Journal of Advances in Mathematics and Computer Science

26(2): 1-20, 2018; Article no.JAMCS.38773

ISSN: 2456-9968

(Past name: British Journal of Mathematics & Computer Science, Past ISSN: 2231-0851)

There is No Standard Model of ZFC and ZFC$_2$

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Authors' contributions

This work was carried out in collaboration between both authors. Author JF designed the study, carried out the model analysis and wrote the first draft of the manuscript. Author EM wrote Section 3 of the manuscript and managed the literature searches. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/JAMCS/2018/38773

Editor(s):

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Complete Peer review History: http://www.sciencedomain.org/review-history/22940

Received: 4th November 2017

Accepted: 8th January 2018

Published: 30th January 2018

Original Research Article

Abstract

In this paper we view the first order set theory ZFC under the canonical first order semantics and the second order set theory ZFC_2 under the Henkin semantics.
Main results are: (i) Let M_{st}^{ZFC} be a standard model of ZFC, then $\neg Con(ZFC + \exists M_{st}^{ZFC})$.
(ii) Let $M_{st}^{ZFC_2}$ be a standard model of ZFC_2 with Henkin semantics, then $\neg Con(ZFC_2 + \exists M_{st}^{ZFC_2})$.
(iii) Let k be inaccessible cardinal then $\neg Con(ZFC + \exists \kappa)$.
In order to obtain the statements (i) and (ii) examples of the inconsistent countable set in a set theory $ZFC + \exists M_{st}^{ZFC}$ and in a set theory $ZFC_2 + \exists M_{st}^{ZFC_2}$ were derived.
It is widely believed that $ZFC + \exists M_{st}^{ZFC}$ and $ZFC_2 + \exists M_{st}^{ZFC_2}$ are inconsistent, i.e. ZFC and ZFC_2 have a standard models. Unfortunately this belief is wrong.

Keywords: *Gödel encoding; Russell's paradox; standard model; Henkin semantics; inaccessible cardinal.*

2010 Mathematics Subject Classification: 53C25; 83C05; 57N16.

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1 Introduction

1.1 Main results

Let us remind that accordingly to naive set theory, any definable collection is a set. Let R be the set of all sets that are not members of themselves. If R qualifies as a member of itself, it would contradict its own definition as a set containing all sets that are not members of themselves. On the other hand, if such a set is not a member of itself, it would qualify as a member of itself by the same definition. This contradiction is the Russell's paradox. In 1908, two ways of avoiding the paradox were proposed, the Russell's type theory and the Zermelo set theory, the first constructed axiomatic set theory. Zermelo's axioms went well beyond Frege's axioms of extensionality and unlimited set abstraction, and evolved into the now-canonical Zermelo–Fraenkel set theory ZFC. *"But how do we know that ZFC is a consistent theory, free of contradictions? The short answer is that we don't; it is a matter of faith (or of skepticism)"*— E. Nelson wrote in his paper [1]. However, it is deemed unlikely that even ZFC_2 which is significantly stronger than ZFC harbors an unsuspected contradiction; it is widely believed that if ZFC and ZFC_2 were inconsistent, that fact would have been uncovered by now. This much is certain —ZFC and ZFC_2 is immune to the classic paradoxes of naive set theory: the Russell's paradox, the Burali-Forti paradox, and Cantor's paradox.

Remark 1.1.1. Note that in this paper we view

(i) the first order set theory ZFC under the canonical first order semantics,

(ii) the second order set theory ZFC_2 under the Henkin semantics [2], [3], [4], [5], [6].

Remark 1.1.2. Second-order logic essantially differs from the usual first-order predicate calculus in that it has variables and quantifiers not only for individuals but also for subsets of the universe and variables for n-ary relations as well [2], [6]. The deductive calculus $\mathbf{DED_2}$ of second order logic is based on rules and axioms which guarantee that the quantifiers range at least over definable subsets [6]. As to the semantics, there are two tipes of models: (i) Suppose $\mathbf{U}$ is an ordinary first-order structure and $\mathbf{S}$ is a set of subsets of the domain A of $\mathbf{U}$. The main idea is that the set-variables range over $\mathbf{S}$, i.e.

$$\langle \mathbf{U}, \mathbf{S} \rangle \models \exists X \Phi\left(X\right) \iff \exists S \left(S \in \mathbf{S}\right) \left[\langle \mathbf{U}, \mathbf{S} \rangle \models \Phi\left(S\right)\right].$$

We call $\langle \mathbf{U}, \mathbf{S} \rangle$ the Henkin model, if $\langle \mathbf{U}, \mathbf{S} \rangle$ satisfies the axioms of $\mathbf{DED_2}$ and truth in $\langle \mathbf{U}, \mathbf{S} \rangle$ is preserved by the rules of $\mathbf{DED_2}$. We call this semantics of second-order logic the Henkin semantics and second-order logic with the Henkin semantics the Henkin second-order logic. There is a special class of Henkin models, namely those $\langle \mathbf{U}, \mathbf{S} \rangle$ where $\mathbf{S}$ is the set of all subsets of A.

We call these full models. We call this semantics of second-order logic the full semantics and second-order logic with the full semantics the full second-order logic.

Remark 1.1.3. We emphasize that the following facts are the main features of second-order logic:

1.The Completeness Theorem: A sentence is provable in $\mathbf{DED_2}$ if and only if it holds in all Henkin models [2], [6].

2.The Löwenheim-Skolem Theorem: A sentence with an infinite Henkin model has a countable Henkin model.

3.The Compactness Theorem: A set of sentences, every finite subset of which has a Henkin model, has itself a Henkin model.

4. The Incompleteness Theorem: Neither $\mathbf{DED_2}$ nor any other effectively given deductive calculus is complete for full models, that is, there are always sentences which are true in all full models but which are unprovable.

5. Failure of the Compactness Theorem for full models.

6. Failure of the Löwenheim-Skolem Theorem for full models.

7. There is a finite second-order axiom system $\mathbb{Z}_2$ such that the semiring $\mathbb{N}$ of natural numbers is the only full model (up to isomorphism) of $\mathbb{Z}_2$.

8. There is a finite second-order axiom system RCF_2 such that the field $\mathbb{R}$ of real numbers is the only (up to isomorphism) full model of RCF_2.

Remark 1.1.4. For let second-order ZFC be, as usual, the theory that results obtained from ZFC when the axiom schema of replacement is replaced by its second-order universal closure, i.e.

$$\forall X\left[Func\left(X\right) \implies \forall u\exists\nu\forall r\left[r\in\nu \iff \exists s\left(s\in u\wedge\left(s,r\right)\in X\right)\right]\right], \quad (1.1.1)$$

where X is a second-order variable, and where $Func\left(X\right)$ abbreviates " X is a functional relation", see [7].

Designation 1.1.1. We will denote

(i) by ZFC_2^{Hs} set theory ZFC_2 with the Henkin semantics,

(ii) by $\overline{ZFC}_2^{Hs}$ set theory $ZFC_2^{Hs}+\exists M_{st}^{ZFC_2^{Hs}}$, and

(iii) by ZFC_{st} set theory $ZFC+\exists M_{st}^{ZFC}$, where M_{st}^{Th} is a standard model of the theory Th.

Axiom $\exists M^{ZFC}$. [8]. There is a set M^{ZFC} and a binary relation $\epsilon \subseteq M^{ZFC}\times M^{ZFC}$ which makes M^{ZFC} a model for ZFC.

Remark 1.1.5. (i) We emphasize that it is well known that axiom $\exists M^{ZFC}$ a single statement in ZFC see [8], Ch.II, section 7. We denote this statement throught all this paper by symbol $Con\left(ZFC;M^{ZFC}\right)$. The completness theorem says that $\exists M^{ZFC} \iff Con\left(ZFC\right)$.

(ii) Obviously there exists a single statement in ZFC_2^{Hs} such that $\exists M^{ZFC_2^{Hs}} \iff Con\left(ZFC_2^{Hs}\right)$.

We denote this statement throught all this paper by symbol $Con\left(ZFC_2^{Hs};M^{ZFC_2^{Hs}}\right)$ and there exists a single statement $\exists M^{Z_2^{Hs}}$ in Z_2^{Hs}. We denote this statement throught all this paper by symbol $Con\left(Z_2^{Hs};M^{Z_2^{Hs}}\right)$.

Axiom $\exists M_{st}^{ZFC}$. [[8]]. There is a set M_{st}^{ZFC} such that if R is

$$\left\{\langle x,y\rangle \,|x\in y\wedge x\in M_{st}^{ZFC}\wedge y\in M_{st}^{ZFC}\right\},$$

then M_{st}^{ZFC} is a model for ZFC under the relation R.

Definition 1.1.1. [8].The model M_{st}^{ZFC} and $M_{st}^{Z_2^{Hs}}$ is called a standard model since the relation $\in$ used is merely the standard $\in$- relation.

Remark 1.1.6. [8]. Note that axiom $\exists M^{ZFC}$ doesn't imply axiom $\exists M_{st}^{ZFC}$.

Remark 1.1.7. Note that in order to deduce:

(i) $\tilde{}Con(ZFC_2^{Hs})$ from $Con(ZFC_2^{Hs})$, and

(ii) $\tilde{}Con(ZFC)$ from $Con(ZFC)$, by using Gödel encoding, one needs something more than the consistency of ZFC_2^{Hs}, e.g., that ZFC_2^{Hs} has an omega-model $M_{\omega}^{ZFC_2^{Hs}}$ or an standard model $M_{st}^{ZFC_2^{Hs}}$ i.e., a model in which the *integers are the standard integers*. To put it another way, why should we believe a statement just because there's a ZFC_2^{Hs}-proof of it? It's clear that if ZFC_2^{Hs} is inconsistent, then we won't believe ZFC_2^{Hs}-proofs. What's slightly more subtle is that the mere consistency of ZFC_2 isn't quite enough to get us to believe arithmetical theorems of ZFC_2^{Hs}; we must also believe that these arithmetical theorems are asserting something about the standard naturals. It is "conceivable" that ZFC_2^{Hs} might be consistent but that the only nonstandard models $M_{Nst}^{ZFC_2^{Hs}}$ it has are those in which the integers are nonstandard, in which case we might not "believe" an arithmetical statement such as "ZFC_2^{Hs} is inconsistent" even if there is a ZFC_2^{Hs}-proof of it.

2 Derivation of the Inconsistent Definable Set in Set Theory $\overline{ZFC}_2^{Hs}$ and in Set Theory ZFC_{st}

2.1 Derivation of the inconsistent definable set in set theory $\overline{ZFC}_2^{Hs}$

We assume now that $Con\left(Z_2^{Hs};M_{st}^{Z_2^{Hs}}\right)$.

Designation 2.1.1. Let Γ_X^{Hs} be the collection of the all 1-place open wff of the set theory $\overline{ZFC}_2^{Hs}$.

Definition 2.1.1. Let $\Psi_1(X), \Psi_2(X)$ be 1-place open wff's of the set theory $\overline{ZFC}_2^{Hs}$.

(i) We define now the equivalence relation $(\cdot \sim_X \cdot) \subset \Gamma_X^{Hs} \times \Gamma_X^{Hs}$ by

$$Psi_1(X) \sim \Psi_2(X) \iff \forall X [\Psi_1(X) \iff \Psi_2(X)] \tag{2.1.1}$$

(ii) A subset Λ_X^{Hs} of Γ_X^{Hs} such that $\Psi_1(X) \sim \Psi_2(X)$ holds for all $\Psi_1(X)$ and $\Psi_2(X)$ in Λ_X^{Hs}, and never for $\Psi_1(X)$ in Λ_X^{Hs} and $\Psi_2(X)$ outside Λ_X^{Hs}, is called an equivalence class of Γ_X^{Hs}.

(iii) The collection of all possible equivalence classes of Γ_X^{Hs} by $\tilde{}_X$, denoted $\Gamma_X^{Hs}/\sim_X$

$$\Gamma_X^{Hs}/\sim_X \triangleq \left\{[\Psi(X)]_{Hs} | \Psi(X) \in \Gamma_X^{Hs}\right\}. \tag{2.1.2}$$

(iv) For any $\Psi(X) \in \Gamma_X^{Hs}$ let

$$[\Psi(X)]_{Hs} \triangleq \left\{\Phi(X) \in \Gamma_X^{Hs} | \Psi(X) \sim \Phi(X)\right\}$$

denotes the equivalence class to which $\Psi(X)$ belongs. All elements of Γ_X^{Hs} equivalent to each other are also elements of the same equivalence class.

Definition 2.1.2. [9]. Let Th be any theory in the recursive language $L_{Th} \supset L_{PA}$, where L_{PA} is a language of Peano arithmetic.

We say that a number-theoretic relation $R(x_1, ..., x_n)$ of n arguments is expressible in Th if and only if there is a wff $\widehat{R}(x_1, ..., x_n)$ of Th with the free variables $x_1, ..., x_n$ such that, for any natural numbers $k_1, ..., k_n$, the following hold:

(i) If $R(k_1, ..., k_n)$ is true, then $\vdash_{Th} \widehat{R}\left(\overline{k}_1, ..., \overline{k}_n\right)$;

(ii) If $R(k_1, ..., k_n)$ is false, then $\vdash_{Th} \neg\widehat{R}\left(\overline{k}_1, ..., \overline{k}_n\right)$.

Designation 2.1.2. (i) Let $g_{ZFC_2^{Hs}}(u)$ be a Gödel number of given an expression u of the set theory $\overline{ZFC}_2^{Hs} \triangleq ZFC_2^{Hs} + \exists M_{st}^{ZFC_2^{Hs}}$.

(ii) Let $\mathbf{Fr}_2^{Hs}(y, v)$ be the relation : y is the Gödel number of a wff of the set theory $\overline{ZFC}_2^{Hs}$ that contains free occurrences of the variable X with Gödel number v [9].

(iii) Note that the relation $\mathbf{Fr}_2^{Hs}(y, v)$ is expressible in $\overline{ZFC}_2^{Hs}$ by a wff $\widehat{\mathbf{Fr}_2^{Hs}}(y, v)$

(iv) Note that for any $y, v \in \mathbb{N}$ by definition of the relation $\mathbf{Fr}_2^{Hs}(y,v)$ follows that

$$\widehat{\mathbf{Fr}}_2^{Hs}(y,v) \iff \exists!\Psi(X)\left[\left(g_{\overline{ZFC}_2^{Hs}}(\Psi(X)) = y\right) \wedge \left(g_{\overline{ZFC}_2^{Hs}}(X) = \nu\right)\right], \qquad (2.1.3)$$

where $\Psi(X)$ is a unique wff of $\overline{ZFC}_2^{Hs}$ which contains free occurrences of the variable X with Gödel number v. We denote a unique wff $\Psi(X)$ defined by using equivalence (1.2.3) by symbol $\Psi_{y,\nu}(X)$, i.e.

$$\widehat{\mathbf{Fr}}_2^{Hs}(y,v) \iff \exists!\Psi_{y,\nu}(X)\left[\left(g_{\overline{ZFC}_2^{Hs}}(\Psi_{y,\nu}(X)) = y\right) \wedge \left(g_{\overline{ZFC}_2^{Hs}}(X) = \nu\right)\right], \qquad (2.1.4)$$

(v) Let $\wp_2^{Hs}(y, v, \nu_1)$ be a Gödel number of the following wff: $\exists!X[\Psi(X) \wedge Y = X]$, where

$$g_{\overline{ZFC}_2^{Hs}}(\Psi(X)) = y, g_{\overline{ZFC}_2^{Hs}}(X) = \nu,\ g_{\overline{ZFC}_2^{Hs}}(Y) = \nu_1.$$

Definition 2.1.3. Let Γ_X^{Hs} be the countable collection of the all 1-place open wff's of the set theory$\overline{ZFC}_2^{Hs}$ that contains free occurrences of the variable X.

Definition 2.1.4. Let $g_{\overline{ZFC}_2^{Hs}}(X) = \nu$.

Let Γ_ν^{Hs} be a set of the all Gödel numbers of the 1-place open wff's of the set theory$\overline{ZFC}_2^{Hs}$ that contains free occurrences of the variable X with Gödel number v, i.e.

$$\Gamma_\nu^{Hs} = \left\{y \in \mathbb{N} | \langle y, \nu\rangle \in \mathbf{Fr}_2^{Hs}(y,v)\right\} \qquad (2.1.5)$$

or in the following equivalent form:

$$\forall y (y \in \mathbb{N})\left[y \in \Gamma_\nu \iff (y \in \mathbb{N}) \wedge \widehat{\mathbf{Fr}}_2^{Hs}(y,v)\right]. \qquad (2.1.6)$$

Remark 2.1.1. Note that from the axiom of separation it follows directly that Γ_ν^{Hs} is a set in the sense of the set theory $\overline{ZFC}_2^{Hs}$.

Definition 2.1.5. (i) We define now the equivalence relation

$$(\cdot \sim_\nu \cdot) \subset \Gamma_\nu^{Hs} \times \Gamma_\nu^{Hs} \qquad (2.1.7)$$

in the sense of the set theory $\overline{ZFC}_2^{Hs}$ by

$$y_1 \sim_\nu y_2 \iff (\forall X[\Psi_{y_1,\nu}(X) \iff \Psi_{y_2,\nu}(X)]). \qquad (2.1.8)$$

Note that from the axiom of separation it follows directly that the equivalence relation $(\cdot \sim_\nu \cdot)$ is a relation in the sense of the set theory $\overline{ZFC}_2^{Hs}$.

(ii) A subset Λ_ν^{Hs} of Γ_ν^{Hs} such that $y_1 \sim_\nu y_2$ holds for all y_1 and y_1 in Λ_ν^{Hs},and never for y_1 in Λ_ν^{Hs} and y_2 outside Λ_ν^{Hs}, is an equivalence class of Γ_ν^{Hs}.

(iii) For any $y \in \Gamma_\nu^{Hs}$ let $[y]_{Hs} \triangleq \left\{z \in \Gamma_\nu^{Hs} | y \sim_\nu z\right\}$ denote the equivalence class to which y belongs. All elements of Γ_ν^{Hs} equivalent to each other are also elements of the same equivalence class.

(iv) The collection of all possible equivalence classes of Γ_ν^{Hs} by $\tilde{}_\nu$, denoted $\Gamma_\nu^{Hs}/\sim_\nu$

$$\Gamma_\nu^{Hs}/\sim_\nu \triangleq \left\{[y]_{Hs} | y \in \Gamma_\nu^{Hs}\right\}. \qquad (2.1.9)$$

Remark 2.1.2. Note that from the axiom of separation it follows directly that $\Gamma_\nu^{Hs}/\sim_\nu$ is a set in the sense of the set theory $\overline{ZFC}_2^{Hs}$.

Definition 2.1.6. Let $\Im_2^{Hs}$ be the countable collection of the all sets definable by 1-place open wff of the set theory $\overline{ZFC}_2^{Hs}$, i.e.

$$\forall Y \left\{ Y \in \Im_2^{Hs} \iff \exists \Psi(X) \left[\left([\Psi(X)]_{Hs} \in \Gamma_X^{Hs} / \sim_X \right) \wedge [\exists! X [\Psi(X) \wedge Y = X]] \right] \right\}. \quad (2.1.10)$$

Definition 2.1.7. We rewrite now (2.1.10) in the following equivalent form

$$\forall Y \left\{ Y \in \Im_2^{Hs} \iff \exists \Psi(X) \left[\left([\Psi(X)]_{Hs} \in \Gamma_X^{*Hs} / \sim_X \right) \wedge (Y = X) \right] \right\}, \quad (2.1.11)$$

where the countable collection $\Gamma_X^{*Hs} / \sim_X$ is defined by

$$\forall \Psi(X) \left\{ [\Psi(X)] \in \Gamma_X^{*Hs} / \sim_X \iff \left[\left([\Psi(X)] \in \Gamma_X^{Hs} / \sim_X \right) \wedge \exists! X \Psi(X) \right] \right\} \quad (2.1.12)$$

Definition 2.1.8. Let $\Re_2^{Hs}$ be the countable collection of the all sets such that

$$\forall X \left(X \in \Im_2^{Hs} \right) \left[X \in \Re_2^{Hs} \iff X \notin X \right]. \quad (2.1.13)$$

Remark 2.1.3. Note that $\Re_2^{Hs} \in \Im_2^{Hs}$ since $\Re_2^{Hs}$ is a collection definable by 1-place open wff is definable by formula

$$\Psi\left(Z, \Im_2^{Hs}\right) \triangleq \forall X \left(X \in \Im_2^{Hs} \right) [X \in Z \iff X \notin X].$$

From (2.1.13) one obtains

$$\Re_2^{Hs} \in \Re_2^{Hs} \iff \Re_2^{Hs} \notin \Re_2^{Hs}. \quad (2.1.14)$$

But (2.1.14) gives a contradiction

$$\left(\Re_2^{Hs} \in \Re_2^{Hs} \right) \wedge \left(\Re_2^{Hs} \notin \Re_2^{Hs} \right). \quad (2.1.15)$$

However contradiction (2.1.15) it is not a contradiction inside $\overline{ZFC}_2^{Hs}$ for the reason that the countable collection $\Im_2^{Hs}$ is not a set in the sense of the set theory $\overline{ZFC}_2^{Hs}$.

In order to obtain a contradiction inside $\overline{ZFC}_2^{Hs}$ we introduce the following definitions.

Definition 2.1.9. We define now the countable set $\Gamma_\nu^{*Hs} / \sim_\nu$ by

$$\forall y \left\{ [y]_{Hs} \in \Gamma_\nu^{*Hs} / \sim_\nu \iff \left([y]_{Hs} \in \Gamma_\nu^{Hs} / \sim_\nu \right) \wedge \widehat{\mathbf{Fr}}_2^{Hs}(y, v) \wedge [\exists! X \Psi_{y,\nu}(X)] \right\}. \quad (2.1.16)$$

Remark 2.1.4. Note that from the axiom of separation it follows directly that $\Gamma_\nu^* /$ is a set in the sense of the set theory $\overline{ZFC}_2^{Hs}$.

Definition 2.1.10. We define now the countable set $\Im_2^{*Hs}$ by formula

$$\forall Y \left\{ Y \in \Im_2^{*Hs} \iff \exists y \left[\left([y] \in \Gamma_\nu^{*Hs} / \sim_\nu \right) \wedge \left(g_{\overline{ZFC}_2^{Hs}}(X) = \nu \right) \wedge Y = X \right] \right\}. \quad (2.1.17)$$

Note that from the axiom schema of replacement (1.1.1) it follows directly that $\Im_2^{*Hs}$ is a set in the sense of the set theory $\overline{ZFC}_2^{Hs}$.

Definition 2.1.11. We define now the countable set $\Re_2^{*Hs}$ by formula

$$\forall X \left(X \in \Im_2^{*Hs} \right) \left[X \in \Re_2^{*Hs} \iff X \notin X \right]. \quad (2.1.18)$$

Note that from the axiom schema of separation it follows directly that $\Re_2^{*Hs}$ is a set in the sense of the set theory $\overline{ZFC}_2^{Hs}$.

Remark 2.1.5. Note that $\Re_2^{*Hs} \in \Im_2^{*Hs}$ since $\Re_2^{*Hs}$ is definable by the following formula

$$\Psi^*(Z) \triangleq \forall X \left(X \in \Im_2^{*Hs} \right) [X \in Z \iff X \notin X]. \quad (2.1.19)$$

Theorem 2.1.1. Set theory $\overline{ZFC}_2^{Hs}$ is inconsistent.

Proof. From (2.1.18) and Remark 2.1.5 we obtain $\Re_2^{*Hs} \in \Re_2^{*Hs} \iff \Re_2^{*Hs} \notin \Re_2^{*Hs}$ from which immediately one obtains a contradiction

$$\left(\Re_2^{*Hs} \in \Re_2^{*Hs}\right) \wedge \left(\Re_2^{*Hs} \notin \Re_2^{*Hs}\right). \quad (2.1.20)$$

2.2 Derivation of the inconsistent definable set in set theory ZFC_{st}

Designation 2.2.1. (i) Let $g_{ZFC_{st}}(u)$ be a Gödel number of given an expression u of the set theory $ZFC_{st} \triangleq ZFC + \exists M_{st}^{ZFC}$.

(ii) Let $\mathbf{Fr}_{st}(y,v)$ be the relation : y is the Gödel number of a wff of the set theoryZFC_{st} that contains free occurrences of the variable X with Gödel number v [9].

(iii) Note that the relation $\mathbf{Fr}_{st}(y,v)$ is expressible in ZFC_{st} by a wff $\widehat{\mathbf{Fr}}_{st}(y,v)$.

(iv) Note that for any $y, v \in \mathbb{N}$ by definition of the relation $\mathbf{Fr}_{st}(y,v)$ follows that

$$\widehat{\mathbf{Fr}}_{st}(y,v) \iff \exists!\Psi(X)\left[\left(g_{ZFC_{st}}(\Psi(X)) = y\right) \wedge \left(g_{ZFC_{st}}(X) = \nu\right)\right], \quad (2.2.1)$$

where $\Psi(X)$ is a unique wff of ZFC_{st} which contains free occurrences of the variable X with Gödel number v. We denote a unique wff $\Psi(X)$ defined by using equivalence (2.2.1) by symbol $\Psi_{y,\nu}(X)$, i.e.

$$\widehat{\mathbf{Fr}}_{st}(y,v) \iff \exists!\Psi_{y,\nu}(X)\left[\left(g_{ZFC_{st}}(\Psi_{y,\nu}(X)) = y\right) \wedge \left(g_{ZFC_{st}}(X) = \nu\right)\right], \quad (2.2.2)$$

(v) Let $\wp_{st}(y, v, \nu_1)$ be a Gödel number of the following wff: $\exists!X\left[\Psi(X) \wedge Y = X\right]$, where

$$g_{ZFC_{st}}(\Psi(X)) = y, g_{ZFC_{st}}(X) = \nu,\ g_{ZFC_{st}}(Y) = \nu_1$$

(2.6) in section 2, see Remark 2.2 and Designation 2.3**,** (see also [8]-[9]).

Definition 2.2.1. Let Γ_X^{st} be the countable collection of the all 1-place open wff's of the set theory ZFC_{st} that contains free occurrences of the variable X.

Definition 2.2.2. Let $g_{ZFC_{st}}(X) = \nu$. Let Γ_ν^{st} be a set of the all Gödel numbers of the 1-place open wff's of the set theory ZFC_{st} that contains free occurrences of the variable X with Gödel number v, i.e.

$$\Gamma_\nu^{st} = \{y \in \mathbb{N} | \langle y, \nu\rangle \in \mathbf{Fr}_{st}(y,v)\}, \quad (2.2.3)$$

or in the following equivalent form:

$$\forall y\,(y \in \mathbb{N})\left[y \in \Gamma_\nu^{st} \iff (y \in \mathbb{N}) \wedge \widehat{\mathbf{Fr}}_{st}(y,v)\right].$$

Remark 2.2.1. Note that from the axiom of separation it follows directly that Γ_ν^{st} is a set in the sense of the set theory ZFC_{st}.

Definition 2.2.3. (i) We define now the equivalence relation $(\cdot \sim_X \cdot) \subset \Gamma_X^{st} \times \Gamma_X^{st}$ by

$$\Psi_1(X) \sim_X \Psi_2(X) \iff \left(\forall X\left[\Psi_1(X) \iff \Psi_2(X)\right]\right) \quad (2.2.4)$$

(ii) A subcollection Λ_X^{st} of Γ_X^{st} such that $\Psi_1(X) \sim_X \Psi_2(X)$ holds for all $\Psi_1(X)$ and $\Psi_2(X)$ in Λ_X^{st}, and never for $\Psi_1(X)$ in Λ_X^{st} and $\Psi_2(X)$ outside Λ_X^{st}, is an equivalence class of Γ_X^{st}.

(iii) For any $\Psi(X) \in \Gamma_X^{st}$ let $[\Psi(X)]_{st} \triangleq \left\{\Phi(X) \in \Gamma_X^{st} | \Psi(X) \sim_X \Phi(X)\right\}$ denote the equivalence class to which $\Psi(X)$ belongs. All elements of Γ_X^{st} equivalent to each other are also elements of the same equivalence class.

(iv) The collection of all possible equivalence classes of Γ_X^{st} by $\tilde{}_X$, denoted $\Gamma_X^{st}/\sim_X$

$$\Gamma_X^{st}/\sim_X \triangleq \left\{[\Psi(X)]_{st} | \Psi(X) \in \Gamma_X^{st}\right\}. \quad (2.2.5)$$

Definition 2.2.4. (i) We define now the equivalence relation $(\cdot \sim_\nu \cdot) \subset \Gamma_\nu^{st} \times \Gamma_\nu^{st}$ in the sense of the set theory ZFC_{st} by

$$y_1 \sim_\nu y_2 \iff (\forall X\, [\Psi_{y_1,\nu}(X) \iff \Psi_{y_2,\nu}(X)]) \tag{2.2.6}$$

Note that from the axiom of separation it follows directly that the equivalence relation $(\cdot \sim_\nu \cdot)$ is a relation in the sense of the set theory ZFC_{st}.

(ii) A subset Λ_ν^{st} of Γ_ν^{st} such that $y_1 \sim_\nu y_2$ holds for all y_1 and y_1 in Λ_ν^{st},and never for y_1 in Λ_ν^{st} and y_2 outside Λ_ν^{st}, is an equivalence class of Γ_ν^{st}.

(iii) For any $y \in \Gamma_\nu^{st}$ let $[y]_{st} \triangleq \{z \in \Gamma_\nu^{st} | y \sim_\nu z\}$ denote the equivalence class to which y belongs. All elements of Γ_ν^{st} equivalent to each other are also elements of the same equivalence class.

(iv) The collection of all possible equivalence classes of Γ_ν^{st} by $\tilde{}_\nu$, denoted $\Gamma_\nu^{st}/\sim_\nu$

$$\Gamma_\nu^{st}/\sim_\nu \triangleq \left\{[y]_{st} \,|y \in \Gamma_\nu^{st}\right\}. \tag{2.2.7}$$

Remark 2.2.2. Note that from the axiom of separation it follows directly that $\Gamma_\nu^{st}/\sim_\nu$ is a set in the sense of the set theory ZFC_{st}.

Definition 2.2.5. Let $\Im_{st}$ be the countable collection of all sets definable by 1-place open wff of the set theory ZFC_{st}, i.e.

$$\forall Y \left\{Y \in \Im_{st} \iff \exists \Psi(X) \left[\left([\Psi(X)]_{st} \in \Gamma_X^{st}/\sim_X\right) \wedge [\exists! X [\Psi(X) \wedge Y = X]]\right]\right\}. \tag{2.2.8}$$

Definition 2.2.6. We rewrite now (2.2.8) in the following equivalent form

$$\forall Y \left\{Y \in \Im_{st} \iff \exists \Psi(X) \left[\left([\Psi(X)]_{st} \in \Gamma_X^{*st}/\sim_X\right) \wedge (Y = X)\right]\right\}, \tag{2.2.9}$$

where the countable collection $\Gamma_X^{*st}/\sim_X$ is defined by

$$\forall \Psi(X) \left\{[\Psi(X)]_{st} \in \Gamma_X^{*st}/\sim_X \iff \left[\left([\Psi(X)]_{st} \in \Gamma_X^{st}/\sim_X\right) \wedge \exists! X \Psi(X)\right]\right\} \tag{2.2.10}$$

Definition 2.2.7. Let $\Re_{st}$ be the countable collection of the all sets such that

$$\forall X\,(X \in \Im_{st})\,[X \in \Re_{st} \iff X \notin X]. \tag{2.2.11}$$

Remark 2.2.3. Note that $\Re_{st} \in \Im_{st}$ since $\Re_{st}$ is a collection definable by 1-place open wff is definable by formula

$$\Psi(Z, \Im_{st}) \triangleq \forall X\,(X \in \Im_{st})\,[X \in Z \iff X \notin X].$$

From (2.2.11) and Remark 2.2.3 one obtains directly

$$\Re_{st} \in \Re_{st} \iff \Re_{st} \notin \Re_{st}. \tag{2.2.12}$$

But (2.2.12) immediately gives a contradiction

$$(\Re_{st} \in \Re_{st}) \wedge (\Re_{st} \notin \Re_{st}). \tag{2.2.13}$$

However contradiction (2.2.13) it is not a true contradiction inside ZFC_{st} for the reason that the countable collection $\Im_{st}$ is not a set in the sense of the set theory ZFC_{st}.

In order to obtain a true contradiction inside ZFC_{st} we introduce the following definitions.

Definition 2.2.8. We define now the countable set $\Gamma_\nu^{*st}/\sim_\nu$ by formula

$$\forall y \left\{[y]_{st} \in \Gamma_\nu^{*st}/\sim_\nu \iff \left([y]_{st} \in \Gamma_\nu^{st}/\sim_\nu\right) \wedge \widehat{\mathbf{Fr}}_{st}(y, v) \wedge [\exists! X \Psi_{y,\nu}(X)]\right\}. \tag{2.2.14}$$

Remark 2.2.4. Note that from the axiom of separation it follows directly that $\Gamma_\nu^{*st}/\sim_\nu$ is a set in the sense of the set theory ZFC_{st}.

Definition 2.2.9. We define now the countable set $\Im^*_{st}$ by formula

$$\forall Y \left\{ Y \in \Im^*_{st} \iff \exists y \left[\left([y]_{st} \in \Gamma^{*st}_{\nu} / \sim_{\nu} \right) \wedge \left(g_{ZFC_{st}}(X) = \nu \right) \wedge Y = X \right] \right\}. \quad (2.2.15)$$

Note that from the axiom schema of replacement it follows directly that $\Im^*_{st}$ is a set in the sense of the set theory ZFC_{st}.

Definition 2.2.10. We define now the countable set $\Re^*_{st}$ by formula

$$\forall X \left(X \in \Im^*_{st} \right) \left[X \in \Re^*_{st} \iff X \notin X \right]. \quad (2.2.16)$$

Note that from the axiom schema of separation it follows directly that $\Re^*_{st}$ is a set in the sense of the set theory ZFC_{st}.

Remark 2.2.5. Note that $\Re^*_{st} \in \Im^*_{st}$ since $\Re^*_{st}$ is definable by the following formula

$$\Psi^*(Z) \triangleq \forall X \left(X \in \Im^*_{st} \right) \left[X \in Z \iff X \notin X \right]. \quad (2.2.17)$$

Theorem 2.2.1. [10]. Set theory ZFC_{st} is inconsistent.

Proof. From (2.2.17) and Remark 2.2.5 we obtain $\Re^*_{st} \in \Re^*_{st} \iff \Re^*_{st} \notin \Re^*_{st}$ from which immediately one obtains a contradiction

$$\left(\Re^*_{st} \in \Re^*_{st} \right) \wedge \left(\Re^*_{st} \notin \Re^*_{st} \right). \quad (2.2.18)$$

Remark 2.2.6. Theorem 2.2.1 originally was proved in papers [10], [11], [12] by using another essentially complicated approach.

2.3 Derivation of the inconsistent definable set in ZFC_{Nst}

Definition 2.3.1. Let $\overline{PA}$ be a first order theory which contain usual postulates of Peano arithmetic [9] and recursive defining equations for every primitive recursive function as desired. So for any $(n+1)$-place function f defined by primitive recursion over any n-place base function g and $(n+2)$-place iteration function h there would be the defining equations:

(i) $f(0, y_1, ..., y_n) = g(y_1, ..., y_n)$,

(ii) $f(x+1, y_1, ..., y_n) = h(x, f(x, y_1, ..., y_n), y_1, ..., y_n)$.

Designation 2.3.1. (i) Let M^{ZFC}_{Nst} be a nonstandard model of ZFC and let $M^{\overline{PA}}_{st}$ be a standard model of $\overline{PA}$. We assume now that $M^{\overline{PA}}_{st} \subset M^{ZFC}_{Nst}$ and denote such nonstandard model of the set theory ZFC by $M^{ZFC}_{Nst}\left[\overline{PA}\right]$.

(ii) Let ZFC_{Nst} be the theory

$$ZFC_{Nst} = ZFC + M^{ZFC}_{Nst}\left[\overline{PA}\right].$$

Designation 2.3.2. (i) Let $g_{ZFC_{Nst}}(u)$ be a Gödel number of given an expression u of the set theory $ZFC_{Nst} \triangleq ZFC + \exists M^{ZFC}_{Nst}\left[\overline{PA}\right]$.

(ii) Let $\mathbf{Fr}_{Nst}(y, v)$ be the relation : y is the Gödel number of a wff of the set theory ZFC_{Nst} that contains free occurrences of the variable X with Gödel number v [9].

(iii) Note that the relation $\mathbf{Fr}_{Nst}(y, v)$ is expressible in ZFC_{Nst} by a wff $\widehat{\mathbf{Fr}}_{Nst}(y, v)$.

(iv) Note that for any $y, v \in \mathbb{N}$ by definition of the relation $\mathbf{Fr}_{Nst}(y, v)$ follows that

$$\widehat{\mathbf{Fr}}_{Nst}(y, v) \iff \exists! \Psi(X) \left[\left(g_{ZFC_{Nst}}(\Psi(X)) = y \right) \wedge \left(g_{ZFC_{Nst}}(X) = \nu \right) \right], \quad (2.3.1)$$

where $\Psi(X)$ is a unique wff of ZFC_{st} which contains free occurrences of the variable X with Gödel number v. We denote a unique wff $\Psi(X)$ defined by using equivalence (2.3.1) by symbol $\Psi_{y,\nu}(X)$, i.e.

$$\widehat{\mathbf{Fr}}_{Nst}(y,v) \iff \exists!\Psi_{y,\nu}(X)\left[(g_{ZFC_{Nst}}(\Psi_{y,\nu}(X))=y)\wedge(g_{ZFC_{Nst}}(X)=\nu)\right]. \quad (2.3.2)$$

(v) Let $\wp_{Nst}(y,v,\nu_1)$ be a Gödel number of the following wff: $\exists!X\left[\Psi(X)\wedge Y=X\right]$, where

$$g_{ZFC_{Nst}}(\Psi(X))=y, g_{ZFC_{Nst}}(X)=\nu,\ g_{ZFC_{Nst}}(Y)=\nu_1.$$

Definition 2.3.2. Let Γ_X^{Nst} be the countable collection of the all 1-place open wff's of the set theory ZFC_{Nst} that contains free occurrences of the variable X.

Definition 2.3.3. Let $g_{ZFC_{Nst}}(X)=\nu$.Let Γ_ν^{Nst} be a set of the all Gödel numbers of the 1-place open wff's of the set theory ZFC_{Nst} that contains free occurrences of the variable X with Gödel number v, i.e.

$$\Gamma_\nu^{Nst}=\{y\in\mathbb{N}|\ \langle y,\nu\rangle\in\mathbf{Fr}_{Nst}(y,v)\}, \quad (2.3.3)$$

or in the following equivalent form

$$\forall y\,(y\in\mathbb{N})\left[y\in\Gamma_\nu^{Nst}\iff(y\in\mathbb{N})\wedge\widehat{\mathbf{Fr}}_{Nst}(y,v)\right].$$

Remark 2.3.1. Note that from the axiom of separation it follows directly that Γ_ν^{st} is a set in the sense of the set theory ZFC_{Nst}.

Definition 2.3.4. (i) We define now the equivalence relation $(\cdot\sim_X\cdot)\subset\Gamma_X^{Nst}\times\Gamma_X^{Nst}$ by

$$\Psi_1(X)\sim_X\Psi_2(X)\iff(\forall X\left[\Psi_1(X)\iff\Psi_2(X)\right]) \quad (2.3.4)$$

(ii) A subcollection Λ_X^{st} of Γ_X^{st} such that $\Psi_1(X)\sim_X\Psi_2(X)$ holds for all $\Psi_1(X)$ and $\Psi_2(X)$ in Λ_X^{st}, and never for $\Psi_1(X)$ in Λ_X^{Nst} and $\Psi_2(X)$ outside Λ_X^{Nst}, is an equivalence class of Γ_X^{Nst}.

(iii) For any $\Psi(X)\in\Gamma_X^{Nst}$ let

$$[\Psi(X)]_{Nst}\triangleq\left\{\Phi(X)\in\Gamma_X^{Nst}|\Psi(X)\sim_X\Phi(X)\right\}$$

denote the equivalence class to which $\Psi(X)$ belongs. All elements of Γ_X^{st} equivalent to each other are also elements of the same equivalence class.

(iv) The collection of all possible equivalence classes of Γ_X^{Nst} by $\tilde{}_X$, denoted $\Gamma_X^{Nst}/\sim_X$

$$\Gamma_X^{Nst}/\sim_X\triangleq\left\{[\Psi(X)]_{Nst}|\Psi(X)\in\Gamma_X^{Nst}\right\}. \quad (2.3.5)$$

Definition 2.3.5. (i) We define now the equivalence relation $(\cdot\sim_\nu\cdot)\subset\Gamma_\nu^{Nst}\times\Gamma_\nu^{Nst}$ in the sense of the set theory ZFC_{Nst} by

$$y_1\sim_\nu y_2\iff(\forall X\left[\Psi_{y_1,\nu}(X)\iff\Psi_{y_2,\nu}(X)\right]) \quad (2.3.6)$$

Note that from the axiom of separation it follows directly that the equivalence relation $(\cdot\sim_\nu\cdot)$ is a relation in the sense of the set theory ZFC_{Nst}.

(ii) A subset Λ_ν^{Nst} of Γ_ν^{Nst} such that $y_1\sim_\nu y_2$ holds for all y_1 and y_1 in Λ_ν^{Nst}, and never for y_1 in Λ_ν^{Nst} and y_2 outside Λ_ν^{Nst}, is an equivalence class of Γ_ν^{Nst}.

(iii) For any $y\in\Gamma_\nu^{Nst}$ let $[y]_{Nst}\triangleq\left\{z\in\Gamma_\nu^{Nst}|y\sim_\nu z\right\}$ denote the equivalence class to which y belongs. All elements of Γ_ν^{Nst} equivalent to each other are also elements of the same equivalence class.

(iv) The collection of all possible equivalence classes of Γ_ν^{Nst} by $\tilde{}_\nu$, denoted $\Gamma_\nu^{Nst}/\sim_\nu$

$$\Gamma_{\nu}^{Nst}/\sim_{\nu} \triangleq \left\{[y]_{Nst} \,|\, y \in \Gamma_{\nu}^{Nst}\right\}. \quad (2.3.7)$$

Remark 2.3.2. Note that from the axiom of separation it follows directly that $\Gamma_{\nu}^{Nst}/\sim_{\nu}$ is a set in the sense of the set theory ZFC_{Nst}.

Definition 2.3.6. Let $\Im_{Nst}$ be the countable collection of the all sets definable by 1-place open wff of the set theory ZFC_{Nst}, i.e.

$$\forall Y \left\{Y \in \Im_{Nst} \iff \exists \Psi(X)\left[\left([\Psi(X)]_{Nst} \in \Gamma_{X}^{Nst}/\sim_{X}\right) \wedge [\exists! X\,[\Psi(X) \wedge Y = X]]\right]\right\}. \quad (2.3.8)$$

Definition 2.3.7. We rewrite now (2.3.8) in the following equivalent form

$$\forall Y \left\{Y \in \Im_{Nst} \iff \exists \Psi(X)\left[\left([\Psi(X)]_{Nst} \in \Gamma_{X}^{*Nst}/\sim_{X}\right) \wedge (Y = X)\right]\right\}, \quad (2.3.9)$$

where the countable collection $\Gamma_{X}^{*Nst}/\sim_{X}$ is defined by formula

$$\forall \Psi(X)\left\{[\Psi(X)]_{Nst} \in \Gamma_{X}^{*Nst}/\sim_{X} \iff \left[\left([\Psi(X)]_{Nst} \in \Gamma_{X}^{Nst}/\sim_{X}\right) \wedge \exists! X \Psi(X)\right]\right\}. \quad (2.3.10)$$

Definition 2.3.8. Let $\Re_{Nst}$ be the countable collection of the all sets such that

$$\forall X\,(X \in \Im_{Nst})\,[X \in \Re_{Nst} \iff X \notin X]. \quad (2.3.11)$$

Remark 2.3.3. Note that $\Re_{Nst} \in \Im_{Nst}$ since $\Re_{Nst}$ is a collection definable by 1-place open wff is definable by formula

$$\Psi(Z, \Im_{Nst}) \triangleq \forall X\,(X \in \Im_{Nst})\,[X \in Z \iff X \notin X].$$

From (2.3.11) one obtains

$$\Re_{Nst} \in \Re_{Nst} \iff \Re_{Nst} \notin \Re_{Nst}. \quad (2.3.12)$$

But (2.3.12) gives a contradiction

$$(\Re_{Nst} \in \Re_{Nst}) \wedge (\Re_{Nst} \notin \Re_{Nst}). \quad (2.3.13)$$

However a contradiction (2.3.13) it is not a true contradiction inside ZFC_{Nst} for the reason that the countable collection $\Im_{Nst}$ is not a set in the sense of the set theory ZFC_{Nst}.

In order to obtain a true contradiction inside ZFC_{Nst} we introduce the following definitions.

Definition 2.3.9. We define now the countable set $\Gamma_{\nu}^{*Nst}/\sim_{\nu}$ by formula

$$\forall y \left\{[y]_{Nst} \in \Gamma_{\nu}^{*Nst}/\sim_{\nu} \iff \left([y]_{Nst} \in \Gamma_{\nu}^{Nst}/\sim_{\nu}\right) \wedge \widehat{\mathbf{Fr}}_{Nst}(y, v) \wedge [\exists! X \Psi_{y,\nu}(X)]\right\}. \quad (2.3.14)$$

Remark 2.3.4. Note that from the axiom of separation it follows directly that $\Gamma_{\nu}^{*Nst}/\sim_{\nu}$ is a set in the sense of the set theory ZFC_{st}.

Definition 2.3.10. We define now the countable set $\Im_{Nst}^{*}$ by formula

$$\forall Y \left\{Y \in \Im_{Nst}^{*} \iff \exists y\left[\left([y]_{Nst} \in \Gamma_{\nu}^{*Nst}/\sim_{\nu}\right) \wedge (g_{ZFC_{Nst}}(X) = \nu) \wedge Y = X\right]\right\}. \quad (2.3.15)$$

Note that from the axiom schema of replacement it follows directly that $\Im_{st}^{*}$ is a set in the sense of the set theory ZFC_{Nst}.

Definition 2.3.11. We define now the countable set $\Re_{Nst}^{*}$ by formula

$$\forall X\,(X \in \Im_{Nst}^{*})\,[X \in \Re_{Nst}^{*} \iff X \notin X]. \quad (2.3.16)$$

Note that from the axiom schema of separation it follows directly that $\Re_{Nst}^{*}$ is a set in the sense of the set theory ZFC_{Nst}.

Remark 2.3.5. Note that $\Re^*_{Nst} \in \Im^*_{Nst}$ since $\Re^*_{Nst}$ is definable by the following formula

$$\Psi^*(Z) \triangleq \forall X \left(X \in \Im^*_{Nst}\right) [X \in Z \iff X \notin X]. \quad (2.3.17)$$

Theorem 2.3.1. Set theory ZFC_{Nst} is inconsistent.

Proof. From (2.3.16) and Remark 2.3.5 we obtain $\Re^*_{Nst} \in \Re^*_{Nst} \iff \Re^*_{Nst} \notin \Re^*_{Nst}$ from which one obtains a contradiction

$$\left(\Re^*_{Nst} \in \Re^*_{Nst}\right) \wedge \left(\Re^*_{Nst} \notin \Re^*_{Nst}\right). \quad (2.3.18)$$

3 Avoiding the Contradictions from Set Theory $\overline{ZFC}_2^{Hs}$ and Set Theory ZFC_{st} Using Quinean Approach

In order to avoid difficulties mentioned above we use well known Quinean approach [13].

3.1 Quinean set theory NF

Remind that the primitive predicates of Russellian unramified typed set theory (TST), a streamlined version of the theory of types, are equality $=$ and membership $\in$. TST has a linear hierarchy of types: type 0 consists of individuals otherwise undescribed. For each (meta-) natural number n, type $n+1$ objects are sets of type n objects; sets of type n have members of type $n-1$. Objects connected by identity must have the same type. The following two atomic formulas succinctly describe the typing rules: $x^n = y^n$ and $x^n \in y^{n+1}$.

The axioms of TST are:

Extensionality: sets of the same (positive) type with the same members are equal.

Axiom schema of comprehension:

If $\Phi(x^n)$ is a formula, then the set $\{x^n \mid \Phi(x^n)\}^{n+1}$ exists i.e., given any formula $\Phi(x^n)$, the formula

$$\exists A^{n+1} \forall x^n [x^n \in A^{n+1} \leftrightarrow \Phi(x^n)] \quad (3.1.1)$$

is an axiom where A^{n+1} represents the set $\{x^n \mid \Phi(x^n)\}^{n+1}$ and is not free in $\Phi(x^n)$.

Quinean set theory [13] (New Foundations) seeks to eliminate the need for such superscripts.

New Foundations has a universal set, so it is a non-well founded set theory. That is to say, it is a logical theory that allows infinite descending chains of membership such as $\ldots\ x_n \in x_{n-1} \in \ldots x_3 \in x_2 \in x_1$. It avoids Russell's paradox by only allowing stratifiable formulae in the axiom of comprehension. For instance $x \in y$ is a stratifiable formula, but $x \in x$ is not (for details of how this works see below).

Definition 3.1.1. In New Foundations (NF) and related set theories, a formula Φ in the language of first-order logic with equality and membership is said to be stratified if and only if there is a function $f(x)$ which sends each variable appearing in Φ [considered as an item of syntax] to a natural number (this works equally well if all integers are used) in such a way that any atomic formula $x \in y$ appearing in Φ satisfies $f(x) + 1 = f(y)$ and any atomic formula $x = y$ appearing in Φ satisfies $f(x) = f(y)$.

Quinean set theory.

Axioms and stratification are:

the well-formed formulas of New Foundations (NF) are the same as the well-formed formulas of TST, but with the type annotations erased. The axioms of NF are.

Extensionality: two objects with the same elements are the same object.

A comprehension schema: all instances of TST Comprehension but with type indices dropped (and without introducing new identifications between variables).

By convention, NF's Comprehension schema is stated using the concept of stratified formula and making no direct reference to types. Comprehension then becomes.

Axiom schema of comprehension:

$\{x \mid \Phi^s\}$ exists for each stratified formula Φ^s.

Even the indirect reference to types implicit in the notion of stratification can be eliminated. Theodore Hailperin showed in 1944 that Comprehension is equivalent to a finite conjunction of its instances, [14] so that NF can be finitely axiomatized without any reference to the notion of type. Comprehension may seem to run afoul of problems similar to those in naive set theory, but this is not the case. For example, the existence of the impossible Russell class $\{x \mid x \notin x\}$ is not an axiom of NF, because $x \notin x$ cannot be stratified.

3.2 Set theory $\overline{ZFC_2}^{Hs}, ZFC_{st}$ and set theory ZFC_{Nst} with stratified axiom schema of replacement

The stratified axiom schema of replacement asserts that the image of a set under any function definable by stratified formula of the theory ZFC_{st} will also fall inside a set.

Stratified Axiom schema of replacement.

Let $\Phi^s(x, y, w_1, w_2, \ldots, w_n)$ be any stratified formula in the language of ZFC_{st} whose free variables are among $x, y, A, w_1, w_2, \ldots, w_n$, so that in particular B is not free in Φ^s. Then

$$\begin{gathered}\forall A \forall w_1 \forall w_2 ... \forall w_n \left[\forall x \left(x \in A \implies \exists! y \Phi^s(x, y, w_1, w_2, \ldots, w_n)\right) \implies \right. \\ \left. \implies \exists B \forall x \left(x \in A \implies \exists y \left(y \in B \wedge \Phi^s(x, y, w_1, w_2, \ldots, w_n)\right)\right)\right],\end{gathered} \tag{3.2.1}$$

i.e., if the relation $\Phi^s(x, y, ...)$ represents a definable function f, A represents its domain, and $f(x)$ is a set for every $x \in A$, then the range of f is a subset of some set B.

Stratified Axiom schema of separation.

Let $\Phi^s(x, w_1, w_2, \ldots, w_n)$ be any stratified formula in the language of ZFC_{st} whose free variables are among $x, A, w_1, w_2, \ldots, w_n$, so that in particular B is not free in Φ^s. Then

$$\forall w_1 \forall w_2 ... \forall w_n \forall A \exists B \forall x \left[x \in B \iff \left(x \in A \wedge \Phi^s(x, w_1, w_2, \ldots, w_n)\right)\right], \tag{3.2.2}$$

Remark 3.2.1. Notice that the stratified axiom schema of separation follows from the stratified axiom schema of replacement together with the axiom of empty set.

Remark 3.2.2. Notice that the stratified axiom schema of replacement (separation) obviously violeted any contradictions (2.1.20), (2.2.18) and (2.3.18) mentioned above. The existence of the countable Russell sets $\mathfrak{R}_2^{*Hs}, \mathfrak{R}_{st}^*$ and $\mathfrak{R}_{Nst}^*$ impossible, because $x \notin x$ cannot be stratified.

4 Second-order Set Theory ZFC_2 with the Full Second-order Semantics

4.1 Second order set theory ZFC_2 with urlogic

Remind that urlogic has the following characteristics [6].

1. Sentences of urlogic are finite strings of symbols. That a string of symbols is a sentence of urlogic, is a non-mathematical judgement.

2. Some sentences are accepted as axioms. That a sentence is an axiom, is a non-mathematical judgement.

3. Derivations are made from axioms. The derivations obey certain rules of proof. That a derivation obeys the rules of proof, is a non-mathematical judgement.

4. Derived sentences can be asserted as facts.

Remark 4.1.1. Let ZFC_2^{Ul} be second order set theory ZFC_2 with urlogic. Note that in ZFC_2^{Ul} by using the rules of $\mathbf{DED}_2$ we dealing without any reference to semantics, i.e. satisfiability in some standard model, validity etc.

Definition 4.1.1. Let Γ_X^{Ul} be the countable collection of the all 1-place open wff's of the set theory ZFC_2^{Ul} that contains free occurrences of the variable X.

Let $\Psi_1(X), \Psi_2(X)$ be 1-place open wff's of the set theory ZFC_2^{Ul}. We define now the equivalence relation $(\cdot \sim_X \cdot) \subset \Gamma_X^{Ul} \times \Gamma_X^{Ul}$ by

$$\Psi_1(X) \sim_X \Psi_2(X) \iff \forall X [\Psi_1(X) \iff \Psi_2(X)] \tag{4.1.1}$$

For any $\Psi(X) \in \Gamma_X^{Ul}$ let $[\Psi(X)]_{Ul} \triangleq \left\{\Phi(X) \in \Gamma_X^{Ul} | \Psi(X) \sim \Phi(X)\right\}$ denote the equivalence class to which $\Psi(X)$ belongs. All elements of Γ_X^{Ul} equivalent to each other are also elements of the same equivalence class. The collection of all possible equivalence classes of Γ_X^{Ul} by $\tilde{}_X$, denoted $\Gamma_X^{Ul}/\sim_X$

$$\Gamma_X^{Ul}/\sim_X \triangleq \left\{[\Psi(X)]_{Ul} | \Psi(X) \in \Gamma_X^{Ul}\right\}. \tag{4.1.2}$$

Let $\mathbf{Fr}_2^{Ul}(y, v)$ be the relation : y is the Gödel number of a wff of the set theoryZFC_2^{Ul} that contains free occurrences of the variable X with Gödel number v [9].

Note that the relation $\mathbf{Fr}_2^{Ul}(y, v)$ is expressible in ZFC_2^{Ul} by a wff $\widehat{\mathbf{Fr}}_2^{Ul}(y, v)$.

Note that for any $y, v \in \mathbb{N}$ by definition of the relation $\mathbf{Fr}_2^{Ul}(y, v)$ follows that

$$\widehat{\mathbf{Fr}}_2^{Ul}(y, v) \iff \exists!\Psi(X) \left[\left(g_{ZFC_2^{Ul}}(\Psi(X)) = y\right) \wedge \left(g_{ZFC_2^{Ul}}(X) = \nu\right)\right], \tag{4.1.3}$$

where $\Psi(X)$ is a unique wff of ZFC_2^{Ul} which contains free occurrences of the variable X with Gödel number v. We denote a unique wff $\Psi(X)$ defined by using equivalence (4.1.3) by symbol $\Psi_{y,\nu}^{Ul}(X)$, i.e.

$$\widehat{\mathbf{Fr}}_2^{Ul}(y, v) \iff \exists!\Psi_{y,\nu}^{Ul}(X) \left[\left(g_{ZFC_2^{Ul}}\left(\Psi_{y,\nu}^{Ul}(X)\right) = y\right) \wedge \left(g_{ZFC_2^{Ul}}(X) = \nu\right)\right]. \tag{4.1.4}$$

Definition 4.1.2. Let $g_{ZFC_2^{Ul}}(X) = \nu$. Let Γ_ν^{Ul} be a set of the all Gödel numbers of the 1-place open wff's of the set theory ZFC_2^{Ul} that contains free occurrences of the variable X with Gödel number v, i.e.

$$\Gamma_\nu^{Ul} = \left\{y \in \mathbb{N} | \langle y, \nu \rangle \in \mathbf{Fr}_2^{Ul}(y, v)\right\}, \tag{4.1.5}$$

or in the following equivalent form:

$$\forall y (y \in \mathbb{N}) \left[y \in \Gamma_\nu^{Ul} \iff (y \in \mathbb{N}) \wedge \widehat{\mathbf{Fr}}_2^{Ul}(y, v)\right]. \tag{4.1.6}$$

Remark 4.1.2. Note that from the axiom of separation it follows directly that Γ_ν^{Ul} is a set in the sense of the set theory ZFC_2^{Ul}.

Definition 4.1.3. (i) We define now the equivalence relation

$$(\cdot \sim_\nu \cdot) \subset \Gamma_\nu^{Ul} \times \Gamma_\nu^{Ul} \qquad (4.1.7)$$

in the sense of the set theory ZFC_2^{Ul} by

$$y_1 \sim_\nu y_2 \iff \left(\forall X \left[\Psi_{y_1,\nu}^{Ul}(X) \iff \Psi_{y_2,\nu}^{Ul}(X)\right]\right). \qquad (4.1.8)$$

For any $y_1 \in \Gamma_v^{Ul}$ let $[y_1]_{Ul} \triangleq \left\{y \in \Gamma_X^{Ul} | y_1 \sim_\nu y_2\right\}$ denote the equivalence class to which y_1 belongs. The collection of all possible equivalence classes of Γ_ν^{Ul} by $\tilde{}_\nu$, denoted $\Gamma_\nu^{Ul}/\sim_\nu$

$$\Gamma_2^{Ul}/\sim_\nu \triangleq \left\{[y]_{Ul} | y \in \Gamma_\nu^{Ul}\right\}. \qquad (4.1.9)$$

Remark 4.1.3. Note that from the axiom of separation it follows directly that $\Gamma_\nu^{Hs}/\sim_\nu$ is a set in the sense of the set theory ZFC_2^{Ul}.

Definition 4.1.4. Let $\Im_2^{Ul}$ be the countable collection of all sets definable by 1-place open wff of the set theory ZFC_2^{Ul}, i.e.

$$\forall Y \left\{Y \in \Im_2^{Ul} \iff \exists \Psi(X) \left[\left([\Psi(X)]_{Ul} \in \Gamma_X^{Ul}/\sim_X\right) \wedge [\exists! X [\Psi(X) \wedge Y = X]]\right]\right\}. \qquad (4.1.10)$$

Definition 4.1.5. We rewrite now (4.1.10) in the following equivalent form

$$\forall Y \left\{Y \in \Im_2^{Ul} \iff \exists \Psi(X) \left[\left([\Psi(X)]_{Ul} \in \Gamma_X^{*Ul}/\sim_X\right) \wedge (Y = X)\right]\right\}, \qquad (4.1.11)$$

where the countable collection $\Gamma_X^{*Ul}/\sim_X$ is defined by

$$\forall \Psi(X) \left\{[\Psi(X)]_{Ul} \in \Gamma_X^{*Ul}/\sim_X \iff \left[\left([\Psi(X)]_{Ul} \in \Gamma_X^{Ul}/\sim_X\right) \wedge \exists! X \Psi(X)\right]\right\}. \qquad (4.1.12)$$

Definition 4.1.6. Let $\Re_2^{Ul}$ be the countable collection of all sets such that

$$\forall X \left(X \in \Im_2^{Ul}\right) \left[X \in \Re_2^{Ul} \iff X \notin X\right]. \qquad (4.1.13)$$

Remark 4.1.4. Note that $\Re_2^{Ul} \in \Im_2^{Ul}$ since $\Re_2^{Ul}$ is a collection definable by 1-place open wff

$$\Psi\left(Z, \Im_2^{Ul}\right) \triangleq \forall X \left(X \in \Im_2^{Ul}\right) [X \in Z \iff X \notin X]. \qquad (4.1.14)$$

From (4.1.13) one obtains

$$\Re_2^{Ul} \in \Re_2^{Ul} \iff \Re_2^{Ul} \notin \Re_2^{Ul}. \qquad (4.1.15)$$

But (4.1.15) gives a contradiction

$$\left(\Re_2^{Ul} \in \Re_2^{Ul}\right) \wedge \left(\Re_2^{Ul} \notin \Re_2^{Ul}\right). \qquad (4.1.16)$$

However contradiction (2.1.16) it is not a contradiction inside ZFC_2^{Ul} for the reason that the countable collection $\Im_2^{Ul}$ is not a set in the sense of the set theory ZFC_2^{Ul}.

In order to obtain a contradiction inside ZFC_2^{Ul} we introduce the following definitions.

Definition 4.1.7. We define now the countable set $\Gamma_\nu^{*Ul}/\sim_\nu$ by

$$\forall y \left\{[y]_{Ul} \in \Gamma_\nu^{*Ul}/\sim_\nu \iff \left([y]_{Ul} \in \Gamma_\nu^{Ul}/\sim_\nu\right) \wedge \widehat{\mathbf{Fr}}_2^{Ul}(y, v) \wedge \left[\exists! X \Psi_{y,\nu}^{Ul}(X)\right]\right\}. \qquad (4.1.17)$$

Remark 4.1.5. Note that from the axiom of separation it follows directly that $\Gamma_\nu^{*Ul}/\sim_\nu$ is a set in the sense of the set theory ZFC_2^{Ul}.

Definition 4.1.8. We define now the countable set $\Im_2^{*Ul}$ by formula

$$\forall Y \left\{Y \in \Im_2^{*Ul} \iff \exists y \left[\left([y]_{Ul} \in \Gamma_\nu^{*Ul}/\sim_\nu\right) \wedge \left(g_{ZFC_2^{Ul}}(X) = \nu\right) \wedge Y = X\right]\right\}. \qquad (4.1.18)$$

Note that from the axiom schema of replacement (1.1.1) it follows directly that $\Im_2^{*Hs}$ is a set in the sense of the set theory ZFC_2^{Ul}.

Definition 4.1.9. We define now the countable set $\Re_2^{*Ul}$ by formula

$$\forall X \left(X \in \Im_2^{*Ul}\right) \left[X \in \Re_2^{*Ul} \iff X \notin X\right]. \qquad (4.1.19)$$

Note that from the axiom schema of separation it follows directly that $\Re_2^{*Ul}$ is a set in the sense of the set theory ZFC_2^{Ul}.

Remark 4.1.6. Note that $\Re_2^{*Ul} \in \Im_2^{*Ul}$ since $\Re_2^{*Ul}$ is definable by the following formula

$$\Psi^*(Z) \triangleq \forall X \left(X \in \Im_2^{*Ul}\right) [X \in Z \iff X \notin X]. \qquad (4.1.20)$$

Theorem 4.1.1. Set theory ZFC_2^{Ul} is inconsistent.

Proof. From (4.1.19) and Remark 4.1.6 we obtain $\Re_2^{*Ul} \in \Re_2^{*Ul} \iff \Re_2^{*Ul} \notin \Re_2^{*Ul}$ from which immediately one obtains a contradiction

$$\left(\Re_2^{*Ul} \in \Re_2^{*Ul}\right) \wedge \left(\Re_2^{*Ul} \notin \Re_2^{*Ul}\right). \qquad (4.1.21)$$

4.2 Second-order set theory ZFC_2 with the full second-order semantics

Remind that the canonical approach of second order logic with full second-order semantics to the foundations of mathematics is that mathematical propositions have the form

$$\mathbf{U} \models \Phi \qquad (4.2.1)$$

where $\mathbf{U}$ is a mathematical structure, such as integers, reals etc., and is a mathematical statement written in second order logic. If A is one of the structures, such as $(\mathbb{N}, +, \times, <)$ or $(\mathbb{R}, +, \times, <)$, for which there is a second order sentence $\Xi_\mathbf{U}$ such that

$$\forall \mathbf{W} \left(\mathbf{W} \models \Xi_\mathbf{U} \iff \mathbf{W} \cong \mathbf{U}\right), \qquad (4.2.2)$$

then (4.2.2) can be expressed as a second order semantic logical truth

$$\models \Xi_\mathbf{U} \implies \Phi. \qquad (4.2.3)$$

Remark 4.2.1. Let ZFC_2^{fss} be second order set theory ZFC_2 with the full second-order semantics.

(1) There is no completeness theorem for second-order logic.

(2) Nor do the axioms of second-order ZFC_2^{fss} imply a reflection principle which ensures that if a sentence of second-order set theory is true, then it is true in some standard model.

Remark 4.2.2. Thus there may be sentences of the language of second-order set theory ZFC_2^{fss} :

(i) that are true but unsatisfiable, or

(ii) sentences that are valid, but false.

Remark 4.2.3. For example let Z be the conjunction of all the axioms of second-order ZFC_2^{fss}. Z is surely true. But the existence of a model for Z requires the existence of strongly inaccessible cardinals. The axioms of ZFC_2^{fss} don't entail the existence of strongly inaccessible cardinals, and hence the satisfiability of Z is independent of ZFC_2^{fss}. Thus, Z is true but its unsatisfiability is consistent with ZFC_2^{fss}.

Definition 4.2.1. Well formed formula Ψ of ZFC_2^{fss} is a well formed formula of the first order (wff_1) if Ψ contain only first-order variables and first-order quantifiers.

Let $\Gamma_X^{\sharp fss}$ be the countable collection of the all 1-place open wff_1's of the set theory ZFC_2^{fss} that contains free occurrences of the first-order variable X.

Let $\Psi_1(X), \Psi_2(X)$ be 1-place open wff_1's of the set theory ZFC_2^{fss}. We define now the equivalence relation $(\cdot \sim_X \cdot) \subset \Gamma_X^{\sharp fss} \times \Gamma_X^{\sharp fss}$ by

$$\Psi_1(X) \sim_X \Psi_2(X) \iff \forall X\left[\Psi_1(X) \iff \Psi_2(X)\right] \quad (4.2.4)$$

For any $\Psi(X) \in \Gamma_X^{\sharp fss}$ let

$$[\Psi(X)]_{\sharp fss} \triangleq \left\{\Phi(X) \in \Gamma_X^{\sharp fss} | \Psi(X) \sim \Phi(X)\right\}$$

denotes the equivalence class to which $\Psi(X)$ belongs. All elements of $\Gamma_X^{\sharp fss}$ equivalent to each other are also elements of the same equivalence class. The collection of all possible equivalence classes of $\Gamma_X^{\sharp fss}$ by $\tilde{}_X$, denoted $\Gamma_X^{\sharp fss}/\sim_X$

$$\Gamma_X^{\sharp fss}/\sim_X \triangleq \left\{[\Psi(X)]_{\sharp fss} | \Psi(X) \in \Gamma_X^{\sharp fss}\right\}. \quad (4.2.5)$$

Let $\mathbf{Fr}_2^{\sharp fss}(y,v)$ be the relation : y is the Gödel number of a wff of the set theory $ZFC_2^{\sharp fss}$ that contains free occurrences of the first-order variable X with Gödel number v [9].

Note that the relation $\mathbf{Fr}_2^{\sharp fss}(y,v)$ is expressible in ZFC_2^{fss} by a wff$_1$ $\widehat{\mathbf{Fr}}_2^{\sharp fss}(y,v)$.

Note that for any $y,v \in \mathbb{N}$ by definition of the relation $\mathbf{Fr}_2^{\sharp fss}(y,v)$ follows that

$$\widehat{\mathbf{Fr}}_2^{\sharp fss}(y,v) \iff \exists!\Psi(X)\left[\left(g_{ZFC_2^{fss}}(\Psi(X)) = y\right) \wedge \left(g_{ZFC_2^{fss}}(X) = \nu\right)\right], \quad (4.2.6)$$ where

$\Psi(X)$ is a unique wff$_1$ of ZFC_2^{fss} which contains free occurrences of the variable X with Gödel number v. We denote a unique wff $\Psi(X)$ defined by using equivalence (4.2.6) by symbol $\Psi_{y,\nu}^{\sharp fss}(X)$, i.e.

$$\widehat{\mathbf{Fr}}_2^{\sharp fss}(y,v) \iff \exists!\Psi_{y,\nu}^{\sharp fss}(X)\left[\left(g_{ZFC_2^{fss}}\left(\Psi_{y,\nu}^{\sharp}(X)\right) = y\right) \wedge \left(g_{ZFC_2^{fss}}(X) = \nu\right)\right]. \quad (4.2.7)$$

Remark 4.2.4. In order to avoid difficulties mentioned above,see Remark 4.2.1-Remark 4.2.3 we dealing with the countable collection $\Gamma_X^{\sharp fss}$ of the all 1-place open wff$_1$'s of the set theory ZFC_2^{fss}.

Definition 4.2.2. Let $g_{ZFC_2^{fss}}(X) = \nu$.Let $\Gamma_\nu^{\sharp fss}$ be a set of all Gödel numbers of 1-place open wff$_1$'s of the set theory ZFC_2^{fss} that contains free occurrences of the first-order variable X with Gödel number v, i.e.

$$\Gamma_\nu^{\sharp fss} = \left\{y \in \mathbb{N} | \langle y,\nu\rangle \in \mathbf{Fr}_2^{\sharp fss}(y,v)\right\}, \quad (4.2.8)$$

or in the following equivalent form

$$\forall y\,(y \in \mathbb{N})\left[y \in \Gamma_\nu^{\sharp fss} \iff (y \in \mathbb{N}) \wedge \widehat{\mathbf{Fr}}_2^{\sharp fss}(y,v)\right]. \quad (4.2.9)$$

Remark 4.2.5. Note that from the axiom of separation it follows directly that $\Gamma_\nu^{\sharp fss}$ is a set in the sense of the set theory ZFC_2^{fss}.

Definition 4.2.3. (i) We define now the equivalence relation

$$(\cdot \sim_\nu \cdot) \subset \Gamma_\nu^{\sharp fss} \times \Gamma_\nu^{\sharp fss} \quad (4.2.10)$$

in the sense of the set theory ZFC_2^{fss} by

$$y_1 \sim_\nu y_2 \iff \left(\forall X\left[\Psi_{y_1,\nu}^{\sharp fss}(X) \iff \Psi_{y_2,\nu}^{\sharp fss}(X)\right]\right). \quad (4.2.11)$$

The collection of all possible equivalence classes of $\Gamma_\nu^{\sharp fss}$ by $\tilde{}_\nu$, denoted $\Gamma_\nu^{\sharp fss}/\sim_\nu$

$$\Gamma_v^{\sharp fss}/\sim_\nu \triangleq \left\{[y]_{\sharp fss} | y \in \Gamma_\nu^{\sharp fss}\right\}. \quad (4.2.12)$$

Remark 4.2.6. Note that from the axiom of separation it follows directly that $\Gamma_\nu^{\sharp fss}/\sim_\nu$ is a set in the sense of the set theory ZFC_2^{fss}.

Definition 4.2.4. Let $\Im_2^{\sharp fss}$ be the countable collection of the all sets definable by 1-place open first order wff of the set theory ZFC_2^{fss}, i.e.

$$\forall Y \left\{ Y \in \Im_2^{\sharp fss} \iff \exists \Psi(X) \left[\left([\Psi(X)]_{\sharp fss} \in \Gamma_X^{\sharp fss} / \sim_X \right) \wedge [\exists! X [\Psi(X) \wedge Y = X]] \right] \right\}. \quad (4.2.13)$$

Definition 4.2.5. We rewrite now (4.2.13) in the following equivalent form

$$\forall Y \left\{ Y \in \Im_2^{*\sharp fss} \iff \exists \Psi(X) \left[\left([\Psi(X)]_{\sharp fss} \in \Gamma_X^{*\sharp fss} / \sim_X \right) \wedge (Y = X) \right] \right\}, \quad (4.2.14)$$

where the countable collection $\Gamma_X^{*\sharp fss} / \sim_X$ is defined by

$$\forall \Psi(X) \left\{ [\Psi(X)]_{\sharp fss} \in \Gamma_X^{*\sharp fss} / \sim_X \iff \left[\left([\Psi(X)]_{\sharp fss} \in \Gamma_X^{\sharp fss} / \sim_X \right) \wedge \exists! X \Psi(X) \right] \right\}. \quad (4.2.15)$$

Definition 4.2.6. Let $\Re_2^{*\sharp fss}$ be the countable collection of all sets such that

$$\forall X \left(X \in \Im_2^{\sharp fss} \right) \left[X \in \Re_2^{*\sharp fss} \iff X \notin X \right]. \quad (4.2.16)$$

Remark 4.2.7. Note that $\Re_2^{*\sharp fss} \in \Im_2^{*\sharp fss}$ since $\Re_2^{*\sharp fss}$ is a collection definable by 1-place open wff$_1$

$$\Psi \left(Z, \Im_2^{*\sharp fss} \right) \triangleq \forall X \left(X \in \Im_2^{*\sharp fss} \right) [X \in Z \iff X \notin X]. \quad (4.2.17)$$

From (4.2.16) one obtains

$$\Re_2^{*\sharp fss} \in \Re_2^{*\sharp fss} \iff \Re_2^{*\sharp fss} \notin \Re_2^{*\sharp fss}. \quad (4.2.18)$$

But (4.2.18) gives a contradiction

$$\left(\Re_2^{*\sharp fss} \in \Re_2^{*\sharp fss} \right) \wedge \left(\Re_2^{*\sharp fss} \notin \Re_2^{*\sharp fss} \right). \quad (4.2.19)$$

However contradiction (2.2.19) it is not a contradiction inside ZFC_2^{fss} for the reason that the countable collection $\Im_2^{*\sharp fss}$ is not a set in the sense of the set theory ZFC_2^{fss}.

In order to obtain a contradiction inside ZFC_2^{fss} we introduce the following definitions.

Definition 4.2.7. We define now the countable set $\Gamma_\nu^{*\sharp fss} / \sim_\nu$ by

$$\forall y \left\{ [y]_{Ul} \in \Gamma_\nu^{*\sharp fss} / \sim_\nu \iff \left([y]_{\sharp fss} \in \Gamma_\nu^{*\sharp fss} / \sim_\nu \right) \wedge \widehat{\mathbf{Fr}}_2^{*\sharp fss}(y, v) \wedge \left[\exists! X \Psi_{y,\nu}^{\sharp fss}(X) \right] \right\}. \quad (4.2.20)$$

Remark 4.2.8. Note that from the axiom of separation it follows directly that $\Gamma_\nu^{*Ul} / \sim_\nu$ is a set in the sense of the set theory ZFC_2^{fss}.

Definition 4.2.8. We define now the countable set $\Im_2^{*\sharp fss}$ by formula

$$\forall Y \left\{ Y \in \Im_2^{*\sharp fss} \iff \exists y \left[\left([y]_{\sharp fss} \in \Gamma_\nu^{*\sharp fss} / \sim_\nu \right) \wedge \left(g_{ZFC_2^{fss}}(X) = \nu \right) \wedge Y = X \right] \right\}. \quad (4.2.21)$$

Note that from the axiom schema of replacement (1.1.1) it follows directly that $\Im_2^{*\sharp fss}$ is a set in the sense of the set theory ZFC_2^{fss}.

Definition 4.2.9. We define now the countable set $\Re_2^{*\sharp fss}$ by formula

$$\forall X \left(X \in \Im_2^{*\sharp fss} \right) \left[X \in \Re_2^{*\sharp fss} \iff X \notin X \right]. \qquad (4.2.22)$$

Note that from the axiom schema of separation it follows directly that $\Re_2^{*\sharp fss}$ is a set in the sense of the set theory ZFC_2^{fss}.

Remark 4.2.9. Note that $\Re_2^{*\sharp fss} \in \Im_2^{*Ul}$ since $\Re_2^{*Ul}$ is definable by the following formula

$$\Psi^* (Z) \triangleq \forall X \left(X \in \Im_2^{*\sharp fss} \right) [X \in Z \iff X \notin X]. \qquad (4.2.23)$$

Theorem 4.2.1. Set theory ZFC_2^{fss} is inconsistent.

Proof. From (4.2.22) and Remark 4.1.6 we obtain $\Re_2^{*\sharp fss} \in \Re_2^{*\sharp fss} \iff \Re_2^{*\sharp fss} \notin \Re_2^{*Ul}$ from which immediately one obtains a contradiction

$$\left(\Re_2^{*\sharp fss} \in \Re_2^{*\sharp fss} \right) \wedge \left(\Re_2^{*\sharp fss} \notin \Re_2^{*\sharp fss} \right). \qquad (4.2.24)$$

5 Conclusions

a In this paper we have proved that set theory $ZFC + \exists M_{st}^{ZFC}$ is inconsistent.

b This result originally was obtained in [10], [15] and [11] by using essentially another complicated approach.

Acknowledgement

The reviewers provided important clarifications.

Competing Interests

Authors have declared that no competing interests exist.

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