In press

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BOOK Set theory INC$^{\#}$ based on intuitionistic logic with restricted modus ponens rule.

Preface

In this book set theory INC$^{\#}$ based on intuitionistic logic with restricted modus ponens rule is proposed. It proved that intuitionistic logic with restricted modus ponens rule can to safe Cantor naive set theory from a triviality. Similar results for paraconsistent set theories were obtained in author papers [13]-[16].

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1.Remarks on Zermelo-Fraenkel set theory ZFC.

Considering only pure sets, the naive set comprehension principle says, for any condition, that there is a set containing all and only the sets satisfying this condition. In first-order logic, this can be formulated as the following schematic principle, where ϕ may be any formula in whichy does not occur freely:

$$\exists y \forall x (x \in y \leftrightarrow \phi). \tag{1.1}$$

Russell's paradox shows that the instance obtained by letting ϕ be $x \notin x$ is inconsistent in classical logic. One response to the paradox is to restrict naive set comprehension by ruling out this and other problematic instances: only for each of some special conditions is it claimed there is a set containing all and only the sets satisfying the condition. Many well known set theories can be understood as instances of this generic response,differing in how they understand special.For example,the axiom schema of separation (1.1) in Zermelo-Fraenkel set theory (ZFC) restricts set comprehension to conditions which contain, as a conjunct, the condition of being a member of some given set:

$$\exists y \forall x(x \in y \leftrightarrow \phi \wedge x \in z). \quad (1.2)$$

Similarly, in Quine's New Foundations (NF) set comprehension is restricted to conditions which are stratified, where ϕ is stratified just in case there is a mapping f from individual variables to natural numbers such that for each subformula of ϕ of the form $x \in y, f(y) = f(x) + 1$ and for each subformula of ϕ of the form $x = y, f(x) = f(y)$. Both of these responses block Russell's paradox by ruling out the condition $x \notin x$. Must every restriction of naive comprehension take the form of simply ruling out certain instances? In this article, I have suggest and explore a different approach. As we have seen,standard set comprehension axioms restrict attention to some special conditions: for each of these special conditions, they provide for the existence of a set containing all and only the sets which satisfy it.

Instead of restricting the conditions one is allowed to consider, we propose restricting the way in which the sets in question satisfy a given condition: for every condition, our comprehension axiom will assert the existence of a set containing all and only the sets satisfying that condition in a special way using intuitionistic first-order logic with restricted modus ponens rule.

2.Russell's paradox resolution using intuitionistic first-order logic with restricted modus ponens rule.

2.1.Russell's paradox

The comprehension principle (1.1) for the condition $x \notin x$ gives

$$\exists \Re \forall x(x \in \Re \Leftrightarrow x \notin x). \quad (2.1)$$

Thus $\Re$ is the set whose members are exactly those sets that are not members of themselves. It follows from (2.1)

$$\Re \in \Re \Leftrightarrow \Re \notin \Re. \quad (2.2)$$

Is $\Re$ a member of itself? If it is,i.e.$\Re \in \Re$ then it must satisfy the condition of not being a member of itself and so it is not,i.e.$\Re \notin \Re$. If it is not, then it must not satisfy the condition of not being a member of itself, and so it must be a member of itself. Since by classical logic only one case or the other one must hold – either $\Re$ is a member of itself or it is not – it follows that the theory implies a contradiction known as Russell's paradox discovered by Bertrand Russell in 1901, see [1]-[6].

Remark 2.1. Remind classical logic mandates that any contradiction trivializes a theory by making every sentence of the theory provable. This is because, in classical logic, the following is a theorem:

$$\text{Ex Falso Quodlibet}: \ A \Rightarrow (\neg A \Rightarrow B). \quad (2.3)$$

Remark 2.2. Now, virtually the only way to avoid EFQ is to give up disjunctive syllogism

also known as disjunction elimination :

$$\frac{P \vee \neg Q}{Q} \tag{2.4}$$

that is, given the usual definitions of the connectives, modus ponens! So altering basic sentential logic in this way is radical indeed – but possible.

Remark 2.3.Unfortunately, even giving up EFQ is not enough to retain a semblance of naive Cantor set theory (NC). One also has to give up the following additional theorem of basic sentential logic:

$$\text{Contraction: } (A \supset (A \supset B)) \supset (A \supset B). \tag{2.5}$$

It can then be argued that NC leads directly, not merely to an isolated contradiction, but to triviality. For the argument that this is so,see Curry's paradox [7].

Thus it seems that the woes of NC are not confined to Russell's paradox but also include a negation-free paradox due to Curry.

Remark 2.4. Another suggestion might be to conclude that the paradox depends upon an instance of the principle of Excluded Middle, that either $\mathfrak{R}$ is a member of $\mathfrak{R}$ or it is not. This is a principle that is rejected by some non-classical approaches to logic, including intuitionism [8].

Remind that in classical logic, we often discuss the truth values that a formula can take the values are usually chosen as the members of a Boolean algebra. The meet and join operations in the Boolean algebra are identified with the $\wedge$ and $\vee$ logical connectives, so that the value of a formula of the form $A \wedge B$ is the meet of the value of A and the value of B in the Boolean algebra. Then we have the useful theorem that a formula is a valid proposition of classical logic if and only if its value is 1 for every valuation-that is, for any assignment of values to its variables. A corresponding theorem is true for intuitionistic logic, but instead of assigning each formula a value from a Boolean algebra, one uses values from an Heyting algebra, of which Boolean algebras are a special case. A formula is valid (or holds) in intuitionistic logic if and only if it receives the value of the top element for any valuation on any Heyting algebra. It can be shown that to recognize valid formulas, it is sufficient to consider a single Heyting algebra whose elements are the open subsets of the real line $\mathbb{R}$ [8]. In this algebra we have: (1) $\mathbf{Value}[\bot] = \varnothing$,(2) $\mathbf{Value}[\top] = \mathbb{R}$, (3) $\mathbf{Value}[A \wedge B] = \mathbf{Value}[A] \cap \mathbf{Value}[B]$,

(4)$\mathbf{Value}[A \vee B] = \mathbf{Value}[A] \cup \mathbf{Value}[B]$, (5) $\mathbf{Value}[A \Rightarrow B] = \mathbf{Int}\left(\mathbf{Value}[A]^{\complement} \cup \mathbf{Value}[B]\right)$,(6) $\mathbf{Value}[\neg A] = \mathbf{Int}\left(\mathbf{Value}[A]^{\complement}\right)$,where $\mathbf{Int}(X)$ is the interior of X and $X^{\complement}$ its complement.

Remark 2.5. With these assignments (1)-(6), intuitionistically valid formulas are precisely

those that are assigned the value of the entire line [8]. For example, the formula $\neg(A \wedge \neg A)$ is valid,since $\mathbf{Value}[\neg(A \wedge \neg A)] = \mathbb{R}$. So the valuation of this formula is true, and indeed the formula is valid. But the law of the excluded middle, $A \vee \neg A$, can be easily

shown to be invalid by using a specific value of the set of positive real numbers for A : $\mathbf{Value}[A] = \{x | x > 0\} = \mathbb{R}_{+}$. For such A one obtains $\mathbf{Value}[\neg(A \wedge \neg A)] \neq \mathbb{R}$.

We do now as follows:

Case I.Assume now that: (a) $\mathfrak{R} \in \mathfrak{R}$ holds,i.e.$\mathbf{Value}[\mathfrak{R} \in \mathfrak{R}] = \mathbb{R}$ and therefore $\mathfrak{R} \notin \mathfrak{R}$ is not holds,since $\mathbf{Value}[\Re \notin \Re] = \varnothing$.

From (2.2) it follows that (b) $\Re \in \Re \Rightarrow \Re \notin \Re$. From (a) and (b) by modus ponens rule it

follows that

$$\Re \in \Re, \Re \in \Re \Rightarrow \Re \notin \Re \vdash \Re \notin \Re. \tag{2.6}$$

From (2.6) and (a) one obtains the following formula $\Re \in \Re \wedge \Re \notin \Re$. But by the Law of

Non-contradiction we know that $\neg(\Re \in \Re \wedge \Re \notin \Re)$. Thus we obtain a contradiction and

therefore $\Re \in \Re$ is not holds.

Case II.Assume now that:

(a) $\Re \notin \Re$ holds,i.e.$\mathbf{Value}[\Re \notin \Re] = \mathbb{R}$ and therefore $\Re \in \Re$ is

not holds,since $\mathbf{Value}[\Re \in \Re] = \varnothing$.

From (2.2) it follows that (b) $\Re \notin \Re \Rightarrow \Re \in \Re$. From (a) and (b) by modus ponens rule it

follows that

$$\Re \notin \Re, \Re \notin \Re \Rightarrow \Re \in \Re \vdash \Re \in \Re. \tag{2.7}$$

From (2.7) and (b) one obtains the following formula $\Re \in \Re \wedge \Re \notin \Re$. But by the Law of

Non-contradiction we know that $\neg(\Re \in \Re \wedge \Re \notin \Re)$. Thus we obtain a contradiction and

therefore $\Re \notin \Re$ is not holds. Thus bouth $\Re \in \Re$ and $\Re \notin \Re$ is not holds, but by absent

the Excluded Middle but by absent the law Excluded Middle this does not pose any problems.

Remark 2.6. However it well known that it is possible to derive the contradiction only from

the statement (2.2) i.e., $\Re \in \Re \Leftrightarrow \Re \notin \Re$. We do so as follows:

Assume now that: $\Re \in \Re \Leftrightarrow \Re \notin \Re$ holds and therefore $\Re \in \Re \Rightarrow \Re \notin \Re$. But

we also know that $\Re \in \Re \Rightarrow \Re \in \Re$. So $\Re \in \Re \Rightarrow \Re \in \Re \wedge \Re \notin \Re$.

But by the Law of Non-contradiction we know that $\neg(\Re \in \Re \wedge \Re \notin \Re)$

So by modus tollens we conclude that $\Re \notin \Re$.

At the same time we also know that $\Re \notin \Re \Rightarrow \Re \in \Re$, and thus by modus ponens

we conclude that $\Re \in \Re$.

So we can deduce both $\Re \in \Re$ and its negation $\Re \notin \Re$ using only intuitionistically

acceptable methods.

Remark 2.7. Another suggestion might be to conclude that the paradox depends upon

an instance of the Law of Non-contradiction, that $\neg(\Re \in \Re \wedge \Re \notin \Re)$. This is a principle

that is rejected by some non-classical approaches to logic,including paraconsistent logic

[9].Nevertheless even paraconsistent logic can not safe NC from a triviality [9].

Da Costa's paraconsistent set theories of type NF_{ω}^{C} and NF_{n}^{C}, $1 \leq n \leq \omega$. has been studying A.I. Arruda [9].

Remind that the main postulates of NF_{ω}^{C} are the following [9]:

I.Extensionality

$$\forall\alpha\forall\beta\forall x[x \in \alpha \Leftrightarrow x \in \beta \Rightarrow \alpha = \beta]. \tag{2.8}$$

II.Abstraction

$$\exists\alpha\forall x[x \in \alpha \Leftrightarrow F(x)], \tag{2.9}$$

where α does not occur free in $F(x)$ and $F(x)$ is stratified or it does not contain any formula of the form $A \Rightarrow B$.

A.I. Arruda has been proved that da Costa's formulation of the axiom schema of abstraction (2.9) for the systems $NF_n,\ 1 \leq n < \omega,$ leads to the trivialization of the systems, see [9].

Remark 2.8.Note that in $NF_\omega^C,$ the restrictions regarding the use of non-stratified formulas obstruct a direct proof of the paradox of Curry. Russell's set $\Re,$ defined as $\hat{x}\neg(x \in x)$, exists as well as many other non-classical sets. The paradox of Russell in the form $\Re \in \Re \wedge \neg(\Re \in \Re)$ is derivable but apparently, it causes no ham to the system.

Due to its weakness, the primitive negation of $NF_\omega^C, \neg,$ is almost useless for set-theoretical purposes. Thus, let us define

$$\sim A \text{ for } A \Rightarrow \forall x\forall y[x \in y \wedge x = y]. \tag{2.10}$$

The universal set $\mathbf{V}$ is defined as $\hat{x}(x = x),$ the empty set $\emptyset$ as $\{x|\sim(x = x)\},$and the complement of a set α, $\overline{\alpha}$, as $\{x|\sim(x \in a)\}$.

Theorem 2.1.[8]. In $NF_\omega^C, \sim$ is a minimal intuitionistic negation.

Corollary 2.1. $\vdash A \Rightarrow (\sim A \Rightarrow \sim B),\ \vdash (A \Rightarrow B) \Rightarrow (\sim B \Rightarrow \sim A).$

Corollary 2.2. All the theorems of NF whose proofs depend only on the laws of the minimal intuitionistic first-order logic with equality and on the postulates of extensionality and abstraction of NF are valid in NF_ω^C.

Theorem 2.2.[9].(Cantor's Theorem) $NF_\omega^C \vdash \sim(\alpha \leq P(\alpha)).$

Corollary2.3[9].(Cantor's Paradox) $NF_\omega^C \vdash (\mathbf{V} \leq P(\mathbf{V}))\ \wedge \sim(\mathbf{V} \leq P(\mathbf{V})).$

Remark 2.9.Note that Cantor's paradox does not trivialize $NF_\omega^C,$ since from A and $\neg A$ we cannot obtain any formula B whatsoever. For instance, apparently, we cannot obtain any formula of the form $\neg B,$ where B is a nonatomic formula.

Theorem 5.3.(Paradox of identity) [9].(i) $NF_\omega^C \vdash \forall\alpha\forall\beta[(\alpha = \beta) \wedge \sim(\alpha = \beta)],$(ii) $NF_\omega^C \vdash [(\alpha \in \beta) \wedge \sim(\alpha \in \beta)],$
(iii) $NF_\omega^C \vdash [(\alpha \in \alpha) \wedge \sim(\alpha \in \alpha)].$

Proof. By the corollaries 2.1 and 2.2, we obtain

$$\begin{gathered} x = x \Rightarrow \delta, \\ \delta \Leftrightarrow \forall\alpha\forall\beta[(\alpha \in \beta) \wedge (\alpha = \beta)] \end{gathered} \tag{2.11}$$

Thus, as $x = x,$ then $\forall\alpha\forall\beta(\alpha = \beta)$. By the same corollaries we also obtain $\forall\alpha\forall\beta[\sim(\alpha = \beta)]$.The proof of part (ii) is similar to that of part (i). Part (iii) is an immediate consequence of part (ii).

Remark 2.10.**The** paradox of identity obviously trivialized paraconsistent set theory NF_ω^C.

Thus paraconsistent logics cannot resolved the problem.

2.2.The restricted rules of inference.

1.The restricted modus ponens rule.

The canonical (unrestricted) modus ponens rule may be written in sequent notation as

$$P, P \Rightarrow Q \ \vdash_{MP} \ Q, \tag{2.8}$$

where P,Q and $P \Rightarrow Q$ are statements (or propositions) in a formal language $\mathcal{L}$ and $\vdash_{MP}$

is a metalogical symbol meaning that Q is a syntactic consequence of P and $P \rightarrow Q$ in some logical system,see [10]-[11].

P	Q	$P \Rightarrow Q$
T	**T**	**T**
T	**F**	**F**
F	**T**	**T**
F	**F**	**T**

Truth table1.

The validity of modus ponens in classical two-valued logic can be clearly demonstrated by use of a truth table.In instances of modus ponens we assume as premises that $P \Rightarrow Q$ is true and P is true. Only one line of the truth table1-the first-satisfies these two conditions: P and $P \Rightarrow Q$. On this line, Q is also true. Therefore, whenever $P \Rightarrow Q$ is true and P is true, Q must also be true.

Let $\mathcal{F}_{\mathbf{wff}} = \mathcal{F}_{\mathbf{wff}}(\mathcal{L})$ be a set of the all wff's corresponding to formal language $\mathcal{L}$.

The restricted modus ponens rule $\vdash_{RMP}$ may be written in sequent notation as

$$P, P \Rightarrow Q \ \vdash_{RMP} \ Q \text{ iff } P \notin \Delta_1 \text{and } (P \Rightarrow Q) \notin \Delta_2, \tag{2.9}$$

where $\Delta_1, \Delta_2 \subsetneqq \mathcal{F}_{\mathbf{wff}}$.Therefore the restricted modus ponens rule $\vdash_{RMP}$ meant that

$$P, P \Rightarrow Q \nvdash_{RMP} \ Q \tag{2.10}$$

if $P \in \Delta_1$or $(P \Rightarrow Q) \in \Delta_2$ or $(P \in \Delta_1) \wedge ((P \Rightarrow Q) \in \Delta_2)$.

2.The restricted disjunction elimination rule.

In propositional logic, canonical (unrestricted) disjunctive syllogism or modus tollendo ponens (MTP) also known as disjunction elimination rule. The rule makes it possible to eliminate a disjunction from a logical proof. It is the rule that:

$$P \vee Q, \neg P \vdash_{MTP} Q \tag{2.11}$$

and can be expressed by truth-functional tautology of propositional logic:

$$((P \vee Q) \wedge \neg P) \Rightarrow Q \tag{2.12}$$

The restricted disjunction elimination rule $\vdash_{RMTP}$ may be written in sequent notation as

$$P \vee Q, \neg P \vdash_{RMTP} Q \text{ iff } P \notin \overline{\Delta}_1 \text{and } Q \notin \overline{\Delta}_2, \tag{2.13}$$

where $\overline{\Delta}_1, \overline{\Delta}_2 \subsetneqq \mathcal{F}_{\mathbf{wff}}$.In additional we set $((P \vee Q) \wedge \neg P) \in \Delta_1$and $(((P \vee Q) \wedge \neg P) \Rightarrow Q) \in \Delta_2$ iff $P \in \overline{\Delta}_1$and $Q \in \overline{\Delta}_2$.Therefore the restricted disjunction elimination rule $\vdash_{RMTP}$ meant that

$$P \vee Q, \neg P \nvdash_{RMTP} Q \tag{2.14}$$

iff $P \in \overline{\Delta}_1$or $Q \in \overline{\Delta}_2$ or $(P \in \overline{\Delta}_1) \wedge (Q \in \overline{\Delta}_2)$.

3.**The restricted modus tollens rule**.

The canonical (unrestricted) modus tollens rule may be written in sequent notation as

$$P \Rightarrow Q, \neg Q \vdash_{MT} \neg P, \tag{2.15}$$

where $\vdash_{MT}$ is a metalogical symbol meaning that $\neg P$ is a syntactic consequence of $P \Rightarrow Q$ and $\neg Q$ in some logical system; or by the statement of a functional tautology of propositional logic:

$$((P \Rightarrow Q) \wedge \neg Q) \Rightarrow \neg P. \tag{2.16}$$

The validity of modus tollens can be clearly demonstrated through a truth table1.

In instances of the canonical modus tollens we assume as premises that $P \Rightarrow Q$ is true and Q is false. There is only one line of the truth table1-the fourth line-which satisfies these two conditions. In this line, P is false. Therefore, in every instance in which $P \Rightarrow Q$ is true and Q is false, P must also be false.The restricted modus tollens rule may be written in sequent notation as

$$P \Rightarrow Q, \neg Q \vdash_{RMT} \neg P \text{ iff } (P \Rightarrow Q) \notin \Delta'_1 \text{and } Q \notin \Delta'_2, \tag{2.17}$$

where $\Delta'_1, \Delta'_2 \subsetneqq \mathcal{F}_{\mathbf{wff}}$.Therefore the restricted modus tollens rule meant that

$$P \Rightarrow Q, \neg Q \nvdash_{RMT} \neg P \tag{2.18}$$

if $(P \Rightarrow Q) \in \Delta'_1$or $Q \in \Delta'_2$ or $((P \Rightarrow Q) \in \Delta'_1) \wedge (Q \in \Delta'_2)$.

2.3.Curry's paradox resolution using bivalent logic with restricted modus ponens rule.

In set theories that allow unrestricted comprehension, we can nevertheless prove any logical statement Φ by examining the set

$X = \{x | x \in x \Rightarrow \Phi\}$.

Assuming that $\in$ takes precedence over both $\Rightarrow$ and $\Leftrightarrow$, the proof proceeds as follows:

1. $X = \{x | x \in x \Rightarrow \Phi\}$ [Definition of X]
2. $x = X \Rightarrow (x \in x \Leftrightarrow X \in X)$ [Substitution of equal sets in membership]
3. $x = X \Rightarrow ((x \in x \Rightarrow \Phi) \Leftrightarrow (X \in X \Rightarrow \Phi))$
[Addition of a consequent to both sides of a biconditional (from 2)]
4. $X \in X \Leftrightarrow (X \in X \Rightarrow \Phi)$ [Law of concretion (from 1 and 3)]
5. $X \in X \Rightarrow (X \in X \Rightarrow \Phi)$ [Biconditional elimination (from 4)]
6. $X \in X \Rightarrow \Phi$ [Contraction (from 5)]
7. $(X \in X \Rightarrow \Phi) \Rightarrow X \in X$ [Biconditional elimination (from 4)]
8. $X \in X$ [Unrestricted modus ponens $\vdash_{UMP}$ (from 6 and 7)]
9.Φ [Unrestricted Modus ponens $\vdash_{UMP}$ (from 8 and 6)] since
$X \in X, X \in X \Rightarrow \Phi \vdash_{UMP} \Phi$.

Curry's paradox violated NC since any Φ statement is provable.Therefore, in a consistent set theory, the set $\{x \mid x \in x \to \Phi\}$ does not exist for false Φ such that $0 = 1$,etc.Some proposals for set theory have attempted to deal with Curry's paradox not by restricting the rule of comprehension, but by restricting the deduction rules of canonical logic [7]. The existence of proofs like the one above shows that at least one of

the deduction rules used in the proof above must be restricted.

It is clear that in order to avoid Curry's paradox only modus ponens rule must be restricted as mentioned above in subsection 2.2.

Let $LP^{\#}$ be bivalent predicate calculus with restricted modus ponens rule. Let $NC^{\#}$ be Cantor set theory with unrestricted comprehension and equiped with bivalent predicate calculus $LP^{\#}$. Let $\mathcal{L}^{\#} = \mathcal{L}^{\#}(NC^{\#})$ be formal lenguage corresponding to set theory $NC^{\#}$. Let $\mathcal{F}^{\#}_{\mathbf{wff}} = \mathcal{F}^{\#}_{\mathbf{wff}}(\mathcal{L}^{\#})$ be a set of the all closed wff's of the lenguage $\mathcal{L}^{\#}$.

Let $X[\Phi]$ be a set $X[\Phi] = \{x|x \in x \Rightarrow \Phi\}$, where $\Phi \in \mathcal{F}^{\#}_{\mathbf{wff}}$.

$X = \{x|x \in x \Rightarrow \Phi\}$.

Assuming that $\in$ takes precedence over both $\Rightarrow$ and $\Leftrightarrow$, the proof proceeds as follows:

1. $X[\Phi] = \{x|x \in x \Rightarrow \Phi\}$ [Definition of X]
2. $x = X[\Phi] \Rightarrow (x \in x \Leftrightarrow X[\Phi] \in X[\Phi])$ [Substitution of equal sets in membership]
3. $x = X[\Phi] \Rightarrow ((x \in x \Rightarrow \Phi) \Leftrightarrow (X[\Phi] \in X[\Phi] \Rightarrow \Phi))$ [Addition of a consequent to both sides of a biconditional (from 2)]
4. $X[\Phi] \in X[\Phi] \Leftrightarrow (X[\Phi] \in X[\Phi] \Rightarrow \Phi)$ [Law of concretion (from 1 and 3)]
5. $X[\Phi] \in X[\Phi] \Rightarrow (X[\Phi] \in X[\Phi] \Rightarrow \Phi)$ [Biconditional elimination (from 4)]
6. $X[\Phi] \in X[\Phi] \Rightarrow \Phi$ [Contraction (from 5)]
7. $(X[\Phi] \in X[\Phi] \Rightarrow \Phi) \Rightarrow X[\Phi] \in X[\Phi]$ [Biconditional elimination (from 4)]
8. $X[\Phi] \in X[\Phi]$ [Unrestricted modus ponens $\vdash_{UMP}$ (from 6 and 7)]

Let $\widetilde{\Delta}_1$be a set of the all closed wff's corresponding to lenguage $\mathcal{F}^{\#}_{\mathbf{wff}}$ such that $\widetilde{\Delta} = \{(0 = 1)\} \cup \{\Phi|\Phi \Leftrightarrow (0 = 1)\}$.

Let $\Delta_1$be a set of the all closed wff's $\Psi[\Phi]$ corresponding to lenguage $\mathcal{F}^{\#}_{\mathbf{wff}}$ such that $\Psi[\Phi] = (X[\Phi] \in X[\Phi])$ with $\Phi \in \widetilde{\Delta}$. Let Δ_2 be a set of the all closed wff's $F[\Phi]$ corresponding to lenguage $\mathcal{F}^{\#}_{\mathbf{wff}}$ such that $F[\Phi] = X[\Phi] \in X[\Phi] \Rightarrow \Phi$ with $\Phi \in \widetilde{\Delta}$.

Thus from $X[\Phi] \in X[\Phi]$ and $X[\Phi] \in X[\Phi] \Rightarrow \Phi$, we conclude Φ if and only if $(X[\Phi] \in X[\Phi]) \notin \Delta_1$and $(X[\Phi] \in X[\Phi] \Rightarrow \Phi) \notin \Delta_2$, where $\Delta_1, \Delta_2 \subsetneqq \mathcal{F}^{\#}_{\mathbf{wff}}$

3.Russell's paradox resolution using intuitionistic first-order logic with restricted modus ponens rule.

3.1. The intuitionistic propositional calculus $\mathbf{Pp}^{\#}$ with restricted modus ponens rule.

The first step in the metamathematical study of any part of logic or mathematics is to specify a formal language $\mathcal{L}$. For propositional or sentential logic, the standard language has denumerably many distinct proposition letters $P_0, P_1, P_2, \ldots$ and symbols $\&, \vee, \rightarrow, \neg, \perp$ for the propositional connectives "and," "or," "if ...then," and "not" respectively, with left and right parentheses (,) (sometimes written "[,]" for ease of reading). Classical logic actually needs only two connectives (since classical $\vee$ and $\rightarrow$ can be defined in terms of & and $\neg$), but the four intuitionistic connectives are independent. The classical language is thus properly contained in the intuitionistic, which is more expressive. The most important tool of metamathematics is generalized induction, a method Brouwer endorsed. The class of wff's (well-formed formulas) of the language $\mathcal{L}(\mathbf{Pp}^{\#})$ of $\mathbf{Pp}^{\#}$ is defined inductively by the rules:

(i) Each proposition letter is a (prime) formula.
(ii) If A, B are formulas so are $(A\&B), (A \vee B), (A \rightarrow B)$ and $(\neg A)$.
(iii) Nothing is a formula except as required by (i) and (ii).

(iv)The class of wff's of the language $\mathcal{L}(\mathbf{Pp}^{\#})$ we will denoted by $\mathcal{F}_{\mathbf{wff}}(\mathbf{Pp}^{\#})$.

As in classical logic, $(A \leftrightarrow B)$ abbreviates $((A \to B)\&(B \to A))$.

The axioms are all formulas of the following forms:

$\mathbf{Pp}^{\#}$ 1. $A \to (B \to A)$.

$\mathbf{Pp}^{\#}$ 2. $(A \to B) \to ((A \to (B \to C)) \ \to (A \to C))$.

$\mathbf{Pp}^{\#}$ 3. $A \to (B \to A\&B)$.

$\mathbf{Pp}^{\#}$ 4. $A\&B \to A$.

$\mathbf{Pp}^{\#}$ 5.$A\&B \to B$.

$\mathbf{Pp}^{\#}$ 6. $A \to A \vee B$.

$\mathbf{Pp}^{\#}$ 7. $B \to A \vee B$.

$\mathbf{Pp}^{\#}$ 8. $(A \to C) \to ((B \to C) \to (A \vee B \to C))$.

$\mathbf{Pp}^{\#}$ 9. $(A \to B) \to ((A \to \neg B) \to \neg A)$.

$\mathbf{Pp}^{\#}$10. $\neg A \to (A \to B)$.

$\mathbf{Pp}^{\#}$11. $\perp \to A$.

Remark 3.1.The system of classical logic is obtained by adding any one of the following axioms: 1.$\phi \vee \neg \phi$ (Law of the excluded middle. May also be formulated as $(\phi \to \chi) \to ((\neg \phi \to \chi) \to \chi))$

2.$\neg \neg \phi \to \phi$ (Double negation elimination)

$((\phi \to \chi) \to \phi) \to \phi$ (Peirce's law)

$(\neg \phi \to \neg \chi) \to (\chi \to \phi)$ (Law of contraposition)

The rules of inference of $\mathbf{Pp}^{\#}$ is

$\mathbf{R}^{\#}$1.RMP (Restricted Modus Ponens).

From A and $A \to B$, conclude B iff $A \notin \Delta_1$and $(A \to B) \notin \Delta_2$,where $\Delta_1, \Delta_2 \subsetneq \mathcal{F}_{\mathbf{wff}}(\mathbf{Pp}^{\#})$

We abbraviate by $A, A \to B \vdash_{RMP} B$.

$\mathbf{R}^{\#}$2.MT (Restricted Modus Tollens)

$P \to Q, \neg Q \vdash_{RMT} \neg P$ iff $P \notin \Delta_1'$and $(P \to Q) \notin \Delta_2'$,where $\Delta_1', \Delta_2' \subsetneq \mathcal{F}_{\mathbf{wff}}(\mathbf{Pp}^{\#})$

If Γ is any collection of formulas and $E_1, \ldots, E_k$ any finite sequence of formulas each of which is a member of Γ, an axiom, or an immediate consequence by RMP of two preceding formulas, then $E_1, \ldots, E_k$ is a derivation in $\mathbf{Pp}^{\#}$ of its last formula E_k from the assumptions Γ. We write Γ ⊢Pp E to denote that such a derivation exists with $E_k = E$. The following theorem is proved by induction over the definition of a derivation; its converse follows from $\mathbf{R}^{\#}$1.

Deduction Theorem. If Γ is any collection of formulas and A, B are any formulas such that $\Gamma \cup \{A\} \vdash_{RMP} B$, then also $\Gamma \vdash_{RMP} (A \to B)$.

3.2. The intuitionistic first-order predicate calculus $\mathbf{Pd}^{\#}$ with restricted modus ponens rule.

The pure firstorder language $\mathcal{L}(\mathbf{Pd}^{\#})$ has individual variables $a_1, a_2, a_3, \ldots,$ and countably infinitely many distinct predicate letters $P_1(\ldots), P_2(\ldots), P_3(\ldots), \ldots$ of arity n for each $n = 0, 1, 2, \ldots,$ including the 0-ary proposition letters. There are two new logical symbols $\forall$ ("for all") and $\exists$ ("there exists"). The terms of the language $\mathcal{L}(\mathbf{Pd}^{\#})$ of $\mathbf{Pd}^{\#}$ are the individual variables. The well formed formulas are defined by by the rules:

(i) If $P(\ldots)$ is an n-ary predicate letter and $t_1, \ldots, t_n$ are terms then $P(t_1, \ldots, t_n)$ is a (prime) formula.

(ii) If A, B are formulas so are $(A\&B), (A \vee B), \quad (A \to B)$ and $(\neg A)$.

(iii) If A is a formula and x an individual variable, then $(\forall x A)$ and $(\exists x A)$ are formulas.

(iv) Nothing else is a formula.

(v) The class of wff's of the language $\mathcal{L}(\mathbf{Pd}^{\#})$ we will denoted by $\mathcal{F}_{\mathbf{wff}}(\mathbf{Pd}^{\#})$

We use $x,y,z,w,x_1,y_1,\ldots$ and $A,B,C,\ldots,A(x),A(x,y),\ldots$ as metavariables for variables and formulas, respectively. Anticipating applications (e.g. to arithmetic), $s,t,s_1,t_1,\ldots$ vary over terms. In omitting parentheses, $\forall x$ and $\exists x$ are treated like $\neg$. The scope of a quantifier, and free and bound occurrences of a variable in a formula, are defined as usual. A formula in which every variable is bound is a sentence or closed formula.

If x is a variable, t a term, and $A(x)$ a formula which may or may not contain x free, then $A(t)$ denotes the result of substituting an occurrence of t for each free occurrence of x in $A(x)$. The substitution is free if no free occurrence in t of any variable becomes bound in $A(t)$; in this case we say t is free for x in $A(x)$.

In addition to $\mathbf{Pp}1$ - $\mathbf{Pp}11$, $\mathbf{Pd}^{\#}$ has two new axiom schemas, where $A(x)$ may be any formula and t any term free for x in $A(x)$:

$\mathbf{Pd}^{\#}12.$ $\forall xA(x) \to A(t)$.

$\mathbf{Pd}^{\#}13.$ $A(t) \to \exists xA(x)$.

The rules of inference are:

$\mathbf{R}^{\#}1$.RMP (Restricted Modus Ponens).

From A and $A \to B$, conclude B iff $A \notin \Delta_1$and $(A \to B) \notin \Delta_2$,where $\Delta_1, \Delta_2 \subsetneqq \mathcal{F}_{\mathbf{wff}}(\mathbf{Pd}^{\#})$

We abbraviate $\mathbf{R}^{\#}1$ by $A, A \to B \vdash_{RMP} B$.

$\mathbf{R}^{\#}2$.MT (Restricted Modus Tollens)

$P \to Q, \neg Q \vdash_{RMT} \neg P$ iff $P \notin \Delta'_1$and $(P \to Q) \notin \Delta'_2$,where $\Delta'_1, \Delta'_2 \subsetneqq \mathcal{F}_{\mathbf{wff}}(\mathbf{Pp}^{\#})$

$\mathbf{R}^{\#}3$. From $C \to A(x)$ where x does not occur free in C, conclude $C \to \forall xA(x)$.

$\mathbf{R}^{\#}4$. From $A(x) \to C$ where x does not occur free in C, conclude $\exists xA(x) \to C$.

A deduction (or derivation) in $\mathbf{Pd}^{\#}$ of a formula E from a collection Γ of assumption formulas is a finite sequence of formulas, each of which is an axiom by $\mathbf{Pd}^{\#}1$ - $\mathbf{Pd}^{\#}13$, or a member of Γ, or follows immediately by $\mathbf{R}^{\#}1$, $\mathbf{R}^{\#}2$ or $\mathbf{R}^{\#}3$ from one or two formulas occurring earlier in the sequence. A proof is a deduction from no assumptions. If Γ is a collection of sentences and E a formula, the notation $\Gamma \vdash_{RMP} E$ means that a deduction of E from Γ exists. If Γ is a collection of formulas, we write $\Gamma \vdash_{RMP} E$ only if there is a deduction of E from Γ in which neither $\mathbf{R}^{\#}2$ nor $\mathbf{R}^{\#}3$ is used with respect to any variable free in Γ. With this restriction, the deduction theorem extends to $\mathbf{Pd}^{\#}$: If $\Gamma \cup \{A\} \vdash_{RMP}$ B then $\Gamma \vdash_{RMP} (A \to B)$.such that $\Gamma \cup \{A\} \vdash_{RMP} B$, then also $\Gamma \vdash_{RMP} (A \to B)$.

3.3.Russell's paradox resolution using first-order predicate calculus $\mathbf{Pd}^{\#}$ with restricted modus ponens rule.

Assume now that: $\Re \in \Re \Leftrightarrow \Re \notin \Re$ holds and therefore $\Re \in \Re \Rightarrow \Re \notin \Re$.

Remark 3.2.We set now $(\Re \notin \Re) \in \Delta_1$and $(\Re \notin \Re \Rightarrow \Re \in \Re) \in \Delta_2$.

We also know that $\Re \in \Re \Rightarrow \Re \in \Re$. So $\Re \in \Re \Rightarrow \Re \in \Re \wedge \Re \notin \Re$.

But by the Law of Non-contradiction we know that $\neg(\Re \in \Re \wedge \Re \notin \Re)$

So by canonical (unrestricted) modus tollens we conclude that $\Re \notin \Re$.

At the same time we also know that $\Re \notin \Re \Rightarrow \Re \in \Re$, and thus by restricted modus ponens we can not conclude that $\Re \in \Re$.

From $\Re \in \Re \Leftrightarrow \Re \notin \Re$ we obtain $\Re \notin \Re \Rightarrow \Re \in \Re$. We also know that $\Re \notin \Re \Rightarrow \Re \notin \Re$. So $\Re \notin \Re \Rightarrow \Re \in \Re \wedge \Re \notin \Re$. But by the Law of Non-contradiction we know that $\neg(\Re \in \Re \wedge \Re \notin \Re)$,

So by unrestricted modus tollens we conclude that $\neg(\Re \notin \Re)$ and therefore we obtain

that $\neg(\Re \notin \Re)$ & $\Re \notin \Re$. We set now $(\Re \notin \Re) \in \Delta_1'$ and $(\Re \notin \Re \Rightarrow \Re \in \Re) \in \Delta_2'$, and thus by restricted modus tollens we can not conclude that $\neg(\Re \notin \Re)$.
Thus by using calculus $\mathbf{Pd}^{\#}$ with restricted modus ponens rule and restricted modus tollens Russell's paradox dissipears.

4.Intuitionistic Set Theory INC$^{\#}$ based on first-order predicate calculus $\mathbf{Pd}^{\#}$ with restricted modus ponens rule.

4.1.Axioms and basic definitions.

Intuitionistic set theory $\mathbf{INC}^{\#}$ is formulated as a system of axioms in the same first order language as its classical counterpart, only based on intuitionistic logic with restricted modus ponens rule.. The language of set theory is a first-order language $\mathcal{L}^{\#}$ with equality $=$, which includes a binary symbol $\in$. We write $x \neq y$ for $\neg(x = y)$ and $x \notin y$ for $\neg(x \neg y)$. Individual variables $x, y, z, \ldots$ of $\mathcal{L}^{\#}$ will be understood as ranging over classical sets. The unique existential quantifier $\exists!$ is introduced by writing, for any formula $\varphi(x), \exists! x\varphi(x)$ as an abbreviation of the formula $\exists x[\varphi(x)\ \&\ \forall y(\varphi(y) \Rightarrow x = y)]$. $\mathcal{L}^{\#}$ will also allow the formation of terms of the form $\{x|\varphi(x)\}$, for any formula φ containing the free variable x. Such terms are called nonclassical sets; we shall use upper case letters $A, B, \ldots$ for such sets. For each nonclassical set $A = \{x|\varphi(x)\}$ the formulas $\forall x[x \in A \Leftrightarrow \varphi(x)]$ and $\forall x[x \in A \Leftrightarrow \varphi(x, A)]$is called the defining axioms for the nonclassical set A.

Remark 4.1. Note that (1) the formula $\forall x[x \in A \Leftrightarrow \varphi(x)]$ and
$\forall x[x \in a \Leftrightarrow \varphi(x) \wedge x \in u]$
is not always asserts that $\forall x[x \in A \vdash_{RMP} \varphi(x)]$ and (or) $\forall x[\varphi(x) \vdash_{RMP} x \in A]$ even for a classical set since for some y possible $y \in A \Rightarrow \varphi(y) \nvdash_{RMP} \varphi(y)$ and (or) $\varphi(y) \Rightarrow y \in A \nvdash_{RMP} x \in A$ and $y \in a \Rightarrow \varphi(y) \wedge y \in u \nvdash_{RMP} \varphi(y) \wedge y \in u$, etc.In order to emphasize this fact we often vrite the defining axioms for the nonclassical set in the following form $\forall x[x \in A \Leftrightarrow_w \varphi(x)]$

Remark 4.2.(1) Two nonclassical sets A, B are defined to be equal and we write $A = B$ if
$\forall x[x \in A \Leftrightarrow x \in B]$. (2) A is a subset of B, and we write $A \subseteqq B$, if $\forall x[x \in A \Rightarrow x \in B]$. (3) We also write $\mathbf{Cl.Set}(A)$ for the formula $\exists u\forall x[x \in A \Leftrightarrow x \in u]$. (4) We also write $\mathbf{NCl.Set}(A)$ for the formulas $\forall x[x \in A \Leftrightarrow \varphi(x)]$ and $\forall x[x \in A \Leftrightarrow \varphi(x, A)]$.

Remark 4.3.$\mathbf{Cl.Set}(A)$) asserts that the set A is a classical set. For any classical set u, it follows from the defining axiom for the classical set $\{x|x \in u\}$ that
$\mathbf{Cl.Set}(\{x : x \in u\})$.
We shall identify $\{x|x \in u\}$ with u, so that sets may be considered as (special sorts of) nonclassical sets and we may introduce assertions such as $u \subset A, u \subseteqq A, u = A$, etc.
If A is a nonclassical set, we write $\exists x \in A\ \varphi(x, A)$ for $\exists x[x \in A \wedge \varphi(x, A)]$ and $\forall x \in A\varphi(x, A)$ for $\forall x[x \in A \Rightarrow \varphi(x, A)]$.
We define now the following sets:
1.$\{u_1, u_2, \ldots, u_n\} = \{x|x = u_1 \vee x = u_2 \vee \ldots \vee x = u_n\}$. 2. $\{A_1, A_2, \ldots, A_n\} =$
$= \{x|x = A_1 \vee x = A_2 \vee \ldots \vee x = A_n\}$. 3.$\cup A = \{x|\exists y[y \in A \wedge x \in y]\}$.
4.$\cap A = \{x|\forall y[y \in A \Rightarrow x \in y]\}$. 5.$A \cup B = \{x|x \in A \vee x \in B\}$.
5.$A \cap B = \{x|x \in A \wedge x \in B\}$. 6.$A - B = \{x|x \in A \wedge x \notin B\}$. 7.$u^+ = u \cup \{u\}$.
8.$\mathbf{P}(A) = \{x|x \subseteq A\}$. 9.$\{x \in A|\varphi(x, A)\} = \{x|x \in A \wedge \varphi(x, A)\}$. 10.$\mathbf{V} = \{x|: x = x\}$.

11. $\varnothing = \{x | x \neq x\}$.

The system $\mathbf{INC}^{\#}$ of set theory is based on the following axioms:

Extensionality1: $\forall u \forall v[\forall x(x \in u \Leftrightarrow x \in v) \Rightarrow u = v]$

Extensionality2: $\forall A \forall B[\forall x(x \in A \Leftrightarrow x \in B) \Rightarrow A = B]$

Universal Set: $\mathbf{NCl.Set}(\mathbf{V})$

Empty Set: $\mathbf{Cl.Set}(\varnothing)$

Pairing1: $\forall u \forall v \ \mathbf{Cl.Set}(\{u, v\})$

Pairing2: $\forall A \forall B \ \mathbf{NCl.Set}(\{A, B\})$

Union1: $\forall u \ \mathbf{Cl.Set}(\cup u)$

Union2: $\forall A \ \mathbf{NCl.Set}(\cup A)$

Powerset1: $\forall u \ \mathbf{Cl.Set}(\mathbf{P}(u))$

Powerset2: $\forall A \ \mathbf{NCl.Set}(\mathbf{P}(A))$

Infinity $\exists a[\varnothing \in a \wedge \forall x \in a(x^+ \in a)]$

Separation1$\forall u_1 \forall u_2, \ldots \forall u_n \mathbf{Cl.Set}(\{x \in a | \varphi(x, u_1, u_2, \ldots, u_n)\})$

Separation2$\forall u_1 \forall u_2, \ldots \forall u_n \mathbf{NCl.Cl.Set}(\{x \in A | \varphi(x, A; u_1, u_2, \ldots, u_n)\})$.

4.2.Cantor paradox resolution

Theorem 4.1.If the domain $\mathbf{T}$ of function F is contained in a set A and if the values of F
are subsets of A, then the set $Z = \{t \in T : t \notin F\{t)\}$ is not a value of the function F.

Proof. We have to show that for every $t \in T, F(t) \neq Z$. From the definition of the set Z it
follows that if $t \in T$, then $t \in Z \Leftrightarrow t \notin F\{t)$. Thus if $F(t) = Z$ one obtains $t \in Z \Leftrightarrow t \notin Z$
and by using unrestricted rules of inference one obtains the
contradiction: $t \in Z \wedge t \notin Z$.

Theorem 4.2. The set $\mathbf{P}(A)$ is not equipollent to A nor to any subset of A.
For otherwise there would exist a one-to-one function whose domain is a subset of A
and whose range is the family of all subsets of A. But this contradicts Theorem 1.

Theorem 4.3. No two of the sets $A, \mathbf{P}(A), \mathbf{P}(\mathbf{P}(A))$, etc. are equipollent, i.e.

$$\overline{\overline{A}} < \overline{\overline{\mathbf{P}(A)}} < \overline{\overline{\mathbf{P}(\mathbf{P}(A))}}, \text{etc.} \tag{4.1}$$

Cantor paradox.For universal set from Theorem 4.3 one obtains $\overline{\overline{\mathbf{V}}} < \overline{\overline{\mathbf{P}(\mathbf{V})}}$ .But other
hand one obtains $\overline{\overline{\mathbf{P}(\mathbf{V})}} \leq \overline{\overline{\mathbf{V}}}$, since $\mathbf{P}(\mathbf{V}) \subset \mathbf{V}$, but this is a contradiction [12].

Remark 4.4.Note that in order to avoid Cantor paradox one needs to avoid the
inequalities (4.1). The canonical proof of the Theorem 4.3 can to blocked only by using
logic with restricted rules of inference.

I.We assume now that there exists a function $\overline{F}(t)$ such that $\exists \bar{t}[\overline{F}(\bar{t}) = \overline{Z}]$,i.e.there exists $\bar{t}$
such that the following statement holds

$$\bar{t} \in \overline{Z} \Leftrightarrow \bar{t} \notin \overline{Z}. \tag{4.2}$$

where $\overline{Z} = \{t \in T : t \notin \overline{F}\{t)\}$.

We set now (i) $(\bar{t} \notin \overline{Z}) \in \Delta_1$and (ii) $(\bar{t} \notin \overline{Z} \Rightarrow \bar{t} \in \overline{Z}) \in \Delta_2$. From (4.1) we know that

$$\bar{t} \in \overline{Z} \Rightarrow \bar{t} \notin \overline{Z}. \tag{4.3}$$

So from (4.3) we obtain

$$\bar{t} \in \bar{Z} \Rightarrow \bar{t} \in \bar{Z} \wedge \bar{t} \notin \bar{Z} \quad (4.4)$$

since $\bar{t} \in Z \Rightarrow \bar{t} \in Z.$ But by the Law of Non-contradiction we know that $\neg(\bar{t} \in \bar{Z} \wedge \bar{t} \notin \bar{Z}).$

So by canonical (unrestricted) modus tollens rule we conclude that

$$\bar{t} \notin \bar{Z}. \quad (4.5)$$

At the same time we also know that $\bar{t} \notin \bar{Z} \Rightarrow \bar{t} \in \bar{Z}$, but by using restricted modus ponens rule [under conditions (i)-(ii) mentioned above] we can not conclude that $\bar{t} \in Z.$

II.From $\bar{t} \notin \bar{Z} \Leftrightarrow \bar{t} \in \bar{Z}$ we obtain $\bar{t} \notin \bar{Z} \Rightarrow \bar{t} \in \bar{Z}.$ We also know that $\bar{t} \notin \bar{Z} \Rightarrow \bar{t} \notin \bar{Z}.$ So $\bar{t} \notin \bar{Z} \Rightarrow \bar{t} \in \bar{Z} \wedge \bar{t} \notin \bar{Z}.$ But by the Law of Non-contradiction we know that $\neg(\bar{t} \in Z \wedge \bar{t} \notin Z).$ Thus by unrestricted modus tollens we conclude that

$$\neg(\bar{t} \notin Z) \quad (4.6)$$

and therefore from (4.5)-(4.6) we obtain that $\neg(\bar{t} \notin Z) \wedge \bar{t} \notin Z$ but this is a contradiction.

In order to avoid the contradiction, we set now $(\bar{t} \notin Z) \in \Delta_1'$ and $(\bar{t} \notin Z \Rightarrow \bar{t} \in Z) \in \Delta_2'$, and thus by restricted modus tollens we can not conclude that $\neg(\bar{t} \notin Z).$ Thus finally we obtain that only (4.5) holds.Thus by using calculus $\mathbf{Pd}^{\#}$ with restricted modus ponens rule and restricted modus tollens Cantor paradox disappears sinse the inequality $\bar{\bar{\mathbf{V}}} < \overline{\overline{\mathbf{P(V)}}}$ no longer holds.

Chapter 2.Set Theory $\mathrm{INC}_{\infty^{\#}}^{\#}$ Based on Infinitary Intuitionistic Logic with Restricted Modus Ponens Rule. Hyper inductive definitions

1.Introduction.

In this chapter intuitionistic set theory $\mathrm{INC}_{\infty^{\#}}^{\#}$ based on infinitary intuitionistic logic with restricted modus ponens rule is considered [1]. External induction principle in nonstandard intuitionistic arithmetic were derived. Non trivial application in number theory is considered. The Goldbach-Euler theorem is obtained without any references to Catalan conjecture.

2.Axiom of nonregularity and axiom of hyperinfinity.

2.1.Axiom of nonregularity.

Remind that a non-empty set u is called regular iff

$$\forall x[x \neq \varnothing \to (\exists y \in x)(x \cap y = \varnothing)]. \quad (2.1)$$

Let's investigate what it says: suppose there were a non-empty x such that

$(\forall y \in x)(x \cap y \neq \varnothing).$ For any $z_1 \in x$ we would be able to get $z_2 \in z_1 \cap x.$ Since $z_2 \in x$ we would be able to get $z_3 \in z_2 \cap x.$ The process continues forever:
$\ldots \in z_{n+1} \in z_n \ldots \in z_4 \in z_3 \in z_2 \in z_1 \in x.$ Thus we wish to rule out such an infinite regress.

2.1.Axiom of hyperinfinity.

Definition 2.1.(i) A non-empty transitive non regular set u is a well formed non regular set iff:

(i) there is unique countable sequence $\{u_n\}_{n=1}^{\infty}$ such that

$$\ldots \in u_{n+1} \in u_n \ldots \in u_4 \in u_3 \in u_2 \in u_1 \in u, \quad (2.2)$$

(ii) for any $n \in \mathbb{N}$ and any $u_{n+1} \in u_n$:

$$u_n = u_{n+1}^{+}, \quad (2.3)$$

where $a^{+} = a \cup \{a\}$.

(ii) we define a function $a^{+[k]}$inductively by $a^{+[k+1]} = (a^{+[k]})^{+}$

Definition 2.2. Let u and w are well formed non regular sets. We write $w \prec u$ iff for any $n \in \mathbb{N}$

$$w \in u_n. \quad (2.4)$$

Definition 2.3. We say that an well formed non regular set u is infinite (or hyperfinite) hypernatural nuber iff:

(I) For any member $w \in u$ one and only one of the following conditions are satified:

(i) $w \in \mathbb{N}$ or

(ii) $w = u_n$ for some $n \in \mathbb{N}$ or

(iii) $w \prec u$.

(II) Let $_{\prec}u$ be a set $_{\prec}u = \{z | z \prec u\}$,then by relation $(\cdot \prec \cdot)$ a set $_{\prec}u$ is densely ordered with no first element.

(III) $\mathbb{N} \subset u$.

Axiom of hyperinfinity

There exists unique set $\mathbb{N}^{\#}$ such that:

(i) $\mathbb{N} \subset \mathbb{N}^{\#}$

(ii) if u is infinite (hypernatural) number then $u \in \mathbb{N}^{\#} \backslash \mathbb{N}$

(iii) if u is infinite (hypernatural) number then there exists infinite (hypernatural) number v such that $v \prec u$

(iv) if u is infinite hypernatural number then there exists infinite (hypernatural) number w such that $u \prec w$

(v) set $\mathbb{N}^{\#} \backslash \mathbb{N}$ is patially ordered by relation $(\cdot \prec \cdot)$ with no first and no last element.

In this paper we introduced a set $\mathbb{N}^{\#} \backslash \mathbb{N}$ of the infinite numbers axiomatically without any references to non-standard model of arithmetic via canonical ultraproduct approach, see [2]-[5].

4.Infinitary and hyperinfinitary logics.

4.1.Classical infinitary logic.

By a vocabulary, we mean a set L of constant symbols, and relation and operation symbols with finitely many argument places. As usual,by an L-structure M , we mean a universe set M with an interpretation for each symbol of L. In cases where the vocabulary L is clear, we may just say structure. For a given vocabulary L and infinite cardinals $\mu \leq \kappa, L_{\kappa\mu}$ is the infinitary logic with κ variables, conjunctions and disjunctions over sets of formulas of size less than κ, and existential and universal quantifiers over sets of variables of size less than μ. All logics that we consider also have equality, and are closed under negation. The equality symbol is always available, but is not counted as an element of the vocabulary L.

During last sentury canonical infinitary logic many developed, see for example [6]-[10].

4.2.Why we need infinitary logic

It well known that some classes of mathematical structures, such as algebraically closed fields of a given characteristic, are characterized by a set of axioms in $L_{\omega\omega}$. Other classes cannot be characterized in this way, but can be axiomatized by a single sentence of $L_{\omega_1\omega}$.

Remark 4.1.In the practice of the contemporary model theory, and in more general mathematics as well, it often becomes necessary to consider structures satisfying certain collections of sentences rather than just single sentences. This consideration leads to the familiar notion of a theory in a logic. For example, in ordinary finitary logic, $L_{\omega\omega}$, if φ_n is a sentence which expresses that there are at least n elements, then the theory $\{\varphi_n | n \in \omega\}$ would express that there are infinitely many elements. Similarly, in the theory of groups, if φn is the sentence $\forall x[x^n \neq 1]$, then $\{\psi_n : n \in \omega\}$ expresses that a group is torsion free.

Remark 4.2.Suppose we want to express the idea that a set is finite, or that a group is torsion. A simple compactness argument would immediately reveal that neither of these notions can be expressed by a theory in $L_{\omega\omega}$. What we need to express in each case is that a certain theory is not satisfied, that is, that at least one of the sentences is false. While theories are able to simulate infinite conjunctions, there is no apparent way to simulate infinite disjunctions–which is just what is needed in this case.

Example 4.1. The Abelian torsion groups are the models of a sentence obtained by taking the conjunction of the usual axioms for Abelian groups (a finite set) and the following infinite disjunction:

$$\forall x \left[\bigvee_{n \in \mathbb{N}} \underbrace{x + x + \ldots + x}_{n} = 0 \right]. \tag{4.1}$$

Example 4.2. The Archimedean ordered fields are the models of a sentence obtained by taking the conjunction of the usual axioms for ordered fields and the following infinite disjunction:

$$\forall x \left[\bigvee_{n \in \mathbb{N}} \underbrace{1 + 1 + \ldots + 1}_{n} > x \right]. \tag{4.2}$$

Example 4.3. Let L be a countable vocabulary. Let T be an elementary first order theory, and let $\Gamma(\bar{x})$ be a set of finitary formulas in a fixed tuple of variables $\bar{x}$. The models of T that omit Γ are the models of the single $L_{\omega_1\omega}$ sentence obtained by taking the conjunction of the sentences of T and the following infinite disjunction:

$$\forall \bar{x} \left[\bigvee_{\gamma \in \Gamma} \neg\gamma(\bar{x}) \right]. \tag{4.3}$$

Example 4.4. The non Archimedean ordered fields are the models of a sentence obtained by taking the conjunction of the usual axioms for non Archimedean ordered fields i.e., the following infinite conjunction:

$$\exists x\left[\bigwedge_{n\in\mathbb{N}}\underbrace{1+1+\ldots+1}_{n}< x\right]. \tag{4.4}$$

4.3. Bivalent hyperinfinitary first-order logic $IL^{\#}_{\infty^{\#}}$ with restricted rules of conclusion.

Hyperinfinitary language $L^{\#}_{\infty^{\#}}$ are defined according to the length of infinitary conjunctions/disjunctions as well as quantification it allows. In that way, assuming a supply of $\kappa < \infty^{\#} = card(\mathbb{N}^{\#})$ variables to be interpreted as ranging over a nonempty domain, one includes in the inductive definition of formulas an infinitary clause for conjunctions and disjunctions, namely, whenever the hypernturals indexed hypersequence $\{A_\delta\}_{\delta\in\mathbb{N}^{\#}}$ of formulas has length less than κ, one can form the hyperfinite conjunction/disjunction of them to produce a formula. Analogously, whenever an hypernaturals indexed sequence of variables has length less than λ, one can introduce one of the quantifiers $\forall$ or $\exists$ together with the sequence of variables in front of a formula to produce a new formula. One also stipulates that the length of any well-formed formula is less than $\infty^{\#}$ itself.

The syntax of bivalent hyperinfinitary first-order logics $L^{\#}_{\infty^{\#}}$ consists of a (ordered) set of sorts and a set of function and relation symbols, these latter together with the corresponding type, which is a subset with less than $\infty^{\#} = card(\mathbb{N}^{\#})$ many sorts. Therefore, we assume that our signature may contain relation and function symbols on $\gamma < \infty^{\#}$ many variables, and we suppose there is a supply of $\kappa < \infty^{\#}$ many fresh variables of each sort. Terms and atomic formulas are defined as usual, and general formulas are defined inductively according to the following rules:If $\phi,\psi,\{\phi_\alpha : \alpha < \gamma\}$ (for each $\gamma < \kappa$) are formulas of $\mathcal{L}_{\kappa,\kappa}$, the following are also formulas: $\bigwedge_{\alpha<\gamma}\phi_\alpha$, $\bigvee_{\alpha<\gamma}\phi_\alpha$, $\phi\to\psi$, $\forall_{\alpha<\gamma}x_\alpha\phi$ (also written $\forall\mathbf{x}_\gamma\phi$ if $\mathbf{x}_\gamma = \{x_\alpha : \alpha < \gamma\}$), $\exists_{\alpha<\gamma}x_\alpha\phi$ (also written $\exists\mathbf{x}_\gamma\phi$ if $\mathbf{x}_\gamma = \{x_\alpha : \alpha < \gamma\}$).

The axioms of hyperinfinitary first-order logic $L^{\#}_{\infty^{\#}}$ consist of the following schemata:

I. **Logical axiom**

1. $A \to [B \to A]$
2. $[A \to [B \to C] \to [[A \to B] \to [A \to C]]]$
3. $[\neg B \to \neg A] \to [A \to B]$
4. $[\bigwedge_{i<\alpha}[A \to A_i]] \to [A \to \bigwedge_{i<\alpha}A_i], \alpha \in \mathbb{N}^{\#}$
5. $[\bigwedge_{i<\alpha}A_i] \to A_j, \alpha \in \mathbb{N}^{\#}$
6. $[\forall\mathbf{x}[A \to B] \to [A \to \forall\mathbf{x}B]]$
   provided no variable in $\mathbf{x}$ occurs free in A;
7. $\forall\mathbf{x}A \to S_f(A)$

where $S_f(A)$ is a substitution based on a function *f* from $\mathbf{x}$ to the terms of the language;

II.**Restricted rules of conclusion**.

$\mathbf{R}^{\#}$1.RMP (Restricted Modus Ponens).

From A and $A \to B$, conclude B iff $A \notin \Delta_1$and $(A \to B) \notin \Delta_2$,where $\Delta_1,\Delta_2 \subsetneqq \mathcal{F}_{\mathbf{wff}}$

We abbraviate by $A, A \to B \vdash_{RMP} B$.

$\mathbf{R}^{\#}$2.MT (Restricted Modus Tollens)
$P \to Q, \neg Q \vdash_{RMT} \neg P$ iff $P \notin \Delta_1'$and $(P \to Q) \notin \Delta_2'$,where $\Delta_1', \Delta_2' \subsetneqq \mathcal{F}_{\mathbf{wff}}$.

III.**Equality axioms**:

(a) $t = t$

(b) $[\bigwedge_{i<\alpha} t_i = t_i'] \to [\phi(t_0, \ldots, t_\xi, \ldots) \ = \phi(t_0', \ldots, t_\xi', \ldots)]$

(c) $[\bigwedge_{i<\alpha} t_i = t_i'] \to [P(t_0, \ldots, t_\xi, \ldots) \ \to P(t_0', \ldots, t_\xi', \ldots)]$

for each $\alpha \in \mathbb{N}^{\#}$, where t, t_i are terms and ϕ is a function symbol of arity α and P a relation symbol of arity $\alpha \in \mathbb{N}^{\#}$.

IV.**Distributivity axiom**:

$$\bigwedge_{i<\gamma} \bigvee_{j<\gamma} \psi_{ij} \to \bigvee_{f \in \gamma^\gamma} \bigwedge_{i<\gamma} \psi_{if(i)} \tag{4.5}$$

V.**Dependent choice axiom**:

$$\bigwedge_{\alpha<\gamma} \forall_{\beta<\alpha} \mathbf{x}_\beta \exists \mathbf{x}_\alpha \psi_\alpha \to \exists_{\alpha<\gamma} \mathbf{x}_\alpha \bigwedge_{\alpha<\gamma} \psi_\alpha \tag{4.6}$$

provided the sets $\mathbf{x}_\alpha$ are pairwise disjoint and no variable in $\mathbf{x}_\alpha$ is free in ψ_β for $\beta < \alpha \in \mathbb{N}^{\#}$.

5.Intuitionistic hyperinfinitary logic $L^{\#}_{\infty^{\#}}$ with restricted rules of conclusion.

We will denote the class of hypernaturals by $\mathbb{N}^{\#}$, the class of binary sequences of hypernatural length by $2^{<\mathbb{N}^{\#}}$, and the class of sets of hypernatural numbers by $\Sigma(\mathbb{N}^{\#})$.

We fix a class of variables x_i for each $i \in \mathbb{N}^{\#}$. Given an $\alpha \in \mathbb{N}^{\#}$, a context of length α is a sequence $\mathbf{x} = \langle x_{i_j} | j < \alpha \rangle$ of variables. In this paper we will use boldface letters, $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$, to denote contexts and light-face letters, $x_i, y_i, z_i, \ldots$, to denote the i-th variable symbol of $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$, respectively.

We will denote the length of a context $\mathbf{x}$ by $l(\mathbf{x})$. The formulas of the hyperinfinitary language $\mathcal{L}^{\#}_{\infty^{\#}}$ of set theory INC$^{\#}_{\infty^{\#}}$ are defined to be the smallest class of formulas closed under the following rules:

1.$\bot$ is a formula,

2.$x_i \in x_j$ is a formula for any variables x_i and x_j,

3.$x_i = x_j$ is a formula for any variables x_i and x_j,

4. if ϕ and ψ are formulas, then $\phi \to \psi$ are formulas,

5. if ϕ_α is a formula for every $\alpha : \alpha \leq \beta \in \mathbb{N}^{\#}$, then

1. $\bigvee_{\alpha \leq \beta} \phi_\alpha$ is a gyperfinite formula, (5.1)

6.if ϕ_α is a formula for every $\alpha : \alpha \leq \beta \in \mathbb{N}^{\#}$, then

$\bigwedge_{\alpha \leq \beta} \phi_\alpha$is a gyperfinite formula, (5.2)

7. if $\mathbf{x}$ is a context of length α, then $\exists^\alpha \mathbf{x} \phi$ is a formula, and,

8. if $\mathbf{x}$ is a context of length α, then $\forall^\alpha \mathbf{x} \phi$ is a formula.

By this definition, our language allows set-sized disjunctions and conjunctions as well as quantification over set-many variables at once. However, note that infinite alternating sequences of existential and universal quantifiers are excluded by this definition.

Remark 5.**1**.Whenever it is clear from the context, we will omit the superscripts from the quantifiers and write $\exists$ and $\forall$ instead of $\exists^\alpha$ and $\forall^\alpha$, respectively. In many situations it will be useful to identify a variable x with the context $\mathbf{x} = \langle x \rangle$ whose unique element is x

such that we can write, for example, “$\exists x\phi$” for “$\exists \mathbf{x}\phi$” and “$\forall x\phi$” for “$\forall \mathbf{x}\phi$”. A variable x_i is called a free variable of a formula ϕ whenever x_i appears in ϕ but not in any quantification of ϕ. As usual, a formula without free variables is called a sentence. We say that $\mathbf{x}$ is a context of the formula ϕ if all free variables of ϕ are among those in $\mathbf{x}$. As usual, we will write $\phi(\mathbf{x})$ in case that ϕ is a formula and $\mathbf{x}$ is a context of φ. Similarly, given two contexts $\mathbf{x}$ and $\mathbf{y}$ with $x_j \neq y_{j'}$ for all $j < \ell(\mathbf{x})$ and $j' < \ell(\mathbf{y})$, we will write $\varphi(\mathbf{x},\mathbf{y})$ in case that the sequence obtained by concatenating $\mathbf{x}$ and $\mathbf{y}$ is a context for φ.

Remark 5.2. We extend the classical abbreviations as follows: Given a formula ϕ and an hypernatural $\alpha \in \mathbb{N}^{\#}$ we introduce the bounded quantifiers as abbreviations, namely,

$$\forall^{\alpha}\mathbf{x} \in y\, \phi \text{ for } \forall^{\alpha}\mathbf{x}(\mathbf{x} \in y \to \phi), \tag{5.3}$$

and

$$\exists^{\alpha}\mathbf{x} \in y\, \phi \text{ for } \exists^{\alpha}\mathbf{x}(\mathbf{x} \in y \wedge \phi). \tag{5.4}$$

Notation 5.1. A sequent $\phi \vdash_{\mathbf{x},\boldsymbol{\alpha}} \psi$ is however equivalent to the formula $\forall^{\alpha}\mathbf{x}(\phi \to \psi)$.
The system of axioms and rules for hyperinfinitary intuitionistic first-order logic consists of the following schemata:

I. **Logical axiom**

1. $A \to [B \to A]$
2. $[A \to [B \to C]] \to [[A \to B] \to [A \to C]]]$
3. $[\bigwedge_{i<\alpha}[A \to A_i]] \to [A \to \bigwedge_{i<\alpha} A_i], \alpha \in \mathbb{N}^{\#}$
4. $[\bigwedge_{i<\alpha} A_i] \to A_j, \alpha \in \mathbb{N}^{\#}$
5. $[\forall\mathbf{x}[A \to B] \to [A \to \forall\mathbf{x}B]]$
   provided no variable in $\mathbf{x}$ occurs free in A.
7. $\forall\mathbf{x}A \to S_f(A)$

where $S_f(A)$ is a substitution based on a function *f* from $\mathbf{x}$ to the terms of the language;

II.**Restricted rules of conclusion**.

$\mathbf{R}^{\#}$1.RMP (Restricted Modus Ponens).
From A and $A \to B$, conclude B iff $A \notin \Delta_1$and $(A \to B) \notin \Delta_2$,where $\Delta_1, \Delta_2 \subsetneqq \mathcal{F}_{\mathbf{wff}}$
We abbraviate by $A, A \to B \vdash_{RMP} B$.
$\mathbf{R}^{\#}$2.MT (Restricted Modus Tollens)
$P \to Q, \neg Q \vdash_{RMT} \neg P$ iff $P \notin \Delta_1'$and $(P \to Q) \notin \Delta_2'$,where $\Delta_1', \Delta_2' \subsetneqq \mathcal{F}_{\mathbf{wff}}$.

III.**Weak distributivity axiom**:

$$\phi \wedge \bigvee_{i<\gamma} \psi_i \vdash_{\mathbf{x}} \bigvee_{i<\gamma} \phi \wedge \psi_i \tag{5.5}$$

for each $\gamma \in \mathbb{N}^{\#}$.

IV.**Frobenius axiom**:

$$\phi \wedge \exists\mathbf{y}\psi \vdash_{\mathbf{x}} \exists\mathbf{y}(\phi \wedge \psi) \tag{5.6}$$

where no variable in $\mathbf{y}$ is in the context $\mathbf{x}$.

V.**Structural rules**:

(a) Identity axiom:

$$\varphi \vdash_{\mathbf{x},\boldsymbol{\alpha}} \varphi \tag{5.7}$$

(b) Substitution rule:

$$\frac{\varphi \vdash_{\mathbf{x},\boldsymbol{\alpha}} \psi}{\varphi[\mathbf{s}/\mathbf{x}] \vdash_{\mathbf{y}} \psi[\mathbf{s}/\mathbf{x}]} \tag{5.8}$$

where $\mathbf{y}$ is a string of variables including all variables occurring in the string of terms $\mathbf{s}$.
(c) Restricted cut rule:

$$\frac{\varphi \vdash_{\mathbf{x},\boldsymbol{\alpha}} \psi, \psi \vdash_{\mathbf{x},\alpha} \theta}{\varphi \vdash_{\mathbf{x},\alpha} \theta} \tag{5.9}$$

iff $\varphi \notin \Delta_1$and $(\psi \vdash_{\mathbf{x},\alpha} \theta) \notin \Delta_2$.
IV.**Equality axioms**:

(a)

$$\top \vdash_{\mathbf{x}} \mathbf{x} = \mathbf{x} \tag{5.10}$$

(b)

$$(\mathbf{x} = \mathbf{y}) \wedge \varphi[\mathbf{x}/\mathbf{w}] \vdash_{\mathbf{z}} \varphi[\mathbf{y}/\mathbf{w}] \tag{5.11}$$

where $\mathbf{x}, \mathbf{y}$ are contexts of the same length and type and $\mathbf{z}$ is any context containing $\mathbf{x}, \mathbf{y}$ and the free variables of φ.
V.**Conjunction axioms and rules**:
(a)

$$\bigwedge_{i<\gamma} \varphi_i \vdash_{\mathbf{x},\alpha} \varphi_j \tag{5.12}$$

for each $\gamma \in \mathbb{N}^{\#}$ and $j < \gamma$

(b)

$$\frac{\{\phi \vdash_{\mathbf{x},\boldsymbol{\alpha}} \psi_i\}_{i<\gamma}}{\phi \vdash_{\mathbf{x},\boldsymbol{\alpha}} \bigwedge_{i<\gamma} \psi_i} \tag{5.13}$$

for each $\gamma \in \mathbb{N}^{\#}$.
VI.**Disjunction axioms and rules**:

(a)

$$\phi_j \vdash_{\mathbf{x},\boldsymbol{\alpha}} \bigvee_{i<\gamma} \phi_i \tag{5.14}$$

for each $\gamma \in \mathbb{N}^{\#}$

(b)

$$\frac{\{\phi_i \vdash_{\mathbf{x},\boldsymbol{\alpha}} \theta\}_{i<\gamma}}{\bigvee_{i<\gamma} \phi_i \vdash_{\mathbf{x},\boldsymbol{\alpha}} \theta} \tag{5.15}$$

for each $\gamma \in \mathbb{N}^{\#}$.
VII. **Implication rule**:

$$\frac{\phi \wedge \psi \vdash_{\mathbf{x},\boldsymbol{\alpha}} \theta}{\phi \vdash_{\mathbf{x},\boldsymbol{\alpha}} \psi \Rightarrow \theta} \tag{5.16}$$

IX.**Existential rule**:

$$\frac{\phi \vdash_{\mathbf{xy}} \psi}{\exists \mathbf{y}(\phi \vdash_{\mathbf{x}} \psi)} \tag{5.17}$$

where no variable in $\mathbf{y}$ is free in ψ.

X.Universal rule:

$$\frac{\varphi \vdash_{\mathbf{xy}} \psi}{\phi \vdash_{\mathbf{x}} \forall \mathbf{y}\psi} \quad (5.18)$$

where no variable in $\mathbf{y}$ is free in φ.

6.Intuitionistic set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ in hyperinfinitary set theoretical language.

6.1.Axioms and basic definitions.

Intuitionistic set theory $\mathbf{INC}^{\#}$ is formulated as a system of axioms in the same first order language as its classical counterpart, only based on intuitionistic logic $IL^{\#}_{\infty^{\#}}$ with restricted modus ponens rule. The language of set theory is a first-order language $L^{\#}_{\infty^{\#}}$ with equality $=$, which includes a binary symbol $\in$. We write $x \neq y$ for $\neg(x = y)$ and $x \notin y$ for $\neg(x \neg y)$. Individual variables $x, y, z, \ldots$ of $L^{\#}_{\infty^{\#}}$ will be understood as ranging over classical sets. The unique existential quantifier $\exists!$ is introduced by writing, for any formula $\varphi(x), \exists! x\varphi(x)$ as an abbreviation of the formula $\exists x[\varphi(x) \ \& \ \forall y(\varphi(y) \Rightarrow x = y)]$. $L^{\#}_{\infty^{\#}}$ will also allow the formation of terms of the form $\{x|\varphi(x)\}$, for any formula φ containing the free variable x. Such terms are called nonclassical sets; we shall use upper case letters $A, B, \ldots$ for such sets. For each nonclassical set $A = \{x|\varphi(x)\}$ the formulas $\forall x[x \in A \Leftrightarrow \varphi(x)]$ and $\forall x[x \in A \Leftrightarrow \varphi(x, A)]$is called the defining axioms for the nonclassical set A.

Remark 6.1.Note that the formula $\forall x[x \in A \Leftrightarrow \varphi(x, A)]$ and $\forall x[x \in a \Leftrightarrow \varphi(x, a) \wedge x \in u]$
is not always asserts that $\forall x[x \in A \vdash_{RMP} \varphi(x, A)]$ and (or) $\forall x[\varphi(x, A) \vdash_{RMP} x \in A]$ even for a classical set since for some y possible $y \in A \Rightarrow \varphi(y) \nvdash_{RMP} \varphi(y)$ and (or) $\varphi(y) \Rightarrow y \in A \nvdash_{RMP} x \in A$ and $y \in a \Rightarrow \varphi(y) \wedge y \in u \nvdash_{RMP} \varphi(y) \wedge y \in u$, etc.In order to emphasize this fact we sometimes vrite the defining axioms for the nonclassical set in the
following form $\forall x[x \in A \Leftrightarrow_w \varphi(x, A)]$

Remark 6.2.(1) Two nonclassical sets A, B are defined to be equal and we write $A = B$ if
$\forall x[x \in A \Leftrightarrow x \in B]$. (2) A is a subset of B, and we write $A \subseteqq B$, if $\forall x[x \in A \Rightarrow x \in B]$. (3) We also write $\mathbf{Cl.Set}(A)$ for the formula $\exists u \forall x[x \in A \Leftrightarrow x \in u]$. (4) We also write $\mathbf{NCl.Set}(A)$ for the formulas $\forall x[x \in A \Leftrightarrow \varphi(x)]$ and $\forall x[x \in A \Leftrightarrow \varphi(x, A)]$.

Remark 6.3.$\mathbf{Cl.Set}(A)$) asserts that the set A is a classical set. For any classical set u, it follows from the defining axiom for the classical set $\{x|x \in u \wedge \varphi(x)\}$ that $\mathbf{Cl.Set}(\{x|x \in u \wedge \varphi(x)\})$.

We shall identify $\{x|x \in u\}$ with u, so that sets may be considered as (special sorts of) nonclassical sets and we may introduce assertions such as $u \subset A, u \subseteqq A, u = A$, etc.

Remark 6.4.If A is a nonclassical set, we write $\exists x \in A \ \varphi(x, A)$ for $\exists x[x \in A \wedge \varphi(x, A)]$ and $\forall x \in A\varphi(x, A)$ for $\forall x[x \in A \Rightarrow \varphi(x, A)]$.

We define now the following sets:

1.$\{u_1, u_2, \ldots, u_n\} = \{x|x = u_1 \vee x = u_2 \vee \ldots \vee x = u_n\}$. 2. $\{A_1, A_2, \ldots, A_n\} =$
$= \{x|x = A_1 \vee x = A_2 \vee \ldots \vee x = A_n\}$. 3.$\cup A = \{x|\exists y[y \in A \wedge x \in y]\}$.

4.$\cap A = \{x|\forall y[y \in A \Rightarrow x \in y]\}$. 5.$A \cup B = \{x|x \in A \vee x \in B\}$.
5.$A \cap B = \{x|x \in A \wedge x \in B\}$. 6.$A - B = \{x|x \in A \wedge x \notin B\}$. 7.$u^+ = u \cup \{u\}$.
8.$\mathbf{P}(A) = \{x|x \subseteq A\}$. 9.$\{x \in A|\varphi(x,A)\} = \{x|x \in A \wedge \varphi(x,A)\}$. 10.$\mathbf{V} = \{x|: x = x\}$.
11.$\varnothing = \{x|x \neq x\}$.
The system $\mathbf{INC}^{\#}_{\infty^{\#}}$ of set theory is based on the following axioms:
Extensionality1: $\forall u \forall v[\forall x(x \in u \Leftrightarrow x \in v) \Rightarrow u = v]$
Extensionality2: $\forall A \forall B[\forall x(x \in A \Leftrightarrow x \in B) \Rightarrow A = B]$
Universal Set: $\mathbf{NCl.Set}(\mathbf{V})$
Empty Set: $\mathbf{Cl.Set}(\varnothing)$
Pairing1: $\forall u \forall v \ \mathbf{Cl.Set}(\{u,v\})$
Pairing2: $\forall A \forall B \ \mathbf{NCl.Set}(\{A,B\})$
Union1: $\forall u \ \mathbf{Cl.Set}(\cup u)$
Union2: $\forall A \ \mathbf{NCl.Set}(\cup A)$
Powerset1: $\forall u \ \mathbf{Cl.Set}(\mathbf{P}(u))$
Powerset2: $\forall A \ \mathbf{NCl.Set}(\mathbf{P}(A))$
Infinity $\exists a[\varnothing \in a \wedge \forall x \in a(x^+ \in a)]$
Separation1$\forall u_1 \forall u_2, \ldots \forall u_n \forall a \exists \mathbf{Cl.Set}(\{x \in a|\varphi(x,u_1,u_2,\ldots,u_n)\})$
Separation2$\forall u_1 \forall u_2, \ldots \forall u_n \mathbf{NCl.Set}(\{x \in A|\varphi(x,A;u_1,u_2,\ldots,u_n)\})$
Comprehension1$\forall u_1 \forall u_2, \ldots \forall u_n \exists A \forall x[x \in A \Leftrightarrow \varphi(x;u_1,u_2,\ldots,u_n)]$
Comprehension 2 $\forall u_1 \forall u_2, \ldots \forall u_n \exists A \forall x[x \in A \Leftrightarrow \varphi(x,A;u_1,u_2,\ldots,u_n)]$
Hyperinfinity: see subsection 2.1.
Remark 6.5.Note that the axiom of hyperinfinity follows from the schemata Comprehension 2.

7.External induction principle and hyperinductive definitions.

7.1.External induction principle in nonstandard intuitionistic arithmetic.

Axiom of infite ω-induction
(i)

$$\forall S(S \subset \mathbb{N})\left\{\left[\bigwedge_{n \in \omega}(n \in S \Rightarrow n^+ \in S)\right] \Rightarrow S = \mathbb{N}\right\}. \tag{7.1}$$

(ii) Let $F(x)$ be a wff of the set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$, then

$$\left[\bigwedge_{n \in \omega}(F(n) \Rightarrow F(n^+))\right] \Rightarrow \forall n(n \in \omega)F(n). \tag{7.2}$$

Definition 7.1.Let β be a hypernatural such that $\beta \in \mathbb{N}^{\#}\backslash\mathbb{N}$. Let $[0,\beta] \subset \mathbb{N}^{\#}$be a set such that $\forall x[x \in [0,\beta] \Leftrightarrow 0 \leq x \leq \beta]$ and $[0,\beta) = [0,\beta]\backslash\{\beta\}$.
Definition 7.2.(i) Let $F(x)$ be a wff of $\mathrm{INC}^{\#}$with unique free variable x.We will say that a wff $F(x)$ is restricted on a set S such that $S \subsetneqq \mathbb{N}^{\#}$ iff the following conditions are satisfied

$$\forall \alpha(\alpha \in \mathbb{N}^{\#})[F(\alpha) \Rightarrow \alpha \in S] \tag{7.3}$$

and

$$\forall\alpha(\alpha\in\mathbb{N}^{\#})[\neg F(\alpha)\Rightarrow\alpha\in\mathbb{N}^{\#}\backslash S]. \quad (7.4)$$

(ii) Let $F(x)$ be a wff of $\mathrm{INC}^{\#}_{\infty^{\#}}$with unique free variable x. We will say that a wff $F(x)$ is a strongly restricted on a set $S\subsetneqq\mathbb{N}^{\#}$ iff the following condition is satisfied

$$\forall\alpha(\alpha\in\mathbb{N}^{\#})[F(\alpha)\Leftrightarrow\alpha\in S] \quad (7.5)$$

Definition 7.2. Let $F(x)$ be a wff of $\mathrm{INC}^{\#}_{\infty^{\#}}$with unique free variable x. We will say that a wff $F(x)$ is unrestricted if wff $F(x)$ is not restricted on any set S such that $S\subsetneqq\mathbb{N}^{\#}$.

Example 7.1.(i) Let $\mathbf{fin}(\alpha),\alpha\in\mathbb{N}^{\#}$be a wff formula such that $\mathbf{fin}(\alpha)\Leftrightarrow\alpha\in\mathbb{N}$. Obviously wff $\mathbf{fin}(\alpha)$ is a strongly restricted on a set $S=\mathbb{N}$ since $\forall\alpha(\alpha\in\mathbb{N}^{\#})[\mathbf{fin}(\alpha)\Leftrightarrow\alpha\in\mathbb{N}]$.

(ii) Let $\mathbf{ifin}(\alpha),\alpha\in\mathbb{N}^{\#}$be a wff formula such that $\mathbf{ifin}(\alpha)\Leftrightarrow\alpha\in\mathbb{N}^{\#}\backslash\mathbb{N}$. Obviously wff $\mathbf{ifin}(\alpha)$ is a strongly restricted on a set $\mathbb{N}^{\#}\backslash\mathbb{N}$.

Axiom of hyperfinite induction 1

$$\forall\beta(\beta\in\mathbb{N}^{\#})\forall S(S\subseteq[0,\beta])\searrow$$
$$\left\{\forall\alpha(\alpha\in[0,\beta])\left[\bigwedge_{0\leq\alpha<\beta}(\alpha\in S\Rightarrow\alpha^{+}\in S)\right]\Rightarrow S=[0,\beta]\right\}. \quad (7.6)$$

Axiom of hyperinfinite induction 1

$$\forall S(S\subset\mathbb{N}^{\#})\left\{\forall\beta(\beta\in\mathbb{N}^{\#})\left[\bigwedge_{0\leq\alpha<\beta}(\alpha\in S\Rightarrow\alpha^{+}\in S)\right]\Rightarrow S=\mathbb{N}^{\#}\right\}. \quad (7.7)$$

Remark 7.1.Note that from comprechesion shemata 2 (see subsection 6.1) follows that

$$\forall\beta\exists S(S\subset[0,\beta])\forall\bar{\beta}(\bar{\beta}\in[0,\beta])\left[\bar{\beta}\in S\Leftrightarrow\bigwedge_{0\leq\alpha<\bar{\beta}}(\alpha\in S\Rightarrow\alpha^{+}\in S)\right]. \quad (7.8)$$

Therefore for any $\bar{\beta}\in[0,\beta]$ from (7.8) it follows that

$$\bigwedge_{0\leq\alpha<\bar{\beta}}(\alpha\in S\Rightarrow\alpha^{+}\in S)\vdash\bar{\beta}\in S. \quad (7.9)$$

Thus axiom of hyperfinite induction 1,i.e., (7.6) holds, since from (7.9) it follows that $\forall\bar{\beta}[\bar{\beta}\in[0,\beta]\Rightarrow\bar{\beta}\in S]$.

Remark 7.2.Note that from comprechesion shemata 2 (see subsection 6.1) follows that

$$\exists S(S\subset\mathbb{N}^{\#})\forall\beta(\beta\in\mathbb{N}^{\#})\left[\beta\in S\Leftrightarrow\bigwedge_{0\leq\alpha<\beta}(\alpha\in S\Rightarrow\alpha^{+}\in S)\right]. \quad (7.10)$$

Therefore for any $\beta\in\mathbb{N}^{\#}$ from (7.10) it follows that

$$\bigwedge_{0\leq\alpha<\beta}(\alpha\in S\Rightarrow\alpha^{+}\in S)\vdash\beta\in S \quad (7.12)$$

Thus axiom of hyperinfinite induction 1,i.e., (7.8) holds, since it follows from (7.12) that $\forall\beta[\beta\in\mathbb{N}^{\#}\Rightarrow\beta\in S]$.

Axiom of hyperfinite induction 2

Let $F(x)$ be a wff of the set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ strongly restricted on a set $[0,\beta]$ then

$$\left[\forall\beta(\beta\in[0,\beta])\left[\bigwedge_{0\leq\alpha<\beta}(F(\alpha)\Rightarrow F(\alpha^{+}))\right]\right]\Rightarrow\forall\alpha(\alpha\in[0,\beta])F(\alpha). \quad (7.13)$$

Axiom of hyperinfinite induction 2

Let $F(x)$ be anrestricted wff of the set theory INC$^{\#}_{\infty^{\#}}$ then

$$\left[\forall\beta(\beta\in\mathbb{N}^{\#})\left[\bigwedge_{0\leq\alpha<\beta}(F(\alpha)\Rightarrow F(\alpha^{+}))\right]\right]\Rightarrow\forall\beta(\beta\in\mathbb{N}^{\#})F(\beta). \quad (7.14)$$

Remark 7.3.Note that from comprechesion shemata 2 (see subsection 6.1) follows that

$$\forall\beta\exists S(S\subset[0,\beta])\forall\overline{\beta}(\overline{\beta}\in[0,\beta])\left[\overline{\beta}\in S\Leftrightarrow\bigwedge_{0\leq\alpha<\overline{\beta}}(F(\alpha)\Rightarrow F(\alpha^{+}))\right]. \quad (7.15)$$

Therefore for any $\overline{\beta}\in[0,\beta]$ from (7.15) it follows that

$$\bigwedge_{0\leq\alpha<\overline{\beta}}(F(\alpha)\Rightarrow F(\alpha^{+}))\vdash\overline{\beta}\in S \quad (7.16)$$

Thus axiom of hyperfinite induction 2,i.e., (7.13) holds, since it follows from (7.16) that $\forall\overline{\beta}[\overline{\beta}\in[0,\beta]\Rightarrow\overline{\beta}\in S]$.

Remark 7.4.Note that from comprechesion shemata 2 (see subsection 6.1) follows that

$$\exists S(S\subset\mathbb{N}^{\#})\forall\overline{\beta}(\overline{\beta}\in\mathbb{N}^{\#})\left[\overline{\beta}\in S\Leftrightarrow\bigwedge_{0\leq\alpha<\overline{\beta}}(F(\alpha)\Rightarrow F(\alpha^{+}))\right]. \quad (7.17)$$

Therefore for any $\overline{\beta}\in\mathbb{N}^{\#}$ from (7.17) it follows that

$$\bigwedge_{0\leq\alpha<\overline{\beta}}(F(\alpha)\Rightarrow F(\alpha^{+}))\vdash\overline{\beta}\in S. \quad (7.18)$$

Thus axiom of hyperinfinite induction 2,i.e., (7.14) holds, since From (7.18) it follows that $\forall\overline{\beta}\left[\overline{\beta}\in\mathbb{N}^{\#}\Rightarrow\overline{\beta}\in S\right]$.

Axiom of hyperfinite induction 3

Let $F(x)$ be a wff of the set theory INC$^{\#}_{\infty^{\#}}$ strongly restricted on inductive set $W_{\mathbf{ind}}$ such that

$\mathbb{N}\subseteq W_{\mathbf{ind}}\subsetneqq\mathbb{N}^{\#}$ then

$$\forall W\left[(\mathbb{N}\subseteq W_{\mathbf{ind}}\subsetneqq\mathbb{N}^{\#})\wedge\left[\bigwedge_{\alpha\in W_{\mathbf{ind}}}(F(\alpha)\Rightarrow F(\alpha^{+}))\right]\right]\Rightarrow\forall\alpha(\alpha\in W_{\mathbf{ind}})F(\alpha). \quad (7.19)$$

Proposition 7.1. (a) For any natural or hypernatural number $k\in\mathbb{N}^{\#}$,

$$\vdash\bigvee_{0\leq m\leq k}(x=m)\Leftrightarrow x\leq k. \quad (7.20)$$

(a$'$) For any hypernatural number κ and any wff B

$$\vdash\bigwedge_{0\leq m\leq k}B(m)\Leftrightarrow\forall x(x\leq k\Rightarrow B(x)). \quad (7.21)$$

(b) For any hypernatural number $k\in\mathbb{N}^{\#}$ such that $k>0$,

$$\vdash\bigvee_{1\leq m\leq k}(x=m-1)\Leftrightarrow x<k. \quad (7.22)$$

(b$'$) For any hypernatural number $k\in\mathbb{N}^{\#}$ such that $k>0$ and any wff $B(x)$,

$$\vdash\bigwedge_{0\leq m\leq k-1}B(m)\Leftrightarrow\forall x(x<k\Rightarrow B(x)). \quad (7.23)$$

(c) $\vdash(\forall x(x<y\Rightarrow B(x)))\wedge(\forall x(x\geq y\Rightarrow E(x)))\Rightarrow\forall x(B(x)\vee E(x))$.

Proof. (a) We prove $\bigvee_{0\leq m\leq k}(x=m)\Leftrightarrow x\leq k$ by hyperfinite induction in the metalanguage on k. The case for $k=0, \vdash x=0\Leftrightarrow x\leq 0,$ is obvious from the definitions.

Assume as inductive hypothesis that

$$\bigvee_{0\leq m\leq k}(x=m)\Leftrightarrow x\leq k. \tag{7.24}$$

Now assume that

$$\left[\bigvee_{0\leq m\leq k}(x=m)\right]\vee(x=k+1). \tag{7.25}$$

But $\vdash x=k+1\Rightarrow x\leq k+1$ and, by the inductive hypothesis,

$$\bigvee_{0\leq m\leq k}(x=m). \tag{7.26}$$

Also $\vdash x\leq k\Rightarrow x<k+1$. Thus, $x\leq k+1$. So,

$$\vdash\bigvee_{0\leq m\leq k+1}(x=m)\Rightarrow x\leq k+1. \tag{7.27}$$

Conversely, assume $x\leq k+1$. Then $x=k+1\vee x<k+1$. If $x=k+1$, then

$$\bigvee_{0\leq m\leq k+1}(x=m). \tag{7.28}$$

If $x<k+1$, then we have $x\leq k$. By the inductive hypothesis,

$$\bigvee_{0\leq m\leq k}(x=m) \tag{7.29}$$

and,therefore,

$$\bigvee_{0\leq m\leq k+1}(x=m). \tag{7.30}$$

Thus in either case,

$$\bigvee_{0\leq m\leq k+1}(x=m). \tag{7.31}$$

This proves

$$\vdash x\leq k+1\Rightarrow\bigvee_{0\leq m\leq k+1}(x=m). \tag{7.32}$$

From the inductive hypothesis, we have derived

$$\bigvee_{0\leq m\leq k+1}(x=m)\Leftrightarrow x\leq k+1 \tag{7.33}$$

and this completes the proof. Note that this proof has been given in an informal manner that we shall generally use from now on. In particular, the deduction theorem, the replacement theorem, and various rules and tautologies will be applied without being explicitly mentioned.

Parts (a′), (b), and (b′) follow easily from part (a). Part (c) follows almost immediately from the statement $t \neq r \Rightarrow (t < r) \vee (r < t)$, using obvious tautologies.

There are several stronger forms of the hyperinfinite induction principles that we can prove at this point.

Theorem 7.1.(Complete hyperinfinite induction) Let $B(x)$ be anrestricted wff of the set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ then

$$\forall x(x \in \mathbb{N}^{\#})[\forall z(z < x \Rightarrow B(z)) \Rightarrow B(x)] \Rightarrow \forall x(x \in \mathbb{N}^{\#})B(x) \qquad (7.34)$$

In ordinary languageI consider a property $B(x)$ such that, for any x, if $B(x)$ holds for all hypernatural numbers less than x, then $B(x)$ holds for x also. Then $B(x)$ holds for all hypernatural numbers $x \in \mathbb{N}^{\#}$.

Proof.Let $E(x)$ be a wff $\forall z(z \leq x \Rightarrow B(z))$.

(i) 1.Assume that $\forall x(x \in \mathbb{N}^{\#})[\forall z(z < x \Rightarrow B(z)) \Rightarrow B(x)]$, then

2.$[\forall z(z < 0 \Rightarrow B(z)) \Rightarrow B(0)]$ it follows from 1.

3. $z \nless 0$, then

4. $\forall z(z < 0 \Rightarrow B(z))$ it follows from 1,

5. $B(0)$ it follows from 2,4 by MP

6. $\forall z(z \leq 0 \Rightarrow B(z))$ i.e.,$E(0)$ holds it follows from Proposition7.1(a$'$)

7.$\forall x(x \in \mathbb{N}^{\#})[\forall z(z < x \Rightarrow B(z)) \Rightarrow B(x)] \vdash E(0)$ it follows from 1,6 by MP

(ii) 1.Assume that: $\forall x(x \in \mathbb{N}^{\#})[\forall z(z < x \Rightarrow B(z)) \Rightarrow B(x)]$.

2.Assume that: $E(x) \equiv \forall z(z \leq x \Rightarrow B(z))$, then

3.$\forall z(z < x^{+} \Rightarrow B(z))$ it follows from 2 since $z \leq x \Rightarrow z < x^{+}$.

4.$\forall x(x \in \mathbb{N}^{\#})[\forall z(z < x^{+} \Rightarrow B(z)) \Rightarrow B(x^{+})]$ it follows from 1 by rule A4:if t is free for x in $B(x)$, then $\forall xB(x) \vdash B(t)$.

5. $B(x^{+})$ it follows from 3,4 by unrestricted MP rule.

6. $z \leq x^{+} \Rightarrow z < x^{+} \vee z = x^{+}$ it follows from definitions.

7. $z < x^{+} \Rightarrow B(z)$ it follows from 3 by rule A4.

8. $z = x^{+} \Rightarrow B(z)$ it follows from 5.

9. $E(x^{+}) \equiv \forall z(z \leq x^{+} \Rightarrow B(z))$ it follows from 6,7,8,rule Gen.

10.$\forall x(x \in \mathbb{N}^{\#})[\forall z(z < x \Rightarrow B(z)) \Rightarrow B(x)] \vdash \forall x(x \in \mathbb{N}^{\#})[E(x) \Rightarrow E(x^{+})]$

it follows from 1,9 by deduction theorem,rule Gen.

Now by (i), (ii) and the induction axiom, we obtain $D \vdash \forall x(x \in \mathbb{N}^{\#})E(x)$ that is $D \vdash \forall x(x \in \mathbb{N}^{\#})[\forall z(z < x \Rightarrow B(z))]$, where $D \equiv \forall x(x \in \mathbb{N}^{\#})[\forall z(z < x \Rightarrow B(z)) \Rightarrow B(x)]$.

Hence, by rule A4 twice, $D \vdash x \leq x \Rightarrow B(x)$. But $\vdash x \leq x$. So,$D \vdash B(x)$, and, by Gen and

the deduction theorem, $D \vdash \forall x(x \in \mathbb{N}^{\#})B(x)$.

Theorem 7.2.(Complete hyperfinite induction) Let $B(x)$ be wff of the set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ strongly restricted on inductive set $W_{\mathbf{ind}}$ such that $\mathbb{N} \subseteq W_{\mathbf{ind}} \subsetneq \mathbb{N}^{\#}$ then

$$\forall x(x \in W_{\mathbf{ind}})[\forall z(z < x \Rightarrow B(z)) \Rightarrow B(x)] \Rightarrow \forall x(x \in W_{\mathbf{ind}})B(x) \qquad (7.35)$$

Proof. Similarly as Theorem 7.1.

Remark 7.5.Remind that the following statement holds in standard bivalent arithmetic [11]:Least-number principle (LNP)

$$\exists xB(x) \Rightarrow \exists y[B(y) \wedge \forall z(z < y \Rightarrow \neg B(z))]. \qquad (7.36)$$

In ordinary language:if a property expressed by wff $B(x)$ holds for some natural number n,

then there is a least number satisfying $B(x)$.Obviously LNP (7.23) is not holds in nonstandard arithmetic, since there is no a least number in a set $\mathbb{N}^{\#}\backslash\mathbb{N}$.

Theorem 7.3.(Weak least-number principle) Let $B(x)$ be a wff of the set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ such that a wff $\neg B(x)$ strongly restricted on inductive set $W_{\mathbf{ind}}$ such that $\mathbb{N} \subseteq W_{\mathbf{ind}} \subsetneqq \mathbb{N}^{\#}$ and $W^{\complement}_{\mathbf{ind}} = \mathbb{N}^{\#}\backslash W_{\mathbf{ind}}$ then

$$\exists x\left(x \in W^{\complement}_{\mathbf{ind}}\right)B(x) \Rightarrow \\ \neg\exists y\left(y \in W^{\complement}_{\mathbf{ind}}\right)[B(y) \wedge \forall z(z < y \Rightarrow \neg B(z))] \Rightarrow \forall y(y \in W_{\mathbf{ind}})[\neg B(y)] \quad (7.37)$$

Proof.We assume now that

1.$\neg\exists y\left(y \in W^{\complement}_{\mathbf{ind}}\right)[B(y) \wedge \forall z(z < y \Rightarrow \neg B(z))]$

2.$\forall y\left(y \in W^{\complement}_{\mathbf{ind}}\right)\neg[B(y) \wedge \forall z(z < y \Rightarrow \neg B(z))]$ it follows from 1.

3.$\forall y\left(y \in W^{\complement}_{\mathbf{ind}}\right)[\forall z(z < y \Rightarrow \neg B(z)) \Rightarrow \neg B(y)]$it follows from 2 by tautology.

4.$\forall y(y \in W_{\mathbf{ind}})[\neg B(y)]$ it follows from 3 by Theorem 7.2 with wff $\neg B(y)$ instead wff $B(y)$

5.$\neg\exists y\left(y \in W^{\complement}_{\mathbf{ind}}\right)[B(y) \wedge \forall z(z < y \Rightarrow \neg B(z))] \Rightarrow \forall y(y \in W_{\mathbf{ind}})[\neg B(y)]$ it follows from 1,4.

Hyperinductive definitions in general.

A function $f : \mathbb{N}^{\#} \to A$ whose domain is the set $\mathbb{N}^{\#}$ is colled an hyperinfinite sequence and denoted by $\{f_n\}_{n\in\mathbb{N}^{\#}}$ or by $\{f(n)\}_{n\in\mathbb{N}^{\#}}$The set of all hyperinfinite sequences whose terms belong to A is clearly $A^{\mathbb{N}^{\#}}$; the set of all hyperfinite sequences of $n \in \mathbb{N}^{\#}$ terms in A is A^n. The set of all hyperfinite sequences with terms in A can be defined as

$$\left\{R \subset \mathbb{N}^{\#} \times A : (R \text{ is a function}) \wedge \bigvee\nolimits_{n\in\mathbb{N}^{\#}}(D_1(R) = n)\right\}, \quad (7.38)$$

where $D_1(R)$ is domain of R.This definition implies the existence of the set of all hyperfinite sequences with terms in A.The simplest case is the inductive definition of a hyperinfinite sequence $\{\varphi(n)\}_{n\in\mathbb{N}^{\#}}$ (with terms belonging to a certain set Z) satisfying the following conditions:

(a) $\varphi(0) = z, \varphi(n^+) = e(\varphi(n), n),$

where $z \in Z$ and e is a function mapping $Z \times \mathbb{N}^{\#}$ into Z.

More generally, we consider a mapping f of the cartesian product $Z \times \mathbb{N}^{\#} \times A$ into Z and seek a function $\varphi \in Z^{\mathbb{N}^{\#}\times A}$ satisfying the conditions :

(b) $\varphi(0, a) = g(a), \varphi(n^+, a) = f(\varphi(n, a), n, a),$

where $g \in Z^A$. This is a definition by induction with parameter a ranging over the set A. Schemes (a) and (b) correspond to induction “from n to $n^+ = n + 1$”,i.e. $\varphi(n^+)$ or $\varphi(n^+, a)$ depends upon $\varphi(n)$ or $\varphi(n, a)$ respectively. More generally, $\varphi(n^+)$ may depend upon all values $\varphi(m)$ where $m \leq n$ (i.e. $m \in n^+$). In the case of induction with parameter, $\varphi(n^+, a)$ may depend upon all values $\varphi(m, a)$, where $m \leq n$; or even upon all values $\varphi(m, a)$, where $m \leq n^+$ and $b \in A$. In this way we obtain the following schemes of definitions by induction:

(c) $\varphi(0) = z, \varphi(n^+) = h(\varphi|n^+, n),$

(d) $\varphi(0, a) = g(a), \quad \varphi(n^+, a) = H(\varphi|(n^+ \times A), n, a).$

In the scheme (c), $z \in Z$ and $h \in Z^{C\times\mathbb{N}^{\#}}$, where C is the set of hyperfinite sequences whose terms belong to Z; in the scheme (d), $g \in Z^A$ and $H \in Z^{T\times\mathbb{N}^{\#}\times A}$, where T is the set of functions whose domains are included in $\mathbb{N}^{\#} \times A$ and whose values belong to Z.

It is clear that the scheme (d) is the most general of all the schemes considered above.

By coise of functions one obtains from (d) any of the schemes (a)-(d). For example, taking the function defined by $H(c,n,a) = f(c(n,a),n,a)$ for $a \in A, n \in \mathbb{N}^{\#}, c \in Z^{\mathbb{N}^{\#}\times A}$ as H in (d), one obtain (b). We shall now show that, conversely, the scheme (d) can be obtained from (a). Let g and H be functions belonging to Z^A and $Z^{T\times\mathbb{N}^{\#}\times A}$ respectively, and let φ be a function satisfying (d). We shall show that the sequence $\Psi = \{\Psi_n\}_{n\in\mathbb{N}^{\#}}$ with $\Psi_n = \varphi|(n^+,A)$ can be defined by (a).Obviously, $\Psi_n \in T$ for every $n \in \mathbb{N}^{\#}$. The first term of the sequence Ψ is equal to $\varphi|(0^+,A)$, i.e. to the set: $z^* = \{\langle\langle 0,a\rangle, g(a)\rangle | a \in A\}$.The relation between Ψ_n, and Ψ_{n^+} is given by the formula:$\Psi_{n^+} = \Psi_n \cup \varphi|(\{n^+\} \times A)$, where the second component is

$$\{\langle\langle n^+,a\rangle, \varphi(n^+,a)\rangle | a \in A\} = \{\langle n^+,a\rangle,, H(\Psi_n,n,a) | a \in A\}. \quad (7.39)$$

Thus we see that the sequence Ψ can be defined by (a) if we substitute T for Z, z^* for z and let $e(c,n) = c \cup \{\langle n^+,a\rangle, H(c,n,a) | a \in A\}$ for $c \in T$.

Now we shall prove the existence and uniqueness of the function satisfying (a). This theorem shows that we are entitled to use definitions by induction of the type (a). According to the remark made above, this will imply the existence of functions satisfying the formulas (b), (c), and (d). Since the uniqueness of such functions can be proved in the same manner as for (a), we shall use in the sequel definitions by induction of any of the types (a)-(d).

Theorem 7.4. If Z is any set $z \in Z$ and $e \in Z^{Z\times\mathbb{N}^{\#}}$, then there exists exactly one hyper sequence φ satisfying formulas (a).

Proof. Uniqueness. Suppose that $\{\varphi_1(n)\}_{n\in\mathbb{N}^{\#}}$ and $\{\varphi_2(n)\}_{n\in\mathbb{N}^{\#}}$ satisfy (a) and let

$$K = \{n | n \in \mathbb{N}^{\#} \wedge \varphi_1(n) = \varphi_2(n)\} \quad (7.40)$$

Then (a) implies that K is hyperinductive. Hence $\mathbb{N}^{\#} \subseteqq K$ and therefore $\varphi_1(n) \equiv \varphi_2(n)$.
Existence. Let $\Phi(z,n,t)$ be the formula $e(z,n) = t$ and let $\Psi(w,z,F)$ be the following formula:

$$(F \text{ is a function}) \wedge (D_1(F) = n^+) \wedge (F(0) = z) \wedge \bigwedge_{m\in n}(F(m), m, F(m^+)). \quad (7.41)$$

In other words, F is a function defined on the set of numbers $\leq n \in \mathbb{N}^{\#}$ such that $F(0) = z$ and $F(m^+) = e(F(m),m)$ for all $m < n \in \mathbb{N}^{\#}$.

We prove by induction that there exists exactly one function F_n such that $\Psi(n,z,F_n)$. The proof of uniqueness of this function is similar to that given in the first part of Theorem 7.4. The existence of F_n can be proved as follows: for $n = 0$ it suffices to take $\{\langle 0,z\rangle\}$ as F_n; if $n \in \mathbb{N}^{\#}$ and F_n satisfies $\Psi(n,z,F_n)$, then $F_{n^+} = F_n \cup \{\langle n^+, e(F_n(n),n)\rangle\}$ satisfies the condition $\Psi(n^+,z,F_{n^+})$.

Now, we take as φ the set of pairs $\langle n,s\rangle$ such that $n \in \mathbb{N}^{\#}, s \in Z$ and

$$\bigvee_F [\Psi(n,z,F) \wedge (s = F(n))]. \quad (7.42)$$

Since F is the unique function satisfying $\Psi(n,z,F)$, it follows that φ is a function. For $n = 0$ we have $\varphi(0) = F_0(0) = z$; if $n \in \mathbb{N}^{\#}$, then $\varphi(n^+) = F_{n^+}(n^+) = e(F_n(n),n)$ by the definition of F_n; hence we obtain $\varphi(n^+) = e(\varphi(0),n)$. Theorem 7.4 is thus proved.

We frequently define not one but several functions (with the same range Z) by a simultaneous induction:

$\varphi(0) = z,\ \psi(0) = t,$

$\varphi(n^+) = f(\varphi(n),\psi(n),n),\ \psi(n^+) = g(\varphi(n),\psi(n),n)$

where $z,t \in Z$ and $f,g \in Z^{Z\times Z\times N}$.

This kind of definition can be reduced to the previous one. It suffices to notice that the hypersequence $\vartheta_n = \langle\varphi(n),\psi(n)\rangle$ satisfies the formulas:$\vartheta_0 = \langle z,t\rangle, \vartheta_{n^+} = e(\vartheta_n,n),$where we set

$$e(u,n) = \langle f(K(u),L(u),n), g(K(u),F(w),n)\rangle, \tag{7.43}$$

and K,L denote functions such that

$K(\langle x,y\rangle)$ and $L(\langle x,y\rangle) = y$ respectively. Thus the function ϑ is defined by induction by means of (a). We now define φ and ψ by $\varphi(n) = K(\vartheta_n), \psi(n) = L(\vartheta_n)$.

8.Useful examples of the hyperinductive definitions.

1.Addition operation of gypernatural numbers

The function $+(m,n) \triangleq m+n : \mathbb{N}^\# \times \mathbb{N}^\# \to \mathbb{N}^\#$ is defined by

$m+0 = m, m+n^+ = (m+n)^+$.

This definition is obtained from (b) by seting $Z = A = \mathbb{N}^\#, g(a) = a, f(p,n,a) = p^+$.

This function satisfies all properties of addition such as: for all $m,n,k \in \mathbb{N}^\#$

(i) $m+0 = m$ (ii) $m+n = n+m$ (iii) $m+(n+k) = (m+n)+k$.

2.Multiplicattion operation of gypernatural numbers

The function $\times(m,n) \triangleq m\times n : \mathbb{N}^\# \times \mathbb{N}^\# \to \mathbb{N}^\#$ is defined by

$m \times 1 = 1, m\times n^+ = m\times n + m$.

(i) $m\times 1 = 1$ (ii) $m\times n = n\times m$ (iii) $m\times(n\times k) = (m\times n)\times k$.

4.Distributivity with respect to multiplication over addition.

$m\times(n+k) = m\times n + m\times k$.

5. Let $Z = A = X^X, g(a) = I_X, f(u,n,a) = u\circ a$ in (b). Then (b) takes on the following form

$$\varphi(0,a) = I_X, \varphi(n^+,a) = \varphi(n,a)\circ a. \tag{8.1}$$

The function $\varphi(n,a)$ is denoted by a^n and is colled n-th iteration of the function a :

$$a^0(x) = x, a^{n^+}(x) = a^n(a(x)), x \in X, a \in X^X, n \in \mathbb{N}^\#. \tag{8.2}$$

6.Let $A = (\mathbb{N}^\#)^{\mathbb{N}^\#}, g(a) = a_0, f(u,n,a) = u + a_{n^+}$.Then (b) takes on the following form

$$\varphi(0,a) = a_0, \varphi(n^+,a) = \varphi(n,a) + a_{n^+} \tag{8.3}$$

The function is defined by the Eqs.(8.3) is denoted by

$$\sum_{i=0}^{n} a_i \tag{8.4}$$

7.Let $A = (\mathbb{N}^\#)^{\mathbb{N}^\#}, g(a) = a_0, f(u,n,a) = u\times a_{n^+}$.Then (b) takes on the following form

$$\varphi(0,a) = a_0, \varphi(n^+,a) = \varphi(n,a)\times a_{n^+} \tag{8.5}$$

The function is defined by the Eqs.(8.5) is denoted by

$$\prod_{i=0}^{n} a_i \tag{8.6}$$

8. Similarly we define $\max_{i\leq n}(a_i), \min_{i\leq n}(a_i), n \in \mathbb{N}^\#$.

Theorem 8.1. The following equalities holds for any $n,k_1,l_1 \in \mathbb{N}^\#$:

(1) using distributivity

$$b\times\sum_{i=0}^{n} a_i = \sum_{i=0}^{n} b\times a_i \tag{8.7}$$

(2) using commutativity and associativity

$$\sum_{i=0}^{n} a_i \pm \sum_{i=0}^{n} b_i = \sum_{i=0}^{n}(a_i \pm b_i) \quad (8.8)$$

(3) splitting a sum, using associativity

$$\sum_{i=0}^{n} a_i = \sum_{i=0}^{j} a_i + \sum_{i=j+1}^{n} a_i \quad (8.9)$$

(4) using commutativity and associativity, again

$$\sum_{i=k_0}^{k_1}\sum_{j=l_0}^{l_1} a_{ij} = \sum_{j=l_0}^{l_1}\sum_{i=k_0}^{k_1} a_{ij} \quad (8.10)$$

(5) using distributivity

$$\left(\sum_{i=0}^{n} a_i\right) \times \left(\sum_{j=0}^{n} b_j\right) = \sum_{i=0}^{n}\sum_{j=0}^{n} a_i \times b_j \quad (8.11)$$

(6)

$$\left(\prod_{i=0}^{n} a_i\right) \times \left(\prod_{i=0}^{n} b_i\right) = \prod_{i=0}^{n} a_i \times b_i \quad (8.12)$$

(7)

$$\left(\prod_{i=0}^{n} a_i\right)^m = \prod_{i=0}^{n} a_i^m \quad (8.13)$$

Proof. Imediately from Theorem 7.4 and hyperinfinite induction principle.

Definition 8.1.A non-empty non regular sequence $\{u_n\}_{n\in\mathbb{Z}}$ is a blok corresponding to gyperfinite number $u = u_0 \in \mathbb{N}^{\#}\backslash\mathbb{N}$ iff there is gyperfinite number u such that $\ldots\in u_{-(n+1)} \in u_{-n}\ldots\in u_{-4} \in u_{-3} \in u_{-2} \in u_{-1} \in u$ and the following conditions are satisfied

$$\ldots\in u_{-(n+1)} \in u_{-n}\ldots\in u_{-4} \in u_{-3} \in u_{-2} \in u_{-1} \in u \in u_1 \in u_2 \in\ldots\in u_n \in u_{n+1} \in\ldots \quad (8.14)$$

where for any $n \in \mathbb{N}$: $u_{-(n+1)} \in u_{-n}$, where $u_{-n} = u^{+}_{-(n+1)}$.

Thus beginning with an infinite integer $u \in \mathbb{N}^{\#}\backslash\mathbb{N}$ we obtain a block (8.20) of infinite integers.However, given a “block,” there is another block consisting of even larger infinite integers. For example, there is the integer $u+u$, where $u+k < u+u$ for each $k \in \mathbb{N}$. And $v = u+u$ is itself part of the block:

$$\ldots< v-3 < v-2 < v-1 < v < v+1 < v-2 <\ldots \quad (8.15)$$

Of course, $v < v+u < v+v$, and so forth. There are even infinite integers $u \times u$ and u^u, and so forth.Proceeding in the opposite direction, if $u \in \mathbb{N}^{\#}\backslash\mathbb{N}$, either u or $u+1$ is of the form $v+v$. Here v must be infinite. So there is no first block, since $v < u$. In fact, the ordering of the blocks is dense. For let the block containing v precede the one containing u, that is,

$$v-2 < v-1 < v < v+1 <\ldots<\ldots< u-2 < u-1 < u < u+1 <\ldots \quad (8.16)$$

Either $u+v$ or $u+v+1$ can be written $z+z$ where $v+k < z < u-l$ for all $k,l \in \mathbb{N}$.

To conclude our consideration: $\mathbb{N}^{\#}$ consists of $\mathbb{N}$ as an initial segment followed by an ordered set of blocks. These blocks are densely ordered with no first or last element. Each block is itself order-isomorphic to the integers

$$-3,-2,-1,0,1,2,3, \qquad (8.17)$$

Although $\mathbb{N}^{\#}\backslash\mathbb{N}$ is a nonempty subset of $\mathbb{N}^{\#}$, as we have just seen it has no least element and likewise for any block.

9.Analisys on nonarchimedian field $\mathbb{Q}^{\#}$.

9.1.Basic properties of the hyperrationals $\mathbb{Q}^{\#}$.

Now that we have the hypernatural numbers, defining hyperintegers and hyperrational numbers is well within reach.

Definition 9.1. Let $Z' = \mathbb{N}^{\#} \times \mathbb{N}^{\#}$. We can define an equivalence relation $\approx$ on Z' by $(a,b) \approx (c,d)$ if and only if $a+d = b+c$. Then we denote the set of all hyperintegers by $\mathbb{Z}^{\#} = Z'/\approx$ (The set of all equivalence classes of Z' modulo $\approx$).

Definition 9.2. Let $Q' = \mathbb{Z}^{\#} \times (\mathbb{Z}^{\#} - \{0\}) = \{(a,b) \in \mathbb{Z}^{\#} \times \mathbb{Z}^{\#} | b \neq 0\}$. We can define an equivalence relation $\approx$ on Q' by $(a,b) \approx (c,d)$ if and only if $a \times d = b \times c$. Then we denote
the set of all hyperrational numbers by $\mathbb{Q}^{\#} = Q'/\approx$ (The set of all equivalence classes of
Q' modulo $\approx$).

Definition 9.3. A linearly ordered set $(P,<)$ is called dense if for any $a,b \in P$ such that $a < b$, there exists $z \in P$ such that $a < z < b$.

Lemma 9.1. $(\mathbb{Q}^{\#},<)$ is dense.

Proof. Let $x = (a,b), y = (c,d) \in \mathbb{Q}^{\#}$ be such that $x < y$. Consider $z = (ad+bc, 2bd) \in \mathbb{Q}^{\#}$.
It is easily shown that $x < z < y$.

Remark 9.1.Consider the ring B of all limited (i.e. finite) elements in $\mathbb{Q}^{\#}$. Then B has a unique maximal ideal $I_{\approx}$, the infinitesimal numbers. The quotient ring $B/I_{\approx}$ gives the field
$\mathbb{R}$ of the classical real numbers.

1.Let $A = (\mathbb{Q}^{\#})^{\mathbb{Q}^{\#}}, g(a) = a_0, f(u,n,a) = u + a_{n^+}$. Then (b) takes on the following form

$$\varphi(0,a) = a_0, \varphi(n^+,a) = \varphi(n,a) + a_{n^+} \qquad (9.1)$$

The function is defined by the Eqs.(9.1) is denoted by

$$\sum_{i=0}^{n} a_i. \qquad (9.2)$$

2.Let $A = (\mathbb{Q}^{\#})^{\mathbb{Q}^{\#}}, g(a) = a_0, f(u,n,a) = u \times a_{n^+}$. Then (b) takes on the following form

$$\varphi(0,a) = a_0, \varphi(n^+,a) = \varphi(n,a) \times a_{n^+} \qquad (9.3)$$

The function is defined by the Eqs.(9.3) is denoted by.

$$\prod_{i=0}^{n} a_i. \qquad (9.3)$$

9.2.Countable summation from gyperfinite sum.

Definition 9.1. Let $\{a_n\}_{n\in\mathbb{N}}$ be $\mathbb{Q}^{\#}$-valued countable sequence. Let $\{a_n\}_k^m$ be any hyperfinite sequence with $m \in \mathbb{N}^{\#}\backslash\mathbb{N}$ and such that $a_n = 0$ if $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$. Then we define summation of the countable sequence $\{a_n\}_{n\in\mathbb{N}}$ by the following hyperfinite sum

$$\sum_{n=k}^{m} a_n \in \mathbb{Q}^{\#} \qquad (9.4)$$

and denote such summ by the symbol

$$\sum_{n=k}^{\omega} a_n. \qquad (9.5)$$

Remark 9.2. Let $\{a_n\}_{n\in\mathbb{N}}$ be $\mathbb{Q}$-valued countable sequence. Note that: (i) for canonical summation we always apply standard notation

$$\sum_{n=k}^{\infty} a_n. \qquad (9.6)$$

(ii) the countable summ (ω-summ) (9.5) in contrast with (9.6) abviously always exists even if a series (9.6) diverges absolutely i.e., $\sum_{n=k}^{\infty}|a_n| = \infty$.

Example 9.1. The ω-summ $\sum_{n=1}^{\omega}\frac{1}{n} \in \mathbb{Q}^{\#}$ exists by Theorem 8.1, however $\sum_{n=1}^{\infty}\frac{1}{n} = \infty$.

Theorem 9.3. Let $\sum_{n=k}^{\omega} a_n = A$ and $\sum_{n=k}^{\omega} b_n = B,$ where $A,B,C \in \mathbb{Q}^{\#}$. Then

$$\sum_{n=k}^{\omega} C \times a_n = C \times \sum_{n=k}^{\omega} a_n \qquad (9.6)$$

and

$$\sum_{n=k}^{\omega}(a_n \pm b_n) = A \pm B. \qquad (9.7)$$

Proof. It follows from Theorem 8.2.

Example 9.2. Consider the countable sum

$$S_\omega(r) = \sum_{n=0}^{\omega} r^n, -1 < r < 1. \qquad (9.5)$$

It follows from (9.5)

$$S_\omega(r) = 1 + \sum_{n=1}^{\omega} r^n = 1 + r\sum_{n=0}^{\omega} r^n = 1 + rS_\omega(r) \qquad (9.6)$$

Thus

$$S_\omega(r) = \frac{1}{1-r}. \qquad (9.7)$$

Remark 9.3. Note that for any $r \in \mathbb{R}$ such that $-1 < r < 1$

$$S_\omega(r) = \sum_{n=0}^{\omega} r^n = \sum_{n=0}^{\infty} r^n \qquad (9.8)$$

since as we know

$$S_\infty(r) = \lim_{n\to\infty}\sum_{n=0}^{n} r^n = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}. \qquad (9.9)$$

10. Euler's proof of the Goldbach-Euler theorem revisited.

Theorem 10.1. (Goldbach-Euler theorem 1738)[]. This infinite series, continued to infinity,

$$\frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \frac{1}{31} + \frac{1}{35} + \ldots \qquad (10.1)$$

the denominators of which are all numbers which are one less than powers of degree two or higher of whole numbers, that is, terms which can be expressed with the formula $(m^n - 1)^{-1}$, where m and n are integers greater than one, then the sum of this series is

$= 1.$

10.1.How Euler did it.

Euler's proof begins with an $18^{\underline{th}}$ century step that treats any infinite sum as a real number which may be infinite large. Such steps became unpopular among rigorous mathematicians about a hundred years later.

Euler takes Σ to be the sum of the harmonic series

$$\Sigma = \sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \ldots \qquad (10.2)$$

Next, Euler subtracts from Eq.(10.2) the geometric series

$$1 = \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots \qquad (10.3)$$

leaving

$$\Sigma - 1 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{9} + \frac{1}{10} + \ldots \qquad (10.4)$$

Subtract from Eq.(10.4) geometric series

$$\frac{1}{2} = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \ldots \qquad (10.5)$$

leaving

$$\Sigma - 1 - \frac{1}{2} = 1 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \frac{1}{11} + \ldots \qquad (10.6)$$

Subtract from Eq.(10.6) geometric series

$$\frac{1}{4} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots \qquad (10.7)$$

leaving

$$\Sigma - 1 - \frac{1}{2} - \frac{1}{4} = 1 + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \ldots \qquad (10.8)$$

Remark 9.1. Note that Euler had to skip subtracting the geometric series

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots \qquad (10.9)$$

because the series of powers of $1/4$ on the right is already a subseries of the series of powers of $1/2$, so those terms have already been subtracted. This happens because 3 is one less than a power, $4.$ It happens again every time we reach a term one less than a power. He will have to skip $7,$ because that is one less than the cube $8,$ and 8 because it is one less than the square $9, 15$ because it is one less than the square $16,$ etc. Continuing formally in this way to infinity, we see that all of the terms on the right except the term 1 can be eliminated, leaving

$$\Sigma - 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{9} - \ldots = 1. \qquad (10.10)$$

Thus

$$\Sigma - 1 - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{9} + \frac{1}{10} + \ldots \right] = 1 \qquad (10.11)$$

so

$$\Sigma - 1 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{9} + \frac{1}{10} + \ldots \qquad (10.12)$$

Remark 10.2. Note that it gets just a little bit tricky. Since Σ is sum of the harmonic series, Euler believes that the 1 on the left must equal the terms of the harmonic series that are missing on the right. Those missing terms are exactly the ones with denominators one less than powers, so finally Euler concludes that

$$1 = \frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \frac{1}{31} + \frac{1}{35} + \ldots \tag{10.13}$$

where the terms on the right have denominators one less than powers.

10.2. **Proof of the Goldbach-Euler theorem using canonical analysis**.

We reproduce the proof here for the sake of completeness.

Lemma 1. For any positive integers n and k with $2 \leq n < k$

$1/n - 1 = 1/(n-1)n + 1/n(n+1) + \cdots + 1/(k-1)k + 1/k$

Lemma 2. For any positive integers n and k with $n \geq 2$

$1/n - 1 = 1/n + 1/n^2 + \cdots + 1/n^k + 1/n^k(n-1)$

We let denote the n-th harmonic number by H_n :

$$H_n = 1 + 1/2 + 1/3 + \ldots + 1/n, \tag{10.14}$$

but we now think of n as either a finite natural number or an infinite nonstandard natural number. Let k_2 be defined by $2^{k_2} \leq n < 2^{k_2+1}$. The existence and uniqueness of k_2 is clear either if we think of n as a finite natural number or as a nonstandard natural number: remember the transfer principle. Using Lemma 2, we can write

$$1 = 1/2 + 1/2^2 + 12^3 + \cdots + 1/2^{k_2} + 1/2^{k_2} \cdot 1,$$

and subtracting this series from (9.14), we obtain

$$H_n - 1 = 1 + 1/3 + 1/5 + 1/6 + 1/7 + 1/9 + \cdots + 1/n - 1/2^{k_2} \cdot 1. \tag{10.15}$$

Hence, all powers of two, including two itself, disappear from the denominators, leaving the rest of integers up to n. If from (10.15) we subtract

$$1/2 = 1/3 + 13^2 + 1/3^3 + \cdots + 1/3^{k_3} + 1/3^{k_3} \cdot 2, \tag{10.16}$$

again obtained from Lemma 2 with k_3 defined by $3^{k_3} \leq n < 3^{k_3+1}$, the result will be

$$H_n - 1 - 1/2 = 1 + 1/5 + 1/6 + 1/7 + 1/10 + \cdots + 1/n - [1/2^{k_2} \cdot 1 + 1/3^{k_3} \cdot 2]. \tag{10.17}$$

Proceeding similarly we end up by deleting all the terms that remain,arriving finally at

$$\begin{aligned} H_n - 1 - 1/2 - 1/4 - 1/5 - 1/6 - 1/7 - 1/10 - \cdots - 1/n = \\ = 1 - [1/2^{k_2} \cdot 1 + 1/3^{k_3} \cdot 2 + \cdots + 1/n \cdot (n-1)]. \end{aligned} \tag{10.18}$$

Notice that $k_2 \geq k_3 \geq \cdots$.In fact,when $m > \sqrt{n}$ we get $k_m = 1$. This last expression has been obtained assuming that n is a nonpower. If n is a power, then $1/n$ will have disappeared at some stage of this process,and the last fraction to beremoved from(10.17) will be $1/(n-1)$, whose denominator is a nonpower unless $n = 9$. (This is Catalan's conjecture that 8 and 9 are the only consecutive powers that exist. The conjecture was recently proved by Mihăilescu []. In fact, it does not matter here whether there are more consecutive powers or not.) The corresponding expression will thus be

$$\begin{aligned} H_n - 1 - 1/2 - 1/4 - 1/5 - 1/6 - 1/7 - 1/10 - \cdots - 1/n - 1 \\ = 1 - [1/2^{k_2} \cdot 1 + 1/3^{k_3} \cdot 2 + \cdots + 1/(n-1) \cdot (n-2)]. \end{aligned} \tag{10.19}$$

Consequently, if we subtract (10.18) from (10.14) we obtain

$$\begin{aligned} 1 - [1/2^{k_2} \cdot 1 + 1/3^{k_3} \cdot 2 + \cdots + 1/n \cdot (n-1)] = \\ 1/3 + 1/7 + 1/8 + 1/15 + 1/24 + 1/26 + \cdots + 1/n - 1 \end{aligned} \tag{10.20}$$

or, correspondingly subtracting (10.19) from (10.14),

$$1 - [12k_2 \cdot 1 + 13k_3 \cdot 2 + \cdots + 1/(n-1)(n-2)] = \\ 1/3 + 1/7 + 1/8 + 1/15 + 1/24 + 1/26 + \cdots + 1/n, \quad (10.21)$$

sums that containin their denominators,increased by one,all the power so fthe integers up to n. We must now take care of the “remainder,” that is, the expression between parentheses above or on the right-hand side of (10.17) (respectively, (10.19)).

Since for each $m \geq 2$ we know by the definition of km that $n < m^{k_m+1} \leq m^{2k_m}$, it follows that $\sqrt{n} < m^{k_m}$ and

$$1/[m^{k_m} \cdot (m-1)] \leq 1/\sqrt{n}\,(m-1). \quad (10.22)$$

This implies that

$$1/2^{k_2} \cdot 1 + 1/3^{k_3} \cdot 2 + \cdots + 1/n \cdot (n-1) \leq H_{n-1}/\sqrt{n} \quad (10.23)$$

or, if n is a power,

$$1/2^{k_2} \cdot 1 + 1/3^{k_3} \cdot 2 + \cdots + 1/(n-1) \cdot (n-2) \leq H_{n-2}/\sqrt{n-1}\,. \quad (10.24)$$

If we have chosen to regard n as a finite integer then we can pass to the limit and use Euler’s asymptotic value for H_n : $\lim_{n\to\infty} H_{n-1}/\sqrt{n} = \lim_{n\to\infty}[\log(n-1)+\gamma]/\sqrt{n} = 0$. The proof is now complete.

10.3.Euler proof revisited using elementary analysis on nonarchimedian field $\mathbb{Q}^{\#}$.

We replace Eq.(10.2) by

$$\Sigma_\omega = \sum_{n=1}^{\omega} \frac{1}{n} = \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \ldots\right]^{\#} \quad (10.22)$$

Remark 10.3.Note that $\Sigma_\omega \in \mathbb{Q}^{\#} \backslash \mathbb{Q}$.

Subtract from Eq.(10.22) the ω-sum

$$1 = \sum_{n=1}^{\omega} \frac{1}{2^n} = \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots\right]^{\#} \quad (10.23)$$

using Theorem 9.3 we obtain

$$\Sigma_\omega - 1 = \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \ldots\right]^{\#} - \\ -\left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots\right]^{\#} = \\ \left[1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{9} + \frac{1}{10} + \ldots\right]^{\#}. \quad (10.24)$$

Subtract from Eq.(10.24) the ω-sum

$$\frac{1}{2} = \sum_{n=1}^{\omega} \frac{1}{3^n} = \left[\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \ldots\right]^{\#} \quad (10.25)$$

using Theorem 9.3 we obtain

$$\Sigma_\omega - 1 - \frac{1}{2} = \left[1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{9} + \frac{1}{10} + \ldots\right]^{\#} -$$
$$-\left[\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \ldots\right]^{\#} = \quad (10.26)$$
$$= \left[1 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \frac{1}{11} + \ldots\right]^{\#}.$$

Subtract from Eq.(10.26) the ω-summ

$$\frac{1}{4} = \left[\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots\right]^{\#} \quad (10.27)$$

using Theorem 9.3 we obtain

$$\Sigma_\omega - 1 - \frac{1}{2} - \frac{1}{4} = \left[1 + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \ldots\right]^{\#} \quad (10.28)$$

Remark 10.4.Note that in calculation above we had skip subtracting the ω-sum (see Remark 9.1)

$$\frac{1}{3} = \left[\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots\right]^{\#} \quad (10.29)$$

because the series of powers of $1/4$ on the right is already a subseries of the ω-sum (10.23) of powers of $1/2$, so those terms have already been subtracted. This happens because 3 is one less than a power, 4. It happens again every time we reach a term one less than a power. He will have to skip 7, because that is one less than the cube 8, and 8 because it is one less than the square 9, 15 because it is one less than the square 16, etc. Continuing in this way to an gyperfinite number $m \in \mathbb{Q}^{\#}\backslash\mathbb{Q}$ by using gyperfinite induction principle, we see that all of the terms on the right except the term 1 can be eliminated, leaving

$$\left[\Sigma_\omega - 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{9} - \frac{1}{10} - \ldots\right]^{\#} = 1. \quad (10.30)$$

Thus by Theorem 9.3 we obtain

$$\Sigma - 1 - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{9} + \frac{1}{10} + \ldots\right]^{\#} = 1. \quad (10.31)$$

Finally we get

$$1 = \left[\frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \frac{1}{31} + \frac{1}{35} + \ldots\right]^{\#}, \quad (10.32)$$

where the terms on the right have denominators one less than powers.

Note that Eq.(10.32) now is obtained without any references to Catalan conjecture [13],[14].

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