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A Note on Harmony

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Abstract In the proof-theoretic semantics approach to meaning, *harmony*, requiring a balance between introduction-rules (I-rules) and elimination rules (E-rules) within a meaning conferring natural-deduction proof-system, is a central notion. In this paper, we consider two notions of harmony that were proposed in the literature: 1. *GE-harmony*, requiring a certain form of the E-rules, given the form of the I-rules. 2. *Local intrinsic harmony*: imposes the existence of certain transformations of derivations, known as *reduction* and *expansion*. We propose a *construction* of the E-rules (in GE-form) from given I-rules, and prove that the constructed rules satisfy also local intrinsic harmony. The construction is based on a classification of I-rules, and constitute an implementation to Gentzen's (and Pawitz') remark, that E-rules can be "read off" I-rules.

Keywords Natural-deduction · Proof-theoretic semantics · Introduction-rules · Elimination-rules · Harmony · Local soundness/completeness

1 Introduction

Within the *proof-theoretic semantics (PTS)* approach to meaning, using a natural-deduction proof-system, the notion of *harmony* [4] occupies a central

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role as an acceptability criterion for the introduction-rules[1] (I-rules) as *conferring meaning* on the introduced "logical constant", with the elimination-rules (E-rules) as using this meaning. Informally, harmony expresses a certain balance between the introduction-rules (I-rules) and the elimination-rules (E-rules), a balance that excludes pathological operators like Prior's *tonk* [16] from counting as meaning conferring. In [4], harmony is viewed solely as a property of the verificationist approach, and the above mentioned balance is maintained via a different criterion, called *stability*, associated with the *pragmatist* approach to PTS, whereby the E-rules constitute definitions, used by the I-rules.

There is no consensus among adherents of the PTS school as to the exact nature of harmony and, consequently, there is no unique, universally acceptable, definition of it. We consider here two approaches to harmony, which we found applicable to an extension of PTS to *natural language* [6, 7]. Both approaches can be viewed as an "implementation", with a certain augmentation, of Prawitz' *inversion principle* (a term coined by Lorenzen [12]), which is itself an elaboration of some comments by Gentzen [10]. The inversion-principle is formulated in [14] as follows:

> Let ρ be an application of an elimination-rule with consequence ψ. Then, the derivation justifying the introduction of the major premiss ϕ of ρ, together with the derivations of minor premisses of ρ "contain" already a derivation of ψ, without the use of ρ.

The augmentation is needed in order to remain within the verificationist approach. The two notions of harmony are the following:

General-Elimination (GE) harmony According to this approach, in order to be harmonious, an E-rule has to have some *specific form*, depending on the corresponding I-rules. This form guarantees that the inversion-principle and its augmentation obtain. Our point of departure is the specific proposal of [17].

Local Intrinsic Harmony Here, in order to be harmonious, no constraints on the form of the E-rules is imposed, but they have to stand in a certain relationship to the I-rules, so as to directly reflect the required balance among them. We consider here a specific proposal by [3, 13], based on two properties known as *local soundness* and *local completeness* (introduced below in a somewhat modified formulation). The original conception in [4] consists more or less of local soundness. We add to it here local completeness as an augmentation replacing an appeal to stability.

In particular, we are interested in the relationship between the two notions. We will show that, under a proper rendering of the first, if it obtains, then the second obtains too. The other direction cannot be shown, as there are

[1] This approach is known as *verificationist*; a dual *pragmatist* approach regards the E-rules as conferring meaning, while the I-rules constitute use of this meaning.

harmonious rules not having the specific form considered bellow. In addition to relating these two notions of harmony, a contribution of this paper is also a study of the structure of I-rules, and an analysis of the effect of this structure on making more precise Gentzen's remarks on the induced E-rules.

2 Preliminaries

We assume that natural-deduction systems are formulated in Gentzen's "logistic form" [9, p. 150], by which the premises and conclusions of a rule are *sequents* of the form $\Gamma \vdash \phi$, where Γ is a *context* (here, a finite set of formulas), and ϕ is a formula, all in some given object-language. This is in contrast to the more frequently used form, as in [10], where premises and conclusion are formulas, and assumptions are implicit. Thus, we deal here only with single-conclusion sequents and rules; in addition, we assume all rules have only finitely many premises. *Discharged assumptions* are in square brackets, where the assumption is indexed,[2] and the index is used by the rule-application discharging that assumption. We assume the usual notion of a tree-shaped ND-derivation, using $\mathcal{D}$ to denote "bare" derivations and $\begin{array}{c}\Gamma\\ \mathcal{D}\\ \psi\end{array}$ for a derivation of ψ from (open) assumptions Γ; and when context is suppressed, we also use $\begin{array}{c}[\phi]_i\\ \mathcal{D}\\ \psi\end{array}$ to depict a derivation $\mathcal{D}$ of ψ under a discharged assumption ϕ. To save space, we occasionally display derivations in the more usual style, where all contexts are omitted, whenever they can be unambiguously recovered.

For the sake of the upcoming discussion, we classify ND-rules according to the following criteria.

Definition (Rule Classification)

1. An ND-rule is *hypothetical* if it allows for at least one premise with *assumptions discharge*; otherwise, it is *categorical*.
2. An ND-rule is *combining* if it has more than one premise; otherwise it is non-combining.
3. An ND-rule is *parameterized* if (at least one of) its premises depend on a free variable; otherwise, it is non-parameterized.
4. An ND-rule is *conditional* if it has some *side condition* on its applicability; otherwise – it is *unconditional*. We restrict ourselves here to side conditions stating *freshness* of a variable for a context, meaning the variable does

[2] In order to avoid complex indices, discharged assumptions in rules are always numbered starting with '1', under the understanding that in rule applications within derivations, indices are always chosen fresh.

not occur *free* in any formula in the context. We assume there is an inexhaustible supply of fresh variables whenever a rule-application needs one. More general side conditions may have to be considered in a more general setting. This issue is discussion further in the Conclusion section.

Note that the classification by the above criteria is orthogonal to the I-rule vs. E-rule classification. Thus, the the conjunction-introduction rule $\frac{\Gamma \vdash \phi \quad \Gamma \vdash \psi}{\Gamma \vdash \phi \wedge \psi}(\wedge I)$ and the conjunction-elimination rules $\frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \phi}(\wedge E_1)$, $\frac{\Gamma \vdash \phi \wedge \psi}{\Gamma \vdash \psi}(\wedge E_2)$ are categorical, while the implication-introduction rule $\frac{\Gamma, [\phi]_1 \vdash \psi}{\Gamma \vdash \phi \rightarrow \psi}(\rightarrow I^1)$ and the disjunction-elimination rule $\frac{\Gamma \vdash \phi \vee \psi \quad \Gamma, [\phi]_1 \vdash \xi \quad \Gamma, [\psi]_2 \vdash \xi}{\Gamma \vdash \xi}(\vee E^{1,2})$ are hypothetical. Also, the $(\wedge I)$ and $(\vee E)$ are combining, while $(\wedge E_i)$ and $(\rightarrow I)$ are non-combining. None of the above rules are parameterized (hence, none are conditional). A categorical, parameterized and conditional rule is the usual I-rule for existential quantification, which has the form

$$\frac{\phi[t/x]}{\exists x.\phi}(\exists I)$$

where t is safe for ϕ. The premise is here parameterized by an explicit substitution of the term t. Here $\phi[t/x]$ stands for ϕ with the term t substituted for the free occurrences of x in ϕ. For simplicity, we do not consider function symbols here, so variables are the only terms.

A hypothetical, parameterized and conditional I-rule with which we will be concerned is that for *restricted universal quantification*, taken from[3] [6, 7], as well as the parameterized and categorial *restricted existential quantification* I-rule, both cast here in 1st-order logic syntax. Here '*e*' stands for 'every', and *s* for 'some'.

$$\frac{\Gamma, [\phi(y)]_1 \vdash \psi(y)}{\Gamma \vdash \forall x.\phi(x) \rightarrow \psi(x)}(eI^1) \quad y \text{ fresh for } \Gamma \tag{1}$$

$$\frac{\Gamma \vdash \phi[t/x] \quad \Gamma \vdash \psi[t/x]}{\Gamma \vdash \exists x.\phi(x) \wedge \psi(x)}(sI) \tag{2}$$

For an example of a hypothetical, parameterized and conditional E-rule, consider the following rule for *restricted existential quantification*.

$$\frac{\Gamma \vdash \exists x.\phi(x) \wedge \psi(x) \quad \Gamma, [\phi(y)]_1, [\psi(y)]_2 \vdash \xi}{\Gamma \vdash \xi}(sE^{1,2}), \qquad y \text{ fresh for } \Gamma, \xi \tag{3}$$

[3] The rule is used in a language having neither implication nor unrestricted universal quantification, hence its derivability in 1st-order logic is besides the point.

Finally, note that all hypothetical I-rules presented so far are non-combining. For an example of a hypothetical and combining I-rule, consider the following I-rule for the ternary connective 'if-then-else' (abbreviated as *ite*)

$$\frac{\Gamma, [\phi]_1 \vdash \psi \quad \Gamma, [\neg\phi]_2 \vdash \chi}{\Gamma \vdash ite(\phi, \psi, \chi)} (iteI^{1,2}) \tag{4}$$

In what follows, we show that each of the considered above dimensions of rule classification leaves its impression on the induced GE-rules. In particular, hypothetical and combining I-rules have an effect that seems to not have been noticed in the literature: they force more than one generalized elimination rule for a connective introduced by such an I-rule.

An important role in relating the two notions of harmony is played by *substitutions* within derivations: of a derivation for a formula within another derivation, and of a term for a fresh variable within a derivation. As shown below, closure of the derivations under these substitutions guarantees the well-formedness of the induced GE-rules.

Definition (Substitutions)

Derivation substitution For a derivation $\overset{\phi}{\mathcal{D}}$ with open assumption ϕ, and a derivation $\overset{\mathcal{D}'}{\phi}$ with consequence ϕ, let $\mathcal{D}[\overset{\mathcal{D}'}{\phi}/\phi]$ denote the result obtained by replacing, within $\mathcal{D}$, every occurrence of ϕ as an open assumption by $\overset{\mathcal{D}'}{\phi}$. Derivation substitution is naturally extended to simultaneous substitutions for more than one formula, and can also be nested.

Term substitution For a term t and a derivation $\begin{matrix}\phi(y)\\ \mathcal{D}\\ \psi\end{matrix}$, where $y \in FV(\phi)$ is[4] *fresh* for (the suppressed) Γ and for ψ, let $\begin{matrix}\phi(t)\\ \mathcal{D}[t/y]\\ \psi\end{matrix}$ denote the result of replacing (in the usual capture-free way) every free occurrence of y in any formula (including an open assumption) in $\mathcal{D}$ by t.

These substitutions are usually defined by induction on the structure of $\mathcal{D}$.

In the sequel, we assume the ND-systems considered satisfy the following two closure assumptions:

Admissibility of derivation substitution For all derivations $\mathcal{D}$, $\mathcal{D}'$ and formula ϕ, if $\overset{\phi}{\mathcal{D}}$ and $\overset{\mathcal{D}'}{\phi}$ are derivations, so is $\mathcal{D}[\overset{\mathcal{D}'}{\phi}/\phi]$.
Admissibility does not always hold; see, for example, [20].

[4] It is convenient to assume Barendregt's *variable convention* [2, p. 26], by which no variable occurs both free and bound in the same formula.

Admissibility of term substitution For all terms t and derivations $\begin{array}{c}\phi(y)\\ \mathcal{D}\\ \psi\end{array}$, where $y \in FV(\phi)$ is *fresh* for (the suppressed) Γ and for ψ, $\begin{array}{c}\phi(t)\\ \mathcal{D}[t/y]\\ \psi\end{array}$ is a derivation.

3 The Two Notions of Harmony

In this section we present the two notions of harmony mentioned above.

3.1 General-Elimination Harmony

Another formulation of the idea behind the inversion-principle is, that any consequence drawn from a formula with an introduced "main operator" δ, say $\phi\delta\psi$, can be drawn from ("is included in") *the grounds of introducing* δ.

What are these grounds?

For a categorical, unconditional I-rule, there is a natural answer: the premises of the I-rule constitute the grounds of introduction (by that rule).

In [17], the following form of an E-rule is presented[5] as embodying harmony with given I-rules. This form will easily be seen as fitting categorical, and non-parameterized I-rules. Suppose the I-rules of an operator δ, the main operator in ϕ, can be schematically presented (omitting all shared contexts) as $\dfrac{\Pi_i}{\phi}\,(\delta I)_i$, $i = 1, \cdots, n$. Here Π_i is the collection of premises (formulas) of (δI_i), and constitutes the grounds of introducing δ via (δI_i). Then, to be harmonious, the E-rule should combine all the grounds of introducing δ (by all of δ's I-rules) and use each of those grounds as (discharged) assumptions for deriving an *arbitrary* conclusion ξ, thus having the following form:

$$\dfrac{\phi \quad \begin{array}{c}[\Pi_1]_1\\ \mathcal{D}'_1\\ \xi\end{array} \quad \cdots \quad \begin{array}{c}[\Pi_n]_n\\ \mathcal{D}'_n\\ \xi\end{array}}{\xi}\,(\delta GE^{1,\cdots,n}) \tag{5}$$

Here $[\Pi]_i$ abbreviates $[\phi_1]_i, \cdots, [\phi_n]_i$, for $\Pi = \{\phi_1, \cdots, \phi_n\}$. Such a rule (and similar ones for the other connectives), under the name of *generalized elimination*, was also proposed by [19] for independent reasons, related to the relationship between normal ND-derivations, and cut-free sequent-calculus derivations.

[5] A similar construction of E-rules from I-rules, fitting the categorical case, is presented in [15], with a more specific form of the assumptions discharged by the E-rule. The construction is used there to define and establish functional completeness of a collection of connectives (without reference to truth-tables).

A simple example is the following GE-rule [11, 18] for conjunction, where $\Pi = \{\phi, \psi\}$ is the grounds for introducing '$\wedge$' via its (only) I-rule $(\wedge I)$:

$$\frac{\phi \wedge \psi \qquad \begin{matrix}[\phi, \psi]_1 \\ \vdots \\ \xi\end{matrix}}{\xi} \ (\wedge GE^1)$$

For a categorical, unconditional and non-parameterized rule, this form of a GE-rule indeed reflects the inversion idea: any arbitrary consequence ξ that can be drawn from (the major premise) ϕ, can already be drawn from each of its grounds of introduction (all of them!) Π_i, $i = 1, \cdots, n$. Note that all those assumed grounds are discharged by the rule. The usual disjunction-elimination rule is of this form to start with.

This leads directly to the well-known reduction, removing a maximal formula, the basis of Prawitz's *proof normalization*:

$$\frac{\dfrac{\begin{matrix}\hat{\mathcal{D}}_i \\ \Pi_i\end{matrix}}{\phi}\,(\delta I)_i \qquad \begin{matrix}[\Pi_1]_1 \\ \mathcal{D}'_1 \\ \xi\end{matrix} \quad \cdots \quad \begin{matrix}[\Pi_n]_n \\ \mathcal{D}'_n \\ \xi\end{matrix}}{\xi}\,(\delta GE^{1,\cdots,n}) \;\leadsto_r\; \begin{matrix}\hat{\mathcal{D}}_i \\ \mathcal{D}'_i[\Pi_i/\Pi_i] \\ \xi\end{matrix} \tag{6}$$

As we see in the next subsection, the availability of such a reduction constitutes part of the definition of local intrinsic harmony. Note that the availability of this reduction rests on the admissibility of derivation substitution, as noted also in [17] (referring to 'cut' instead of derivation substitution admissibility).

This leaves us with a question, what happens in the hypothetical, conditional and parameterized cases? As a first approximation, we first consider the effect of hypotheticalness on the harmoniously induced GE-rule. The effect of parameterization is deferred to the next stage. In the case of a hypothetical I-rule, that for implication, the generalized E-rule is [5, 17, 19]:

$$\frac{\phi \rightarrow \psi \quad \phi \qquad \begin{matrix}[\psi]_1 \\ \vdots \\ \xi\end{matrix}}{\xi} \ (\rightarrow GE^1) \tag{7}$$

An analysis of the structure of this rule reveals, that one minor premise (ϕ) is the assumption of the corresponding (single) I-rule, while the "categorical part" of the I-rule (ψ) serves as an assumption of the GE-rule, discharged by it. We continue to refer to the categorical part as the '*grounds*', and refer to the assumptions on which the grounds depend (in the I-rule) as the ground's *support*. We now propose this structure as the *general harmonious form* of the E-rule, based on the following reformulation of (the shorter version of) the inversion-principle, that can serve as a semantic justification of the proposed GE-rule:

> Any consequence drawn from a formula with an introduced main operator δ, say $\phi\delta\psi$, can be drawn from ("is included in") the grounds of its

introduction (all of them), *together with their respective supports (all of them)*.

Below is our proposal for a first approximation of the *harmoniously-induced* GE-rule (ignoring, for a while, conditionality and parameterization). Suppose the I-rules for an operator δ, the main operator in ϕ, are of the following form (contexts omitted).

$$
\frac{\Pi_i \quad \begin{array}{c}[\Sigma_1^i]_1\\ \vdots\\ \alpha_1^i\end{array} \quad \cdots \quad \begin{array}{c}[\Sigma_{m_i}^i]_{m_i}\\ \vdots\\ \alpha_{m_i}^i\end{array}}{\phi}\ (\delta I^{1,\cdots,m_i})_i \tag{8}
$$

for $1 \leq i \leq n$. The i'th rule has $l_i = |\Pi_i|$ premises not discharging assumptions, and m_i discharging premises, each discharging a collection Σ of assumptions. These I-rules generate m_i GE-rules per I-rule, each GE-rule corresponding to one premise discharging discharged assumptions in the I-rule. The general form of (the first approximation of) the GE-rules is as follows.

$$
\frac{\phi \quad \Sigma_k^i \quad \begin{array}{c}[\Pi_1, \alpha_k^i]_1\\ \vdots\\ \xi\end{array} \quad \cdots \quad \begin{array}{c}[\Pi_n, \alpha_k^i]_n\\ \vdots\\ \xi\end{array}}{\xi}\ (\delta GE_{i,k}^{1,\cdots,n}) \tag{9}
$$

$1 \leq i \leq n, \ 1 \leq k \leq m_i.$

Thus, the contribution of hypotheticality in an I-rule is two-folded. Each of the supports becomes a premise (in the corresponding GE-rule), and all the grounds become dischargeable assumptions (in all GE-rules). This leads to the following reductions, where each $(\delta I)_i$ is confronted against each $(\delta E)_{i,k}$, for $1 \leq i \leq n$ and $1 \leq k \leq m_i$.

$$
\frac{\dfrac{\begin{array}{c}\hat{\mathcal{D}}_i\\ \Pi_i\end{array} \quad \begin{array}{c}[\Sigma_1^i]_1\\ \mathcal{D}_1^i\\ \alpha_1^i\end{array} \quad \cdots \quad \begin{array}{c}[\Sigma_{m_i}^i]_{m_i}\\ \mathcal{D}_{m_i}^i\\ \alpha_{m_i}^i\end{array}}{\phi}\ (\delta I^{1,\cdots,m_i})_i \quad \begin{array}{c}\mathcal{D}_k^i\\ \Sigma_k^i\end{array} \quad \begin{array}{c}[\Pi_1, \alpha_k^i]_1\\ \mathcal{D}_1^*\\ \xi\end{array} \quad \cdots \quad \begin{array}{c}[\Pi_n, \alpha_k^i]_n\\ \mathcal{D}_n^*\\ \xi\end{array}}{\xi}\ (\delta GE^{1,\cdots,n})_{i,k}
$$

$$
\rightsquigarrow_r \quad \begin{array}{c} \mathcal{D}_i^*[\Pi_i/\Pi_i, \ \begin{array}{c}\mathcal{D}_k^i\\ \mathcal{D}_k^i[\Sigma_k^i/\Sigma_k^i]\\ \alpha_k^i\end{array} /\alpha_k^i] \\ \xi \end{array} \quad (\text{with } \hat{\mathcal{D}}_i \text{ above } \Pi_i) \tag{10}
$$

Note the vectored notation for the categorical part in the (δI_i) application. The availability of this reduction depends on our assumption of closure of derivations under derivation substitution.

Returning to the example of (*ite*) (cf. Eq. 4), $n = 1$, and the single I-rule has no categorical premise, and has two premises discharging assumptions; thus, the two following GE-rules are induced.

$$\frac{ite(\phi, \psi, \chi) \quad \phi \quad \begin{array}{c}[\psi]_1\\ \vdots\\ \xi\end{array}}{\xi}\ (iteGE^1)_1 \qquad \frac{ite(\phi, \psi, \chi) \quad \neg\phi \quad \begin{array}{c}[\chi]_1\\ \vdots\\ \xi\end{array}}{\xi}\ (iteGE^1)_2 \tag{11}$$

The reductions are as follows

$$\frac{\dfrac{\begin{array}{c}[\phi]_1\\ \mathcal{D}_1\\ \psi\end{array} \quad \begin{array}{c}[\neg\phi]_2\\ \mathcal{D}_2\\ \chi\end{array}}{ite(\phi, \psi, \chi)}\ (iteI^{1,2}) \quad \begin{array}{c}\mathcal{D}_1\\ \phi\end{array} \quad \begin{array}{c}[\psi]_3\\ \mathcal{D}_1^*\\ \xi\end{array}}{\xi}\ (iteGE^3)_1 \rightsquigarrow_r \begin{array}{c}\mathcal{D}_1\\ \mathcal{D}_1^*[\mathcal{D}_1[\ \phi\ /\phi]/\psi]\\ \xi\end{array} \tag{12}$$

and

$$\frac{\dfrac{\begin{array}{c}[\phi]_1\\ \mathcal{D}_1\\ \psi\end{array} \quad \begin{array}{c}[\neg\phi]_2\\ \mathcal{D}_2\\ \chi\end{array}}{ite(\phi, \psi, \chi)}\ (iteI^{1,2}) \quad \begin{array}{c}\mathcal{D}_2\\ \neg\phi\end{array} \quad \begin{array}{c}[\chi]_3\\ \mathcal{D}_2^*\\ \xi\end{array}}{\xi}\ (iteGE^3)_2 \rightsquigarrow_r \begin{array}{c}\mathcal{D}_2\\ \mathcal{D}_2^*[\mathcal{D}_2[\neg\phi/\neg\phi]/\chi]\\ \xi\end{array} \tag{13}$$

Clearly, the harmoniously-induced GE-rule for a categorical I-rule is the assumption-less special case, yielding the original formulation[6] of [17]. This construction explains the form of the generalized implication GE-rule (Eq. 7), the I-rule of which is hypothetical: The one minor premise, ϕ, is the support of the I-rule, while the other minor premise is a derivation of ξ under the assumption of the ground ψ.

We next refine the above approximation of the harmoniously induced GE-rule into its final form, by incorporating the effects of conditionality (freshness restriction on free variables) and parameterization. To simplify the notation, we assume one free variable and one parameter only. Passing to any finite number involves vectoring the notation. Thus, suppose the i'th I-rule has the following form.

$$\frac{\Pi_i(x_i) \quad \begin{array}{c}[\Sigma_1^i(X_1^i)]_1\\ \vdots\\ \alpha_1^i(X_1^i)\end{array} \quad \cdots \quad \begin{array}{c}[\Sigma_{m_i}^i(X_{m_i}^i)]_{m_i}\\ \vdots\\ \alpha_{m_i}^i(X_{m_i}^i)\end{array}}{\phi}\ (\delta I^{1,\cdots,m_i})_i \tag{14}$$

[6]The hypothetical case is treated there too, by appealing to heuristics drawn from Gentzen's sequent calculi, not fully formalized and justified.

X^i_j fresh for $1 \leq i \leq n,\ 1 \leq j \leq m_i$. Here x_i are the parameters, and X_i are the subjects of conditionalization (freshness requirement).

The harmoniously-induced GE-rule reflects the incorporation of variables by substitutions, that preserve validity by the freshness assumption.

$$\dfrac{\phi \quad \Sigma^i_k[Y^i_k/X^i_k] \quad \begin{array}{c}[\Pi_1[y^i_1/x^i], \alpha^i_k[Y^i_1/X^i_1]]_1\\ \vdots\\ \xi\end{array} \quad \cdots \quad \begin{array}{c}[\Pi_n[y^i_n/x^i_n], \alpha^i_k[Y^i_1/X^i_1]]_n\\ \vdots\\ \xi\end{array}}{\xi}\ (\delta GE^{1,\cdots,n}_{i,k}) \tag{15}$$

$1 \leq i \leq n,\ 1 \leq k \leq m_i$, and y^i_j are fresh, for $1 \leq j \leq k$. The resulting reductions are as follows.

$$\dfrac{\dfrac{\begin{array}{c}\hat{\mathcal{D}}_i\\ \Pi_i(x_i)\end{array} \quad \begin{array}{c}[\Sigma^i_1(X^i_1)]_1\\ \mathcal{D}^i_1\\ \alpha^i_1(X^i_1)\end{array} \quad \cdots \quad \begin{array}{c}[\Sigma^i_{m_i}(X^i_{m_i})]_{m_i}\\ \mathcal{D}^i_{m_i}\\ \alpha^i_{m_i}(X^i_{m_i})\end{array}}{\phi}\ (\delta I)_i \quad \begin{array}{c}\mathcal{D}^i_k\\ \Sigma^i_k(Y^i_k/X^i_k)\end{array} \quad \begin{array}{c}[\Pi_1[y^i_1/x^i_1], \alpha^i_k[Y^i_k/X^i_k]]_1\\ \mathcal{D}^*_1\\ \xi\end{array} \quad \cdots \quad \begin{array}{c}[\Pi_n[y^i_n/x^i_n], \alpha^i_k[Y^i_k/X^i_k]]_n\\ \mathcal{D}^*_n\\ \xi\end{array}}{\xi}\ (\delta GE^{\{1,\cdots,n\}})_{i,k}$$

$$\leadsto_r$$

$$\begin{array}{c} \hat{\mathcal{D}}_i[y_i/x_i] \qquad\qquad \begin{array}{c}\mathcal{D}^i_k\\ \mathcal{D}^i_k[\Sigma^i_k[Y^i_k/X^i_k]/\Sigma^i_k(X^i_k)]\end{array}\\ \mathcal{D}^*_i[\Pi_{[}y_i/x_i]/\Pi_i(x_i), \qquad \alpha^i_k[Y^i_k/X^i_k] \qquad /\alpha^i_k X^i_k)]\\ \xi \end{array} \tag{16}$$

The availability of this reduction rests on both substitution-admissibility assumptions. Clearly, for the categorical, unconditional, non-parameterized case, the reduction above yields the one in Eq. 6 observed in [17].

For the restricted universal quantification I-rule (Eq. 1), the generalized GE-rule harmoniously induced by this construction is

$$\dfrac{\forall x.\phi(x)\rightarrow\psi(x) \quad \phi(y/x) \quad \begin{array}{c}[\psi(y/x)]_1\\ \vdots\\ \xi\end{array}}{\xi}\ (eE^1), \quad y \text{ fresh} \tag{17}$$

Indeed, this is the rule employed in [6, 7], but here presented in 1st-order logic syntax.

As an example of the effect of rule parameterization, consider the premises of the I-rule for restricted existential quantification, presented in Eq. 2, which are of the parameterized form (with the same parameter t). Indeed, the harmoniously-induced E-rule has the form

$$\dfrac{\exists x.\phi(x)\wedge\psi(x) \quad \begin{array}{c}[\phi[y/x]]_1, [\psi[y/x]]_2\\ \vdots\\ \xi\end{array}}{\xi}\ (sE^{1,2}) \quad , \quad y \text{ fresh} \tag{18}$$

The rules for the usual intuitionistic quantifications stand in a similar relationship.

Definition (GE-harmony) For an operator δ, its I-rules and E-rules are *GE-harmonious* iff the E-rules are the GE-rules harmoniously induced by the I-rules.

Note[7] that while the generation of harmoniously-induced GE-rules are functional, this function is not injective, and different sets of I-rules may lead to the same set of harmoniously induced GE-rules; however, there is no reason to expect a one-one relationship.

3.2 Local Intrinsic Harmony

As already mentioned, this notion of harmony does not depend directly on the form of the rules; it requires a certain balance between the I-rules and E-rules (for any operator δ), in that *the E-rules are neither too strong nor too weak w.r.t. the I-rules*. The following two properties were proposed in [3, 13] to capture this requirement.

- **Local Soundness** *Every* introduction followed *directly* by an elimination can be *reduced*.
  This shows that the E-rules are not *too strong* w.r.t. the I-rules. This is one aspect of Prawitz's normalizability (the other aspects being weak and strong termination). Clearly, the rules for '*tonk*' in [16] fail local soundness. The maximal formula in the following derivation cannot be removed.

$$\dfrac{\dfrac{\phi}{\phi \ tonk \ \psi}\,(tonkI)}{\psi}\,(tonkE) \tag{19}$$

- **Local Completeness** *There is* a way to eliminate[8] and reintroduce, reconstructing the eliminated operator. This process is referred to as *expansion*. Since 'reintroduce' might allude to *ordering*[9] of the applications of I-rules/E-rules within the reconstructed derivation, we prefer the following order-neutral formulation:
  Every derivation of a formula ϕ with principal operator δ *can be* expanded to one containing an application of an E-rule of δ, and applications of all I-rules of δ each with conclusion ϕ.
  Failure of local completeness is related to the E-rules being too weak w.r.t. the corresponding I-rules. For example, if one of the two conjunction-elimination rules of the intuitionistic conjunction is omitted, the remaining rule cannot expand and is not locally complete. However, note that local

[7] We thank a JPL referee for this observation.

[8] 'Eliminate' here means applying an elimination-rule, not necessarily actually eliminating the operator occurrence at hand.

[9] Clearly, the conjunction expansion and disjunction expansion below exhibit different relative ordering of I-rules and E-rules applications.

> completeness is only[10] *necessary* (and not sufficient) for the E-rules not being to weak. As was noted already in [4] (leading to the discussion of stability), the I-rule for disjunction in quantum logic, not allowing collateral assumptions in its minor premises, *does* expand, in spite of being too weak. It does fail stability w.r.t. the usual I-rule for intuitionistic disjunction.
>
> As mentioned already, we add local completeness as an augmentation to local soundness for harmony, in order to stay within the verificationist framework (and not deal also with stability).

Definition (Local Intrinsic Harmony) for an operator δ, its I-rules (if any) and E-rules are *locally intrinsically harmonious* iff they satisfy local soundness and local completeness.

The usual ND-rules for minimal logic are easily shown to be locally intrinsically harmonious. For example:

Implication The well known β-reduction and η-expansion.

$$\dfrac{\dfrac{\begin{matrix}[\phi]_1\\ \mathcal{D}_1\\ \psi\end{matrix}}{\phi\to\psi}\,(\to I^1) \qquad \begin{matrix}\mathcal{D}_2\\ \phi\end{matrix}}{\psi}\,(\to E) \;\rightsquigarrow_r\; \begin{matrix}\mathcal{D}_2\\ \mathcal{D}_1[\ \phi\ /\phi]\\ \psi\end{matrix} \qquad \begin{matrix}\mathcal{D}\\ \phi\to\psi\end{matrix} \;\rightsquigarrow_e\; \dfrac{\dfrac{\begin{matrix}\mathcal{D}\\ \phi\to\psi\end{matrix} \quad [\phi]_1}{\psi}\,(\to E)}{\phi\to\psi}\,(\to I^1) \tag{20}$$

Disjunction We show reduction for one rule, the other being similar.

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \phi\end{matrix}}{\phi\vee\psi}\,(\vee I)_1 \quad \begin{matrix}[\phi]_1\\ \mathcal{D}_2\\ \xi\end{matrix} \quad \begin{matrix}[\phi]_2\\ \mathcal{D}_3\\ \xi\end{matrix}}{\xi}\,(\vee E^{1,2}) \;\rightsquigarrow_r\; \begin{matrix}\mathcal{D}_1\\ \mathcal{D}_2[\ \phi\ /\phi]\\ \xi\end{matrix} \qquad \begin{matrix}\mathcal{D}\\ \phi\vee\psi\end{matrix} \;\rightsquigarrow_e\; \dfrac{\begin{matrix}\mathcal{D}\\ \phi\vee\psi\end{matrix} \quad \dfrac{[\phi]_1}{\phi\vee\psi}\,(\vee I)_1 \quad \dfrac{[\psi]_2}{\phi\vee\psi}\,(\vee I)_2}{\phi\vee\psi}\,(\vee E^{1,2}) \tag{21}$$

Conjunction is locally intrinsically harmonious too. Although there is no I-rule for $\bot$ (falsehood), the rules for Intuitionistic logic are locally intrinsically harmonious too, because the boundary case of *no* I-rules vacuously satisfied the requirement. The expansion obtained for $\bot$ is $\begin{matrix}\mathcal{D}\\ \bot\end{matrix} \rightsquigarrow_e \dfrac{\begin{matrix}\mathcal{D}\\ \bot\end{matrix}}{\bot}\,(\bot E)$, containing, indeed, *all* the (non-existing) $\bot$ I-rules.

[10] Thanks to the anonymous JPL referee who drew our attention to this point.

4 Relating Local Intrinsic and Generalized-Elimination Harmony

4.1 GE-harmony Implies Local Intrinsic Harmony

In this section, we establish the relationship between the two notions of harmony considered, and show that GE-harmony (under our extended GE-rule construction) is stronger than locally intrinsic harmony: the form of the GE-rules *guarantees* both local soundness and local completeness w.r.t. to the I-rules.

Theorem (Harmony Implication) *For any operator δ, its GE-rules harmoniously-induced by its I-rules are locally intrinsically-harmonious.*

Proof Assume the GE-rules for δ are harmoniously-induced by its I-rules.

> **Local soundness** The reduction of a δ-maximal formula was already presented in Eq. 16.
> **Local completeness** We have to show *some* way to expand a derivation of ϕ (with main operator δ). The way we do it is to choose, for each $1 \leq i \leq n$, the "arbitrary consequence" ξ in applications of the E-rule $\delta GE_{i,k}$ as $\alpha^i_{k,\ 1 \leq k \leq m_i}$ as well as Π_i itself, and take all of the supports, Σ^i_k, as assumptions (to be discharged by the δI-rule applications). We get the following expansion. For simplicity of notation, we treat each Π and Σ as a single formula, and the potential parameterization is suppressed.

$$\begin{array}{l}\overset{\mathcal{D}}{\phi} \rightsquigarrow_e \\ \dfrac{\dfrac{\overset{\mathcal{D}}{\phi} \quad [\hat{\Sigma}^i_1]\ [\hat{\Pi}_1, \hat{\alpha}^i_1] \cdots [\hat{\Pi}_n, \hat{\alpha}^i_1]}{\hat{\Pi}_i}(\delta E_{i,1}) \quad \dfrac{\overset{\mathcal{D}}{\phi}\ [\hat{\Sigma}^i_1]\ [\hat{\Pi}_1, \hat{\alpha}^i_1] \cdots [\hat{\Pi}_n, \hat{\alpha}^i_1]}{\hat{\alpha}^i_1}(\delta E_{i,1}) \quad \cdots \quad \dfrac{\overset{\mathcal{D}}{\phi}\ [\hat{\Sigma}^i_{m_i}]\ [\hat{\Pi}_1, \hat{\alpha}^i_{m_i}] \cdots [\hat{\Pi}_n, \hat{\alpha}^i_{m_i}]}{\hat{\alpha}^i_{m_i}}(\delta E_{i,m_i})}{\phi}(\delta I^{1,\cdots,n}_i)\end{array} \tag{22}$$

> where $\hat{\Pi}_i$ is $\Pi_i[y_i/x_i]$, $\hat{\Sigma}^i_j$ is $\Sigma^i_j[Y^i_j/X^i_j]$ and $\hat{\alpha}^i_j$ is $\alpha^i_j[Y^i_j/X^i_j]$, for fresh Ys and ys (in case the I-rule is conditional and/or parameterized).

As an example, consider the expansion for (*ite*), where $n = 1$ (and Π_1 is empty), the I-rules of which are given in Eq. 4, and E-rules in Eq. 11.

$$\dfrac{\dfrac{ite(\phi, \psi, \chi) \quad [\phi]_1 \quad [\psi]_2}{\psi}(iteGE^2_1) \quad \dfrac{ite(\phi, \psi, \chi) \quad [\neg\phi]_3 \quad [\chi]_4}{\chi}(iteGE^4_2)}{ite(\phi, \psi, \chi)}(iteI^{1,3}) \tag{23}$$

4.2 More Examples

We present some more examples of harmonically-induced E-rules and their reductions and expansions. The obtained reduction for the implication is:

$$\frac{\dfrac{\begin{matrix}[\phi]_1\\ \hat{\mathcal{D}}\\ \psi\end{matrix}}{\phi\rightarrow\psi}\,(\rightarrow I^1) \quad \begin{matrix}\mathcal{D}\\ \phi\end{matrix} \quad \begin{matrix}[\psi]_2\\ \mathcal{D}'\\ \xi\end{matrix}}{\xi}\,(\rightarrow GE^2) \quad \rightsquigarrow_r \quad \begin{matrix} \mathcal{D}\\ \hat{\mathcal{D}}[\phi/\phi] \\ \mathcal{D}'[\quad \psi \quad /\psi]\\ \xi\end{matrix} \tag{24}$$

The expansion of the implication is given by

$$\begin{matrix}\mathcal{D}\\ \phi\rightarrow\psi\end{matrix} \quad \rightsquigarrow_e \quad \frac{\dfrac{\begin{matrix}\mathcal{D}\\ \phi\rightarrow\psi\end{matrix} \quad [\phi]_1 \quad [\psi]_2}{\psi}\,(\rightarrow GE^2)}{\phi\rightarrow\psi}\,(\rightarrow I^1) \tag{25}$$

As another example, the reduction for restricted universal quantification is:

$$\frac{\dfrac{\begin{matrix}[\phi(x)]_1\\ \hat{\mathcal{D}}\\ \psi(x)\end{matrix}}{\forall x.\phi(x)\rightarrow\psi(x)}\,(eI_1) \quad \begin{matrix}\mathcal{D}_1\\ \phi(y)\end{matrix} \quad \begin{matrix}[\psi(y)]_i\\ \mathcal{D}'\\ \xi\end{matrix}}{\xi}\,(eE^i) \quad \rightsquigarrow_r \quad \begin{matrix} \mathcal{D}[y/x]\\ \hat{\mathcal{D}}[\ \phi(y)\ /\phi(x)] \\ \mathcal{D}'[\qquad \psi(y) \qquad /\psi(y)]\\ \xi\end{matrix} \quad d \tag{26}$$

(since x is fresh in $\hat{\mathcal{D}}$, its replacement by y does not affect any other assumption used in $\mathcal{D}$, and preserves correctness). The expansion for restricted universal quantification is given by

$$\begin{matrix}\mathcal{D}\\ \forall x.\phi(x)\rightarrow\psi(x)\end{matrix} \quad \rightsquigarrow_e \quad \frac{\dfrac{\forall x.\phi(x)\rightarrow\psi(x) \quad \phi(y) \quad [\psi(y)]_1}{\phi(y)}\,(eGE^1) \quad \dfrac{\forall x.\phi(x)\rightarrow\psi(x) \quad \phi(y) \quad [\psi(y)]_2}{\psi(y)}\,(eGE^2)}{\forall x.\phi(x)\rightarrow\psi(x)}\,(eI) \tag{27}$$

5 Conclusion

The paper considers two notions of harmony, an acceptability criterion for rules being meaning conferring along the conception of the proof-theoretic semantics school. A construction is presented for elimination-rules being "automatically" harmonious (under both notions) w.r.t. given I-rules. This is done also for rules for quantifiers.

A weakness of the GE-harmony is that it is not sensitive[11] enough to side conditions. This weakness stands in the way of regarding GE-harmony as a proper formalization of the pre-theoretic notion of harmony. For example, the only difference between the I-rule of the intuitionistic implication and the (strict relevant) implication of **E** [1] is, that in the latter, of formulas in the context Γ have themselves to be implications, no such side condition applies

[11]Thanks to an anonymous JPL referee for stressing this issue.

to the former. While this specific side-condition, of restricting the context to a sub-language,[12] is easily incorporable in the proposed transformation, by requiring the appropriate premisses of the induced E-rule to also belong to this sub-language, this clearly ad-hoc. What is really needed is some *general* scheme for side conditions. This is certainly a non-trivial task, deserving further study. Witness the variety of side conditions implementing various "discharge policies" on (copies of) the discharged premise, or the wealth of possibilities introduced by considering labelled deductive systems [8].

A major point deserving further research is finding proof-theoretic conditions for *global* soundness and completeness, namely existence of β-normal form and long η-normal form, as well as *strong* normalization.

A point we intend to investigate is applying the methodology presented here to the dual "pragmatist" approach to proof-theoretic semantics, whereby the E-rules serve as meaning conferring, and the I-rules use this meaning. In this case, *stability* was proposed by [4] as the acceptability criterion. Thus, it is interesting to see whether there is a construction, that will induce I-rules "automatically" for given E-rules, stable w.r.t. these given E-rules. We believe this is possible.

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[12] A similar restriction as side condition occurs in the $\Box$ I-rule for necessity in various modal logics.

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