# Remarks on Möller mistaken famous paper from 1943

**Jaykov Foukzon**

jaykovfoukzon@list.ru

*Center for Mathematical Sciences, Israel Institute of Technology,Haifa,Israel*

**Abstract** Einstein field equations was originally derived by Einstein in 1915 in respect with canonical formalism of Riemann geometry,i.e. by using the classical sufficiently smooth metric tensor, smooth Riemann curvature tensor, smooth Ricci tensor,smooth scalar curvature, etc.. However have soon been found singular solutions of the Einstein field equations with *degenerate and singular* metric tensor and singular Riemann curvature tensor. These *degenerate and singular* solutions of the Einstein field equations was formally accepted by main part of scientific community beyond rigorous canonical formalism of Riemannian geometry.

## 1.The breakdown of canonical formalism of Riemann geometry for the singular solutions of the Einstein field equations

Einstein field equations was originally derived by Einstein in 1915 in respect with canonical formalism of Riemann geometry,i.e. by using the classical sufficiently smooth metric tensor, smooth Riemann curvature tensor, smooth Ricci tensor,smooth scalar curvature, etc.. However have soon been found singular solutions of the Einstein field equations with *degenerate and singular* metric tensor and singular Riemann curvature tensor. These *degenerate and singular* solutions of the Einstein field equations was formally accepted by main part of scientific community beyond rigorous canonical formalism of Riemannian geometry.

## 1.1.A. Einstein and N. famous paper from May 8,1935

In famous paper from May 8,1935 [1],(see [1],sec.1,p.74) A.Einstein originally emphasized that *degenerate (singular)* solutions of the Einstein field equations are problematic: "The first step to the general theory of relativity was to be found in the so-called "Principle of Equivalence": If in a space free from gravitation a reference system is uniformly accelerated, the reference system can be treated as being "at rest, " provided one interprets the condition of the space with respect to it as a homogeneous gravitational field. As is well known the latter is exactly described by the metric field

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + \alpha^2 x_1^2 dx_4^2 \qquad (1.1.1)$$

The $g_{\mu\nu}$ of this field satisfy in general the equations

$$R^i_{klm} = 0, \qquad (1.1.2)$$

and hence the equations

$$R_{kl} = R^m_{klm} = 0, \qquad (1.1.3)$$

The $g_{\mu\nu}$ corresponding to (1.1.1) are regular for all finite (i.e. nonzero) points of space-time. Nevertheless one cannot assert that Eqs.(1.1.3) are satisfied by (1.1.1) for *all* finite values of $x_1, \ldots, x_4$. This is due to the fact that the determinant $g$ of the $g_{\mu\nu}$ vanishes for $x_1 = 0$. The contravariant $g^{\mu\nu}$ therefore become infinite and the tensors $R^i_{klm}$ and $R_{kl}$ take on the form

$$0/0. \qquad (1.1.3')$$

From the standpoint of Eqs.(1.1.3) the hyperplane $x_1 = 0$ then represents a singularity of the field.

We now ask whether the field law of gravitation (and later on the field law of gravitation and electricity) could not be modified in a natural way without essential change so that the solution (1.1) would satisfy the field equations for all finite points, i.e., also for $x_1 = 0$.

W. Mayer has called our attention to the fact that one can make $R^i_{klm}$ and $R_{kl}$ into rational functions of the $g_{\mu\nu}$, and their first two derivatives by multiplying them by suitable powers of $g$. It is easy to show that in $g^2 R_{kl}$ there is no longer any denominator. If then we replace (1.1.3) by

$$R^*_{kl} = g^2 R_{kl} = 0, \qquad (1.1.3.a)$$

this system of equations is satisfied by (1.1.1) at all finite points. This amounts to introducing in place of the $g^{\mu\nu}$ the cofactors $[g_{\mu\nu}]$ of the $g_{\mu\nu}$ in $g$ in order to avoid the occurrence of denominators. One is therefore operating with tensor densities of a suitable weight instead of with tensors. In this way one succeeds in avoiding singularities of that special kind which is characterized by the vanishing of $g$.

The solution (1.1.1) naturally has no deeper physical significance insofar as it extends into spatial infinity. It allows one to see however to what extent the regularization of the hypersurfaces $g = 0$ leads to a theoretical representation of matter, regarded from the standpoint of the original theory. Thus, in the framework of the original theory one has the gravitational equations

$$R_{ik} - \frac{1}{2} g_{ik} R = -T_{ik}, \qquad (1.1.4)$$

where $T_{ik}$ is the tensor of mass or energy density. To interpret (1.1.1) in the framework of this theory we must approximate the line element by a slightly different one which avoids the singularity $g = 0$. Accordingly we introduce a small constant $\sigma$ and let

$$ds^2 = -dx_1^2 - dx_2^2 - dx_3^2 + (\alpha^2 x_1^2 + \sigma) dx_4^2 \qquad (1.1.1.a)$$

the smaller $\sigma$ ($> 0$) is chosen, the nearer does this gravitational field come to that of (1.1.1). If one calculates from this the (fictitious) energy tensor $T_{ik}$ one obtains as nonvanishing components

$$T_{22} = T_{23} = \frac{\alpha^2 \sigma}{(\alpha^2 x_1^2 + \sigma)^2}. \qquad (1.1.4.a)$$

We see then that the smaller one takes $\sigma$ the more is the tensor concentrated in the neighborhood of the hypersurface $x_1 = 0$. From the standpoint of the original theory the solution (1.1.1) contains a singularity which corresponds to an energy or mass concentrated in the surface $x_1 = 0$; from the standpoint of the modified theory, however, (1.1.1) is a solution of (1.1.3.a), free from singularities, which describes the "field-producing mass, " without requiring for this the introduction of any new field quantities.

It is clear that all equations of the absolute differential calculus can be written in a form free from denominators, whereby the tensors are replaced by tensor densities of suitable weight.

It is to be noted that in the case of the solution (1.1.1) the whole field consists of two equal halves, separated by the surface of symmetry $x_1 = 0$, such that for the corresponding points $(x_1, x_2, x_3, x_4)$ and $(-x_1, x_2, x_3, x_4)$ the $g_{ik}$ are equal. As a result we find that, although we are permitting the determinant g to take on the value $0$ $(x_1 = 0)$, no change of sign of $g$ and in general no change in the “inertial index” of the quadratic form (1.1.1) occurs. These features are of fundamental importance from the point of view of the physical interpretation, and will be encountered again in the solutions to be considered later."

## 1.2.Remarks on Möller ubnormal famous paper from 1943

Recall that the classical Cartan’s structural equations show in a compact way the relation
between a connection and its curvature, and reveals their geometric interpretation in
terms of moving frames. In order to study the mathematical properties of singularities, we
need to study the geometry of manifolds endowed on the tangent bundle with a symmetric
bilinear form which is allowed to become degenerate (singular). But if the fundamental
tensor is allowed to be degenerate (singular), there are some obstructions in constructing
the geometric objects normally associated to the fundamental tensor. Also, local
orthonormal frames and co-frames no longer exist, as well as the metric connection and
its curvature operator [2].

As an important example of the geometry with the fundamental tensor which is allowed to
be degenerate, we consider now Möller’s uniformly accelerated frame given by Möller’s
line element (1.2.1).

Recall that Möller dealing with the following line element [3]:

$$ds^2 = -\Delta(x)dt^2 + dx^2 + dy^2 + dz^2, \qquad (1.2.1)$$

where $\Delta(x) = (a + gx)^2$.

**Remark 1.2.1**. Of course Möller’s metric (1.2.1) degenerate at Möller horizon $x_{hor} = -a/g$.

However in contrast with A.Einstein paper [1],in famous but ubnormal paper [3] Möller
mistakenly argue that metric field (1.2.1) is an global vaccuum solution of the A.Einstein
field equations (1.1.4), i.e. the $g_{\mu\nu}$ of this field for all values of $t, x, y, z$ satisfy the

equations

$$R_{ik} - \frac{1}{2} g_{ik} R = 0. \tag{1.2.2}$$

**Remark 1.2.2**. In physical literature this Möller's ubnormal mistake holds from Möller's time until nowadays.

**Remark 1.2.3**. Note that formally corresponding to the Möller's metric (1.2.1) classical Levi-Civit'a connection is

$$\Gamma^1_{44}(x) = a + gx, \Gamma^4_{14}(x) = \Gamma^4_{41}(x) = g(a + gx)^{-1} \tag{1.2.3}$$

and therefore classical Levi-Civit'a connection (1.2.3) of course is not available at Möller

horizon since at horizon formal expressions (1.2.3) becomes infinity:

$$\Gamma^4_{14}\left(-\frac{a}{g}\right) = \Gamma^4_{41}\left(-\frac{a}{g}\right) = \infty. \tag{1.2.4}$$

**Remark 1.2.4**.Note that Möller dealing with Einstein's field equations in the following form

$$G^k_i = R^k_i - \frac{1}{2}\delta^k_i R, \tag{1.2.5}$$

where $R^k_i$ is the contracted Riemann-Christoffel tensor, formally calculated by canonical

way by using classical Levi-Civit'a connection (1.2.3) and $R = R^i_i$

By using the following ansatz

$$ds^2 = -D(x)dt^2 + dx^2 + dy^2 + dz^2, \tag{1.2.6}$$

Möller finally obtain

$$G^2_2 = G^3_3 = -\frac{1}{2D}\left[ D'' - \frac{(D')^2}{2D} \right] = -\frac{(D^{1/2})''}{D^{1/2}}. \tag{1.2.7}$$

where $D' = dD(x)/dx$.

**Remark 1.2.5**.From Eq.(1.2.7) Möller mistakenly obtain the following equation

$$(D^{1/2})'' = 0, \tag{1.2.8}$$

since it was mistakenly assumed that $G^2_2$ and $G^3_3$ for all values of $t,x,y,z$ satisfy the equations

$$G^2_2(x) = G^3_3(x) \equiv 0. \tag{1.2.9}$$

The equation (1.2.8) obviously has the following trivial general solution

$$G(x) = (a + gx)^2. \tag{1.2.10}$$

**Remark 1.2.6**.Note that at horizon $G^2_2$ and $G^3_3$ ofcourse is not zero but becomes uncertainty since

$$G^2_2(-a/g) = G^3_3(-a/g) = -\frac{\left([D(-a/g)]^{1/2}\right)''}{[D(-a/g)]^{1/2}} = \frac{0}{0} \tag{1.2.11}$$

in acordance with (1.1.3′) in A.Einstein paper [1].

Thus solution (1.2.10) obviously holds only except horizon $x_{hor} = -a/g$ as A.Einstein emphasize in paper [1]. For better explanation see [4].

# References


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