcal{F}(H^\#)$ itself, but two of its subspaces which are used in quantum field theory. These two hyper infinite-dimensional subspaces are constructed as follows: Let $P_n$ be the permutation group on $n \in {}^*\mathbb{N}$ elements and let $\{\varphi_k\}_{k=1}^{*\infty}$be a basis for a space $H^\#$. For each $\sigma \in P_n$ we define an operator (which we also denote by $\sigma$) on basis elements of $H^{\#(n)}$ by $\sigma\left(Ext\text{-}\otimes_{i=1}^n \varphi_{k_i}\right) = Ext\text{-}\otimes_{i=1}^n \varphi_{k_{\sigma(i)}}$. The operator extends by linearity to a bounded operator (of #-norm one) on $H^\#$ and we can define $\breve{\mathbf{S}}_n^\# = \left(\frac{1}{n!}\right)\left(Ext\text{-}\sum_{\sigma\in P_n}\sigma\right)$. It is easily to show by definitions that $\breve{\mathbf{S}}_n^{\#2} = \breve{\mathbf{S}}_n^\#$ and $\breve{\mathbf{S}}_n^{\#*} = \breve{\mathbf{S}}_n^\#$ so $\breve{\mathbf{S}}_n^\#$ is an orthogonal projection. The range of $\breve{\mathbf{S}}_n^\#$ is called the $n$-fold symmetric tensor product of $H^\#$. We now define $\mathcal{F}_s^\#(H^\#) = Ext\text{-}\oplus_{n\in{}^*\mathbb{N}}\breve{\mathbf{S}}_n^\# H^{\#(n)}$. Non-Archimedean Hilbert space $\mathcal{F}_s^\#(H^\#)$ is called the symmetric Fock space over non-Archimedean Hilbert space $H^\#$ or the Boson Fock space over non-Archimedean Hilbert space $H^\#$.

## SEGAL QUANTIZATION OVER NONARCHIMEDEAN HILBERT SPACE

Let $H^\#$ be a complex non-Archimedean Hilbert space over field ${}^*\mathbb{C}_c^\#$ and let $\mathcal{F}(H^\#) = Ext\text{-}\oplus_{n\in{}^*\mathbb{N}}\left(H^{\#(n)}\right)$, where $H^{\#(n)} = Ext\text{-}\otimes_{k=1}^n H^\#$ be the Fock space over $H^\#$and let $\mathcal{F}_s(H^\#)$ be the Boson subspace of $\mathcal{F}(H^\#)$. Let $f \in H^\#$ be fixed. For vectors in $H^{\#(n)}$ of the form $\eta = Ext\text{-}\otimes_{i=1}^n \psi_i, n \in {}^*\mathbb{N}$ we define a map $b^-(f): H^{\#(n)} \to H^{\#(n-1)}$ by $b^-(f)\eta = (f, \psi_1)(Ext\text{-}\otimes_{i=2}^n \psi_i)$ and $b^-(f)$ extends by linearity to finite and hyperfinite linear combinations of such $\eta$, the extension is well defined, and $\|b^-(f)\eta\|_\# \leq \|f\|_\#\|\eta\|_\#$. Thus $b^-(f)$ extends to a bounded map (of #-norm $\|f\|_{\#}$) of $H^{\#(n)}$ into $H^{\#(n-1)}$. Since this holds for each $n \in {}^*\mathbb{N}$ (except for $n = 0$ in which case we define $b^-(f): H^{\#(0)} \to \{0\}$), $b^-(f)$ is a bounded operator of #-norm $\|f\|_{\#}$ from $\mathcal{F}(H^{\#})$ to $\mathcal{F}(H^{\#})$. It is easy to check that operator $b^+(f) = (b^-(f))^*$ takes each subspace $H^{\#(n)}$ into $H^{\#(n+1)}$ with the action $b^+(f)\eta = f \otimes Ext\text{-}\otimes_{i=1}^{n}\psi_i$ on product vectors. Note that the map $f \to b^+(f)$ is linear and the map $f \to b^-(f)$ is antilinear. Let $S_n$ be the symmetrization operators introduced in previous section and then the operator $\breve{\mathbf{S}}^{\#} = Ext\text{-}\oplus_{n \in {}^*\mathbb{N}}\breve{\mathbf{S}}_n^{\#}$ is the projection onto the symmetric Fock space $\mathcal{F}_s(H^{\#}) = Ext\text{-}\oplus_{n \in {}^*\mathbb{N}}\breve{\mathbf{S}}_n^{\#}H^{\#(n)}$, we will write $\breve{\mathbf{S}}_n^{\#}H^{\#(n)} = H_s^{\#(n)}$ and call $H_s^{\#(n)}$ the $n$- particle subspace of $\mathcal{F}_s(H^{\#})$. Note that operator $b^-(f)$ takes space $\mathcal{F}_s(H^{\#})$ into itself, but the operator $b^+(f)$ does not. A vector $\psi = \{\psi^{(n)}\}_{n=1}^{*\infty}$ with $\psi^{(n)} = 0$ for all except finite or hyperfinite set of number $n$ is called a finite or hyperfinite particle vector correspondingly. We will denote the set of hyperfinite particle vectors by $F_0$. The vector $\Omega_0 = \langle 1,0,0,\ldots\rangle$ is called the vacuum vector. Let $A$ be any self-adjoint operator on $H^{\#}$ with domain of essential self-#-adjointness $D = D(A)$. Let $D_A = \{\psi \in F_0 | \psi^{(n)} \in Ext\text{-}\otimes_{i=1}^{n}D, n \in {}^*\mathbb{N}\}$ and define operator $d\Gamma^{\#}(A)$ on $D_A \cap H_s^{\#(n)}$ by $d\Gamma^{\#}(A) = A \otimes I \cdots \otimes I + I \otimes A \otimes \cdots \otimes I + \cdots + \otimes I \cdots \otimes I \otimes A$. Note that $d\Gamma^{\#}(A)$ is essentially self-#-adjoint on $D_A$ . Operator $d\Gamma^{\#}(A)$ is called the second quantization of the operator $A$. For example, let $A = I$, then its second quantization $N^{\#} = d\Gamma^{\#}(I)$ is essentially self-#-adjoint on $F_0$ and for $\psi \in H_s^{\#(n)}, N^{\#}\psi = n\psi$. $N^{\#}$ is called the number operator. If $U$ is a unitary operator on space $H^{\#}$, we define $d\Gamma^{\#}(U)$ to be the unitary operator on $\mathcal{F}_s(H^{\#})$ which equals $Ext\text{-}\otimes_{i=1}^{n}U$ when restricted to $H_s^{\#(n)}$ for $n > 0$, and which equals the identity on $H_s^{\#(0)}$. If $Ext\text{-}\exp(itA)$ is a #-continuous unitary group on $H^{\#}$, then $\Gamma^{\#}(Ext\text{-}\exp(itA))$ is the group generated by $d\Gamma^{\#}(A)$, i.e., that expressed by the formula $\Gamma^{\#}(Ext\text{-}\exp(itA)) = Ext\text{-}\exp(itd\Gamma^{\#}(A))$.

**Definition 9.1** We define the annihilation operator $a^-(f)$ on $\mathcal{F}_s(H^{\#})$ with domain $F_0$ by the formula

$$a^-(f) = \sqrt{N+1}\,b^-(f). \tag{86}$$

Operator $a^-(f)$ is called an annihilation operator because it takes each $(n+1)$-particle subspace into the $n$-particle subspace. For each $\psi$ and $\eta$ in $F_0$, $(\sqrt{N+1}\,b^-(f)\psi, \eta) = (\psi, S^{\#}b^+(f)\sqrt{N+1})$, then we get

$$(a^-(f))^* \upharpoonright F_0 = S^{\#}b^+(f)\sqrt{N+1}\,. \tag{87}$$

The operator $(a^-(f))^*$ is called a creation operator. Both $a^-(f)$ and $(a^-(f))^*$ #-closable; we denote their #-closures by $a^-(f)$ and $(a^-(f))^*$ also. The equation (1) implies that the Segal field operator $\Phi_S^{\#}(f)$ on $F_0$ defined by $\Phi_S^{\#}(f) = \frac{1}{\sqrt{2}}[a^-(f) + (a^-(f))^*]$ is symmetric and essentially self-#-adjoint. The mapping from $H^{\#}$ to the self-#-adjoint operators on $\mathcal{F}_s(H^{\#})$ given by $f \to \Phi_S^{\#}(f)$ is called the Segal quantization over $H^{\#}$. Note that the Segal quantization is a real linear map.

**Theorem 9.1** Let $H^{\#}$ be hyper infinite dimensional Hilbert space over complex field ${}^*\mathbb{C}_c^{\#} = {}^*\mathbb{R}_c^{\#} + \mathrm{i}{}^*\mathbb{R}_c^{\#}$ and $\Phi_S^{\#}(f)$ the corresponding Segal quantization. Then:

(a) (self-#-adjointness) for each $f \in H^{\#}$ the operator $\Phi_S^{\#}(f)$ is essentially self-#-adjoint on $F_0$, the hyperfinite particle vectors;

(b) (cyclicity of the vacuum) the vector $\Omega_0$ is in the domain of all hyperfinite products $Ext\text{-}\prod_{i=1}^{n}\Phi_S^{\#}(f_i), n \in {}^*\mathbb{N}$ and the set $\{Ext\text{-}\prod_{i=1}^{n}\Phi_S^{\#}(f_i) \,|\, f_i \in H^{\#}, n \in {}^*\mathbb{N}\}$ is #-total in $\mathcal{F}_s(H^{\#})$;

(c) (commutation relations) for each $\psi \in F_0$ and $f, g \in H^{\#}$: $[\Phi_S^{\#}(f)\Phi_S^{\#}(g) - \Phi_S^{\#}(g)\Phi_S^{\#}(f)]\psi = i\mathrm{Im}(f,g)_{H^{\#}}\psi$; $(c')$ (generalized commutation relations) assuming that $(f,g)_{H^{\#}} \approx 0$ and $\psi \in F$ is a near standard vector we get $[\Phi_S^{\#}(f)\Phi_S^{\#}(g) - \Phi_S^{\#}(g)\Phi_S^{\#}(f)]\psi \approx 0$ and therefore $\mathrm{st}([\Phi_S^{\#}(f)\Phi_S^{\#}(g) - \Phi_S^{\#}(g)\Phi_S^{\#}(f)]\psi) = 0$;

(d) let $W(f)$ denotes the external unitary operator $Ext\text{-}\exp(i\Phi_S^{\#}(f))$ then $W(f+g) = \left[Ext\text{-}\exp\left(-\frac{i}{2}\mathrm{Im}(f,g)_{H^{\#}}\right)\right]W(f)W(g)$;

(e) (#-continuity) if $\{f_n\}_{n=1}^{*\infty}$ is hyper infinite sequence such as $\#\text{-}\lim_{n\to{}^*\infty} f_n = f$ in $H^{\#}$ then:

1) $\#\text{-}\lim_{n\to{}^*\infty} W(f_n)\psi$ exists for all $\psi \in \mathcal{F}_s(H^{\#})$ and $\#\text{-}\lim_{n\to{}^*\infty} W(f_n)\psi = W(f)\psi$

2) $\#\text{-}\lim_{n\to{}^*\infty} \Phi_S^{\#}(f_n)\psi$ exists for all $\psi \in F_0$ and $\#\text{-}\lim_{n\to{}^*\infty} \Phi_S^{\#}(f_n)\psi = \Phi_S^{\#}(f)\psi$

(e) For every unitary operator $U$ on $H^{\#}, \Gamma^{\#}(U): D(\#\text{-}\overline{\Phi_S^{\#}(f)}) \to D(\#\text{-}\overline{\Phi_S^{\#}(Uf)})$ and for all $\psi \in D(\#\text{-}\overline{\Phi_S^{\#}(Uf)})$, $\Gamma^{\#}(U)(\#\text{-}\overline{\Phi_S^{\#}(f)})\Gamma^{\#-1}(U)\psi = \#\text{-}\overline{\Phi_S^{\#}(Uf)}\psi$ for all $\psi \in F_0$ and $f \in H^{\#}$.

**Remark 9.1** Henceforth we use $\Phi_S^{\#}(f)$ to denote the #-closure $\#\text{-}\overline{\Phi_S^{\#}(f)}$ of $\Phi_S^{\#}(f)$.

**Definition 9.2** For each $m > 0, m \in \mathbb{R}$ let $H_m^{\#} = \{p \in {}^*\mathbb{R}_c^{\#4} | p \cdot \tilde{p} = m^2, p_0 > 0\}$, where $\tilde{p} = (p^0, -p^1, -p^2, -p^3)$, the sets $H_m^{\#}$, are called mass hyperboloids, are invariant under canonical Lorentz group${}^{\sigma}L_{+}^{\uparrow}$. Let $j_m$ be the #-homeomorphism of $H_m^{\#}$ onto ${}^{*}\mathbb{R}_c^{\#3}$ given by $j_m\colon \langle p_0,p_1,p_2,p_3\rangle \to \langle p_1,p_2,p_3\rangle = \boldsymbol{p}$. Define a #-measure $\Omega_m^{\#}$ on $H_m^{\#}$ for any #-measurable set $E \subset H_m^{\#}$by

$$\Omega_m^{\#}(E) = Ext\text{-}\int_{j_m(E)} \frac{d^{\#3}\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}\,. \qquad (88).$$

**Theorem 9.2** Let $\mu^{\#}$ be a polynomially bounded #-measure with support in

$$\#\text{-}\bar{V}_{+}$$

. If $\mu^{\#}$ is ${}^{\sigma}L_{+}^{\uparrow} = L_{+}^{\uparrow}$- invariant, there exists a polynomially bounded #-measure $\rho^{\#}$ on $[0,\infty^{\#})$ and a constant $c$ so that for any $f \in S^{\#}({}^{*}\mathbb{R}_c^{\#4})$

$$Ext\text{-}\int_{{}^{*}\mathbb{R}_c^{\#4}} f\, d^{\#}\mu^{\#} = cf(0) + Ext\text{-}\int_0^{{}^{*}\infty} d^{\#}\rho^{\#}(m)\left(Ext\text{-}\int_{{}^{*}\mathbb{R}_c^{\#3}} \frac{f\left(\sqrt{|\boldsymbol{p}|^2+m^2},p_1,p_2,p_3\right)d^{\#3}\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}\right). \qquad (89)$$

**Definition 9.3** Let $\boldsymbol{\mathcal{F}}(f)$ be a linear #-continuous functional $\mathcal{F}\colon S_{\text{fin}}^{\#}({}^{*}\mathbb{R}_c^{\#4}) \to {}^{*}\mathbb{R}_c^{\#}$. Functional $\mathcal{F}$ is $L_{+}^{\uparrow}$- $\approx$ -invariant if for any $\Lambda \in L_{+}^{\uparrow}$ the following property holds $\mathcal{F}\big(f(\Lambda \mathrm{x})\big) \approx \mathcal{F}(f)$ for all $f \in S_{\text{fin}}^{\#}({}^{*}\mathbb{R}_c^{\#4})$.

**Theorem 9.3** Let $\mu^{\#}$ be a polynomially bounded $L_{+}^{\uparrow}$- invariant #-measure with support in $\#\text{-}\bar{V}_{+}$. Let $\mathcal{F}(f)$ be a linear #-continuous functional $\mathcal{F}\colon S_{\text{fin}}^{\#}({}^{*}\mathbb{R}_c^{\#4}) \to {}^{*}\mathbb{R}_{c,\text{fin}}^{\#}$ defined by $Ext\text{-}\int_{{}^{*}\mathbb{R}_c^{\#4}} f\, d^{\#}\mu^{\#}$ and there exists a polynomially bounded #-measure $\rho^{\#}$ on $[0,\infty^{\#})$ such that $\int_0^{{}^{*}\infty} d^{\#}\rho^{\#}(m) \in {}^{*}\mathbb{R}_{c,\text{fin}}^{\#}$ and a constant $c \in {}^{*}\mathbb{R}_{c,\text{fin}}^{\#}$ so that (1) holds. Then for any $f \in S_{\text{fin}}^{\#}({}^{*}\mathbb{R}_c^{\#4})$ and for any $\varkappa \in {}^{*}\mathbb{R}_{c,\infty}^{\#}$ the following property holds

$$\mathcal{F}(f) \approx cf(0) + Ext\text{-}\int_0^{{}^{*}\infty} d^{\#}\rho^{\#}(m)\left(Ext\text{-}\int_{|p|\leq\varkappa} \frac{f\left(\sqrt{|\boldsymbol{p}|^2+m^2},p_1,p_2,p_3\right)d^{\#3}\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}\right). \qquad (90)$$

**Definition 9.4** Let $\chi(\varkappa,\boldsymbol{p})$ be a function such that: $\chi(\varkappa,\boldsymbol{p}) \equiv 1$ if $|\boldsymbol{p}| \leq \varkappa, \chi(\varkappa,\boldsymbol{p}) \equiv 0$ if $|\boldsymbol{p}| > \varkappa$. Define a #-measure $\Omega_{m,\varkappa}^{\#}$ on $H_m^{\#}$ by

$$\Omega_{m,\varkappa}^{\#}(E) = Ext\text{-}\int_{j_m(E)} \frac{\chi(\varkappa,\boldsymbol{p})d^{\#3}\boldsymbol{p}}{\sqrt{|\boldsymbol{p}|^2+m^2}}. \qquad (91)$$

We use the Segal quantization to define the free Hermitian scalar field of mass $m$. We take $H^{\#} = L_2^{\#}\big(H_m^{\#}, d^{\#}\Omega_{m,\varkappa}^{\#}\big)$. For each $f \in S_{\text{fin}}^{\#}({}^{*}\mathbb{R}_c^{\#4})$ we define $Ef \in H^{\#}$ by $Ef = 2\pi\big(Ext\text{-}\hat{f}\big) \upharpoonright H_m^{\#}$ where the Fourier transform is defined in terms of the Lorentz invariant inner product $p \cdot \tilde{x}$: $Ext\text{-}\hat{f} = \frac{1}{4\pi^2}\Big(Ext\text{-}\int_{{}^{*}\mathbb{R}_c^{\#4}} Ext\text{-}\exp\left[i(p \cdot \tilde{x})\right]d^{\#4}x\Big)$. If $\Phi_{S,\varkappa}^{\#}(\cdot)$ is the Segal quantization over $L_2^{\#}\big(H_m^{\#}, d^{\#}\Omega_{m,\varkappa}^{\#}\big)$, we define for each ${}^{*}\mathbb{R}_c^{\#}$- valued $f \in S^{\#}({}^{*}\mathbb{R}_c^{\#4})$: $\Phi_{m,\varkappa}^{\#}(f) = \Phi_{S,\varkappa}^{\#}(Ef)$ and for each ${}^{*}\mathbb{C}_c^{\#}$- valued $f \in S^{\#}({}^{*}\mathbb{R}_c^{\#4})$ we define $\Phi_{m,\varkappa}^{\#}(f) = \Phi_{m,\varkappa}^{\#}(\mathrm{Re}f) + \mathrm{i}\Phi_{m,\varkappa}^{\#}(\mathrm{Im}f)$.

**Definition 9.5** The mapping $f \to \Phi_{m,\varkappa}^{\#}(f)$ is called the free non-Archimedean Hermitian scalar field of mass $m$.

**Definition 9.6** On $L_2^{\#}\big(H_m^{\#}, d^{\#}\Omega_{m,\varkappa}^{\#}\big)$ we define the following unitary representation of the restricted Poincare group $L_{+}^{\uparrow}$: $(U_m(a,\Lambda)\psi)(p) = (Ext\text{-}\exp[i(p \cdot \tilde{a})])\psi(\Lambda^{-1}p)$ where we are using $\Lambda$ to denote both an element of the abstract restricted Lorentz group and the corresponding element in the standard representation on ${}^{\sigma}\mathbb{R}^4$.

**Remark 9.2** Note that by Theorem 9.1(e) for all $\psi \in F_0$ and $f \in L_2^{\#}\big(H_m^{\#}, d^{\#}\Omega_{m,\varkappa}^{\#}\big)$ we get

$$\Gamma^{\#}\big(U_m(a,\Lambda)\big)\big(\#\text{-}\overline{\Phi_{m,\varkappa}^{\#}(f)}\big)\Gamma^{\#-1}\big(U_m(a,\Lambda)\big)\psi = \Gamma^{\#}\big(U_m(a,\Lambda)\big)\big(\#\text{-}\overline{\Phi_S^{\#}(Ef)}\big)\Gamma^{\#-1}\big(U_m(a,\Lambda)\big)\psi =$$
$$\#\text{-}\overline{\Phi_S^{\#}(U_m(a,\Lambda)Ef)}\psi.$$

A change of variables for all $f \in S_{\text{fin}}^{\#}({}^{*}\mathbb{R}_c^{\#4})$ gives that

$$U_m(a,\Lambda)Ef \approx EU_m(a,\Lambda)f.$$

Therefore for all $\psi \in D_{S_{\text{fin}}^{\#}} \subset F_0$ such that $\|\psi\|_{\#} \in {}^{*}\mathbb{R}_{c,\text{fin}}^{\#}$ and for ${}^{*}\mathbb{R}_{c,\text{fin}}^{\#}$-valued function $f$ such that $f \in S_{\text{fin}}^{\#}({}^{*}\mathbb{R}_c^{\#4})$ we obtain that

$$\Gamma^{\#}\big(U_m(a,\Lambda)\big)\left(\#\text{-}\Phi_{m,\varkappa}^{\#}(f)\right)\Gamma^{\#-1}\big(U_m(a,\Lambda)\big)\psi \approx \#\text{-}\Phi_{m,\varkappa}^{\#}(U_m(a,\Lambda)f)\psi.$$

**Definition 9.7** The #-conjugation on a non-Archimedean Hilbert space $H^{\#}$ is an antilinear #-isometry $\mathbf{C}^{\#}$ so that the following equality holds $\mathbf{C}^{\#2} = I$.

**Definition 9.8** Let $H^{\#}$ be a non-Archimedean Hilbert space over field ${}^{*}\mathbb{C}_c^{\#}$, $\Phi_S^{\#}(\cdot)$ the associated Segal quantization. Let $H_{\mathbf{C}^{\#}}^{\#} = \{f|\mathbf{C}^{\#}f = f\}$. For each $f \in H_{\mathbf{C}^{\#}}^{\#}$ we define $\varphi^{\#}(f) = \Phi_S^{\#}(f)$ and $\pi^{\#}(f) = \Phi_S^{\#}(if)$, the map $f \to \varphi^{\#}(f)$ is called the canonical free field over the doublet $\langle \boldsymbol{H}^{\#}, \mathbf{C}^{\#}\rangle$ and the map $f \to \pi^{\#}(f)$ is called the canonical conjugate momentum.

**Theorem 9.4** Let $H^{\#}$ be a non-Archimedean Hilbert space over field ${}^{*}\mathbb{C}_{c}^{\#}$ with #-conjugation $\mathbf{C}^{\#}$. Let $\varphi^{\#}(\cdot)$ and $\pi^{\#}(\cdot)$ be the corresponding canonical fields. Then: (a) For each $f \in H_{\mathbf{C}^{\#}}^{\#}$, $\varphi^{\#}(f)$ is essentially self-#-adjoint on $F_0$. (b) $\{\varphi^{\#}(f)|f \in H_{\mathbf{C}^{\#}}^{\#}\}$ is a commuting family of self-#-adjoint operators. (c) $\Omega_0$ is a #-cyclic vector for the family $\{\varphi^{\#}(f)|f \in H_{\mathbf{C}^{\#}}^{\#}\}$. (d) If $\{f_n\}_{n=1}^{*\infty}$ is hyper infinite sequence such as $\#\text{-}\lim_{n\to{}^*\infty} f_n = f$ in $H_{\mathbf{C}^{\#}}^{\#}$, then $\#\text{-}\lim_{n\to{}^*\infty} \varphi^{\#}(f_n)\psi$ exists for all $\psi \in F_0$ and $\#\text{-}\lim_{n\to{}^*\infty} \varphi^{\#}(f_n)\psi = \varphi^{\#}(f)\psi$. (e) $\#\text{-}\lim_{n\to{}^*\infty}(Ext\text{-}\exp[i\varphi^{\#}(f_n)]\psi) = Ext\text{-}\exp[i\varphi^{\#}(f)]\psi$ for all $\psi \in \mathcal{F}_s(H^{\#})$. (f) Properties (a)-(e) hold with $\varphi^{\#}(f)$ replaced by $\pi^{\#}(f)$. (g) If $f,g \in H_{\mathbf{C}^{\#}}^{\#}$, then $[\varphi^{\#}(f)\varphi^{\#}(g) - \varphi^{\#}(g)\varphi^{\#}(f)]\psi = i(f,g)$ for all $\psi \in \mathcal{F}_s(H^{\#})$ and $(Ext\text{-}\exp[i\varphi^{\#}(f)])(Ext\text{-}\exp[i\pi^{\#}(f)]) = (Ext\text{-}\exp[i(f,g)])(Ext\text{-}\exp[i\pi^{\#}(f)])(Ext\text{-}\exp[i\varphi^{\#}(f)])$.

**Definition 9.9** We write now $f \in L_2^{\#}(H_m^{\#}, d^{\#}\Omega_{m,\varkappa}^{\#})$ as $f(p_0,\boldsymbol{p})$ and define the #-conjugation $\mathbf{C}^{\#}$ by $\mathbf{C}^{\#}(f)(p_0,\boldsymbol{p}) = \overline{f(p_0,-\boldsymbol{p})}$. Note that $\mathbf{C}^{\#}$ is well-defined on $f \in L_2^{\#}(H_m^{\#}, d^{\#}\Omega_{m,\varkappa}^{\#})$ since $\langle p_0,-\boldsymbol{p}\rangle \in H_m^{\#}$ if and only if $\langle p_0,\boldsymbol{p}\rangle \in H_m^{\#}$.

**Definition 9.10** We denote the canonical fields corresponding to $\mathbf{C}^{\#}$ by $\varphi^{\#}(\cdot)$ and $\pi^{\#}(\cdot)$ and define $\varphi_{m,\varkappa}^{\#}(f) = \varphi^{\#}(Ef)$ and $\pi_{m,\varkappa}^{\#}(f) = \pi^{\#}(\mu(\boldsymbol{p})Ef), \mu(\boldsymbol{p}) = \sqrt{\boldsymbol{p}^2 + m^2}$ for ${}^{*}\mathbb{R}_{c}^{\#}$- valued $f \in L_2^{\#}({}^{*}\mathbb{R}_{c}^{\#4})$, extending to all of $L_2^{\#}({}^{*}\mathbb{R}_{c}^{\#4})$ by linearity. We let now $D_{S_{\text{fin}}^{\#}} = \{\psi|\psi \in F_0, \psi^{(n)} \in S_{\text{fin}}^{\#}({}^{*}\mathbb{R}_{c}^{\#3n})\}$ and for each $p \in {}^{*}\mathbb{R}_{c}^{\#3}$ we define the operator $a(p)$ on $\mathcal{F}_s\left(L_2^{\#}({}^{*}\mathbb{R}_{c}^{\#3})\right)$ with domain $D_{S_{\text{fin}}^{\#}}$ by $(a(p)\psi)^{(n)} = \sqrt{n+1}\,\psi^{(n+1)}(p,k_1,\ldots k_n)$ and therefore the formal #-adjoint of the operator $a(p)$ reads $(a^{\dagger}(p)\psi)^{(n)} = \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\delta^{(3)}(p-k_l)\psi^{(n-1)}(k_1,\ldots,k_{l-1},k_{l+1},\ldots,k_n)$. Note that the formulas

$$a(g) = Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#3}} a(p)g(-p)d^{\#3}p, \tag{92}$$

$$a^{\dagger}(g) = Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#3}} a^{\dagger}(p)g(p)d^{\#3}p \tag{93}$$

hold for all $g \in S_{\text{fin}}^{\#}({}^{*}\mathbb{R}_{c}^{\#3})$ if the equalities (92)-(93) are understood in the sense of quadratic forms. That is, (92) means that for $\psi_1,\psi_2 \in D_{S_{\text{fin}}^{\#}}$ : $(\psi_1,a(g)\psi_2) = Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#3}}(\psi_1,a(p)\psi_2)g(-p)d^{\#3}p$ and similarly (93) means that for $\psi_1,\psi_2 \in D_{S_{\text{fin}}^{\#}}$ : $(\psi_1,a(g)\psi_2) = Ext\text{-}\int_{{}^{*}\mathbb{R}_{c}^{\#3}}(\psi_1,a^{\dagger}(p)\psi_2)g(p)d^{\#3}p$. The particles number operator reads

$$N_{0,\varkappa} = Ext\text{-}\int_{|p|\le\varkappa} a^{\dagger}(p)a(p)\,d^{\#3}p. \tag{94}$$

The generator of time translations in the free scalar field theory of mass $m$ is given by

$$H_{0,\varkappa} = Ext\text{-}\int_{|p|\le\varkappa} \mu(p)a^{\dagger}(p)a(p)\,d^{\#3}p. \tag{95}$$

We express the free scalar field and the time zero fields in terms of $a^{\dagger}(p)$ and $a(p)$ as quadratic forms on $D_{S_{\text{fin}}^{\#}} \times D_{S_{\text{fin}}^{\#}}$ by

$$\Phi_{0,m,\varkappa}^{\#}(x,t) = (2\pi)^{-3/2}Ext\text{-}\int_{|p|\le\varkappa}\{(Ext\text{-}\exp(\mu(p)t - ipx))a^{\dagger}(p) + (Ext\text{-}\exp(\mu(p)t + ipx))a(p)\}\frac{d^{\#3}p}{\sqrt{2\mu(p)}}, \tag{96}$$

$$\Phi_{0,m,\varkappa}^{\#}(x) = (2\pi)^{-3/2}Ext\text{-}\int_{|p|\le\varkappa}\{(Ext\text{-}\exp(-ipx))a^{\dagger}(p) + (Ext\text{-}\exp(ipx))a(p)\}\frac{d^{\#3}p}{\sqrt{2\mu(p)}}, \tag{97}$$

$$\pi_{0,m,\varkappa}^{\#}(x) = (2\pi)^{-3/2}Ext\text{-}\int_{|p|\le\varkappa}\{(Ext\text{-}\exp(-ipx))a^{\dagger}(p) + (Ext\text{-}\exp(ipx))a(p)\}\frac{d^{\#3}p}{\sqrt{\mu(p)/2}}. \tag{98}$$

**Abbreviation 9.1** We shall write for the sake of brevity through this paper $\Phi_{0,\varkappa}^{\#}(x,t)$, $\Phi_{0,\varkappa}^{\#}(x)$ and $\pi_{0,\varkappa}^{\#}(x)$ instead $\Phi_{0,m,\varkappa}^{\#}(x,t)$, $\Phi_{0,m,\varkappa}^{\#}(x)$ and $\pi_{0,m,\varkappa}^{\#}(x)$ correspondingly.

**Theorem 9.5** Let $n_1,n_2 \in \mathbb{N}$ and suppose that $W(k_1,\ldots k_{n_1},p_1,\ldots,p_{n_2}) \in L_2^{\#}\left({}^{*}\mathbb{R}_{c}^{\#3(n_1+n_2)}\right)$ where $W(k_1,\ldots k_{n_1},p_1,\ldots,p_{n_2})$ is a ${}^{*}\mathbb{C}_{c,\text{fin}}^{\#}$-valued function on ${}^{*}\mathbb{R}_{c}^{\#3(n_1+n_2)}$. Then there is a unique operator $T_W$ on $\mathcal{F}_s\left(L_2^{\#}({}^{*}\mathbb{R}_{c}^{\#3})\right)$ so that $D_{S_{\text{fin}}^{\#}} \subset D(T_W)$ is a #- core for $T_W$.

1) As ${}^{*}\mathbb{C}_{c}^{\#}$-valued quadratic forms on $D_{S_{\text{fin}}^{\#}} \times D_{S_{\text{fin}}^{\#}}$

$$T_W = Ext\text{-}\int_{{}^{*}\mathbb{R}^{3(n_1+n_2)}} W(k_1,\ldots k_{n_1},p_1,\ldots,p_{n_2})\left(\prod_{i=1}^{n_1}a^{\dagger}(k_i,\varepsilon)\right)\left(\prod_{i=1}^{n_2}a(p_i,\varepsilon)\right)d^{\#3n_1}k\,d^{\#3n_2}p.$$

2) As ${}^{*}\mathbb{C}_{c}^{\#}$-valued quadratic forms on $D_{S_{\text{fin}}^{\#}} \times D_{S_{\text{fin}}^{\#}}$

$$T_W^{*} = Ext\text{-}\int_{{}^{*}\mathbb{R}^{3(n_1+n_2)}} \overline{W(k_1,\ldots k_{n_1},p_1,\ldots,p_{n_2})}\left(\prod_{i=1}^{n_1}a^{\dagger}(k_i,\varepsilon)\right)\left(\prod_{i=1}^{n_2}a(p_i,\varepsilon)\right)d^{\#3n_1}k\,d^{\#3n_2}p.$$

3) If $m_1$ and $m_2$ are nonnegative integers so that $m_1+m_2=n_1+n_2$, then

$$(1+N^{\#})^{-m_1/2}T_W(1+N^{\#})^{-m_2/2}\leq C(m_1,m_2)\|W\|_{L_2^{\#}}.$$

4) On vectors in $F_0$ the operators $T_W$ and $T_W^{*}$ are given by the explicit formulas

$$\left(T_W(\psi)\right)^{l-n_2+n_1}=$$

$$K(l,n_1,n_2)\check{\mathbf{S}}\left[Ext\text{-}\int_{|p_1|\leq\varkappa}\dots Ext\text{-}\int_{|p_{n_2}|\leq\varkappa}W(k_1,\dots k_{n_1},p_1,\dots,p_{n_2})\,\psi^{(l)}(p_1,\dots,p_{n_2},k_1,\dots k_{n_1})d^{\#3n_2}p\right],$$

$$\left(T_W(\psi)\right)^{n}=0\text{ if }n<n_1-n_2,$$

$$\left(T_W^{*}(\psi)\right)^{l-n_1+n_2}=$$

$$K(l,n_2,n_1)\check{\mathbf{S}}\left[Ext\text{-}\int_{|p_1|\leq\varkappa}\dots\ Ext\text{-}\int_{|p_{n_2}|\leq\varkappa}\overline{W(k_1,\dots k_{n_1},p_1,\dots,p_{n_2})}\,\psi^{(l)}(p_1,\dots,p_{n_2},k_1,\dots k_{n_1})d^{\#3n_1}k\right]$$

$\left(T_W^{*}(\psi)\right)^{n}=0$ if and only if $n<n_2-n_1$. Here $\check{\mathbf{S}}$ is the symmetrization operator.

## $\mathbf{Q^{\#}}$-SPACE REPRESENTATION OF THE FOCK SPACE STRUCTURES

In this section the construction of a non-Archimedean $Q^{\#}$-space and $L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})$ , another representation of the Fock space structures are presented. In analogy with the one degree of freedom case where $\mathcal{F}^{\#}({}^{*}\mathbb{R}_c^{\#})$ is isomorphic to $L_2^{\#}({}^{*}\mathbb{R}_c^{\#},d^{\#}x)$ in such a way that $\Phi_S^{\#}(1)$ becomes multiplication by $x$, we will construct a $\sigma^{\#}$-measure space $\langle Q^{\#},\mu^{\#}\rangle$, with $\mu^{\#}(Q^{\#})=1$, and a unitary map $\mathrm{S}^{\#}:\mathcal{F}_s^{\#}(H^{\#})\to L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})$ so that for each $f\in H_C^{\#}$, $\mathrm{S}^{\#}\phi_{\varkappa}^{\#}(f)$ $\mathrm{S}^{\#-1}$ acts on $L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})$ by multiplication by a $\mu^{\#}$-measurable function. We can then show that in the case of the free scalar field of mass $m$ in 4-dimensional space-time $M_4^{\#}$, $V=\mathrm{S}^{\#}H_{I,\varkappa}^{\#}(g)\mathrm{S}^{\#-1}$ is just multiplication by a function $V(q)$ which is in $L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})$ for each $p\in{}^{*}\mathbb{N}$. Let $\{g_n\}_{n=1}^{{}^{*}\infty}$ be an orthonormal basis for $H^{\#}$ so that each $g\in H_C^{\#}$ and let $\{g_n\}_{n=1}^{N}, N\in{}^{*}\mathbb{N}$ be a finite or hyperfinite subcollection of the set $\{f_n\}_{n=1}^{{}^{*}\infty}$ .Let $P_N$ be a set of the all external finite and hyperfinite polynomials $Ext\text{-}P[u_1,\dots,u_N]$ and $\mathcal{F}_N^{\#}$ be the #-closure of the set $\{Ext\text{-}P[\varphi_{\varkappa}^{\#}(g_1),\dots,\varphi_{\varkappa}^{\#}(g_N)]|P\in P_N\}$ in $\mathcal{F}_s^{\#}(H^{\#})$ and define a set $F_0^N=\mathcal{F}_N^{\#}\cap F_0$. From Theorem 55 it follows that $\varphi_{\varkappa}^{\#}(g_k)$ and $\pi_{\varkappa}^{\#}(g_k)$, for all $1\leq k,l\leq N$ are essentially self-#-adjoint on $F_0^N$ and that

$$(Ext\text{-}\exp[it\varphi_{\varkappa}^{\#}(g_k)])(Ext\text{-}\exp[it\pi_{\varkappa}^{\#}(g_l)])=$$

$$\left(Ext\text{-}\exp\left[-ist\delta_{kl}\right]\right)(Ext\text{-}\exp[it\pi_{\varkappa}^{\#}(g_l)])(Ext\text{-}\exp[it\varphi_{\varkappa}^{\#}(g_k)])\ .$$

Therefore we have a representation of the generalized Weyl relations in which the vector $\Omega_0$ satisfies the equality $([\varphi_{\varkappa}^{\#}(g_k)]^2+[\pi_{\varkappa}^{\#}(g_l)]^2-1)\Omega_0=0$ and is cyclic for the operators $\{\varphi_{\varkappa}^{\#}(g_k)\}_{k=1}^{N}$. Therefore there is a unitary map $\mathrm{S}^{\#(N)}:\mathcal{F}_N^{\#}\to L_2^{\#}({}^{*}\mathbb{R}_c^{\#N})$ such that: 1) $\mathrm{S}^{\#(N)}\varphi_{\varkappa}^{\#}(g_k)\left(\mathrm{S}^{\#(N)}\right)^{-1}=x_k$, 2) $\mathrm{S}^{\#(N)}\pi_{\varkappa}^{\#}(g_k)\left(\mathrm{S}^{\#(N)}\right)^{-1}=-\frac{1}{i}\frac{d^{\#}}{d^{\#}x_k}$ and 3) $\mathrm{S}^{\#(N)}\Omega_0=\pi^{-N/4}\left[Ext\text{-}\exp\left(-Ext\text{-}\sum_{k=1}^{N}\frac{x_k^2}{2}\right)\right]$. It is convenient to use the non-Archimedean Hilbert space $L_2^{\#}\left({}^{*}\mathbb{R}_c^{\#N},\pi^{-N/4}\left(Ext\text{-}\exp\left(-Ext\text{-}\sum_{k=1}^{N}\frac{x_k^2}{2}\right)\right)\right)d^{\#N}x$ instead of $L_2^{\#}({}^{*}\mathbb{R}_c^{\#N})$ so we let $d^{\#}\mu_k^{\#}=Ext\text{-}\exp\left(-\frac{x_k^2}{2}\right)d^{\#}x_k$ and define the operator $(Tf)(x)=\pi^{N/4}\left(Ext\text{-}\exp\left(Ext\text{-}\sum_{k=1}^{N}\frac{x_k^2}{2}\right)\right)$, Then $T$ is a unitary map of $L_2^{\#}({}^{*}\mathbb{R}_c^{\#N})$ onto $L_2^{\#}\left({}^{*}\mathbb{R}_c^{\#N},Ext\text{-}\prod_{k=1}^{N}d^{\#}\mu_k^{\#}\right)$ and if we let $\mathrm{S}_1^{\#(N)}=T\mathrm{S}^{\#(N)}$ we get: 1) $\mathrm{S}_1^{\#(N)}:\mathcal{F}_N^{\#}\to L_2^{\#}\left({}^{*}\mathbb{R}_c^{\#N},Ext\text{-}\prod_{k=1}^{N}d^{\#}\mu_k^{\#}\right)$, 2) $\mathrm{S}_1^{\#(N)}\varphi_{\varkappa}^{\#}(g_k)\left(\mathrm{S}_1^{\#(N)}\right)^{-1}=x_k$, 3) $\mathrm{S}_1^{\#(N)}\pi_{\varkappa}^{\#}(g_k)\left(\mathrm{S}_1^{\#(N)}\right)^{-1}=-\frac{x_k}{i}+\frac{1}{i}\frac{d^{\#}}{d^{\#}x_k}$ and 4) $\mathrm{S}_1^{\#(N)}\Omega_0=1$, where 1 is the function identically one. Note that each #- measure $\mu_k^{\#}$ has mass one, which implies that

$$\langle\Omega_0,\left(Ext\text{-}\prod_{k=1}^{N}P_k\left(\varphi_{\varkappa}^{\#}(g_k)\right)\right)\Omega_0\rangle=\int_{{}^{*}\mathbb{R}_c^{\#N}}(Ext\text{-}\prod_{k=1}^{N}P_k(x_k))\left(Ext\text{-}\prod_{k=1}^{N}d^{\#}\mu_k^{\#}\right)= \quad (99)$$

$$=Ext\text{-}\prod_{k=1}^{N}\int_{{}^{*}\mathbb{R}_c^{\#N}}P_k(x_k)\,d^{\#}\mu_k^{\#}=Ext\text{-}\prod_{k=1}^{N}\int_{{}^{*}\mathbb{R}_c^{\#N}}\langle\Omega_0,P_k(\varphi_{\varkappa}^{\#}(g_k)\Omega_0)\rangle.$$

Here $P_1,\therefore,P_N$ are external finite and hyperfinite polynomials. Now we can to construct directly the $\sigma^{\#}$-measure space $\langle Q^{\#},\mu^{\#}\rangle$. We define a space $Q^{\#}=\times_{k=1}^{{}^{*}\infty}{}^{*}\mathbb{R}_c^{\#}$. Take the $\sigma^{\#}$-algebra generated by hyper infinite products of #-measurable sets in ${}^{*}\mathbb{R}_c^{\#}$ and set $\mu^{\#}=\bigotimes_{k=1}^{{}^{*}\infty}\mu_k^{\#}$. We denote the points of $Q^{\#}$ symbolically by $q=\langle q_1,q_2,\dots\rangle$, then $\langle Q^{\#},\mu^{\#}\rangle$ is a $\sigma^{\#}$- measure space and the set of functions of the form $P(q_1,q_2,\dots)$, where $P$ is a polynomial and

$n\in {}^*\mathbb{N}$ is arbitrary, is #-dense in $L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})$. Let $P$ be a polynomial in $N\in {}^*\mathbb{N}$ variables $P(x_1,x_2,\dots,x_N)=Ext\text{-}\sum_{l_1,\dots,l_N} c_{l_1,\dots,l_N} x_{k_1}^{l_1}\cdots x_{k_N}^{l_N}$ and define $\mathbf{S}^{\#}\colon P\left(\varphi_{\varkappa}^{\#}(g_{k_1}),\dots,\varphi_{\varkappa}^{\#}(g_{k_N})\right)\Omega_0\to P(q_{k_1},q_{k_2},\dots,q_{k_N})$. Then we get

$$\left(\varphi_{\varkappa}^{\#}(g_{k_1}),\dots,\varphi_{\varkappa}^{\#}(g_{k_N})\right)\Omega_0=Ext\text{-}\sum\nolimits_{\boldsymbol{l},\boldsymbol{m}} c_{\boldsymbol{l}}\bar{c}_{\boldsymbol{m}}\left(\Omega_0,\varphi_{\varkappa}^{\#}(g_{k_1})^{l_1+m_1},\dots,\varphi_{\varkappa}^{\#}(g_{k_N})^{l_N+m_N}\Omega_0\right)=$$

$$Ext\text{-}\sum\nolimits_{\boldsymbol{l},\boldsymbol{m}} c_{\boldsymbol{l}}\bar{c}_{\boldsymbol{m}}\int_{{}^*\mathbb{R}_c^{\#N}} q_{k_1}^{l_1+m_1}\times\dots\times q_N^{l_N+m_N}\left(Ext\text{-}\prod\nolimits_{i=1}^{N} d^{\#}\mu_{k_i}^{\#}\right)=Ext\text{-}\int_{Q^{\#}}\left|P(x_{k_1},x_{k_2},\dots,x_{k_N})\right|^2 d^{\#}\mu^{\#}.$$

**By** the equation (99) and the fact that each measure $\mu_{k_i}^{\#}$ has mass one. Since $\Omega_0$ is cyclic for polynomials in the fields, $\mathbf{S}^{\#}$extends to a unitary map of $\mathcal{F}_s^{\#}(H^{\#})$ onto $L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})$.

**Theorem 10.1** [15] Let $\varphi_{m,\varkappa}^{\#}(x)$, $\varkappa\in {}^*\mathbb{R}_{c,\infty}^{\#}$be the free scalar field of mass $m$ (in 4-dimensional space-time) at time zero. Let $g\in L_1^{\#}({}^*\mathbb{R}_c^{\#3})\cap L_2^{\#}({}^*\mathbb{R}_c^{\#3})$ and define $H_{I,\varkappa,\lambda(\varkappa)}(g)=\lambda(\varkappa)\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} g(x)\colon\varphi_{m,\varkappa}^{\#4}(x)\colon d^{\#3}x\right)$, where $\lambda(\varkappa)\in {}^*\mathbb{R}_{c,\approx}^{\#}$. Let $\mathbf{S}^{\#}$ denote the unitary map $\mathbf{S}^{\#}\colon\mathcal{F}_s^{\#}(H^{\#})\to L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})$ constructed above. Then $V=\mathbf{S}^{\#}H_{I,\varkappa,\lambda}(g)\mathbf{S}^{\#-1}$is multiplication by a function $V_{\varkappa,\lambda}(q)$ which satisfies: (a) $V_{\varkappa,\lambda}(q)\in L_p^{\#}(Q^{\#},d^{\#}\mu^{\#})$ for all $p\in {}^*\mathbb{N}$. (b) $Ext\text{-}\exp\left(-tV_{\varkappa,\lambda}(q)\right)\in L_1^{\#}(Q^{\#},d^{\#}\mu^{\#})$ for all $t\in[0,{}^*\infty)$.

**Proof.** (a) Note that $\varphi_{m,\varkappa}^{\#}(x)$ is a well-defined operator-valued function of $x\in {}^*\mathbb{R}_c^{\#3}$. We define now $\colon\varphi_{m,\varkappa}^{\#4}(x)\colon$by moving all the $a^{\dagger}$'s to the left in the formal expression for $\varphi_{m,\varkappa}^{\#4}(x)$. By Theorem 59 $\colon\varphi_{m,\varkappa}^{\#4}(x)\colon$ is also a well-defined operator for each $x\in {}^*\mathbb{R}_c^{\#3}$. Notice that for each $x\in {}^*\mathbb{R}_c^{\#3}$ operator $\colon\varphi_{m,\varkappa}^{\#4}(x)\colon$ takes $F_0$ into itself. Thus for each $x\in {}^*\mathbb{R}_c^{\#3}$ operator $\colon\varphi_{m,\varkappa}^{\#4}(x)\colon$ reads $\colon\varphi_{m,\varkappa}^{\#4}(x)\coloneqq\varphi_{m,\varkappa}^{\#4}(x)+d_2(\varkappa)\,\varphi_{m,\varkappa}^{\#2}(x)+d_1(\varkappa)$ where the coefficients $d_1(\varkappa)$ and $d_2(\varkappa)$ are hyperfinite constant independent of $x$. For each $x\in {}^*\mathbb{R}_c^{\#3}$, $\mathbf{S}^{\#}\varphi_{m,\varkappa}^{\#}(x)(g)\mathbf{S}^{\#-1}$is the operator on #-measurable space $L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})$ which acts by multiplying by the function $Ext\text{-}\sum_{k=1}^{{}^*\infty} c_k(x,\varkappa)q_k$ where $c_k(x,\varkappa)=\quad(2\pi)^{-3/2}\left(g_k,\left(Ext\text{-}\exp(ipx)\right)\chi(\varkappa,p)\mu(p)^{-1/2}\right)$ and $\chi(\varkappa,p)\equiv 1$ if $|p|\le\varkappa,\chi(\varkappa,p)\equiv 0$ if $|p|>\varkappa$. Note that

$$Ext\text{-}\sum\nolimits_{k=1}^{{}^*\infty}|c_k(x,\varkappa)|^2=(2\pi)^{-3/2}\|\chi(\varkappa,p)\mu(p)\|_{\#2}^2,\tag{100}$$

so the functions $\mathbf{S}^{\#}\varphi_{m,\varkappa}^{\#4}(x)(g)\mathbf{S}^{\#-1}$ and $\mathbf{S}^{\#}\varphi_{m,\varkappa}^{\#2}(x)(g)\mathbf{S}^{\#-1}$ are in $L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})$ and the $L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})$ norms are uniformly bounded in $x$. Therefore, since $g\in L_1^{\#}({}^*\mathbb{R}_c^{\#3})$, $\mathbf{S}^{\#}H_{I,\varkappa,\lambda(\varkappa)}(g)\mathbf{S}^{\#-1}$operates on $L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})$ by multiplication by some $L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})$-function which we denote by $V_{I,\varkappa,\lambda(\varkappa)}(q)$. Consider now the expression for $H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0$. This is a vector $(0,0,0,0,\psi^{\#4},0,\dots)$ with

$$\psi^{\#4}(p_1,p_2,p_3,p_4)=Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\frac{\lambda(\varkappa)g(x)\chi(\varkappa,p)\left(Ext\text{-}\exp\left(-ix\sum_{i=1}^{i=4}p_i\right)\right)d^3x}{(2\pi)^{3/2}\prod_{i=1}^{4}[2\mu(p_i)]^{1/2}}=\frac{\lambda(\varkappa)\prod_{i=1}^{4}\chi(\varkappa,p_i)\left(Ext\text{-}\hat{g}\left(\sum_{i=1}^{i=4}p_i\right)\right)}{(2\pi)^{9/2}\prod_{i=1}^{4}[2\mu(p_i)]^{1/2}}\tag{101}$$

Here $|p_i|\le\varkappa,1\le i\le 4$. We choose now the parameter $\lambda=\lambda(\varkappa)\approx 0$ such that $\|\psi^{\#4}\|_{\#2}^2\in\mathbb{R}$ and therefore we obtain $\left\|H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0\right\|_{\#2}^2\in\mathbb{R}$, since $\left\|H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0\right\|_{\#2}^2=\|\psi^{\#4}\|_{\#2}^2$. But, since $\mathbf{S}^{\#}\Omega_0=1$, we get the equalities

$$\left\|H_{I,\varkappa,\lambda(\varkappa)}(g)\Omega_0\right\|_{\#2}=\left\|\mathbf{S}^{\#}H_{I,\varkappa,\lambda(\varkappa)}(g)\mathbf{S}^{\#-1}\right\|_{L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})}=\left\|V_{I,\varkappa,\lambda(\varkappa)}(q)\right\|_{L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})}.\tag{102}$$

From (101)-(102) we get that $\left\|V_{I,\varkappa,\lambda(\varkappa)}(q)\right\|_{L_2^{\#}(Q^{\#},d^{\#}\mu^{\#})}\in\mathbb{R}$. It is easily verify that each polynomial $P(q_1,q_2,\dots,q_n),n\in {}^*\mathbb{N}$ is in the domain of the operator $V_{I,\varkappa,\lambda(\varkappa)}(q)$ and $\mathbf{S}^{\#}H_{I,\varkappa,\lambda(\varkappa)}(g)\mathbf{S}^{\#-1}\equiv V_{I,\varkappa,\lambda(\varkappa)}(q)$ on that domain. Since $\Omega_0$ is in the domain of $H^{p}{}_{I,\varkappa,\lambda(\varkappa)}(g),p\in {}^*\mathbb{N}$, 1 is in the domain of the operator $V^{p}{}_{I,\varkappa,\lambda(\varkappa)}(q)$ for all $p\in {}^*\mathbb{N}$. Thus, for all $p\in {}^*\mathbb{N}$ $V_{I,\varkappa,\lambda(\varkappa)}(q)\in L_{2p}^{\#}(Q^{\#},d^{\#}\mu^{\#})$, since $\mu^{\#}(Q^{\#})$ is finite, we conclude that $V_{I,\varkappa,\lambda(\varkappa)}(q)\in L_p^{\#}(Q^{\#},d^{\#}\mu^{\#})$ for all $p\in {}^*\mathbb{N}$. (b) Remind Wick's theorem asserts that $\colon\varphi_{m,\varkappa}^{\#j}(x)\coloneqq\sum_{i=0}^{[j/2]}(-1)^i\frac{j!}{(j-2i)!i!}c_{\varkappa}^i\varphi_{m,\varkappa}^{\#(j-2i)}(x)$ with $c_{\varkappa}=\left\|\varphi_{m,\varkappa}^{\#}(x)\Omega_0\right\|_{\#2}^2$. For $j=4$ we get $-O(c_{\varkappa}^2)\le\colon\varphi_{m,\varkappa}^{\#4}(x)\colon$ and therefore $-\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} g(x)\,d^{\#3}x\right)O(c_{\varkappa}^2)\le H_{I,\varkappa,\lambda(\varkappa)}(g)$.Finally we obtain $Ext\text{-}\int_{Q^{\#}} Ext\text{-}\exp\left(-t\left(\colon\varphi_{m,\varkappa}^{\#4}(x)\colon\right)\right)d^{\#}\mu^{\#}\le Ext\text{-}\exp\left(O(c_{\varkappa}^2)\right)$ and this inequality finalized the proof.

## GENERALIZED HAAG KASTLER AXIOMS

**Definition 11.1** [15] A non- Archimedean Banach algebra $A_{\#}$ is a complex #-algebra over field ${}^*\mathbb{C}_c^{\#}$ (or ${}^*\mathbb{C}_{c,\mathrm{fin}}^{\#}={}^*\mathbb{R}_{c,\mathrm{fin}}^{\#}+\mathrm{i}\,{}^*\mathbb{R}_{c,\mathrm{fin}}^{\#}$ ) which is a non-Archimedean Banach space under a ${}^*\mathbb{R}_c^{\#}$ -valued -norm which is sub multiplicative, i.e., $\|xy\|_{\#}\le\|x\|_{\#}\|y\|_{\#}$for all $x,y\in A_{\#}$. An involution on a non- Archimedean Banach algebra $A_{\#}$

is a conjugate-linear isometric antiautomorphism of order two denoted by $x \mapsto x^*$, i.e.,$(x+y)^* = x^* + y^*$, and for all $x, y \in A_\#$: $(xy)^* = y^* x^*$, $(\lambda x)^* = \bar{\lambda} x$,$(x^*)^* = x$, $\|x^*\|_\# = x$, $\lambda \in {}^*\mathbb{C}_c^\#$. A Banach #- algebra is a non-Archimedean Banach algebra with an involution.

**Definition 11.2** An $C_\#^*$-algebra is a Banach #-algebra $A_\#$ satisfying the $C_\#^*$-axiom: for all $x \in A_\#$, $\|x^* x\|_\# = \|x\|_\#^2$.

**Definition 11.3** 1) A linear operator $a: H_\# \to H_\#$ on a non-Archimedean Hilbert space $H_\#$ is said to be bounded if there is a number $K \in {}^*\mathbb{R}_c^\#$ with $\|a\xi\|_\# \leq K\|\xi\|_\#$ for all $\xi \in H_\#$. 2) A linear operator $a: H_\# \to H_\#$ a non-Archimedean Hilbert space $H_\#$ is said to be finitely bounded if there is a number $K \in {}^*\mathbb{R}_{c,\mathrm{fin}}^\#$ with $\|a\xi\|_\# \leq K\|\xi\|_\#$ for all $\xi \in H_\#$. The infimum of all such $K$ if exists, is called the #-norm of $a$, written $\|a\|_\#$.

**Abbreviation 11.1** The set of all finitely bounded operators $a: H_\# \to H_\#$ we will be denoting by $\mathcal{B}^\#(H_\#)$.

**Abbreviation 11.2** The set of all finitely bounded operators $a: H_\# \to H_\#$ we will be denoting by $\mathcal{B}_\#(H_\#)$.

**Remark 11.1** Note that $\mathcal{B}_\#(H_\#)$ is a $C_\#^*$-algebra over field ${}^*\mathbb{C}_{c,\mathrm{fin}}^\#$.

**Definition 11.4** If $S \subseteq \mathcal{B}^\#(H_\#)$ (or $\mathcal{B}_\#(H_\#)$ ) then the commutant $S'$ of $S$ is $S' = \{x \in \mathcal{B}^\#(H_\#) | \forall a \in S(xa = ax)\}$.

**Remark 11.2** The algebra $\mathcal{B}^\#(H_\#)$ of bounded linear operators on a non-Archimedean Hilbert space $H_\#$ is a $C_\#^*$-algebra with involution $T \to T^*, T \in \mathcal{B}^\#(H_\#)$. Clearly, any #-closed #-selfadjoint subalgebra of $\mathcal{B}^\#(H_\#)$ is also a $C_\#^*$-algebra.

**Remark 11.3** We will be especially concerned with #-separable Hilbert Spaces where there is an orthonormal basis, i.e. a hyper infinite sequence ,$\{\xi_i\}_{i=1}^{*\infty}$ of unit vectors with $\langle \xi_i, \xi_j \rangle = 0$ for $i \neq j$ and such that 0 is the only element of $H_\#$ orthogonal to all the $\xi_i$.

**Definition 11.5** 1) The topology on $\mathcal{B}^\#(H_\#)$ (or $\mathcal{B}_\#(H_\#)$ of pointwise #-convergence on $H_\#$ is called the strong operator topology. A basis of neighbourhoods of $a \in \mathcal{B}^\#(H_\#)$ (or $a \in \mathcal{B}_\#(H_\#)$ is formed by the following way

$$N(a, \{\xi_i\}_{i=1}^n, \varepsilon) = \{b | \|(b-a)\xi_i\|_\# < \varepsilon, \forall i (1 \leq i \leq n)\}.$$

2) The weak operator topology is formed by the basic neighbourhoods

$$N(a, \{\xi_i\}_{i=1}^n, \{\eta_i\}_{i=1}^n, \varepsilon) = \{b | \langle (b-a)\xi_i, \eta_i \rangle < \varepsilon, \forall i (1 \leq i \leq n)\}.$$

**Theorem 11.1** If $M = M^*$ is subalgebra of $\mathcal{B}^\#(H_\#)$ (or $\mathcal{B}_\#(H_\#)$ with $1 \in M$, then the following statements are equivalent: 1) $M = M''$ ; 2) $M$ is strongly #-closed; 3) $M$ is weakly #-closed.

**Definition 11.6** A subalgebra of $\mathcal{B}^\#(H_\#)$ (or $\mathcal{B}_\#(H_\#)$ satisfying the conditions of Theorem 61is called a von Neumann #-algebra.

**Theorem 11.2** [15] (Generalized Gelfand-Naimark theorem) Let $A$ be a $C_\#^*$-algebra with unit. Then there exist a non-Archimedean Hilbert space $H_\#$ and an #-isometric homomorphism $U$ of $A$ into $B(H_\#)$ such that $Ux^* = Ux^*$, $x \in A$.

**Abbreviation 11.3** We denote by $M_4^\# = \{{}^*\mathbb{R}_c^{\#4}, (\cdot,\cdot)\}$, the vector space ${}^*\mathbb{R}_c^{\#4}$ with the Minkowski product: $(x,y) = x_0 y_0 - x_i y_i$, $i = 1,2,3$.

***Statement of the Axioms*** [15]. Let $M_4^\#$ be Minkowski space over field ${}^*\mathbb{R}_c^\#$ of four space-time dimensions.

1. ***Algebras of Local Observables.*** To each finitely bounded #-open set $O \subset M_4^\#$ we assign a unital $C_\#^*$ -algebra

$$O \to \mathcal{B}_\#(O)$$

2. ***Isotony***. If $O_1 \subset O_2$ , then $\mathcal{B}(O_1)$ is the unital $C_\#^*$ -subalgebra of the unital $C_\#^*$-algebra $\mathcal{B}(O_2)$ :

$$\mathcal{B}_\#(O_1) \subset \mathcal{B}_\#(O_2).$$

This axiom allow us to form the algebra of all local observables

$$\mathcal{B}_{\#\mathrm{loc}} = \cup_{O \subset M_4^\#} \mathcal{B}_\#(O).$$

The algebra $\mathcal{B}_{\#\mathrm{loc}}$ is a well-defined $C_\#^*$ -algebra because given any $O_1, O_2 \subset M_4^\#$, both $\mathcal{B}_\#(O_1)$ and $\mathcal{B}_\#(O_2)$ are subalgebras of the $C_\#^*$ -algebra$\mathcal{B}_\#(O_1 \cup O_2)$. From there one can take the #-norm completion to obtain

$$\mathcal{B}_\# = \#\text{-}\overline{\mathcal{B}_{\#\mathrm{loc}}}\ ,$$

called the algebra of quasi-local observables. This gives a $C_\#^*$ -algebra in which all the local observable $C_\#^*$ -algebras are embedded.

3. ***Poincare ≈ -Covariance.*** For each Poincare transformation $g \in {}^\sigma P_+^\uparrow$ , there is a $C_\#^*$- isomorphism $\alpha_g : \mathcal{B}_\# \to \mathcal{B}_\#$ such that

$$\alpha_g(\mathcal{B}_\#(O)) \approx \mathcal{B}_\#(g(O)),$$

for all bounded #-open $O \subset M_4^\#$. For fixed $g \in \mathcal{B}_\#$ , the map $g \to \alpha_g(A)$ is required to be #-continuous.

3′. For each Poincare transformation $g \in {}^\sigma P_+^\uparrow$ , there is a $C_\#^*$- isomorphism $\alpha_g : \mathcal{B}_\# \to \mathcal{B}_\#$ such that

$$\mathrm{st}\left(\alpha_g\left(\mathcal{B}_\#(O)\right)\right)=\mathrm{st}\left(\mathcal{B}_\#\left(g(O)\right)\right),$$

for all bounded #-open $O \subset M_4^\#$. For fixed $g \in \mathcal{B}_\#$ , the map $g \to \alpha_g(A)$ is required to be #-continuous.

4. $\approx$-***Causality.*** If $O_1$ and $O_2$ are spacelike separated, then all elements of $\mathcal{B}_\#(O_1) \approx$ -commute with all elements of a $C_\#^*$ -algebra $\mathcal{B}_\#(O_2)$

$$[\mathcal{B}_\#(O_1), \mathcal{B}_\#(O_2)] \approx 0.$$

4′. If $O_1$ and $O_2$ are space-like separated, then the standard part of the all elements of $C_\#^*$ -algebra $\mathcal{B}_\#(O_1)$ commute with the standard part of the all elements of $C_\#^*$ -algebra $\mathcal{B}_\#(O_2)$

$$\mathrm{st}\big(\mathcal{B}_\#(O_1), \mathcal{B}_\#(O_2)\big) = 0.$$

**Definition 11.7** If $O \subset M_4^\#$, we say $x$ belongs to the future causal shadow of $O$ if every past directed time-like or light-like trajectory beginning at x intersects with $O$. Essentially, $O$ separates the past light cone of $x$.Likewise, we say $x$ belongs to the past causal shadow of $O$ if every future-directed timelike or lightlike trajectory beginning at $x$ inter-sects with $O$. The causal completion or causal envelope $\hat{O}$ of $O$ is the union of its future and past directed causal shadows. This definition of the causal completion $\hat{O}$ can be reformulated in terms of “causal complements,” which are computationally easier to deal with. If $O \subset M_4^\#$, we define the causal complement $O'$ of $O$ to be the set of all points with are spacelike to all points in $O$. Then $O''=\hat{O}$ is the causal completion of $O$. One expects the observables localized to $\hat{O}$ to be completely determined by the observables localized to $O$, carrying the same information.

5. Time Evolution.

$$\mathcal{B}_\#(\hat{O}\,) = \mathcal{B}_\#(O).$$

6. ***Vacuum state and positive spectrum.*** There exists a faithful irreducible representation $\pi_0 : \mathcal{B}_\# \to B(H_\#)$ with a unique (up to a factor) vector $\Omega \in H_\#$ such that $\Omega$ is cyclic and Poincaré invariant, and such that unitary representation of translations, given by

$$U(x)\pi_0\,(A)\Omega\ = \pi(\alpha_x(A))\Omega,$$

where $A \in \mathcal{B}_\#$ and $\alpha_x(\cdot)$ is the $C_\#^*$-isomorphism from Axiom 3 associated with translation by $x \in M_4^\#$, has Hermitian generators $P^\mu, \mu = 1,2,3$ whose joint spectrum lies in the forward light cone. The last phrase is the most physically important here; it simply states that we have energy-momentum operators whose spectrum satisfies $E^2 - \mathbf{P}^2 \gg 0$, i.e, or in other words, that the energy $E \geq 0$ and nothing can move faster than the speed of light. The vector $\Omega$ is the vacuum state This axiom does not appear to be purely algebraic; we have had to introduce an non-Archimedean Hilbert space $H_\#$ . In fact, we can rewrite the axiom in a completely algebraic but less transparent way as follows. We postulate that there exists an vacuum state $\omega_0$ on the $C_\#^*$ -algebra (i.e., a normalized, positive, bounded linear functional) such that the following holds $\omega_0(Q^*Q) = 0$ for all $Q \in \mathcal{B}_\#$ of the form

$$Q(f, A) = Ext\text{-}\int f(x)\alpha_x(A)\, d^{\#4}x$$

where $A \in \mathcal{B}_\#$ and $f(x)$ is a #-smooth function whose Fourier transform has bounded support disjoint from the forward light-cone centered at the origin in $M_4^\#$.

Remind that in a quantum system with a Hamiltonian $H$, the Heisenberg picture dynamics is given by the canonical formula

$$A(t) = \{Ext\text{-}\exp[itH]\}A(0)\{Ext\text{-}\exp[-itH]\}.$$

Then $A(t)$ is the observable at time $t$ corresponding to the time zero observable $A(0)$. In our model we have hyper finitely locally correct Hamiltonians $H(g)$ but no hyper infinitely global Hamiltonian, and we construct the Heisenberg picture dynamics nonetheless. We do this by restricting the observables to lie in the local algebras $\mathcal{B}_\#(O)$ and by using the finite propagation speed implicit in axiom 3.

**Definition 11.8** Let $\mathcal{F}_n^\#$ be the space of symmetric $L_2^\#({}^*\mathbb{R}_c^{\#3n})$ functions defined on ${}^*\mathbb{R}_c^{\#3n}$, $\mathcal{F}_0^\# = {}^*\mathbb{C}_c^\#$ and let $\boldsymbol{\mathcal{F}}^\# = Ext\text{-}\bigoplus_{n=0}^{{}^*\infty}\mathcal{F}_n^\#$, $\Omega_0 = 1 \in {}^*\mathbb{C}_c^\# \subset \mathcal{F}^\#$. Let $S_n$ be the projection of $L_2^\#({}^*\mathbb{R}_c^{\#3n})$ onto $\mathcal{F}_n^\#$and let $D_\#$ be the #-dense domain in $\mathcal{F}^\#$ spanned algebraically by $\Omega_0$ and vectors of the form $S_n(Ext\text{-}\prod_{k=1}^n f_k(k_n))$ where $f_k \in S_{\mathrm{fin}}^\#\,({}^*\mathbb{R}_c^{\#3}, {}^*\mathbb{R}_c^{\#3}), n \in {}^*\mathbb{N}$.

**Definition 11.9** We set now

$$H_{0,\varkappa} = Ext\text{-}\int \tfrac{1}{2}\!:\left(\pi_\varkappa^2(\boldsymbol{x}) + \nabla^\#\varphi_\varkappa^2(\boldsymbol{x}) + m^2\varphi_\varkappa^2(\boldsymbol{x})\right)\!: d^{\#3}\boldsymbol{x}. \tag{103}$$

**Theorem 11.3** As the bilinear form on the domain $D_\# \times D_\#$

$$H_{0,\varkappa} = Ext\text{-}\int_{|\boldsymbol{k}|\leq\varkappa} \mu(k)\, a^\dagger(\boldsymbol{k})a(\boldsymbol{k})d^{\#3}\boldsymbol{k}. \tag{104}$$

**Theorem 11.4** (1) The operator $H_0 = H_{0,\varkappa}$ leaves each subdomain $D_\# \cap \mathcal{F}_n^\#$ invariant. (2) The operator $H_0 = H_{0,\varkappa}$ is essentially self-#-adjoint as an operator on the domain $D_\#$.

**Definition 11.10** We set now

$$\varphi^{\#}_{\varkappa,0}(x,t) = Ext\text{-}\exp(itH_0)\varphi^{\#}_{\varkappa}(x)Ext\text{-}\exp(-itH_0) \quad (105)$$

$$\pi^{\#}_{\varkappa,0}(x,t) = Ext\text{-}\exp(itH_0)\pi^{\#}_{\varkappa}(x)Ext\text{-}\exp(-itH_0) \quad (106)$$

$$\varphi^{\#}_{\varkappa,0}(f,t) = Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}_c} \varphi^{\#}_{\varkappa,0}(x,t)\, f(x) d^{\#3}x \quad (107)$$

$$\pi^{\#}_{\varkappa,0}(f,t) = Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}_c} \pi^{\#}_{\varkappa,0}(x,t)\, f(x) d^{\#3}x. \quad (108)$$

Here $\varphi^{\#}_{\varkappa}(x)$ and $\pi^{\#}_{\varkappa}(x)$ is given by formulas (97) and (98) respectively.

**Remark 11.4** Note that $\varphi^{\#}_{\varkappa,0}(x,t)$ and $\pi^{\#}_{\varkappa,0}(x,t)$ are bilinear forms defined on $D_{\#} \times D_{\#}$.

**Theorem 11.5** As bilinear forms on $D_{\#} \times D_{\#}$.

$$\varphi^{\#}_{\varkappa,0}(x,t) = Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}_c} \Delta_{\#}(x-y,t)\, \pi^{\#}_{\varkappa}(x) d^{\#3}y + Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}_c} \frac{\partial^{\#}}{\partial^{\#}t}\Delta_{\#}(x-y,t)\, \varphi^{\#}_{\varkappa}(x) d^{\#3}y \quad (109)$$

$$\pi^{\#}_{\varkappa,0}(x,t) = Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}_c} \frac{\partial^{\#}}{\partial^{\#}t}\Delta_{\#}(x-y,t)\, \pi^{\#}_{\varkappa}(x) d^{\#3}y + Ext\text{-}\int_{{}^*\mathbb{R}^{\#3}_c} \frac{\partial^{\#2}}{\partial^{\#}t^2}\Delta_{\#}(x-y,t)\, \pi^{\#}_{\varkappa}(x) d^{\#3}y \quad (110)$$

**Remark 11.5** Here $\Delta_{\#}(x-y,t)$ is the solution of the generalized Klein-Gordon equation

$$\frac{\partial^{\#2}}{\partial^{\#}t^2}\Delta_{\#}(x,t) - \frac{\partial^{\#2}}{\partial^{\#}x_1^2}\Delta_{\#}(x,t) - \frac{\partial^{\#2}}{\partial^{\#}x_2^2}\Delta_{\#}(x,t) - \frac{\partial^{\#2}}{\partial^{\#}x_3^2}\Delta_{\#}(x,t) + m^2\Delta_{\#}(x,t) = 0 \quad (111)$$

with Cauchy data $\Delta_{\#}(x,0) = 0, \frac{\partial^{\#}}{\partial^{\#}t}\Delta_{\#}(x,0) = \delta(x)$.

**Remark 11.6** Note the distribution $\Delta_{\#}(x,t)$ has support in the double light-cone $|x| \leq |t|$.

**Theorem 11.6** Let $f_1, f_2 \in S^{\#}({}^*\mathbb{R}^{\#3}_c, {}^*\mathbb{R}^{\#3}_c)$. The operator $\varphi^{\#}_{\varkappa,0}(f,t) + \pi^{\#}_{\varkappa,0}(f,t)$ is essentially self-#-adjoint on the domain $D_{\#}$.

**Definition 11.11** We introduce now the class $\mathfrak{J}\big(S^{\#}({}^*\mathbb{R}^{\#3}_c)\big)$ of bilinear forms on $D_{\#} \times D_{\#}$ expressible as a linear combination of the forms

$$V = \sum_{j=0}^{n} \binom{n}{j} Ext\text{-}\int_{{}^*\mathbb{R}^{\#3n}_c} v(k)\, a^{\dagger}(k_1) \cdots a^{\dagger}(k_j) a(k_{j+1}) \cdots a(k_n) d^{\#3n}k \quad (112)$$

with symmetric kernels $v(k) \in S^{\#}({}^*\mathbb{R}^{\#3}_c)$ having real Fourier transforms.

**Theorem 11.7** Let $V \in \mathfrak{J}\big(S^{\#}({}^*\mathbb{R}^{\#3}_c)\big)$. Then $V$is essentially self-#-adjoint on $D_{\#}$.

**Theorem 11.8** Let $O$ be a bounded #-open region of vector space${}^*\mathbb{R}^{\#3}_c$ and let $\mathcal{M}_{\#}(O)$ be the von Neumann algebra generated by the field operators $Ext\text{-}\exp[i\varphi^{\#}_{\varkappa}(f)]$ with $f \in S^{\#}({}^*\mathbb{R}^{\#3}_c, {}^*\mathbb{R}^{\#3}_c)$ and $\operatorname{supp} f \subset O$. Let $g(x) = 0$ on ${}^*\mathbb{R}^{\#3}_c \backslash O$. Then $Ext\text{-}\exp[itH_I(g)] \in \mathcal{M}_{\#}(O)$ for all $t \in {}^*\mathbb{R}^{\#}_c$.

**Definition 11.12** Let $O$ be a bounded #-open region of space and let $\mathcal{B}_{\#}(O)$ be the von Neumann algebra generated by the operators $Ext\text{-}\exp\big[i\big(\varphi^{\#}_{\varkappa}(f_1) + \pi^{\#}_{\varkappa}(f_2)\big)\big]$ with $f_1, f_2 \in S^{\#}({}^*\mathbb{R}^{\#3}_c, {}^*\mathbb{R}^{\#3}_c)$ and $\operatorname{supp} f_1, \operatorname{supp} f_2 \subset O$. Let $O_t$ be the set of points with distance less than $|t|$ to $O$ for any instant of the time $t$.

**Theorem 11.9** $Ext\text{-}\exp(itH_0)\mathcal{B}_{\#}(O)Ext\text{-}\exp(-itH_0) \subset \mathcal{B}_{\#}(O_t)$.

**Theorem 11.10** If $O_1$ and $O_2$ are disjoint bounded open regions of vector space${}^*\mathbb{R}^{\#3}_c$ then the standard part of the operators in $\mathcal{B}_{\#}(O_1)$ commute with the standard part of the operators in operators in $\mathcal{B}_{\#}(O_2)$.

**Theorem 11.11** Let $g \in L_2^{\#}\big(({}^*\mathbb{R}^{\#3}_c)\big)$, and let $g = 0$ on open region $O$, then $Ext\text{-}\exp[itH_I(g)] \in \mathcal{B}_{\#}(O)'$ for all $t \in {}^*\mathbb{R}^{\#}_c$.

**Theorem 11.12** [15] (Free field $\approx$-Causality) Let $f_1, f_2 \in S^{\#}_{\text{fin}}({}^*\mathbb{R}^{\#4}_c, {}^*\mathbb{R}^{\#4}_c)$ with $\operatorname{supp} f_1 \subset O_1, \operatorname{supp} f_2 \subset O_2$. We set now $\varphi^{\#}_{\varkappa,0}(f_1) = Ext\text{-}\int_{{}^*\mathbb{R}^{\#4}_c} \varphi^{\#}_{\varkappa,0}(x,t)\, f_1(x,t) d^{\#4}x$ and $\varphi^{\#}_{\varkappa,0}(f_2) = Ext\text{-}\int_{{}^*\mathbb{R}^{\#4}_c} \varphi^{\#}_{\varkappa,0}(x,t)\, f_2(x,t) d^{\#4}x$. If region $O_1$ and region $O_2$ are space-like separated, then $\big[\varphi^{\#}_{\varkappa,0}(f_1),\ \varphi^{\#}_{\varkappa,0}(f_2)\big]\psi \approx 0$ for all near standard vector $\psi \in H_{\#}$.

**Proof.** The commutator $\big[\varphi^{\#}_{\varkappa,0}(f_1),\ \varphi^{\#}_{\varkappa,0}(f_2)\big]$ reads

$$\big[\varphi^{\#}_{\varkappa,0}(f_1),\ \varphi^{\#}_{\varkappa,0}(f_2)\big] = Ext\text{-}\int_{{}^*\mathbb{R}^{\#4}_c} d^{\#3}x_1 d^{\#}t_1 Ext\text{-}\int_{{}^*\mathbb{R}^{\#4}_c} d^{\#3}x_2 d^{\#}t_1 \Delta^{\#}_{\varkappa}(x_1 - x_2, t_1 - t_2) f_1(x_1,t_1) f_2(x_1,t_1),$$

$$\Delta^{\#}_{\varkappa}(x_1 - x_2, t_1 - t_2) = \Xi_1(x_1 - x_2, t_1 - t_2; \varkappa) - \Xi_2(x_1 - x_2, t_1 - t_2; \varkappa), \text{ where}$$

$$\Xi_1(x_1 - x_2, t_1 - t_2; \varkappa) = Ext\text{-}\int_{|\boldsymbol{p}| \leq \varkappa} \{\exp\{[i\boldsymbol{p}(x_1 - x_2)] - i\omega(\boldsymbol{p})(t_1 - t_2)\}\} \frac{d^{\#3}\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2 + m^2}},$$

$$\Xi_2(x_1 - x_2, t_1 - t_2; \varkappa) = Ext\text{-}\int_{|\boldsymbol{p}| \leq \varkappa} \big\{-\exp\big[[i\boldsymbol{p}(x_1 - x_2)] + i\omega(\boldsymbol{p})(t_1 - t_2)\big]\big\} \frac{d^{\#3}\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2 + m^2}}.$$

Here $\varkappa \in {}^*\mathbb{R}^{\#}_{c,\infty}$, $\omega(p) = \sqrt{\boldsymbol{p}^2 + m^2}$. Define $\breve{\Xi}_1(x_1 - x_2, t_1 - t_2; \varkappa)$ and $\breve{\Xi}_2(x_1 - x_2, t_1 - t_2; \varkappa)$ by

$$\breve{\Xi}_1(x_1 - x_2, t_1 - t_2; \varkappa) = Ext\text{-}\int_{|\boldsymbol{p}| > \varkappa} \{\exp\{[i\boldsymbol{p}(x_1 - x_2)] - i\omega(\boldsymbol{p})(t_1 - t_2)\}\} \frac{d^{\#3}\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2 + m^2}},$$

$$\breve{\Xi}_2(x_1 - x_2, t_1 - t_2; \varkappa) = Ext\text{-}\int_{|\boldsymbol{p}| > \varkappa} \big\{-\exp\big[[i\boldsymbol{p}(x_1 - x_2)] + i\omega(\boldsymbol{p})(t_1 - t_2)\big]\big\} \frac{d^{\#3}\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2 + m^2}}.$$

Note that: (a) $\breve{\Xi}_1(x_1 - x_2, t_1 - t_2; \varkappa) \approx 0$ and $\Xi_2(x_1 - x_2, t_1 - t_2; \varkappa) \approx 0$, (b) $\Xi_1(x_1 - x_2, t_1 - t_2; \varkappa)$ and $\Xi_2(x_1 - x_2, t_1 - t_2; \varkappa)$ are Lorentz $\approx$-invariant tempered distribution (see definition 4), since the distributions $\Xi_1(x_1 - x_2, t_1 - t_2)$ and $\Xi_2(x_1 - x_2, t_1 - t_2)$ defined by

$$\Xi_1(x_1 - x_2, t_1 - t_2; \varkappa) + \breve{\Xi}_1(x_1 - x_2, t_1 - t_2; \varkappa) = \textit{Ext-}\int \left\{\exp\left[[i\boldsymbol{p}(x_1 - x_2)] - i\omega(\boldsymbol{p})(t_1 - t_2)\right]\right\} \frac{d\#^3\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2+m^2}}$$

$$\Xi_2(x_1 - x_2, t_1 - t_2; \varkappa) + \breve{\Xi}_2(x_1 - x_2, t_1 - t_2; \varkappa) = \textit{Ext-}\int \left\{\exp\left[[-i\boldsymbol{p}(x_1 - x_2)] + i\omega(\boldsymbol{p})(t_1 - t_2)\right]\right\} \frac{d\#^3\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2+m^2}}$$

are Lorentz invariant by Theorem 56. From expression of the distribution $\Xi_2(x_1 - x_2, t_1 - t_2; \varkappa)$ by replacement $\boldsymbol{p} \to -\boldsymbol{p}$ we obtain

$$\Xi_2(x_1 - x_2, t_1 - t_2; \varkappa) = -\textit{Ext-}\int_{|\boldsymbol{p}|>\varkappa} \left\{\exp\left[[i\boldsymbol{p}(x_1 - x_2)] + i\omega(\boldsymbol{p})(t_1 - t_2)\right]\right\} \frac{d\#^3\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2+m^2}}.$$

And therefore finally we get

$$\Delta_\varkappa^\#(x_1 - x_2, t_1 - t_2) = \textit{Ext-}\int_{|\boldsymbol{p}|\leq\varkappa} \sin[\omega(\boldsymbol{p})(t_1 - t_2)]\exp[i\boldsymbol{p}(x_1 - x_2)] \frac{d\#^3\boldsymbol{p}}{\sqrt{\boldsymbol{p}^2+m^2}}.$$

Thus for any points $(x_1, t_1)$ and $(x_2, t_2)$ separated by space-like interval we obtain that $\Delta_\varkappa^\#(x_1 - x_2, t_1 - t_2) \approx 0$, since $\Delta_\varkappa^\#(x_1 - x_2, t_1 - t_2)$ is a Lorentz $\approx$-invariant tempered distribution.

**Theorem 11.13** (Time zero free field $\approx$-locality) Let $f_1, f_2 \in S^{\#}_{\text{fin}}({}^*\mathbb{R}_c^{\#3}, {}^*\mathbb{R}_c^{\#3})$ with $\operatorname{supp} f_1 \subset O_1$, and $\operatorname{supp} f_2 \subset O_2$ are disjoint bounded open regions of vector space ${}^*\mathbb{R}_c^{\#3}$, then $\left[\varphi_{\varkappa,0}^\#(f_1, 0), \varphi_{\varkappa,0}^\#(f_2, 0)\right] \approx 0$.

**Proof.** It follows immediately from Theorem 11.12.

**Theorem 11.14** Let $O$ be a bounded #-open region of vector space ${}^*\mathbb{R}_c^{\#3}$, let $t \in {}^*\mathbb{R}_c^\#$, let $g$ be a nonnegative function in $L_1^\#({}^*\mathbb{R}_c^{\#3}) \cap L_2^\#({}^*\mathbb{R}_c^{\#3})$ and let $g$ be identically equal to one on $O_t$.For $A \in \mathcal{B}_\#(O)$, then

$$\sigma_t(A) = \{\textit{Ext-}\exp[itH(g)]\}A\{\textit{Ext-}\exp[-itH(g)]\}$$

is independent of $g$ and $\sigma_t(A) \in \mathcal{B}_\#(O_t)$.

**Proof.** Let $\sigma_t^0(A) = \{\textit{Ext-}\exp[itH_0]\}A\{\textit{Ext-}\exp[-itH_0]\}$ and $\sigma_t^I(A) = \{\textit{Ext-}\exp[itH_I]\}A\{\textit{Ext-}\exp[-itH_I]\}$. Notice that generalized Trotter's product formula is valid for the unitary group $\textit{Ext-}\exp\left[it\left(H_0 + H_I(g)\right)\right]$. Thus we get the following product formula for the associated automorphism group:

$$\sigma_t(A) = \#\text{-}\lim_{n\to{}^*\infty}\left[\left(\sigma_{t/n}^0\sigma_{t/n}^I\right)^n(A)\right]. \tag{113}$$

Each automorphism $\sigma_t^I$ maps each $\mathcal{B}_\#(O_s)$ into itself and is independent of $g$ on $\mathcal{B}_\#(O_s)$ for $|s| \ll |t|$. To see this, let $\chi(O_s)$ be the characteristic function of a set $O_s$. We assert that

$$\sigma_{t/n}^I(C) = \left\{\textit{Ext-}\exp\left[i(t/n)H_I\left(\chi(O_s)\right)\right]\right\}C\left\{\textit{Ext-}\exp\left[-i(t/n)H_I\left(\chi(O_s)\right)\right]\right\} \tag{114}$$

for any $C \in \mathcal{B}_\#(O_s)$ and that $\sigma_t^I(C) \in \mathcal{B}_\#(O_s)$. In other words the interaction automorphism has propagation speed zero and is independent of $g$ on $\mathcal{B}_\#(O_s)$ for $|s| \ll |t|$. The theorem follows from (113), (114) and Theorem 11.9. To prove (113), we rewrite $H_I(g) = H_I\left(\chi(O_s)\right) + H_I(g[1 - \chi(O_s)])$ as a sum of commuting self-#-adjoint operators. By **Theorem 11.15** $\textit{Ext-}\exp\left[itH_I\left(\chi(O_s)\right)\right] \in \mathcal{B}_\#(O_s)$ and so the right side of (8.3) belongs to $\mathcal{B}_\#(O_s)$. By Theorem 70,

$$\textit{Ext-}\exp[itH_I(g[1 - \chi(O_s)])] \in \mathcal{B}_\#(O_s)'$$

and (114) follows.

**Definition 11.13** Let $B$ be a bounded #-open region of spacetime $M_4^\#$ and for any time $t$, let $B(t) = \{x|\ x, t \in B\}$ be the time $t$ time slice of $B$. We define $\mathcal{B}_\#(B)$ to be the von Neumann algebra generated by

$$\cup_s \sigma_s\left(\mathcal{B}_\#\left(B(t)\right)\right). \tag{115}$$

**Theorem 11.16** The generalized Haag-Kastler axioms (1)-(5) are valid for all these local algebras $\mathcal{B}_\#(B)$.

**Proof** (Except Lorentz rotations) The axioms (1) and (2) are obvious, while (4) follows easily from the finite propagation speed, Theorem 11.10, together with the time zero $\approx$-locality, Theorem 11.12. Because the time zero fields coincide with the time zero free fields, and because the time zero fields generate $\mathcal{B}_\#$ by Theorem 11.12 and the definition of the local algebras, the free field result carries over to our scalar model with interaction $H_I \neq 0$. In the Poincaré covariance axiom (3), the time translation is given by $\sigma_t$. Let $B + t$ be the time translate of the space time region $B \subset M_4^\#$. Then $(B + t)(s) = B(s - t)$ and so

$$\sigma_t\left[\cup_s \sigma_s\left(\mathcal{B}_\#\left(B(s)\right)\right)\right] = \cup_s \sigma_{s+t}\left(\mathcal{B}_\#\left(B(s)\right)\right) = \cup_s \sigma_s\left(\mathcal{B}_\#\left(B(s - t)\right)\right) = \cup_s \sigma_{s+t}\left(\mathcal{B}_\#\left(B(s + t)\right)\right) \tag{116}$$

Thus $\sigma_t\left(\mathcal{B}_\#(B)\right) = \mathcal{B}_\#(B + t)$ and axiom (3) is verified for time translations. Since the local algebras are #-norm dense in $\mathcal{B}_\#$ and since automorphisms of $\mathrm{C}_\#^*$-algebras preserve the #-norm, $\sigma_t$ extends to an automorphism of algebra $\mathcal{B}_\#$.

**Definition 11.14** To define the space translation automorphism $\sigma_s$, we set now

$$P^{\mu} = Ext\text{-}\int_{\|p\|\ll\varkappa} p^{\mu} a^{\dagger}(p) a(p)\, d^{\#4}p, \mu = 1,2,3;\ \sigma_t(A) = \{Ext\text{-}\exp[-ixP]\}A\{Ext\text{-}\exp[ixP]\}. \quad (117)$$

Then we get $\{Ext\text{-}\exp[-ixP]\}\varphi_{\varkappa}(x)\{Ext\text{-}\exp[ixP]\} = \varphi_{\varkappa}(x+y)$, $\{Ext\text{-}\exp[-ixP]\}\pi_{\varkappa}(x)\{Ext\text{-}\exp[ixP]\} = \varphi(x+y)$.

The following theorem completes the proof of Theorem 11.16 except for Lorentz rotations.

**Theorem 11.17** The automorphism $\sigma_x(\mathcal{B}_{\#}(B)) = \mathcal{B}_{\#}(B+x), \mathrm{st}(\sigma_x)$ extends up to $\mathbb{C}^*_{\#}$-automorphism of $\mathcal{B}_{\#}$, and $\langle x,t\rangle \to \mathrm{st}(\sigma_x)\mathrm{st}(\sigma_t) = = \mathrm{st}(\sigma_t)\mathrm{st}(\sigma_x)$ defines a 4-parameter abelian automorphism group of $\mathcal{B}_{\#}$.

**Theorem 11.18** Let $O$ be a bounded #-open region of space and let $\mathcal{B}_{\#}(O)$ be the von Neumann algebra generated by the operators $Ext\text{-}\exp[i(\varphi_{\varkappa}(f_1)+\pi_{\varkappa}(f_2))]$ where $f_1, f_2 \in \mathcal{E}^{\#}_{\mathrm{fin}}({}^*\mathbb{R}^{\#}_c)$ and $\mathrm{supp}f_1 \subset B, \mathrm{supp}f_2 \subset B$. Then

$$Ext\text{-}\exp(itH_0)\mathcal{B}_{\#}(O)Ext\text{-}\exp(-itH_0) \subset \mathcal{B}_{\#}(O_t).$$

**Remark 11.7** We reformulate the theorem by saying that $H_0$ has propagation speed at most one.

In order to obtain automorphisms for the full Lorentz group and to complete the proof of Theorem 11.16, there are four separate steps.

1. The first step is to construct a self-#-adjoint locally correct generator for Lorentz rotations. This generator then defines a locally correct unitary group and automorphism group.
2. The second step is to prove this statement for the fields, by showing that the field $\varphi_{\varkappa}(x,t)$, considered as a non-standard operator valued function on a suitable domain, and is transformed locally correctly by our unitary group.
3. The third step is to show that the local algebras $\mathcal{B}_{\#}(B)$ are also transformed correctly.
4. The fourth final step is to reconstruct the Lorentz group automorphisms from the locally correct pieces given by the first three steps. This final step is not difficult as in in the case of the two dimensional spacetime $d = 2$, see [16-18].

Let $H_{0,\varkappa}(x)$ denote the integrand in (103), where

$$H_{0,\varkappa} = Ext\text{-}\int H_{0,\varkappa}(\boldsymbol{x})d^{\#3}\boldsymbol{x} = Ext\text{-}\int \tfrac{1}{2}\!:\!\left(\pi^2_{\varkappa}(\boldsymbol{x}) + \nabla^{\#}\varphi^2_{\varkappa}(\boldsymbol{x}) + m^2\varphi^2_{\varkappa}(\boldsymbol{x})\right)\!:\!d^{\#3}\boldsymbol{x}\,. \quad (118)$$

The formal generator of classical Lorentz rotations is

$$M^{0k}_{\varkappa} = M^{0k}_{0,\varkappa} + M^{0k}_{I,\varkappa} = Ext\text{-}\int x^k H_{0\varkappa,}(\boldsymbol{x})d^{\#3}\boldsymbol{x} + Ext\text{-}\int x^k\!:\!P\,(\varphi_{\varkappa}(\boldsymbol{x}))\!:\!d^{\#3}\boldsymbol{x},\ k = 1,2,3. \quad (119)$$

The local Lorentzian rotations are

$$M^{0k}_{\varkappa}\left(g^{(k)}_1, g^{(k)}_2\right) = \varepsilon H_{0,\varkappa} + H_{0,\varkappa}\left(g^{(k)}_1\right) + H_{I,\varkappa}\left(, g^{(k)}_2\right), H_{0,\varkappa}\left(g^{(k)}_1\right) = Ext\text{-}\int H_{0,\varkappa}(\boldsymbol{x})g^{(k)}_1(\boldsymbol{x})d^{\#3}\boldsymbol{x}. \quad (120)$$

We require that $0 < \varepsilon$ and that: $g^{(k)}_1(x_1,x_2,x_3), g^{(k)}_2(x_1,x_2,x_3)$, $k = 1,2,3$ be nonnegative $C_0^{*\infty}$ functions. In the second step we require more, for example that $\varepsilon + g^{(k)}_1(x_1,x_2,x_3) = x_k$ and $g^{(k)}_2(x_1,x_2,x_3) = x_k, k = 1,2,3$ in some local space region. This region is contained in the Cartesian product $[\varepsilon,{}^*\infty) \times [\varepsilon,{}^*\infty) \times [\varepsilon,{}^*\infty)$. By using decomposing $H_{0,\varkappa}\left(g^{(k)}_1\right)$ into a sum of a diagonal and an off-diagonal term we obtain $H_{0,\varkappa}\left(g^{(k)}_1\right) =$

$$Ext\text{-}\int v^{(k)}_{D,\varkappa}(\boldsymbol{k},\boldsymbol{l})\, a^*(\boldsymbol{k})a(\boldsymbol{l})d^{\#3}\boldsymbol{k}d^{\#3}\boldsymbol{l} + Ext\text{-}\int v^{(k)}_{0D,\varkappa}(\boldsymbol{k},\boldsymbol{l})\,[a^*(\boldsymbol{k})a^*(\boldsymbol{l}) + a(-\boldsymbol{k})a(-\boldsymbol{l})]d^{\#3}\boldsymbol{k}d^{\#3}\boldsymbol{l} =$$
$$= H^{D}_{0,\varkappa}\left(g^{(k)}_1\right) + H^{0D}_{0,\varkappa}\left(g^{(k)}_1\right),$$

where

$$v^{(k)}_{D,\varkappa}(\boldsymbol{k},\boldsymbol{l}) = c_1\chi(\boldsymbol{k},\boldsymbol{l},\varkappa)(\mu(\boldsymbol{k})\mu(\boldsymbol{l}) + \langle \boldsymbol{k}\,,\boldsymbol{l}\,\rangle + m^2)[\mu(\boldsymbol{k})\mu(\boldsymbol{l})]^{-1/2}\hat{g}^{(k)}_1(-k_1+l_1,-k_2+l_2,-k_3+l_3),$$
$$v^{(k)}_{0D,\varkappa}(\boldsymbol{k},\boldsymbol{l}) = c_2\chi(\boldsymbol{k},\boldsymbol{l},\varkappa)(-\mu(\boldsymbol{k})\mu(\boldsymbol{l}) - \langle \boldsymbol{k}\,,\boldsymbol{l}\,\rangle + m^2)[\mu(\boldsymbol{k})\mu(\boldsymbol{l})]^{-1/2}\hat{g}^{(1)}_1(-k_1-l_1,-k_2-l_2,-k_3-l_3),$$

and where $\boldsymbol{k} = (k_1,k_2,k_3), \boldsymbol{l} = (l_1,l_2,l_3), \langle \boldsymbol{k}\,,\boldsymbol{l}\,\rangle = \sum_{i=1}^{3} k_i\, l_i$, $\chi(\boldsymbol{k},\boldsymbol{l},\varkappa) =1$ if $|\boldsymbol{k}| \le \varkappa$ and $|\boldsymbol{l}| \le \varkappa$, otherwise $\chi(\boldsymbol{k},\boldsymbol{l},\varkappa) = 0$.

**Theorem 11.19** (a) $v^{(k)}_{0D,\varkappa} \in L^{\#}_2({}^*\mathbb{R}^{\#3}_c)$. (b) Function $v^{(k)}_{D,\varkappa}$ is the kernel of a nonnegative operator and $\varepsilon\mu(\boldsymbol{k})\delta(\boldsymbol{k}-\boldsymbol{l}) + \beta v^{(k)}_{D,\varkappa}$ is the kernel of a positive self-#-adjoint operator, for $\beta \ge 0$, these operators are real in configuration space.

**Proof.** The statement (a) is obvious. The statement (b) is proved by using a finite sequence of Kato perturbations. Let$v^{(k)}_{\beta} = \varepsilon\mu(\boldsymbol{k})\delta(\boldsymbol{k}-\boldsymbol{l}) + \beta v^{(k)}_{D,\varkappa}$ and let $V_{\beta}$ and $V_D$ denote the operators with kernels $v^{(k)}_{\beta}$ and $v^{(k)}_{D,\varkappa}$ correspondingly. The operator $V_D$ is a sum of three terms of the form $A^*M_{g_1}A$ in configuration space, where $M_{g_1}$is multiplication by $g_1 \ge 0$. Thus $0 \le V_D$. Moreover for $\gamma$ sufficiently small, but chosen independently of $\beta$, we obtain $\gamma V_D \le \frac{1}{2}V_0 \le \frac{1}{2}(V_0 + \beta V_D) = \frac{1}{2}V_{\beta}$ and therefore $V_{\beta+\gamma} = V_{\beta} + \gamma V_D$ is a Kato perturbation, in the sense of bilinear forms. Consequently if the operator $V_\beta$ is self-#-adjoint, so is $V_{\beta+\gamma}$ and $D\left(V_{\beta+\gamma}^{1/2}\right) = D\left(V_\gamma^{1/2}\right)$. Thus canonical finite induction starting from $V_0 = V_0^*$ shows that $V_\beta$ is self-adjoint, for all $\beta \geq 0$.

**Theorem 11.20** The operator $H_0^D\left(g_1^{(k)}\right)$ is nonnegative and $\varepsilon H_0 + \beta H_0^D\left(g_1^{(k)}\right)$ is self-#-adjoint, for all $\beta > 0$.

The main purpose of the third step is to give a covariant definition of the local algebras $\mathcal{B}_\#(B)$. Le $f \in \mathcal{E}_{\text{fin}}^\#(B)$ be the ${}^*\mathbb{R}_c^{\#3}$-valued function with support in $B$. Let $\{\alpha_i\}_{i=1}^n, n \in {}^*\mathbb{N}$ be finite hyperreal numbers and consider the expressions

$$\varphi_\varkappa^\#(f) = Ext\text{-}\int \varphi_\varkappa^\#(x,t)\, f(x,t) d^{\#3}x d^\# t \tag{121}$$

$$\varphi_\varkappa^\#(f,t) = Ext\text{-}\int \varphi_\varkappa^\#(x,t)\, f(x,t) d^{\#3}x \tag{122}$$

$$\Re(f) = Ext\text{-}\sum_{i=1}^n \alpha_i \varphi_\varkappa^\#(f,t_i) \tag{123}$$

$$\pi_\varkappa^\#(f,t) = Ext\text{-}\int \pi_\varkappa^\#(x,t)\, f(x,t) d^{\#3}x. \tag{124}$$

For $g \equiv 1$ on a sufficiently large set (the domain of dependence of the region $B$), the time integration in (1) #-converges strongly, and all four operators above are symmetric and defined on $D\big(H(g)\big)$.

**Theorem 11.21** The operators (1)-(4) are essentially self-#-adjoint on any #-core for $H(g)^{1/2}$.

**Theorem 11.22** The algebra $\mathcal{B}_\#(B)$ is the von Neumann algebra generated by finitely bounded functions of operators of the form (121).

**Proof.** Note that if a hyper infinite sequence $\{A_n\}$ of self-#-adjoins operators #-converges strongly to a self #-adjoint #-limit $A$ on a core for $A$ then the unitary operators $Ext\text{-}\exp(itA_n)$ #-converge strongly to $Ext\text{-}\exp(itA)$. Using this fact, one can easily show that the operators (1) and (4) generate the same von Neumann algebra, $\mathcal{B}_{\#1}(B)$ and that $\mathcal{B}_{\#1}(B) \supset \mathcal{B}_\#(B)$. To show that $\mathcal{B}_{\#1}(B) \subset \mathcal{B}_\#(B)$, recall that a self- #-adjoint operator $A$ commutes with a finitely bounded operator $C$ provided $CD \subset D(A)$ and $CA = AC$ on $D$, for some core $D$ of $A$. Equivalently is the condition that the operator $C$ commutes with all finitely bounded functions of $A$. Also equivalent is the relation $CA = AC$ on $D(A)$. We choose $D = D(H(g))$. If the operator $C$ commutes with all operators of the form (122), it also commutes on $D(H(g))$ with all operators of the form (123). Hence we get $\mathcal{B}_\#(B)' \subset \mathcal{B}_{\#1}(B)'$ and so $\mathcal{B}_{\#1}(B) = \mathcal{B}_{\#1}(B)'' \subset \mathcal{B}_\#(B)'' = \mathcal{B}_\#(B)''$.

**Remark 11.8** The Poincare group ${}^\sigma P_+^\uparrow$ is the semidirect product of the space-time translations group $\mathbb{R}^{1,3}$ with the Lorentz group $\mathrm{O}(1,3)$ such that $\{a_1 + \Lambda_1\}\{a_2 + \Lambda_2\} = \{a_1 + \Lambda_1 a_2, \Lambda_1\Lambda_2\}$. Here $a \in \mathbb{R}^{1,3}$ and $\Lambda(\beta): (x_i, t) \to \big(x_i \times \cosh(\beta) + t \times \sinh(\beta), x_i \times \sinh(\beta) + t \times \cosh(\beta)\big), i = 1,2,3$. We prove that there exists a representation $\sigma(a,\Lambda)$ of the Poincare group ${}^\sigma P_+^\uparrow$ by $*$- automorphisms of $\mathcal{B}_\#$, such that $\sigma(a,\Lambda)\big(\mathcal{B}_\#(O)\big) = \mathcal{B}_\#(\{a,\Lambda\}O)$ for all bounded open sets $O$ and all $\{a,\Lambda\} \in {}^\sigma P_+^\uparrow$. The Lorentz group composition law gives $\sigma(a,\Lambda) = \sigma(a,I)\sigma(0,\Lambda)$. Obviously the existence of the automorphism representation $\sigma(a,\Lambda)$ follows directly from the construction of the pure Lorentz transformation $\sigma(0,\Lambda) = \sigma(\Lambda)$. One obtains $\sigma(\Lambda)$ by constructing locally correct infinitesimal generators. Formally, the operators,

$$M_\varkappa^{0k} = M_{0,\varkappa}^{0k} + M_{I,\varkappa}^{0k} = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \frac{1}{2}\left\{:\pi_\varkappa(x)^2: + :\big(\nabla\varphi_\varkappa(x)\big)^2: + m^2 :\varphi_\varkappa(x)^2:\right\} x^k d^{\#3}x + H_{I,\varkappa}(x^k g) \tag{125}$$

$k = 1,2,3$ s infinitesimal generators of Lorentz transformations in a region $O$ if the cutoff function $g$ equals one on a sufficiently large interval. We consider now the regions $O_1$ contained in the sets $\{x \in {}^*\mathbb{R}_c^{\#3} |\, x_1, x_2, x_3 > |t| + 1\}$. Thus for such regions $O_1$ we may replace (1) by $M^{0k} = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} H(x)\, x^k g(x) d^{\#3}x$, with a nonnegative functions $x^k g(x)$, $k = 1,2,3$. Here $H(x)$ is the formally positive energy density:

$$H(x) = \frac{1}{2}\left\{:\pi_\varkappa(x)^2: + :\big(\nabla^\#\varphi_\varkappa(x)\big)^2: + m^2 :\varphi_\varkappa(x)^2:\right\} + H_{I,\varkappa}(x) = H_{0,\varkappa}(x) + H_{I,\varkappa}(x).$$

Therefore $M^{0k}$ is formally positive. In fact it is technically convenient to use different spatial cutoffs in the free and the interaction part of $M^{0k}$, $k = 1,2,3$. Final formulas for $M_\varkappa^{0k}$ reads

$$M_\varkappa^{0k} = M_\varkappa^{0k}\big(g_0^k, g^k\big) = \alpha H_{0,\varkappa} + H_{0,\varkappa}(x^k g_0^k) + H_{I,\varkappa}(x^k g). \tag{126}$$

Here $0 < \alpha$ and $0 \leq x^k g_0^k(x), 0 \leq x^k g(x), k = 1,2,3$ and in order that (126) be formally correct, we assume that: $\alpha + x^k g_0^k = x^k = x^k g$ on $[1,R]^3 = [1,R] \times [1,R] \times [1,R]$ with $R$ sufficiently large. For technical reasons we assume that: $\alpha + x^k g_0^k(x) = x^k, k = 1,2,3$ on $\operatorname{supp}(g)$. By above restrictions on $g_0^k$ and $g^k$ we have that $\operatorname{supp}(g_0^k), \operatorname{supp}(g) \subset \{x | \alpha \leq x^k, k = 1,2,3\}$ and we show that the operator $M_\varkappa^{0k}$ is essentially self- #-adjoint and it generates Lorentz rotations in an algebra $\mathcal{B}_\#(O_1)$

$$Ext\text{-}\exp(i\beta M_\varkappa^{0k})\mathcal{B}_\#(O_1) Ext\text{-}\exp(-i\beta M_\varkappa^{0k}) \subset \mathcal{B}_\#(\{a,\Lambda(\beta)\}O_1) \tag{127}$$

provided that $O_1$ and $\{a,\Lambda(\beta)\}O_1$ are contained in the region

$$\{x \in {}^*\mathbb{R}_c^{\#3}, t \in {}^*\mathbb{R}_c^\# |\, |t| + 1 < x_k < R - |t|, k = 1,2,3\}, \tag{128}$$

where $M^{0k}$ is formally correct. These results permit us to define the Lorentz rotation automorphism $\sigma(\Lambda)$ on an arbitrary local algebra $\mathcal{B}_{\#}(O)$. Using a space time translation $\sigma(a), a \in {}^*\mathbb{R}_c^{\#4}$ we can translate $O$ into a region $O + a = O_1 \subset \{x \in {}^*\mathbb{R}_c^{\#3}, t \in {}^*\mathbb{R}_c^{\#} | \, x_1 > |t| + 1\}$ and for $R \in {}^*\mathbb{R}_c^{\#}$ large enough, $O_1$ and $\{a, \Lambda(\beta)\}O_1$ are contained in the region (1) we define $\sigma(0, \Lambda(\beta)) = \sigma(\Lambda(\beta))$ by

$$\sigma(\Lambda(\beta)) \upharpoonright \mathcal{B}_{\#}(O) = \sigma(\{-\Lambda(\beta)a, I\})\sigma(\{0, \Lambda(\beta)\})\sigma(\{a, I\}) \upharpoonright \mathcal{B}_{\#}(O).$$

**Theorem 11.23** Let $M^{0k}(g_0, g), k = 1,2,3$ be given by (126), with $\alpha, g_0(x), g(x)$ restricted as mentioned above. Then $M^{0k}(g_0, g)$ is essentially self #-adjoint on $C^{*\infty}(H \cap H_0)$.

**Theorem 11.24** Let $O_1$ and $\{0, \Lambda(\beta)\}O_1$ be contained in the set (1). Then the following identity holds between self- #-adjoint operators:

$$Ext\text{-}\exp(i\beta M^{0k})\varphi_{\varkappa}^{\#}(f)Ext\text{-}\exp(i\beta M^{0k}) \approx \varphi_{\varkappa}^{\#}(f(\{0, \Lambda(\beta)\}x)) = \int_{{}^*\mathbb{R}_c^{\#4}} \varphi_{\varkappa}^{\#}\left(f(\{0, \Lambda(\beta)\}(x, t))\right) d^{\#3}x d^{\#}t. \quad (129)$$

Here provided $\operatorname{supp}(f) \subset O_1$.

The proof of the Theorem 11.24 is reduced to the verification of the following equations

$$\left\{x_k \frac{\partial^{\#}}{\partial^{\#} t} + t \frac{\partial^{\#}}{\partial^{\#} x_k}\right\} \varphi_{\varkappa}^{\#}(x, t) = [iM^{0k}, \varphi_{\varkappa}^{\#}(x, t)], k = 1,2,3. \quad (130)$$

Here (130) that is equation for bilinear forms on an appropriate domain. Since $M^{0k}$ is self #-adjoint, we can integrate (130), thus we compute formally for $H = H_{0,\varkappa} + H_{I,\varkappa}(g)$,

$$[iM^{0k}, \varphi_{\varkappa}^{\#}(x, t)] = [iM^{0k}, Ext\text{-}\exp(itH)\varphi_{\varkappa}^{\#}(x, t)Ext\text{-}\exp(-itH)] =$$
$$Ext\text{-}\exp(itH)[iM^{0k}(-t), \varphi_{\varkappa}^{\#}(x, 0)]Ext\text{-}\exp(-itH). \quad (131)$$

Here $M^{0k}(-t) = Ext\text{-}\exp(-itH)M^{0k}Ext\text{-}\exp(itH)$. Formally one obtains that

$$M^{0k}(-t) = Ext\text{-}\sum_{n=0}^{*\infty} \frac{(-t)^n}{n!} ad^n(iH)(M^{0k}), k = 1,2,3.$$

Note that if $M^{0k}$ and $H$ were the correct global Lorentzian generators and Hamiltonian they would satisfy

$$[iH, M^{0k}] = ad\,(iH)(M^{0k}) = P^k, [iH, [iH, M^{0k}]] = 0, M^{0k}(-t) = M^{0k} - P^k t. \quad (132)$$

Here $P^k, k = 1,2,3$ are the generators of space translations. Thus from (131) we get

$$[\, iM^{0k}, \varphi_{\varkappa}^{\#}(x, 0)] = [iM_0^{0k}] = x\pi_{\varkappa}^{\#}(x, 0), [iP^k, \varphi_{\varkappa}^{\#}(x, 0)] = -\nabla^{\#}(\varphi_{\varkappa}^{\#})(x, 0).$$

Formally we have (130).However the difficulty with this formal argument is that $H$ and $M^{0k}$ do not obey (132) exactly, since they are correct only in $O_1$. We have instead (132) the equations

$$[iH, M^{0k}] = P_{loc}^k, [iH, [iH, M^{0k}]] = R_k^{loc}, k = 1,2,3. \quad (133)$$

Here $P_{loc}^k$ acts like the momentum operators only in the region $O_1$, i.e.

$$[P_{loc}^k, \varphi_{\varkappa}^{\#}(x, t)] = [P^k, \varphi_{\varkappa}^{\#}(x, t)], (x, t) \in O_1.$$

Hence $[iH, P_{loc}^k] = R_k^{loc}, k = 1,2,3$ is not identically zero, but commutes with $\mathcal{B}_{\#}(O_1)$. Formally, further commutators of $R_k^{loc}, k = 1,2,3$ with $H$ are localized outside region $O_1$, and (130) follows formally even for our approximate, but locally correct $H$ and $M^{0k}$. In order to convert this formal argument into a rigorous mathematical result, we apply now generalized Taylor series expansion [12] for the quantities

$$E_k(-t) = \langle \Omega, [\, iM^{0k}(-t), \varphi_{\varkappa}^{\#}(x, 0)]\Omega \rangle, k = 1,2,3. \quad (134)$$

Here $\Omega \in C^{*\infty}(H)$ and thus we obtain

$$E_k(-t) = E_k(0) - t\frac{d^{\#}E_k(0)}{d^{\#}t} + \frac{t^2}{2}\frac{d^{\#2}E_k(\xi)}{d^{\#}t^2}, \text{ where } \xi \in [-t, t].$$

From (133) we obtain

$$\frac{d^{\#2}E_k(-\xi)}{d^{\#}t^2} = \langle Ext\text{-}\exp(i\xi H)\Omega, [iR_k^{loc}, \varphi_{\varkappa}^{\#}(x, \xi)]Ext\text{-}\exp(i\xi H)\Omega \rangle.$$

Note that $(x, t) \in O_1$, so that with $\xi \in [-t, t]$, $(x, \xi) \in O_1$ and therefore

$$[R_k^{loc}, \varphi_{\varkappa}^{\#}(x, \xi)] \equiv 0. \quad (135)$$

After integration over $x \in {}^*\mathbb{R}_c^{\#3}$ with a function $f \in S_{\mathrm{fin}}^{\#}({}^*\mathbb{R}_c^{\#3})$ we obtain the operator identity:

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} [R_k^{loc}, \varphi_{\varkappa}^{\#}(x, \xi)] f(x) d^{\#3}x \equiv 0, k = 1,2,3. \quad (136)$$

Therefore $\frac{d^{\#2}E_k(\xi)}{d^{\#}t^2} \equiv 0$ if $|\xi| \leq |t|$ and

$$E_k(-t) = E_k(0) - t\frac{d^{\#}E_k(0)}{d^{\#}t} = \langle \Omega, \{[\, iM^{0k}, \varphi_{\varkappa}^{\#}(x, 0)] - t[P_{loc}^k, \varphi_{\varkappa}^{\#}(x, 0)]\}\Omega \rangle =$$
$$= \langle \Omega, \{x\pi_{\varkappa}^{\#}(x, 0) + t\nabla^{\#}(\varphi_{\varkappa}^{\#})(x, 0)\}\, \Omega \,\rangle.$$

Thus we get

$$[\, iM^{0k}(-t), \varphi_{\varkappa}^{\#}(x, 0)] = x\pi_{\varkappa}^{\#}(x, 0) + t\nabla^{\#}\varphi_{\varkappa}^{\#}(x, 0) \quad (137)$$

Inserting the relation (137) in (131) finally we obtain (130).This completes the proof of Lorentz covariance.

**Definition 11.14** For the local free field energy we set $T_0(g) = T_0^1(g) + T_0^2(g)$, where

$$T_0^1(g) = c_1 Ext\text{-}\int_{|\boldsymbol{k}_1|\leq\varkappa} d^{\#3}\,\boldsymbol{k}_1 Ext\text{-}\int_{|\boldsymbol{k}_2|\leq\varkappa} d^{\#3}\,\boldsymbol{k}_2 \hat{g}(k_1^1-k_2^1,k_1^2-k_2^3,k_1^3-k_2^3)\left\{\frac{\mu(\boldsymbol{k}_1)\mu(\boldsymbol{k}_2)+\langle\boldsymbol{k}_1,\boldsymbol{k}_2\rangle+m^2}{\sqrt{\mu(\boldsymbol{k}_1)\mu(\boldsymbol{k}_2)}}\right\}\times a^{\dagger}(\boldsymbol{k}_1)a(\boldsymbol{k}_2), \quad (138)$$

$$T_0^2(g) = c_2 Ext\text{-}\int_{|\boldsymbol{k}_1|\leq\varkappa} d^{\#3}\,\boldsymbol{k}_1 Ext\text{-}\int_{|\boldsymbol{k}_2|\leq\varkappa} d^{\#3}\,\boldsymbol{k}_2 \hat{g}(k_1^1-k_2^1,k_1^2-k_2^3,k_1^3-k_2^3)\left\{\frac{-\mu(\boldsymbol{k}_1)\mu(\boldsymbol{k}_2)+\langle\boldsymbol{k}_1,\boldsymbol{k}_2\rangle+m^2}{\sqrt{\mu(\boldsymbol{k}_1)\mu(\boldsymbol{k}_2)}}\right\}\times \times\{a^{\dagger}(\boldsymbol{k}_1)a^{\dagger}(-\boldsymbol{k}_2)+a(-\boldsymbol{k}_1)a(\boldsymbol{k}_2)\}. \quad (139)$$

Here $\boldsymbol{k}_1 = (k_1^1,k_1^2,k_1^3)$, $\boldsymbol{k}_2 = (k_2^1,k_2^2,k_2^3)$, $\langle\boldsymbol{k}_1,\boldsymbol{k}_2\rangle = \sum_{i=1}^3 k_1^i\,k_2^i$, $\hat{g}(\boldsymbol{p}) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}(Ext\text{-}[i\langle\boldsymbol{p},\boldsymbol{x}\rangle])g(x)\,d^{\#3}\boldsymbol{x}$.

Similarly, for the local momentum we set $P^i\,(g) = P^{i1}(g) + P^{i2}(g)$, $i = 1,2,3$ where

$$P^{i1}(g) = c_1 Ext\text{-}\int_{|\boldsymbol{k}_1|\leq\varkappa} d^{\#3}\,\boldsymbol{k}_1 Ext\text{-}\int_{|\boldsymbol{k}_2|\leq\varkappa} d^{\#3}\,\boldsymbol{k}_2 \hat{g}(k_1^1-k_2^1,k_1^2-k_2^3,k_1^3-k_2^3)\times \times\left\{\frac{(k_1^1+k_1^2+k_1^3)\mu(\boldsymbol{k}_2)+(k_2^1+k_2^2+k_2^3)\mu(\boldsymbol{k}_1)}{\sqrt{\mu(\boldsymbol{k}_1)\mu(\boldsymbol{k}_2)}}\right\}a^{\dagger}(\boldsymbol{k}_1)a(\boldsymbol{k}_2), \quad (140)$$

$$P^2(g) = c_2 Ext\text{-}\int_{|\boldsymbol{k}_1|\leq\varkappa} d^{\#3}\,\boldsymbol{k}_1 Ext\text{-}\int_{|\boldsymbol{k}_2|\leq\varkappa} d^{\#3}\,\boldsymbol{k}_2 \hat{g}(k_1^1-k_2^1,k_1^2-k_2^3,k_1^3-k_2^3)\times \times\left\{\frac{(k_1^1+k_1^2+k_1^3)\mu(\boldsymbol{k}_2)-(k_2^1+k_2^2+k_2^3)\mu(\boldsymbol{k}_1)}{\sqrt{\mu(\boldsymbol{k}_1)\mu(\boldsymbol{k}_2)}}\right\}\{-a^{\dagger}(\boldsymbol{k}_1)a^{\dagger}(-\boldsymbol{k}_2)+a(-\boldsymbol{k}_1)a(\boldsymbol{k}_2)\}. \quad (141)$$

**Definition 11.15** Let $\breve{P}_{\varkappa}(f)$ be the local operator, defined for $f\in S_{\mathrm{fin}}^{\#}({}^*\mathbb{R}_c^{\#3})$ by

$$\breve{P}_{\varkappa}(f) = H_{0\varkappa,}(f) - m^2\int_{{}^*\mathbb{R}_c^{\#3}} :\varphi_{\varkappa}^{\#2}(x): f(x)d^{\#3}x \quad (142)$$

**Theorem 11.25** Let the operators $M^{0k}, k = 1,2,3$ are given by $M^{0k} = \alpha H_0 + T_0\big(x_k g_0^{(k)}\big) + T_I\big(x_k g_1^{(k)}\big)$, $H \triangleq H_{0,\varkappa} +$ where $H_0 \triangleq H_{0,\varkappa}$ and $T_I \triangleq H_{I,\varkappa}$. Then the following statements hold.

(1) For $k = 1,2,3$, $D((M^{0k})^2) \subset D(H\,)$, $D(H^2) \subset D(M^{0k})$.

(2) For $k = 1,2,3$, $D(M^{0k}) \subset D\left((H\,+b)^{\frac{1}{2}}\right)$, $D(H\,) \subset D\left((M^{0k}\,+b)^{\frac{1}{2}}\right)$.

**Theorem 11.26** Let the operators $M^{0k}, k = 1,2,3$ are given by $M^{0k} = \alpha H_0 + T_0\big(x_k g_0^{(k)}\big) + T_I\big(x_k g_1^{(k)}\big)$, where $H_0 \triangleq H_{0,\varkappa}$ and $T_I \triangleq H_{I,\varkappa}$. Then the following statements hold.

(1) For $l = 2,3,4$, $M\colon D(H^l) \to D(H^{l-2})$.

(2) As operator equalities on $D(H^3)$ for $k = 1,2,3$,

$$[iH,M^{0k}] = P\left(\frac{d^{\#}\big(x_k g_0^{(k)}\big)}{d^{\#}x_k}\right). \quad (143)$$

(3) As operator equalities on $D(H^4)$, for $k = 1,2,3$,

$$\big[iH,[iH,M^{0k}]\big] = \breve{P}_{\varkappa}\left(\sum_{i=1}^{i=3}\frac{d^{\#2}\big(x_k g_0^{(k)}\big)}{d^{\#}x_i^2}\right) - T_I\left(\sum_{i=1}^{i=3}\frac{d^{\#}g_1^{(k)}}{d^{\#}x_i}\right). \quad (144)$$

(4) For $l = 2,3,4$, $H\colon D((M^{0k})^l) \to D((M^{0k})^{l-2})$.

The equalities (143) hold on the domain $D((M^{0k})^3)$, and on the domain $D((M^{0k})^4)$, for $k = 1,2,3$,

$$\big[iM^{0k},[iM^{0k},H]\big] = T_0\left(\left(\frac{d^{\#}}{d^{\#}x_k}\big(x_k g_0^{(k)}\big)\right)^2\right) + T_I\left(\left(\frac{d^{\#}}{d^{\#}x_k}\big(x_k g_1^{(k)}\big)\right)\right) - \breve{P}_{\varkappa}\left(\big(\alpha + x_k g_0^{(k)}\big)\frac{d^{\#2}}{d^{\#}x_k^2}\big(x_k g_0^{(k)}\big)\right) \quad (145)$$

**Theorem 11.27** As bilinear forms on $D(H_0)\times D(H_0)$ for $f,g\in S_{\mathrm{fin}}^{\#}({}^*\mathbb{R}_c^{\#3})$

$$[iT_0(f),T_0(g)] = P\left(f\left(\sum_{i=1}^{i=3}\frac{d^{\#}g}{d^{\#}x_i}\right) - g\left(\sum_{i=1}^{i=3}\frac{d^{\#}f}{d^{\#}x_i}\right)\right), \quad (146)$$

$$[iT_0(f),P(g)] = \breve{P}\left(f\left(\sum_{i=1}^{i=3}\frac{d^{\#}g}{d^{\#}x_i}\right)\right) - T_0\left(g\left(\sum_{i=1}^{i=3}\frac{d^{\#}f}{d^{\#}x_i}\right)\right). \quad (147)$$

The equalities (146)-(147) also hold if $f = 1$ or $g = 1$. In particular from (147) we get

$$[iH_0(f),P(g)] = \breve{P}\left(\sum_{i=1}^{i=3}\frac{d^{\#}g}{d^{\#}x_i}\right). \quad (148)$$

**Proof.** The operators $T_0, P, \breve{P}$ are #-closable (symmetric), defined on $D(H_0)$ and bounded as operators relative to $H_0 + I$. Therefore (146)-(147) are defined as bilinear forms on $D(H_0)\times D(H_0)$ and it suffices to establish equality on a core for $H_0$, e.g. on $D^{\#} = \big\{\psi\in\mathcal{F}^{\#}\,|\,\psi^{(n)}\in S_{\mathrm{fin}}^{\#}({}^*\mathbb{R}_c^{\#3n}), \psi^{(m)} = 0 \text{ for all sufficiently large m}\big\}$. By direct calculations on $D^{\#}\times D^{\#}$ one obtains the equalities (146)-(147). For example

$$[iH_0,T_0^1(g)] = c_1 Ext\text{-}\int_{|\boldsymbol{k}_1|\leq\varkappa} d^{\#3}\,\boldsymbol{k}\,Ext\text{-}\int_{|\boldsymbol{k}_2|\leq\varkappa} d^{\#3}\,\boldsymbol{p}\hat{g}(k_1-p_1,k_2-2,k_3-p_3)\left\{\frac{\mu(\boldsymbol{k})\mu(\boldsymbol{p})+\langle\boldsymbol{k},\boldsymbol{p}\rangle+m^2}{\sqrt{\mu(\boldsymbol{k})\mu(\boldsymbol{p})}}\right\}\times [H_0,a^{\dagger}(\boldsymbol{k})a(\boldsymbol{p})] = \quad (149)$$

$$ic_1 Ext\text{-}\int_{|k_1|\le\varkappa} d^{\#3}\,\boldsymbol{k}\, Ext\text{-}\int_{|k_2|\le\varkappa} d^{\#3}\,\boldsymbol{p}\hat{g}(k_1-p_1,k_2-2,k_3-p_3)\big(\mu(\boldsymbol{k})-\mu(\boldsymbol{p})\big)\left\{\frac{\mu(\boldsymbol{k})\mu(\boldsymbol{p})+\langle\boldsymbol{k},\boldsymbol{p}\rangle+m^2}{\sqrt{\mu(\boldsymbol{k})\mu(\boldsymbol{p})}}\right\}a^{\dagger}(\boldsymbol{k})a(\boldsymbol{p})$$

$$=c_1 Ext\text{-}\int_{|k_1|\le\varkappa} d^{\#3}\,\boldsymbol{k}\, Ext\text{-}\int_{|k_2|\le\varkappa} d^{\#3}\,\boldsymbol{p}i\left(\sum_{i=1}^{i=3}(k_i-p_i)\right)\hat{g}(k_1-p_1,k_2-2,k_3-p_3)\left\{\frac{(k_1+k_2+k_3)\mu(\boldsymbol{p})+(p_1+p_2+p_3)\mu(\boldsymbol{k})}{\sqrt{\mu(\boldsymbol{k})\mu(\boldsymbol{p})}}\right\}$$

$$=P^{(1)}\left(\left(\sum_{i=1}^{i=3}\frac{d^{\#}g}{d^{\#}x_i}\right)\right)$$

By a similar calculation on $D^{\#}\times D^{\#}$ one obtains

$$\left[iT_0^{(1)}(f),T_0^{(1)}(g)\right]+\left[iT_0^{(2)}(f),T_0^{(2)}(g)\right]=P^{(1)}\left(f\left(\sum_{i=1}^{i=3}\frac{d^{\#}g}{d^{\#}x_i}\right)-g\left(\sum_{i=1}^{i=3}\frac{d^{\#}f}{d^{\#}x_i}\right)\right).$$

**Theorem 11.28** As bilinear forms on $D(H_{0,\varkappa}N_{\varkappa})\times D(H_{0,\varkappa}N_{\varkappa})$

$$[iT_I(h),T_0(f)]=-4\lambda Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} f(x)\,h(x){:}\,\varphi_{\varkappa}^{\#3}(x)\pi_{\varkappa}^{\#}(x){:}\,d^{\#3}x, \tag{150}$$

$$[iT_I(h),P(f)]=-T_I\left(\sum_{i=1}^{i=3}\frac{d^{\#}(fh)}{d^{\#}x_i}\right). \tag{151}$$

**Proof.** The operators $T_0,T_I,P$ are #-closable, defined on $D(H_{0,\varkappa}N_{\varkappa})$, and are bounded as operators relative to $(H_{0,\varkappa}N_{\varkappa}+I)$. Note that the right hand side of (150) is a bilinear form on $D(H_{0,\varkappa}N_{\varkappa})\times D(H_{0,\varkappa}N_{\varkappa})$, and that $(H_{0,\varkappa}N_{\varkappa}+I)^{-1}\left[Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} f(x)\,h(x){:}\,\varphi_{\varkappa}^{\#3}(x)\pi_{\varkappa}^{\#}(x){:}\,d^{\#3}x\right](H_{0,\varkappa}N_{\varkappa}+I)^{-1}$ is a bounded operator. Hence each term in (150)-(151) is a bilinear form on $D(H_{0,\varkappa}N_{\varkappa})\times D(H_{0,\varkappa}N_{\varkappa})$. It suffices to establish equality on $D^{\#}\times D^{\#}$, as in the proof of the **Theorem 84**, since $D^{\#}$ is a #-core for $H_{0,\varkappa}N_{\varkappa}$. Note that on the domain $D^{\#}\times D^{\#}$, the equalities (150)-(151) are seen to hold by direct computation in momentum space similarly to proof of the Theorem 11.27.

**Remark 11.9** We assume now the relations:

$$0<\alpha,\, x_k g_i^{(k)}(x_1,x_2,x_3)=\left[h_i^{(k)}(x_1,x_2,x_3)\right]^2, k=1,2,3; i=0,1; h_i^{(k)}\in S_{\text{fin}}^{\#}({}^*\mathbb{R}_c^{\#3}). \tag{152}$$

On a neighbourhood of a polyhedron $[a,b]^3\subset{}^*\mathbb{R}_c^{\#}$, we assume for $k=1,2,3$

$$\alpha+x_k g_0^{(k)}(x_1,x_2,x_3)=x_k=x_k g_1(x_1,x_2,x_3). \tag{153}$$

For all $x_k\in{}^*\mathbb{R}_c^{\#3}, k=1,2,3$, we assume

$$x_k g_1(x_1,x_2,x_3)=\left(\alpha+x_k g_0^{(k)}(x_1,x_2,x_3)\right)g_1(x_1,x_2,x_3). \tag{154}$$

The conditions (154) are satisfied if $\alpha+x_k g_0^{(k)}(x_1,x_2,x_3)=x_k$ is valid on the support of $g_1$ for $k=1,2,3$. The condition (154) makes the required commutators densely defined operators, rather than bilinear forms.

**Definition 11.16** Let $\mathfrak{R}_{[a,b]}^4$ be a set

$$\mathfrak{R}_{[a,b]}^4=\{(x_1,x_2,x_3,t)\in{}^*\mathbb{R}_c^{\#4}\,|\,a+|t|<x_k<b-|t| \text{ for all } k=1,2,3\}. \tag{155}$$

**Remark 11.10** Note that the operators $M^{0k}$, $k=1,2,3$ are formally a Lorentz generators for the space-time region $\mathfrak{R}_{[a,b]}^4$, also note that (152) implies that interval $I=[a,b]$ lies in the positive half line. Of course, we can also consider the operators $\widetilde{M}^{0k}=-\alpha H_0+T_0\left(x_k\tilde{g}_0^{(k)}\right)+T_I\left(x_k\tilde{g}_1^{(k)}\right)$ with $\tilde{g}_i^{(k)}(x)=g_i^{(k)}(-x)$ and therefore the operators $\widetilde{M}^{0k}$, $k=1,2,3$ are locally correct generators for $\widetilde{\mathfrak{R}}_{[a,b]}^4=\mathfrak{R}_{[-a,-b]}^4$.

**Definition 11.17** We also write $\mathfrak{R}_I^4$ instead $\mathfrak{R}_{[a,b]}^4$ for $I=[a,b]$ and we write $I_s^3$ for $I^3=[a-s,b+s]^3$. The conditions (152)-(1544) are satisfied since we can choose $g_i^{(k)}$ so that for some $\varepsilon, 0<\varepsilon<a/3$,

$$\operatorname{supp}g_1\subset I_{2\varepsilon}^3;\operatorname{supp}g_0^{(k)}\subset I_{3\varepsilon}^3,\ k=1,2,3 \tag{156}$$

and $\alpha+x_k g_0^{(k)}(x_1,x_2,x_3)=x_k$, $x_k\in I_{2\varepsilon}^3$. Hence the conditions (154) hold. We can also let$g_1=1, x_k\in I_{\varepsilon}^3$; so the conditions (153) hold on $I_{\varepsilon}^3$. The Hamiltonian

$$H=H_{0\varkappa}+T_I(g_1) \tag{157}$$

is correct in the region $\mathfrak{R}_I^4$. We shall work as above with this particular choice of the Hamiltonian.

**Theorem 11.29** For the operators $M^{0k}$ in Theorem 11.25 and $H$ in (157) the following hold:

(1) $D((M^{0k})^2)\subset D(H), D(H^2)\subset D(M^{0k}), k=1,2,3$

(2) $D(M^{0k})\subset D\left((H+b)^{\frac{1}{2}}\right), D(H)\subset D\left((M^{0k}+b)^{\frac{1}{2}}\right) k=1,2,3$

where $b$ is an constant sufficiently large so that the operators $H+b$ and $M^{0k}+b$ are positive.

T**heorem 11.30** Ander the conditions (152) and (154) the equalities (143)-(145) hold as bilinear forms on $D(H^2)\times D(H^2)$ and on $D((M^{0k})^2)\times D((M^{0k})^2)$.

**Proof.** As bilinear forms on $D(H^2)\times D(H^2)$ or $D((M^{0k})^2)\times D((M^{0k})^2)$ for $k=1,2,3$ the following equalities hold $[iH,M^{0k}]=\left[iH_0,T_0\left(x_k g_0^{(k)}\right)\right]+\left\{[iH_0,T_I(x_k g_1)]+[iT_I(g_1),\alpha H_0]+\left[iT_I(g_1),\,T_0\left(x_k g_0^{(k)}\right)\right]\right\}$. In order to compute these commutators we apply Theorem 11.27 and Theorem 11.28.

$$[iH,M^{0k}]=P\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_k}\right)+4\lambda Ext\text{-}\int_{^*\mathbb{R}_c^{\#3}}\left\{x_k g_1(x)-\alpha g_1(x)-x_k g_1(x)g_0^{(k)}(x)\right\}:\varphi_\varkappa^{\#3}(x)\pi_\varkappa^{\#}(x):d^{\#3}x=$$
$$P\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_k}\right),$$

This equality holds by the conditions (154). Hence the equality (143) holds on $D(H^2)\times D(H^2)$ and on the domain $D((M^{0k})^2)\times D((M^{0k})^2)$.

**Theorem 11.31** If n≥2, $D(H^n)$ is a #-core for $M$ and $D((M^{0k})^n)$ is a #-core for $H$.

**Theorem 11.32** Let $f\in S_{\text{fin}}^{\#}(^*\mathbb{R}_c^{\#3})$ and supp$f\subset\mathfrak{R}_{[a,b]}^4$, then the operator $\varphi^{\#}(f)$ is defined on $D((M^{0k})^2)$, $\varphi^{\#}(f):((M^{0k})^2)\to D(M^{0k}),k=1,2,3$ and, as the operator equalities on $D(M^{0k}),k=1,2,3$

$$[iM^{0k},\varphi_\varkappa^{\#}(f)]=-\varphi_\varkappa^{\#}\left(t\frac{\partial^{\#}f}{\partial^{\#}x_k}+x_k\frac{\partial^{\#}f}{\partial^{\#}t}\right).\tag{158}$$

**Remark 11.11** Note that for $f$ real, the operator $\varphi_\varkappa^{\#}(f)$ is essentially self #-adjoint on $D(H^n)$ for any $n\geq 1/2$ and

$$\varphi_\varkappa^{\#}(f):D((H+b)^n)\to D\left((H+b)^{n-\frac{1}{2}}\right).\tag{159}$$

**Proof** The terms in (158) are operators on $D(H^3)$ since $\varphi_\varkappa^{\#}(f)D(H^3)\subset D(H^2)\subset D(M^{0k}),k=1,2,3$ and $M^{0k}D(H^3)\subset D(H)\subset D\left(\varphi_\varkappa^{\#}(f)\right)$ by (157) and Theorem 11.26**.** Note that by Theorem 11.40 (158) holds on the domain $D(H^5)$. Assuming this, we now can to prove the theorem. Let $\psi\in D((M^{0k})^2),k=1,2,3$. By Theorem 11.29, $D((M^{0k})^2)\subset D(H)$ and by (159) we get $\psi\in D\left(\varphi_\varkappa^{\#}(f)\right)$. Let us prove now that

$$\varphi_\varkappa^{\#}(f)\psi\in D(M^{0k}),k=1,2,3.\tag{160}$$

Note that $M^{0k}\psi\in D(M^{0k})\subset D\left((H+b)^{\frac{1}{2}}\right)\subset D\left(\varphi_\varkappa^{\#}(f)\right)$ by Theorem 11.29 and (159), also for $k=1,2,3$

$$\psi\in D\left(\varphi_\varkappa^{\#}\left(t\frac{\partial^{\#}f}{\partial^{\#}x_k}+x_k\frac{\partial^{\#}f}{\partial^{\#}t}\right)\right).$$

Therefore by the assumption mentioned above that (158) holds on domain $D(H^5)$, we get for all $k=1,2,3$ and for all $\chi\in D(H^5)$ that

$$\langle M^{0k}\chi,\varphi_\varkappa^{\#}(f)M^{0k}\psi\rangle=\langle\chi,\varphi_\varkappa^{\#}(f)M^{0k}\psi\rangle+i\langle\chi,\varphi_\varkappa^{\#}\left(t\frac{\partial^{\#}f}{\partial^{\#}x_k}+x_k\frac{\partial^{\#}f}{\partial^{\#}t}\right)\psi\rangle.\tag{161}$$

So $\varphi_\varkappa^{\#}(f)\psi\in D\left(\left(M^{0k}\upharpoonright D(H^5)\right)^*\right)$ for $k=1,2,3$. By Theorem 11.31, $D(H^5)$ is a #-core for the $M^{0k}$, $k=1,2,3$ and therefore we get inclusion (160). By using (160) we can rewrite (161) in the following equivalent form

$$\langle\chi,[M^{0k},\varphi_\varkappa^{\#}(f)]\psi\rangle=\langle\chi,i\varphi_\varkappa^{\#}\left(t\frac{\partial^{\#}f}{\partial^{\#}x_k}+x_k\frac{\partial^{\#}f}{\partial^{\#}t}\right)\psi\rangle.\tag{162}$$

Since $D(H^5)$ is #-dense, we get$[M^{0k},\varphi_\varkappa^{\#}(f)]\psi=i\varphi_\varkappa^{\#}\left(t\frac{\partial^{\#}f}{\partial^{\#}x_k}+x_k\frac{\partial^{\#}f}{\partial^{\#}t}\right)\psi$, proving (158) on the stated domain $D(M^{0k}),k=1,2,3$.

**Remark 11.12** Let us consider the self #-adjoint operators $M^{0k}(t)=Ext\text{-}\exp(-itH)M^{0k}Ext\text{-}\exp(itH),k=1,2,3$. Since the operator $Ext\text{-}\exp(itH)$ leaves $D(H^n)$ invariant, we have by Theorem 11.29 and Theorem 11.26 that $D(H^2)\subset D\left(M^{0k}(t)\right)$, $k=1,2,3$. And for $l=2,3,4$ we have that

$$M^{0k}(t):D(H^l)\to D(H^{l-2}),k=1,2,3.\tag{163}$$

Let $f\in S_{\text{fin}}^{\#}(^*\mathbb{R}_c^{\#4})$ with supp$f\subset\mathfrak{R}_I^4$ for $I=[a,b]$. By (159) and (160) we can to conclude that $\varphi^{\#}(f)D(H^3)\subset D(H^2)\subset D\left(M^{0k}(t)\right),k=1,2,3$ and $M^{0k}(t)D(H^3)\subset D(H)\subset D(\varphi_\varkappa^{\#}(f))$ or more generally, we can replace the operator $\varphi_\varkappa^{\#}(f)$ by $Ext\text{-}\exp(itH)\varphi_\varkappa^{\#}(f)Ext\text{-}\exp(-itH)$. Thus for $\psi\in D(H^3)$ and $f\in S_{\text{fin}}^{\#}(^*\mathbb{R}_c^{\#4})$ with supp$f\subset\mathfrak{R}_\Delta^4$, we can to define the functions

$$F_k(t)=\langle\psi,[iM^{0k}(t),\varphi_\varkappa^{\#}(f)]\psi\rangle=\langle\psi(t),[iM^{0k},Ext\text{-}\exp(itH)\varphi_\varkappa^{\#}(f)Ext\text{-}\exp(-itH)]\psi(t)\rangle,\tag{164}$$

$$\psi(t)=Ext\text{-}\exp(itH)\psi.\tag{165}$$

Let $I=[a,b]$, $I_\delta=[a-\delta,b+\delta]$ and let $\mathfrak{R}_{\Delta_\delta}$ be the causal shadow of $\Delta_\delta=I_\delta\times I_\delta\times I_\delta$. Let $\mathfrak{R}_s^4$ be a set

$$\mathfrak{R}_s^4=\mathfrak{R}_{\Delta_{-|s|}}\cap\left\{(x,t)||t|<\tfrac{1}{2}\varepsilon\right\}=\left\{(x,t)||t|<\tfrac{1}{2}\varepsilon,a+|s|+|t|<b-|s|-|t|\right\}.\tag{166}$$

Note that the points of $\mathfrak{R}_s^4$ have small times, and $\mathfrak{R}_s^4$ translated by times less than $|s|$ lies in $\mathfrak{R}_\Delta^4$.

**Theorem 11.33** Let $\psi \in D(H^5)$, then $F_k(t), k = 1,2,3$ in (161) is twice #-continuously differentiable. If function $f$ has #-compact support in $\Re_s$, then for $|t| \leq |s|$, $\frac{d^{\#2}F_k(t)}{d^{\#}t^2} \equiv 0$.

**Proof** First we prove the differentiability of $F_k(t), k = 1,2,3$. Let $\Delta_n$ be the difference quotient for the $n$-derivative of $Ext$-$\exp(itH)$ at $t = 0$. For instance, $\Delta_1(\varepsilon) = \varepsilon^{-1}(Ext\text{-}\exp(i\varepsilon H) - I)$. Note that for a given vector $\psi \in D(H^n)$, and $m + j \leq n$, as $\varepsilon \to_{\#} 0$, we get $\left\|H^m\{\Delta_j(\varepsilon) - (iH)^j\}\psi\right\|_{\#} = \left\|\{\Delta_j(\varepsilon) - (iH)^j\}H^m\psi\right\|_{\#} \to_{\#} 0$. Hence, for $\psi \in D(H^n)$, the operator valued functions $M^{0k}(Ext\text{-}\exp(itH))$ is $n-2$ times #-differentiable, since for $j \leq n-2$ we get $\left\|M^{0k}(Ext\text{-}\exp(itH))\{\Delta_j(\varepsilon) - (iH)^j\}\psi\right\|_{\#} \leq \left\|\{\Delta_j(\varepsilon) - (iH)^j\}(H+b)^2\psi\right\|_{\#} \to_{\#} 0$. All these functions $F_k(t)$ has the following form

$$F_k(t) = i\langle M^{0k}(Ext\text{-}\exp(itH))\psi, Ext\text{-}\exp(itH)\varphi_{\varkappa}^{\#}(f)\psi\rangle - i\langle Ext\text{-}\exp(itH)\varphi_{\varkappa}^{\#}(f)\psi, M^{0k}(Ext\text{-}\exp(itH))\psi\rangle.$$

For a given vector $\psi \in D(H^5)$, $\varphi_{\varkappa}^{\#}(f)\psi \in D(H^4)$ and $F_k(t)$ is three times #-continuously #-differentiable. Note that

$$\frac{d^{\#}F_k(t)}{d^{\#}t} = \langle M^{0k}H\psi(t), Ext\text{-}\exp(itH)\varphi_{\varkappa}^{\#}(f)\psi\rangle - \langle M^{0k}\psi(t), H(Ext\text{-}\exp(itH))\psi\rangle - \langle Ext\text{-}\exp(itH)\varphi_{\varkappa}^{\#}(f)\psi, HM^{0k}\psi(t)\rangle + \langle Ext\text{-}\exp(itH)\varphi_{\varkappa}^{\#}(f)\psi, M^{0k}H\psi(t)\rangle. \quad (167)$$

By rearranging the terms in (167) and using the domain relations of Theorem 11.26.1) we obtain by (143) that

$$\begin{aligned}\frac{d^{\#}F_k(t)}{d^{\#}t} &= \langle\psi, [H, M^{0k}(t)]\varphi_{\varkappa}^{\#}(f)\psi\rangle - \langle\varphi_{\varkappa}^{\#}(f)\psi, [H, M^{0k}(t)]\psi\rangle = \\ &-i\,\langle\psi, (Ext\text{-}\exp(-itH))P\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_k}\right)(Ext\text{-}\exp(itH))\varphi_{\varkappa}^{\#}(f)\psi\rangle + \\ &i\,\langle\varphi_{\varkappa}^{\#}(f)\psi, (Ext\text{-}\exp(-itH))P\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_k}\right)(Ext\text{-}\exp(itH))\psi\rangle.\end{aligned} \quad (168)$$

By #-differentiating (168) and writing $P_k$ for the operator $P\left(\frac{d^{\#}\left(x_k g_0^{(k)}\right)}{d^{\#}x_k}\right)$ we obtain

$$\begin{aligned}\frac{d^{\#2}F_k(t)}{d^{\#}t^2} &= -\langle\psi, (Ext\text{-}\exp(-itH))[H, P_k](Ext\text{-}\exp(itH))\psi\rangle + \\ &\langle\varphi_{\varkappa}^{\#}(f)\psi, (Ext\text{-}\exp(-itH))[H, P_k](Ext\text{-}\exp(itH))\psi\rangle = \\ &i\,\langle\psi(t), \left[\breve{P}\left(\frac{d^{\#2}\left(x_k g_0^{(k)}\right)}{d^{\#}x_k^2}\right) - T_I\left(\frac{d^{\#}(g_1)}{d^{\#}x_k}\right), (Ext\text{-}\exp(itH))\varphi_{\varkappa}^{\#}(f)(Ext\text{-}\exp(-itH))\psi\right]\rangle.\end{aligned} \quad (169)$$

Note that the all terms in (169) are well defined. For instance, $HP_k(Ext\text{-}\exp(itH))\varphi_{\varkappa}^{\#}(f)\psi$ is well defined since, for a given vector $\psi \in D(H^5)$, $(Ext\text{-}\exp(itH))\varphi_{\varkappa}^{\#}(f)\psi \in D(H^4)$, and by Theorem 11.26 for all $k = 1,2,3$ we obtain

$$P_k(Ext\text{-}\exp(itH))\varphi_{\varkappa}^{\#}(f)\psi = [iH, M^{0k}](Ext\text{-}\exp(itH))\varphi_{\varkappa}^{\#}(f)\psi.$$

Note that $HM^{0k}\left(D(H^4)\right) \subset D(H)$ and $M^{0k}H\left(D(H^4)\right) \subset D(H)$, so $HP_k(Ext\text{-}\exp(itH))\varphi_{\varkappa}^{\#}(f)\psi$ is well defined. Now, assuming that $\mathrm{supp}f \subset \Re_s^4$, $|t| \leq |s|$ we can to show that $\frac{d^{\#2}F_k(t)}{d^{\#}t^2} \equiv 0, k = 1,2,3$, this proof is based on the locality of the operators $S_k, k = 1,2,3$

$$S_k = \breve{P}_{\varkappa}\left(\sum_{i=1}^{i=3}\frac{d^{\#2}\left(x_k g_0^{(k)}\right)}{d^{\#}x_i^2}\right) - T_I\left(\sum_{i=1}^{i=3}\frac{d^{\#}g_1}{d^{\#}x_i}\right). \quad (170)$$

The operators $S_k$ are symmetric on $D(H_0N)$ and by (153) for $k = 1,2,3$ and $i = 1,2,3$ $\frac{d^{\#2}\left(x_k g_0^{(k)}\right)}{d^{\#}x_i^2} = 0 = \frac{d^{\#}g_1}{d^{\#}x_i}$ in a neighbourhood of $\Delta = [a,b]^3$. We prove that $S_k, k = 1,2,3$ commutes with the von Neumann algebra $W(I) = \left\{Ext\text{-}\exp\left(i\varphi_{\varkappa}^{\#}(h_1) + i\pi_{\varkappa}^{\#}(h_2)\right)|h_i = \overline{h_i} \in S_{\mathrm{fin}}^{\#}({}^*\mathbb{R}_c^{\#3}), \mathrm{supp}h_i \subset \Re_I\right\}''$ generated by the spectral projections of the time zero fields $Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\varphi_{\varkappa}^{\#}(x)\,h_1(x)d^{\#3}x$ and $Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\pi_{\varkappa}^{\#}(x)\,h_2(x)d^{\#3}x$, $h_i = \overline{h_i} \in S_{\mathrm{fin}}^{\#}({}^*\mathbb{R}_c^{\#3}), \mathrm{supp}h_i \subset \Re_I$.

**Theorem 11.34** On the domain $D(H^2)$ for $k = 1,2,3$ the equalities hold

$$[S_k, W(I)]D(H^2) = 0. \quad (171)$$

**Proof** Let $D^{\#}$ be the domain of well-behaved vectors.

$$D^{\#} = \left\{\psi \in \mathcal{F}^{\#}|\psi^{(n)} \in S_{\mathrm{fin}}^{\#}({}^*\mathbb{R}_c^{\#3n}), \psi^{(m)} = 0 \text{ for all sufficiently large m}\right\}. \quad (172)$$

For $\chi_1, \chi_2 \in D^{\#}$, direct momentum space computation gives for all $n \in {}^*\mathbb{N}$

$$\langle S_k\chi_1, \left(\varphi_{\varkappa}^{\#}(h_1) + \pi_{\varkappa}^{\#}(h_2)\right)^n\chi_2\rangle = \langle\left(\varphi_{\varkappa}^{\#}(h_1) + \pi_{\varkappa}^{\#}(h_2)\right)^n\chi_1, S_k\chi_2\rangle \quad (173)$$

By easy computation we get the inequality $\left\|\left(\varphi_{\varkappa}^{\#}(h_1)+\pi_{\varkappa}^{\#}(h_2)\right)^n\chi\right\| \leq c_1 c_2^n (n!)^{\frac{1}{2}}$ for constants $c_1$ and $c_2$ depending on vector $\chi \in D^{\#}$. Therefore $\chi \in D^{\#}$ are entire vectors for the operator $\left(\varphi_{\varkappa}^{\#}(h_1)+\pi_{\varkappa}^{\#}(h_2)\right)$, and the sum

$$U\chi = Ext\text{-}\sum_{n=0}^{*\infty}\frac{\left(i\varphi_{\varkappa}^{\#}(h_1)+i\pi_{\varkappa}^{\#}(h_2)\right)^n}{n!}\chi = Ext\text{-}\exp\left[i\left(\varphi_{\varkappa}^{\#}(h_1)+\pi_{\varkappa}^{\#}(h_2)\right)\right]\chi \qquad (174)$$

#-converges strongly. Now, we multiply (173) by $i^n(n!)^{-1}$ and by summation over $n$ using the #-convergence of the hyper infinite series (174) we get for all $k = 1,2,3$ that $\langle S_k\chi_1, U\chi_2\rangle = \langle U^*\chi_1, S_k\chi_2\rangle = \langle \chi_1, US_k\chi_2\rangle$ for $\chi_i \in D^{\#}$, $i = 1,2$. Note that this equality extends to $\chi_i \in D(H_{0\varkappa}N), i = 1,2$ since $D^{\#}$ is a core for operators $H_{0\varkappa}N$ and $S_k$ and $\|S_k\chi\|_{\#} \leq \mu\|(H_{0\varkappa}N + I)\chi\|_{\#}$ where $\mu$ is finite constant. Therefore for $\chi \in D(H_{0\varkappa}N)$, we have proved that $U\chi \in D(S_k^*)$ and $S_k^*U\chi = US_k\chi, k = 1,2,3$. For the next step we now prove that $\chi \in D(H_{0\varkappa}N) \Rightarrow U\chi \in D(H_{0\varkappa}N)$, so that $S_kU\chi = US_k\chi, k = 1,2,3$, since the operators $S_k$ are symmetric on $D(H_{0\varkappa}N)$. We define on $D(H_{0\varkappa}N)$ a #-norm by $\|\chi\|_{\#} = \|(H_{0\varkappa}N + I)\chi\|_{\#}$; Note that the corresponding scalar product makes $D(H_{0\varkappa}N)$ a non-Archimedean Hubert space, say $H_{\#1}$. For the next step we now prove that the operator $\mathcal{B} = \varphi_{\varkappa}^{\#}(h_1)+\pi_{\varkappa}^{\#}(h_2)$ generates a one parameter group $U(\alpha) = Ext\text{-}\exp(i\alpha\mathcal{B}) = Ext\text{-}\exp\left[i\alpha\left(\mathcal{B} = \varphi_{\varkappa}^{\#}(h_1)+\pi_{\varkappa}^{\#}(h_2)\right)\right]$ on $H_{\#1}$ and therefore we need to prove that the operator

$$\widehat{\mathcal{B}} = (H_{0\varkappa}N + I)\mathcal{B}(H_{0\varkappa}N + I)^{-1} \qquad (175)$$

is a generator to one parameter group on a corresponding Fock space. Since $\widehat{\mathcal{B}}$ is essentially self #-adjoint on $D^{\#}$, and on this domain we have that

$$\widehat{\mathcal{B}} = \mathcal{B} + [H_{0\varkappa}N, \mathcal{B}](H_{0\varkappa}N + I)^{-1} = \mathcal{B} + [N, \mathcal{B}]H_{0\varkappa}(H_{0\varkappa}N + I)^{-1} + N[H_{0\varkappa}, \mathcal{B}](H_{0\varkappa}N + I)^{-1} = \mathcal{B} + A.$$

Hear $A$ is bounded operator. Note that $\widehat{\mathcal{B}} \upharpoonright D^{\#}$ is a bounded perturbation of an essentially self #-adjoint operator. Hence it #- closure #- $\overline{\left(\widehat{\mathcal{B}} \upharpoonright D^{\#}\right)}$ generates a one parameter group on Fock space $\mathcal{F}^{\#}$, and operator $\mathcal{B} \upharpoonright (H_{0\varkappa}N + I)D^{\#}$ has a #- closure in $H_{\#1}$ that generates a one parameter group on $H_{\#1}$. Since the topology of $H_{\#1}$ is stronger than that of $\mathcal{F}^{\#}$, the #-closure of $\mathcal{B} \upharpoonright (H_{0\varkappa}N + I)D^{\#}$ in $H_{\#1}$ is a restriction of #- $\overline{\mathcal{B}}$ in $\mathcal{F}^{\#}$ and the one parameter group in $H_{\#1}$ is a restriction of the one parameter group generated by #- $\overline{\mathcal{B}}$ in $\mathcal{F}^{\#}$. This proves that

$$U: D(H_{0\varkappa}N) \to D(H_{0\varkappa}N)$$

Therefore we have proved that $S_kU\chi = US_k\chi, k = 1,2,3$. Now by passing to strong limits of linear combinations of such operators $U$ we obtain (165) on restricting to the domain $D(H^2) \subset D(H_{0\varkappa}N)$. This makes precise the statement that operators $S_k, k = 1,2,3$ are localized outside $\Delta = [a,b]^3$.

**Remark 11.13** Note that for each $t_1$, $|t_1| \leq |s_1|$, the spectral projections of $Ext\text{-}\int_{^*\mathbb{R}_c^{\#3}}\varphi_{\varkappa}^{\#}(x)f(x,t_1)d^{\#3}x$ belong to $W\left(\#\text{-int}\left(\Delta_{-|s|}\right)\right)$, where $\#\text{-int}\left(\Delta_{-|s|}\right)$ is the #-interior of $\Delta_{-|s|} = \{x|(x,t_1) \in \mathfrak{R}_s^4\} = \{(x_1,x_2,x_3)|a + |s| < x_k < b - |s|\}$. Note that $\operatorname{supp} f \subset \mathfrak{R}_s^4$, hence the spectral projections of

$$Ext\text{-}\exp[iH(t+t_1)]\left(Ext\text{-}\int_{^*\mathbb{R}_c^{\#3}}\varphi_{\varkappa}^{\#}(x)f(x,t_1)d^{\#3}x\right)Ext\text{-}\exp[-iH(t+t_1)] \qquad (176)$$

belong to $W\left(\#\text{-int}\left(\Delta_{|t|-|s|}\right)\right)$. For $|t| \leq |s|, \#\text{-}int\left(\Delta_{|t|-|s|}\right) \subset \Delta$; so the spectral projections of (170) belong to $W(\Delta)$. Now we use the locality property of the operators $S_k, k = 1,2,3$. Note that for vector $\chi \in D(H^2), \psi \in D(H^3)$ we have that $\psi \in D\left(Ext\text{-}\int_{^*\mathbb{R}_c^{\#3}}\varphi_{\varkappa}^{\#}(x,0)f(x,t_1)d^{\#3}x\right)$, and for $\varphi_{\varkappa}^{\#}(f) = Ext\text{-}\int_{^*\mathbb{R}_c^{\#3}}\varphi_{\varkappa}^{\#}(x,t)f(x,t)d^{\#3}x\, d^{\#}t$, by (159) it follows

$$Ext\text{-}\exp[itH]\varphi_{\varkappa}^{\#}(f)Ext\text{-}\exp[itH]\psi \in D(H^2). \qquad (177)$$

Therefore by (171) and the localization of (176) for all $k = 1,2,3$ we get

$$\langle S_k\chi, Ext\text{-}\exp[iH(t+t_1)]\left(Ext\text{-}\int_{^*\mathbb{R}_c^{\#3}}\varphi_{\varkappa}^{\#}(x)f(x,t_1)d^{\#3}x\right)Ext\text{-}\exp[-iH(t+t_1)]\psi\rangle = \qquad (178)$$
$$\langle Ext\text{-}\exp[iH(t+t_1)]\left(Ext\text{-}\int_{^*\mathbb{R}_c^{\#3}}\varphi_{\varkappa}^{\#}(x)f(x,t_1)d^{\#3}x\right)Ext\text{-}\exp[-iH(t+t_1)]\chi, S_k\psi\rangle.$$

Note that for $|t| \leq |s|$ and $f \in S_{\text{fin}}^{\#}(^*\mathbb{R}_c^{\#4})$ with $\operatorname{supp} f \subset \mathfrak{R}_s^4$ we can integrate the equality (178) over $t_1$ to obtain

$$\langle S_k\chi, Ext\text{-}\exp[iH(t)]\varphi_{\varkappa}^{\#}(f)Ext\text{-}\exp[-iH(t)]\psi\rangle = \langle Ext\text{-}\exp[iH(t)]\varphi_{\varkappa}^{\#}(f)Ext\text{-}\exp[-iH(t)]\chi, S_k\psi\rangle = \qquad (179)$$
$$\langle \chi, S_kExt\text{-}\exp[iH(t)]\varphi_{\varkappa}^{\#}(f)Ext\text{-}\exp[-iH(t)]\psi\rangle.$$

Here the last equality in (179) follows by (177) and the fact that $S_k$ is a symmetric operator on $D(H_{0\varkappa}N) \supset D(H^2)$. From (179) we obtain that $S_k\psi \in D\left(\left((Ext\text{-}\exp[iH(t)]\varphi_{\varkappa}^{\#}(f)Ext\text{-}\exp[-iH(t)]) \upharpoonright D(H^2)\right)^*\right)$ and therefore that $S_k\psi \in D(Ext\text{-}\exp[iH(t)]\varphi_{\varkappa}^{\#}(f)Ext\text{-}\exp[-iH(t)])$, since $D(H^2)$ is a #-core for $\varphi_{\varkappa}^{\#}(f)$. Finally from (179) we get for $|t| \leq |s|$ and $f \in S_{\text{fin}}^{\#}(^*\mathbb{R}_c^{\#4})$ with $\operatorname{supp} f \subset \mathfrak{R}_s^4$ for all $k = 1,2,3$ that

$$S_k Ext\text{-}\exp[iH(t)]\varphi_\varkappa^\#(f)Ext\text{-}\exp[-iH(t)]\psi = Ext\text{-}\exp[iH(t)]\varphi_\varkappa^\#(f)Ext\text{-}\exp[-iH(t)]S_k\psi. \quad (180)$$

We apply the relation (180) to (169). In that case $\psi(t) \in D(H^5) \subset D(H^3)$, so $\frac{d^{\#2}F_k(t)}{d^\# t^2} \equiv 0$, for $|t| \le |s|$.

**Theorem 11.35** [15] Let $f \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#4})$ and supp$f \subset \Re_s^4$, then on domain $D(H^5)$ the operator equalities hold for all $k = 1,2,3$

$$[iM^{0k}(s), \varphi_\varkappa^\#(f)\,] = [iM^{0k}, \varphi_\varkappa^\#(f)\,] - s\left[P\left(\frac{d^\#\left(x_k g_0^{(k)}\right)}{d^\# x_k}\right), \varphi_\varkappa^\#(f)\right]. \quad (181)$$

The next step in the proof of Theorem 11.32 is to pass to the sharp time #-limit of Theorem 11.35, thus we need to choose a hyper infinite sequence of functions $f_n \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#4}), n \in {}^*\mathbb{N}$ which pick out a time zero contribution in the #-limit. Let us define now

$$A_\varkappa(f,t) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_\varkappa^\#(x) f(x,t) d^{\#3}x, \quad (182)$$

$$B_\varkappa(f,t) = Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \pi_\varkappa^\#(x) f(x,t) d^{\#3}x. \quad (183)$$

Where $\varphi_\varkappa^\#(x)$ and $\pi_\varkappa^\#(x)$ the canonical time-zero fields. For real $f \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#4})$, with #-compact support, $A_\varkappa(f,t)$ and $B_\varkappa(f,t)$ are essentially self-#-adjoint on $D\left((H+b)^{\frac{1}{2}}\right)$. Let $f \in C_0^{*\infty}(\Re_I^4)$ and let $f_n(x,t) \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#4}), n \in {}^*\mathbb{N}$ be a hyper infinite sequence of functions of the following form $f_n(x,t) = f_n(x,s)\delta_n(t)$ with support in $\Re_s^4$ and #-converging in the weak sense to $f_n(x,s)\delta(t)$ as $n \to {}^*\infty$. For the vector $\psi \in D(H^5)$, the vectors $M^{0k}(s)\psi, k = 1,2,3$, and the vectors $M^{0k}\psi$, $P\left(\frac{d^\#\left(x_k g_0^{(k)}\right)}{d^\# x_k}\right)\psi$ the same as in the proof of Theorem 11.35. Note that the bilinear form $\varphi_\varkappa^\#(x,t)$ for $(x,t) \in \Re_I^4$ determines a bounded operator

$$G(x,t) = (H+b)^{\frac{1}{2}}\varphi_\varkappa^\#(x,t)(H+b)^{-\frac{1}{2}}. \quad (184)$$

Note that the operator valued function $G(x,t)$ is #-continuous in variable $(x,t)$.

**Theorem 11.36** Let $f \in S_{\text{fin}}^\#({}^*\mathbb{R}_c^{\#4})$ and supp$f \subset \Re_\Delta^4$. Then, in the sense of bilinear forms on $D(H^5)$, for all $k = 1,2,3$

$$[iM^{0k}(s), A_\varkappa(f,s)] = [iM^{0k}, A_\varkappa(f,s)] - s[iP_k, A_\varkappa(f,s)] \quad (185)$$

Here $P_k = P\left(\frac{d^\#\left(x_k g_0^{(k)}\right)}{d^\# x_k}\right)$.

**Theorem 11.37** [15] Let $f \in C_0^{*\infty}(\Re_\Delta^4)$. As an equality of bilinear forms on $D(H\,) \times D(H\,)$

$$[i\,P_k, A_\varkappa(f,s)] = A\left(\frac{d^\# f}{d^\# x_k}, s\right). \quad (186)$$

And where $P_k$ is defined in Theorem 11.36.

**Theorem 11.38** As the equalities of bilinear forms on $D(H^2) \times D(H^2)$ for all $k = 1,2,3$

$$[iM^{0k}, A_\varkappa(f,s)] = [iH, A_\varkappa(x_k f, s)] = B_\varkappa(x_k f, s). \quad (187)$$

**Theorem 11.39** [15] Let $|f|_{\#1}$ be the #-norm $|f|_{\#1} = c\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\left\{\|f(\cdot,t)\|_{\#2} + \sum_{i=1}^3 \left\|\partial_{x_i}^\# f(\cdot,t)\right\|_{\#2}\right\} d^\# t\right)$.

Let $|f|_{\#1}$ is finite. Then on the domain $D\left((H+b)^{\frac{3}{2}}\right)$, ), the field $\varphi_\varkappa^\#(f)$ satisfies the following equation

$$(\partial_t^\# \varphi_\varkappa^\#)(f) = -\varphi_\varkappa^\#(\partial_t^\# f) = \pi_\varkappa^\#(f) = [iH, \varphi_\varkappa^\#(f)]. \quad (188)$$

**Proof** Note that the first equality in (188) is the definition of a distribution #-derivative. The out the difference quotient $\Delta_\varepsilon f(x,t)$ to #-derivative $\partial_t^\# f$ reads $\Delta_\varepsilon f(x,t) = \frac{[f(x+\varepsilon,t) - f(x,t)]}{\varepsilon}$, note that #-$\lim_{\varepsilon \to_\# 0} \Delta_\varepsilon f(x,t) = \partial_t^\# f(x,t)$. Note that for any vector $\psi$ such that $\psi \in D\left((H+b)^{\frac{1}{2}}\right)$ by canonical consideration we get

$$\#\text{-}\lim_{\varepsilon \to_\# 0} \left\|\varphi_\varkappa^\#(\partial_t^\# f)\psi - \varphi_\varkappa^\#\left(\Delta_\varepsilon f(x,t)\right)\psi\right\|_\# = 0.$$

We have for $\psi \in D\left((H+b)^{\frac{3}{2}}\right)$ that

$$\varphi_\varkappa^\#\left(\Delta_\varepsilon f(x,t)\right)\psi = \varepsilon^{-1}(I - Ext\text{-}\exp[i\varepsilon H])\left\{Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_\varkappa^\#(x,t-\varepsilon) f(x,t) d^{\#3}x\psi d^\# t\right\} +$$

$$\varepsilon^{-1}\left\{Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} A_\varkappa(f,t)(Ext\text{-}\exp[i\varepsilon H] - I)\psi d^\# t\right\}.$$

Here the last term #-converges as $\varepsilon \to_\# 0$ and it #-limit is: $i\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} A_\varkappa(f,t) H\psi d^\# t\right)$. Since $\varphi_\varkappa^\#\left(\Delta_\varepsilon f(x,t)\right)\psi$ #-converges as $\varepsilon \to_\# 0$, the remaining term in expression for $\varphi_\varkappa^\#\left(\Delta_\varepsilon f(x,t)\right)\psi$ #-converges also to a #-limit $\psi_1$. For $\chi \in D(H)$ we obtain that

$$\langle \chi, \psi_1 \rangle = \#\text{-}\lim_{\varepsilon \to_\# 0} \langle \chi, \varepsilon^{-1}(I - Ext\text{-}\exp[i\varepsilon H]) \left\{ Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_\varkappa^\#(x, t - \varepsilon) f(x, t) d^{\#3}x \psi d^\# t \right\} \rangle = \langle iH\chi, \varphi_\varkappa^\#(f)\psi \rangle.$$

Since $H = H^*$, it follows that $\varphi_\varkappa^\#(f)\psi \in D(H)$ and $\psi_1 = iH\varphi_\varkappa^\#(f)\psi$ and therefore: $-\varphi_\varkappa^\#(\partial_t^\# f)\psi = [iH, \varphi_\varkappa^\#(f)]\psi$. From the above equation we obtain

$$\langle \psi, \varphi_\varkappa^\#(\partial_t^\# f)\psi \rangle = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} \langle H\psi(t), Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_\varkappa^\#(x, 0) f(x, t) d^{\#3}x \psi(t) \rangle \, d^\# t -$$

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} \langle Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_\varkappa^\#(x, 0) f(x, t) d^{\#3}x \psi(t),\ H\psi(t) \rangle \, d^\# t.$$

Here $\psi(t) = Ext\text{-}\exp[itH]\psi$. Note that $\psi(t) \in D(H_{0\varkappa}) \cap D(H_{I,\varkappa})$, and $\left\| H_{I,\varkappa}(\psi(t) - \psi(s)) \right\|_\# \leq a \left\| (H + b)(\psi(t) - \psi(s)) \right\|_\# \to_\# 0$, as $|t - s| \to_\# 0$. Therefore we may substitute $H_{0\varkappa} + H_{I,\varkappa}$ for $H$ and consider each term separately. Note that the operators $H_{I,\varkappa}$ and $Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_\varkappa^\#(x, 0) f(x, t) d^{\#3}x$ commute and therefore $H_{I,\varkappa}$ contribute zero to equality above. The following identity by canonical computation holds for any $\psi \in D(H_{0\varkappa})$, in particular for $\psi(t) = Ext\text{-}\exp[itH]\psi \in D(H_{0\varkappa})$

$$\langle H_{0\varkappa}\psi, Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_\varkappa^\#(x, 0) f(x, t) d^{\#3}x \psi \rangle - \langle Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \varphi_\varkappa^\#(x, 0) f(x, t) d^{\#3}x \psi\ , H_{0\varkappa}\psi \rangle =$$

$$\langle \psi, -iExt\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \pi_\varkappa^\#(x, 0) f(x, t) d^{\#3}x \psi \rangle.$$

Therefore finally we get

$$i\langle \psi, \varphi_\varkappa^\#(\partial_t^\# f)\psi \rangle = Ext\text{-}\int_{{}^*\mathbb{R}_c^\#} \langle \psi(t), -iExt\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \pi_\varkappa^\#(x, 0) f(x, t) d^{\#3}x \psi \rangle \, d^\# t = \langle \psi, -i\pi_\varkappa^\#(f)\psi \rangle.$$

This equality finalized the proof.

**Theorem 11.40** As the operator equalities on $D(H^5)$ for all $k = 1,2,3$

$$[iM^{0k}, \varphi_\varkappa^\#(f)] = -\varphi_\varkappa^\# \left( t \frac{\partial^\# f}{\partial^\# x_k} + x_k \frac{\partial^\# f}{\partial^\# t} \right). \tag{189}$$

**Proof** We first prove (189) as equalities of bilinear forms on $D(H^5) \times D(H^5)$. Let $\psi$ is a near standard vector and $\psi \in D(H^5)$. By Theorems 11.37-11.39, for all $k = 1,2,3$ we get

$$\langle \psi, iM^{0k}(s), A_\varkappa(f, s)\psi \rangle = \langle \psi, B_\varkappa(x_k f, s)\psi, \rangle - \langle \psi, A\left( \frac{d^\# f}{d^\# x_k}, s \right)\psi \rangle.$$

Substituting $Ext\text{-}\exp(iHs)$ for $\psi$, we obtain that

$$\langle \psi, [iM^{0k}, Ext\text{-}\exp(iHs)A_\varkappa(f, s)Ext\text{-}\exp(-iHs)]\psi \rangle = \tag{190}$$

$$\langle \psi, Ext\text{-}\exp(iHs) \left\{ B_\varkappa(x_k f, s) - A\left( s\frac{d^\# f}{d^\# x_k}, s \right) \right\} Ext\text{-}\exp(-iHs)\psi \rangle.$$

From (188) we get

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#4}} Ext\text{-}\exp(iHt)\, \pi_\varkappa^\#(x) Ext\text{-}\exp(iHt) f(x, t) d^{\#3}x d^\# t = -\varphi_\varkappa^\#\left( \frac{\partial^\# f}{\partial^\# t} \right). \tag{191}$$

Using (191) we integrate (190) over s to obtain for all $k = 1,2,3$ the equalities of bilinear forms

$$\langle \psi, iM^{0k}, \varphi_\varkappa^\#(f)\psi \rangle = -\langle \psi, \varphi_\varkappa^\# \left( t \frac{\partial^\# f}{\partial^\# x_k} + x_k \frac{\partial^\# f}{\partial^\# t} \right)\psi \rangle. \tag{192}$$

Since $M^{0k}\varphi_\varkappa^\#(f), \varphi_\varkappa^\#(f)M^{0k}$, and $\varphi_\varkappa^\# \left( t \frac{\partial^\# f}{\partial^\# x_k} + x_k \frac{\partial^\# f}{\partial^\# t} \right)$ are operators on $D(H^5)$ for all $k = 1,2,3$, the operator equalities (189) follows by polarization and the #-density of $D(H^5)$. This final remark completes the proof of the theorem and hence it completes the proof of Theorem 11.32.

**Theorem 11.41** [15] Let $\mathfrak{R} \subset {}^*\mathbb{R}_{c,\text{fin}}^{\#4}$ be an bounded region in ${}^*\mathbb{R}_{c,\text{fin}}^{\#4}$ and let $F_k(\beta, x, t), k = 1,2,3$ be a functions such that $F_k(\beta, x, t), \beta \in {}^*\mathbb{R}_{c,\text{fin}}^\#$ and $\frac{\partial^\# F_k(\beta, x, t)}{\partial^\# \beta}$ are #- continuous in $(\beta, x, t)$, where the partial #-derivative exists for each point $(x, t) \in {}^*\mathbb{R}_{c,\text{fin}}^{\#4}$. Assume that for all $f(x, t) \in C_{0,\text{fin}}^{{}^*\infty}(\mathfrak{R})$ the following equalities hold for all $k = 1,2,3$,

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} \frac{\partial^\# F_k(\beta, x, t)}{\partial^\# \beta} f(x, t) d^{\#3}x d^\# t = -Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} F_k(\beta, x, t) \left[ x_k \frac{\partial^\# f}{\partial^\# t} + t \frac{\partial^\# f}{\partial^\# x_k} \right] d^{\#3}x d^\# t. \tag{193}$$

Then for all $(\beta, x, t)$ such that $\Lambda_{\gamma\beta}(x, t) \in \mathfrak{R}$ for $0 \leq \gamma \leq 1, k = 1,2,3$

$$F_k(\beta, x, t) = F_k\left(0, \Lambda_{\gamma\beta}(x, t)\right) + \delta(\beta, x, t) = \tag{194}$$

$$F_k(0, x_k \cosh\beta + t \sinh\beta, x_k \sinh\beta + t \cosh\beta) + \delta(\beta, x, t).$$

Here $\delta(\beta, x, t)$ is a nonzero function such that $\delta(\beta, x, t) \neq 0$ and $\delta(\beta, x, t)$ is #- differentiable with zero partial #-derivatives $\delta_\beta^{\#'}(\beta, x, t) \equiv 0, \delta_{x_k}^{\#'}(\beta, x, t) \equiv 0, \delta_t^{\#'}(\beta, x, t) \equiv 0$.

**Proof** Obviously (194) is a solution to the equations (193). Thus we need prove uniqueness (194) for a given function $\delta(\beta,x,t)$ and for all $k=1,2,3$ and it is sufficient to prove uniqueness for the case $F_k(0,x,t)=\delta(0,x,t)$. Let $A_k$ be the operator $A_k=x_k\frac{\partial^\#}{\partial^\# t}+t\frac{\partial^\#}{\partial^\# x_k}$. Note that by (177), provided $\operatorname{supp} f\left(\Lambda_{\gamma\beta'}(x,t)\right)\subset\mathfrak{R}$ we get

$$\frac{\partial^\#}{\partial^\#\beta'}\left(Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}F_k(\beta',x,t)f\left(\Lambda_{\gamma\beta'}(x,t)\right)d^{\#3}xd^\# t\right)= \tag{195}$$

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}\left\{\frac{\partial^\# F_k(\beta',x,t)}{\partial^\#\beta'}f\left(\Lambda_{\gamma\beta'}(x,t)\right)+F_k(\beta',x,t)A_kf\left(\Lambda_{\gamma\beta'}(x,t)\right)\right\}d^{\#3}xd^\# t=0.$$

Let $\breve{\mathfrak{R}}=\bigcap_{0\leq\gamma\leq 1}\Lambda_{\gamma\beta}\,\mathfrak{R}$ and $f(x,t)\in C^{*\infty}_{0,\mathrm{fin}}(\breve{\mathfrak{R}})$, then (195) holds for all $\beta'$ such that $0\leq\beta'\leq\beta$. Note that for all functions $f(x,t)\in C^{*\infty}_{0,\mathrm{fin}}(\mathfrak{R})$ the following equalities (196) hold for all $k=1,2,3$,

$$Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}}F_k(\beta,x,t)f\left(\Lambda_{\gamma\beta'}(x,t)\right)d^{\#3}xd^\# t=0. \tag{196}$$

Thus, in the sense of distributions we obtain that

$$F_k(\beta,x,t)=0,(x,t)\in\breve{\mathfrak{R}}. \tag{197}$$

Since $F_k(\beta,x,t)$ is #-continuous, (197) holds in usual sense everywhere in $\breve{\mathfrak{R}}$. This establishes required uniqueness, and completes the proof of the theorem.

**Definition 11.18** (1) Let $(H_\#,\|\cdot\|_\#)$ be a linear normed space over field ${}^*\mathbb{C}_c^\#$. An element $x\in H_\#$ is called finite or norm finite if $\|x\|_\#\in{}^*\mathbb{R}^\#_{c,\mathrm{fin}}$ and we let $\mathrm{Fin}(H_\#)$ denote the set of the all finite elements of $H_\#$; the element $x\in H_\#$ is called infinitesimal if $\|x\|_\#\approx 0$ and we write $x\approx y$ for $\|x-y\|_\#\approx 0$. (2)Let $(H_\#,\langle\cdot,\cdot\rangle_\#)$ be a non-Archimedean Hilbert space over field ${}^*\mathbb{C}_c^\#$ endowed with a canonical #-norm $\|x\|_\#=\sqrt{\langle x,x\rangle_\#}$, then we apply the same definition as in (1).

**Definition 11.19** Let $A$ be a linear operator $A\colon H_\#\to H_\#$ with domain $D(A)$. Let $D_{\mathrm{fin}}(A)\subset D(A)$ be a subdomain such that for all $\psi\in D(A)$: $\psi\in D_{\mathrm{fin}}(A)\Leftrightarrow\|x\|_\#\in{}^*\mathbb{R}^\#_{c,\mathrm{fin}}$ and let $D^\#_{\mathrm{fin}}(A)$ be a subdomain $D^\#_{\mathrm{fin}}(A)\subset D_{\mathrm{fin}}(A)$ such that for all $\psi\in D_{\mathrm{fin}}(A)$: $\psi\in D^\#_{\mathrm{fin}}(A)\Leftrightarrow\|Ax\|_\#\in{}^*\mathbb{R}^\#_{c,\mathrm{fin}}$.

**Definition 11.20** Let $q(\cdot,\cdot)$ be a bilinear form with domain $D(q)\times D(q)$ on $H_\#$ such that $D(q)\times D(q)\subsetneq H_\#\times H_\#$ and $D(q)\times D(q)\to{}^*\mathbb{C}_c^\#$. Let $D_{\mathrm{fin}}(q)\times D_{\mathrm{fin}}(q)\subset D(q)\times D(q)$ be a subdomain such that for all $\{\psi_1,\psi_2\}\in D_{\mathrm{fin}}(q)\times D_{\mathrm{fin}}(q)\Leftrightarrow|\langle\psi_1,\psi_2\rangle_\#|\in{}^*\mathbb{R}^\#_{c,\mathrm{fin}}$. Let $D^\#_{\mathrm{fin}}(q)\times D^\#_{\mathrm{fin}}(q)\subset D_{\mathrm{fin}}(q)\times D_{\mathrm{fin}}(q)$ be a subdomain such that for all $\{\psi_1,\psi_2\}\in D_{\mathrm{fin}}(q)\times D_{\mathrm{fin}}(q)$: $\{\psi_1,\psi_2\}\in D^\#_{\mathrm{fin}}(q)\times D^\#_{\mathrm{fin}}(q)\Leftrightarrow q(\psi_1,\psi_2)\in{}^*\mathbb{C}^\#_{c,\mathrm{fin}}$.

**Theorem 11.42** [15] Assume that the operators $M^{0k}=M^{0k}_\varkappa=M^{0k}_{0,\varkappa}+M^{0k}_{I,\varkappa}$, $k=1,2,3$ satisfy conditions (152)-(154) and where the operators $M^{0k}_{0,\varkappa}$ are defined by (125). We set now $\delta(\beta,x,t)\approx 0$.

(1) If $f\in S^\#_{\mathrm{fin}}({}^*\mathbb{R}_c^{\#4})$, $\operatorname{supp} f\subset\#\text{-int}(\mathfrak{R}^4_\Delta)$, $\Delta=[a,b]^3$ and $\operatorname{supp} f_{\Lambda(\beta)}\subseteq\#\text{-int}(\mathfrak{R}^4_\Delta)=\wp^4_\Delta$, then for all $k=1,2,3$ on domains $D_{\mathrm{fin}}((M^{0k})^2)$

$$Ext\text{-}\exp(iM^{0k}\beta)\varphi^\#_\varkappa(f)Ext\text{-}\exp(-iM^{0k}\beta)\approx\varphi^\#_\varkappa(f_{\Lambda(\beta)}). \tag{198}$$

Here the $\approx$ - equalities (198) hold as $\approx$ -equalites for self #-adjoint operators.

(2) If $(x,t)\in\mathfrak{R}^4_\Delta$ and $\Lambda_\beta(x,t)\in\mathfrak{R}^4_\Delta$, then for all $k=1,2,3$

$$Ext\text{-}\exp(iM^{0k}\beta)\varphi^\#_\varkappa(x,t)Ext\text{-}\exp(-iM^{0k}\beta)\approx\varphi^\#_\varkappa\left(\Lambda_\beta(x,t)\right) \tag{199}$$

Here the $\approx$ - equalities (199) hold in the sense of ${}^*\mathbb{R}^\#_{c,\mathrm{fin}}$- valued bilinear forms on domains $D^\#_{\mathrm{fin}}(M^{0k})\times D^\#_{\mathrm{fin}}(M^{0k})$ and on domains $D^\#_{\mathrm{fin}}(M^{0k})\times D^\#_{\mathrm{fin}}(M^{0k})$.

**Remark 11.15** Note that (1) for real-valued $f\in S^\#_{\mathrm{fin}}({}^*\mathbb{R}_c^{\#4})$ is a self-#-adjoint operator $\varphi^\#_\varkappa(f)$, essentially self -#-adjoint operator on a variety of appropriate domains. It is for this self #-adjoint operator that (198) is valid; (2) on the subdomains $D^\#_{\mathrm{fin}}((M^{0k})^2)$ $\approx$ -equalites (198) entail for all $k=1,2,3$ the equalities

$$\mathrm{st}\left(Ext\text{-}\exp(iM^{0k}\beta)\varphi^\#_\varkappa(x,t)Ext\text{-}\exp(-iM^{0k}\beta)\right)=\mathrm{st}\left(\varphi^\#_\varkappa\left(\Lambda_\beta(x,t)\right)\right);$$

(3) on the subdomains $D^\#_{\mathrm{fin}}((M^{0k})^2)$ the $\approx$ -equalites (198) entail for all $k=1,2,3$ the equalities

$$\mathrm{st}\left(Ext\text{-}\exp(iM^{0k}\beta)\varphi^\#_\varkappa(f)Ext\text{-}\exp(-iM^{0k}\beta)\right)=\mathrm{st}\left(\varphi^\#_\varkappa(f_{\Lambda(\beta)})\right).$$

**Proof** Let $\psi\in D(M^{0k})$ and let $F_k(\beta,x,t)$ be the function is defined by

$$F_k(\beta,x,t)=\langle Ext\text{-}\exp(-iM^{0k}\beta)\psi,\varphi^\#_\varkappa(x,t)(Ext\text{-}\exp(-iM^{0k}\beta)\psi)\rangle. \tag{200}$$

For all $(\beta,x,t)\in{}^*\mathbb{R}^\#_{c,\mathrm{fin}}\times{}^*\mathbb{R}^{\#4}_{c,\mathrm{fin}}$ and for $f\in S^\#_{\mathrm{fin}}({}^*\mathbb{R}_c^{\#4})$, let $F_k(\beta,f)$ be the function is defined by

$$F_k(\beta,f)=\langle Ext\text{-}\exp(-iM^{0k}\beta)\psi,\varphi^\#_\varkappa(f)(Ext\text{-}\exp(-iM^{0k}\beta)\psi)\rangle=$$

$$Ext\text{-}\int_{\wp_\Delta^4} F_k(\beta,x,t)f(x,t)d^{\#3}xd^{\#}t. \tag{201}$$

Note that $\varphi_\varkappa^\#(x,t)$ is a bilinear form defined on $D\left((H+b)^{\frac{3}{2}}\right)\times D\left((H+b)^{\frac{3}{2}}\right)$, #-continuous in $(x,t)\in {}^*\mathbb{R}_{c,\mathrm{fin}}^{\#4}$. By Theorem 11.29 $D(M^{0k})\subset D\left((H+b)^{\frac{1}{2}}\right)$ and therefore $F_k(\beta,x,t)$ is well defined and #-continuous in $(x,t)$. Note that a function $F_k(\beta,x,t)$ is #-continuously #-differentiable in $\beta\in {}^*\mathbb{R}_{c,\mathrm{fin}}^{\#}$ and for all $k=1,2,3$

$$\frac{\partial^\# F_k(\beta,x,t)}{\partial^\#\beta} = -\langle Ext\text{-}\exp(-iM^{0k}\beta)iM^{0k}\psi,\varphi_\varkappa^\#(f)(Ext\text{-}\exp(-iM^{0k}\beta)\psi)\rangle \tag{202}$$
$$-\langle Ext\text{-}\exp(-iM^{0k}\beta)\psi,\varphi_\varkappa^\#(f)(Ext\text{-}\exp(-iM^{0k}\beta)iM^{0k}\psi)\rangle.$$

By the canonical argument, we have for all $k=1,2,3$ that

$$\frac{\partial^\# F_k(\beta,f)}{\partial^\#\beta} = \langle Ext\text{-}\exp(-iM^{0k}\beta)\psi,[iM^{0k},\varphi_\varkappa^\#(f)](Ext\text{-}\exp(-iM^{0k}\beta)\psi)\rangle = \tag{203}$$
$$Ext\text{-}\int_{\wp_\Delta^4} F_k(\beta,x,t)f(x,t)d^{\#3}xd^{\#}t.$$

By Theorem 11.40 under the condition $\mathrm{supp}f\subset \#\text{-int}(\mathfrak{R}_\Delta^4)$ we have for all $k=1,2,3$ that

$$\frac{\partial^\# F_k(\beta,f)}{\partial^\#\beta} = -\langle Ext\text{-}\exp(-iM^{0k}\beta)\psi,\varphi_\varkappa^\#\left(x_k\frac{\partial^\# f}{\partial^\# t}+t\frac{\partial^\# f}{\partial^\# x_k}\right)Ext\text{-}\exp(-iM^{0k}\beta)\psi\rangle =$$
$$-Ext\text{-}\int_{{}^*\mathbb{R}_c^{\#3}} F_k(\beta,x,t)\left(x_k\frac{\partial^\# f}{\partial^\# t}+t\frac{\partial^\# f}{\partial^\# x_k}\right)f(x,t)d^{\#3}xd^{\#}t. \tag{204}$$

Therefore by Theorem 11.40 under the condition

$$\bigcup_{0\le\gamma\le 1}\Lambda_{\gamma\beta}(x,t)\in\mathfrak{R}_\Delta^4 \tag{205}$$

we have for all $k=1,2,3$ that

$$F_k(\beta,x,t)=F_k\left(0,\ \Lambda_{\gamma\beta}(x,t)\right)+\delta(\beta,x,t) \tag{206}$$

That is, if (205) holds, then (206) also holds for all $k=1,2,3$ and finally we get

$$Ext\text{-}\exp(iM^{0k}\beta)\varphi_\varkappa^\#(x,t)Ext\text{-}\exp(-iM^{0k}\beta)=\varphi_\varkappa^\#\left(\Lambda_\beta(x,t)\right)+\delta(\beta,x,t). \tag{207}$$

Here the equations (207) hold in the sense of bilinear forms on $D((M^{0k})^2)\times D((M^{0k})^2)$, i.e.

$$\langle\psi_1,Ext\text{-}\exp(iM^{0k}\beta)\varphi_\varkappa^\#(x,t)Ext\text{-}\exp(-iM^{0k}\beta)\psi_2\rangle=\langle\psi_1,\varphi_\varkappa^\#\left(\Lambda_\beta(x,t)\right)\psi_2\rangle+\delta(\beta,x,t)\langle\psi_1,\psi_2\rangle. \tag{208}$$

From (208) on the domain $D_{\mathrm{fin}}^\#((M^{0k})^2)\times D_{\mathrm{fin}}^\#((M^{0k})^2)\subset D_{\mathrm{fin}}((M^{0k})^2)\times D_{\mathrm{fin}}((M^{0k})^2)\subset D((M^{0k})^2)\times D((M^{0k})^2)$ we get the $\approx$ -equality

$$\langle\psi_1,Ext\text{-}\exp(iM^{0k}\beta)\varphi_\varkappa^\#(x,t)Ext\text{-}\exp(-iM^{0k}\beta)\psi_2\rangle\approx\langle\psi_1,\varphi_\varkappa^\#\left(\Lambda_\beta(x,t)\right)\psi_2\rangle, \tag{209}$$

since $\langle\psi_1,\psi_2\rangle$ is finite and therefore $\delta(\beta,x,t)\langle\psi_1,\psi_2\rangle\approx 0$.

Note that in the #-limit $\lambda\to_\# 0$ by (125) we get

$$\#\text{-}\lim_{\to_\# 0} M^{0k} = M_\varkappa^{0k}. \tag{210}$$

Therefore in the #-limit $\lambda\to_\# 0$ from (208) and (210) we obtain that

$$\lim_{\lambda\to_\# 0}\langle\psi_1,Ext\text{-}\exp(iM^{0k}\beta)\varphi_\varkappa^\#(x,t)Ext\text{-}\exp(-iM^{0k}\beta)\psi_2\rangle = \tag{211}$$
$$\langle\psi_1,Ext\text{-}\exp(iM_\varkappa^{0k}\beta)\varphi_{0,\varkappa}^\#(x,t)Ext\text{-}\exp(-iM_\varkappa^{0k}\beta)\psi_2\rangle =$$
$$\mathrm{Lim}_{\lambda\to_\# 0}\langle\psi_1,\varphi_\varkappa^\#\left(\Lambda_\beta(x,t)\right)\psi_2\rangle+\delta(\beta,x,t)\langle\psi_1,\psi_2\rangle=\langle\psi_1,\varphi_{0,\varkappa}^\#\left(\Lambda_\beta(x,t)\right)\psi\rangle+\delta(\beta,x,t)\langle\psi_1,\psi_2\rangle.$$

From (211) on the domain $D_{\mathrm{fin}}^\#((M^{0k})^2)\times D_{\mathrm{fin}}^\#((M^{0k})^2)\subset D_{\mathrm{fin}}((M^{0k})^2)\times D_{\mathrm{fin}}((M^{0k})^2)\subset D((M^{0k})^2)\times D((M^{0k})^2)$ we get the $\approx$ -equality for free quantum field $\varphi_{0,\varkappa}^\#(x,t)$

$$\langle\psi_1,Ext\text{-}\exp(iM_\varkappa^{0k}\beta)\varphi_{0,\varkappa}^\#(x,t)Ext\text{-}\exp(-iM_\varkappa^{0k}\beta)\psi_2\rangle\approx\langle\psi_1,\varphi_{0,\varkappa}^\#\left(\Lambda_\beta(x,t)\right)\psi_2\rangle. \tag{212}$$

**Remark 11.16** Note that the $\approx$ -equality required by (212) is necessary, see Remark 9.2.

The $\approx$ -equality (209) extends by #-closure to $D_{\mathrm{fin}}^\#(M)\times D_{\mathrm{fin}}^\#(M)$, since $D_{\mathrm{fin}}^\#(M)\subset D_{\mathrm{fin}}^\#\left((H+b)^{1/2}\right)$ by Theorem 11.29, and the estimate

$$|\langle\psi,Ext\text{-}\exp(iM^{0k}\beta)\varphi_\varkappa^\#(x,t)Ext\text{-}\exp(-iM^{0k}\beta)\psi\rangle|\approx \tag{213}$$
$$\left|\langle\psi,\varphi_\varkappa^\#\left(\Lambda_\beta(x,t)\right)\psi\rangle\right|\le c\left\|(H+b)^{1/2}\psi\right\|^2.$$

Here $c$ is finite constant. Furthermore $D((M^{0k})^2)$ for any $k=1,2,3$ is a #-core for $H$, by Theorem 11.31, and therefore a #-core for $(H+b)^{\frac{1}{2}}$. Thus (208) extends to $D((M^{0k})^2)\times D((M^{0k})^2)$ and on this domain we also have #-continuity of the form in $(x,t)\in {}^*\mathbb{R}_{c,\mathrm{fin}}^{\#4}$. Note that it is necessary to assume that $\bigcup_{0\le\gamma\le 1}\Lambda_{\gamma\beta}(x,t)\in\mathfrak{R}_\Delta^4$.

However for the regions $\Re_{\Delta}^{4}$ this statement follows from the condition $(x,t) \in \Re_{\Delta}^{4} \Rightarrow \Lambda_{\beta}(x,t) \in \Re_{\Delta}^{4}$. This final remark completes the proof of this theorem part (2). Now we go to prove the operator $\approx$ -equality (198) for the case $f \in S_{\text{fin}}^{\#}({}^{*}\mathbb{R}_{c}^{\#4})$, $\text{supp}f \cup \text{supp}f_{\Lambda_{\beta}}$. By Theorem 11.29, the operators $\varphi_{\varkappa}^{\#}(f)$ and $\varphi_{\varkappa}^{\#}\left(f_{\Lambda_{\beta}}\right)$ are defined on domain $D((M^{0k})^{2})$. Integrating (207) against $f(x,t)$, we get the equalities

$$Ext\text{-}\exp(iM^{0k}\beta)\varphi_{\varkappa}^{\#}(f)Ext\text{-}\exp(-iM^{0k}\beta) = \varphi_{\varkappa}^{\#}\left(f_{\Lambda_{\beta}}\right) + Ext\text{-}\int_{\Re_{\Delta}^{4}} \delta(\beta,x,t)f(x,t)d^{\#3}xd^{\#}t. \quad (214)$$

Obviously the equalities (213) hold on the domains $D((M^{0k})^{2})$ with $k = 1,2,3$ correspondingly. For any vector $\psi$ such that $\psi \in D((M^{0k})^{2})$ from (207) we obtain the equalities

$$\varphi_{\varkappa}^{\#}(f)Ext\text{-}\exp(-iM^{0k}\beta)\psi = Ext\text{-}\exp(-iM^{0k}\beta)\varphi_{\varkappa}^{\#}\left(f_{\Lambda_{\beta}}\right)\psi + \left(Ext\text{-}\int_{\Re_{\Delta}^{4}} \delta(\beta,x,t)f(x,t)d^{\#3}xd^{\#}t\right)\psi. \quad (215)$$

Since $\left\|\varphi_{\varkappa}^{\#}\left(f_{\Lambda_{\beta}}\right)\psi\right\| \leq c_{1}\left\|(H+b)^{\frac{1}{2}}\psi\right\|$ and $D((M^{0k})^{2})$ for any $k = 1,2,3$ is a #-core for $H$, by Theorem 11.31, the equalities (215) extends by #-closure to $D(H)$ and (215) holds for $\psi \in D(H)$. Since the domain $D(H)$ is a #-core for the operator $\varphi_{\varkappa}^{\#}\left(f_{\Lambda_{\beta}}\right)$, we conclude that (214) extends by #-closure to $D\left(\varphi_{\varkappa}^{\#}\left(f_{\Lambda_{\beta}}\right)\right)$ and therefore the equalities (215) hold for all $k = 1,2,3$ and for any $\psi$ such that $\psi \in D\left(\varphi_{\varkappa}^{\#}\left(f_{\Lambda_{\beta}}\right)\right)$. Thus we have proved that

$$Ext\text{-}\exp(-iM^{0k}\beta)D\left(\varphi_{\varkappa}^{\#}\left(f_{\Lambda_{\beta}}\right)\right) \subset D\left(\varphi_{\varkappa}^{\#}(f)\right).$$

By similar consideration one obtains that

$$Ext\text{-}\exp(-iM^{0k}\beta)D\left(\varphi_{\varkappa}^{\#}\left(f_{\Lambda_{\beta}}\right)\right) \subset D\left(\varphi_{\varkappa}^{\#}(f)\right).$$

This proves (214) as an equality between self- #-adjoint operators, completing the proof of the theorem.

## CONCLUSION

A new non-Archimedean approach to interacted quantum fields is presented. In proposed approach, a field operator $\varphi(x,t)$ no longer a standard tempered operator-valued distribution, but a non-classical operator-valued function. We prove using this novel approach that the quantum field theory with Hamiltonian $P(\varphi)_{4}$ exists and that the canonical $C^{*}$- algebra of bounded observables corresponding to this model satisfies all the Haag-Kastler axioms except Lorentz covariance. We prove that the $\lambda(\varphi^{4})_{4}$ quantum field theory model is Lorentz covariant. For each Poincare transformation $a, \Lambda$ and each bounded region $O$ of Minkowski space we obtain a unitary operator $U$ which correctly transforms the field bilinear forms $\varphi(x,t)$ for $(x,t) \in O$. The von Neumann algebra $\mathfrak{C}(O)$ of local observables is obtained as standard part of external nonstandard algebra $\mathcal{B}_{\#}(O)$.

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