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MATHEMATICAL NECESSITY AND REALITY

Einstein said:

> As far as the propositions of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality. ([2]).

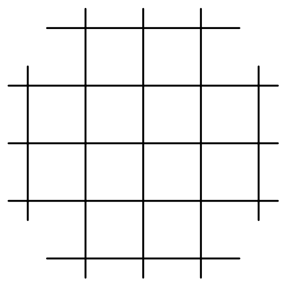
It is clear that by 'certain' he meant 'necessary', and philosophers of this century have mostly agreed with him that there cannot be mathematical truths that are at once necessary and about reality.

This proposition is denied.

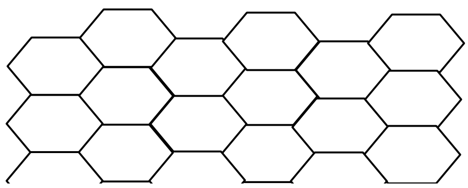
If the proposition were affirmed, it would be natural to begin by explaining away either the apparent necessity of mathematics or its apparent reference to reality. The case is different with a denial. Here the correct course is to exhibit some examples of mathematics which appear to be both necessary and about reality, and then reply to the standard objections. The aim, of course, is to choose examples that make the replies easy.

Example 1.
It is impossible to tile my bathroom floor with (equally-sized) regular pentagonal lines.

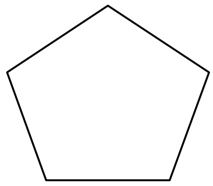
It is a proposition of geometry that 'it is impossible to tile the Euclidean plane with regular pentagons'. That is, although it is possible to fit together (equally-sized) squares or regular hexagons so as to cover the whole space, thus:



and



it is impossible to do this with regular pentagons:



No matter how they are put on the plane, there is space left over between them.

Now the 'Euclidean plane' is no doubt an abstraction, or a Platonic form, or an idealisation, or a mental being – in any case it is not 'reality'. If the 'Euclidean plane' is something that could have real instances, my bathroom floor is not one of them, and it may be that there are no exact real instances of it at all. It is a further fact of mathematics, however, that the proposition has 'stability', in the sense that it remains true if the terms in it are varied slightly. That is, it is impossible to tile a (substantial part of) an almost Euclidean-plane with shapes that are nearly regular pentagons. (The qualification 'substantial part of' is simply to avoid the possibility of taking a part that is exactly the shape and size of one tile; such a part could of course be tiled). This proposition has the same status, as far as reality goes, as the original one, since 'being an almost-Euclidean-plane' and 'being a nearly-regular pentagon' are as purely abstract or mathematical as 'being an exact Euclidean plane' and 'being an exactly regular pentagon'. The proposition has the consequence that if anything, real or abstract, does have the shape of a nearly-Euclidean-plane, then it cannot be tiled with nearly-regular-pentagons. But my bathroom floor does have, exactly, the shape of a nearly-Euclidean-plane. Therefore, it cannot be tiled with tiles which are, nearly or exactly, regular pentagons.

The 'cannot' in the last sentence is a necessity at once mathematical and about reality.

Example 2.

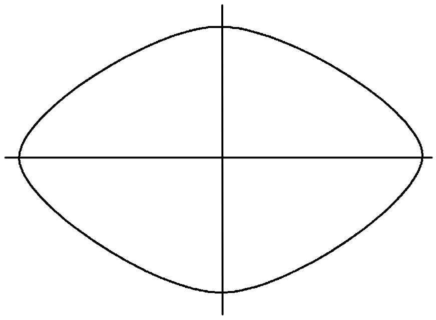
It is impossible to build a circular or nearly-circular staircase that goes up all the way round and ends at its starting point. (The famous Escher drawings which seem to show this kind of thing happening are thus impossible to realise.) The impossibility is not just empirical, since no change in the laws of nature would make such a staircase possible. There is a purely mathematical fact underlying the impossibility, namely, that there is no continuous function from the circle to the real numbers which is increasing all the way round. The proposition has therefore nothing to do with the Euclideanness or otherwise of space; in any space where 'up' makes sense, the statement is true.)

If a staircase as described were to be built, there would be a real thing which violated the mathematical theorem. So the existence of the real thing is mathematically impossible.

The two examples given were examples of impossibility. The last one will be an example of necessity in the full sense.

Example 3.

For simplicity, let us restrict ourselves to two dimensions, though there are similar examples in three dimensions. A body is said to be symmetrical about an axis when a point is in the body if and only if the point opposite it across the axis is also in the body. Thus a square is symmetrical about a vertical axis, a horizontal axis and both its diagonals. A body is said to be symmetrical about a point P when a point is in the body if and only if the point directly opposite is across P is also in the body. Thus a square is symmetrical about its centre. The following is a necessarily true statement about real bodies: All bodies symmetrical about both a horizontal and a vertical axis are also symmetrical about the point of intersection of the axes:



Again, the space need not be Euclidean for this proposition to be true. All that is needed is a space in which the terms make sense.

These examples appear to be necessarily true mathematical propositions which are about reality. It remains to defend this appearance against some well-known objections.

Objection 1.

The proposition 7+ 5 = 12 appears at first both to be necessary and to say something about reality. For example, it appear to have the consequence that if I put seven apples in a bowl and then put in another five, there will be twelve apples in the bowl. A standard objection begins by noting that it would be different for raindrops, since they may coalesce. So in order to say something about reality, the mathematical proposition must need at least to be conjoined with some proposition such as, 'Apples don't coalesce', which is plainly contingent. This consideration is reinforced by the suspicion that the proposition 7 + 5 = 12 is tautological, or almost so, in some sense.

Perhaps these objections can be answered, but there is plainly at least a *prima facie* case for a divorce between the necessity of the mathematical proposition and its application to reality. The application seems to be at the cost of introducing stipulations about bodies which may be empirically false.

Examples 1-3 above are not susceptible to this objection. Being nearly-pentagonal, being symmetrical and so on are properties that real things can have, and the mathematical propositions say something about things with these properties, without the need for any empirical assumptions.

Objection 2.

This objection is perhaps in effect the same as the first one, but historically it has been posed separately. It does at least cast more light on how the examples given escape objections of this kind.

The objection goes as follows: Geometry does not study the shapes of real things. The theory of spheres, for example, cannot apply to bronze spheres, since bronze spheres are not perfectly spherical ([1], [4], pp. 10-11). Those who thought along these lines postulated a relation of 'idealisation' variously understood, between the perfect spheres of geometry and the bronze sphere of mundane reality. Any such thinking, even if not leading to fully Platonist conclusions, will result in a contrast between the ideal (and hence necessary) realm of mathematics and the physical (and contingent) world.

It has been found that the problem was simply a result of the primitive state of Greek mathematics. Ancient mathematics could only deal with simple shapes such as perfect spheres. Modern mathematics, by studying continuous variation, has been able to extend its activities to more complex shapes such as imperfect spheres. That is, there are results not about particular imperfect spheres, but about the ensemble of imperfect spheres of various kinds. For example, consider all imperfect spheres which differ little from a sphere of radius one metre – say which do not deviate by more than one centimetre from the sphere anywhere. Then the volume of any such imperfect sphere differs from the volume of the perfect sphere by less than one tenth of a cubic metre. So imperfect-sphere shapes can be studied mathematically just as well as – though with more difficulty than – perfect spheres. But real bronze things do have imperfect-sphere shapes, without any 'idealisation' or 'simplification'. So mathematical results about imperfect spheres can apply directly to the real shapes of real things.

The examples above involved no idealisations. They therefore escape any problems from objection 2.

Objection 3.

The third objection proceeds from the supposed hypothetical nature of mathematics. Bertrand Russell's dictum, 'Pure mathematics consists entirely of assertions to the effect that, if such and such a proposition is true of anything, then such and such another proposition is true of that thing.' ([5], p. 75) suggests a connection between hypotheticality and lack of content. Even those who have not gone so far as to think that mathematics is just logic have thought that mathematics is not about reality, but only, like logic, relates statements which may happen to be about reality. Physicists, Einstein included, have been especially prone to speak in this way, since for them mathematics is primarily a bag of tricks used to deduce consequences from theories.

The answer to this objection consists fundamentally in a denial that mathematics is more hypothetical than any other science. The examples given above do not look hypothetical, but they could easily be cast in hypothetical form. But the fact that mathematical statements are often written in if-then form is not in itself an argument that mathematics is especially hypothetical. Any science, even a purely classificatory one, contains universally quantified statements, and any 'All A's are B's' statement can equally well be expressed hypothetically, as 'If anything is an A, it is a B'. A hypothetical statement may be convenient, especially in a complex situation, but it is just as much about real A's and B's as 'All A's are B's'.

No-one argues that

> All applications of 550 mls/hectare Igran are effective against normal infestations of capeweed

is not about reality because it can be expressed hypothetically as

> If 550 mls/hectare Igran is applied to a normal infestation of capeweed, the weed will die.

Neither should mathematical propositions such as those in the examples be thought to be not about reality because they can be expressed hypothetically. Real portions of liquid can be (approximately) 550 mls of Igran. Real tables can be (approximately) symmetrical about axes. Real bathroom floors can be (nearly) flat and real tiles (nearly) regular pentagons (see [3], §5).

The impact of this argument is not lessened even if the process of recasting mathematics into if-then form goes as far as axiomatisation. Einstein thought it was: his quotation with which the article began continues:

> As far as the propositions of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality. It seems to me that complete clarity as to this state of things became common property only through that trend in mathematics which is known by the name of 'axiomatics'. ([2] p. 233)

Einstein goes on to argue that deductive axiomatised geometry is mathematics, is certain and is 'purely formal', that is, uninterpreted; while applied geometry, which includes the proposition that solid bodies are related as bodies in three-dimensional Euclidean space, is a branch of physics. Granted that it is a contingent physical proposition that solid bodies are related in this way, and granted that an uninterpreted system of deductive 'geometry' is possible, there remain two main problems about Einstein's conclusion that 'mathematics as such cannot predicate anything about … real objects' ([2], p. 234)

Firstly, non-mathematical topics, such as special relativity, can be axiomatised without thereby ceasing to be about real things. This remains so even if one sets up a parallel system of 'purely formal axiomatised special relativity' which one pretends not to interpret.

Secondly, even if some of the propositions of 'applied geometry' are contingent, not all are, as the examples above showed. Doubtless there is a 'proposition' of 'purely formal geometry' corresponding to 'It is impossible to tile my bathroom floor with regular pentagonal tiles'; the point is that the modality, 'impossible', is still there when it is interpreted.

In theory this completes the reply to the objection that mathematics is necessary only because it is hypothetical. Unfortunately it does nothing to explain the strong feeling among ordinary users of mathematics, such as physicists and engineers, that mathematics is a kind of tool kit for getting one scientific proposition out of another. If an electrical engineer is accustomed to working out currents by reaching for his table of Laplace transforms, he will inevitably see this mathematical method as a tool whose 'necessity', if any, is because mathematics is not about anything, but is only a kind of theoretical juice extractor.

It must be admitted that a certain amount of applicable mathematics really does consist of tricks or calculatory devices. Tricks, in mathematics or anywhere else, are not *about* anything, and any real mathematics that concerns them will be in explaining why and when they work; this is a problem the engineer has little interest in, except perhaps for the final answer. The difficulty is to explain how mathematics can have both necessity and application to reality, without appearing to do so to many of its users.

The short answer to this lies in the mind's tendency to think of relations as not really existing. Since mathematics is so tied up with relations of certain kinds, its subject matter is easy to overlook. A familiar example of how mathematics applies in physics will make this clearer.

Newton postulated the inverse square law of gravitation, and derived from it the proposition that the orbits of the planets are elliptical. Let us look a little more closely at the derivation, to see whether the mathematical reasoning is in some way about reality or is only a logical device for deriving one scientific law from another.

First of all, Newton did not derive the shape of the orbits from the law of gravitation alone. An orbit is a path along which a planet moves, so there needs to be a proposition connecting the law of force with movement; the link is, of course,

$$\text{force} = \text{mass} \times \text{acceleration}$$

Then there must be an assertion that net accelerations other than those caused by the gravitation of the sun are negligible. Ideally this should be accompanied by a stability analysis showing that small extra net forces will only produce small deviations from the calculated paths. Adding the necessary premises has not, however, introduced any ellipses. What the premises give is the local change of motion of a planet at any point; given any planet at any point with any speed, the laws give the force, and hence the acceleration – change of speed – that the planet undergoes. The job of the mathematics – the only job of the mathematics – is to add together these changes of motion at all the points of the path, and reveal that the resulting path must be an ellipse. The mathematics must track the path, that is, it must extract the global motion from the local motions.

There are two ways to do this mathematics. In this particular case, there are some neat tricks available with angular momentum. They are remarkable enough, but are still purely matters of technique that luckily allow an exact solution to the problem with little work. The other method is more widely applicable and is here more revealing because more direct; it is to use a computer to approximate the path by cutting it into small pieces. At the initial point the acceleration is calculated and the motion of the planet calculated for a short distance, then the new acceleration is calculated for the new position, and so on. The smaller the pieces the path is cut into, the more accurate the calculation. This is the method actually used for calculating planetary orbits, since it can easily take account of small extra forces, such as the gravitational interaction of the planets, which render special tricks useless. The absence of computational tricks exposes what the mathematics is actually doing – extracting global structure from local.

The example is typical of how mathematics is applied, as is clear from the large proportion of applied mathematics that is concerned one way or another with the solution of differential equations. Solving a differential equation is entirely a matter of getting global structure from local – the equation gives what is happening in the neighbourhood of each point; the solution is the global behaviour that results. (see [6]) A good deal of mathematical modelling and operations research also deals with calculating the overall effects of local causes. Examples 1-3 above all involved some kind of interaction of local with global structure.

Though it is notoriously difficult to say what 'structure' is, it is at least something to do with relations, especially internal part-whole relations. If an orbit is elliptical globally, its curvature at each point is necessarily that given by the inverse square law, and vice versa. In general the connections between local and global structure are necessary, though it seems to make the matter more obscure rather than less to call the necessity 'logical'. Seen this way, there is little temptation to regard the function of mathematics as merely the deducing of consequences, like a logical engine. It is easy to see, though, why mathematics has been seen as having no subject matter – the western mind has had enormous difficulty focussing on the reality of relations at all ([7]), let alone such abstract relations as structural ones. Nevertheless, symmetry, continuity and the rest are just as real as relations that can be measured, such as ratios of masses; bought and sold, such as interest rate futures; and litigated over, such as paternity.

Typically, then, a scientist will postulate or observe some simple local behaviour in a system, such as the inverse square law of attraction or a population growth rate proportional to the size of the population. The mathematical work, whether by hand or computer, will put the pieces together to find out the global effect of the continued operation of the proposed law – in these cases elliptical orbits and exponential growth. There are bad reasons for thinking the mathematics is just 'turning the handle' – for example it costs less than experiment, and many scientists' expertise runs to only simple mathematical techniques. But there are no good reasons. The mathematics investigates the necessary interconnections between the parts of the global structure, which are as real properties of the system studied as any other.

This completes the explanation of why mathematics seems to many to be just a deduction engine, or to be purely hypothetical, even though it is not.

Objection 4.

Certain schools of philosophy have thought there can be no necessary truths that are genuinely about reality, so that any necessary truth must be vacuous. 'There can be no necessary connections between distinct existences.'

Answer: The philosophy of mathematics has enough to do dealing with mathematics, without taking upon itself the refutation of outmoded metaphysical dogmas. Mathematics must be appreciated on its own terms, and wider metaphysical theories adjusted to take account of whatever is found.

Nevertheless something can be said about the exact point where this objection fails to make contact with examples 1-3. The clue is the word 'distinct'. The word suggests a kind of logical atomism, as if relations can be thought of as strings joining point particulars. One need not be F.H. Bradley to find that view too simple. It is especially inappropriate when treating things with internal structure, as above. In an infinitely divisible thing that the surface of a bathroom floor, where are the point particulars with purely external relations? (The points of space, perhaps? But the relations between tile-sized parts of space and the whole space, as in example 1, either have nothing to do with points at all, or are properties of the whole system of relations between points.)

All the objections are thus answered. The conclusion stands, therefore, that the three examples are, as they appear to be, mathematical, necessary and about reality.

The thesis defended has been that *some* necessary mathematical statements refer directly to reality. The stronger thesis that *all* mathematical truths refer to reality seems too strong. It would indeed follow, if there were no relevant differences between the examples above and other mathematical truths. But there are differences. In particular, there are more things dreamed of in mathematics than could possibly be in reality. Some mathematical entities are just too big; even if something in reality could have the structure of an infinite dimensional vector space, it would be too big for us to know it did. Other mathematical entities seem obviously fictions from the way they are introduced, such as negative numbers. Statements about negative numbers can refer to reality in some way, since one can make true conclusions about debts by using negative numbers. But the references is indirect, in the way that statements about the average wage-earner refer to reality, but not in the direct sense of asserting something about an entity, ‘the average wage-earner’. Indirect reference of this kind is not in principle mysterious, though it needs to be explained in each particular case. So it can be conceded that many of the entities mentioned in mathematics are fictional, without any admission that this makes mathematics unique; minus-1 can be seen as like fictional entities elsewhere, such as the typical Londoner, holes, the national debt, the Zeitgeist and so on.

What has been asserted is that there are properties, such as symmetry, continuity, divisibility, increase, order, part and whole which are possessed by real things and are studies directly by mathematics, resulting in necessary propositions about them.

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