Set Theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ Based on Infinitary Intuitionistic Logic with Restricted Modus Ponens Rule. Hyper Inductive Definitions. Application in transcendental number theory. Generalized Lindemann-Weierstrass theorem.

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Abstract

In this paper intuitionistic set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ in infinitary set theoretical language is considered. External induction principle in nonstandard intuitionistic arithmetic were derived. Non trivial application in number theory is considered.The Goldbach-Euler theorem is obtained without any references to Catalan conjecture. Main results are: (i) number e^e is transcendental; (ii) the both numbers $e+\pi$ and $e-\pi$ are irrational.

Keywords: Infinitary Intuitionistic logic; Nonstandard Arithmetic; Goldbach and Euler theorem;

Nonstandard Analisys; Lindemann-Weierstrass theorem.

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Content

1.Introduction

In this paper intuitionistic set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ based on infinitary intuitionistic logic with restricted modus ponens rule is considered [1]. External induction principle in nonstandard intuitionistic arithmetic were derived. Non trivial application in number theory is considered. The Goldbach-Euler theorem is obtained without any references to Catalan conjecture.

2.Axiom of nonregularity and axiom of hyperinfinity

2.1.Axiom of nonregularity

Remind that a non-empty set u is called regular iff

$$\forall x[x \neq \varnothing \to (\exists y \in x)(x \cap y = \varnothing)]. \tag{2.1}$$

Let's investigate what it says: suppose there were a non-empty x such that $(\forall y \in x)(x \cap y \neq \varnothing)$. For any $z_1 \in x$ we would be able to get $z_2 \in z_1 \cap x$. Since $z_2 \in x$ we would be able to get $z_3 \in z_2 \cap x$. The process continues forever: $\ldots \in z_{n+1} \in z_n \ldots \in z_4 \in z_3 \in z_2 \in z_1 \in x$. Thus we wish to rule out such an infinite regress.

2.1.Axiom of hyper infinity.

Definition 2.1.(i) A non-empty transitive non regular set u is a well formed non regular set iff:
(i) there is unique countable sequence $\{u_n\}_{n=1}^{\infty}$ such that

$$\ldots \in u_{n+1} \in u_n \ldots \in u_4 \in u_3 \in u_2 \in u_1 \in u, \tag{2.2}$$

(ii) for any $n \in \mathbb{N}$ and any $u_{n+1} \in u_n$:

$$u_n = u_{n+1}^{+}, \tag{2.3}$$

where $a^{+} = a \cup \{a\}$.

(ii) we define a function $a^{+[k]}$inductively by $a^{+[k+1]} = (a^{+[k]})^{+}$

Definition 2.2. Let u and w are well formed non regular sets. We write $w \prec u$ iff for any $n \in \mathbb{N}$

$$w \in u_n. \tag{2.4}$$

Definition 2.3. We say that an well formed non regular set u is infinite (or hyperfinite) hypernatural nuber iff:

(I) For any member $w \in u$ one and only one of the following conditions are satified:

(i) $w \in \mathbb{N}$ or

(ii) $w = u_n$ for some $n \in \mathbb{N}$ or

(iii) $w \prec u$.

(II) Let ${}_{\prec}u$ be a set ${}_{\prec}u = \{z | z \prec u\}$, then by relation $(\bullet \prec \bullet)$ a set ${}_{\prec}u$ is densely ordered with no first element.

(III) $\mathbb{N} \subset u$.

Axiom of hyper infinity

There exists unique set $\mathbb{N}^{\#}$ such that:

(i) $\mathbb{N} \subset \mathbb{N}^{\#}$

(ii) if u is infinite (hypernatural) number then $u \in \mathbb{N}^{\#} \backslash \mathbb{N}$

(iii) if u is infinite (hypernatural) number then there exists infinite (hypernatural) number v such that $v \prec u$

(iv) if u is infinite hypernatural number then there exists infinite (hypernatural) number w such that $u \prec w$

(v) set $\mathbb{N}^{\#} \backslash \mathbb{N}$ is patially ordered by relation $(\bullet \prec \bullet)$ with no first and no last element.

In this paper we introduced a set $\mathbb{N}^{\#} \backslash \mathbb{N}$ of the infinite numbers axiomatically without any references to non-standard model of arithmetic via canonical ultraproduct approach, see [2]-[5].

3.Infinitary logic.

3.1.Classical infinitary logic.

By a vocabulary, we mean a set L of constant symbols, and relation and operation symbols with finitely many argument places. As usual,by an L-structure M , we mean a universe set M with an interpretation for each symbol of L. In cases where the vocabulary L is clear, we may just say structure. For a given vocabulary L and infinite cardinals $\mu \leq \kappa, L_{\kappa\mu}$ is the infinitary logic with κ variables, conjunctions and disjunctions over sets of formulas of size less than κ, and existential and universal quantifiers over sets of variables of size less than μ. All logics that we consider also have equality, and are closed under negation. The equality symbol is always available, but is not counted as an element of the vocabulary L.

During last sentury canonical infinitary logic many developed, see for example [6]-[10].

3.2.Why we need infinitary logic.

It well known that some classes of mathematical structures, such as algebraically closed fields of a given characteristic, are characterized by a set of axioms in $L_{\omega\omega}$. Other classes cannot be characterized in this way, but can be axiomatized by a single sentence of $L_{\omega_1\omega}$.

Remark 3.1.In the practice of the contemporary model theory, and in more general mathematics as well, it often becomes necessary to consider structures satisfying certain collections of sentences rather than just single sentences. This consideration leads to the familiar notion of a theory in a logic. For example, in ordinary finitary logic, $L_{\omega\omega}$, if φ_n is a sentence which expresses that there are at least n elements, then the theory $\{\varphi_n | n \in \omega\}$ would express that there are infinitely many elements. Similarly, in the theory of groups, if φn is the sentence $\forall x[x^n \neq 1]$, then $\{\psi_n : n \in \omega\}$ expresses that a group is torsion free.

Remark 3.2.Suppose we want to express the idea that a set is finite, or that a group is torsion. A simple compactness argument would immediately reveal that neither of these notions can be expressed by a theory in $L_{\omega\omega}$. What we need to express in each case is that a certain theory is not satisfied, that is, that at least one of the sentences is false. While theories are able to simulate infinite conjunctions, there is no apparent way to simulate infinite disjunctions–which is just what is needed in this case.

Example 3.1. The Abelian torsion groups are the models of a sentence obtained by taking the conjunction of the usual axioms for Abelian groups (a finite set) and the following infinite disjunction:

$$\forall x\left[\bigvee_{n\in\mathbb{N}} \underbrace{x+x+\ldots+x}_{n} = 0\right]. \tag{3.1}$$

Example 3.2. The Archimedean ordered fields are the models of a sentence obtained by taking the conjunction of the usual axioms for ordered fields and the following infinite disjunction:

$$\forall x\left[\bigvee_{n\in\mathbb{N}} \underbrace{1+1+\ldots+1}_{n} > x\right]. \tag{3.2}$$

Example 4.3. Let L be a countable vocabulary. Let T be an elementary first order theory, and let $\Gamma(\bar{x})$ be a set of finitary formulas in a fixed tuple of variables $\bar{x}$. The models of T that omit Γ are the models of the single $L_{\omega_1\omega}$ sentence obtained by taking the conjunction of the sentences of T and the following infinite disjunction:

$$\forall \bar{x}\left[\bigvee_{\gamma\in\Gamma} \neg\gamma(\bar{x})\right]. \tag{3.3}$$

Example 4.4. The non Archimedean ordered fields are the models of a sentence obtained by taking the conjunction of the usual axioms for non Archimedean ordered fields i.e., the following infinite conjunction:

$$\exists x\left[\bigwedge_{n\in\mathbb{N}} \underbrace{1+1+\ldots+1}_{n} < x\right]. \tag{3.4}$$

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4.Hyper Infinitary logics.

4.1. Bivalent Hyper Infinitary first-order logic ${}^{2}L^{\#}_{\infty^{\#}}$ with restricted rules of conclusion.

Hyper infinitary language $L^{\#}_{\infty^{\#}}$ are defined according to the length of hyper infinitary conjunctions/disjunctions as well as quantification it allows. In that way, assuming a supply of $\kappa < \aleph^{\#}_{0} = card(\mathbb{N}^{\#})$ variables to be interpreted as ranging over a nonempty domain, one includes in the inductive definition of formulas an infinitary clause for conjunctions and disjunctions, namely, whenever the hypernturals indexed hypersequence $\{A_{\delta}\}_{\delta\in\mathbb{N}^{\#}}$ of formulas has length less than κ, one can form the hyperfinite conjunction/disjunction of them to produce a formula. Analogously, whenever an hypernaturals indexed sequence of variables has length less than λ, one can introduce one of the quantifiers $\forall$ or $\exists$ together with the sequence of variables in front of a formula to produce a new formula. One also stipulates that the length of any well-formed formula is less than $\aleph^{\#}_{0}$ itself.

The syntax of bivalent hyperinfinitary first-order logics $L^{\#}_{\infty^{\#}}$ consists of a (ordered) set of sorts and a set of function and relation symbols, these latter together with the corresponding type, which is a subset with less than $\aleph^{\#}_{0} = card(\mathbb{N}^{\#})$ many sorts. Therefore, we assume that our signature may contain relation and function symbols on $\gamma < \aleph^{\#}_{0}$ many variables, and we suppose there is a supply of $\kappa < \aleph^{\#}_{0}$ many fresh variables of each sort. Terms and atomic formulas are defined as usual, and general formulas are defined inductively according to the following rules:If $\phi,\psi,\{\phi_{\alpha} : \alpha < \gamma\}$ (for each $\gamma < \kappa$) are formulas of $\mathcal{L}_{\kappa,\kappa}$, the following are also formulas: $\bigwedge_{\alpha<\gamma}\phi_{\alpha}$, $\bigvee_{\alpha<\gamma}\phi_{\alpha}$, $\phi \to \psi$, $\forall_{\alpha<\gamma}x_{\alpha}\phi$ (also written $\forall\mathbf{x}_{\gamma}\phi$ if $\mathbf{x}_{\gamma} = \{x_{\alpha} : \alpha < \gamma\}$), $\exists_{\alpha<\gamma}x_{\alpha}\phi$ (also written $\exists\mathbf{x}_{\gamma}\phi$ if $\mathbf{x}_{\gamma} = \{x_{\alpha} : \alpha < \gamma\}$).

The axioms of hyperinfinitary first-order logic ${}^{2}L^{\#}_{\infty^{\#}}$ consist of the following schemata:

I. **Logical axiom**

1. $A \to [B \to A]$
2. $[A \to [B \to C]] \to [[A \to B] \to [A \to C]]]$
3. $[\neg B \to \neg A] \to [A \to B]$
4. $[\bigwedge_{i<\alpha}[A \to A_{i}]] \to [A \to \bigwedge_{i<\alpha}A_{i}], \alpha \in \mathbb{N}^{\#}$
5. $[\bigwedge_{i<\alpha}A_{i}] \to A_{j}, \alpha \in \mathbb{N}^{\#}$
6. $[\forall\mathbf{x}[A \to B] \to [A \to \forall\mathbf{x}B]]$

 provided no variable in $\mathbf{x}$ occurs free in A;

7. $\forall\mathbf{x}A \to S_{f}(A)$

where $S_{f}(A)$ is a substitution based on a function f from $\mathbf{x}$ to the terms of the language;

II.**Restricted rules of conclusion**.

$\mathbf{R}^{\#}$1.RMP (Restricted Modus Ponens).

From A and $A \to B$, conclude B iff $A \notin \Delta_{1}$and $(A \to B) \notin \Delta_{2}$,where $\Delta_{1},\Delta_{2} \subsetneqq \mathcal{F}_{\mathbf{wff}}$

We abbraviate by $A, A \to B \vdash_{RMP} B$.

$\mathbf{R}^{\#}$2.MT (Restricted Modus Tollens)

$P \to Q, \neg Q \vdash_{RMT} \neg P$ iff $P \notin \Delta_1'$ and $(P \to Q) \notin \Delta_2'$, where $\Delta_1', \Delta_2' \subsetneqq \mathcal{F}_{\mathbf{wff}}$.

III.**Equality axioms**:

(a) $t = t$

(b) $[\bigwedge_{i<\alpha} t_i = t_i'] \to [\phi(t_0,\ldots,t_\xi,\ldots) = \phi(t_0',\ldots,t_\xi',\ldots)]$

(c) $[\bigwedge_{i<\alpha} t_i = t_i'] \to [P(t_0,\ldots,t_\xi,\ldots) \to P(t_0',\ldots,t_\xi',\ldots)]$

for each $\alpha \in \mathbb{N}^{\#}$, where t, t_i are terms and ϕ is a function symbol of arity α and P a relation symbol of arity $\alpha \in \mathbb{N}^{\#}$.

IV.**Distributivity axiom**:

$$\bigwedge_{i<\gamma} \bigvee_{j<\gamma} \psi_{ij} \to \bigvee_{f\in\gamma^\gamma} \bigwedge_{i<\gamma} \psi_{if(i)} \tag{4.1}$$

V.**Dependent choice axiom**:

$$\bigwedge_{\alpha<\gamma} \forall_{\beta<\alpha}\mathbf{x}_\beta \exists \mathbf{x}_\alpha \psi_\alpha \to \exists_{\alpha<\gamma}\mathbf{x}_\alpha \bigwedge_{\alpha<\gamma} \psi_\alpha \tag{4.2}$$

provided the sets $\mathbf{x}_\alpha$ are pairwise disjoint and no variable in $\mathbf{x}_\alpha$ is free in ψ_β for $\beta < \alpha \in \mathbb{N}^{\#}$.

4.2.Why we need hyper infinitary logic.

Definition 4.**1**.A set $S \subset \mathbb{N}^{\#}$ is a hyper inductive if the following statement holds

$$\bigwedge_{\alpha\in\mathbb{N}^{\#}} (\alpha \in S \Rightarrow \alpha^+ \in S). \tag{4.3}$$

Obviously a set $\mathbb{N}^{\#}$ is a hyper inductive. As we see later there is just one hyper inductive

subset of $\mathbb{N}^{\#}$, namely $\mathbb{N}^{\#}$ itself.

In this paper we apply the following hyper inductive definitions of the sets

$$\exists\, S \forall \beta \left[\ \beta \in S \Leftrightarrow \bigwedge_{0\leq\alpha<\beta} (\alpha \in S \Rightarrow \alpha^+ \in S)\ \right], \tag{4.4}$$

see cection 7. Note that a statement

$$\bigwedge_{0\leq\alpha<\beta} (\alpha \in S \Rightarrow \alpha^+ \in S) \tag{4.5}$$

cannot be expressed in finitary set theoretical language. See also section 11, subsect 11.1.

5.Intuitionistic hyper infinitary logic $IL^{\#}_{\infty^{\#}}$ with restricted rules of conclusion.

We will denote the class of hypernaturals by $\mathbb{N}^{\#}$, the class of binary sequences of hypernatural length by $2^{<\mathbb{N}^{\#}}$, and the class of sets of hypernatural numbers by $\Sigma(\mathbb{N}^{\#})$.

We fix a class of variables x_i for each $i \in \mathbb{N}^{\#}$. Given an $\alpha \in \mathbb{N}^{\#}$, a context of length α is a sequence $\mathbf{x} = \langle x_{i_j} | j < \alpha \rangle$ of variables. In this paper we will use boldface letters, $\mathbf{x}, \mathbf{y}, \mathbf{z}, \ldots$, to denote contexts and light-face letters, $x_i, y_i, z_i, \ldots$, to denote the i-th variable symbol of $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$, respectively.

We will denote the length of a context $\mathbf{x}$ by $l(\mathbf{x})$. The formulas of the hyperinfinitary language $\mathcal{L}^{\#}_{\infty^{\#}}$ of set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ are defined to be the smallest class of formulas closed under the following rules:

1.$\perp$ is a formula,

2.$x_i \in x_j$ is a formula for any variables x_i and x_j,
3.$x_i = x_j$ is a formula for any variables x_i and x_j,
4. if ϕ and ψ are formulas, then $\phi \to \psi$ are formulas,
5. if ϕ_α is a formula for every $\alpha : \alpha \leq \beta \in \mathbb{N}^\#$, then

1.
$$\bigvee_{\alpha\leq\beta} \phi_\alpha \text{ is a gyperfinite formula,} \tag{5.1}$$

6.if ϕ_α is a formula for every $\alpha : \alpha \leq \beta \in \mathbb{N}^\#$, then

$$\bigwedge_{\alpha\leq\beta} \phi_\alpha \text{is a gyperfinite formula,} \tag{5.2}$$

7. if $\mathbf{x}$ is a context of length α, then $\exists^\alpha \mathbf{x}\phi$ is a formula, and,
8. if $\mathbf{x}$ is a context of length α, then $\forall^\alpha \mathbf{x}\phi$ is a formula.

By this definition, our language allows set-sized disjunctions and conjunctions as well as quantification over set-many variables at once. However, note that infinite alternating sequences of existential and universal quantifiers are excluded by this definition.

Remark 5.**1**.Whenever it is clear from the context, we will omit the superscripts from the quantifiers and write $\exists$ and $\forall$ instead of $\exists^\alpha$ and $\forall^\alpha$, respectively. In many situations it will be useful to identify a variable x with the context $\mathbf{x} = \langle x \rangle$ whose unique element is x such that we can write, for example, "$\exists x\phi$" for "$\exists \mathbf{x}\phi$" and "$\forall x\phi$" for "$\forall \mathbf{x}\phi$". A variable x_i is called a free variable of a formula ϕ whenever x_i appears in ϕ but not in any quantification of ϕ. As usual, a formula without free variables is called a sentence. We say that $\mathbf{x}$ is a context of the formula ϕ if all free variables of ϕ are among those in $\mathbf{x}$. As usual, we will write $\phi(\mathbf{x})$ in case that ϕ is a formula and $\mathbf{x}$ is a context of φ. Similarly, given two contexts $\mathbf{x}$ and $\mathbf{y}$ with $x_j \neq y_{j'}$ for all $j < \ell(\mathbf{x})$ and $j' < \ell(\mathbf{y})$, we will write $\varphi(\mathbf{x},\mathbf{y})$ in case that the sequence obtained by concatenating $\mathbf{x}$ and $\mathbf{y}$ is a context for φ.

Remark 5.**2**. We extend the classical abbreviations as follows: Given a formula ϕ and an hypernatural $\alpha \in \mathbb{N}^\#$ we introduce the bounded quantifiers as abbreviations, namely,

$$\forall^\alpha \mathbf{x} \in y\, \phi \text{ for } \forall^\alpha \mathbf{x}(\mathbf{x} \in y \to \phi), \tag{5.3}$$

and

$$\exists^\alpha \mathbf{x} \in y\, \phi \text{ for } \exists^\alpha \mathbf{x}(\mathbf{x} \in y \wedge \phi). \tag{5.4}$$

Notation 5.**1**. A sequent $\phi \vdash_{\mathbf{x},\boldsymbol{\alpha}} \psi$ is however equivalent to the formula $\forall^\alpha \mathbf{x}(\phi \to \psi)$.
The system of axioms and rules for hyperinfinitary intuitionistic first-order logic consists of the following schemata:

I. **Logical axiom**

1. $A \to [B \to A]$
2. $[A \to [B \to C]] \to [[A \to B] \to [A \to C]]]$
3. $[\bigwedge_{i<\alpha}[A \to A_i]] \ \to [A \to \bigwedge_{i<\alpha} A_i], \alpha \in \mathbb{N}^\#$
4. $[\bigwedge_{i<\alpha} A_i] \to A_j, \alpha \in \mathbb{N}^\#$
5. $[\forall \mathbf{x}[A \to B] \to [A \to \forall \mathbf{x} B]]$
   provided no variable in $\mathbf{x}$ occurs free in A.
7. $\forall \mathbf{x} A \to S_f(A)$

where $S_f(A)$ is a substitution based on a function f from $\mathbf{x}$ to the terms of the language;

II.**Restricted rules of conclusion**.

$\mathbf{R}^\#$1.RMP (Restricted Modus Ponens).
From A and $A \to B$, conclude B iff $A \notin \Delta_1$and $(A \to B) \notin \Delta_2$,where $\Delta_1, \Delta_2 \subsetneqq \mathcal{F}_{\mathbf{wff}}$
We abbraviate by $A, A \to B \vdash_{RMP} B$.

$\mathbf{R}^{\#}$2.MT (Restricted Modus Tollens)
$P \to Q, \neg Q \vdash_{RMT} \neg P$ iff $P \notin \Delta_1'$and $(P \to Q) \notin \Delta_2'$,where $\Delta_1', \Delta_2' \subsetneqq \mathcal{F}_{\mathbf{wff}}$.
III.**Weak distributivity axiom**:

$$\phi \wedge \bigvee_{i<\gamma} \psi_i \vdash_{\mathbf{x}} \bigvee_{i<\gamma} \phi \wedge \psi_i \tag{5.5}$$

for each $\gamma \in \mathbb{N}^{\#}$.
IV.**Frobenius axiom**:

$$\phi \wedge \exists \mathbf{y} \psi \vdash_{\mathbf{x}} \exists \mathbf{y}(\phi \wedge \psi) \tag{5.6}$$

where no variable in $\mathbf{y}$ is in the context $\mathbf{x}$.
V.**Structural rules**:

(a) Identity axiom:

$$\varphi \vdash_{\mathbf{x},\boldsymbol{\alpha}} \varphi \tag{5.7}$$

(b) Substitution rule:

$$\frac{\varphi \vdash_{\mathbf{x},\boldsymbol{\alpha}} \psi}{\varphi[\mathbf{s}/\mathbf{x}] \vdash_{\mathbf{y}} \psi[\mathbf{s}/\mathbf{x}]} \tag{5.8}$$

where $\mathbf{y}$ is a string of variables including all variables occurring in the string of terms $\mathbf{s}$.
(c) Restricted cut rule:

$$\frac{\varphi \vdash_{\mathbf{x},\boldsymbol{\alpha}} \psi, \psi \vdash_{\mathbf{x},\alpha} \theta}{\varphi \vdash_{\mathbf{x},\alpha} \theta} \tag{5.9}$$

iff $\varphi \notin \Delta_1$and $(\psi \vdash_{\mathbf{x},\alpha} \theta) \notin \Delta_2$.
IV.**Equality axioms**:

(a)

$$\top \vdash_{\mathbf{x}} \mathbf{x} = \mathbf{x} \tag{5.10}$$

(b)

$$(\mathbf{x} = \mathbf{y}) \wedge \varphi[\mathbf{x}/\mathbf{w}] \vdash_{\mathbf{z}} \varphi[\mathbf{y}/\mathbf{w}] \tag{5.11}$$

where $\mathbf{x}, \mathbf{y}$ are contexts of the same length and type and $\mathbf{z}$ is any context containing $\mathbf{x}, \mathbf{y}$ and the free variables of φ.
V.**Conjunction axioms and rules**:
(a)

$$\bigwedge_{i<\gamma} \varphi_i \vdash_{\mathbf{x},\alpha} \varphi_j \tag{5.12}$$

for each $\gamma \in \mathbb{N}^{\#}$ and $j < \gamma$

(b)

$$\frac{\{\phi \vdash_{\mathbf{x},\boldsymbol{\alpha}} \psi_i\}_{i<\gamma}}{\phi \vdash_{\mathbf{x},\boldsymbol{\alpha}} \bigwedge_{i<\gamma} \psi_i} \tag{5.13}$$

for each $\gamma \in \mathbb{N}^{\#}$.
VI.**Disjunction axioms and rules**:

(a)

$$\phi_j \vdash_{\mathbf{x},\boldsymbol{\alpha}} \bigvee_{i<\gamma} \phi_i \tag{5.14}$$

for each $\gamma \in \mathbb{N}^{\#}$

(b)

$$\frac{\{\phi_i \vdash_{\mathbf{x},\boldsymbol{\alpha}} \theta\}_{i<\gamma}}{\bigvee_{i<\gamma} \phi_i \vdash_{\mathbf{x},\boldsymbol{\alpha}} \theta} \tag{5.15}$$

for each $\gamma \in \mathbb{N}^{\#}$.

VII. **Implication rule**:

$$\frac{\phi \wedge \psi \vdash_{\mathbf{x},\boldsymbol{\alpha}} \theta}{\phi \vdash_{\mathbf{x},\boldsymbol{\alpha}} \psi \Rightarrow \theta} \tag{5.16}$$

IX.**Existential rule**:

$$\frac{\phi \vdash_{\mathbf{xy}} \psi}{\exists \mathbf{y}(\phi \vdash_{\mathbf{x}} \psi)} \tag{5.17}$$

where no variable in $\mathbf{y}$ is free in ψ.

X.**Universal rule**:

$$\frac{\varphi \vdash_{\mathbf{xy}} \psi}{\phi \vdash_{\mathbf{x}} \forall \mathbf{y}\psi} \tag{5.18}$$

where no variable in $\mathbf{y}$ is free in φ.

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6.Set theory in hyper infinitary set theoretical languages.

6.1.Intuitionistic set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ in hyper infinitary set theoretical language.

Axioms and basic definitions.

Intuitionistic set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ is formulated as a system of axioms in the same first order language as its classical counterpart, only based on intuitionistic logic $IL^{\#}_{\infty^{\#}}$ with restricted modus ponens rule [1]. The language of set theory is a first-order language $L^{\#}_{\infty^{\#}}$ with equality $=$, which includes a binary symbol $\in$. We write $x \neq y$ for $\neg(x = y)$ and $x \notin y$ for $\neg(x \in y)$. Individual variables $x, y, z, \ldots$ of $L^{\#}_{\infty^{\#}}$ will be understood as ranging over classical sets. The unique existential quantifier $\exists!$ is introduced by writing, for any formula $\varphi(x)$, $\exists! x\varphi(x)$ as an abbreviation of the formula $\exists x[\varphi(x) \;\&\; \forall y(\varphi(y) \Rightarrow x = y)]$. $L^{\#}_{\infty^{\#}}$ will also allow the formation of terms of the form $\{x|\varphi(x)\}$, for any formula φ containing the free variable x. Such terms are called nonclassical sets; we shall use upper case letters $A, B, \ldots$ for such sets. For each nonclassical set $A = \{x|\varphi(x)\}$ and $A = \{x|\varphi(x,A)\}$ the formulas

$$\forall x[x \in A \Leftrightarrow \varphi(x)] \tag{6.1}$$

and more general formulas

$$\forall x[x \in A \Leftrightarrow \varphi(x,A)] \quad (6.2)$$

is called the defining axioms for the nonclassical set A.

Remark 6.1. Remind that in intuitionistic logic $IL^{\#}_{\infty^{\#}}$ with restricted modus ponens rule the statement $\alpha \wedge (\alpha \Rightarrow \beta)$ does not always guarantee that

$$\alpha, \alpha \Rightarrow \beta \vdash_{RMP} \beta \quad (6.3)$$

since for some α and β possible

$$\alpha, \alpha \Rightarrow \beta \nvdash_{RMP} \beta \quad (6.4)$$

even if the statement $\alpha \wedge (\alpha \Rightarrow \beta)$ holds [1].

Abbreviation 6.1. We often write for the sake of brevity instead (6.3) by

$$\alpha \Rightarrow_s \beta \quad (6.5)$$

and we often write instead (6.4) by

$$\alpha \Rightarrow_w \beta. \quad (6.6)$$

Remark 6.2. Let A be an nonclassical set. Note that in set theory $\text{INC}^{\#}_{\infty^{\#}}$ the following true formula

$$\exists A \forall x[x \in A \Leftrightarrow \varphi(x,A)] \quad (6.7)$$

does not always guarantee that

$$x \in A, x \in A \Rightarrow \varphi(x,A) \vdash_{RMP} \varphi(x,A) \quad (6.8)$$

even if $x \in A$ holds and (or)

$$\varphi(x,A), \varphi(x,A) \Rightarrow x \in A \vdash_{RMP} x \in A; \quad (6.9)$$

even $\varphi(x,A)$ holds, since for nonclassical set A for some y possible

$$y \in A,\ y \in A \Rightarrow \varphi(y,A) \nvdash_{RMP} \varphi(y,A) \quad (6.10)$$

and (or)

$$\varphi(y,A),\ \varphi(y,A) \Rightarrow y \in A \nvdash_{RMP} y \in A. \quad (6.11)$$

Remark 6.3. Note that in this paper the formulas

$$\exists a \forall x[x \in a \Leftrightarrow \varphi(x) \wedge x \in u] \quad (6.12)$$

and more general formulas

$$\exists a \forall x[x \in a \Leftrightarrow \varphi(x,a)] \quad (6.13)$$

is considered as the defining axioms for the classical set a.

Remark 6.4. Let a be an classical set. Note that in $\text{INC}^{\#}_{\infty^{\#}}$: (i) the following true formula

$$\exists a \forall x[x \in a \Leftrightarrow \varphi(x,a) \wedge x \in u] \quad (6.14)$$

always guarantee that

$$x \in a, x \in a \Rightarrow \varphi(x,a) \vdash_{RMP} \varphi(x) \quad (6.15)$$

if $x \in a$ holds and

$$\varphi(x), \varphi(x) \Rightarrow x \in a \vdash_{RMP} x \in a; \quad (6.16)$$

if $\varphi(x)$ holds;

Remark 6.4. In order to emphasize this fact mentioned above in Remark 6.1-6.3, we rewrite the defining axioms in general case for the nonclassical sets in the following form

$$\exists A \forall x\{[x \in A \Leftrightarrow_s \varphi(x,A)] \vee [x \in A \Leftrightarrow_w \varphi(x,A)]\} \quad (6.17)$$

and similarly we rewrite the defining axioms in general case for the classical sets in the
following form

$$\forall x[x \in a \Leftrightarrow_s \varphi(x,a) \wedge (x \in u)]. \quad (6.18)$$

Abbreviation 6.2.We write instead (6.17) by

$$\forall x\{[x \in A \Leftrightarrow_{s,w} \varphi(x,A)]\} \quad (6.19)$$

Definition 6.1. (1) Let A be a nonclassical set defined by formula (6.1) or by formula (6.2).Assum that: (i) for some y statement $\varphi(y)$ and statement $\varphi(y) \Rightarrow y \in A$ holds and (ii) $\varphi(y), \varphi(y) \Rightarrow y \in A \nvdash_{RMP} y \in A$, $y \in A, y \in A \Rightarrow \varphi(y) \nvdash_{RMP} \varphi(y)$.
Then we say that y is a <u>weak member</u> of non-classical set A and abbreviate $y \in_w A$.

Abbreviation 6.3. Let A be a nonclassical set defined by formula (6.1) or by formula (6.2). We abbreviate $x \in_{s,w} A$ if the following statement $x \in_s A \vee x \in_w A$ holds, i.e.

$$x \in A \leftrightarrow_{def} (x \in_s A \vee x \in_w A). \quad (6.20)$$

Definition 6.2.(1) Two nonclassical sets A, B are defined to be equal and we write $A = B$ if $\forall x[x \in_{s,w} A \Leftrightarrow_s x \in_{s,w} B]$. (2) A is a subset of B, and we often write $A \subset_{s,v} B$, if $\forall x[x \in_{s,w} A \Rightarrow_s x \in_{s,w} B]$.(3) We also write $\mathbf{Cl.Set}(A)$ for the formula $\exists u \forall x[x \in A \Leftrightarrow x \in u]$. (4) We also write $\mathbf{NCl.Set}(A)$ for the formulas $\forall x[x \in_{s,v} A \Leftrightarrow_{s,v} \varphi(x)]$ and $\forall x[x \in_{s,v} A \Leftrightarrow_{s,v} \varphi(x,A)]$.

Remark 6.5.$\mathbf{Cl.Set}(A)$) asserts that the set A is a classical set. For any classical set u, it follows from the defining axiom for the classical set $\{x|x \in u \wedge \varphi(x)\}$ that $\mathbf{Cl.Set}(\{x|x \in u \wedge \varphi(x)\})$.
We shall identify $\{x|x \in u\}$ with u, so that sets may be considered as (special sorts of) nonclassical sets and we may introduce assertions such as $u \subset A, u \subsetneqq A, u = A$, etc.

Remark 6.6.If A is a nonclassical set, we write $\exists x \in A\ \varphi(x,A)$ for $\exists x[x \in A \wedge \varphi(x,A)]$ and $\forall x \in A \varphi(x,A)$ for $\forall x[x \in A \Rightarrow \varphi(x,A)]$.
We define now the following sets:
1.$\{u_1, u_2, \ldots, u_n\} = \{x|x = u_1 \vee x = u_2 \vee \ldots \vee x = u_n\}$. 2. $\{A_1, A_2, \ldots, A_n\} =$
$= \{x|x = A_1 \vee x = A_2 \vee \ldots \vee x = A_n\}$. 3.$\cup A = \{x|\exists y[y \in A \wedge x \in y]\}$.
4.$\cap A = \{x|\forall y[y \in A \Rightarrow x \in y]\}$. 5.$A \cup B = \{x|x \in A \vee x \in B\}$.
5.$A \cap B = \{x|x \in A \wedge x \in B\}$. 6.$A - B = \{x|x \in A \wedge x \notin B\}$. 7.$u^+ = u \cup \{u\}$.
8.$\mathbf{P}(A) = \{x|x \subseteq A\}$. 9.$\{x \in A|\varphi(x,A)\} = \{x|x \in A \wedge \varphi(x,A)\}$. 10.$\mathbf{V} = \{x|: x = x\}$.
11.$\varnothing = \{x|x \neq x\}$.
The system $\mathbf{INC}^{\#}_{\infty^{\#}}$ of set theory is based on the following axioms:
Extensionality1: $\forall u \forall v[\forall x(x \in u \Leftrightarrow x \in v) \Rightarrow u = v]$
Extensionality2: $\forall A \forall B[\forall x(x \in A \Leftrightarrow_{s,w} x \in B) \Rightarrow A = B]$
Universal Set: $\mathbf{NCl.Set}(\mathbf{V})$
Empty Set: $\mathbf{Cl.Set}(\varnothing)$
Pairing1: $\forall u \forall v\ \mathbf{Cl.Set}(\{u,v\})$
Pairing2: $\forall A \forall B\ \mathbf{NCl.Set}(\{A,B\})$
Union1: $\forall u\ \mathbf{Cl.Set}(\cup u)$
Union2: $\forall A\ \mathbf{NCl.Set}(\cup A)$
Powerset1: $\forall u\ \mathbf{Cl.Set}(\mathbf{P}(u))$

Powerset2: $\forall A\ \mathbf{NCl.Set}(\mathbf{P}(A))$
Infinity $\exists a[\varnothing \in a \land \forall x \in a(x^{+} \in a)]$
Separation1$\forall u_1 \forall u_2, \ldots \forall u_n \forall a \exists \mathbf{Cl.Set}(\{x \in_s a | \varphi(x, u_1, u_2, \ldots, u_n)\})$
Separation2$\forall u_1 \forall u_2, \ldots \forall u_n \mathbf{NCl.Set}(\{x \in_{s,w} A | \varphi(x, A; u_1, u_2, \ldots, u_n)\})$
Comprehension1$\forall u_1 \forall u_2, \ldots \forall u_n \exists A \forall x[x \in A \Leftrightarrow_{s,w} \varphi(x; u_1, u_2, \ldots, u_n)]$
Comprehension 2 $\forall u_1 \forall u_2, \ldots \forall u_n \exists A \forall x[x \in A \Leftrightarrow_{s,w} \varphi(x, A; u_1, u_2, \ldots, u_n)]$
Comprehension 3 $\forall u_1 \forall u_2, \ldots \forall u_n \exists a \forall x[x \in a \Leftrightarrow_{s} \varphi(x, a; u_1, u_2, \ldots, u_n)]$
Hyperinfinity: see subsection 2.1.
Remark 6.7.Note that the axiom of hyper infinity follows from the schemata Comprehension 3.
Definition 6.3. The ordered pair of two sets u, v is defined as usual by

$$\langle u, v \rangle = \{\{u\}, \{u, v\}\}. \tag{6.21}$$

Definition 6.4. We define the Cartesian product of two nonclassical sets A and B as usual by

$$A \times_{s,w} B = \{\langle x, y \rangle | x \in_{s,w} A \land y \in_{s,w} B\} \tag{6.22}$$

Definition 6.5. A binary relation between two nonclassical sets A, B is a subset $R \subseteq_{s,w} A \times_{s,w} B$. We also write $aR_{s,w}b$ for $< a, b >\in_{s,w} R$. The doman $\mathbf{dom}(R)$ and the range $\mathbf{ran}(R)$ of R are defined by

$$\mathbf{dom}(R) = \{x | \exists y(xR_{s,w}y)\}, \mathbf{ran}(R) = \{y : \exists x(xR_{s,w}y)\}. \tag{6.23}$$

Definition 6.6.A relation $F_{s,w}$ is a function, or map, written $\mathbf{Fun}(F_{s,w})$, if for each $a \in_{s,w} \mathbf{dom}(F)$ there is a unique b for which $aF_{s,w}b$. This unique b is written $F(a)$ or Fa. We write $F_{s,w} : A \to B$ for the assertion that $F_{s,w}$ s a function with $\mathbf{dom}(F_{s,w}) = A$ and $\mathbf{ran}(F_{s,w}) = B$. In this case we write $a \mapsto F_{s,w}(a)$ for $F_{s,w}a$.
Definition 6.7.The identity map $\mathbf{1}_A$ on A is the map $A \to A$ given by $a \mapsto a$. If $X \subseteq_{s,w} A$, the
map $x \mapsto x : X \to A$ is called the insertion map of X into A.
Definition 6.8.If $F_{s,w} : A \to B$ and $X \subseteq_{s,w} A$, the restriction $F_{s,w}|X$ of $F_{s,w}$ to X is the map $X \to A$ given by $x \mapsto F_{s,w}(x)$. If $Y \subseteq_{s,w} B$, the inverse image of Y under $F_{s,w}$ is the set

$$F^{-1}_{s,w}[Y] = \{x \in_{s,w} A : F_{s,w}(x) \in_{s,w} Y\}. \tag{6.24}$$

Given two functions $F_{s,w} : A \to B, G_{s,w} : B \to C$, we define the composite function $G_{s,w} \circ F_{s,w} : A \to C$ to be the function $a \mapsto G_{s,w}(F_{s,w}(a))$. If $F_{s,w} : A \to A$, we write $F^2_{s,w}$ for $F_{s,w} \circ F_{s,w}, F^3_{s,w}$ for $F_{s,w} \circ F_{s,w} \circ F_{s,w}$ etc.
Definition 6.9.A function $F_{s,w} : A \to B$ is said to be monic if for all $x, y \in_{s,w} A, F_{s,w}(x) = F_{s,w}(y)$ implies $x = y$, epi if for any $b \in_{s,w} B$ there is $a \in_{s,w} A$ for which $b = F_{s,w}(a)$, and bijective, or a bijection, if it is both monic and epi. It is easily shown that
$F_{s,w}$ is bijective if and only if $F_{s,w}$ has an inverse, that is, a map $G_{s,w} : B \to A$ such that $F_{s,w} \circ G_{s,w} = \mathbf{1}_B$ and $G_{s,w} \circ F_{s,w} = \mathbf{1}_A$.
Definition 6.10.Two sets X and Y are said to be equipollent, and we write $X \approx_{s,w} Y$, if there is a bijection between them.
Definition 6.11.Suppose we are given two sets I, A and an epi map $F_{s,w} : I \to A$. Then $A = \{F_{s,w}(i) | i \in I\}$ and so, if, for each $i \in_{s,w} I$, we write a_i for $F_{s,w}(i)$, then A can be presented in the form of an indexed set $\{a_i : i \in_{s,w} I\}$. If A is presented as an indexed

set of sets $\{X_i|i \in_{s,w} I\}$, then we write $\bigcup_{i\in I} X_i$ and $\bigcap_{i\in I} X_i$ for $\cup A$ and $\cap A$, respectively.

Definition 6.12.The projection maps $\pi_1 : A \times_{s,w} B \to A$ and $\pi_2 : A \times_{s,w} B \to B$ are defined to be the maps <a, b>↦ a and <a, b> ↦ b respectively.

Definition 6.13.For sets A,B, the exponential B^A is defined to be the set of all functions
from A to B.

6.2.Set theory $\mathrm{NC}^{\#}_{\infty^{\#}}$ in bivalent hyper infinitary set theoretical language.

Set theory $\mathrm{NC}^{\#}_{\infty^{\#}}$ is formulated as a system of axioms in the same first order language as its classical counterpart, only based on bivalent hyper infinitary logic ${}^2L^{\#}_{\infty^{\#}}$ with restricted modus ponens rule [1]. The language of set theory is a first-order hyper infinitary language $L^{\#}_{\infty^{\#}}$ with equality $=$, which includes a binary symbol $\in$. We write $x \neq y$ for $\neg\ (x = y)$ and $x \notin y$ for $\neg(x \in y)$. Individual variables $x,y,z,\ldots$ of $L^{\#}_{\infty^{\#}}$ will be understood as ranging over classical sets. The unique existential quantifier $\exists!$ is introduced by writing, for any formula $\varphi(x), \exists!x\varphi(x)$ as an abbreviation of the formula $\exists x[\varphi(x)\ \&\ \forall y(\varphi(y) \Rightarrow x = y)]$. $L^{\#}_{\infty^{\#}}$ will also allow the formation of terms of the form $\{x|\varphi(x)\}$, for any formula φ containing the free variable x. Such terms are called non-classical sets; we shall use upper case letters $A,B,\ldots$ for such sets. For each non-classical set $A = \{x|\varphi(x)\}$ the formulas $\forall x[x \in A \Leftrightarrow \varphi(x)]$ and $\forall x[x \in A \Leftrightarrow \varphi(x,A)]$ is called the defining axioms for the non-classical set A.

Remark 6.8.Remind that in intuitionistic logic $IL^{\#}_{\infty^{\#}}$ with restricted modus ponens rule the statement $\alpha \wedge (\alpha \Rightarrow \beta)$ does not always guarantee that

$$\alpha, \alpha \Rightarrow \beta \vdash_{RMP} \beta \qquad (6.25)$$

since for some α and β possible

$$\alpha, \alpha \Rightarrow \beta \nvdash_{RMP} \beta \qquad (6.26)$$

even if the statement $\alpha \wedge (\alpha \Rightarrow \beta)$ holds [1].

The system $\mathbf{NC}^{\#}_{\infty^{\#}}$ of set theory is based on the following axioms:

Extensionality1: $\forall u\forall v[\forall x(x \in u \Leftrightarrow x \in v) \Rightarrow u = v]$

Extensionality2: $\forall A\forall B[\forall x(x \in A \Leftrightarrow_{s,w} x \in B) \Rightarrow A = B]$

Universal Set: $\mathbf{NCl.Set}(\mathbf{V})$

Empty Set: $\mathbf{Cl.Set}(\varnothing)$

Pairing1: $\forall u\forall v\ \mathbf{Cl.Set}(\{u,v\})$

Pairing2: $\forall A\forall B\ \mathbf{NCl.Set}(\{A,B\})$

Union1: $\forall u\ \mathbf{Cl.Set}(\cup u)$

Union2: $\forall A\ \mathbf{NCl.Set}(\cup A)$

Powerset1: $\forall u\ \mathbf{Cl.Set}(\mathbf{P}(u))$

Powerset2: $\forall A\ \mathbf{NCl.Set}(\mathbf{P}(A))$

Infinity $\exists a[\varnothing \in a \wedge \forall x \in a(x^+ \in a)]$

Separation1$\forall u_1\forall u_2,\ldots\forall u_n\forall a\exists\mathbf{Cl.Set}(\{x \in_s a|\varphi(x,u_1,u_2,\ldots,u_n)\})$

Separation2$\forall u_1\forall u_2,\ldots\forall u_n\mathbf{NCl.Set}(\{x \in_{s,w} A|\varphi(x,A;u_1,u_2,\ldots,u_n)\})$

Comprehension1$\forall u_1\forall u_2,\ldots\forall u_n\exists A\forall x[x \in A \Leftrightarrow_{s,w} \varphi(x;u_1,u_2,\ldots,u_n)]$

Comprehension 2 $\forall u_1\forall u_2,\ldots\forall u_n\exists A\forall x[x \in A \Leftrightarrow_{s,w} \varphi(x,A;u_1,u_2,\ldots,u_n)]$

Comprehension 3 $\forall u_1\forall u_2,\ldots\forall u_n\exists a\forall x[x \in a \Leftrightarrow_s \varphi(x,a;u_1,u_2,\ldots,u_n)]$

Hyperinfinity: see subsection 2.1.
Remark 6.7.Note that the axiom of hyper infinity follows from the schemata Comprehension 3.

7.External induction principle and hyper inductive definitions.

7.1.External induction principle in nonstandard intuitionistic arithmetic.

Axiom of infite ω-induction
(i)

$$\forall S(S \subset \mathbb{N})\left\{\left[\bigwedge_{n\in\omega}(n \in S \Rightarrow n^{+} \in S)\right] \Rightarrow S = \mathbb{N}\right\}. \tag{7.1}$$

(ii) Let $F(x)$ be a wff of the set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$, then

$$\left[\bigwedge_{n\in\omega}(F(n) \Rightarrow F(n^{+}))\right] \Rightarrow \forall n(n \in \omega)F(n). \tag{7.2}$$

Definition 7.1.Let β be a hypernatural such that $\beta \in \mathbb{N}^{\#}\backslash\mathbb{N}$. Let $[0,\beta] \subset \mathbb{N}^{\#}$be a set such that $\forall x[x \in [0,\beta] \Leftrightarrow 0 \leq x \leq \beta]$ and $[0,\beta) = [0,\beta]\backslash\{\beta\}$.
Definition 7.2.(i) Let $F(x)$ be a wff of $\mathrm{INC}^{\#}$with unique free variable x. We will say that a wff $F(x)$ is restricted on a set S_F such that $S_F \subsetneqq \mathbb{N}^{\#}$ iff the following conditions are satisfied

$$\forall\alpha(\alpha \in \mathbb{N}^{\#})[F(\alpha) \Rightarrow \alpha \in S_F] \tag{7.3}$$

and

$$\forall\alpha(\alpha \in \mathbb{N}^{\#})[\neg F(\alpha) \Rightarrow \alpha \in \mathbb{N}^{\#}\backslash S_F]. \tag{7.4}$$

Definition 7.3. Let $F(x)$ be a wff of $\mathrm{INC}^{\#}_{\infty^{\#}}$with unique free variable x. We will say that a wff $F(x)$ is unrestricted on variable x if wff $F(x)$ is not restricted on any set S such that $S \subsetneqq \mathbb{N}^{\#}$. This definition meant

$$\bigwedge_{\alpha\in\mathbb{N}^{\#}}(F(\alpha) \nvdash \alpha \notin \mathbb{N}^{\#}). \tag{7.5}$$

Axiom of hyperfinite induction 1

$$\forall\beta(\beta \in \mathbb{N}^{\#}\backslash\mathbb{N})\forall S(S \subseteq [0,\beta]) \searrow$$
$$\left\{\forall\alpha(\alpha \in [0,\beta])\left[\bigwedge_{0\leq\alpha<\beta}(\alpha \in S \Rightarrow \alpha^{+} \in S)\right] \Rightarrow S = [0,\beta]\right\}. \tag{7.7}$$

Axiom of hyper infinite induction 1

$$\forall S(S \subset \mathbb{N}^{\#})\left\{\forall\beta(\beta \in \mathbb{N}^{\#})\left[\bigwedge_{0\leq\alpha<\beta}(\alpha \in S \Rightarrow \alpha^{+} \in S)\right] \Rightarrow S = \mathbb{N}^{\#}\right\}. \tag{7.8}$$

Remark 7.1.Note that from comprechesion shemata 2 (see subsection 6.1) follows that

$$\forall\beta\exists S(S \subset [0,\beta])\forall\bar{\beta}(\bar{\beta} \in [0,\beta])\left[\bar{\beta} \in S \Leftrightarrow \bigwedge_{0\leq\alpha<\bar{\beta}}(\alpha \in S \Rightarrow \alpha^{+} \in S)\right]. \tag{7.9}$$

Therefore for any $\bar{\beta} \in [0,\beta]$ from (7.9) it follows that

$$\bigwedge_{0\leq\alpha<\overline{\beta}}(\alpha\in S\Rightarrow\alpha^{+}\in S)\vdash\overline{\beta}\in S. \tag{7.10}$$

Thus axiom of hyperfinite induction 1,i.e., (7.6) holds, since from (7.10) it follows that $\forall\overline{\beta}[\overline{\beta}\in[0,\beta]\Rightarrow\overline{\beta}\in S]$.

Remark 7.2.Note that from comprechesion shemata 2 (see subsection 6.1) it follows that

$$\exists S(S\subset\mathbb{N}^{\#})\forall\beta(\beta\in\mathbb{N}^{\#})\left[\beta\in S\Leftrightarrow\bigwedge_{0\leq\alpha<\beta}(\alpha\in S\Rightarrow\alpha^{+}\in S)\right]. \tag{7.11}$$

Therefore for any $\beta\in\mathbb{N}^{\#}$ from (7.11) it follows that

$$\bigwedge_{0\leq\alpha<\beta}(\alpha\in S\Rightarrow\alpha^{+}\in S)\vdash\beta\in S \tag{7.12}$$

Thus axiom of hyper infinite induction 1, i.e., (7.8) holds, since it follows from (7.12) that $\forall\beta[\beta\in\mathbb{N}^{\#}\Rightarrow\beta\in S]$.

Axiom of hyperfinite induction 2

Let $F(x)$ be a wff of the set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ restricted on a set $[0,\beta]$ then

$$\left[\forall\beta(\beta\in\mathbb{N}^{\#}\backslash\mathbb{N})\left[\bigwedge_{0\leq\alpha<\beta}(F(\alpha)\Rightarrow F(\alpha^{+}))\right]\right]\Rightarrow\forall\alpha(\alpha\in[0,\beta])F(\alpha). \tag{7.11}$$

Axiom of hyper infinite induction 2

Let $F(x)$ be unrestricted wff of the set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ then

$$\left[\forall\beta(\beta\in\mathbb{N}^{\#})\left[\bigwedge_{0\leq\alpha<\beta}(F(\alpha)\Rightarrow F(\alpha^{+}))\right]\right]\Rightarrow\forall\beta(\beta\in\mathbb{N}^{\#})F(\beta). \tag{7.12}$$

Remark 7.3.Note that from comprechesion shemata 2 (see subsection 6.1) follows that

$$\forall\beta\exists S(S\subset[0,\beta])\forall\overline{\beta}(\overline{\beta}\in[0,\beta])\left[\overline{\beta}\in S\Leftrightarrow\bigwedge_{0\leq\alpha<\overline{\beta}}(F(\alpha)\Rightarrow F(\alpha^{+}))\right]. \tag{7.13}$$

Therefore for any $\overline{\beta}\in[0,\beta]$ from (7.13) it follows that

$$\bigwedge_{0\leq\alpha<\overline{\beta}}(F(\alpha)\Rightarrow F(\alpha^{+}))\vdash\overline{\beta}\in S \tag{7.14}$$

Thus axiom of hyperfinite induction 2,i.e., (7.13) holds, since it follows from (7.16) that $\forall\overline{\beta}[\overline{\beta}\in[0,\beta]\Rightarrow\overline{\beta}\in S]$.

Remark 7.4.Note that from comprechesion shemata 2 (see subsection 6.1) follows that

$$\exists S(S\subset\mathbb{N}^{\#})\forall\overline{\beta}(\overline{\beta}\in\mathbb{N}^{\#})\left[\overline{\beta}\in S\Leftrightarrow\bigwedge_{0\leq\alpha<\overline{\beta}}(F(\alpha)\Rightarrow F(\alpha^{+}))\right]. \tag{7.15}$$

Therefore for any $\overline{\beta}\in\mathbb{N}^{\#}$ from (7.15) it follows that

$$\bigwedge_{0\leq\alpha<\overline{\beta}}(F(\alpha)\Rightarrow F(\alpha^{+}))\vdash\overline{\beta}\in S. \tag{7.16}$$

Thus axiom of hyper infinite induction 2,i.e., (7.12) holds, since From (7.16) it follows that $\forall\overline{\beta}\left[\overline{\beta}\in\mathbb{N}^{\#}\Rightarrow\overline{\beta}\in S\right]$.

Axiom of hyperfinite induction 3

Let $F(x)$ be a wff of the set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ restricted on inductive set $W_{\mathbf{ind}}$ such that

$\mathbb{N} \subseteq W_{\mathbf{ind}} \subsetneqq \mathbb{N}^{\#}$ then

$$\forall W\left[(\mathbb{N} \subseteq W_{\mathbf{ind}} \subsetneqq \mathbb{N}^{\#}) \wedge \left[\bigwedge_{\alpha \in W_{\mathbf{ind}}} (F(\alpha) \Rightarrow F(\alpha^{+})) \right] \right] \Rightarrow \forall \alpha (\alpha \in W_{\mathbf{ind}}) F(\alpha). \tag{7.17}$$

Proposition 7.1. (a) For any natural or hypernatural number $k \in \mathbb{N}^{\#}$,

$$\vdash \bigvee_{0 \leq m \leq k} (x = m) \Leftrightarrow x \leq k. \tag{7.18}$$

(a′) For any hypernatural number κ and any wff B

$$\vdash \bigwedge_{0 \leq m \leq k} B(m) \Leftrightarrow \forall x (x \leq k \Rightarrow B(x)). \tag{7.19}$$

(b) For any hypernatural number $k \in \mathbb{N}^{\#}$ such that $k > 0$,

$$\vdash \bigvee_{1 \leq m \leq k} (x = m - 1) \Leftrightarrow x < k. \tag{7.20}$$

(b′) For any hypernatural number $k \in \mathbb{N}^{\#}$ such that $k > 0$ and any wff $B(x)$,

$$\vdash \bigwedge_{0 \leq m \leq k-1} B(m) \Leftrightarrow \forall x (x < k \Rightarrow B(x)). \tag{7.21}$$

(c) $\vdash (\forall x (x < y \Rightarrow B(x))) \wedge (\forall x (x \geq y \Rightarrow E(x))) \Rightarrow \forall x (B(x) \vee E(x))$.

Proof. (a) We prove $\bigvee_{0 \leq m \leq k} (x = m) \Leftrightarrow x \leq k$ by hyperfinite induction in the metalanguage on k. The case for $k = 0, \vdash x = 0 \Leftrightarrow x \leq 0$, is obvious from the definitions.

Assume as inductive hypothesis that

$$\bigvee_{0 \leq m \leq k} (x = m) \Leftrightarrow x \leq k. \tag{7.22}$$

Now assume that

$$\left[\bigvee_{0 \leq m \leq k} (x = m) \right] \vee (x = k + 1). \tag{7.25}$$

But $\vdash x = k + 1 \Rightarrow x \leq k + 1$ and, by the inductive hypothesis,

$$\bigvee_{0 \leq m \leq k} (x = m). \tag{7.26}$$

Also $\vdash x \leq k \Rightarrow x < k + 1$. Thus, $x \leq k + 1$. So,

$$\vdash \bigvee_{0 \leq m \leq k+1} (x = m) \Rightarrow x \leq k + 1. \tag{7.27}$$

Conversely, assume $x \leq k + 1$. Then $x = k + 1 \vee x < k + 1$. If $x = k + 1$, then

$$\bigvee_{0 \leq m \leq k+1} (x = m). \tag{7.28}$$

If $x < k + 1$, then we have $x \leq k$. By the inductive hypothesis,

$$\bigvee_{0 \leq m \leq k} (x = m) \tag{7.29}$$

and,therefore,

$$\bigvee_{0 \leq m \leq k+1} (x = m). \tag{7.30}$$

Thus in either case,

$$\bigvee_{0\leq m\leq k+1}(x=m). \qquad (7.31)$$

This proves

$$\vdash x\leq k+1\Rightarrow\bigvee_{0\leq m\leq k+1}(x=m). \qquad (7.32)$$

From the inductive hypothesis, we have derived

$$\bigvee_{0\leq m\leq k+1}(x=m)\Leftrightarrow x\leq k+1 \qquad (7.33)$$

and this completes the proof. Note that this proof has been given in an informal manner that we shall generally use from now on. In particular, the deduction theorem, the replacement theorem, and various rules and tautologies will be applied without being explicitly mentioned.

Parts (a′), (b), and (b′) follow easily from part (a). Part (c) follows almost immediately from the statement $t\neq r\Rightarrow(t<r)\vee(r<t)$, using obvious tautologies.

There are several stronger forms of the hyperinfinite induction principles that we can prove at this point.

Theorem 7.1.(Complete hyperinfinite induction) Let $B(x)$ be anrestricted wff of the set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ then

$$\forall x(x\in\mathbb{N}^{\#})[\forall z(z<x\Rightarrow B(z))\Rightarrow B(x)]\Rightarrow\forall x(x\in\mathbb{N}^{\#})B(x) \qquad (7.34)$$

In ordinary languageI consider a property $B(x)$ such that, for any x, if $B(x)$ holds for all hypernatural numbers less than x, then $B(x)$ holds for x also. Then $B(x)$ holds for all hypernatural numbers $x\in\mathbb{N}^{\#}$.

Proof.Let $E(x)$ be a wff $\forall z(z\leq x\Rightarrow B(z))$.

(i) 1.Assume that $\forall x(x\in\mathbb{N}^{\#})[\forall z(z<x\Rightarrow B(z))\Rightarrow B(x)]$, then

2.$[\forall z(z<0\Rightarrow B(z))\Rightarrow B(0)]$ it follows from 1.

3. $z\nless 0$, then

4. $\forall z(z<0\Rightarrow B(z))$ it follows from 1,

5. $B(0)$ it follows from 2,4 by MP

6. $\forall z(z\leq 0\Rightarrow B(z))$ i.e.,$E(0)$ holds it follows from Proposition7.1(a′)

7.$\forall x(x\in\mathbb{N}^{\#})[\forall z(z<x\Rightarrow B(z))\Rightarrow B(x)]\vdash E(0)$ it follows from 1,6 by MP

(ii) 1.Assume that: $\forall x(x\in\mathbb{N}^{\#})[\forall z(z<x\Rightarrow B(z))\Rightarrow B(x)]$.

2.Assume that: $E(x)\equiv\forall z(z\leq x\Rightarrow B(z))$, then

3.$\forall z(z<x^{+}\Rightarrow B(z))$ it follows from 2 since $z\leq x\Rightarrow z<x^{+}$.

4.$\forall x(x\in\mathbb{N}^{\#})[\forall z(z<x^{+}\Rightarrow B(z))\Rightarrow B(x^{+})]$ it follows from 1 by rule A4:if t is free for x in $B(x)$, then $\forall xB(x)\vdash B(t)$.

5. $B(x^{+})$ it follows from 3,4 by unrestricted MP rule.

6. $z\leq x^{+}\Rightarrow z<x^{+}\vee z=x^{+}$ it follows from definitions.

7. $z<x^{+}\Rightarrow B(z)$ it follows from 3 by rule A4.

8. $z=x^{+}\Rightarrow B(z)$ it follows from 5.

9. $E(x^{+})\equiv\forall z(z\leq x^{+}\Rightarrow B(z))$ it follows from 6,7,8,rule Gen.

10.$\forall x(x\in\mathbb{N}^{\#})[\forall z(z<x\Rightarrow B(z))\Rightarrow B(x)]\vdash\forall x(x\in\mathbb{N}^{\#})[E(x)\Rightarrow E(x^{+})]$

it follows from 1,9 by deduction theorem,rule Gen.

Now by (i), (ii) and the induction axiom, we obtain $D \vdash \forall x(x \in \mathbb{N}^{\#})E(x)$ that is $D \vdash \forall x(x \in \mathbb{N}^{\#})[\forall z(z < x \Rightarrow B(z))]$, where $D \equiv \forall x(x \in \mathbb{N}^{\#})[\forall z(z < x \Rightarrow B(z)) \Rightarrow B(x)]$.

Hence, by rule A4 twice, $D \vdash x \leq x \Rightarrow B(x)$. But $\vdash x \leq x$. So,$D \vdash B(x)$, and, by Gen and

the deduction theorem, $D \vdash \forall x(x \in \mathbb{N}^{\#})B(x)$.

Theorem 7.2.(Complete hyperfinite induction) Let $B(x)$ be wff of the set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ strongly restricted on inductive set $W_{\mathbf{ind}}$ such that $\mathbb{N} \subseteq W_{\mathbf{ind}} \subsetneqq \mathbb{N}^{\#}$ then

$$\forall x(x \in W_{\mathbf{ind}})[\forall z(z < x \Rightarrow B(z)) \Rightarrow B(x)] \Rightarrow \forall x(x \in W_{\mathbf{ind}})B(x) \qquad (7.35)$$

Proof. Similarly as Theorem 7.1.

Remark 7.5.Remind that the following statement holds in standard bivalent arithmetic [11]:Least-number principle (LNP)

$$\exists xB(x) \Rightarrow \exists y[B(y) \wedge \forall z(z < y \Rightarrow \neg B(z))]. \qquad (7.36)$$

In ordinary language:if a property expressed by wff $B(x)$ holds for some natural number n,

then there is a least number satisfying $B(x)$.Obviously LNP (7.23) is not holds in nonstandard arithmetic, since there is no a least number in a set $\mathbb{N}^{\#}\backslash\mathbb{N}$.

Theorem 7.3.(Weak least-number principle) Let $B(x)$ be a wff of the set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ such that a wff $\neg B(x)$ restricted on inductive set $W_{\mathbf{ind}}$ such that $\mathbb{N} \subseteq W_{\mathbf{ind}} \subsetneqq \mathbb{N}^{\#}$ and $W^{\complement}_{\mathbf{ind}} = \mathbb{N}^{\#}\backslash W_{\mathbf{ind}}$ then

$$\begin{gathered}\exists x\left(x \in W^{\complement}_{\mathbf{ind}}\right)B(x) \Rightarrow \\ \neg\exists y\left(y \in W^{\complement}_{\mathbf{ind}}\right)[B(y) \wedge \forall z(z < y \Rightarrow \neg B(z))] \Rightarrow \forall y(y \in W_{\mathbf{ind}})[\neg B(y)]\end{gathered} \qquad (7.37)$$

Proof.We assume now that

1.$\neg\exists y\left(y \in W^{\complement}_{\mathbf{ind}}\right)[B(y) \wedge \forall z(z < y \Rightarrow \neg B(z))]$

2.$\forall y\left(y \in W^{\complement}_{\mathbf{ind}}\right)\neg[B(y) \wedge \forall z(z < y \Rightarrow \neg B(z))]$ it follows from 1.

3.$\forall y\left(y \in W^{\complement}_{\mathbf{ind}}\right)[\forall z(z < y \Rightarrow \neg B(z)) \Rightarrow \neg B(y)]$it follows from 2 by tautology.

4.$\forall y(y \in W_{\mathbf{ind}})[\neg B(y)]$ it follows from 3 by Theorem 7.2 with wff $\neg B(y)$ instead wff $B(y)$

5.$\neg\exists y\left(y \in W^{\complement}_{\mathbf{ind}}\right)[B(y) \wedge \forall z(z < y \Rightarrow \neg B(z))] \Rightarrow \forall y(y \in W_{\mathbf{ind}})[\neg B(y)]$ it follows from 1,4.

Remark 7.6.Note that: (i) the statement

$$(\mathbf{I}) : \ \exists y(y \in W_{\mathbf{ind}})[\neg\neg B(y)] \Rightarrow \neg\neg\exists y\left(y \in W^{\complement}_{\mathbf{ind}}\right)[B(y) \wedge \forall z(z < y \Rightarrow \neg B(z))]$$

is unprovable in $\mathrm{INC}^{\#}_{\infty^{\#}}$ from the statement

$$(\mathbf{II}) : \neg\exists y\left(y \in W^{\complement}_{\mathbf{ind}}\right)[B(y) \wedge \forall z(z < y \Rightarrow \neg B(z))] \Rightarrow \forall y(y \in W_{\mathbf{ind}})[\neg B(y)]$$

$$(\mathbf{II}) : \neg\exists y\left(y \in W^{\complement}_{\mathbf{ind}}\right)[B(y) \wedge \forall z(z < y \Rightarrow \neg B(z))] \Rightarrow \neg\exists y(y \in W_{\mathbf{ind}})[\neg\neg B(y)]$$

since the law of contraposition is not holds in intuitionistic hyperinfinitary logic $L^{\#}_{\infty^{\#}}$;

(ii) similarly it unprovable in $\mathrm{NC}^{\#}_{\infty^{\#}}$ by the restricted modus ponens rule.

Example 7.1. We set now $W_{\mathbf{ind}} = \mathbb{N}$ and $B(y) \Leftrightarrow y \in \mathbb{N}^{\#}_{\infty}$.The statement (**I**) reads

$$(\mathbf{I}^{*}) : \ \exists y(y \in \mathbb{N})[y \notin \mathbb{N}^{\#}_{\infty}] \Rightarrow \exists y(y \in \mathbb{N}^{\#}_{\infty})[B(y) \wedge \forall z(z < y \Rightarrow z \notin \mathbb{N}^{\#}_{\infty})]$$

and the statement (**II**) reads

$$(\mathbf{II}^*) : \neg\exists y(y \in \mathbb{N}^{\#}_{\infty})[(y \in \mathbb{N}^{\#}_{\infty}) \wedge \forall z(z < y \Rightarrow z \notin \mathbb{N}^{\#}_{\infty})] \Rightarrow \forall y(y \in \mathbb{N})[\neg(y \notin \mathbb{N}^{\#}_{\infty})].$$

Note that the statement $\mathbf{I}^*$is unprovable in $\mathrm{INC}^{\#}_{\infty^{\#}}$ from the statement $\mathbf{II}^*$since the law of contraposition is not holds in intuitionistic hyperinfinitary logic $L^{\#}_{\infty^{\#}}$;

Hyper inductive definitions in general.

A function $f : \mathbb{N}^{\#} \to A$ whose domain is the set $\mathbb{N}^{\#}$ is colled an hyper infinite sequence and denoted by $\{f_n\}_{n\in\mathbb{N}^{\#}}$ or by $\{f(n)\}_{n\in\mathbb{N}^{\#}}$The set of all hyperinfinite sequences whose terms belong to A is clearly $A^{\mathbb{N}^{\#}}$; the set of all hyperfinite sequences of $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ terms in A is A^n. The set of all hyperfinite sequences with terms in A can be defined as

$$\left\{R \subset \mathbb{N}^{\#} \times A : (R \text{ is a function}) \wedge \bigvee\nolimits_{n\in\mathbb{N}^{\#}}(D_1(R) = n)\right\}, \qquad (7.38)$$

where $D_1(R)$ is domain of R.This definition implies the existence of the set of all hyperfinite sequences with terms in A.The simplest case is the inductive definition of a hyperinfinite sequence $\{\varphi(n)\}_{n\in\mathbb{N}^{\#}}$ (with terms belonging to a certain set Z) satisfying the following conditions:

(a)

$$\varphi(0) = z, \varphi(n^+) = e(\varphi(n), n), \qquad (7.39)$$

where $z \in Z$ and e is a function mapping $Z \times \mathbb{N}^{\#}$ into Z.

More generally, we consider a mapping f of the cartesian product $Z \times \mathbb{N}^{\#} \times A$ into Z and seek a function $\varphi \in Z^{\mathbb{N}^{\#}\times A}$ satisfying the conditions :

(b)

$$\varphi(0,a) = g(a), \varphi(n^+,a) = f(\varphi(n,a), n, a), \qquad (7.40)$$

where $g \in Z^A$. This is a definition by induction with parameter a ranging over the set A. Schemes (a) and (b) correspond to induction “from n to $n^+ = n + 1$”,i.e. $\varphi(n^+)$ or $\varphi(n^+,a)$ depends upon $\varphi(n)$ or $\varphi(n,a)$ respectively. More generally, $\varphi(n^+)$ may depend upon all values $\varphi(m)$ where $m \leq n$ (i.e. $m \in n^+$). In the case of induction with parameter, $\varphi(n^+,a)$ may depend upon all values $\varphi(m,a)$, where $m \leq n$; or even upon all values $\varphi(m,a)$, where $m \leq n^+$ and $b \in A$. In this way we obtain the following schemes of definitions by hyper infinite induction:

(c) $\varphi(0) = z, \varphi(n^+) = h(\varphi|n^+, n)$,

(d) $\varphi(0,a) = g(a), \quad \varphi(n^+,a) = H(\varphi|(n^+ \times A), n, a)$.

In the scheme (c), $z \in Z$ and $h \in Z^{C\times\mathbb{N}^{\#}}$, where C is the set of hyperfinite sequences whose terms belong to Z; in the scheme (d), $g \in Z^A$ and $H \in Z^{T\times\mathbb{N}^{\#}\times A}$, where T is the set of functions whose domains are included in $\mathbb{N}^{\#} \times A$ and whose values belong to Z.

It is clear that the scheme (d) is the most general of all the schemes considered above.

By coise of functions one obtains from (d) any of the schemes (a)-(d). For example, taking the function defined by $H(c,n,a) = f(c(n,a),n,a)$ for $a \in A, n \in \mathbb{N}^{\#}, c \in Z^{\mathbb{N}^{\#}\times A}$ as H in (d), one obtain (b). We shall now show that, conversely, the scheme (d) can be obtained from (a). Let g and H be functions belonging to Z^A and $Z^{T\times\mathbb{N}^{\#}\times A}$ respectively, and let φ be a function satisfying (d). We shall show that the sequence $\Psi = \{\Psi_n\}_{n\in\mathbb{N}^{\#}}$ with $\Psi_n = \varphi|(n^+,A)$ can be defined by (a).Obviously, $\Psi_n \in T$ for every $n \in \mathbb{N}^{\#}$. The first term of the sequence Ψ is equal to $\varphi|(0^+,A)$, i.e. to the set: $z^* = \{\langle\langle 0,a\rangle, g(a)\rangle | a \in A\}$.The relation between Ψ_n, and Ψ_{n^+} is given by the formula:$\Psi_{n^+} = \Psi_n \cup \varphi|(\{n^+\} \times A)$, where the second component is

$$\{\langle\langle n^+,a\rangle,\varphi(n^+,a)\rangle|a \in A\} = \{\langle n^+,a\rangle,H(\Psi_n,n,a)|a \in A\}. \quad (7.41)$$

Thus we see that the sequence Ψ can be defined by (a) if we substitute T for Z,z^* for z and let $e(c,n) = c \cup \{\langle n^+,a\rangle,H(c,n,a)|a \in A\}$ for $c \in T$.

Now we shall prove the existence and uniqueness of the function satisfying (a). This theorem shows that we are entitled to use definitions by induction of the type (a). According to the remark made above, this will imply the existence of functions satisfying the formulas (b), (c), and (d). Since the uniqueness of such functions can be proved in the same manner as for (a), we shall use in the sequel definitions by induction of any of the types (a)-(d).

Theorem 7.4. If Z is any set $z \in Z$ and $e \in Z^{Z\times\mathbb{N}^\#}$, then there exists exactly one hyper infinite sequence φ satisfying formulas (a).

Proof. Uniqueness. Suppose that $\{\varphi_1(n)\}_{n\in\mathbb{N}^\#}$ and $\{\varphi_2(n)\}_{n\in\mathbb{N}^\#}$ satisfy (a) and let

$$K = \{n|n \in \mathbb{N}^\# \wedge \varphi_1(n) = \varphi_2(n)\} \quad (7.42)$$

Then (a) implies that K is hyperinductive. Hence $\mathbb{N}^\# \subseteq K$ and therefore $\varphi_1(n) \equiv \varphi_2(n)$.

Existence. Let $\Phi(z,n,t)$ be the formula $e(z,n) = t$ and let $\Psi(w,z,F_n)$ be the following formula:

$$(F_n \text{ is a function}) \wedge (D_1(F) = n^+) \wedge (F(0) = z) \wedge \bigwedge_{m\in n} \Phi(F_n(m),m,F_n(m^+)). \quad (7.43)$$

In other words, F is a function defined on the set of numbers $\leq n \in \mathbb{N}^\#$ such that $F(0) = z$ and $F(m^+) = e(F(m),m)$ for all $m < n \in \mathbb{N}^\#$.

Remark 7.7.We assume now that predicate $\Psi(w,z,F_n)$ is unrestricted on variable $n \in \mathbb{N}^\#$, see Definition 7.3.

We prove by induction that there exists exactly one function F_n such that $\Psi(n,z,F_n)$. The proof of uniqueness of this function is similar to that given in the first part of Theorem 7.4. The existence of F_n can be proved as follows: for $n = 0$ it suffices to take $\{\langle 0,z\rangle\}$ as F_n; if $n \in \mathbb{N}^\#$ and F_n satisfies $\Psi(n,z,F_n)$, then $F_{n^+}=$ $F_n \cup \{\langle n^+,e(F_n(n),n)\rangle\}$ satisfies the condition $\Psi(n^+,z,F_{n^+})$.

Now, we take as φ the set of pairs $\langle n,s\rangle$ such that $n \in \mathbb{N}^\#, s \in Z$ and

$$\exists F[\Psi(n,z,F) \wedge (s = F(n))]. \quad (7.44)$$

Since F is the unique function satisfying $\Psi(n,z,F)$, it follows that φ is a function. For $n = 0$ we have $\varphi(0) = F_0(0) = z$; if $n \in \mathbb{N}^\#$, then $\varphi(n^+) = F_{n^+}(n^+) = e(F_n(n),n)$ by the definition of F_n; hence we obtain $\varphi(n^+) = e(\varphi(0),n)$. Theorem 7.4 is thus proved.

We frequently define not one but several functions (with the same range Z) by a simultaneous induction:

$\varphi(0) = z, \qquad \psi(0) = t,$

$\varphi(n^+) = f(\varphi(n),\psi(n),n), \ \psi(n^+) = g(\varphi(n),\psi(n),n)$

where $z,t \in Z$ and $f,g \in Z^{Z\times Z\times\mathbb{N}^\#}$.

This kind of definition can be reduced to the previous one. It suffices to notice that the hypersequence $\vartheta_n = \langle\varphi(n),\psi(n)\rangle$ satisfies the formulas: $\vartheta_0 = \langle z,t\rangle, \vartheta_{n^+} = e(\vartheta_n,n)$, where we set

$$e(u,n) = \langle f(K(u),L(u),n),g(K(u),F(w),n)\rangle, \quad (7.45)$$

and K,L denote functions such that

$K(\langle x,y\rangle)$ and $L(\langle x,y\rangle) = y$ respectively. Thus the function ϑ is defined by induction by means of (a). We now define φ and ψ by $\varphi(n) = K(\vartheta_n), \psi(n) = L(\vartheta_n)$.

Remark 7.8.We assume now that predicate $\Psi(w,z,F_n)$ is restricted on variable $n \in \mathbb{N}^{\#}$, on a set $[0,\beta]$, see Definition 7.2, then there exists exactly one hyperfinite sequence φ satisfying formulas (a).

The theorem 7.4 on hyper inductive definitions can be generalized to the case of operations. We shall discuss only one special case. Let $\Phi(z,n,t)$ be a formula such that

$$\forall z \forall n(n \in \mathbb{N}^{\#})\forall t_1 \forall t_2[\Phi(z,n,t_1) \wedge \Phi(z,n,t_2) \Rightarrow t_1 = t_2]. \tag{7.46}$$

Theorem 7.5. For any set S there exists exactly one hyperinfinite sequence $\varphi_n, n \in \mathbb{N}^{\#}$ such that $\varphi_0 = S$ and

$$\forall n(n \in \mathbb{N}^{\#})\Phi(\varphi_n, n, \varphi_{n^+}). \tag{7.47}$$

Proof. Uniqueness can be proved as in Theorem 7.4 above.

To prove the existence of φ_n, let us consider the following formula $\Psi(n,S,F)$:

$$(F \text{ is a function})(D_1(F) = n^+) \wedge (F(0) = S) \wedge \forall m(m \in n)\Phi(F(m),m,F(m')), \tag{7.48}$$

where $D_1(F)$ is domain of F.

As in the proof of Theorem 7.4, it can be shown that there exists exactly one function F_n such that $\Psi(n,S,F_n)$. To proceed further we must make certain that there exists a set containing all the elements of the form $F_n(n)$ where $n \in \mathbb{N}^{\#}$. (In the case considered in Theorem 7.4 this set is Z for the domain of the last variable of the formula Φ which we used in the proof of Theorem 7.4 was limited to the set Z.) In the case under consideration, the existence of the required set Z follows from the axiom of replacement.

In fact, the uniqueness of F_n implies that the formula

$$\exists F_n[\Psi(n,S,F_n) \wedge (y = F_n(n))] \tag{7.49}$$

satisfies the assumption of axiom of replacement. Hence by means of axiom of replacement the image of $\mathbb{N}^{\#}$ obtained by this formula exists. This image is the required set Z containing all the elements $F_n(n)$.

The remainder of the proof is analogous to that of Theorem 7.4.

Example 7.1. Let $\Phi(S,\mathbf{t})$ be the formula $\mathbf{t} = \mathbf{P}(S)$. Thus for any set S there exists exactly one hyper infinite sequence $\{\varphi_n\}_{n\in\mathbb{N}^{\#}}$ such that $\varphi_0 = S$ and $\varphi_{n^+} = \mathbf{P}(\varphi_n)$ for every number $n \in \mathbb{N}^{\#}$.

8.Useful examples of the hyper inductive definitions.

1.Addition operation of hypernatural numbers

The function $+(m,n) \triangleq m+n : \mathbb{N}^{\#} \times \mathbb{N}^{\#} \to \mathbb{N}^{\#}$ is defined by

$m+0 = m, m+n^+ = (m+n)^+$.

This definition is obtained from (b) by seting $Z = A = \mathbb{N}^{\#}, g(a) = a, f(p,n,a) = p^+$.

This function satisfies all properties of addition such as: for all $m,n,k \in \mathbb{N}^{\#}$

(i) $m+0 = m$ (ii) $m+n = n+m$ (iii) $m+(n+k) = (m+n)+k$.

2.Multiplicattion operation of gypernatural numbers

The function $\times(m,n) \triangleq m \times n : \mathbb{N}^{\#} \times \mathbb{N}^{\#} \to \mathbb{N}^{\#}$ is defined by

$m\times 1=1, m\times n^{+}=m\times n+m.$
(i) $m\times 1=1$ (ii) $m\times n=n\times m$ (iii) $m\times(n\times k)=(m\times n)\times k.$
4.Distributivity with respect to multiplication over addition.
$m\times(n+k)=m\times n+m\times k.$
5. Let $Z=A=X^{X}, g(a)=I_X, f(u,n,a)=u\circ a$ in (b). Then (b) takes on the following form

$$\varphi(0,a)=I_X, \varphi(n^{+},a)=\varphi(n,a)\circ a. \tag{8.1}$$

The function $\varphi(n,a)$ is denoted by a^{n} and is colled n-th iteration of the function a :

$$a^{0}(x)=x, a^{n^{+}}(x)=a^{n}(a(x)), x\in X, a\in X^{X}, n\in\mathbb{N}^{\#}. \tag{8.2}$$

6.Let $A=(\mathbb{N}^{\#})^{\mathbb{N}^{\#}}, g(a)=a_0, f(u,n,a)=u+a_{n^{+}}$. Then (b) takes on the following form

$$\varphi(0,a)=a_0, \varphi(n^{+},a)=\varphi(n,a)+a_{n^{+}} \tag{8.3}$$

The function is defined by the Eqs.(8.3) is denoted by

$$\sum_{i=0}^{n}a_i \tag{8.4}$$

7.Let $A=(\mathbb{N}^{\#})^{\mathbb{N}^{\#}}, g(a)=a_0, f(u,n,a)=u\times a_{n^{+}}$. Then (b) takes on the following form

$$\varphi(0,a)=a_0, \varphi(n^{+},a)=\varphi(n,a)\times a_{n^{+}} \tag{8.5}$$

The function is defined by the Eqs.(8.5) is denoted by

$$\prod_{i=0}^{n}a_i \tag{8.6}$$

8. Similarly we define $\max_{i\leq n}(a_i), \min_{i\leq n}(a_i), n\in\mathbb{N}^{\#}.$
Theorem 8.1. The following equalities holds for any $n,k_1,l_1\in\mathbb{N}^{\#}$:
(1) using distributivity

$$b\times\sum_{i=0}^{n}a_i=\sum_{i=0}^{n}b\times a_i \tag{8.7}$$

(2) using commutativity and associativity

$$\sum_{i=0}^{n}a_i\pm\sum_{i=0}^{n}b_i=\sum_{i=0}^{n}(a_i\pm b_i) \tag{8.8}$$

(3) splitting a sum, using associativity

$$\sum_{i=0}^{n}a_i=\sum_{i=0}^{j}a_i+\sum_{i=j+1}^{n}a_i \tag{8.9}$$

(4) using commutativity and associativity, again

$$\sum_{i=k_0}^{k_1}\sum_{j=l_0}^{l_1}a_{ij}=\sum_{j=l_0}^{l_1}\sum_{i=k_0}^{k_1}a_{ij} \tag{8.10}$$

(5) using distributivity

$$\left(\sum_{i=0}^{n}a_i\right)\times\left(\sum_{j=0}^{n}b_j\right)=\sum_{i=0}^{n}\sum_{j=0}^{n}a_i\times b_j \tag{8.11}$$

(6)

$$\left(\prod_{i=0}^{n}a_i\right)\times\left(\prod_{i=0}^{n}b_i\right)=\prod_{i=0}^{n}a_i\times b_i \tag{8.12}$$

(7)

$$\left(\prod_{i=0}^{n} a_i\right)^m = \prod_{i=0}^{n} a_i^m \qquad (8.13)$$

Proof. Imediately from Theorem 7.4 and hyperinfinite induction principle.

Definition 8.1.A non-empty non regular sequence $\{u_n\}_{n\in\mathbb{Z}}$ is a blok corresponding to gyperfinite number $u = u_0 \in \mathbb{N}^{\#}\backslash\mathbb{N}$ iff there is gyperfinite number u such that $\ldots\in u_{-(n+1)} \in u_{-n}\ldots\in u_{-4} \in u_{-3} \in u_{-2} \in u_{-1} \in u$ and the following conditions are satisfied

$$\ldots\in u_{-(n+1)} \in u_{-n}\ldots\in u_{-4} \in u_{-3} \in u_{-2} \in u_{-1} \in u \in u_1 \in u_2 \in\ldots\in u_n \in u_{n+1} \in\ldots \quad (8.14)$$

where for any $n \in \mathbb{N}$: $u_{-(n+1)} \in u_{-n}$, where $u_{-n} = u^{+}_{-(n+1)}$.

Thus beginning with an infinite integer $u \in \mathbb{N}^{\#}\backslash\mathbb{N}$ we obtain a block (8.20) of infinite integers.However, given a “block,” there is another block consisting of even larger infinite integers. For example, there is the integer $u + u$, where $u + k < u + u$ for each $k \in \mathbb{N}$. And $v = u + u$ is itself part of the block:

$$\ldots< v - 3 < v - 2 < v - 1 < v < v + 1 < v - 2 <\ldots \qquad (8.15)$$

Of course, $v < v + u < v + v$, and so forth. There are even infinite integers $u \times u$ and u^u, and so forth.Proceeding in the opposite direction, if $u \in \mathbb{N}^{\#}\backslash\mathbb{N}$, either u or $u + 1$ is of the form $v + v$. Here v must be infinite. So there is no first block, since $v < u$. In fact, the ordering of the blocks is dense. For let the block containing v precede the one containing u, that is,

$$v - 2 < v - 1 < v < v + 1 <\ldots<\ldots< u - 2 < u - 1 < u < u + 1 <\ldots \qquad (8.16)$$

Either $u + v$ or $u + v + 1$ can be written $z + z$ where $v + k < z < u - l$ for all $k, l \in \mathbb{N}$.

To conclude our consideration: $\mathbb{N}^{\#}$ consists of $\mathbb{N}$ as an initial segment followed by an ordered set of blocks. These blocks are densely ordered with no first or last element. Each block is itself order-isomorphic to the integers

$$-3, -2, -1, 0, 1, 2, 3, \qquad (8.17)$$

Although $\mathbb{N}^{\#}\backslash\mathbb{N}$ is a nonempty subset of $\mathbb{N}^{\#}$, as we have just seen it has no least element and likewise for any block.

9.Analisys on nonarchimedian field $\mathbb{Q}^{\#}$.

9.1.Basic properties of the hyperrationals $\mathbb{Q}^{\#}$.

Now that we have the hypernatural numbers, defining hyperintegers and hyperrational numbers is well within reach.

Definition 9.1. Let $Z' = \mathbb{N}^{\#} \times \mathbb{N}^{\#}$. We can define an equivalence relation $\approx$ on Z' by $(a,b) \approx (c,d)$ if and only if $a + d = b + c$. Then we denote the set of all hyperintegers by $\mathbb{Z}^{\#} = Z'/\approx$ (The set of all equivalence classes of Z' modulo $\approx$).

Definition 9.2. Let $Q' = \mathbb{Z}^{\#} \times (\mathbb{Z}^{\#} - \{0\}) = \{(a,b) \in \mathbb{Z}^{\#} \times \mathbb{Z}^{\#} | b \neq 0\}$. We can define an equivalence relation $\approx$ on Q' by $(a,b) \approx (c,d)$ if and only if $a \times d = b \times c$.Then we denote the set of all hyperrational numbers by $\mathbb{Q}^{\#} = Q'/\approx$ (The set of all equivalence classes of Q' modulo $\approx$).

Definition 9.3. A linearly ordered set $(P,<)$ is called dense if for any $a,b \in P$ such that $a < b,$ there exists $z \in P$ such that $a < z < b.$

Lemma 9.1. $(\mathbb{Q}^{\#},<)$ is dense.

Proof. Let $x = (a,b), y = (c,d) \in \mathbb{Q}^{\#}$ be such that $x < y.$ Consider $z = (ad + bc, 2bd) \in \mathbb{Q}^{\#}.$

It is easily shown that $x < z < y.$

Remark 9.1.Consider the ring B of all limited (i.e. finite) elements in $\mathbb{Q}^{\#}$. Then B has a unique maximal ideal $I_{\approx}$, the infinitesimal numbers. The quotient ring $B/I_{\approx}$ gives the field $\mathbb{R}$ of the classical real numbers.

1.Let $A = (\mathbb{Q}^{\#})^{\mathbb{Q}^{\#}}, g(a) = a_0, f(u,n,a) = u + a_{n^+}.$ Then (b) takes on the following form

$$\varphi(0,a) = a_0, \varphi(n^+,a) = \varphi(n,a) + a_{n^+} \tag{9.1}$$

The function is defined by the Eqs.(9.1) is denoted by

$$\sum_{i=0}^{n} a_i. \tag{9.2}$$

2.Let $A = (\mathbb{Q}^{\#})^{\mathbb{Q}^{\#}}, g(a) = a_0, f(u,n,a) = u \times a_{n^+}.$ Then (b) takes on the following form

$$\varphi(0,a) = a_0, \varphi(n^+,a) = \varphi(n,a) \times a_{n^+} \tag{9.3}$$

The function is defined by the Eqs.(9.3) is denoted by.

$$\prod_{i=0}^{n} a_i. \tag{9.3}$$

9.2.Countable summation from hyperfinite sum.

Definition 9.1. Let $\{a_n\}_{n\in\mathbb{N}}$ be $\mathbb{Q}^{\#}$-valued countable sequence. Let $\{a_n\}_k^m$ be any hyperfinite sequence with $m \in \mathbb{N}^{\#}\backslash\mathbb{N}$ and such that $a_n = 0$ if $n \in \mathbb{N}^{\#}\backslash\mathbb{N}.$ Then we define summation of the countable sequence $\{a_n\}_{n\in\mathbb{N}}$ by the following hyperfinite sum

$$\sum_{n=k}^{m} a_n \in \mathbb{Q}^{\#} \tag{9.4}$$

and denote such sum by the symbol

$$\sum_{n=k}^{\omega} a_n. \tag{9.5}$$

Remark 9.2. Let $\{a_n\}_{n\in\mathbb{N}}$ be $\mathbb{Q}$-valued countable sequence. Note that: (i) for canonical summation we always apply standard notation

$$\sum_{n=k}^{\infty} a_n. \tag{9.6}$$

(ii) the countable sum (ω-sum) (9.5) in contrast with (9.6) abviously always exists even if a series (9.6) diverges absolutely i.e., $\sum_{n=k}^{\infty}|a_n| = \infty.$

Example 9.1. The ω-summ $\sum_{n=1}^{\omega} \frac{1}{n} \in \mathbb{Q}^{\#}$ exists by Theorem 8.1, however $\sum_{n=1}^{\infty} \frac{1}{n} = \infty.$

Theorem 9.3. Let $\sum_{n=k}^{\omega} a_n = A$ and $\sum_{n=k}^{\omega} b_n = B,$ where $A,B,C \in \mathbb{Q}^{\#}.$ Then

$$\sum_{n=k}^{\omega} C \times a_n = C \times \sum_{n=k}^{\omega} a_n \tag{9.6}$$

and

$$\sum_{n=k}^{\omega} (a_n \pm b_n) = A \pm B. \tag{9.7}$$

Proof. It follows from Theorem 8.2.

Example 9.2. Consider the countable sum

$$S_{\omega}(r) = \sum_{n=0}^{\omega} r^n, -1 < r < 1. \tag{9.5}$$

It follows from (9.5)

$$S_{\omega}(r) = 1 + \sum_{n=1}^{\omega} r^n = 1 + r \sum_{n=0}^{\omega} r^n = 1 + r S_{\omega}(r) \tag{9.6}$$

Thus

$$S_{\omega}(r) = \frac{1}{1-r}. \tag{9.7}$$

Remark 9.3. Note that

$$S_{\omega}(r) = \sum_{n=0}^{\omega} r^n = \sum_{n=0}^{\infty} r^n \tag{9.8}$$

since as we know

$$S_{\infty}(r) = \lim_{n\to\infty} \sum_{n=0}^{n} r^n = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}. \tag{9.9}$$

Definition 9.2.An element $x \in \mathbb{Q}^{\#}$ is called finite if $|x| < r$ for some $r \in \mathbb{Q}, r > 0$.

Abbreviation 9.1.For $x \in \mathbb{Q}^{\#}$ we abbreviate $x \in \mathbb{Q}^{\#}_{\mathbf{fin}}$ if x is finite.

Remark 9.4. Let $x \in \mathbb{Q}^{\#}$ be finite. Let D_1, be the set of $r \in \mathbb{Q}$ such that $r < x$ and D_2 the set of $r' \in \mathbb{Q}$ such that $x < r'$. The pair (D_1, D_2) forms a Dedekind cut in $\mathbb{R}_d$, hence determines a unique $r_0 \in \mathbb{R}_d$. A simple argument shows that $|x - r_0|$ is infinitesimal, i.e., $|x - r_0| \approx 0$.

Definition 9.3.This unique r_0 is called the standard part of x and is denoted by ${}^{\circ}x$.

Theorem 9.4. If $x \in \mathbb{R}_d$, then ${}^{\circ}x = x$; if $x, y \in \mathbb{Q}^{\#}$ are both finite, then

$${}^{\circ}(x+y) = {}^{\circ}(x) + {}^{\circ}(y), {}^{\circ}(x-y) = {}^{\circ}(x) - {}^{\circ}(y). \tag{9.10}$$

Definition 9.4.Let $\{a_i\}_{i=0}^{\infty}$ be countable $\mathbb{Q}^{\#}_{\mathbf{fin}}$-valued sequence. We say that a sequence $\{a_i\}_{i=0}^{\infty}$ converges to standard limit $\bar{a} \in \mathbb{R}_d$ and abbreviate $\bar{a} = \text{st-}\lim_{i\to\infty} a_i$ if for every $\epsilon > 0, \epsilon \not\approx 0$ there is an integer $N \in \mathbb{N}$ such that $|a_i - \bar{a}| < \epsilon$ if $i \geq N$.

Theorem 9.5. Let $\{a_i\}_{i=0}^{n}$, $n \in \mathbb{N}^{\#} \backslash \mathbb{N}$ be a hyperfinite $\mathbb{Q}^{\#}_{\mathbf{fin}}$-valued sequence such that: (i) ${}^{\circ}a_i = a_i$ for any $i \leq n$ and (ii) for any $m \leq n$: $Ext\text{-}\sum_{i=0}^{m} |a_i| < \mu \in \mathbb{Q}^{\#}_{\mathbf{fin}}$, then

$${}^{\circ}\left(Ext\text{-}\sum_{i=0}^{n} a_i \right) = Ext\text{-}\sum_{i=0}^{n} a_i. \tag{9.11}$$

Proof. From Eq.(9.10) by the condition (ii) and hyper infinite induction we get

$${}^{\circ}\left(Ext\text{-}\sum_{i=0}^{n} a_i \right) = Ext\text{-}\sum_{i=0}^{n} {}^{\circ}a_i. \tag{9.12}$$

From Eq.(9.12) by the condition (i) we obtain Eq.(9.11).

Theorem 9.6. Let $\{a_i\}_{i\in\mathbb{N}}$ be a countable $\mathbb{Q}$-valued sequence, i.e. $^\circ a_i = a_i \in \mathbb{Q}$ for any $i \leq n$ and $\sum_{i=0}^{\infty}|a_i| < \infty$, thus there exists st-$\lim_{m\to\infty}\sum_{i=0}^{m} a_i$, then

$$^\circ\left(Ext\text{-}\sum_{i=0}^{\omega} a_i\right) \equiv Ext\text{-}\sum_{i=0}^{\omega} a_i = \sum_{i=0}^{\infty} a_i. \quad (9.13)$$

Proof. It follows directly from Theorem 9.5 for the case if for any $i \in \mathbb{N}^{\#}\backslash\mathbb{N} : a_i \equiv 0$.

Theorem 9.7. Let $\{b_i\}_{i=0}^{\infty}$, be a countable $\mathbb{Q}$-valued sequence such that $\lim_{m\to\infty}\sum_{i=0}^{m}|b_i|$ exists.Then

$$\sum_{i=0}^{\infty} b_i \equiv Ext\text{-}\sum_{i=0}^{\omega} b_i. \quad (9.14)$$

Proof. It follows directly from Theorem 9.6 and Eq.(9.13).

10.Euler's proof of the Goldbach-Euler theorem revisited.

Theorem 10.1. (Goldbach-Euler theorem 1738)[12]-[13]. This infinite series, continued to infinity,

$$\frac{1}{3}+\frac{1}{7}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{26}+\frac{1}{31}+\frac{1}{35}+\ldots \quad (10.1)$$

the denominators of which are all numbers which are one less than powers of degree two or higher of whole numbers, that is, terms which can be expressed with the formula $(m^n - 1)^{-1}$, where m and n are integers greater than one, then the sum of this series is $= 1$.

10.1.How Euler did it.

Euler's proof begins with an $18^{\underline{\text{th}}}$ century step that treats any infinite sum as a real number which may be infinite large. Such steps became unpopular among rigorous mathematicians about a hundred years later.

Euler takes Σ to be the sum of the harmonic series

$$\Sigma = \sum_{n=1}^{\infty}\frac{1}{n} = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\ldots. \quad (10.2)$$

Next, Euler subtracts from Eq.(10.2) the geometric series

$$1 = \sum_{n=1}^{\infty}\frac{1}{2^n} = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+\ldots \quad (10.3)$$

leaving

$$\Sigma - 1 = 1+\frac{1}{3}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{9}+\frac{1}{10}+\ldots \quad (10.4)$$

Subtract from Eq.(10.4) geometric series

$$\frac{1}{2} = \frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\frac{1}{81}+\frac{1}{243}+\ldots \quad (10.5)$$

leaving

$$\Sigma - 1 - \frac{1}{2} = 1+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{10}+\frac{1}{11}+\ldots \quad (10.6)$$

Subtract from Eq.(10.6) geometric series

$$\frac{1}{4} = \frac{1}{5}+\frac{1}{25}+\frac{1}{125}+\ldots \quad (10.7)$$

leaving

$$\Sigma - 1 - \frac{1}{2} - \frac{1}{4} = 1 + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \ldots \tag{10.8}$$

Remark 10.1.Note that Euler had to skip subtracting the geometric series

$$\frac{1}{3} = \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots \tag{10.9}$$

because the series of powers of $1/4$ on the right is already a subseries of the series of powers of $1/2$, so those terms have already been subtracted. This happens because 3 is one less than a power, 4. It happens again every time we reach a term one less than a power. He will have to skip 7, because that is one less than the cube 8, and 8 because it is one less than the square 9, 15 because it is one less than the square 16, etc. Continuing formally in this way to infinity, we see that all of the terms on the right except the term 1 can be eliminated, leaving

$$\Sigma - 1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{5} - \frac{1}{6} - \frac{1}{9} - \ldots = 1. \tag{10.10}$$

Thus

$$\Sigma - 1 - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{9} + \frac{1}{10} + \ldots \right] = 1 \tag{10.11}$$

so

$$\Sigma - 1 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{9} + \frac{1}{10} + \ldots \tag{10.12}$$

Remark 10.2.Note that it gets just a little bit tricky. Since Σ is sum of the harmonic series, Euler believes that the 1 on the left must equal the terms of the harmonic series that are missing on the right. Those missing terms are exactly the ones with denominators one less than powers, so finally Euler concludes that

$$1 = \frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \frac{1}{31} + \frac{1}{35} + \ldots \tag{10.13}$$

where the terms on the right have denominators one less than powers.

10.2. Proof of the Goldbach-Euler theorem using canonical analysis.

We reproduce the proof here for the sake of completeness.

Lemma 1. For any positive integers n and k with $2 \leq n < k$
$1/n - 1 = 1/(n-1)n + 1/n(n+1) + \cdots + 1/(k-1)k + 1/k$
Lemma 2. For any positive integers n and k with $n \geq 2$
$1/n - 1 = 1/n + 1/n^2 + \cdots + 1/n^k + 1/n^k(n-1)$
We let denote the n-th harmonic number by H_n :

$$H_n = 1 + 1/2 + 1/3 + \ldots + 1/n, \tag{10.14}$$

but we now think of n as either a finite natural number or an infinite nonstandard natural number. Let k_2 be defined by $2^{k_2} \leq n < 2^{k_2+1}$. The existence and uniqueness of k_2 is clear either if we think of n as a finite natural number or as a nonstandard natural number: remember the transfer principle. Using Lemma 2, we can write

$$1 = 1/2 + 1/2^2 + 12^3 + \cdots + 1/2^{k_2} + 1/2^{k_2} \cdot 1,$$

and subtracting this series from (9.14), we obtain

$$H_n - 1 = 1 + 1/3 + 1/5 + 1/6 + 1/7 + 1/9 + \cdots + 1/n - 1/2^{k_2} \cdot 1. \qquad (10.15)$$

Hence, all powers of two, including two itself, disappear from the denominators, leaving the rest of integers up to n. If from (10.15) we subtract

$$1/2 = 1/3 + 13^2 + 1/3^3 + \cdots + 1/3^{k_3} + 1/3^{k_3} \cdot 2, \qquad (10.16)$$

again obtained from Lemma 2 with k_3 defined by $3^{k_3} \le n < 3^{k_3+1}$, the result will be

$$H_n - 1 - 1/2 = 1 + 1/5 + 1/6 + 1/7 + 1/10 + \cdots + 1/n - [1/2^{k_2} \cdot 1 + 1/3^{k_3} \cdot 2]. \qquad (10.17)$$

Proceeding similarly we end up by deleting all the terms that remain,arriving finally at

$$\begin{aligned} H_n - 1 - 1/2 - 1/4 - 1/5 - 1/6 - 1/7 - 1/10 - \cdots - 1/n &= \\ = 1 - [1/2^{k_2} \cdot 1 + 1/3^{k_3} \cdot 2 + \cdots + 1/n \cdot (n-1)]. \end{aligned} \qquad (10.18)$$

Notice that $k_2 \ge k_3 \ge \cdots$.In fact,when $m > \sqrt{n}$ we get $k_m = 1$. This last expression has been obtained assuming that n is a nonpower. If n is a power, then $1/n$ will have disappeared at some stage of this process,and the last fraction to beremoved from(10.17) will be $1/(n-1)$, whose denominator is a nonpower unless $n = 9$. (This is Catalan's conjecture that 8 and 9 are the only consecutive powers that exist. The conjecture was recently proved by Mihǎilescu [14]. In fact, it does not matter here whether there are more consecutive powers or not.) The corresponding expression will thus be

$$\begin{aligned} H_n - 1 - 1/2 - 1/4 - 1/5 - 1/6 - 1/7 - 1/10 - \cdots - 1/n - 1 \\ = 1 - [1/2^{k_2} \cdot 1 + 1/3^{k_3} \cdot 2 + \cdots + 1/(n-1) \cdot (n-2)]. \end{aligned} \qquad (10.19)$$

Consequently, if we subtract (10.18) from (10.14) we obtain

$$\begin{aligned} 1 - [1/2^{k_2} \cdot 1 + 1/3^{k_3} \cdot 2 + \cdots + 1/n \cdot (n-1)] &= \\ 1/3 + 1/7 + 1/8 + 1/15 + 1/24 + 1/26 + \cdots + 1/n - 1 \end{aligned} \qquad (10.20)$$

or, correspondingly subtracting (10.19) from (10.14),

$$\begin{aligned} 1 - [12k_2 \cdot 1 + 13k_3 \cdot 2 + \cdots + 1/(n-1)(n-2)] &= \\ 1/3 + 1/7 + 1/8 + 1/15 + 1/24 + 1/26 + \cdots + 1/n, \end{aligned} \qquad (10.21)$$

sums that containin their denominators,increased by one,all the power so fthe integers up to n. We must now take care of the "remainder," that is, the expression between parentheses above or on the right-hand side of (10.17) (respectively, (10.19)).

Since for each $m \ge 2$ we know by the definition of k_m that $n < m^{k_m+1} \le m^{2k_m}$, it follows that $\sqrt{n} < m^{k_m}$ and

$$1/[m^{k_m} \cdot (m-1)] \le 1/\sqrt{n}\,(m-1). \qquad (10.22)$$

This implies that

$$1/2^{k_2} \cdot 1 + 1/3^{k_3} \cdot 2 + \cdots + 1/n \cdot (n-1) \le H_{n-1}/\sqrt{n} \qquad (10.23)$$

or, if n is a power,

$$1/2^{k_2} \cdot 1 + 1/3^{k_3} \cdot 2 + \cdots + 1/(n-1) \cdot (n-2) \le H_{n-2}/\sqrt{n-1}. \qquad (10.24)$$

If we have chosen to regard n as a finite integer then we can pass to the limit and use Euler's asymptotic value for H_n : $\lim_{n\to\infty} H_{n-1}/\sqrt{n} = \lim_{n\to\infty}[\log(n-1) + \gamma]/\sqrt{n} = 0$. The proof is now complete.

10.3.Euler proof revisited using elementary analysis on nonarchimedian field $\mathbb{Q}^{\#}$.

We replace Eq.(10.2) by

$$\Sigma_\omega = \sum_{n=1}^{\omega} \frac{1}{n} \triangleq \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \ldots\right]^{\#}, \qquad (10.25)$$

where we write symbolically for convenience

$\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \ldots\right]^{\#}$ instead ω-sum $\sum_{n=1}^{\omega} \frac{1}{n}$.

Remark 10.3.Remind that ω-sum $\sum_{n=1}^{\omega} \frac{1}{n}$ is defined as hyperfinite sum $\sum_{n=1}^{\mathbf{m}} a_n$, where $a_n = n^{-1}$if $n \in \mathbb{Q}$ and $a_n = 0$ if $n \in \mathbb{Q}^{\#} \backslash \mathbb{Q}$.

Remark 10.4.Note that $\Sigma_\omega \in \mathbb{Q}^{\#} \backslash \mathbb{Q}$.

Subtract from Eq.(10.25) the ω-sum

$$1 = \sum_{n=1}^{\omega} \frac{1}{2^n} = \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots\right]^{\#} \qquad (10.26)$$

using Theorem 9.3 we obtain

$$\begin{aligned}\Sigma_\omega - 1 = \left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \ldots\right]^{\#} - \\ -\left[\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots\right]^{\#} = \\ \left[1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{9} + \frac{1}{10} + \ldots\right]^{\#}.\end{aligned} \qquad (10.27)$$

Subtract from Eq.(10.27) the ω-sum

$$\frac{1}{2} = \sum_{n=1}^{\omega} \frac{1}{3^n} = \left[\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \ldots\right]^{\#} \qquad (10.28)$$

using Theorem 9.3 we obtain

$$\begin{aligned}\Sigma_\omega - 1 - \frac{1}{2} = \left[1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{9} + \frac{1}{10} + \ldots\right]^{\#} - \\ -\left[\frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \frac{1}{243} + \ldots\right]^{\#} = \\ = \left[1 + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \frac{1}{11} + \ldots\right]^{\#}.\end{aligned} \qquad (10.29)$$

Subtract from Eq.(10.29) the ω-sum

$$\frac{1}{4} = \left[\frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \ldots\right]^{\#} \qquad (10.30)$$

using Theorem 9.3 we obtain

$$\Sigma_\omega - 1 - \frac{1}{2} - \frac{1}{4} = \left[1 + \frac{1}{6} + \frac{1}{7} + \frac{1}{10} + \ldots\right]^{\#} \qquad (10.31)$$

Remark 10.5.Note that in calculation above we had skip subtracting the ω-sum (see Remark 9.1)

$$\frac{1}{3} = \left[\frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \frac{1}{256} + \ldots\right]^{\#} \qquad (10.32)$$

because the series of powers of $1/4$ on the right is already a subseries of the ω-sum

(10.28) of powers of $1/2$, so those terms have already been subtracted. This happens because 3 is one less than a power, $4.$ It happens again every time we reach a term one less than a power. He will have to skip $7,$ because that is one less than the cube $8,$ and 8 because it is one less than the square $9, 15$ because it is one less than the square $16,$ etc. Continuing in this way to gyperfinite number $\mathbf{m} \in \mathbb{Q}^{\#} \backslash \mathbb{Q}$ by using gyperfinite induction principle, we see that all of the terms on the right except the term 1 can be eliminated. Thus by Theorem 9.3 and Remark 10.5 we obtain

$$\Sigma_{\omega} - 1 - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{9} + \frac{1}{10} + \ldots \right]^{\#} =$$

$$\Sigma_{\omega} - \left[\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \ldots . \right]^{\#} = \tag{10.33}$$

$$= \Sigma_{\omega} - \sum_{n=2}^{\omega} \frac{1}{n} = 1.$$

From Eq.(10.33) we obtain

$$\Sigma_{\omega} - 1 - \left[\frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{9} + \frac{1}{10} + \ldots \right]^{\#} = 1. \tag{10.34}$$

Finally we get

$$1 = \left[\frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \frac{1}{31} + \frac{1}{35} + \ldots \right]^{\#}, \tag{10.35}$$

where the terms on the right have denominators one less than powers. From Eq.(10.32) by Theorem 9.7 we obtain

$$1 = \frac{1}{3} + \frac{1}{7} + \frac{1}{8} + \frac{1}{15} + \frac{1}{24} + \frac{1}{26} + \frac{1}{31} + \frac{1}{35} + \ldots, \tag{10.36}$$

where the terms on the right have denominators one less than powers.
Note that Eq.(10.36) now is obtained without any references to Catalan conjecture [13],[14].

11.External induction principle and hyper inductive definitions in nonstandard analysis.

11.1.Internal induction principle in Robinson nonstandard analysis.

Remind that in Robinson nonstandard analysis [2]-[5] each member of ${}^{*}P(\mathbb{N})$ is colled to be an internal subset of ${}^{*}\mathbb{N};$ any other subset of ${}^{*}\mathbb{N}$ is colled an external subset of ${}^{*}\mathbb{N}$.

The importance of internal sets versus external sets rests on the theorem which says that each statement which is true for $\mathbb{N}$ is true for ${}^{*}\mathbb{N}$if and only if its quantifiers are restricted on internal subset of ${}^{*}\mathbb{N}.$ Thus the induction postulate reads

$$\forall S[S \in {}^{*}P(\mathbb{N})]\{1 \in S \wedge \forall x[x \in S \Rightarrow x+1 \in S] \Rightarrow S = {}^{*}\mathbb{N}\}. \tag{11.1}$$

Remind that a set S is inductive if $1 \in S \wedge \forall x[x \in S \Rightarrow x+1 \in S].$ The induction postulate (11.1) is not holds for inductive set S which is not internal. For example the induction postulate (11.1) is not holds for inductive set $S = \mathbb{N}$ since $\mathbb{N} \neq {}^{*}\mathbb{N}$.

We emphasize that in contrast with ZFC in set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ notion of internal subset of ${}^{*}\mathbb{N}$is not important since the induction postulate (11.1) holds for any hyper inductive set S which is not initially defined as internal.

Definition 11.1.A set $S \subset {}^*\mathbb{N}$is a hyper inductive if the following statement holds

$$\bigwedge_{\alpha \in {}^*\mathbb{N}} (\alpha \in S \Rightarrow \alpha^+ \in S). \tag{11.2}$$

Obviously a set ${}^*\mathbb{N}$ is a hyper inductive. As we see later there is just one hyper inductive

subset of ${}^*\mathbb{N}$, namely ${}^*\mathbb{N}$itself.

11.2.External induction principle in nonstandard analysis based on set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$.

Definition 11.2.Let β be a hypernatural such that $\beta \in {}^*\mathbb{N}\backslash\mathbb{N}$. Let $[0,\beta] \subset {}^*\mathbb{N}$be a set such that $\forall x[x \in [0,\beta] \Leftrightarrow 0 \leq x \leq \beta]$ and $[0,\beta) = [0,\beta]\backslash\{\beta\}$.

Definition 11.3.(i) Let $F(x)$ be a wff of $\mathrm{INC}^{\#}$with unique free variable x. We will say that a wff $F(x)$ is restricted on a set S_F such that $S_F \subsetneqq {}^*\mathbb{N}$ iff the following conditions are satisfied

$$\forall \alpha(\alpha \in {}^*\mathbb{N})[F(\alpha) \Rightarrow \alpha \in S_F] \tag{11.3}$$

and

$$\forall \alpha(\alpha \in {}^*\mathbb{N})[\neg F(\alpha) \Rightarrow \alpha \in {}^*\mathbb{N}\backslash S_F]. \tag{11.4}$$

Definition 11.4. Let $F(x)$ be a wff of $\mathrm{INC}^{\#}_{\infty^{\#}}$with unique free variable x. We will say that a

wff $F(x)$ is unrestricted on variable x if wff $F(x)$ is not restricted on any set S such that $S \subsetneqq {}^*\mathbb{N}$. This definition meant

$$\bigwedge_{\alpha \in \mathbb{N}^{\#}} (F(\alpha) \nvdash \alpha \notin \mathbb{N}^{\#}). \tag{11.5}$$

Axiom of hyperfinite induction 1

$$\forall \beta(\beta \in {}^*\mathbb{N}\backslash\mathbb{N})\forall S(S \subseteq [0,\beta]) \searrow$$
$$\left\{\forall \alpha(\alpha \in [0,\beta])\left[\bigwedge_{0 \leq \alpha < \beta} (\alpha \in S \Rightarrow \alpha^+ \in S)\right] \Rightarrow S = [0,\beta]\right\}. \tag{11.6}$$

Axiom of hyper infinite induction 1

$$\forall S(S \subset {}^*\mathbb{N})\left\{\forall \beta(\beta \in {}^*\mathbb{N})\left[\bigwedge_{0 \leq \alpha < \beta} (\alpha \in S \Rightarrow \alpha^+ \in S)\right] \Rightarrow S = {}^*\mathbb{N}\right\}. \tag{11.6}$$

Remark 11.1.Note that from comprechesion shemata 2 (see subsection 6.1) follows that

$$\forall \beta \exists S(S \subset [0,\beta])\forall \bar{\beta}(\bar{\beta} \in [0,\beta])\left[\bar{\beta} \in S \Leftrightarrow \bigwedge_{0 \leq \alpha < \bar{\beta}} (\alpha \in S \Rightarrow \alpha^+ \in S)\right]. \tag{11.7}$$

Therefore for any $\bar{\beta} \in [0,\beta]$ from (11.7) it follows that

$$\bigwedge_{0 \leq \alpha < \bar{\beta}} (\alpha \in S \Rightarrow \alpha^+ \in S) \vdash \bar{\beta} \in S. \tag{11.8}$$

Thus axiom of hyperfinite induction 1,i.e., (11.5) holds, since from (11.8) it follows that $\forall \bar{\beta}[\bar{\beta} \in [0,\beta] \Rightarrow \bar{\beta} \in S]$.

Remark 11.2.Note that from comprechesion shemata 2 (see subsection 6.1) follows that

$$\exists S(S \subset {}^*\mathbb{N})\forall\beta(\beta \in {}^*\mathbb{N})\left[\beta \in S \Leftrightarrow \bigwedge_{0\leq\alpha<\beta}(\alpha \in S \Rightarrow \alpha^+ \in S)\right]. \qquad (11.9)$$

Therefore for any $\beta \in {}^*\mathbb{N}$ from (11.9) it follows that

$$\bigwedge_{0\leq\alpha<\beta}(\alpha \in S \Rightarrow \alpha^+ \in S) \vdash \beta \in S \qquad (11.10)$$

Thus axiom of hyperinfinite induction 1, i.e., (7.6) holds, since it follows from (7.10) that $\forall\beta[\beta \in {}^*\mathbb{N} \Rightarrow \beta \in S]$.

Axiom of hyperfinite induction 2

Let $F(x)$ be a wff of the set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ restricted on a set $[0,\beta]$ then

$$\left[\forall\beta(\beta \in {}^*\mathbb{N}\backslash\mathbb{N})\left[\bigwedge_{0\leq\alpha<\beta}(F(\alpha) \Rightarrow F(\alpha^+))\right]\right] \Rightarrow \forall\alpha(\alpha \in [0,\beta])F(\alpha). \qquad (11.11)$$

Axiom of hyper infinite induction 2

Let $F(x)$ be unrestricted wff of the set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ then

$$\left[\forall\beta(\beta \in {}^*\mathbb{N})\left[\bigwedge_{0\leq\alpha<\beta}(F(\alpha) \Rightarrow F(\alpha^+))\right]\right] \Rightarrow \forall\beta(\beta \in {}^*\mathbb{N})F(\beta). \qquad (11.12)$$

Remark 11.3.Note that from comprechesion shemata 2 (see subsection 6.1) follows that

$$\forall\beta\exists S(S \subset [0,\beta])\forall\overline{\beta}(\overline{\beta} \in [0,\beta])\left[\overline{\beta} \in S \Leftrightarrow \bigwedge_{0\leq\alpha<\overline{\beta}}(F(\alpha) \Rightarrow F(\alpha^+))\right]. \qquad (11.13)$$

Therefore for any $\overline{\beta} \in [0,\beta]$ from (11.13) it follows that

$$\bigwedge_{0\leq\alpha<\overline{\beta}}(F(\alpha) \Rightarrow F(\alpha^+)) \vdash \overline{\beta} \in S \qquad (11.14)$$

Thus axiom of hyperfinite induction 2,i.e., (11.12) holds, since it follows from (11.14) that $\forall\overline{\beta}[\overline{\beta} \in [0,\beta] \Rightarrow \overline{\beta} \in S]$.

12.Hyper inductive definitions corresponding to Robinson hyperreals ${}^*\mathbb{R}$.

12.1.Hyper inductive definitions corresponding to Robinson hyperreals ${}^*\mathbb{R}$ in general.

A function $f : {}^*\mathbb{N} \to A$ whose domain is the set ${}^*\mathbb{N}$ is colled an hyper infinite sequence and denoted by $\{f_n\}_{n\in{}^*\mathbb{N}}$ or by $\{f(n)\}_{n\in{}^*\mathbb{N}}$The set of all hyper infinite sequences whose terms belong to A is clearly $A^{{}^*\mathbb{N}}$; the set of all hyperfinite sequences of $n \in {}^*\mathbb{N}\backslash\mathbb{N}$ terms in A is A^n. The set of all hyperfinite sequences with terms in A can be defined as

$$\left\{R \subset {}^*\mathbb{N} \times A : (R \text{ is a function}) \wedge \bigvee_{n\in{}^*\mathbb{N}}(D_1(R) = n)\right\}, \qquad (12.1)$$

where $D_1(R)$ is domain of R.This definition implies the existence of the set of all hyperfinite sequences with terms in A.The simplest case is the hyper inductive definition of a hyperinfinite sequence $\{\varphi(n)\}_{n\in{}^*\mathbb{N}}$ (with terms belonging to a certain set Z) satisfying the following conditions:

(**a**)

$$\varphi(0) = z, \varphi(n^+) = e(\varphi(n), n), \tag{12.2}$$

where $z \in Z$ and e is a function mapping $Z \times {}^*\mathbb{N}$ into Z.

More generally, we consider a mapping f of the cartesian product $Z \times {}^*\mathbb{N} \times A$ into Z and seek a function $\varphi \in Z^{{}^*\mathbb{N}\times A}$ satisfying the conditions :

(**b**)

$$\varphi(0,a) = g(a), \varphi(n^+,a) = f(\varphi(n,a),n,a), \tag{12.3}$$

where $g \in Z^A$. This is a definition by induction with parameter a ranging over the set A. Schemes (a) and (b) correspond to induction "from n to $n^+ = n+1$",i.e. $\varphi(n^+)$ or $\varphi(n^+,a)$ depends upon $\varphi(n)$ or $\varphi(n,a)$ respectively. More generally, $\varphi(n^+)$ may depend upon all values $\varphi(m)$ where $m \leq n$ (i.e. $m \in n^+$). In the case of induction with parameter, $\varphi(n^+,a)$ may depend upon all values $\varphi(m,a)$, where $m \leq n$; or even upon all values $\varphi(m,a)$, where $m \leq n^+$ and $b \in A$. In this way we obtain the following schemes of definitions by induction:

(**c**) $\varphi(0) = z, \varphi(n^+) = h(\varphi|n^+, n)$,

(**d**) $\varphi(0,a) = g(a), \quad \varphi(n^+,a) = H(\varphi|(n^+ \times A), n, a)$.

In the scheme (c), $z \in Z$ and $h \in Z^{C\times {}^*\mathbb{N}}$, where C is the set of hyperfinite sequences whose terms belong to Z; in the scheme (d), $g \in Z^A$ and $H \in Z^{T\times {}^*\mathbb{N}\times A}$, where T is the set of functions whose domains are included in ${}^*\mathbb{N} \times A$ and whose values belong to Z.

It is clear that the scheme (d) is the most general of all the schemes considered above.

By coise of functions one obtains from (d) any of the schemes (a)-(d). For example, taking the function defined by $H(c,n,a) = f(c(n,a),n,a)$ for $a \in A, n \in {}^*\mathbb{N}, c \in Z^{{}^*\mathbb{N}\times A}$ as H in (d), one obtain (b). We shall now show that, conversely, the scheme (d) can be obtained from (a). Let g and H be functions belonging to Z^A and $Z^{T\times {}^*\mathbb{N}\times A}$ respectively, and let φ be a function satisfying (d). We shall show that the sequence $\Psi = \{\Psi_n\}_{n\in {}^*\mathbb{N}}$ with $\Psi_n = \varphi|(n^+, A)$ can be defined by (a).Obviously, $\Psi_n \in T$ for every $n \in \mathbb{N}^\#$. The first term of the sequence Ψ is equal to $\varphi|(0^+, A)$, i.e. to the set: $z^* = \{\langle\langle 0,a\rangle, g(a)\rangle | a \in A\}$.The relation between Ψ_n, and Ψ_{n^+} is given by the formula:$\Psi_{n^+} = \Psi_n \cup \varphi|(\{n^+\} \times A)$, where the second component is

$$\{\langle\langle n^+,a\rangle, \varphi(n^+,a)\rangle | a \in A\} = \{\langle n^+,a\rangle,, H(\Psi_n,n,a) | a \in A\}. \tag{12.4}$$

Thus we see that the sequence Ψ can be defined by (a) if we substitute T for Z, z^* for z and let $e(c,n) = c \cup \{\langle n^+,a\rangle, H(c,n,a) | a \in A\}$ for $c \in T$.

Now we shall prove the existence and uniqueness of the function satisfying (a). This theorem shows that we are entitled to use definitions by induction of the type (a). According to the remark made above, this will imply the existence of functions satisfying the formulas (b), (c), and (d). Since the uniqueness of such functions can be proved in the same manner as for (a), we shall use in the sequel definitions by induction of any of the types (a)-(d).

Theorem 12.1. If Z is any set $z \in Z$ and $e \in Z^{Z\times {}^*\mathbb{N}}$, then there exists exactly one hyper infinite sequence φ satisfying formulas (a).

Proof. Uniqueness. Suppose that $\{\varphi_1(n)\}_{n\in {}^*\mathbb{N}}$ and $\{\varphi_2(n)\}_{n\in {}^*\mathbb{N}}$ satisfy (a) and let

$$K = \{n | n \in {}^*\mathbb{N} \wedge \varphi_1(n) = \varphi_2(n)\} \tag{12.5}$$

Then (a) implies that K is hyperinductive. Hence ${}^*\mathbb{N} \subseteqq K$ and therefore $\varphi_1(n) \equiv \varphi_2(n)$.

Existence. Let $\Phi(z,n,t)$ be the formula $e(z,n) = t$ and let $\Psi(w,z,F)$ be the following formula:

$$(F \text{ is a function}) \wedge (D_1(F) = n^+) \wedge (F(0) = z) \wedge \bigwedge_{m \in n}(F(m), m, F(m^+)). \qquad (12.6)$$

In other words, F is a function defined on the set of numbers $\leq n \in {}^*\mathbb{N}$ such that $F(0) = z$ and $F(m^+) = e(F(m), m)$ for all $m < n \in {}^*\mathbb{N}$.

Remark 11.4.We assume now that predicate $\Psi(w, z, F_n)$ is unrestricted on variable n, see Definition 11.4.

We prove by hyper infinite induction that there exists exactly one function F_n such that $\Psi(n, z, F_n)$. The proof of uniqueness of this function is similar to that given in the first part of Theorem 12.1. The existence of F_n can be proved as follows: for $n = 0$ it suffices to take $\{\langle 0, z\rangle\}$ as F_n; if $n \in {}^*\mathbb{N}$ and F_n satisfies $\Psi(n, z, F_n)$, then $F_{n^+} = F_n \cup \{\langle n^+, e(F_n(n), n)\rangle\}$ satisfies the condition $\Psi(n^+, z, F_{n^+})$.

Now, we take as φ the set of pairs $\langle n, s\rangle$ such that $n \in {}^*\mathbb{N}, s \in Z$ and

$$\exists F[\Psi(n, z, F) \wedge (s = F(n))]. \qquad (12.7)$$

Since F is the unique function satisfying $\Psi(n, z, F)$, it follows that φ is a function. For $n = 0$ we have $\varphi(0) = F_0(0) = z$; if $n \in {}^*\mathbb{N}$, then $\varphi(n^+) = F_{n^+}(n^+) = e(F_n(n), n)$ by the definition of F_n; hence we obtain $\varphi(n^+) = e(\varphi(0), n)$. Theorem 12.1 is thus proved.

We frequently define not one but several functions (with the same range Z) by a simultaneous induction:

$\varphi(0) = z, \qquad \psi(0) = t,$

$\varphi(n^+) = f(\varphi(n), \psi(n), n), \; \psi(n^+) = g(\varphi(n), \psi(n), n)$

where $z, t \in Z$ and $f, g \in Z^{Z \times Z \times {}^*\mathbb{N}}$.

This kind of definition can be reduced to the previous one. It suffices to notice that the hypersequence $\vartheta_n = \langle \varphi(n), \psi(n)\rangle$ satisfies the formulas:$\vartheta_0 = \langle z, t\rangle, \vartheta_{n^+} = e(\vartheta_n, n),$where we set

$$e(u, n) = \langle f(K(u), L(u), n), g(K(u), F(w), n)\rangle, \qquad (12.8)$$

and K, L denote functions such that $K(\langle x, y\rangle)$ and $L(\langle x, y\rangle) = y$ respectively. Thus the function ϑ is defined by induction by means of (a). We now define φ and ψ by $\varphi(n) = K(\vartheta_n), \psi(n) = L(\vartheta_n)$.

12.2.Summation of the hyperfinite external ${}^*\mathbb{R}$-valued sequences.

1.Addition operation of Robinson hypernatural numbers.

The function $+(m, n) \triangleq m + n : {}^*\mathbb{N} \times {}^*\mathbb{N} \to {}^*\mathbb{N}$ is defined hyper inductively by $m + 0 = m, m + n^+ = (m + n)^+$.

This definition is obtained from conditions (12.3) by seting

$Z = A = {}^*\mathbb{N}, g(a) = a, f(p, n, a) = p^+, p^+ = p + 1$

This function satisfies all properties of addition such as: for all $m, n, k \in {}^*\mathbb{N}$

(i) $m + 0 = m$ (ii) $m + n = n + m$ (iii) $m + (n + k) = (m + n) + k$.

2.Multiplicattion operation of Robinson hypernatural numbers.

The function $\times(m, n) \triangleq m \times n : {}^*\mathbb{N} \times {}^*\mathbb{N} \to {}^*\mathbb{N}$ is defined by

$m \times 1 = 1, m \times n^+ = m \times n + m$.

(i) $m \times 1 = 1$ (ii) $m \times n = n \times m$ (iii) $m \times (n \times k) = (m \times n) \times k$.

4.Distributivity with respect to multiplication over addition.

$m \times (n + k) = m \times n + m \times k$.

5. Let $Z = A = X^X, g(a) = I_X, f(u,n,a) = u \circ a$ in (b). Then (12.3) takes on the following form

$$\varphi(0,a) = I_X, \varphi(n^+,a) = \varphi(n,a) \circ a. \tag{12.9}$$

The external function $\varphi(n,a)$ is denoted by a^n and is colled n-th iteration of the function a

$$a^0(x) = x, a^{n^+}(x) = a^n(a(x)), x \in X, a \in X^X, n \in {}^*\mathbb{N}. \tag{12.10}$$

6.Let $A = ({}^*\mathbb{N})^{\mathbb{N}^*}, g(a) = a_0, f(u,n,a) = u + a_{n^+}$.Then (12.3) takes on the following form

$$\varphi(0,a) = a_0, \varphi(n^+,a) = \varphi(n,a) + a_{n^+} \tag{12.11}$$

The external function is defined by the Eqs.(12.11) is denoted by

$$Ext\text{-}\sum_{i=0}^{n} a_i \tag{12.12}$$

7.Let $A = ({}^*\mathbb{N})^{\mathbb{N}^*}, g(a) = a_0, f(u,n,a) = u \times a_{n^+}$.Then (12.3) takes on the following form

$$\varphi(0,a) = a_0, \varphi(n^+,a) = \varphi(n,a) \times a_{n^+} \tag{12.13}$$

The external function is defined by the Eqs.(12.13) is denoted by

$$Ext\text{-}\prod_{i=0}^{n} a_i \tag{12.14}$$

8. Similarly we define $\max_{i\leq n}(a_i), \min_{i\leq n}(a_i), n \in \mathbb{N}^*$.

Theorem 12.2. For any hyperfinite ${}^*\mathbb{N}$-valued sequences $\{a_i\}_{i=1}^n,\ \{b_i\}_{i=1}^n, \{c_i\}_{i=1}^n, n \in \mathbb{N}^\#$ the following equalities holds for any $n, k_1, l_1 \in \mathbb{N}^*$:

(1) distributivity

$$b \times \left(Ext\text{-}\sum_{i=0}^{n} a_i\right) = Ext\text{-}\sum_{i=0}^{n} b \times a_i \tag{12.15}$$

(2)

$$Ext\text{-}\sum_{i=0}^{n} a_i \pm Ext\text{-}\sum_{i=0}^{n} b_i = Ext\text{-}\sum_{i=0}^{n}(a_i \pm b_i) \tag{12.16}$$

(3) splitting a sum

$$Ext\text{-}\sum_{i=0}^{n} a_i = Ext\text{-}\sum_{i=0}^{j} a_i + Ext\text{-}\sum_{i=j+1}^{n} a_i \tag{12.17}$$

(4)

$$Ext\text{-}\sum_{i=k_0}^{k_1}\left(Ext\text{-}\sum_{j=l_0}^{l_1} a_{ij}\right) = Ext\text{-}\sum_{j=l_0}^{l_1}\left(Ext\text{-}\sum_{i=k_0}^{k_1} a_{ij}\right) \tag{12.18}$$

(5)

$$\left(Ext\text{-}\sum_{i=0}^{n} a_i\right) \times \left(Ext\text{-}\sum_{j=0}^{n} b_j\right) = Ext\text{-}\sum_{i=0}^{n}\left(Ext\text{-}\sum_{j=0}^{n} a_i \times b_j\right) \tag{12.19}$$

(6)

$$\left(Ext\text{-}\prod_{i=0}^{n} a_i\right) \times \left(Ext\text{-}\prod_{i=0}^{n} b_i\right) = Ext\text{-}\prod_{i=0}^{n} a_i \times b_i \tag{12.20}$$

(7)

$$\left(Ext\text{-}\prod_{i=0}^{n} a_i\right)^m = Ext\text{-}\prod_{i=0}^{n} a_i^m \qquad (12.21)$$

Proof. Imediately from Theorem 11.1 and hyperinfinite induction principle.

9.Let $A = ({}^*\mathbb{Q})^{\mathbb{N}^*}, g(a) = a_0, f(u,n,a) = u + a_{n^+}$. Then (12.3) takes on the following form

$$\varphi(0,a) = a_0, \varphi(n^+,a) = \varphi(n,a) + a_{n^+} \qquad (12.22)$$

The external function is defined by the Eqs.(12.22) is denoted by

$$Ext\text{-}\sum_{i=0}^{n} a_i \qquad (12.23)$$

10.Let $A = ({}^*\mathbb{Q})^{\mathbb{N}^*}, g(a) = a_0, f(u,n,a) = u \times a_{n^+}$. Then (12.3) takes on the following form

$$\varphi(0,a) = a_0, \varphi(n^+,a) = \varphi(n,a) \times a_{n^+} \qquad (12.24)$$

The external function is defined by the Eqs.(12.24) is denoted by

$$Ext\text{-}\prod_{i=0}^{n} a_i \qquad (12.25)$$

11. Similarly we define $\max_{i \leq n}(a_i), \min_{i \leq n}(a_i), n \in \mathbb{N}^*$.

Theorem 12.3. For any ${}^*\mathbb{Q}$-valued hyperfinite sequences $\{a_i\}_{i=1}^{n}, \{b_i\}_{i=1}^{n}, \{c_i\}_{i=1}^{n}$, $n \in \mathbb{N}^*$

the following equalities holds for any $n, k_1, l_1 \in \mathbb{N}^*$:

(1) distributivity

$$b \times \left(Ext\text{-}\sum_{i=0}^{n} a_i\right) = Ext\text{-}\sum_{i=0}^{n} b \times a_i \qquad (12.26)$$

(2)

$$Ext\text{-}\sum_{i=0}^{n} a_i \pm Ext\text{-}\sum_{i=0}^{n} b_i = Ext\text{-}\sum_{i=0}^{n}(a_i \pm b_i) \qquad (12.27)$$

(3) splitting a sum

$$Ext\text{-}\sum_{i=0}^{n} a_i = Ext\text{-}\sum_{i=0}^{j} a_i + Ext\text{-}\sum_{i=j+1}^{n} a_i \qquad (12.28)$$

(4)

$$Ext\text{-}\sum_{i=k_0}^{k_1}\left(Ext\text{-}\sum_{j=l_0}^{l_1} a_{ij}\right) = Ext\text{-}\sum_{j=l_0}^{l_1}\left(Ext\text{-}\sum_{i=k_0}^{k_1} a_{ij}\right) \qquad (12.29)$$

(5)

$$\left(Ext\text{-}\sum_{i=0}^{n} a_i\right) \times \left(Ext\text{-}\sum_{j=0}^{n} b_j\right) = Ext\text{-}\sum_{i=0}^{n}\left(Ext\text{-}\sum_{j=0}^{n} a_i \times b_j\right) \qquad (12.30)$$

(6)

$$\left(Ext\text{-}\prod_{i=0}^{n} a_i\right) \times \left(Ext\text{-}\prod_{i=0}^{n} b_i\right) = Ext\text{-}\prod_{i=0}^{n} a_i \times b_i \qquad (12.31)$$

(7)

$$\left(Ext\text{-}\prod_{i=0}^{n} a_i\right)^m = Ext\text{-}\prod_{i=0}^{n} a_i^m \qquad (12.32)$$

Proof. Imediately from Theorem 12.1 and hyperinfinite induction principle.

12.Let $A = ({}^*\mathbb{R})^{\mathbb{N}^*}, g(a) = a_0, f(u,n,a) = u + a_{n^+}$. Then (12.3) takes on the following form

$$\varphi(0,a) = a_0, \varphi(n^+,a) = \varphi(n,a) + a_{n^+} \quad (12.33)$$

The external function is defined by the Eqs.(12.33) is denoted by

$$Ext\text{-}\sum_{i=0}^{n} a_i \quad (12.34)$$

13.Let $A = ({}^*\mathbb{R})^{\mathbb{N}^*}, g(a) = a_0, f(u,n,a) = u \times a_{n^+}$. Then (7.40) takes on the following form

$$\varphi(0,a) = a_0, \varphi(n^+,a) = \varphi(n,a) \times a_{n^+} \quad (12.35)$$

The external function is defined by the Eqs.(12.35) is denoted by

$$Ext\text{-}\prod_{i=0}^{n} a_i \quad (12.36)$$

14. Similarly we define $\max_{i \leq n}(a_i), \min_{i \leq n}(a_i), n \in \mathbb{N}^*$.

Theorem 12.4. For any ${}^*\mathbb{R}$-valued hyperfinite sequences $\{a_i\}_{i=1}^{n}$, $\{b_i\}_{i=1}^{n}, \{c_i\}_{i=1}^{n}$, $n \in \mathbb{N}^*$

the following equalities holds for any $n, k_1, l_1 \in \mathbb{N}^*$:

(1) distributivity

$$b \times \left(Ext\text{-}\sum_{i=0}^{n} a_i \right) = Ext\text{-}\sum_{i=0}^{n} b \times a_i \quad (12.37)$$

(2)

$$Ext\text{-}\sum_{i=0}^{n} a_i \pm Ext\text{-}\sum_{i=0}^{n} b_i = Ext\text{-}\sum_{i=0}^{n} (a_i \pm b_i) \quad (12.38)$$

(3) splitting a sum

$$Ext\text{-}\sum_{i=0}^{n} a_i = Ext\text{-}\sum_{i=0}^{j} a_i + Ext\text{-}\sum_{i=j+1}^{n} a_i \quad (12.39)$$

(4)

$$Ext\text{-}\sum_{i=k_0}^{k_1} \left(Ext\text{-}\sum_{j=l_0}^{l_1} a_{ij} \right) = Ext\text{-}\sum_{j=l_0}^{l_1} \left(Ext\text{-}\sum_{i=k_0}^{k_1} a_{ij} \right) \quad (12.40)$$

(5)

$$\left(Ext\text{-}\sum_{i=0}^{n} a_i \right) \times \left(Ext\text{-}\sum_{j=0}^{n} b_j \right) = Ext\text{-}\sum_{i=0}^{n} \left(Ext\text{-}\sum_{j=0}^{n} a_i \times b_j \right) \quad (12.41)$$

(6)

$$\left(Ext\text{-}\prod_{i=0}^{n} a_i \right) \times \left(Ext\text{-}\prod_{i=0}^{n} b_i \right) = Ext\text{-}\prod_{i=0}^{n} a_i \times b_i \quad (12.42)$$

(7)

$$\left(Ext\text{-}\prod_{i=0}^{n} a_i \right)^{m} = Ext\text{-}\prod_{i=0}^{n} a_i^m \quad (12.43)$$

Proof. Imediately from Theorem 21.1 and hyper infinite induction principle.

Remark 12.1.Note that in general case

$$Ext\text{-}\sum_{i=0}^{n} a_i \neq Ext\text{-}\sum_{k=0}^{n}(a_{2k} + a_{2k+1}), \qquad (12.44)$$

where $n \in {}^*\mathbb{N}\backslash\mathbb{N}$.

Remark 12.2.We remind that there exists an natural embedding [5]:

$${}^*[\bullet] : \mathbb{R} \hookrightarrow {}^*\mathbb{R}. \qquad (12.45)$$

For any real number $r \in \mathbb{R}$ let $\bar{r}$ denote the constant function with value r in $\mathbb{R}^{\mathbb{N}}$, i.e., $\bar{r}(n) = r,$ for all $n \in \mathbb{N}$. We then have embedding (11.30).We denote ${}^*[\bullet]$-image of $\mathbb{R}$ in ${}^*\mathbb{R}$ by ${}^*[\mathbb{R}] = {}^*\mathbb{R}_{\mathbf{st}}$.

Remark 12.3.We remind that the following statement holds [5].

EXTENSION PRINCIPLE: ${}^*\mathbb{R}$ is a proper extension of $\mathbb{R}$ and ${}^*r \equiv r$ for all $r \in \mathbb{R}$. This means that we identify $\mathbb{R}$ with its $*$-image ${}^*\mathbb{R}_{\mathbf{st}}$ in ${}^*\mathbb{R}$.

Remark 12.4.We remind that [5]:(i) an element $x \in {}^*\mathbb{R}$ is called finite if $|x| < {}^*r$ for some $r > 0,$ (ii) every finite $x \in {}^*\mathbb{R}$ is infinitely close to some (unique) ${}^*r \in {}^*\mathbb{R}_{\mathbf{st}}$ in the sense that $|x - {}^*r|$ is either 0 or positively infinitesimal in ${}^*\mathbb{R}$.This unique ${}^*r$ is called the standard part of x and is denoted by ${}^\circ x$. If ${}^*r \in {}^*\mathbb{R}_{\mathbf{st}}$, then ${}^\circ({}^*r) = r$; if $x,y \in {}^*\mathbb{R}$ are both finite, then

$${}^\circ(x+y) = {}^\circ(x) + {}^\circ(y), {}^\circ(x-y) = {}^\circ(x) - {}^\circ(y). \qquad (12.46)$$

Definition 12.1.Let $\{a_i\}_{i=0}^{\infty}$ be a countable $\mathbb{R}$-valued sequence and let $\{{}^*a_i\}_{i=0}^{\infty}$ be corresponding countable ${}^*\mathbb{R}_{\mathbf{st}}$-valued sequence, where ${}^*a_i = {}^*[a_i]$. A sequence $\{{}^*a_i\}_{i=0}^{\infty}$ converges to standard limit $\bar{a} \in {}^*\mathbb{R}_{\mathbf{st}}$ and abbraviate $\bar{a} = \text{st-}\lim_{i\to\infty}{}^*a_i$ if for every $\epsilon > 0, \epsilon \not\approx 0$ there is an integer $N \in \mathbb{N}$ such that $|{}^*a_i - \bar{a}| < \epsilon$ if $i \geq N$. Note that $\bar{a} = {}^*a,$ where $a = \lim_{i\to\infty} a_i$.

Theorem 12.4. (i) Let $\{a_i\}_{i=0}^{n}$, $n \in \mathbb{N}$ be a countable $\mathbb{R}$-valued sequence such that a limit
$a = \lim_{i\to\infty} a_i, a \in \mathbb{R}$ exists.Then a countable ${}^*\mathbb{R}_{\mathbf{st}}$-valued sequence converges to standard
limit ${}^*a$: ${}^*a = \text{st-}\lim_{i\to\infty}{}^*a_i$.

Proof. (i) Immediately from defininition12.1.

Example 12.1.$\lim_{i\to\infty} \sum_{n=0}^{i} \frac{(-1)^n \pi^{2n+1}}{2^{2n+1}(2n+1)!} = \sin\left(\frac{\pi}{2}\right) = 1$.Then by Theorem 11.4 we get: $\text{st-}\lim_{i\to\infty}{}^*\left(\sum_{n=0}^{i} \frac{(-1)^n \pi^{2n+1}}{2^{2n+1}(2n+1)!}\right) = {}^*1$.

Theorem 12.5. Let $\{a_i\}_{i=0}^{n}$, $n \in \mathbb{N}^{\#}\backslash\mathbb{N}$ be a hyperfinite sequence such that:
(i) ${}^\circ a_i = a_i$ for any $i \leq n$ and (ii) for any $m \leq n$: $Ext\text{-}\sum_{i=0}^{m} a_i < \mu \in {}^*\mathbb{R}_{\mathbf{st}}$, then

$${}^\circ\left(Ext\text{-}\sum_{i=0}^{n} a_i\right) = Ext\text{-}\sum_{i=0}^{n} a_i. \qquad (12.47)$$

Proof. From Eq.(12.46) by the condition (ii) and hyper infinite induction we get

$${}^\circ\left(Ext\text{-}\sum_{i=0}^{n} a_i\right) = Ext\text{-}\sum_{i=0}^{n} {}^\circ a_i. \qquad (12.48)$$

From Eq.(12.48) by the condition (i) we obtain Eq.(12.47).

12.3.Summation of the cauntable ${}^*\mathbb{R}$-valued sequences.

Definition 12.2. Let $\{a_n\}_{n\in\mathbb{N}}$ be ${}^*\mathbb{R}$-valued countable sequence. Let $\{a_n\}_k^m$ be any hyperfinite sequence with $m \in {}^*\mathbb{N}\backslash\mathbb{N}$ and such that $a_n = 0$ if $n \in {}^*\mathbb{N}\backslash\mathbb{N}$. Then we define summation of the countable sequence $\{a_n\}_{n\in\mathbb{N}}$ by the following hyperfinite sum

$$Ext\text{-}\sum_{n=k}^{m} a_n \in {}^*\mathbb{R} \qquad (12.49)$$

and denote such sum by the symbol

$$Ext\text{-}\sum_{n=k}^{\omega} a_n. \qquad (12.50)$$

Remark 12.5. Let $\{a_n\}_{n\in\mathbb{N}}$ be $\mathbb{R}$-valued countable sequence. Note that: (i) for canonical summation we always apply standard notation

$$\sum_{n=k}^{\infty} a_n. \qquad (12.51)$$

(ii) the countable external sum (ω-summ) (12.50) in contrast with countable external sum

(12.51) obviously always exists even if a series (12.51) diverges absolutely i.e., $\sum_{n=k}^{\infty}|a_n| = \infty$.

Example 12.2. The ω-sum $Ext\text{-}\sum_{n=1}^{\omega}\frac{1}{n} \in {}^*\mathbb{R}\backslash\mathbb{R}$ exists by Theorem 12.1, however

$$\sum_{n=1}^{\infty}\frac{1}{n} = \infty. \qquad (12.52)$$

Theorem 12.6. Let $Ext\text{-}\sum_{n=k}^{\omega} a_n = A$ and $Ext\text{-}\sum_{n=k}^{\omega} b_n = B$, where $A, B, C \in {}^*\mathbb{R}$. Then

$$Ext\text{-}\sum_{n=k}^{\omega} C \times a_n = C \times \left(Ext\text{-}\sum_{n=k}^{\omega} a_n\right) \qquad (12.53)$$

and

$$Ext\text{-}\sum_{n=k}^{\omega}(a_n \pm b_n) = A \pm B. \qquad (12.54)$$

Proof. It follows directly from Theorem 12.4.

Theorem 12.7. Let $\{a_i\}_{i=0}^{n}$, $n \in \mathbb{N}$ be a countable $\mathbb{R}$-valued sequence such that a siries $\sum_{i=0}^{\infty} a_i$ converges absolutely. Assum that: $\text{st-}\lim_{m\to\infty}\left(Ext\text{-}\sum_{i=m}^{\omega}|{}^*a_i|\right) = 0$. Then

$$\text{st-}\lim_{m\to\infty}\sum_{i=0}^{m}{}^*a_i = Ext\text{-}\sum_{i=0}^{\omega}{}^*a_i \qquad (12.55)$$

Proof. Note that

$$\left|\sum_{i=0}^{m}{}^*a_i - Ext\text{-}\sum_{i=0}^{\omega}{}^*a_i\right| = \left|Ext\text{-}\sum_{i=m+1}^{\omega}{}^*a_i\right| \le Ext\text{-}\sum_{i=m+1}^{\omega}|{}^*a_i|. \qquad (12.56)$$

From (12.56) we get

$$\text{st-}\lim_{m\to\infty}\left|\sum_{i=0}^{m}{}^*a_i - Ext\text{-}\sum_{i=0}^{\omega}{}^*a_i\right| \le \text{st-}\lim_{m\to\infty}\left(Ext\text{-}\sum_{i=m+1}^{\omega}|{}^*a_i|\right) = 0. \qquad (12.57)$$

Eq.(12.55) follows directly from Eq.(12.57).

Example 12.2. Consider the countable sum

$$S_\omega({}^*r) = Ext\text{-}\sum_{n=0}^{\omega}{}^*r^n, -{}^*1 < {}^*r < {}^*1. \qquad (12.58)$$

it follows from (12.55)

$$S_\omega({}^*r) = {}^*1 + Ext\text{-}\sum_{n=1}^{\omega}{}^*r^n = {}^*1 + {}^*r\sum_{n=0}^{\omega}{}^*r^n = {}^*1 + {}^*rS_\omega({}^*r) \qquad (12.59)$$

Thus

$$S_\omega({}^*r) = \frac{{}^*1}{{}^*1 - {}^*r}. \qquad (12.60)$$

Remark 12.6. Note that

$$S_\omega({}^*r) = Ext\text{-}\sum_{n=0}^{\omega}{}^*r^n = \text{st-}\lim_{m\to\infty}\sum_{n=0}^{m}{}^*r^n \triangleq \sum_{n=0}^{\infty}{}^*r^n \qquad (12.61)$$

since as we know

$$S_\infty(r) = \lim_{n\to\infty}\sum_{n=0}^{n} r^n = \sum_{n=0}^{\infty} r^n = \frac{1}{1-r}. \qquad (12.62)$$

Theorem 12.8. Let $\{a_i\}_{i\in\mathbb{N}}$ be a countable ${}^*\mathbb{R}_{\mathbf{st}}$-valued sequence, i.e. ${}^\circ a_i = a_i$ for any $i \leq n$ and st-$\lim_{m\to\infty}\left(Ext\text{-}\sum_{i=0}^{m} a_i\right) = 0,$ then

$$^\circ\left(Ext\text{-}\sum_{i=0}^{\omega} a_i\right) \equiv Ext\text{-}\sum_{i=0}^{\omega} a_i. \qquad (12.63)$$

Proof. It follows directly from Theorem 12.5 for the case if for any $i \in \mathbb{N}^\#\backslash\mathbb{N} : a_i \equiv 0.$

Theorem 12.9. Let $\{b_i\}_{i=0}^{\infty}$, be a countable $\mathbb{R}$-valued sequence such that a limit $s = \lim_{m\to\infty}\sum_{i=0}^{m} b_i$ exists.Then

$${}^*s \equiv Ext\text{-}\sum_{i=0}^{\omega}{}^*b_i. \qquad (12.64)$$

Proof. It follows directly from Theorem 12.7 and Eq.(12.63).

13.e^e is transcendental number

13.1.e is #-transcendental number

Definition 13.1. Let $g(x) : \mathbb{R} \to \mathbb{R}$ be any real analytic function such that: (i)

$$g_{\mathbb{Q}}(x) = \sum_{n=0}^{\infty} a_n x^n, |x| < r, \forall n[a_n \in \mathbb{Q}], \qquad (13.1)$$

and where (ii) the sequence $\{a_n\}_{n\in\mathbb{N}}$ is recursive.

We will call any function given by Eq.(13.1) constructive $\mathbb{Q}$-analytic function and denoted

such function by $g_{\mathbb{Q}}(x)$.

Definition 13.2. A transcendental number $z \in \mathbb{R}$ is called #-transcendental number over field $\mathbb{Q}$, if there does not exists constructive $\mathbb{Q}$-analytic function $g_{\mathbb{Q}}(x)$ such that $g_{\mathbb{Q}}(z) = 0$, i.e., for every constructive $\mathbb{Q}$-analytic function $g_{\mathbb{Q}}(x)$ the inequality $g_{\mathbb{Q}}(z) \neq 0$ is satisfied.

Definition 13.3.A transcendental number z is called w-transcendental number over field $\mathbb{Q}$,if z is not #-transcendental number over field $\mathbb{Q}$,i.e., there exists an constructive $\mathbb{Q}$-analytic function $g_{\mathbb{Q}}(x)$ such that $g_{\mathbb{Q}}(z) = 0$.

Notation 13.1.We will call for a short any constructive $\mathbb{Q}$-analytic function $g_{\mathbb{Q}}(x)$ simply $\mathbb{Q}$-analytic function.

Example 13.1. Number π is transcendental but number π is not #-transcendental number over field $\mathbb{Q}$ since:(i) function $\sin x$ is a $\mathbb{Q}$-analytic and (ii) $\sin\left(\frac{\pi}{2}\right) = 1$ i.e.,

$$-1 + \frac{\pi}{2} - \frac{\pi^3}{2^3 3!} + \frac{\pi^5}{2^5 5!} - \frac{\pi^7}{2^7 7!} + \ldots + \frac{(-1)^n \pi^{2n+1}}{2^{2n+1}(2n+1)!} + \ldots = 0. \quad (13.2)$$

Note that the sequence $a_n = \frac{(-1)^n \pi^{2n+1}}{2^{2n+1}(2n+1)!}, n = 0,1,2.\ldots.$ obviously is primitive recursive.To prove that e is #-transcendental number we need to show that e is not w-transcendental i.e., there does not exist real $\mathbb{Q}$-analytic function $g_{\mathbb{Q}}(x) = \sum_{n=0}^{\infty} a_n x^n$ with rational coefficients $a_0, a_1, \ldots, a_n, \ldots \in \mathbb{Q}$ such that

$$\sum_{n=0}^{\infty} a_k e^n = 0, \sum_{n=0}^{\infty} |a_k| e^n \neq \infty. \quad (13.3)$$

Suppose that e is w-transcendental, i.e., there exists an $\mathbb{Q}$-analytic function $\breve{g}_{\mathbb{Q}}(x) = \sum_{n=0}^{\infty} \breve{a}_n x^n$, with rational coefficients:

$$\breve{a}_0 = \frac{k_0}{m_0}, \breve{a}_1 = \frac{k_1}{m_1}, \ldots, \breve{a}_n = \frac{k_n}{m_n}, \ldots \in \mathbb{Q}, \left| \breve{a}_0 \right| > 0, \quad (13.4)$$

such that the following equality is satisfied:

$$\sum_{n=0}^{\infty} \breve{a}_n e^n \equiv 0, \sum_{n=0}^{\infty} |\breve{a}_k| e^n \neq \infty. \quad (13.5)$$

In this subsection we obtain an reduction of the equality is given by Eq.(13.5) to equivalent equality given by Eq.(13.15). The main tool of such reduction that external countable sum defined in subsection 12.2 above.
From Eq.(13.5) by Theorem 12.7 one obtains the equality

$$^*\breve{a}_0 + \sum_{n=1}^{\infty} {}^*\breve{a}_n \times {}^*e^n \equiv 0, \quad (13.6)$$

where we abbreviate $\sum_{n=1}^{\infty} {}^*\breve{a}_n \triangleq \text{st-}\lim_{m\to\infty} \sum_{n=1}^{m} {}^*\breve{a}_n$ Note that from Eq.(13.6) by Theorem 12.9 one obtains the equality

$$^*\breve{a}_0 + \left[Ext\text{-}\sum_{n=1}^{\omega} {}^*\breve{a}_n \times {}^*e^n \right]_{/\approx} \equiv 0. \quad (13.7)$$

Theorem 12.1.[4] The equality (13.6) is inconsistent.
Proof.Let $\Im$ be a hypernatural number $\Im \in {}^*\mathbb{N}\backslash\mathbb{N}$ defined by countable sequence

$$\begin{aligned} \Im &= (m_0, m_0 \times m_1, \ldots, m_0 \times m_1 \times \ldots \times m_n, \ldots) = \\ &= (r_0, r_1, \ldots, r_n, \ldots) \end{aligned} \quad (13.8)$$

where $r_n = m_0 \times m_1 \times \ldots \times m_n$. From Eq.(13.7) and Eq.(13.8) one obtains

$$\frac{\Im^{*}\breve{a}_0}{\Im} + \frac{Ext\text{-}\sum_{n=1}^{\omega}\Im^{*}\breve{a}_n \times {}^{*}e^{n}}{\Im} = 0. \quad (13.9)$$

From Eq.(12.9) one obtains

$$\frac{\Im_0}{\Im} + \frac{Ext\text{-}\sum_{n=1}^{\omega}\Im_n \times {}^{*}e^{n}}{\Im} \equiv 0, \quad (13.10)$$

where $\Im_n = \Im \times \breve{a}_n, n = 0,1,2,\ldots$ Note that

$${}^{*}e^{n} = {}^{*}e^{n} = \frac{{}^{*}M_n(\mathbf{n},\mathbf{p})}{{}^{*}M_0(\mathbf{n},\mathbf{p})} + \frac{{}^{*}\varepsilon_n(\mathbf{n},\mathbf{p})}{{}^{*}M_0(\mathbf{n},\mathbf{p})}, \quad (13.11)$$

$n = 1,2,\ldots,k \in \mathbb{N}, \mathbf{n},\mathbf{p} \in {}^{*}\mathbb{N}_{\infty}$, see Appendix A,Eq.(30). From Eq.(13.10) and Eq.(13.11) by Theorem 12.6 we obtain

$$\frac{\Im_0}{\Im} + \frac{Ext\text{-}\sum_{n=1}^{\omega}\Im_n \times {}^{*}e^{n}}{\Im} =$$
$$\frac{\Im_0}{\Im} + Ext\text{-}\sum_{n=1}^{\omega}\left[\frac{\Im_n \times {}^{*}M_n(\mathbf{n},\mathbf{p})}{\Im \times {}^{*}M_0(\mathbf{n},\mathbf{p})} + \frac{\Im_n \times {}^{*}\varepsilon_n(\mathbf{n},\mathbf{p})}{\Im \times {}^{*}M_0(\mathbf{n},\mathbf{p})}\right] = \quad (13.12)$$
$$= \frac{\Im_0}{\Im} + Ext\text{-}\sum_{n=1}^{\omega}\frac{\Im_n \times {}^{*}M_n(\mathbf{n},\mathbf{p})}{\Im \times {}^{*}M_0(\mathbf{n},\mathbf{p})} + Ext\text{-}\sum_{n=1}^{\omega}\frac{\Im_n \times {}^{*}\varepsilon_n(\mathbf{n},\mathbf{p})}{\Im \times {}^{*}M_0(\mathbf{n},\mathbf{p})} \equiv 0.$$

We abbreviate now

$$\Delta(\mathbf{n},\mathbf{p}) = \frac{\Im_0}{\Im} + Ext\text{-}\sum_{n=1}^{\omega}\frac{\Im_n \times {}^{*}M_n(\mathbf{n},\mathbf{p})}{\Im \times {}^{*}M_0(\mathbf{n},\mathbf{p})} =$$
$$\frac{\Im_0 \times {}^{*}M_0(\mathbf{n},\mathbf{p}) + Ext\text{-}\sum_{n=1}^{\omega}\Im_n \times {}^{*}M_n(\mathbf{n},\mathbf{p})}{\Im \times {}^{*}M_0(\mathbf{n},\mathbf{p})} \quad (13.13)$$

and

$$\Upsilon(\mathbf{n},\mathbf{p}) = Ext\text{-}\sum_{n=1}^{\omega}\frac{\Im_n \times {}^{*}\varepsilon_n(\mathbf{n},\mathbf{p})}{\Im \times {}^{*}M_0(\mathbf{n},\mathbf{p})} =$$
$$\frac{Ext\text{-}\sum_{n=1}^{\omega}\Im_n \times {}^{*}\varepsilon_n(\mathbf{n},\mathbf{p})}{\Im \times {}^{*}M_0(\mathbf{n},\mathbf{p})} \quad (13.14)$$

From the Eq.(13.12) and Eq.(13.13)-Eq.(13.14) we get

$$\Delta(\mathbf{n},\mathbf{p}) + \Upsilon(\mathbf{n},\mathbf{p}) \equiv 0. \quad (13.15)$$

Note that

$${}^{*}\varepsilon_n(\mathbf{n},\mathbf{p}) \leq \frac{\mathbf{n}({}^{*}g(\mathbf{n}))\left([{}^{*}a(\mathbf{n})]^{\mathbf{p}-1}\right)}{(\mathbf{p}-1)!}, \quad (13.16)$$

$n = 1,2,\ldots,k \in \mathbb{N}, \mathbf{n},\mathbf{p} \in {}^{*}\mathbb{N}_{\infty}$, see Appendix,Eq.(29). From Eq.(13.14) and (13.16) one obtains

$$|\Upsilon(\mathbf{n},\mathbf{p})| = \left|\frac{Ext\text{-}\sum_{n=1}^{\omega}\Im_n \times {}^*\varepsilon_n(\mathbf{n},\mathbf{p})}{\Im \times {}^*M_0(\mathbf{n},\mathbf{p})}\right| \leq$$
$$\leq \frac{\mathbf{n}({}^*g(\mathbf{n}))\left([{}^*a(\mathbf{n})]^{\mathbf{p}-1}\right)}{(\mathbf{p}-1)!}\left|\frac{Ext\text{-}\sum_{n=1}^{\omega}\Im_n}{\Im \times {}^*M_0(\mathbf{n},\mathbf{p})}\right|. \quad (13.17)$$

Let $\mathbf{p}$ be a hyperfinite prime integer $\mathbf{p} \in {}^*\mathbb{N}\backslash\mathbb{N}$ defined by countable sequence

$$\mathbf{p} = (p_0, p_1, \ldots, p_n, \ldots), \quad (13.18)$$

where any $p_n \in \mathbb{N}$ is a prime integer such that $p_n > r_n$. Notice we willing to choose a sequence $\{p_n\}_{n\in\mathbb{N}}$ such that any inequality $p_n > r_n, n \in \mathbb{N}$ is decidable, i.e.

$$\forall n[\mathbf{Val}(p_n > r_n) = \mathbb{R}], \quad (13.19)$$

since the sequence $\{r_n\}_{n\in\mathbb{N}}$ is recursive.

We willing to choose now hyperfinite prime integer $\mathbf{p}$ in Eq.(13.13) $\mathbf{p} = \widetilde{\mathbf{p}} \in {}^*\mathbb{N}\backslash\mathbb{N}$ such that

$$\widetilde{\mathbf{p}} > \mathbf{max}(|\Im_0|, \mathbf{n}.) \quad (13.20)$$

From the Appendix Eq.(27) it follows

$$\widetilde{\mathbf{p}} \nmid [{}^*M_0(\mathbf{n},\widehat{\mathbf{p}})]. \quad (13.21)$$

From the inequality (13.20) and (13.21) it follows

$$\widetilde{\mathbf{p}} \nmid [{}^*M_0(\mathbf{n},\widetilde{\mathbf{p}})] \times \Im_0. \quad (13.22)$$

From the Appendix A, Eq.(28) one obtains

$$\widetilde{\mathbf{p}} \mid [{}^*M_n(\mathbf{n},\widetilde{\mathbf{p}})], n = 1,2,\ldots. \quad (13.23)$$

From (13.22)-(13.23) we get the inequality

$$\left|\Im_0 \times {}^*M_0(\mathbf{n},\widetilde{\mathbf{p}}) + Ext\text{-}\sum_{n=0}^{\omega}\Im_n \times {}^*M_n(\mathbf{n},\widetilde{\mathbf{p}})\right| \geq 1 \quad (13.24)$$

and therefore from Eq.(13.13) we get

$$|\Delta(\mathbf{n},\widetilde{\mathbf{p}})| \geq \frac{1}{|\Im \times {}^*M_0(\mathbf{n},\widetilde{\mathbf{p}})|}. \quad (13.25)$$

We willing to choose now hyperfinite prime integer $\widetilde{\mathbf{p}}$ in Eq.(13.16) such that in additional the inequality is satisfied

$$\frac{\mathbf{n}({}^*g(\mathbf{n}))\left([{}^*a(\mathbf{n})]^{\widetilde{\mathbf{p}}-1}\right)Ext\text{-}\sum_{n=1}^{\omega}\Im_n}{(\widetilde{\mathbf{p}}-1)!} < 1. \quad (13.26)$$

From Eq.(13.17) and the inequality (13.26) we get

$$|\Upsilon(\mathbf{n},\widetilde{\mathbf{p}})| = \left|\frac{Ext\text{-}\sum_{n=1}^{\omega}\Im_n \times {}^*\varepsilon_n(\mathbf{n},\widetilde{\mathbf{p}})}{\Im \times {}^*M_0(\mathbf{n},\widetilde{\mathbf{p}})}\right| < \frac{1}{|\Im \times {}^*M_0(\mathbf{n},\widetilde{\mathbf{p}})|}. \quad (13.27)$$

From the inequalities (13.25) and (13.27) finally we get the inequality

$$\Delta(\mathbf{n},\widetilde{\mathbf{p}}) + \Upsilon(\mathbf{n},\widetilde{\mathbf{p}}) \neq 0. \quad (13.28)$$

But the inequality (13.28) contradicts with Eq.(13.15).This contradiction completed the proof.

14.Generalized Lindemann-Weierstrass theorem.

Theorem 14.1.Let $f_l(z), l = 1,2,\ldots,$ be a polynomials with coefficients in $\mathbb{Z}$. Assume that

for any $l \in \mathbb{N}$ algebraic numbers over the field $\mathbb{Q}$: $\beta_{1,l},\ldots,\beta_{k_l,l},\ k_l \geq 1, l = 1,2,\ldots$ form a complete set of the roots of $f_l(z)$ such that

$$f_l(z) \in \mathbb{Z}[z], \deg f_l(z) = k_l, l = 1,2,\ldots \tag{14.1}$$

and $a_l \in \mathbb{Q}, a_0 \neq 0, l = 1,2,\ldots,$. We assume now that

$$\sum_{l=1}^{\infty} |a_l| \sum_{k=1}^{k_l} |e^{\beta_{k,l}}| < \infty. \tag{14.2}$$

Then

$$a_0 + \sum_{l=1}^{\infty} a_l \sum_{k=1}^{k_l} e^{\beta_{k,l}} \neq 0. \tag{14.3}$$

Note that from assumption above by Robinson transfer it follows that algebraic numbers

${}^*\beta_{1,l},\ldots,{}^*\beta_{k_l,l},\ k_l \geq 1, l = 1,2,\ldots,$over field ${}^*\mathbb{Q}$ for any $l = 1,2,\ldots,$ form a complete set of

the roots of ${}^*f_l(z)$ such that

$${}^*f_l(z) \in {}^*\mathbb{Z}[z], \deg({}^*f_l(z)) = k_l, l = 1,2,\ldots \,. \tag{14.4}$$

Assumption 14.1. We assume now that there exists an recursive sequence

$$\breve{a}_l = \frac{q_l}{m_l} \in \mathbb{Q}, l = 1,2,\ldots; r = 1,2,\ldots \tag{14.5}$$

and rational number

$$\breve{a}_0 = \frac{q_0}{m_0} \in \mathbb{Q}, \tag{14.6}$$

such that

$$\sum_{l=1}^{\infty} |\breve{a}_l| \sum_{k=1}^{k_l} |e^{\beta_{k,l}}| < \infty. \tag{14.7}$$

and

$$\breve{a}_0 + \sum_{l=1}^{\infty} \breve{a}_l \sum_{k=1}^{k_l} e^{\beta_{k,l}} \equiv 0. \tag{14.8}$$

Assumption 14.2. We assume now that the all roots ${}^*\beta_{1,l},\ldots,{}^*\beta_{k_l,l}, k_l \geq 1, l = 1,2,\ldots$of ${}^*f_l(z)$ are real.

From Eq.(14.8) by Theorem 12.7 one obtains the equality

$${}^*\breve{a}_0 + \left[\sum_{l=1}^{\infty} {}^*\breve{a}_l \sum_{k=1}^{k_l} {}^*e^{{}^*\beta_{k,l}} \right] \equiv 0, \tag{14.9}$$

where we abbreviate

$$\sum_{l=1}^{\infty} {}^*\breve{a}_l \sum_{k=1}^{k_l} {}^*e^{{}^*\beta_{k,l}} \triangleq \text{st-}\lim_{m\to\infty} \sum_{l=1}^{m} {}^*\breve{a}_n \sum_{k=1}^{k_l} {}^*e^{{}^*\beta_{k,l}}.$$

Note that from Eq.(14.9) by Theorem 12.9 one obtains the equality

$$
{}^*\breve{a}_0 + \left[Ext\text{-}\sum_{l=1}^{\omega} {}^*\breve{a}_l \sum_{k=1}^{k_l} {}^*e^{{}^*\beta_{k,l}} \right]_{/\approx} \equiv 0. \quad (14.10)
$$

Theorem 14.1.The equality (14.10) is inconsistent.
Proof.Let us considered hypernatural number $\Im \in {}^*\mathbb{N}_\infty$ defined by countable sequence

$$
\Im = (m_0, m_0 \times m_1, \ldots, m_0 \times m_1 \times \ldots \times m_n, \ldots) = (r_0, r_1, \ldots, r_n, \ldots) \quad (14.11)
$$

where $r_n = m_0 \times m_1 \times \ldots \times m_n$. From Eq.(14.10) and Eq.(14.11) one obtains

$$
\begin{gathered}
\frac{\Im^*\breve{a}_0}{\Im} + \frac{\Im}{\Im} \times \left[Ext\text{-}\sum_{l=1}^{\omega} {}^*\breve{a}_l \sum_{k=1}^{k_l} {}^*e^{{}^*\beta_{k,l}} \right] = \\
= \frac{\Im_0}{\Im} + \frac{Ext\text{-}\sum_{l=1}^{\omega} \Im_l \sum_{k=1}^{k_l} {}^*e^{{}^*\beta_{k,l}}}{\Im} \equiv 0,
\end{gathered} \quad (14.12)
$$

where

$$
\Im_0 = \Im\breve{a}_0 = \frac{\Im q_0}{m_0}, \Im_l = \Im\breve{a}_l = \frac{\Im q_l}{m_l}. \quad (14.13)
$$

Note that

$$
\left[{}^*e^{{}^*\beta_{k,l}} = \frac{{}^*M_{k,l}(\mathbf{N},\mathbf{p}) + {}^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p})}{{}^*M_0(\mathbf{N},\mathbf{p})} \right], \quad (14.14)
$$

where $k = 1, \ldots, {}^*k_l, l = 1, \ldots, \mathbf{r}$, see Apendix C,Eq.(15).From Eq.(14.12) and Eq.(14.14) we get

$$
\begin{gathered}
\frac{\Im_0}{\Im} + \frac{Ext\text{-}\sum_{l=1}^{\omega} \Im_l \sum_{k=1}^{k_l} \left[{}^*M_{k,l}(\mathbf{N},\mathbf{p}) + {}^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p}) \right]}{\Im^*M_0(\mathbf{N},\mathbf{p})} = \\
\frac{\Im_0{}^*M_0(\mathbf{N},\mathbf{p})}{\Im^*M_0(\mathbf{N},\mathbf{p})} + \frac{Ext\text{-}\sum_{l=1}^{\omega} \Im_l \sum_{k=1}^{k_l} {}^*M_{k,l}(\mathbf{N},\mathbf{p})}{\Im^*M_0(\mathbf{N},\mathbf{p})} + \\
+ \frac{Ext\text{-}\sum_{l=1}^{\omega} \Im_l \sum_{k=1}^{k_l} {}^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p})}{\Im^*M_0(\mathbf{N},\mathbf{p})} = \\
\frac{\Im_0{}^*M_0(\mathbf{N},\mathbf{p}) + Ext\text{-}\sum_{l=1}^{\omega} \Im_l \sum_{k=1}^{k_l} {}^*M_{k,l}(\mathbf{N},\mathbf{p})}{\Im^*M_0(\mathbf{N},\mathbf{p})} + \\
+ \frac{Ext\text{-}\sum_{l=1}^{\omega} \Im_l \sum_{k=1}^{k_l} {}^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p})}{\Im^*M_0(\mathbf{N},\mathbf{p})} \equiv 0
\end{gathered} \quad (14.15)
$$

We abbreviate now

$$
\Delta(\mathbf{N},\mathbf{p}) = \frac{\Im_0{}^*M_0(\mathbf{N},\mathbf{p}) + Ext\text{-}\sum_{l=1}^{\omega} \Im_l \sum_{k=1}^{k_l} {}^*M_{k,l}(\mathbf{N},\mathbf{p})}{\Im^*M_0(\mathbf{N},\mathbf{p})} \quad (14.16)
$$

and

$$
\Upsilon(\mathbf{N},\mathbf{p}) = \frac{Ext\text{-}\sum_{l=1}^{\omega} \Im_l \sum_{k=1}^{k_l} {}^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p})}{\Im^*M_0(\mathbf{N},\mathbf{p})}. \quad (14.17)
$$

From Eq.(14.15) and Eq.(14.16)-Eq.(14.17) we get

$$\Delta(\mathbf{N},\mathbf{p})+\Upsilon(\mathbf{N},\mathbf{p})\equiv 0. \qquad (14.18)$$

Note that

$$|{}^{*}\varepsilon_{k,l}(\mathbf{N},\mathbf{p})|\leq\leq\frac{[{}^{*}g_0(\mathbf{r})][{}^{*}g^{\mathbf{p}-1}(\mathbf{r})]}{(\mathbf{p}-1)!}, \qquad (14.19)$$

where $k=1,\ldots,{}^{*}k_l, l=1,\ldots,\mathbf{r},\mathbf{N},\mathbf{p}\in{}^{*}\mathbb{N}_\infty$, see Appendix C,Eq.(12). From Eq.(14.17) and

(14.19) one obtains

$$|\Upsilon(\mathbf{N},\mathbf{p})|=\left|\frac{Ext\text{-}\sum_{l=1}^{\omega}\Im_l\sum_{k=1}^{k_l}{}^{*}\varepsilon_{k,l}(\mathbf{N},\mathbf{p})}{\Im^{*}M_0(\mathbf{N},\mathbf{p})}\right|\leq\frac{Ext\text{-}\sum_{l=1}^{\omega}\Im_l[{}^{*}g_0(\mathbf{r})][{}^{*}g^{\mathbf{p}-1}(\mathbf{r})]}{(\mathbf{p}-1)!} \qquad (14.20)$$

Note that $\forall\epsilon(\epsilon\in{}^{*}\mathbb{R}_+)[\epsilon\approx 0]$, there exists $\mathbf{p}=\mathbf{p}(\epsilon)$

$$\frac{[{}^{*}g_0(\mathbf{r})][{}^{*}g^{\mathbf{p}-1}(\mathbf{r})]}{(\mathbf{p}-1)!}\leq\epsilon. \qquad (14.21)$$

We will choose now infinite prime integer $\mathbf{p}$ in Eq.(3.56) $\mathbf{p}=\widetilde{\mathbf{p}}\in{}^{*}\mathbb{N}\backslash\mathbb{N}$ such that

$$\widetilde{\mathbf{p}}>\max(|a_0|,\mathbf{b}_{\mathbf{N}},|\mathbf{b}_0|,\Im_0). \qquad (14.22)$$

Hence from the Appendix C, Eq.(8) it follows

$$\widetilde{\mathbf{p}}\nmid M_0(\mathbf{N},\widetilde{\mathbf{p}}). \qquad (14.23)$$

From (14.22) and (14.23) one obtains:

$$\widetilde{\mathbf{p}}\nmid M_0(\mathbf{N},\widetilde{\mathbf{p}},r)\times\Im_0. \qquad (14.24)$$

From the Appendix C, Eq.(10) it follows

$$\widetilde{\mathbf{p}}\mid M_{k,l}(\mathbf{N},\widetilde{\mathbf{p}}),k,l=1,2,\ldots. \qquad (14.25)$$

From (14.24)-(14.25) we get the inequality

$$\Im_0{}^{*}M_0(\mathbf{N},\mathbf{p})+Ext\text{-}\sum_{l=1}^{\omega}\Im_l\sum_{k=1}^{k_l}{}^{*}M_{k,l}(\mathbf{N},\mathbf{p})\geq 1 \qquad (14.26)$$

and therefore from Eq.(14.16) we get

$$|\Delta(\mathbf{n},\widetilde{\mathbf{p}})|\geq\frac{1}{|\Im\times{}^{*}M_0(\mathbf{n},\widetilde{\mathbf{p}})|}. \qquad (14.27)$$

We willing to choose now hyperfinite prime integer $\widetilde{\mathbf{p}}$ in Eq.(14.18) such that in additional the inequality is satisfied

$$|\Upsilon(\mathbf{N},\mathbf{p})|<\frac{1}{|\Im\times{}^{*}M_0(\mathbf{n},\widetilde{\mathbf{p}})|}. \qquad (14.28)$$

From the inequalities (14.27) and (14.28) finally we get the inequality

$$\Delta(\mathbf{n},\widetilde{\mathbf{p}})+\Upsilon(\mathbf{n},\widetilde{\mathbf{p}})\neq 0. \qquad (14.29)$$

But the inequality (14.29) contradicts with Eq.(14.18).This contradiction completed the proof.

Conclusion

In this paper intuitionistic set theory $\mathrm{INC}^{\#}_{\infty^{\#}}$ in infinitary set theoretical language is considered. External induction principle in nonstandard intuitionistic arithmetic were derived. Non trivial application in number theory is considered. Main results are:

number e^e is transcendental; (ii) the both numbers $e+\pi$ and $e-\pi$ are irrational [16].

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Appendix A.The basic definitions of the Shidlovsky quantities

In this apendix we remind the basic definitions of the Shidlovsky quantities [15].Let $M_0(n,p), M_k(n,p)$ and $\varepsilon_k(n,p)$ be the Shidlovsky quantities:

$$M_0(n,p) = \int_0^{+\infty}\left[\frac{x^{p-1}[(x-1)\ldots(x-n)]^p e^{-x}}{(p-1)!}\right]dx \neq 0, \tag{1}$$

$$M_k(n,p) = e^k\int_k^{+\infty}\left[\frac{x^{p-1}[(x-1)\ldots(x-n)]^p e^{-x}}{(p-1)!}\right]dx, k = 1,2,\ldots \tag{2}$$

$$\varepsilon_k(n,p) = e^k\int_0^{k}\left[\frac{x^{p-1}[(x-1)\ldots(x-n)]^p e^{-x}}{(p-1)!}\right]dx, k = 1,2,\ldots \tag{3}$$

where $p \in \mathbb{N}$ this is any prime number.Using Eqs.(1)-(3) by simple calculation one obtains:

$$M_k(n,p) + \varepsilon_k(n,p) = e^k M_0(n,p) \neq 0, k = 1,2,\ldots. \tag{4}$$

and consequently

$$e^k = \frac{M_k(n,p)+\varepsilon_k(n,p)}{M_0(n,p)} \quad k = 1,2,\ldots \tag{5}$$

Lemma 3.1.[15]. Let p be a prime number. Then $M_0(n,p) = (-1)^n(n!)^p + p\Theta_1, \Theta_1 \in \mathbb{Z}$.

Proof. ([15], p.128) By simple calculation one obtains the equality

$$x^{p-1}[(x-1)\ldots(x-n)]^p = (-1)^n(n!)^p x^{p-1} + \sum_{\mu=p+1}^{(n+1)\times p} c_{\mu-1}x^{\mu-1}, \quad c_\mu \in \mathbb{Z}, \mu = p,p+1,\ldots,[(n+1)\times p]-1, n > 0, \tag{6}$$

where p is a prime. By using equality $\Gamma(\mu) = \int_0^\infty x^{\mu-1}e^{-x}dx = (\mu-1)!$, where $\mu \in \mathbb{N}$, from Eq.(1) and Eq.(6) one obtains

$$\begin{aligned} M_0(n,p) &= (-1)^n(n!)^p\frac{\Gamma(p)}{(p-1)!} + \sum_{\mu=p+1}^{(n+1)\times p} c_{\mu-1}\frac{\Gamma(\mu)}{(p-1)!} = \\ &= (-1)^n(n!)^p + c_p p + c_{p+1}p(p+1) + \ldots = \\ &= (-1)^n(n!)^p + p\times\Theta_1, \Theta_1 \in \mathbb{Z}. \end{aligned} \tag{7}$$

Thus

$$M_0(n,p) = (-1)^n(n!)^p + p\cdot\Theta_1(n,p), \Theta_1(n,p) \in \mathbb{Z}. \tag{8}$$

Lemma 3.2.[15]. Let p be a prime number. Then $M_k(n,p) = p\cdot\Theta_2(n,p)$, $\Theta_2(n,p) \in \mathbb{Z}$,

$k = 1, 2, \ldots, n$.

Proof.([15], p.128) By subsitution $x = k + u \Rightarrow dx = du$ from Eq.(3.3) one obtains

$$M_k(n,p) = \int_0^{+\infty} \left[\frac{(u+k)^{p-1}[(u+k-1) \times \ldots \times u \times \ldots \times (u+k-n)]^p e^{-u}}{(p-1)!} \right] du \tag{9}$$

$$k = 1, 2, \ldots$$

By using equality

$$(u+k)^{p-1}[(u+k-1) \times \ldots \times u \times \ldots \times (u+k-n)]^p = \sum_{\mu=p+1}^{(n+1)\times p} d_{\mu-1} u^{\mu-1}, \tag{10}$$

$$d_\mu \in \mathbb{Z}, \mu = p, p+1, \ldots, [(n+1) \times p] - 1,$$

and by subsitution Eq.(3.10) into RHS of the Eq.(3.9) one obtains

$$M_k(n,p) = \frac{1}{(p-1)!} \int_0^{+\infty} \sum_{\mu=p+1}^{(n+1)\times p} d_{\mu-1} u^{\mu-1} du = p \bullet \Theta_2(n,p), \tag{11}$$

$$\Theta_2(n,p) \in \mathbb{Z}, k = 1, 2, \ldots \ .$$

Lemma 1.3.[15]. (i) There exists sequences $a(n), n \in \mathbb{N}$ and $g(n), n \in \mathbb{N}$ such that

$$|\varepsilon_k(n,p)| \le \frac{n \bullet g(n) \bullet [a(n)]^{p-1}}{(p-1)!}, \tag{12}$$

where sequences $a(n), n \in \mathbb{N}$ and $g(n), n \in \mathbb{N}$ does not depend on number p. (ii) For any $n \in \mathbb{N} : \varepsilon_k(n,p) \to 0$ if $p \to \infty$.

Proof.([15], p.129) Obviously there exists sequences $a(n), n \in \mathbb{N}$ and $g(n), k \in \mathbb{N}, n \in \mathbb{N}$ such that $a(n), n \in \mathbb{N}$ and $g(n), n \in \mathbb{N}$ does not depend on number p

$$|x(x-1)\ldots(x-n)| < a(n), 0 \le x \le n \tag{13}$$

and

$$|(x-1)\ldots(x-n)e^{-x+k}| < g(n), 0 \le x \le n, k = 1, 2, \ldots, n. \tag{14}$$

Substitution inequalities (13)-(14) into RHS of the Eq.(3) by simple calculation gives

$$\varepsilon_k(n,p) \le g(n) \frac{[a(n)]^{p-1}}{(p-1)!} \int_0^k dx \le \frac{n \bullet g(n) \bullet [a(n)]^{p-1}}{(p-1)!}. \tag{15}$$

Statement (i) follows from (15). Statement (ii) immediately follows from a statement (ii).

Lemma 1.4.[15]. For any $k \le n$ and for any δ such that $0 < \delta < 1$ there exists $p \in \mathbb{N}$ such that

$$\left| e^k - \frac{M_k(n,p)}{M_0(n,p)} \right| < \delta. \tag{16}$$

Proof.From Eq.(1.5) one obtains

$$\left|e^k - \frac{M_k(n,p)}{M_0(n,p)}\right| = \frac{|\varepsilon_k(n,p)|}{M_0(n,p)}. \tag{17}$$

From Eq.(17) by using Lemma 1.3.(ii) one obtains (3.17).

Remark 1.1.We remind now the proof of the transcendence of e following Shidlovsky proof is given in his book [8].

Theorem 1.1. The number e is transcendental.

Proof.([8], pp.126-129) Suppose now that e is an algebraic number; then it satisfies some relation of the form

$$a_0 + \sum_{k=1}^{n} a_k e^k = 0, \tag{18}$$

where $a_0, a_1, \ldots, a_n \in \mathbb{Z}$ integers and where $a_0 > 0.$ Having substituted RHS of the Eq.(3.5) into Eq.(18) one obtains

$$a_0 + \sum_{k=1}^{n} a_k \frac{M_k(n,p) + \varepsilon_k(n,p)}{M_0(n,p)} = a_0 + \sum_{k=1}^{n} a_k \frac{M_k(n,p)}{M_0(n,p)} + \sum_{k=1}^{n} a_k \frac{\varepsilon_k(n,p)}{M_0(n,p)} = 0. \tag{19}$$

From Eq.(19) one obtains

$$a_0 M_0(n,p) + \sum_{k=1}^{n} a_k M_k(n,p) + \sum_{k=1}^{n} a_k \varepsilon_k(n,p) = 0. \tag{20}$$

We rewrite the Eq.(20) for short in the form

$$a_0 M_0(n,p) + \sum_{k=1}^{n} a_k M_k(n,p) + \sum_{k=1}^{n} a_k \varepsilon_k(n,p) =$$
$$= a_0 M_0(n,p) + \Xi(n,p) + \sum_{k=1}^{n} a_k \varepsilon_k(n,p) = 0, \tag{21}$$
$$\Xi(n,p) = \sum_{k=1}^{n} a_k M_k(n,p).$$

We choose now the integers $M_1(n,p), M_2(n,p), \ldots, M_n(n,p)$ such that:

$$\begin{cases} p|M_1(n,p), p|M_2(n,p), \ldots, p|M_n(n,p) \\ \text{where } p > |a_0| \end{cases} \tag{22}$$

and $p \nmid M_0(n,p).$ Note that $p|\ \Xi(n,p).$ Thus one obtains

$$p \nmid \ a_0 M_0(n,p) + \Xi(n,p) \tag{23}$$

and therefore

$$a_0 M_0(n,p) + \Xi(n,p) \in \mathbb{Z}, \tag{24}$$

where $a_0 M_0(n,p) + \Xi(n,p) \neq 0.$ By using Lemma 3.4 for any δ such that $0 < \delta < 1$ we can choose a prime number $p = p(\delta)$ such that:

$$\left|\sum_{k=1}^{n} a_k \varepsilon_k(n,p)\right| < \delta \sum_{k=1}^{n} |a_k| = \epsilon < 1. \tag{25}$$

From (25) and Eq.(21) we obtain

$$a_0 M_0(n,p) + \Xi(n,p) + \epsilon = 0. \tag{26}$$

From (26) and Eq.(24) one obtains the contradiction.This contradiction finalized the proof.

The Robinson transfer of the Shidlovsky quantities $M_0(n,p),M_k(n,p),\varepsilon_k(n,p)$

In this subsection we will replace using Robinson transfer [5], the Shidlovsky quantities $M_0(n,p),M_k(n,p),\varepsilon_k(n,p)$ by corresponding nonstandard quantities ${}^*M_0(\mathbf{n},\mathbf{p}),{}^*M_k(\mathbf{n},\mathbf{p})$, ${}^*\varepsilon_k(\mathbf{n},\mathbf{p})$. The properties of the nonstandard quantities ${}^*M_0(\mathbf{n},\mathbf{p}),{}^*M_k(\mathbf{n},\mathbf{p})$, ${}^*\varepsilon_k(\mathbf{n},\mathbf{p})$ one obtains directly from the properties of the standard quantities $M_0(n,p),M_k(n,p),\varepsilon_k(n,p)$ using Robinson transfer[4],[5].

1.Using Robinson transfer principle [4],[5] from Eq.(8) one obtains directly

$$\begin{gathered} {}^*M_0(\mathbf{n},\mathbf{p}) = (-1)^{\mathbf{n}}(\mathbf{n}!)^{\mathbf{p}} + \mathbf{p}\times {}^*\Theta_1(\mathbf{n},\mathbf{p}), \\ {}^*\Theta_1(\mathbf{n},\mathbf{p}) \in^* \mathbb{Z}_\infty \triangleq {}^*\mathbb{Z}/\mathbb{Z}, \mathbf{n},\mathbf{p} \in {}^*\mathbb{N}_\infty. \\ \mathbb{N}_\infty \triangleq {}^*\mathbb{N}\backslash\mathbb{N}. \end{gathered} \tag{27}$$

From Eq.(11) using Robinson transfer principle one obtains $\forall k(k \in \mathbb{N})$:

$$\begin{gathered} {}^*M_k(\mathbf{n},\mathbf{p}) = \mathbf{p}\times\left({}^*\Theta_2(\mathbf{n},\mathbf{p})\right), \\ {}^*\Theta_2(\mathbf{n},\mathbf{p}) \in {}^*\mathbb{Z}_\infty, k = 1,2,\ldots,k \in \mathbb{N}, \mathbf{n},\mathbf{p} \in {}^*\mathbb{N}_\infty. \end{gathered} \tag{28}$$

Using Robinson transfer principle from inequality (3.15) one obtains $\forall k(k \in \mathbb{N})$:

$$\begin{gathered} {}^*\varepsilon_k(\mathbf{n},\mathbf{p}) \leq \frac{\mathbf{n}\cdot({}^*g(\mathbf{n}))\cdot\left([{}^*a(\mathbf{n})]^{\mathbf{p}-1}\right)}{(\mathbf{p}-1)!}, \\ k = 1,2,\ldots,k \in \mathbb{N}, \mathbf{n},\mathbf{p} \in {}^*\mathbb{N}_\infty. \end{gathered} \tag{29}$$

Using Robinson transfer principle, from Eq.(3.5) one obtains $\forall k(k \in \mathbb{N})$:

$$\begin{cases} {}^*(e^k) = ({}^*e)^k = \dfrac{{}^*M_k(\mathbf{n},\mathbf{p})}{{}^*M_0(\mathbf{n},\mathbf{p})} + \dfrac{{}^*\varepsilon_k(\mathbf{n},\mathbf{p})}{{}^*M_0(\mathbf{n},\mathbf{p})}, \\ k = 1,2,\ldots,k \in \mathbb{N}, \mathbf{n},\mathbf{p} \in {}^*\mathbb{N}_\infty. \end{cases} \tag{30}$$

Lemma 5. Let $\mathbf{n} \in {}^*\mathbb{N}_\infty$, then for any $k \in \mathbb{N}$ and for any $\delta \approx 0, \delta \in {}^*\mathbb{R}_+$ there exists $\mathbf{p} \in {}^*\mathbb{N}_\infty$ such that

$$\left|{}^*e^k - \frac{{}^*M_k(\mathbf{n},\mathbf{p})}{{}^*M_0(\mathbf{n},\mathbf{p})}\right| < \delta. \tag{31}$$

Proof. From Eq.(30) we obtain $\forall k(k \in \mathbb{N})$:

$$\begin{cases} \left|{}^*e^k - \dfrac{{}^*M_k(\mathbf{n},\mathbf{p})}{{}^*M_0(\mathbf{n},\mathbf{p})}\right| = \dfrac{|{}^*\varepsilon_k(\mathbf{n},\mathbf{p})|}{|{}^*M_0(\mathbf{n},\mathbf{p})|}, \\ k \in \mathbb{N}, \mathbf{n},\mathbf{p} \in {}^*\mathbb{N}_\infty. \end{cases} \tag{32}$$

Appendix B.Generalized Shidlovsky quantities

In this apendix we remind the basic definitions of the Shidlovsky quantities,see [15] p.132-134.

Theorem 1.[15] Let $f_l(z), l = 1,2,\ldots,r$ be a polynomials with coefficients in $\mathbb{Z}$. Assume that for any $l = 1,2,\ldots,r$ algebraic numbers over the field $\mathbb{Q}$: $\beta_{1,l},\ldots,\beta_{k_l,l}$,

$k_l \geq 1, l = 1,2,\ldots,r$ form a complete set of the roots of $f_l(z)$ such that

$$f_l(z) \in \mathbb{Z}[z], \deg f_l(z) = k_l, l = 1,2,\ldots,r \tag{1}$$

and $a_l \in \mathbb{Z}, l = 1,2,\ldots,r, a_0 \neq 0.$ We assume now that

$$\sum_{l=1}^{\infty} |a_l| \sum_{k=1}^{k_l} |e^{\beta_{k,l}}| < \infty. \tag{2}$$

Then

$$a_0 + \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} e^{\beta_{k,l}} \neq 0. \tag{3}$$

Let $M_0(N_r,p), M_{k,l}(N_r,p)$ and $\varepsilon_{k,l}(N_r,p)$ be the quantities

$$M_0(N_r,p) = \int_0^{+\infty} \frac{b_{N_r}^{(N_r-1)p-1} z^{p-1} f_r^p(z) e^{-z} dz}{(p-1)!}, \tag{4}$$

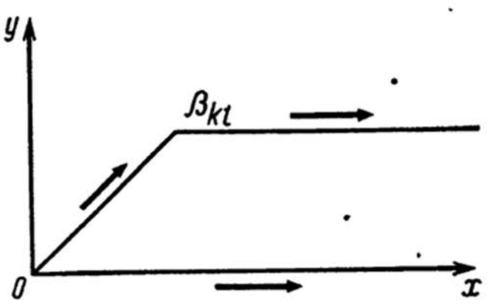
where in (4) we integrate in complex plane $\mathbb{C}$ along line $[0,+\infty]$, see Pic.1.

$$M_{k,l}(N_r,p) = e^{\beta_{k,l}} \int_{\beta_{k,l}}^{+\infty} \frac{b_{N_r}^{(N_r-1)p-1} z^{p-1} f_r^p(z) e^{-z} dz}{(p-1)!}, \tag{5}$$

where $k = 1,\ldots,k_l$ and where in (5) we integrate in complex plane $\mathbb{C}$ along line with initial point $\beta_{k,l} \in \mathbb{C}$ and which are parallel to real axis of the complex plane $\mathbb{C}$, see Pic.1.

$$\varepsilon_{k,l}(N_r,p) = e^{\beta_{k,l}} \int_0^{\beta_{k,l}} \frac{b_{N_r}^{(N_r-1)p-1} z^{p-1} f_r^p(z) e^{-z} dz}{(p-1)!}, \tag{6}$$

where $k = 1,\ldots,k_l$ and where in (6) we integrate in complex plane $\mathbb{C}$ along contour $[0,\beta_{k,l}]$, see Pic.1.



Pic.1.Contour $[0,\beta_{k,l}]$ in complex plane $\mathbb{C}$.

From Eq.(3) one obtains

$$b_{N_r}^{(N_r-1)p-1} z^{p-1} f_r^p(z) = b_{N_r}^{(N_r-1)p-1} b_0^p z^{p-1} + \sum_{s=p+1}^{(N_r+1)p} c_{s-1} z^{s-1}, \tag{7}$$

where $b_{N_r} b_0 \neq 0, c_s \in \mathbb{Z}, s = p,\ldots,(N_r-1)p-1.$ Now from Eq.(4) and Eq.(7) using formula

$$\Gamma(s) = \int_0^{\infty} x^{s-1}e^{-x}dx = (s-1)!, s \in \mathbb{N}$$

one obtains

$$M_0(N_r,p) = \frac{b_{N_r}^{(N_r-1)p-1}b_0^p}{(p-1)!}\int\limits_0^{+\infty} z^{p-1}e^{-z}dz + \sum_{s=p+1}^{(N_r+1)p}\frac{c_{s-1}}{(p-1)!}\int\limits_0^{+\infty} z^{s-1}e^{-z}dz = \\ b_{N_r}^{(N_r-1)p-1}b_0^p + \sum_{s=p+1}^{(N_r-1)p}\frac{(s-1)!}{(p-1)!}c_{s-1} = b_{N_r}^{(N_r-1)p-1}b_0^p + pC, \tag{8}$$

where $b_{N_r}b_0 \neq 0, C \in \mathbb{Z}$. We choose now a prime p such that $p > \max(|a_0|, b_{N_r}, |b_0|)$. Then from Eq.(4.8) follows that

$$p \nmid a_0M_0(N_r,p). \tag{9}$$

From Eq.(4.3) and Eq.(4.5) one obtains

$$M_{k,l}(N_r,p) = \frac{e^{\beta_{k,l}}}{(p-1)!}\int\limits_{\beta_{k,l}}^{+\infty}\left\{b_{N_r}^{N_rp-1}z^{p-1}z^{p-1}\left[\prod_{j=1}^{r}\prod_{i=1}^{k_j}(z-\beta_{i,j})^p\right]\right\}e^{-z+\beta_{k,l}}dz, \tag{10}$$

where $k = 1,\ldots,k_l, l = 1,\ldots,r$. By change of the variable integration $z = u + \beta_{k,l}$ in RHS of the Eq.(10) we obtain

$$M_{k,l}(N_r,p) = \frac{1}{(p-1)!}\int\limits_0^{+\infty}\left\{b_{N_r}^{N_rp-1}(u+\beta_{k,l})^{p-1}u^pe^{-u}\left[\prod_{\substack{j=1\\ j\neq l}}^{r}\prod_{\substack{i=1\\ i\neq k}}^{k_j}(z+\beta_{k,l}-\beta_{i,j})^p\right]\right\}du, \tag{11}$$

where $k = 1,\ldots,k_l, l = 1,\ldots,r$. Let us rewrite now Eq.(11) in the following form

$$M_{k,l}(N_r,p) = \\ \frac{1}{(p-1)!}\int\limits_0^{+\infty}\left\{(b_{N_r}u+b_{N_r}\beta_{k,l})^{p-1}u^pe^{-u}\left[\prod_{\substack{j=1\\ j\neq l}}^{r}\prod_{\substack{r_i=1\\ i\neq k}}^{k_j}(b_{N_r}u+b_{N_r}\beta_{k,l}-b_{N_r}\beta_{i,j})^p\right]\right\}du \tag{12}$$

Let $\mathbb{Z}_{\mathbf{A}}$ be a ring of the all algebraic integers. Note that [8]

$$\alpha_{i,j} = b_{N_r}\beta_{i,j} \in \mathbb{Z}_{\mathbf{A}}, i = 1,\ldots,k_j, j = 1,\ldots,r. \tag{13}$$

Let us rewrite now Eq.(12) in the following form

$$M_{k,l}(N_r,p) = \frac{1}{(p-1)!}\int\limits_0^{+\infty}(b_{N_r}u+\alpha_{k,l})^{p-1}u^pe^{-u}\prod_{\substack{j=1\\ j\neq l}}^{r}\prod_{\substack{i=1\\ i\neq k}}^{k_j}(b_{N_r}u+\alpha_{k,l}-\alpha_{i,j})^pdu \tag{14}$$

where $k = 1,\ldots,k_l, l = 1,\ldots,r$. From Eq.(14) one obtains

$$\sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} M_{k,l}(N_r,p) = \int_0^\infty \frac{u^p e^{-u}\Phi_r(u)}{(p-1)!}du,$$
$$\Phi_r(u) = \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} (b_{N_r}u+\alpha_{k,l})^{p-1}u^p e^{-u} \prod_{\substack{j=1\\ j\neq l}}^{r}\prod_{\substack{i=1\\ i\neq k}}^{k_j}(b_{N_r}u+\alpha_{k,l}-\alpha_{i,j})^p \quad (15)$$

The polynomial $\Phi_r(u)$ is a symmetric polynomial on any system Δ_l of variables $\alpha_{1,l},\alpha_{2,l},\ldots,\alpha_{k_l,l}$, where

$$\Delta_l = \{\alpha_{1,l},\alpha_{2,l},\ldots,\alpha_{k_l,l}\}, l = 1,\ldots,r.$$
$$\alpha_{1,l},\alpha_{2,l},\ldots,\alpha_{k_l,l} \in \mathbb{Z}_{\mathbf{A}}, l = 1,\ldots,r. \quad (16)$$

It well known that $\Phi_r(u) \in \mathbb{Z}[u]$ (see [8] p.134) and therefore

$$u^p\Phi_r(u) = \sum_{s=p+1}^{(N_r+1)p} c_{s-1}u^{s-1}, c_s \in \mathbb{Z}. \quad (17)$$

From Eq.(15) and Eq.(17) one obtains

$$\sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} M_{k,l}(N_r,p) = \int_0^\infty \frac{u^p e^{-u}\Phi_r(u)}{(p-1)!}du =$$
$$\sum_{s=p+1}^{(N_r+1)p} \frac{c_{s-1}}{(p-1)!}\int_0^\infty u^{s-1}e^{-u}du = \sum_{s=p+1}^{(N_r+1)p} c_{s-1}\frac{(s-1)!}{(p-1)!} = pC, C \in \mathbb{Z}. \quad (18)$$

Therefore

$$\Xi(N_r,p) = \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} M_{k,l}(N_r,p) \in \mathbb{Z},$$
$$p|\Xi(N_r,p). \quad (19)$$

Let $O_R \subset \mathbb{C}$ be a circle wth the centre at point $(0,0)$. We assume now that $\forall k\forall l(\beta_{k,l} \in O_R)$. We will designate now

$$g_{k,l}(r) = \max_{|z|\leq R} |b_{N_r}^{-1}f_r(z)e^{-z+\beta_{k,l}}|,$$
$$g_0(r) = \max_{1\leq k\leq k_l, 1\leq l\leq r} g_{k,l}(r), g(r) = \max_{|z|\leq R} |b_{N_r}^{-1}zf_r(z)|. \quad (20)$$

From Eq.(6) and Eq.(20) one obtains

$$|\varepsilon_{k,l}(N_r,p)| = \left|\int_0^{\beta_{k,l}} \frac{b_{N_r}^{(N_r-1)p-1}z^{p-1}f_r^p(z)e^{-z+\beta_{k,l}}dz}{(p-1)!}\right| \leq$$
$$\frac{1}{(p-1)!}\int_0^{\beta_{k,l}} |b_{N_r}^{-1}f(z)e^{-z+\beta_{k,l}}|[|b_{N_r}^{-1}zf_r(z)|]^{p-1}dz \leq \frac{g_0(r)g^{p-1}(r)|\beta_{k,l}|}{(p-1)!} \leq \frac{g_0(r)g^{p-1}(r)R}{(p-1)!}, \quad (21)$$

where $k = 1,\ldots,k_l, l = 1,\ldots,r.$ Note that

$$\frac{g_0(r)g^{p-1}(r)R}{(p-1)!} \to 0 \text{ if } p \to \infty. \tag{22}$$

From (4.22) follows that for any $\epsilon \in [0,\delta]$ there exists a prime number p such that

$$\sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} \varepsilon_{k,l}(N_r,p) = \epsilon(p) < 1. \tag{23}$$

where $k = 1,\ldots,k_l, l = 1,\ldots,r.$ From Eq.(4)-Eq.(6) follows

$$e^{\beta_{k,l}} = \frac{M_{k,l}(N_r,p) + \varepsilon_{k,l}(N_r,p)}{M_0(N_r,p)} \tag{24}$$

where $k = 1,\ldots,k_l, l = 1,\ldots,r.$ Assume now that

$$a_0 + \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} e^{\beta_{k,l}} = 0. \tag{25}$$

Having substituted RHS of the Eq.(24) into Eq.(25) one obtains

$$\begin{gathered} a_0 + \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} \frac{M_{k,l}(N_r,p) + \varepsilon_{k,l}(N_r,p)}{M_0(N,p)} = \\ a_0 + \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} \frac{M_{k,l}(N_r,p)}{M_0(N_r,p)} + \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} \frac{\varepsilon_{k,l}(N_r,p)}{M_0(N_r,p)} = 0. \end{gathered} \tag{26}$$

From Eq.(26) by using Eq.(19) one obtains

$$a_0 + \Xi(N_r,p) + \sum_{l=1}^{r} a_l \sum_{k=1}^{k_l} \varepsilon_{k,l}(N_r,p) = 0. \tag{27}$$

We choose now a prime $p \in \mathbb{N}$ such that $p > \max(|a_0|,|b_0|,|b_{N_r}|)$ and $\epsilon(p) < 1.$ Note that $p|\Xi(N_r,p)$ and therefore from Eq.(19) and Eq.(27) one obtains the contradiction. This contradiction completed the proof.

Appendix.C.The Robinson transfer of the Shidlovsky quantities

Let $f(z) = f_{\mathbf{r}}(z) \in {}^*\mathbb{Z}[z], z \in {}^*\mathbb{C}, l = 1,2,\ldots,\mathbf{r},\mathbf{r} \in {}^*\mathbb{N}_\infty$ be a nonstandard polynomial such that

$$\begin{gathered} f(z) = f_{\mathbf{r}}(z) = \prod_{l=1}^{\mathbf{r}} f_l(z) = \mathbf{b}_0 + \mathbf{b}_1 z + \ldots + \mathbf{b}_{\mathbf{N}} z^{\mathbf{N}} = \\ = \mathbf{b}_{\mathbf{N}} \prod_{l=1}^{\mathbf{r}} \prod_{k=1}^{{}^*k_l} (z - ({}^*\beta_{k,l})), \mathbf{b}_0 \neq 0, \mathbf{b}_{\mathbf{N}} > 0, \\ \mathbf{N} = \mathbf{N}_{\mathbf{r}} = \textstyle\sum_{l=1}^{\mathbf{r}} ({}^*k_l) \in {}^*\mathbb{N}_\infty. \end{gathered} \tag{4}$$

Let ${}^*M_0(\mathbf{N},\mathbf{p}), {}^*M_{k,l}(\mathbf{N},\mathbf{p})$ and ${}^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p})$ be the quantities:

$$^*M_0(\mathbf{N},\mathbf{p}) = \int\limits_0^{^*(+\infty)} \frac{b_{\mathbf{N}}^{(\mathbf{N}-1)\mathbf{p}-1} z^{\mathbf{p}-1} f^{\mathbf{p}}(z)[^*e^{-z}]dz}{(\mathbf{p}-1)!}, \tag{5}$$
$$\mathbf{N},\mathbf{p} \in {}^*\mathbb{N}_\infty,$$

where in (5) we integrate in nonstandard complex plaine $^*\mathbb{C}$ along line $^*[0,+\infty]$, see Pic.1.

$$^*M_{k,l}(\mathbf{N},\mathbf{p}) = (^*e^{^*\beta_{k,l}}) \int\limits_{^*\beta_{k,l}}^{^*(+\infty)} \frac{\mathbf{b}_{\mathbf{N}}^{(\mathbf{N}-1)\mathbf{p}-1} z^{\mathbf{p}-1} f^{\mathbf{p}}(z)[^*e^{-z}]dz}{(\mathbf{p}-1)!}, \tag{6}$$
$$\mathbf{N},\mathbf{p} \in {}^*\mathbb{N}_\infty,$$

where $k = 1,\ldots,{}^*k_l$ and where in (5.6) we integrate in nonstandard complex plain $^*\mathbb{C}$

along line with initial point $^*\beta_{k,l} \in {}^*\mathbb{C}$ and which are parallel to real axis of the complex plane $^*\mathbb{C}$, see Pic.1.

$$^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p}) = (^*e^{^*\beta_{k,l}}) \int\limits_0^{^*\beta_{k,l}} \frac{b_{\mathbf{N}}^{(\mathbf{N}-1)\mathbf{p}-1} z^{\mathbf{p}-1} f^{\mathbf{p}}(z)[^*e^{-z}]dz}{(\mathbf{p}-1)!}, \tag{7}$$
$$\mathbf{N},\mathbf{p} \in {}^*\mathbb{N}_\infty,$$

where $k = 1,\ldots,{}^*k_l$ and where in (5.7) we integrate in nonstandard complex plain $^*\mathbb{C}$ along contour $^*[0,{}^*\beta_{k,l}]$.

1.Using Robinson transfer principle [4],[5],[6] from Eq.(5) and Eq.(8) one obtains directly

$$^*M_0(\mathbf{N},\mathbf{p}) = b_{\mathbf{N}}^{(\mathbf{N}-1)\mathbf{p}-1}\mathbf{b}_0^{\mathbf{p}} + \mathbf{p}\mathbf{C}, \tag{8}$$

where $\mathbf{b}_{\mathbf{N}}\mathbf{b}_0 \neq 0, \mathbf{C} \in {}^*\mathbb{Z}_\infty$. We choose now infinite prime $\mathbf{p} \in {}^*\mathbb{N}_\infty$ such that

$$\mathbf{p} > \max(|\mathbf{a}_0|, \mathbf{b}_{\mathbf{N}}, |\mathbf{b}_0|). \tag{9}$$

2.Using Robinson transfer principle from Eq.(6) and Eq.(19) one obtains directly

$$\forall r(r \in \mathbb{N}) :$$
$$^*\Xi(\mathbf{N},\mathbf{p},r) = \sum_{l=1}^{r}(^*a_l)\sum_{k=1}^{k_l}{}^*M_{k,l}(\mathbf{N},\mathbf{p}) = \mathbf{p}\mathbf{C}_r \in {}^*\mathbb{Z}_\infty. \tag{10}$$

and therefore

$$\forall r(r \in \mathbb{N})[\mathbf{p}|^*\Xi(\mathbf{N},\mathbf{p},r)]. \tag{11}$$

3.Using Robinson transfer principle from Eq.(7) and Eq.(21) one obtains directly

$$|{}^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p})| = \left|({}^*e^{{}^*\beta_{k,l}}) \int_0^{{}^*\beta_{k,l}} \frac{\mathbf{b}_{\mathbf{N}}^{(\mathbf{N}-1)\mathbf{p}-1} z^{\mathbf{p}-1} f^{\mathbf{p}}(z)[{}^*e^{-z}]dz}{(\mathbf{p}-1)!}\right| \leq$$

$$\frac{1}{(\mathbf{p}-1)!} \int_0^{{}^*\beta_{k,l}} |b_{\mathbf{N}}^{-1} f(z)({}^*e^{-z+({}^*\beta_{k,l})})|[|b_{N_r}^{-1} z f(z)|]^{\mathbf{p}-1} dz \leq \frac{[{}^*g_0(\mathbf{r})][{}^*g^{\mathbf{p}-1}(\mathbf{r})]|{}^*\beta_{k,l}|}{(\mathbf{p}-1)!} \tag{12}$$

$$\leq \frac{[{}^*g_0(\mathbf{r})][{}^*g^{\mathbf{p}-1}(\mathbf{r})]}{(\mathbf{p}-1)!},$$

where $k = 1,\ldots,{}^*k_l, l = 1,\ldots,\mathbf{r}$. Note that $\forall \epsilon(\epsilon \in {}^*\mathbb{R})[\epsilon \approx 0]$, there exists $\mathbf{p} = \mathbf{p}(\epsilon)$

$$\frac{[{}^*g_0(\mathbf{r})][{}^*g^{\mathbf{p}-1}(\mathbf{r})]}{(\mathbf{p}-1)!} \leq \epsilon. \tag{13}$$

4. From (13) follows that for any $\epsilon \in [0,\delta]$ there exists an infinite prime $\mathbf{p} \in {}^*\mathbb{N}_\infty$ such that $\forall r(r \in \mathbb{N})$:

$$\sum_{l=1}^{r}({}^*a_l)\sum_{k=1}^{k_l}({}^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p})) = \epsilon(\mathbf{p}) < 1 \tag{14}$$

where $k = 1,\ldots,{}^*k_l, l = 1,\ldots,\mathbf{r}$..

5. From Eq.(5)-Eq.(7) we obtain

$${}^*e^{{}^*\beta_{k,l}} = \frac{{}^*M_{k,l}(\mathbf{N},\mathbf{p}) + ({}^*\varepsilon_{k,l}(\mathbf{N},\mathbf{p}))}{{}^*M_0(\mathbf{N},\mathbf{p})}, \tag{15}$$

where $k = 1,\ldots,{}^*k_l, l = 1,\ldots,\mathbf{r}$.

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