DEFINITION AND ESSENCE FROM ARISTOTLE TO KANT

Edited by
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ROUTLEDGE

2 Definition and Demonstration in the Category of Quantity and the Ancient Search for the Definition of Ratio

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2.1 Introduction

The Aristotelian tradition has seen definitions as paradigmatically in the category of substance, such as the definition 'rational animal' that states what humans really are. But in practice definition has been much more successful in mathematics, that is (in Aristotelian language) in the category of quantity. It is Euclid and his mathematical successors who have most convincingly – indeed almost solely – realised the ideal of the *Posterior Analytics*, of a science laid out as a deductive structure of theorems proved from a set of absolutely clear definitions laying out the essential features of the subject matter.

Why that should be so is far from obvious. We examine how definitions in the category of quantity differ from those in substance, and consider in some detail the ancient definitions of two concepts central to mathematics, the circle and ratio. It becomes clear that the structural nature of quantity – quantity's being essentially related to parts – enables definitions of quantitative concepts to support complex proofs of theorems.

2.2 Definitions in the Category of Substance Versus Definitions in Other Categories

In the Aristotelian tradition, the examples of definitions discussed are almost all either in the category of substance or the category of quantity. Definitions work very differently in the two categories.

The Aristotelian/Porphyrian/Linnean classification of substances builds on and systematises the metaphysics implicit in at least Indo-European grammars, with their clear distinction between common nouns and adjectives. According to that metaphysics, the world is, by and large, uniquely divided into things, and those things are uniquely classified into species. The point of a definition (of a substance) is to state what attributes of a thing are characteristic of its species (Figure 2.1).

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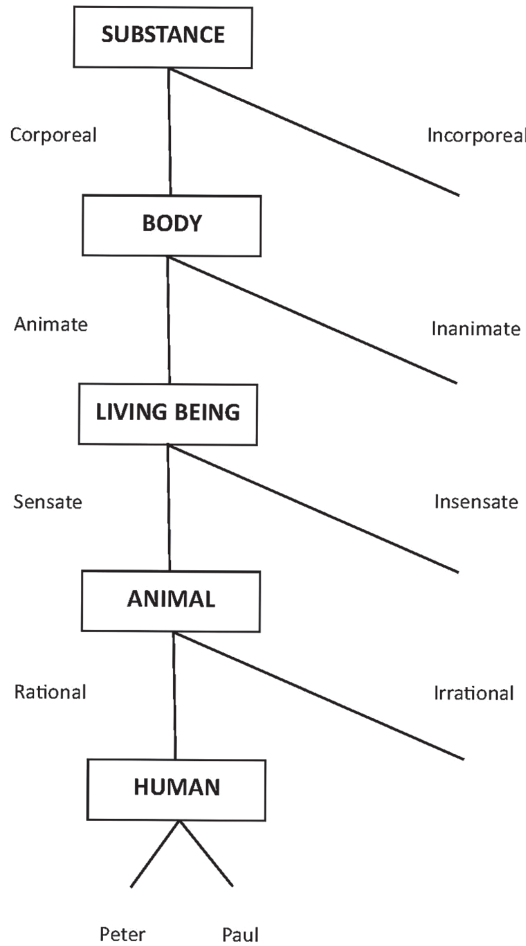

Figure 2.1 The traditional Tree of Porphyry.

In more detail:

1 The world is **uniquely divided** into **things** – for example (live) human hands are not things but parts of a unique human person

(But there are also **stuffs**, which are natural but whose division into individual things is artificial; wood is natural but its division into artefacts like beds, salad bowls and fragments of the True Cross is artificial)

Things (and stuffs) have **attributes** like colour and shape, contain parts, and stand in relations, all of which facts are real and independent of human response.

2 Things (and stuffs) are **uniquely divided** into **species**
(which appear in the classification of the Tree of Porphyry)
3 The uniqueness of division into species implies that some of a thing's attributes are **essential** (characteristic of the species, such as rationality for humans) and some accidental (such as whiteness for humans); the definition of a thing, reached by division down the tree of classification, states its **essence** which it shares fully with all other members of the species

Essentialism, the commitment to the reality of the essential/accidental distinction among properties, is thus a thesis additional to the mere realism about universals defended by Armstrong and others.[1]

The Porphryrean/Linnean tree classification of species suggests that species are distributed discretely (unlike for instance colours, which vary continuously). However, that is not a central part of the theory, and Aristotle himself admitted continuous variation between species among the more primitive life forms.[2]

It is certainly possible to deny that full Aristotelian metaphysics to varying degrees, from radical views that deny any reality to things[3] to those that take a realist view of things and their properties but deny the essential/accidental distinction (so there is only prime matter clothed with various properties such as shape, whiteness, and rationality).

For present purposes, the significance is that these debates, and the place of definitions in them, only make sense in the category of substance. They concern things (or stuffs) – full-bodied existing substances, not properties of substances.

2.3 Definitions in Categories Other than Substance

In principle, all categories other than substance ought to have Porphyrian-style classification trees. For example, the different genera of relations ought to be laid out (spatial, logical, social, mereological, kind-of, and so on), and then subclassified. It is mysterious why one never sees such trees (except to some small degree in recent Artificial Intelligence-related efforts at 'formal upper ontology'[4]).

A fundamental difference between classification of substances versus the other categories is that there seems to be no essential–accidental distinction in the other categories. While an individual person can be rational essentially but white accidentally, surely it makes no sense to say that an individual piece of the world is green essentially but triangular accidentally? If on the other hand, we consider properties of green itself, like resembling blue, those seem all necessarily true of green and not in any sense detachable in the way accidental properties are. It is true that 'being favoured by Titian' could be a relation that a certain shade of green enters into accidentally, but that does not correspond to the way in which 'white' is a genuinely inherent property of a human yet accidental.

Given that definition is possible in categories other than substance, it follows that definition in those categories must be fulfilling some purpose other than distinguishing between essential and accidental properties. There are both 'Aristotelian' and 'Euclidean' possibilities as to what that purpose could be. The 'Aristotelian' purpose would be for a definition to grasp what is 'deep' about an entity, in the way that in the category of substance 'rational animal' is a better definition of 'human' than is 'featherless biped'; it would give a philosophically satisfying account of what something 'really is'. That would contrast with a merely nominal definition, which might explain our linguistic usage and might be a useful starting point for investigation but would not get to grips with reality. As W. D. Ross explains Aristotle's thinking, using the definition of an eclipse (which is fundamentally geometrical though about the properties of physical bodies),

> If the moon is eclipsed because the light of the sun is shut off from the moon by the interposition of the earth, the definition of lunar eclipse is 'the moon's deprivation of light owing to the interposition of the earth'. The true definition, the only definition which is more than the mere account of the usage of a word, is a definition which states the efficient or final cause of the attribute's occurrence … If we are to reach a definition by the aid of demonstration, we must start with a partial knowledge of the nature of the *definiendum*, i.e. with the nominal definition of it such as the definition of eclipse as a loss of light.
>
> (Ross 1995, 49)

The 'Euclidean' possibility is that definition should identify axioms from which all truths about the entity should follow conveniently as theorems. One might surmise that a truly good definition would fulfil both purposes, as a 'deep' definition that explained what an entity really was might be the very one to generate theorems. But there seems no a priori guarantee of that. As we will see, a classic definition of 'circle' does fulfil both tasks well, but it proved hard to find a definition of 'ratio' that did both.

Since later sections deal just with definitions in the category of quantity, it should be mentioned that there are two puzzles in that category that do not apply to the other (non-substance) categories. They set the scene for the examples to be considered later.

The first puzzle is that there is something 'substancy' about quantities, reflected in the prominence of common nouns in mathematics – geometry is full of lines and circles, arithmetic of units and expressions like 'three tens', modern mathematics of sets, vectors, and the like. (That is different from the use of abstract nouns for properties like 'green' as in 'green resembles blue more than orange' – there the nouns refer to the universal itself, like 'triangularity', but triangles and sets are individuals.) That is evident in Aristotle's (and Euclid's) own language, which has been said to treat mathematical objects as 'quasi-substances'.[5] David Bronstein writes:

> Aristotle's theory of science is built around a fundamental ontological distinction between subjects (e.g., moon, human being, triangle) and their attributes (e.g., being eclipsed, being two-footed, having interior angles equal to two right angles).
>
> (Bronstein 2016, 6)

Triangles are subjects, but they are not substances. They are shapes. That difference was the source of tangled and unresolved ancient debates about whether mathematics needed some sort of 'intelligible matter' in which to draw figures.[6]

The same closeness between the categories of substance and quantity appears in the way they cut across the important division between two kinds of entity in Aristotle. Bronstein summarises them in this table:[7]

A	*B*
Those whose causes are the same	Things whose causes are different.
Primary and subordinate subject-kinds	Things that have a middle term – i.e. demonstrable attributes
Things whose essences are in no way demonstrable	Things whose essences are in some way demonstrable
Essences discovered by induction or division	Essences discovered by demonstration
Causally simple essences	Causally complex essences
E.g., human, moon, god, soul, unit.	E.g., eclipse, harmony, thunder, ice, 2R [angle sum in triangle is two right angles].

The kind of intellectual operation described in the left-hand column involves a deepening of understanding of, for example, the nature of humans, resulting in a definition of humans by division (that is, by reaching a leaf of Porphyry's Tree). The kind of intellectual operation in the right-hand column involves putting together connections in a syllogism, to demonstrate the cause of a complex phenomenon like an eclipse. It is to be expected that definitions of substances should be on the left and definitions of complex phenomena on the right. That is approximately so (noting that ice means a state of a substance rather than a substance itself), but it is significant that items from the category of quantity appear in both columns. A unit, or a circle, is in some way a 'thing', whose definition ideally should be reached by induction or division.

The second puzzle about the category of quantity, related to the 'Euclidean' purpose of definitions, is why axiomatisation has been so successful in that category compared to anywhere else.

These two puzzles are not easily solved. We will explain the issues and clarify them by looking in detail at two ancient examples, the circle and ratio.

2.4 Definitions in the Category of Quantity and the Success of Axiomatisation in Mathematics

What mathematicians want from a definition is theorems. That is premised on their acceptance of the Aristotelian-Euclidean model of a deductive structure for their science, with definitions and axioms supporting a superstructure of theorems deduced from them by strict logic.[8]

Aristotle hoped that all sciences would follow mathematics in becoming so organised, but that has not come to pass. One of the most salient facts of the history of science is that while axiomatisation has gone from strength to strength in mathematics, and in allied mathematisable disciplines such as logic and Newtonian mechanics, it has proved close to useless everywhere else. Spinoza's failed attempt to lay out ethics *more geometrico* is an icon of many similar wild goose chases after convincing axiomatisations of this body of knowledge or that. Why is that so?

And even where demonstration is available in sciences other than mathematics, the demonstration often happens in virtue of mathematical properties. In Aristotle's example of eclipses, the opacity of the Earth and the travelling of light in straight lines are physical facts, but those facts translate the problem into geometry and it is there that the demonstration proceeds. It is the same in the rare cases where demonstration is possible in biology: 'to know that circular wounds heal more slowly belongs to the doctor, but to know the reason why belongs to the geometrician', Aristotle says, presumably because wounds heal from the outside in.[9] No doubt his example with broad-leaved trees shedding their leaves[10] works somehow the same way; it is unclear quite how but 'broad' is a geometrical property.

So the question is: Is there something about the category of quantity that makes definitions in it especially apt for supporting a deductive structure?

Let us, in Aristotelian fashion, begin with an example. We will look at the classic case, Euclid's definition of a circle and its use in proving proposition 1 of Book I, and see if we can generalise.

What is a circle? (Figure 2.2)

The *Oxford English Dictionary* reports, surely correctly, the common understanding of what a circle is:

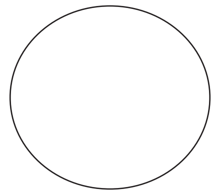

Figure 2.2 A circle, as naively conceived.

> *Circle*: **1.a**. A perfectly round plane figure.

'Plane figure' correctly situates the circle in its category and subcategory, that is, it begins a definition by division. But what is 'round'? According to the *OED* again,

> *Round*: **1**. Having the form of a circle or ring; shaped like a circle; circular.

That is not helpful. It makes the definition *circular*. At best it is a rough first-cut or nominal definition, as in Aristotle's first definition of an eclipse as the loss of light.

Euclid's definition is quite different:

> A *circle* is a plane figure contained by one line such that all the straight lines falling upon it from one point among those lying within the figure are equal to one another.[11]

Bernard Lonergan's *Insight* rightly praises this definition as a significant intellectual achievement.[12] The first-cut definition above does not refer to the centre at all – it needs a conscious intellectual act to understand that the equality of lines from the centre determines and is determined by the perfect roundness of the circle (Figure 2.3).

The advantages of this definition for deriving theorems become apparent immediately in the very first proof (Figure 2.4).

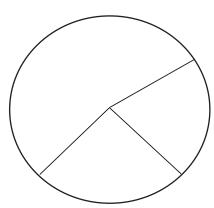

Figure 2.3 A circle, with radii illustrating Euclid's definition.

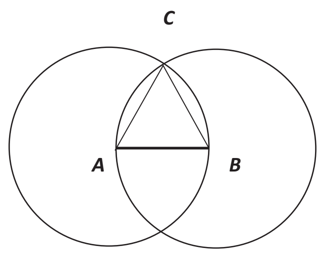

Figure 2.4 Euclid I.1: To construct an equilateral triangle on a given straight line AB.

Euclid's proof is (with some names of points omitted):

> Describe the circle with centre A and radius AB. [*Postulate 3: to describe a circle with any centre and radius*]
>
> Again describe the circle with centre B and radius BA. Join the straight lines CA and CB from the point C at which the circles cut one another to the points A and B. [*Postulate 1: to draw a straight line from any point to any point*]
>
> Now, since the point A is the centre of the first circle, therefore AC equals AB. Again, since the point B is the centre of the second, therefore BC equals BA.
>
> But AC was proved equal to AB, therefore each of the straight lines AC and BC equals AB.
>
> And things which equal the same thing also equal one another, [*common notion 1*] therefore AC also equals BC.
>
> Therefore the three straight lines AC, AB, and BC equal one another.
>
> Therefore the triangle ABC is equilateral [*definition 20: an equilateral triangle is one with three sides equal*], and it has been constructed on the given finite straight line AB.

(The very elaborate geometry of circles that the definition makes possible is then developed in *Elements* Book III.[13])

What that brings to the fore is the *structural* nature of two-dimensional space (the quantity being studied here) – that is, its parts, their properties such as equality (e.g., of length), and their overlapping. That is what allows the parts and their properties to be 'followed around' in the different steps of the proof. The role of partedness is announced in Euclid's first definition: 'A *point* is that which has no part'. In contrast, everything else does have parts (lines, circles, planes, and so on) and indeed consists of many identical parts (e.g., the parts of a line are lines or points). The definition of the circle works well in the proof because the radii are parts of space equal in length – the proof works by laying out well-chosen equal circles, then equal radii.

That is not the case for definitions reached by division. ‘A human is a rational animal’, or ‘A circle is a plane figure which is perfectly round’ or (see below) ‘A ratio is a sort of relation in respect of size between two magnitudes of the same kind’, however true and philosophically helpful they may be, do not give proofs much to latch onto. Given such a definition, where could a proof start?

Proof has to proceed through certain steps. So, it must be able to follow through a chain of necessary connections. It can therefore be thought of as exposing connections between the per se attributes of the subject of demonstration.[14] The syllogistic schema, where A’s being necessarily B and B’s being necessarily C implies A’s being necessarily C, is reasonable when all of A, B, and C are per se attributes of something. The Euclidean model of demonstration, which involves in addition the construction of geometrical entities with parts whose necessary connections can be followed through from part to part, still fits that model but adapts it to the parted nature of the category of quantity.

The relation of quantity and partedness is especially clear for ‘extensive’ quantities, the paradigmatic quantities like length and mass that distribute over parts – if something has a length of two metres, it consists of two parts of length of one metre. Those are the kinds of quantities most easily measured (e.g., by laying out unit rods) and are most suited to mathematical treatment generally. It is harder with ‘intensive’ quantities like velocity and temperature, which are not easily related to the quantity of parts – a velocity of two metres per second does not consist of two parts of one metre per second each.[15] Not surprisingly, measurement and demonstration for intensive quantities is difficult, whereas for extensive quantities it is relatively easy, being just geometry.

The complex and parted nature of quantity is different not only from substance but from the other categories such as quality. That is brought out by a comment from D. C. Williams:

> Experience contains some characters, such as yellowness, which are in some sense phenomenally simple, as well as other characters, such as triangularity, which are in the corresponding sense phenomenally complex. A triangle may be analytically described as a plane closed rectilinear figure with three sides and three angles. But, as a matter of fact, most persons actually recognize the proper occasions for the application of the word ‘triangular’ by a massed effect, a gestalt-quality, without ever having verbally formulated or being cognitively aware of the single details of this elaborate constitution. As Ducasse has it, they know the meaning of ‘triangularity’ intuitively but not discursively.
>
> (Williams 1937, 417)

It is the fact that triangularity, unlike yellowness, is 'phenomenally complex' that allows the complexity, initially confused and gestalt-like, to be teased out discursively, that is, expressed in a definition.

2.5 The Concept of Ratio and Incidents in the History of Ratio

All these themes – of quantity and relation, of demonstration needing a complex definition not simply arising from division, of complexity and partedness – are evident in the ancient search for a satisfactory definition of ratio.

Ratio or proportion is one of the most basic concepts of mathematics. It is the fundamental relation in which quantities of the same kind stand: a length can be double or triple or one and a third times another length, and given any two lengths whatsoever they stand in some ratio. It is the same for two areas or two velocities. As Euclid puts it, 'a *ratio (logos)* is a sort of relation in respect of size between two magnitudes of the same kind'.[16] Indeed, in Aristotle it is the paradigmatic example of relation.[17] A ratio is not the same thing as a number: the number two is how many, the double is a ratio. It is true that if one clones an apple, the ratio of the mass of the pair to the mass of one is the double. But 'two' counts, whereas 'double' compares size.

Ratio is in a sense abstract in that the same ratio is realisable between different quantities: the ratio the triple can be found between heights, between masses, between time intervals, and so on. It is not 'substancy' like a circle, because it is a relation. But it is easily visible (or ponderable, or estimable by time sense, as the case may be), as in the following diagram where the ratio of height to width is immediately perceptible[18] (Figure 2.5).

John Bigelow writes, concerning the reality of ratio and the relation between different kinds of ratio, as well as explaining how ratio is unlike a simple physical property:

> Physical objects, like elephants and Italians, humming-birds and Hottentots, have many physical properties and relations: volume and surface area, for example. And the physical properties of these objects stand in important relations to one another. In particular, such physical properties stand in relations of proportion to one another. There is a relation between the surface area of the humming-bird and that of the Hottentot; and this may or may not be the same as the relationship that holds between the surface areas of an Italian and an elephant. Relationships such as proportion will hold not only between surface areas but also between volumes …[19]

Mice can scurry but elephants can't because mass scales up differently from the muscle cross-section that determines strength: if animal B is twice the

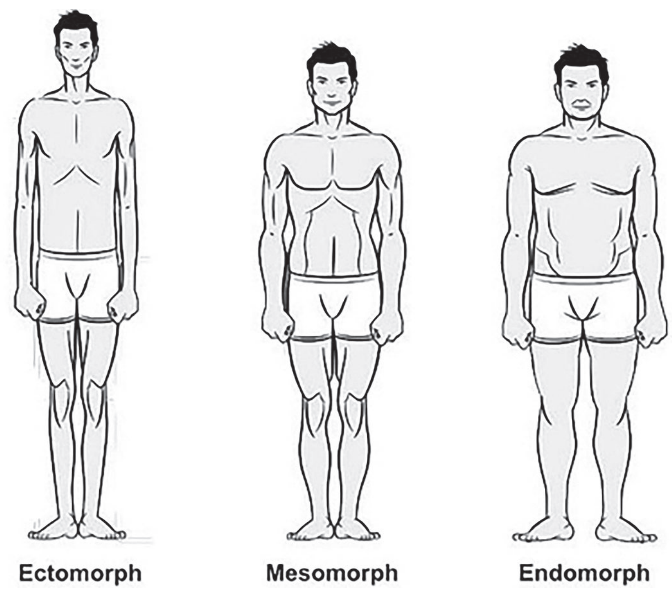

Figure 2.5 Sheldon's 'somatotypes' of human body shapes. https://en.wikipedia.org/wiki/Somatotype_and_constitutional_psychology#/media/File:Bodytypes.jpg

length of animal A (and otherwise in proportion), its mass is eight times but its muscle cross-section only four times. In such examples, ratio, though abstract, is intimately involved in scientific explanation.

A few incidents in the history of ratio will indicate its ubiquity in mathematical science. It can lay claim to be *the* key to the mathematisation of nature. These incidents will also give some sense of the varied perspectives on ratio that might need to be captured in a definition.

As soon as mathematical skills developed, there were demands to fix the calendar. The calendar problem is generated by the fact that the ratios of the day, (mean) lunar month and year appear to have been designed to make the calendar as difficult as possible. If 12 months of 30 days each fitted in the year, the calendar would be easy, but that is not the case. Old Babylonian astronomy addressed the problem and calculated vigorously in sexagesimals, which is equivalent to regarding ratios as all representable as indefinitely long decimals.[20]

Ratio is conceived quite differently in the earliest surviving Greek mathematical proof, the quadrature of lunes by Hippocrates of Chios (about 450 BCE) – not by coincidence, the oldest text in which the deductive structure of Greek geometry is visible. Hippocrates proved that, in the following diagram, the area of the lune (the shaded curved area between the outer and inner semicircles) is exactly the same as the area of the shaded triangle.

No reference to π or formula for the area of a circle is involved – indeed, it is impossible to do anything similar with a circle (that is, construct a figure with straight sides which has the same area as the circle) (Figure 2.6).[21]

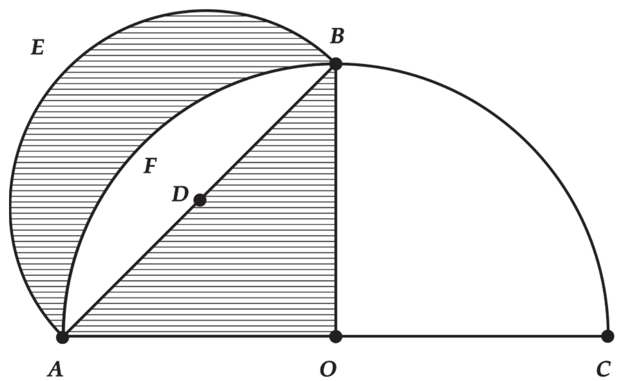

Figure 2.6 Hippocrates' quadrature of the lune After https://en.wikipedia.org/wiki/Lune_of_Hippocrates#/media/File:Lune.svg

Simplicius's text transmitting Hippocrates' proof begins by stating: 'Similar segments of circles have the same ratio to one another as the squares on their bases'.[22] So ratio appears in the first step, and it means a ratio of the areas of perfect geometrical figures. The justification for that is not stated, but the claim is very close to obvious: if any shape, such as a segment of a circle, is thought of as filled up with little squares, then if it is scaled up by some factor, the squares expand by the square of that factor, so the whole area does too. (A similar argument in Euclid XII.2 may or may not go back to Hippocrates.)

But the square on the base AOC of the large semicircle is (in area) twice the square on the base ADB of the smaller semicircle. That is by Pythagoras' theorem applied to the right triangle AOB – but in this simple case the result is exactly the one proved directly by the slave boy in Plato's *Meno*: if we delete the semicircles from Hippocrates' diagram and add the squares he mentions, we get exactly the diagram of the *Meno*. The result there is also about exact ratios of perfect figures (Figure 2.7):

It follows that in the lunes diagram, the small semicircle (ADBE) has the same area as the quarter of the big semicircle (AOBF). On subtracting from both their common area, the unshaded segment (ADBF), we conclude that the shaded lune is the same area as the shaded right triangle. The proof has been accomplished solely in terms of ratios and equality of areas.

The other most celebrated discovery of early Greek mathematical science was the role of ratios in music. Myles Burnyeat writes of

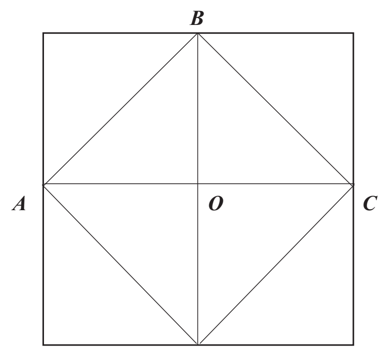

Figure 2.7 The diagram of Plato's Meno (In the *Meno* diagram, the large square, with diameter **AC**, is twice the area of the inner square in 'diamond position', with side **AB**.).

> … the all-pervasive role of ratio in Greek mathematics. From arithmetic through plane and solid geometry to astronomy, ratio and proportion keep turning up in the proofs. Harmonics, though mathematically simpler than advanced geometry and astronomy, is the first discipline to take ratio itself as the primary object of study.[23]

That led to perhaps the most extraordinary event in the history of ratio, the attempt by Plato and in some circumstances Aristotle to actually identify the Good with ratio. Plato conceives the Good on the model of harmony in music.[24] Harmony can be heard and can sound beautiful, but the superior science of music involves an intellectual study of why that is so, of the mathematics of integer ratios such as the octave and fifth that explain (heard) harmony.[25] His 'ethics' is then identified with harmony – that is, with the mathematical system of ratios that stands behind heard harmonies and so can be realised in other categories (such as planetary motions in the 'music of the spheres'). Justice is *defined* as an attunement, in the first instance in the soul[26] and later in the structure of the perfect state. Thus, in Plato, 'the Good is described formally, even mathematically, it is ratio'. Or at least, if the Good is strictly speaking ineffable, 'Measure is the Good in so far as it can be grasped by reason'.[27]

Aristotle's mathematical account of 'justice' betrays the same close conceptual connection between the 'just' and ratio as in Plato, even though it does discuss cases of justice that are closer to our ethical meaning of the word, such as compensation for fraud. His argument that a just division is a proportional one does not rely on any ethical premises. The reason he gives is simply that the just *is* a species of the proportional, and he feels that needs no further justification.[28]

Discussions of a similar tendency on proportion in art, architecture, and the human body, often involving the Golden Ratio, have of course been vast.[29]

The later history of ratio is too extensive for a survey to be attempted here, but a simple list of the laws of nature characteristic of the Scientific Revolution will indicate the crucial role that ratio plays at the core of modern science. It is notable in these laws that a proportion (sameness of ratio) is asserted between magnitudes of different kinds – for example, force and acceleration are quite different kinds of quantities, but twice the force causes twice the acceleration:

- **Kepler's** Second Law: The area swept out by a radius from the sun to a planet is proportional to the time taken
- **Snell's** Law: The sine of the angle of refraction is proportional to the sine of the angle of incidence
- **Galileo's** Law of Uniform Acceleration: The speed of a heavy body falling from rest is proportional to the time from dropping
- **Pascal's** Law: The pressure in an incompressible fluid is proportional to depth
- **Hooke's** Law: The extension of a spring is proportional to the force exerted to stretch it
- **Boyle's** Law: For a fixed quantity of gas at constant temperature, pressure is inversely proportional to volume
- **Newton's** proposition on the prism: there is some kind of proportionality between refrangibility and colour of light
- **Newton's Second** Law of Motion: The acceleration of a body is proportional to the total force acting on it
- **Newton's Law of Gravity**: The force of gravity exerted by one body on another is proportional to the masses of each and inversely proportional to the square of the distance between them
- **Newton's Law of Cooling**: The rate of temperature loss from a body is proportional to the difference in temperature between the body and its surroundings.

2.6 Ancient Definitions of Ratio

That leaves the question of how to define ratio. Ideally, it should be defined in a way like the definition of a circle, suitable for supporting the deduction of theorems about it. The typical sort of theorem that one might want to deduce is the alternation of ratios: supposing a, b, c, and d to be quantities of the same kind, if the ratio of a to b equals the ratio of c to d, then the ratio of a to c equals the ratio of b to d. (A favourite example of Aristotle's.[30])

The previously mentioned definition of Euclid, that 'A *ratio (logos)* is a sort of relation in respect of size between two magnitudes of the same kind' (Euclid V.3) is a true description but incomplete as a definition. What sort of relation? It correctly approaches the question by division: the category of ratio is relation and the subcategory relation between quantities. But that characterisation also fits, for example, 'greater than', and if we look for some differentia to explain how 'greater than' differs from 'ratio', nothing springs to mind and Euclid does not offer anything. It is like defining humans as 'a sort of animal'. The process of division has not reached a definition useful for proving theorems, which explains the fact that it does not appear in any of Euclid's proofs.

A natural thought, implicit in the Babylonian approach of reducing everything to calculation with sexagesimals, is that all ratios of quantities should be reduced to ratios of whole numbers by choosing a suitable unit for each quantity. It should be easy to *compute* the ratio of two quantities of the same kind, say two lengths, by finding a unit small enough to measure both and counting how many of the unit is in each length.

That still needs some explication of the ratio of whole numbers, but that seems reasonably clear: the ratio of 9 to 6 is explained by saying that the unit 3 fills up 6 twice and 9 three times, hence the ratio is 3 to 2. The ratio of two numbers is definable and computable exactly via their relation to the common unit. Equality, at least, of such ratios is as laid out in the definition of Euclid Book VII (on number theory), Definition 20: 'Numbers are *proportional* when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth'.

Furthermore, given two lengths, there is a straightforward computational procedure for finding their common measure. The procedure is variously called 'reciprocal subtraction'/*anthyphairesis*/the Euclidean algorithm, or in Aristotle *antanairesis*.[31]

Given two lengths A and B, we see how many times the smaller one (say B) fits into the larger one A (in the example of Figure 2.8, 3 times). If it does not fit exactly a whole number of times, there is a remainder R that is smaller than both A and B. (If B does fit exactly, then of course B itself is the unit that measures both A and B.) Any unit that measures both A and B must also measure R (since R is just A minus a whole number of B's).

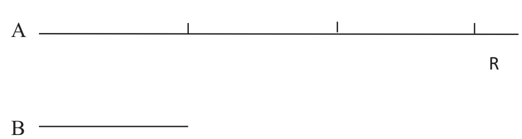

Figure 2.8 Anthyphairesis of lengths A and B to find their common measure.

So we can repeat the process with R and B, either finding that R measures B (and hence A as well), or that there is a smaller remainder R′, which must also be measured by any unit that measures A and B. And so on. Since we always get smaller remainders at each step, we work our way down until the last remainder is the unit that measures all previous remainders and hence also measures A and B.

Now, what happens if we apply *anthyphairesis* to those two very naturally occurring lengths, the side and diagonal of a square?

The side fits once into the diagonal, with a remainder left over (in bold in Figure 2.9). That remainder appears three times in the diagram.

It fits twice into the (original) side (Figure 2.10), and when we take the (small) side length out of the (small) diagonal (the diagonal of the small square in 'diamond position'), we are in the same position as we were originally with the larger square: taking a side out of a diagonal. Thus, the small square, with its diagonal, is a repeat of (the same shape as) the large square with its diagonal, so *anthyphairesis* goes into a loop and keeps repeating: at each stage, one side-length is taken out of one diagonal. Therefore, the remainders just keep getting smaller and smaller and the process never ends. There is thus *no* unit that measures the original diagonal and side. The diagonal and side of a square are 'incommensurable'.[32]

The later legend that the discovery of incommensurability created a foundational crisis in mathematics is not supported by ancient evidence.[33] Incommensurability is first mentioned in Plato without any suggestion that it created a crisis. Nevertheless, there is something surprising about it. Aristotle remarks that 'the incommensurability of the diagonal of a square with the side … that there is a thing which cannot be measured even by the smallest unit' is surprising to the uninformed but the opposite would be even more surprising for experts who understand the reason.[34] It does

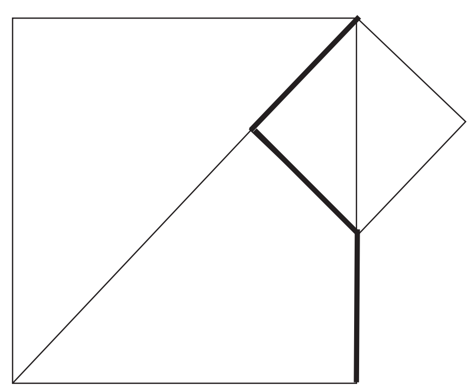

Figure 2.9 Anthyphairesis of diagonal and side of a square.

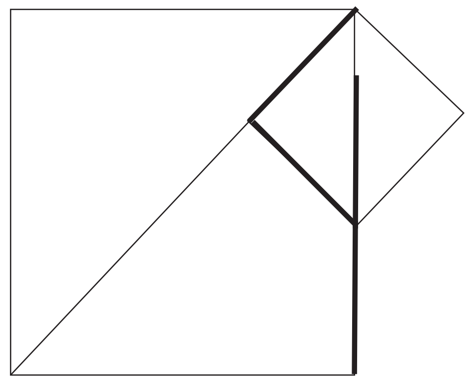

Figure 2.10 Anthyphairesis of diagonal and side: second stage.

suggest two natural conclusions. One is that the discrete and the continuous are fundamentally different mathematically, and must each be approached on their own terms; continuous versus discrete has remained a fundamental theme of mathematics since.[35]

The second conclusion is that some definition of ratio is needed that will apply to continuous quantities as well as discrete ones, and hence it cannot rely on units.[36] And if *anthyphairesis* has created the problem, it can also supply the solution. Sameness of ratio can be defined by sameness of *anthyphairesis*. That is what Aristotle says:

> But it also resembles what happens in mathematics where some things are not easily proved due to a deficiency of definition, e.g., that the line parallel to the side that cuts the plane similarly divides both the line and the area. But when the definition is stated it is straight-away obvious what is meant. For the areas and the lines have the same *antanairesis*. But this is the definition of the same ratio.
>
> (Aristotle, *Topics* 8.3, 158b29–35)[37]

Figure 2.11 makes that clear: whatever operations of subtraction are undertaken with the two lines at the bottom can also be undertaken with the parallelograms standing on them, so (the areas of) the parallelograms must stand in the same ratio (as defined by *anthyphairesis*) as the lines (bases).

Sameness of *anthyphairesis* is however an awkward definition of sameness of ratio. It suits computation with individual pairs of quantities, but it does not easily support proofs of general truths about ratio, such as the principle of alternation of ratios.[38]

That brings us to Book V of Euclid's *Elements*.[39] It is a self-contained treatise on ratios in general and does not require results from the earlier

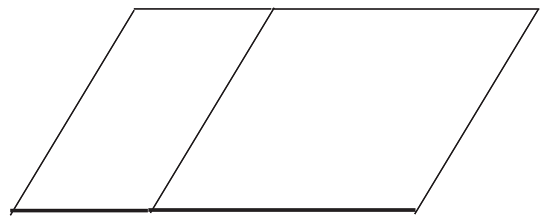

Figure 2.11 Two parallelograms have the same anthyphairesis as their bases.

books on geometry. After the statement that 'A ratio is a sort of relation in respect of size between two magnitudes of the same kind' (Definition 3), which is not used in any proof, there is no further attempt to define ratio as such. Instead, Euclid states the crucial Definition 5, of *equality* of ratio:

> Magnitudes are said to be *in the same ratio*, the first to the second and the third to the fourth, when, if any equimultiples whatever are taken of the first and third, and any equimultiples whatever of the second and fourth, the former equimultiples alike exceed, are alike equal to, or alike fall short of, the latter equimultiples respectively taken in corresponding order.

Note that here the 'first and second' are magnitudes of the same kind, the 'third and fourth' magnitudes of a possibly different kind, while 'equimultiples' refers to multiplying by whole numbers (whereas the magnitudes themselves may be numerical, geometrical or other). In that context, it would not be possible to define equality of ratios in the modern fashion as 'a to b equals c to d if $ad = bc$', since it is impossible in general to multiply quantities – for example, if they are all areas, it is impossible to multiply two areas. It is hard to see how a simpler definition could be thought of that was capable of dealing with the ratios of arbitrary quantities.

That is certainly an awkward definition to use, but just feasible. Its first use is in Proposition 4, 'If a first magnitude has to a second the same ratio as a third to a fourth, then any equimultiples whatever of the first and third also have the same ratio to any equimultiples whatever of the second and fourth respectively, taken in corresponding order'.

In Book VII, on number theory, Euclid starts again by defining proportionality, or equality of ratio, by Definition 20:

> Numbers are proportional when the first is the same multiple, or the same part, or the same parts, of the second that the third is of the fourth.

That is simpler than the definition in Book V since all quantities involved are whole numbers. There is no need for the awkward interleaving of numbers with other kinds of quantities nor for the use of many equimultiples to cope with the possibility that the quantities are incommensurable.

Those definitions may be more complicated than might be desirable, but as Euclid proceeds to show, they support the theorems. It must be admitted however that these two definitions, driven as they are by mathematical requirements (to generate theorems) have diverged from what is philosophically satisfying. Only Euclid's incomplete first definition, that ratio is 'a sort of relation in respect of size' addresses the question of what ratio really is. Unlike in the case of the circle, it seems to be impossible to find a definition of ratio that is both philosophically satisfying and mathematically productive.

2.7 Modern Times

From late antique through Arabic to medieval and early modern times, there were many complaints about Euclid's definitions of ratio, both about their complexity and there being three of them.[40] Attempts to do better were not notably successful, but there was a general tendency to move in the direction of arithmetisation, that is, to translate all quantities into numbers.

From the mid-seventeenth century, ratios came to be identified with numbers, including irrational numbers, instead of as relations between various kinds of quantities.[41] It proved, in the end, possible to unite ratio theory of magnitudes and of numbers, but only at enormous mathematical expense and philosophical unclarity. The costs appear in two places, in the pure mathematical theory of the 'construction of the continuum' and in the 'theory of measurement' that connects pure mathematics to extra-mathematical quantities.

The continuum is the set of all possible ratios, if we identify ratios with numbers, rational and irrational.[42] Officially, in the modern foundations of mathematics, the continuum is the (infinite) set of (infinite) Cauchy sequences (themselves infinite sets) of rational numbers, while rational numbers themselves are (infinite) equivalence classes of pair of integers.[43] It is possible to believe in those infinite sets of infinite sets with practice, and no contradictions have been found to arise. But the procedure does hide the intuitions by which we recognise which set of infinite sets does have the structure that we recognised a priori as the continuum, that is, the real line.[44]

Additionally, to apply the theory of the continuum to ratios between real-world quantities like lengths, one needs a (Platonist) theory of measurement to associate numbers with parts of the world. 'Measurement theory officially takes homomorphisms of empirical domains into (intended) models of mathematical systems as its subject matter', as one recent writer puts it.[45] That is enjoyable mathematics, but it is philosophically in vain as the system of ratios has to exist in the magnitudes for the homomorphism to exist. As Joel Michell puts in, in language reflecting the Aristotelian realism about ratios of quantities that lies at the heart of Euclid's Book V definitions:

> The commitment that measurable attributes sustain ratios has a further implication, viz., that *the real numbers are spatiotemporally located relations*. It commits us to a realist view of number. If Smith's weight is 90 kg, then this is equivalent to asserting that the real number, 90, is a kind of relation, viz., the kind of relation holding between Smith's weight and the weight of the standard kilogram. Since these weights are real, spatiotemporally located instances of the attribute, any relation holding between them will likewise be real and spatiotemporally located. This kind of relation is what was referred to above as a ratio. So the realist view of measurement implies that *real numbers are ratios*. By way of contrast, the standard view within the philosophy of mathematics is that numbers are abstract entities of some kind, entities not intrinsic to the empirical context of measurement, but related externally to features of that situation by human convention.
>
> (Michell 2005, 287)[46]

The significance of those complexities is to lay bare the difficulty of the task that Euclid and his predecessors faced in searching for a correct definition of ratio. From the philosophical point of view, Euclid's definitions of ratio (and the debates leading up to them) remain at least as illuminating as their modern replacement in the complex superstructure of infinite sets of the 'construction of the continuum'.

2.8 Conclusion

The history of the definitions of 'circle' and 'ratio' confirms the power of the Aristotelian ideal that definitions should explain 'what a thing is' and as a result should support a superstructure of demonstrated theorems about that thing. Definitions in the category of quantity, and the resulting mathematics, have turned out to confirm that ideal more convincingly than definitions in other categories. And even though mathematical requirements and philosophical requirements can sometimes come apart, the ideal is still a live one in mathematics. Jamie Tappenden writes:

> The idea that discovering the proper definition can be a significant advance in knowledge has overtones of a classical distinction between 'real' and 'nominal' definition. 'Real definition' has fallen on hard times in recent decades. Enriques' *The Historic Development of Logic*,

> published in 1929, addresses the topic throughout, in a whiggish recounting of the emergence of the idea that 'real definition' is empty and that all definitions are nominal. That seems to be where things stand now. We might need to rework too many entrenched presuppositions to revive the distinction in its traditional form, but it would help to reconstruct a minimal doctrine to support the distinctions we want to draw and connections we want to make. The core motivation is that in mathematics (and elsewhere) finding the proper principles of classification can be an advance in knowledge.
>
> (Tappenden 2008, 269)[47]

The 'distinction in its traditional form' remains live, and a study of Euclid as much as a study of modern mathematical definitions shows why.

Notes

1 Recent defences in Oderberg 2009 and Feser 2014.
2 Franklin 1986.
3 For example, Ladyman and Ross 2007.
4 For example, Mascardi, Cordi, and Rosso 2007.
5 Wilck 2020, 365; also Katz 2018, Scholz 1963, section 4.
6 Gaukroger 1980.
7 Bronstein 2016, 138.
8 McKirahan 1992, chs. 11–12; Acerbi 2013.
9 Aristotle, *Posterior Analytics* 1.13, 79a14–16.
10 *Posterior Analytics* 2.16, 98b35–9.
11 Euclid, *Elements* Book 1 Definition 15; all translations of Euclid are from the slightly simplified version of Heath 1956 which is available at http://aleph0.clarku.edu/~djoyce/elements/bookI/bookI.html.
12 Lonergan 1970, ch. 2.
13 Artmann 2001, ch. 9.
14 Bronstein 2019.
15 Franklin 2014b.
16 Euclid, *Elements*, Book V, Definition 3.
17 Aristotle, *Metaphysics* Δ.15, 1020b26–1021a10.
18 Aristotle's thoughts on the role of ratios in perception are discussed in Barker 1981.
19 Bigelow 1993, 74–5.
20 Neugebauer 1975, 353–7.
21 On the difficult ancient evidence on what exactly Hippocrates thought he had done, see Lloyd 1987.
22 Heath 1921, 187.
23 Burnyeat 2000, 73.
24 Burnyeat 2000, section 10.
25 See Plato, *Republic* 7, 531.
26 Plato, *Republic* 4, 443d–e.
27 Demos 1937, 261.
28 *Nicomachean Ethics* 5.2–4; Balinski 2005.

29 Livio 2003.
30 For example, *Nicomachean Ethics* 5.3, 1131b11; proved in Euclid V.16.
31 Aristotle, *Topics* 8.3, 158b29–35.
32 No ancient text exactly says this so it is a reconstruction: the late texts of Theon of Smyrna and Proclus that hint at it are given in Fowler 1979, 819, 845 and Fowler 1999, section 2.4(e); discussion also in Knorr 1975, ch. 2.
33 The lack of evidence is surveyed in Fowler 1999, 356–69.
34 Aristotle, *Metaphysics* A.2, 983a12–20.
35 Franklin 2017.
36 Discussion of the possible strategies at this point in Rusnock and Thagard 1995.
37 Translation and discussion in Mendell 2007.
38 Rabouin 2016.
39 Artmann 2001, ch. 14.
40 Plooij 1950, De Young 1996, De Risi 2016.
41 Brief survey with references in De Risi 2014, 22–3, 326, some aspects in Barbin 2010, Goldstein 2000.
42 Discussion on the relation with Euclid in Mueller 1981, section 3.1 and Stein 1990.
43 For example, Sohrab 2003, Appendix A.
44 Franklin 2014a, 108.
45 Azzouni 2004, 16.
46 Michell 2005, 287.
47 A similar message by prizewinning mathematician Claire Voisin in Castelvecchi 2024.

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