



## Set Theory INC# Based on Intuitionistic Logic with Restricted Modus Ponens Rule (Part. I)

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*The sole author designed, analyzed, interpreted and prepared the manuscript.*

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### Abstract

In this article Russell's paradox and Cantor's paradox resolved successfully using intuitionistic logic with restricted modus ponens rule.

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## 1 Introduction

Considering only pure sets, the naive set comprehension principle says, for any condition, that there is a set containing all and only the sets satisfying this condition. In first-order logic, this can be formulated as the following schematic principle, where  $\phi$  may be any formula in whichy does not occur freely:

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$$\exists y \forall x (x \in y \leftrightarrow \phi). \tag{1.1}$$

Russell's paradox shows that the instance obtained by letting  $\phi$  be  $x \notin x$  is inconsistent in classical logic. One response to the paradox is to restrict naive set comprehension by ruling out this and other problematic instances: only for each of some special conditions is it claimed there is a set containing all and only the sets satisfying the condition. Many well known set theories can be understood as instances of this generic response, differing in how they understand special. For example, the axiom schema of separation (1.1) in Zermelo-Fraenkel set theory (*ZF*) restricts set comprehension to conditions which contain, as a conjunct, the condition of being a member of some given set:

$$\exists y \forall x (x \in y \leftrightarrow \phi \wedge x \in z). \tag{1.2}$$

Similarly, in Quine's New Foundations (*NF*) set comprehension is restricted to conditions which are stratified, where  $\phi$  is stratified just in case there is a mapping  $f$  from individual variables to natural numbers such that for each subformula of  $\phi$  of the form  $x \in y$ ,  $f(y) = f(x) + 1$  and for each subformula of  $\phi$  of the form  $x = y$ ,  $f(x) = f(y)$ . Both of these responses block Russell's paradox by ruling out the condition  $x \notin x$ . Must every restriction of naive comprehension take the form of simply ruling out certain instances? In this article, I have suggest and explore a different approach. As we have seen, standard set comprehension axioms restrict attention to some special conditions: for each of these special conditions, they provide for the existence of a set containing all and only the sets which satisfy it.

Instead of restricting the conditions one is allowed to consider, we propose restricting the way in which the sets in question satisfy a given condition: for every condition, our comprehension axiom will assert the existence of a set containing all and only the sets satisfying that condition in a special way using intuitionistic first-order logic with restricted modus ponens rule.

## 2 Russell's Paradox Resolution using Intuitionistic First-order Logic with Restricted Modus Ponens Rule

### 2.1 Russell's Paradox

The comprehension principle (1.1) for the condition  $x \notin x$  gives

$$\exists \mathfrak{R} \forall x (x \in \mathfrak{R} \iff x \notin x). \tag{2.1}$$

Thus  $\mathfrak{R}$  is the set whose members are exactly those sets that are not members of themselves. It follows from (2.1)

$$\mathfrak{R} \in \mathfrak{R} \iff \mathfrak{R} \notin \mathfrak{R}. \tag{2.2}$$

Is  $\mathfrak{R}$  a member of itself? If it is, i.e.  $\mathfrak{R} \in \mathfrak{R}$  then it must satisfy the condition of not being a member of itself and so it is not, i.e.  $\mathfrak{R} \notin \mathfrak{R}$ . If it is not, then it must not satisfy the condition of not being a member of itself, and so it must be a member of itself. Since by classical logic only one case or the other one must hold – either  $\mathfrak{R}$  is a member of itself or it is not – it follows that the theory implies a contradiction known as Russell's paradox discovered by Bertrand Russell in 1901, see [1]-[6].

**Remark 2.1.** Remind classical logic mandates that any contradiction trivializes a theory by making every sentence of the theory provable. This is because, in classical logic, the following is a theorem:

$$\text{Ex Falso Quodlibet : } A \implies (\neg A \implies B). \quad (2.3)$$

**Remark 2.2.** Now, virtually the only way to avoid EFQ is to give up disjunctive syllogism also known as disjunction elimination :

$$\frac{P \vee \neg Q}{Q} \quad (2.4)$$

that is, given the usual definitions of the connectives, modus ponens! So altering basic sentential logic in this way is radical indeed – but possible.

**Remark 2.3.** Unfortunately, even giving up EFQ is not enough to retain a semblance of naive Cantor set theory (NC). One also has to give up the following additional theorem of basic sentential logic:

$$\text{Contraction: } (A \supset (A \supset B)) \supset (A \supset B). \quad (2.5)$$

It can then be argued that NC leads directly, not merely to an isolated contradiction, but to triviality. For the argument that this is so, see Curry’s paradox [7].

Thus it seems that the woes of NC are not confined to Russell’s paradox but also include a negation-free paradox due to Curry.

**Remark 2.4.** Another suggestion might be to conclude that the paradox depends upon an instance of the principle of Excluded Middle, that either  $\mathfrak{R}$  is a member of  $\mathfrak{R}$  or it is not. This is a principle that is rejected by some non-classical approaches to logic, including intuitionism [8].

Remind that in classical logic, we often discuss the truth values that a formula can take the values are usually chosen as the members of a Boolean algebra. The meet and join operations in the Boolean algebra are identified with the  $\wedge$  and  $\vee$  logical connectives, so that the value of a formula of the form  $A \wedge B$  is the meet of the value of  $A$  and the value of  $B$  in the Boolean algebra. Then we have the useful theorem that a formula is a valid proposition of classical logic if and only if its value is 1 for every valuation—that is, for any assignment of values to its variables. A corresponding theorem is true for intuitionistic logic, but instead of assigning each formula a value from a Boolean algebra, one uses values from an Heyting algebra, of which Boolean algebras are a special case. A formula is valid (or holds) in intuitionistic logic if and only if it receives the value of the top element for any valuation on any Heyting algebra. It can be shown that to recognize valid formulas, it is sufficient to consider a single Heyting algebra whose elements are the open subsets of the real line  $\mathbb{R}$  [8]. In this algebra we have:

- (1) **Value**  $[\perp] = \emptyset$ ,
- (2) **Value**  $[\top] = \mathbb{R}$ ,
- (3) **Value**  $[A \wedge B] = \text{Value}[A] \cap \text{Value}[B]$ ,
- (4) **Value**  $[A \vee B] = \text{Value}[A] \cup \text{Value}[B]$ ,

$$(5) \text{ Value } [A \implies B] = \text{Int} \left( \text{Value } [A]^c \cup \text{Value } [B] \right),$$

$$(6) \text{ Value } [\neg A] = \text{Int} \left( \text{Value } [A]^c \right), \text{ where } \text{Int}(X) \text{ is the interior of } X \text{ and } X^c \text{ its complement.}$$

**Remark 2.5.** With these assignments (1)-(6), intuitionistically valid formulas are precisely those that are assigned the value of the entire line [8]. For example, the formula  $\neg(A \wedge \neg A)$  is valid, since  $\text{Value}[\neg(A \wedge \neg A)] = \mathbb{R}$ . So the valuation of this formula is true, and indeed the formula is valid. But the law of the excluded middle,  $A \vee \neg A$ , can be easily shown to be invalid by using a specific value of the set of positive real numbers for  $A$ :  $\text{Value}[A] = \{x|x > 0\} = \mathbb{R}_+$ . For such  $A$  one obtains  $\text{Value}[\neg(A \wedge \neg A)] \neq \mathbb{R}$ .

We do now as follows:

**Case I.** Assume now that: (a)  $\mathfrak{R} \in \mathfrak{R}$  holds, i.e.  $\text{Value}[\mathfrak{R} \in \mathfrak{R}] = \mathbb{R}$  and therefore  $\mathfrak{R} \notin \mathfrak{R}$  is not holds, since  $\text{Value}[\mathfrak{R} \notin \mathfrak{R}] = \emptyset$ .

From (2.2) it follows that (b)  $\mathfrak{R} \in \mathfrak{R} \implies \mathfrak{R} \notin \mathfrak{R}$ . From (a) and (b) by modus ponens rule it follows that

$$\mathfrak{R} \in \mathfrak{R}, \mathfrak{R} \in \mathfrak{R} \implies \mathfrak{R} \notin \mathfrak{R} \vdash \mathfrak{R} \notin \mathfrak{R}. \quad (2.6)$$

From (2.6) and (a) one obtains the following formula  $\mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R}$ . But by the Law of Non-contradiction we know that  $\neg(\mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R})$ . Thus we obtain a contradiction and therefore  $\mathfrak{R} \in \mathfrak{R}$  is not holds.

**Case II.** Assume now that:

(a)  $\mathfrak{R} \notin \mathfrak{R}$  holds, i.e.  $\text{Value}[\mathfrak{R} \notin \mathfrak{R}] = \mathbb{R}$  and therefore  $\mathfrak{R} \in \mathfrak{R}$  is not holds, since  $\text{Value}[\mathfrak{R} \in \mathfrak{R}] = \emptyset$ .

From (2.2) it follows that (b)  $\mathfrak{R} \notin \mathfrak{R} \implies \mathfrak{R} \in \mathfrak{R}$ . From (a) and (b) by modus ponens rule it follows that

$$\mathfrak{R} \notin \mathfrak{R}, \mathfrak{R} \notin \mathfrak{R} \implies \mathfrak{R} \in \mathfrak{R} \vdash \mathfrak{R} \in \mathfrak{R}. \quad (2.7)$$

From (2.7) and (b) one obtains the following formula  $\mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R}$ . But by the Law of Non-contradiction we know that  $\neg(\mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R})$ . Thus we obtain a contradiction and therefore  $\mathfrak{R} \notin \mathfrak{R}$  is not holds. Thus both  $\mathfrak{R} \in \mathfrak{R}$  and  $\mathfrak{R} \notin \mathfrak{R}$  is not holds, but by absent the Excluded Middle but by absent the law Excluded Middle this does not pose any problems.

**Remark 2.6.** However it well known that it is possible to derive the contradiction only from the statement (2.2) i.e.,  $\mathfrak{R} \in \mathfrak{R} \iff \mathfrak{R} \notin \mathfrak{R}$ . We do so as follows:

Assume now that:  $\mathfrak{R} \in \mathfrak{R} \iff \mathfrak{R} \notin \mathfrak{R}$  holds and therefore  $\mathfrak{R} \in \mathfrak{R} \implies \mathfrak{R} \notin \mathfrak{R}$ .

But we also know that  $\mathfrak{R} \in \mathfrak{R} \implies \mathfrak{R} \in \mathfrak{R}$ . So  $\mathfrak{R} \in \mathfrak{R} \implies \mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R}$ .

But by the Law of Non-contradiction we know that  $\neg(\mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R})$

So by modus tollens we conclude that  $\mathfrak{R} \notin \mathfrak{R}$ .

At the same time we also know that  $\mathfrak{R} \notin \mathfrak{R} \implies \mathfrak{R} \in \mathfrak{R}$ , and thus by modus ponens we conclude that  $\mathfrak{R} \in \mathfrak{R}$ .

So we can deduce both  $\mathfrak{R} \in \mathfrak{R}$  and its negation  $\mathfrak{R} \notin \mathfrak{R}$  using only intuitionistically acceptable methods.

**Remark 2.7.** Another suggestion might be to conclude that the paradox depends upon an instance of the Law of Non-contradiction, that  $\neg(\mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R})$ . This is a principle that is rejected by some non-classical approaches to logic, including paraconsistent logic [9]. Nevertheless even paraconsistent logic can not safe  $NC$  from a triviality [9].

Da Costa's paraconsistent set theories of type  $NF_\omega^C$  and  $NF_n^C$ ,  $1 \leq n \leq \omega$ . has been studying A.I. Arruda [9].

Remind that the main postulates of  $NF_\omega^C$  are the following [9]:

I. Extensionality

$$\forall \alpha \forall \beta \forall x [x \in \alpha \iff x \in \beta \implies \alpha = \beta]. \quad (2.8)$$

II. Abstraction

$$\exists \alpha \forall x [x \in \alpha \iff F(x)], \quad (2.9)$$

where  $\alpha$  does not occur free in  $F(x)$  and  $F(x)$  is stratified or it does not contain any formula of the form  $A \implies B$ .

A.I. Arruda has been proved that da Costa's formulation of the axiom schema of abstraction (2.9) for the systems  $NF_n$ ,  $1 \leq n < \omega$ , leads to the trivialization of the systems, see [9].

**Remark 2.8.** Note that in  $NF_\omega^C$ , the restrictions regarding the use of non-stratified formulas obstruct a direct proof of the paradox of Curry. Russell's set  $\mathfrak{R}$ , defined as  $\widehat{x} \neg(x \in x)$ , exists as well as many other non-classical sets. The paradox of Russell in the form  $\mathfrak{R} \in \mathfrak{R} \wedge \neg(\mathfrak{R} \in \mathfrak{R})$  is derivable but apparently, it causes no ham to the system.

Due to its weakness, the primitive negation of  $NF_\omega^C$ ,  $\neg$ , is almost useless for set-theoretical purposes. Thus, let us define

$$\sim A \text{ for } A \implies \forall x \forall y [x \in y \wedge x = y]. \quad (2.10)$$

The universal set  $\mathbf{V}$  is defined as  $\widehat{x}(x = x)$ , the empty set  $\emptyset$  as  $\{x | \sim(x = x)\}$ , and the complement of a set  $\alpha$ ,  $\bar{\alpha}$ , as  $\{x | \sim(x \in \alpha)\}$ .

**Theorem 2.1.**[8]. In  $NF_\omega^C$ ,  $\sim$  is a minimal intuitionistic negation.

**Corollary 2.1.**  $\vdash A \implies (\sim A \implies \sim B), \vdash (A \implies B) \implies (\sim B \implies \sim A)$ .

**Corollary 2.2.** All the theorems of  $NF$  whose proofs depend only on the laws of the minimal intuitionistic first-order logic with equality and on the postulates of extensionality and abstraction of  $NF$  are valid in  $NF_\omega^C$ .

**Theorem 2.2.**[9].(Cantor's Theorem)  $NF_\omega^C \vdash \sim(\alpha \leq P(\alpha))$ .

**Corollary 2.3**[9].(Cantor's Paradox)  $NF_\omega^C \vdash (\mathbf{V} \leq P(\mathbf{V})) \wedge \sim(\mathbf{V} \leq P(\mathbf{V}))$ .

**Remark 2.9.**Note that Cantor's paradox does not trivialize  $NF_\omega^C$ , since from  $A$  and  $\neg A$  we cannot

obtain any formula  $B$  whatsoever. For instance, apparently, we cannot obtain any formula of the form  $\neg B$ , where  $B$  is a nonatomic formula.

**Theorem 5.3.**(Paradox of identity) [9].(i)  $NF_{\omega}^C \vdash \forall\alpha\forall\beta [(\alpha = \beta) \wedge \sim (\alpha = \beta)]$ ,

(ii)  $NF_{\omega}^C \vdash [(\alpha \in \beta) \wedge \sim (\alpha \in \beta)]$ ,

(iii)  $NF_{\omega}^C \vdash [(\alpha \in \alpha) \wedge \sim (\alpha \in \alpha)]$ .

**Proof.** By the corollaries 2.1 and 2.2, we obtain

$$\begin{aligned} x = x &\implies \delta, \\ \delta &\iff \forall\alpha\forall\beta [(\alpha \in \beta) \wedge (\alpha = \beta)] \end{aligned} \tag{2.11}$$

Thus, as  $x = x$ , then  $\forall\alpha\forall\beta (\alpha = \beta)$ . By the same corollaries we also obtain  $\forall\alpha\forall\beta [\sim (\alpha = \beta)]$ . The proof of part (ii) is similar to that of part (i). Part (iii) is an immediate consequence of part (ii).

**Remark 2.10.** The paradox of identity obviously trivialized paraconsistent set theory  $NF_{\omega}^C$ .

Thus paraconsistent logics cannot resolved the problem.

## 2.2 The restricted rules of inference

### The restricted modus ponens rule

The canonical (unrestricted) modus ponens rule may be written in sequent notation as

$$P, P \implies Q \vdash_{MP} Q, \tag{2.12}$$

where  $P, Q$  and  $P \implies Q$  are statements (or propositions) in a formal language and  $\vdash_{MP}$  is a metalogical symbol meaning that  $Q$  is a syntactic consequence of  $P$  and  $P \implies Q$  in some logical system, see [10]-[11].

**Truth Table 1.**

$P$	$Q$	$P \implies Q$
<b>T</b>	<b>T</b>	<b>T</b>
<b>T</b>	<b>F</b>	<b>F</b>
<b>F</b>	<b>T</b>	<b>T</b>
<b>F</b>	<b>F</b>	<b>T</b>

The validity of modus ponens in classical two-valued logic can be clearly demonstrated by use of a truth table. In instances of modus ponens we assume as premises that  $P \implies Q$  is true and  $P$  is true. Only one line of the truth table—the first—satisfies these two conditions:  $P$  and  $P \implies Q$ . On this line,  $Q$  is also true. Therefore, whenever  $P \implies Q$  is true and  $P$  is true,  $Q$  must also be true.

Let  $\mathbf{wff} = \mathbf{wff}()$  be a set of the all wff's corresponding to formal language.

The restricted modus ponens rule  $\vdash_{RMP}$  may be written in sequent notation as

$$P, P \implies Q \vdash_{RMP} Q \text{ iff } P \notin \Delta_1 \text{ and } (P \implies Q) \notin \Delta_2, \tag{2.13}$$

where  $\Delta_1, \Delta_2 \subset \mathbf{wff}$ . Therefore the restricted modus ponens rule  $\vdash_{RMP}$  meant that

$$P, P \implies Q \not\vdash_{RMP} Q \quad (2.14)$$

if  $P \in \Delta_1$  or  $(P \implies Q) \in \Delta_2$  or  $(P \in \Delta_1) \wedge ((P \implies Q) \in \Delta_2)$ .

**The restricted disjunction elimination rule.**

In propositional logic, canonical (unrestricted) disjunctive syllogism or modus tollendo ponens (MTP) also known as disjunction elimination rule. The rule makes it possible to eliminate a disjunction from a logical proof. It is the rule that:

$$P \vee Q, \neg P \vdash_{MTP} Q \quad (2.15)$$

and can be expressed by truth-functional tautology of propositional logic:

$$((P \vee Q) \wedge \neg P) \implies Q \quad (2.16)$$

The restricted disjunction elimination rule  $\vdash_{RMTP}$  may be written in sequent notation as

$$P \vee Q, \neg P \vdash_{RMTP} Q \text{ iff } P \notin \overline{\Delta}_1 \text{ and } Q \in \overline{\Delta}_2, \quad (2.17)$$

where  $\overline{\Delta}_1, \overline{\Delta}_2 \subseteq_{\mathbf{wff}}$ . In addition we set  $((P \vee Q) \wedge \neg P) \in \Delta_1$  and

$$(((P \vee Q) \wedge \neg P) \implies Q) \in \Delta_2 \text{ iff } P \in \overline{\Delta}_1 \text{ and } Q \in \overline{\Delta}_2.$$

Therefore the restricted disjunction elimination rule  $\vdash_{RMTP}$  meant that

$$P \vee Q, \neg P \not\vdash_{RMTP} Q \quad (2.18)$$

iff  $P \in \overline{\Delta}_1$  or  $Q \in \overline{\Delta}_2$  or  $(P \in \overline{\Delta}_1) \wedge (Q \in \overline{\Delta}_2)$ .

**The restricted modus tollens rule.**

The canonical (unrestricted) modus tollens rule may be written in sequent notation as

$$P \implies Q, \neg Q \vdash_{MT} \neg P, \quad (2.19)$$

where  $\vdash_{MT}$  is a metalogical symbol meaning that  $\neg P$  is a syntactic consequence of  $P \implies Q$  and  $\neg Q$  in some logical system; or by the statement of a functional tautology of propositional logic:

$$((P \implies Q) \wedge \neg Q) \implies \neg P. \quad (2.20)$$

The validity of modus tollens can be clearly demonstrated through a truth table1.

In instances of the canonical modus tollens we assume as premises that  $P \implies Q$  is true and  $Q$  is false. There is only one line of the truth table1-the fourth line-which satisfies these two conditions. In this line,  $P$  is false. Therefore, in every instance in which  $P \implies Q$  is true and  $Q$  is false,  $P$  must also be false. The restricted modus tollens rule may be written in sequent notation as

$$P \implies Q, \neg Q \vdash_{RMT} \neg P \text{ iff } (P \implies Q) \notin \Delta'_1 \text{ and } Q \notin \Delta'_2, \quad (2.21)$$

where  $\Delta'_1, \Delta'_2 \subseteq_{\mathbf{wff}}$ . Therefore the restricted modus tollens rule meant that

$$P \implies Q, \neg Q \not\vdash_{RMT} \neg P \quad (2.22)$$

if  $(P \implies Q) \in \Delta'_1$  or  $Q \in \Delta'_2$  or  $((P \implies Q) \in \Delta'_1) \wedge (Q \in \Delta'_2)$ .

## 2.3 Curry's paradox resolution using bivalent logic with restricted modus ponens rule

In set theories that allow unrestricted comprehension, we can nevertheless prove any logical statement  $\Phi$  by examining the set

$$X = \{x | x \in x \implies \Phi\}.$$

Assuming that  $\in$  takes precedence over both  $\implies$  and  $\iff$ , the proof proceeds as follows:

1.  $X = \{x | x \in x \implies \Phi\}$  [Definition of  $X$ ]
2.  $x = X \implies (x \in x \iff X \in X)$  [Substitution of equal sets in membership]
3.  $x = X \implies ((x \in x \implies \Phi) \iff (X \in X \implies \Phi))$   
[Addition of a consequent to both sides of a biconditional (from 2)]
4.  $X \in X \iff (X \in X \implies \Phi)$  [Law of concretion (from 1 and 3)]
5.  $X \in X \implies (X \in X \implies \Phi)$  [Biconditional elimination (from 4)]
6.  $X \in X \implies \Phi$  [Contraction (from 5)]
7.  $(X \in X \implies \Phi) \implies X \in X$  [Biconditional elimination (from 4)]
8.  $X \in X$  [Unrestricted modus ponens  $\vdash_{UMP}$  (from 6 and 7)]
9.  $\Phi$  [Unrestricted Modus ponens  $\vdash_{UMP}$  (from 8 and 6)] since  
 $X \in X, X \in X \implies \Phi \vdash_{UMP} \Phi$ .

Curry's paradox violated  $NC$  since any  $\Phi$  statement is provable. Therefore, in a consistent set theory, the set  $\{x | x \in x \rightarrow \Phi\}$  does not exist for false  $\Phi$  such that  $0 = 1$ , etc. Some proposals for set theory have attempted to deal with Curry's paradox not by restricting the rule of comprehension, but by restricting the deduction rules of canonical logic [7]. The existence of proofs like the one above shows that at least one of the deduction rules used in the proof above must be restricted.

It is clear that in order to avoid Curry's paradox only modus ponens rule must be restricted as mentioned above in subsection 2.2.

Let  $LP^\#$  be bivalent predicate calculus with restricted modus ponens rule. Let  $NC^\#$  be Cantor set theory with unrestricted comprehension and equipped with bivalent predicate calculus  $LP^\#$ . Let  $\# =^\# (NC^\#)$  be formal language corresponding to set theory  $NC^\#$ . Let  $\#_{\mathbf{wff}} =^\#_{\mathbf{wff}} (\#)$  be a set of the all closed wff's of the language  $\#$ .

Let  $X[\Phi]$  be a set  $X[\Phi] = \{x | x \in x \implies \Phi\}$ , where  $\Phi \in \#_{\mathbf{wff}}$ .

$$X = \{x | x \in x \implies \Phi\}.$$

Assuming that  $\in$  takes precedence over both  $\implies$  and  $\iff$ , the proof proceeds as follows:

1.  $X[\Phi] = \{x | x \in x \implies \Phi\}$  [Definition of  $X$ ]
2.  $x = X[\Phi] \implies (x \in x \iff X[\Phi] \in X[\Phi])$  [Substitution of equal sets in membership]
3.  $x = X[\Phi] \implies ((x \in x \implies \Phi) \iff (X[\Phi] \in X[\Phi] \implies \Phi))$   
[Addition of a consequent to both sides of a biconditional (from 2)]
4.  $X[\Phi] \in X[\Phi] \iff (X[\Phi] \in X[\Phi] \implies \Phi)$  [Law of concretion (from 1 and 3)]



5.  $X[\Phi] \in X[\Phi] \implies (X[\Phi] \in X[\Phi] \implies \Phi)$  [Biconditional elimination (from 4)]
6.  $X[\Phi] \in X[\Phi] \implies \Phi$  [Contraction (from 5)]
7.  $(X[\Phi] \in X[\Phi] \implies \Phi) \implies X[\Phi] \in X[\Phi]$  [Biconditional elimination (from 4)]
8.  $X[\Phi] \in X[\Phi]$  [Unrestricted modus ponens  $\vdash_{UMP}$  (from 6 and 7)]

Let  $\tilde{\Delta}_1$  be a set of the all closed wff's corresponding to a set  $\overset{\#}{\text{wff}}$  such that

$$\tilde{\Delta} = \{(0 = 1)\} \cup \{\Phi | \Phi \iff (0 = 1)\}.$$

Let  $\Delta_1$  be a set of the all closed wff's  $\Psi[\Phi]$  corresponding to a set  $\overset{\#}{\text{wff}}$  such that  $\Psi[\Phi] = (X[\Phi] \in X[\Phi])$  with  $\Phi \in \tilde{\Delta}$ . Let  $\Delta_2$  be a set of the all closed wff's  $F[\Phi]$  corresponding to a set  $\overset{\#}{\text{wff}}$  such that  $F[\Phi] = X[\Phi] \in X[\Phi] \implies \Phi$  with  $\Phi \in \tilde{\Delta}$ .

Thus from  $X[\Phi] \in X[\Phi]$  and  $X[\Phi] \in X[\Phi] \implies \Phi$ , we conclude  $\Phi$  if and only if

$$(X[\Phi] \in X[\Phi]) \notin \Delta_1 \text{ and } (X[\Phi] \in X[\Phi] \implies \Phi) \notin \Delta_2, \text{ where } \Delta_1, \Delta_2 \overset{\#}{\not\subseteq} \text{wff}$$

### 3 Russell's Paradox Resolution using Intuitionistic First-order Logic with Restricted Modus Ponens Rule

#### 3.1 The intuitionistic propositional calculus $\mathbf{Pp}^\#$ with restricted modus ponens rule

The first step in the metamathematical study of any part of logic or mathematics is to specify a formal language . For propositional or sentential logic, the standard language has denumerably many distinct proposition letters  $P_0, P_1, P_2, \dots$  and symbols  $\&, \vee, \rightarrow, \neg, \perp$  for the propositional connectives “and,” “or,” “if ...then,” and “not” respectively, with left and right parentheses (,) (sometimes written “[, ]” for ease of reading). Classical logic actually needs only two connectives (since classical  $\vee$  and  $\rightarrow$  can be defined in terms of  $\&$  and  $\neg$ ), but the four intuitionistic connectives are independent. The classical language is thus properly contained in the intuitionistic, which is more expressive. The most important tool of metamathematics is generalized induction, a method Brouwer endorsed. The class of wff's (well-formed formulas) of the language ( $\mathbf{Pp}^\#$ ) of  $\mathbf{Pp}^\#$  is defined inductively by the rules:

- (i) Each proposition letter is a (prime) formula.
- (ii) If  $A, B$  are formulas so are  $(A \& B), (A \vee B), (A \rightarrow B)$  and  $(\neg A)$ .
- (iii) Nothing is a formula except as required by (i) and (ii).
- (iv) The class of wff's of the language ( $\mathbf{Pp}^\#$ ) we will denote by  $\overset{\#}{\text{wff}}(\mathbf{Pp}^\#)$ .

As in classical logic,  $(A \leftrightarrow B)$  abbreviates  $((A \rightarrow B) \& (B \rightarrow A))$ .

The axioms are all formulas of the following forms:

$$\mathbf{Pp}^\# \text{ 1. } A \rightarrow (B \rightarrow A).$$

- Pp<sup>#</sup>** 2.  $(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C)).$   
**Pp<sup>#</sup>** 3.  $A \rightarrow (B \rightarrow A \& B).$   
**Pp<sup>#</sup>** 4.  $A \& B \rightarrow A.$   
**Pp<sup>#</sup>** 5.  $A \& B \rightarrow B.$   
**Pp<sup>#</sup>** 6.  $A \rightarrow A \vee B.$   
**Pp<sup>#</sup>** 7.  $B \rightarrow A \vee B.$   
**Pp<sup>#</sup>** 8.  $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow (A \vee B \rightarrow C)).$   
**Pp<sup>#</sup>** 9.  $(A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A).$   
**Pp<sup>#</sup>** 10.  $\neg A \rightarrow (A \rightarrow B).$   
**Pp<sup>#</sup>** 11.  $\perp \rightarrow A.$

**Remark 3.1.** The system of classical logic is obtained by adding any one of the following axioms:

1.  $\phi \vee \neg \phi$  (Law of the excluded middle. May also be formulated as

- $(\phi \rightarrow \chi) \rightarrow ((\neg \phi \rightarrow \chi) \rightarrow \chi)$   
 2.  $\neg \neg \phi \rightarrow \phi$  (Double negation elimination)  
 $((\phi \rightarrow \chi) \rightarrow \phi) \rightarrow \phi$  (Peirce's law)  
 $(\neg \phi \rightarrow \neg \chi) \rightarrow (\chi \rightarrow \phi)$  (Law of contraposition)

The rules of inference of **Pp<sup>#</sup>** is

**R<sup>#</sup>1.RMP** (Restricted Modus Ponens).

From  $A$  and  $A \rightarrow B$ , conclude  $B$  iff  $A \notin \Delta_1$  and  $(A \rightarrow B) \notin \Delta_2$ , where  $\Delta_1, \Delta_2 \not\subseteq_{\text{wff}} (\mathbf{Pp}^\#)$

We abbreviate by  $A, A \rightarrow B \vdash_{RMP} B$ .

**R<sup>#</sup>2.MT** (Restricted Modus Tollens)

$P \rightarrow Q, \neg Q \vdash_{RMT} \neg P$  iff  $P \notin \Delta'_1$  and  $(P \rightarrow Q) \notin \Delta'_2$ , where  $\Delta'_1, \Delta'_2 \not\subseteq_{\text{wff}} (\mathbf{Pp}^\#)$

If  $\Gamma$  is any collection of formulas and  $E_1, \dots, E_k$  any finite sequence of formulas each of which is a member of  $\Gamma$ , an axiom, or an immediate consequence by RMP of two preceding formulas, then  $E_1, \dots, E_k$  is a derivation in **Pp<sup>#</sup>** of its last formula  $E_k$  from the assumptions  $\Gamma$ . We write  $\Gamma \vdash_{Pp^\#} E$  to denote that such a derivation exists with  $E_k = E$ . The following theorem is proved by induction over the definition of a derivation; its converse follows from **R<sup>#</sup>1**.

**Deduction Theorem.** If  $\Gamma$  is any collection of formulas and  $A, B$  are any formulas such that  $\Gamma \cup \{A\} \vdash_{RMP} B$ , then also  $\Gamma \vdash_{RMP} (A \rightarrow B)$ .

### 3.2 The intuitionistic first-order predicate calculus **Pd<sup>#</sup>** with restricted modus ponens rule

The pure first order language (**Pd<sup>#</sup>**) has individual variables  $a_1, a_2, a_3, \dots$ , and countably infinitely many distinct predicate letters  $P_1(\dots), P_2(\dots), P_3(\dots), \dots$  of arity  $n$  for each  $n = 0, 1, 2, \dots$ , including the 0-ary proposition letters. There are two new logical symbols  $\forall$  ("for all") and  $\exists$  ("there exists"). The terms of the language (**Pd<sup>#</sup>**) of **Pd<sup>#</sup>** are the individual variables. The well formed formulas are defined by the rules:

- (i) if  $P(\dots)$  is an  $n$ -ary predicate letter and  $t_1, \dots, t_n$  are terms then  $P(t_1, \dots, t_n)$  is a (prime) formula.

- (ii) If  $A, B$  are formulas so are  $(A \& B), (A \vee B), (A \rightarrow B)$  and  $(\neg A)$ .
- (iii) If  $A$  is a formula and  $x$  an individual variable, then  $(\forall x A)$  and  $(\exists x A)$  are formulas.
- (iv) Nothing else is a formula.
- (v) The class of wff's of the language  $(\mathbf{Pd}^\#)$  we will denote by  $\mathbf{wff}(\mathbf{Pd}^\#)$

We use  $x, y, z, w, x_1, y_1, \dots$  and  $A, B, C, \dots, A(x), A(x, y), \dots$  as metavariables for variables and formulas, respectively. Anticipating applications (e.g. to arithmetic),  $s, t, s_1, t_1, \dots$  vary over terms. In omitting parentheses,  $\forall x$  and  $\exists x$  are treated like  $\neg$ . The scope of a quantifier, and free and bound occurrences of a variable in a formula, are defined as usual. A formula in which every variable is bound is a sentence or closed formula.

If  $x$  is a variable,  $t$  a term, and  $A(x)$  a formula which may or may not contain  $x$  free, then  $A(t)$  denotes the result of substituting an occurrence of  $t$  for each free occurrence of  $x$  in  $A(x)$ . The substitution is free if no free occurrence in  $t$  of any variable becomes bound in  $A(t)$ ; in this case we say  $t$  is free for  $x$  in  $A(x)$ .

In addition to  $\mathbf{Pp1} - \mathbf{Pp11}$ ,  $\mathbf{Pd}^\#$  has two new axiom schemas, where  $A(x)$  may be any formula and  $t$  any term free for  $x$  in  $A(x)$  :

$\mathbf{Pd}^\#12.$   $\forall x A(x) \rightarrow A(t)$ .

$\mathbf{Pd}^\#13.$   $A(t) \rightarrow \exists x A(x)$ .

The rules of inference are:

$\mathbf{R}^\#1.$  RMP (Restricted Modus Ponens).

From  $A$  and  $A \rightarrow B$ , conclude  $B$  iff  $A \notin \Delta_1$  and  $(A \rightarrow B) \notin \Delta_2$ , where  $\Delta_1, \Delta_2 \subseteq_{\mathbf{wff}}(\mathbf{Pd}^\#)$

We abbreviate  $\mathbf{R}^\#1$  by  $A, A \rightarrow B \vdash_{RMP} B$ .

$\mathbf{R}^\#2.$  MT (Restricted Modus Tollens)

$P \rightarrow Q, \neg Q \vdash_{RMT} \neg P$  iff  $P \notin \Delta'_1$  and  $(P \rightarrow Q) \notin \Delta'_2$ , where  $\Delta'_1, \Delta'_2 \subseteq_{\mathbf{wff}}(\mathbf{Pd}^\#)$

$\mathbf{R}^\#3.$  From  $C \rightarrow A(x)$  where  $x$  does not occur free in  $C$ , conclude  $C \rightarrow \forall x A(x)$ .

$\mathbf{R}^\#4.$  From  $A(x) \rightarrow C$  where  $x$  does not occur free in  $C$ , conclude  $\exists x A(x) \rightarrow C$ .

A deduction (or derivation) in  $\mathbf{Pd}^\#$  of a formula  $E$  from a collection  $\Gamma$  of assumption formulas is a finite sequence of formulas, each of which is an axiom by  $\mathbf{Pd}^\#1 - \mathbf{Pd}^\#13$ , or a member of  $\Gamma$ , or follows immediately by  $\mathbf{R}^\#1, \mathbf{R}^\#2$  or  $\mathbf{R}^\#3$  from one or two formulas occurring earlier in the sequence. A proof is a deduction from no assumptions. If  $\Gamma$  is a collection of sentences and  $E$  a formula, the notation  $\Gamma \vdash_{RMP} E$  means that a deduction of  $E$  from  $\Gamma$  exists. If  $\Gamma$  is a collection of formulas, we write  $\Gamma \vdash_{RMP} E$  only if there is a deduction of  $E$  from  $\Gamma$  in which neither  $\mathbf{R}^\#2$  nor  $\mathbf{R}^\#3$  is used with respect to any variable free in  $\Gamma$ . With this restriction, the deduction theorem extends to  $\mathbf{Pd}^\#$ : If  $\Gamma \cup \{A\} \vdash_{RMP} B$  then  $\Gamma \vdash_{RMP} (A \rightarrow B)$ . such that  $\Gamma \cup \{A\} \vdash_{RMP} B$ , then also  $\Gamma \vdash_{RMP} (A \rightarrow B)$ .

### 3.3 Russell's paradox resolution using first-order predicate calculus $\mathbf{Pd}^\#$ with restricted modus ponens rule

Assume now that:  $\mathfrak{R} \in \mathfrak{R} \iff \mathfrak{R} \notin \mathfrak{R}$  holds and therefore  $\mathfrak{R} \in \mathfrak{R} \implies \mathfrak{R} \notin \mathfrak{R}$ .

**Remark 3.2.** We set now  $(\mathfrak{R} \notin \mathfrak{R}) \in \Delta_1$  and  $(\mathfrak{R} \notin \mathfrak{R} \implies \mathfrak{R} \in \mathfrak{R}) \in \Delta_2$ .

We also know that  $\mathfrak{R} \in \mathfrak{R} \implies \mathfrak{R} \in \mathfrak{R}$ . So  $\mathfrak{R} \in \mathfrak{R} \implies \mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R}$ .

But by the Law of Non-contradiction we know that  $\neg(\mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R})$

So by canonical (unrestricted) modus tollens we conclude that  $\mathfrak{R} \notin \mathfrak{R}$ .

At the same time we also know that  $\mathfrak{R} \notin \mathfrak{R} \implies \mathfrak{R} \in \mathfrak{R}$ , and thus by restricted modus ponens we can not conclude that  $\mathfrak{R} \in \mathfrak{R}$ .

From  $\mathfrak{R} \in \mathfrak{R} \iff \mathfrak{R} \notin \mathfrak{R}$  we obtain  $\mathfrak{R} \notin \mathfrak{R} \implies \mathfrak{R} \in \mathfrak{R}$ . We also know that  $\mathfrak{R} \notin \mathfrak{R} \implies \mathfrak{R} \notin \mathfrak{R}$ . So  $\mathfrak{R} \notin \mathfrak{R} \implies \mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R}$ . But by the Law of Non-contradiction we know that  $\neg(\mathfrak{R} \in \mathfrak{R} \wedge \mathfrak{R} \notin \mathfrak{R})$ ,

So by unrestricted modus tollens we conclude that  $\neg(\mathfrak{R} \notin \mathfrak{R})$  and therefore we obtain that  $\neg(\mathfrak{R} \notin \mathfrak{R}) \ \& \ \mathfrak{R} \notin \mathfrak{R}$ . We set now  $(\mathfrak{R} \notin \mathfrak{R}) \in \Delta'_1$  and  $(\mathfrak{R} \notin \mathfrak{R} \implies \mathfrak{R} \in \mathfrak{R}) \in \Delta'_2$ , and thus by restricted modus tollens we can not conclude that  $\neg(\mathfrak{R} \notin \mathfrak{R})$ .

Thus by using calculus  $\mathbf{Pd}^\#$  with restricted modus ponens rule and restricted modus tollens Russell's paradox disappears.

## 4 Intuitionistic Set Theory $\mathbf{INC}^\#$ Based on First-order Predicate Calculus $\mathbf{Pd}^\#$ with Restricted Modus Ponens Rule

### AXIOMS AND BASIC DEFINITIONS

Intuitionistic set theory  $\mathbf{INC}^\#$  is formulated as a system of axioms in the same first order language as its classical counterpart, only based on intuitionistic logic with restricted modus ponens rule.. The language of set theory is a first-order language  $^\#$  with equality  $=$ , which includes a binary symbol  $\in$ . We write  $x \neq y$  for  $\neg(x = y)$  and  $x \notin y$  for  $\neg(x \in y)$ . Individual variables  $x, y, z, \dots$  of  $^\#$  will be understood as ranging over classical sets. The unique existential quantifier  $\exists!$  is introduced by writing, for any formula  $\varphi(x)$ ,  $\exists!x\varphi(x)$  as an abbreviation of the formula  $\exists x[\varphi(x) \ \& \ \forall y(\varphi(y) \implies x = y)]$ .  $^\#$  will also allow the formation of terms of the form  $\{x|\varphi(x)\}$ , for any formula  $\varphi$  containing the free variable  $x$ . Such terms are called nonclassical sets; we shall use upper case letters  $A, B, \dots$  for such sets. For each nonclassical set  $A = \{x|\varphi(x)\}$  the formulas  $\forall x[x \in A \iff \varphi(x)]$  and  $\forall x[x \in A \iff \varphi(x, A)]$  is called the defining axioms for the nonclassical set  $A$ .

**Remark 4.1.** Note that (1) the formula  $\forall x[x \in A \iff \varphi(x)]$  and  $\forall x[x \in a \iff \varphi(x) \wedge x \in u]$  is not always asserts that  $\forall x[x \in A \vdash_{RMP} \varphi(x)]$  and (or)  $\forall x[\varphi(x) \vdash_{RMP} x \in A]$  even for a classical set since for some  $y$  possible  $y \in A \implies \varphi(y) \not\vdash_{RMP} \varphi(y)$  and (or)

$\varphi(y) \implies y \in A \not\vdash_{RMP} x \in A$  and  $y \in a \implies \varphi(y) \wedge y \in u \not\vdash_{RMP} \varphi(y) \wedge y \in u$ , etc. In order to

emphasize this fact we often write the defining axioms for the nonclassical set in the following form  $\forall x [x \in A \iff {}_w\varphi(x)]$

**Remark 4.2.**(1) Two nonclassical sets  $A, B$  are defined to be equal and we write  $A = B$  if  $\forall x [x \in A \iff x \in B]$ . (2)  $A$  is a subset of  $B$ , and we write  $A \subseteq B$ , if  $\forall x [x \in A \implies x \in B]$ .

(3) We also write **Cl.Set**( $A$ ) for the formula  $\exists u \forall x [x \in A \iff x \in u]$ . (4) We also write **NCl.Set**( $A$ ) for the formulas  $\forall x [x \in A \iff \varphi(x)]$  and  $\forall x [x \in A \iff \varphi(x, A)]$ .

**Remark 4.3.****Cl.Set**( $A$ ) asserts that the set  $A$  is a classical set. For any classical set  $u$ , it follows from the defining axiom for the classical set  $\{x|x \in u\}$  that **Cl.Set**( $\{x : x \in u\}$ ).

We shall identify  $\{x|x \in u\}$  with  $u$ , so that sets may be considered as (special sorts of) nonclassical sets and we may introduce assertions such as  $u \subset A, u \subseteq A, u = A$ , etc.

If  $A$  is a nonclassical set, we write  $\exists x \in A \varphi(x, A)$  for  $\exists x [x \in A \wedge \varphi(x, A)]$  and  $\forall x \in A \varphi(x, A)$  for  $\forall x [x \in A \implies \varphi(x, A)]$ .

We define now the following sets:

1.  $\{u_1, u_2, \dots, u_n\} = \{x|x = u_1 \vee x = u_2 \vee \dots \vee x = u_n\}$ .
2.  $\{A_1, A_2, \dots, A_n\} = \{x|x = A_1 \vee x = A_2 \vee \dots \vee x = A_n\}$ .
3.  $\cup A = \{x|\exists y [y \in A \wedge x \in y]\}$ .
4.  $\cap A = \{x|\forall y [y \in A \implies x \in y]\}$ . 5.  $A \cup B = \{x|x \in A \vee x \in B\}$ .
5.  $A \cap B = \{x|x \in A \wedge x \in B\}$ .
6.  $A - B = \{x|x \in A \wedge x \notin B\}$ .
7.  $u^+ = u \cup \{u\}$ .
8.  $\mathbf{P}(A) = \{x|x \subseteq A\}$ .
9.  $\{x \in A|\varphi(x, A)\} = \{x|x \in A \wedge \varphi(x, A)\}$ .
10.  $\mathbf{V} = \{x| : x = x\}$ .
11.  $\emptyset = \{x|x \neq x\}$ .

The system **INC**<sup>#</sup> of set theory is based on the following axioms:

**Extensionality1:**  $\forall u \forall v [\forall x (x \in u \iff x \in v) \implies u = v]$

**Extensionality2:**  $\forall A \forall B [\forall x (x \in A \iff x \in B) \implies A = B]$

**Universal Set:** **NCl.Set**(**V**)

**Empty Set:** **Cl.Set**( $\emptyset$ )

**Pairing1:**  $\forall u \forall v$  **Cl.Set**( $\{u, v\}$ )

**Pairing2:**  $\forall A \forall B$  **NCl.Set**( $\{A, B\}$ )

**Union1:**  $\forall u$  **Cl.Set**( $\cup u$ )

**Union2:**  $\forall A$  **NCl.Set**( $\cup A$ )

**Powerset1:**  $\forall u$  **Cl.Set**(**P**( $u$ ))

**Powerset2:**  $\forall A$  **NCl.Set**(**P**( $A$ ))

**Infinity**  $\exists a [\emptyset \in a \wedge \forall x \in a (x^+ \in a)]$

**Separation1**  $\forall u_1 \forall u_2, \dots \forall u_n$  **Cl.Set**( $\{x \in a|\varphi(x, u_1, u_2, \dots, u_n)\}$ )

**Separation2**  $\forall u_1 \forall u_2, \dots \forall u_n$  **NCl.Cl.Set**( $\{x \in A|\varphi(x, A; u_1, u_2, \dots, u_n)\}$ ).

## 4.1 Cantor paradox resolution

**Theorem 4.1.** If the domain  $\mathbf{T}$  of function  $F$  is contained in a set  $A$  and if the values of  $F$  are subsets of  $A$ , then the set  $Z = \{t \in T : t \notin F\{t\}\}$  is not a value of the function  $F$ .

**Proof.** We have to show that for every  $t \in T, F(t) \neq Z$ . From the definition of the set  $Z$  it follows that if  $t \in T$ , then  $t \in Z \iff t \notin F\{t\}$ . Thus if  $F(t) = Z$  one obtains  $t \in Z \iff t \notin Z$  and by using unrestricted rules of inference one obtains the contradiction:  $t \in Z \wedge t \notin Z$ .

**Theorem 4.2.** The set  $\mathbf{P}(A)$  is not equipollent to  $A$  nor to any subset of  $A$ .

For otherwise there would exist a one-to-one function whose domain is a subset of  $A$  and whose range is the family of all subsets of  $A$ . But this contradicts Theorem 1.

**Theorem 4.3.** No two of the sets  $A, \mathbf{P}(A), \mathbf{P}(\mathbf{P}(A)),$  etc. are equipollent, i.e.

$$\overline{\overline{A}} < \overline{\overline{\mathbf{P}(A)}} < \overline{\overline{\mathbf{P}(\mathbf{P}(A))}}, \text{ etc.} \quad (4.1)$$

**Cantor paradox.** For universal set from Theorem 4.3 one obtains  $\overline{\overline{\mathbf{V}}} < \overline{\overline{\mathbf{P}(\mathbf{V})}}$ . But other hand one obtains  $\overline{\overline{\mathbf{P}(\mathbf{V})}} \leq \overline{\overline{\mathbf{V}}}$ , since  $\mathbf{P}(\mathbf{V}) \subset \mathbf{V}$ , but this is a contradiction [12].

**Remark 4.4.** Note that in order to avoid Cantor paradox one needs to avoid the inequalities (4.1). The canonical proof of the Theorem 4.3 can be blocked only by using logic with restricted rules of inference.

**I.** We assume now that there exists a function  $\overline{F}(t)$  such that  $\exists \bar{t} [\overline{F}(\bar{t}) = \overline{Z}]$ , i.e. there exists  $\bar{t}$  such that the following statement holds

$$\bar{t} \in \overline{Z} \iff \bar{t} \notin \overline{Z}. \quad (4.2)$$

where  $\overline{Z} = \{t \in T : t \notin \overline{F}\{t\}\}$ .

We set now (i)  $(\bar{t} \notin \overline{Z}) \in \Delta_1$  and (ii)  $(\bar{t} \notin \overline{Z} \implies \bar{t} \in \overline{Z}) \in \Delta_2$ . From (4.1) we know that

$$\bar{t} \in \overline{Z} \implies \bar{t} \notin \overline{Z}. \quad (4.3)$$

So from (4.3) we obtain

$$\bar{t} \in \overline{Z} \implies \bar{t} \in \overline{Z} \wedge \bar{t} \notin \overline{Z} \quad (4.4)$$

since  $\bar{t} \in Z \implies \bar{t} \in Z$ . But by the Law of Non-contradiction we know that  $\neg(\bar{t} \in \overline{Z} \wedge \bar{t} \notin \overline{Z})$ .

So by canonical (unrestricted) modus tollens rule we conclude that

$$\bar{t} \notin \overline{Z}. \quad (4.5)$$

At the same time we also know that  $\bar{t} \notin \overline{Z} \implies \bar{t} \in \overline{Z}$ , but by using restricted modus ponens rule [under conditions (i)-(ii) mentioned above] we can not conclude that  $\bar{t} \in Z$ .

**II.** From  $\bar{t} \notin \overline{Z} \iff \bar{t} \in \overline{Z}$  we obtain  $\bar{t} \notin \overline{Z} \implies \bar{t} \in \overline{Z}$ . We also know that  $\bar{t} \notin \overline{Z} \implies \bar{t} \notin \overline{Z}$ . So  $\bar{t} \notin \overline{Z} \implies \bar{t} \in \overline{Z} \wedge \bar{t} \notin \overline{Z}$ . But by the Law of Non-contradiction we know that  $\neg(\bar{t} \in Z \wedge \bar{t} \notin Z)$ . Thus by unrestricted modus tollens we conclude that

$$\neg(\bar{t} \notin Z) \quad (4.6)$$

and therefore from (4.5)-(4.6) we obtain that  $\neg(\bar{t} \notin Z) \wedge \bar{t} \notin Z$  but this is a contradiction. In order to avoid the contradiction, we set now  $(\bar{t} \notin Z) \in \Delta'_1$  and  $(\bar{t} \notin Z \implies \bar{t} \in Z) \in \Delta'_2$ , and thus by restricted modus tollens we can not conclude that  $\neg(\bar{t} \notin Z)$ . Thus finally we obtain that only (4.5) holds. Thus by using calculus  $\mathbf{Pd}^\#$  with restricted modus ponens rule and restricted modus tollens Cantor paradox disappears since the inequality  $\overline{\mathbf{V}} < \overline{\mathbf{P}(\mathbf{V})}$  no longer holds.

## 5 Conclusion

In this paper set theory  $\text{INC}^\#$  based on intuitionistic logic with restricted modus ponens rule is proposed. It proved that intuitionistic logic with restricted modus ponens rule can to safe Cantor naive set theory from a triviality. Similar results for paraconsistent set theories were obtained in papers [13]-[16].

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## Competing Interests

Author has declared that no competing interests exist.

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