

Quantity and Number

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Quantity is the first category that Aristotle lists after substance. More than any other category, it has an extraordinary epistemological clarity. “ $2 + 2 = 4$ ” is the paradigm of objective and irrefutable knowledge, and “2 million + 2 million = 4 million” is not far behind in certainty, despite its distance from immediate perception. Indeed, certainties about quantity extend to the infinite – for example, we know that the counting numbers do not run out. Nor does this certainty come at the expense of application to reality. If we put two rabbits and two rabbits in a box, and later find five rabbits in there, it is our absolute certainty that $2 + 2 = 4$ that allows us to infer that the rabbits must have bred. Continuous quantities are no less open to perfection of knowledge: the quantity π , the ratio of the circumference of any circle to its diameter, is calculable to any degree of precision that computers can cope with (currently claimed to be 10 trillion decimal places¹). The mathematics of quantity delivers certainty about reality, to the envy of other disciplines including philosophy.

Despite its clarity, quantity is subject to some philosophical subtleties and unresolved puzzles. Let us start with two crucial distinctions that organize the types of quantity: extensive (or divisible) versus intensive quantity, and continuous versus discrete quantity.

Extensive versus intensive quantities

Modern physics makes a basic distinction between extensive quantities like length and mass, and intensive ones like temperature and speed.²

If a body has length 2 metres, it consists of two parts, each of length one metre. It is the same with mass or volume: a two-unit mass or volume consists (in many different ways) of two parts of unit mass or volume. A time of 2 seconds consists of

¹ http://www.numberworld.org/misc_runs/pi-10t/details.html

² M.R. Cohen and E. Nagel, *An Introduction to Logic and Scientific Method*, abridged ed, Routledge, London, 1939, 183-7.

two parts, each of one second. Such a quantity is called “extensive”. In the language of the International Union of Pure and Applied Chemistry, “a quantity that is additive for independent, non-interacting subsystems is called *extensive*.”³

Extensive quantities are easy to measure since a unit can be repeated to fill up the quantity to be measured. For example, a length can be measured by concatenating identical rods, because the length occupied by the rods is the sum of the lengths of each one.

“Quantity”, in the definition of Aristotle and hence of the scholastics, meant only extensive quantity. Aristotle writes:

‘Quantum’ means that which is divisible into two or more constituent parts of which each is by nature a ‘one’ and a ‘this’. A quantum is a plurality if it is numerable, a magnitude if it is measurable. ‘Plurality’ means that which is divisible potentially into non-continuous parts, ‘magnitude’ that which is divisible into continuous parts.⁴

“Intensive” quantities are very different. Modern science includes among paradigm quantities measurable intensities such as temperature and speed, which do not distribute over parts like length and mass do. A body with speed two metres per second does not consist of two parts with speeds one metre per second each, nor does a body of temperature 100 degrees consist of two parts of 50 degrees each.

“Intensive” quantities were not recognised as quantities by ancient science and philosophy. Aristotle classifies them in the category of quality, and allows only that they may be (qualitatively) more or less intense.⁵ In that he agreed with ancient science, which had no units of speed or temperature. The later scholastics, however, did come to recognise that such intensities were quantifiable, and their discussions of the “intension and remission of forms” laid the basis for the measurement of such quantities in modern physics.

³ International Union of Pure and Applied Chemistry (IUPAC), *Quantities, Units and Symbols in Physical Chemistry* (“The Green Book”), Third Edition, IUPAC & RSC Publishing, Cambridge, 2007, 6.

⁴ Aristotle, *Metaphysics* bk 5 ch. 13, 1020a7-12.

⁵ Aristotle, *Categories* 8 (10b27-29). To translate as “admit of variations of degree”, as is often done, may suggest to us a numerical scale of degrees, a suggestion not present in the original language of “admit the more and the less”. Discussion in J.-L. Solère, The question of intensive magnitudes according to some Jesuits in the sixteenth and seventeenth centuries, *Monist* 84 (2001), 582-616, at 583-4.

Within intensive quantities, there is a significant distinction between those like speed which are measurable on a ratio scale, which are more essentially quantitative, and than those like temperature which are not. Speed is a rate, that is, a derivative in the sense of calculus, of one extensive quantity (length travelled) with respect to another extensive quantity (time taken). Thus two miles per hour is a speed which is necessarily twice one mile per hour – the measurability of length and time implies the measurability of the rate of one with respect to the other, and also the possibility of adding speeds and multiplying them by numbers. This was a discovery of the fourteenth century scholastic writers of the Merton School, who, although they did not measure speed in any units, realised that speed could be said to be uniform or not, depending on how distance travelled varied with time taken.⁶ One of them writes: “Of local motions, then, that motion is called uniform in which an equal distance is continuously traversed with equal velocity in an equal part of time.”⁷ Their French contemporary Nicole Oresme invented graphs to display the possible uniform and non-uniform ways in which one quality can vary with another. His graphs are conceived of as drawn across the object subject to the variation, and the vertical axis indicates the intensity of the quality. Oresme says the heights measure the ratios of intensities, hence presupposing that intensities are on a ratio scale.⁸ His graphs do not have scales on either axis, as their purpose is simply to indicate the overall “shape” of the variation:

⁶ E. Sylla, Medieval quantifications of qualities: The “Merton School”, *Archive for History of Exact Sciences* 8 (1/2) (1971), 9-39.

⁷ William Heytesbury, *Rules for Solving Sophisms*, c. 1335, excerpted in E. Grant, *A Source Book in Medieval Science*, Harvard University Press, 1974, 238.

⁸ M. Clagett, *Nicole Oresme and the Medieval Geometry of Qualities and Motions*, Madison, Wisc: University of Wisconsin Press, 1978, 179, 199-201.



Fig 1. A page of graphs from the printed edition of Oresme[?]'s *Tractatus de latitudinibus formarum* (1505)

(from http://commons.wikimedia.org/wiki/File:Oresmes_diagrams_full_page.gif)

Forces too admit ratios between them, although forces are not rates – applying two forces of 1 newton at the same place and in the same direction results in a 2-newton force, so forces are additive and stand in ratios in a straightforward way. That is not the case with quantities that are more essentially qualitative, like temperature, hardness, intensity of hue, and IQ and other psychological variables. Such quantities

have no natural zero, nor does it usually make sense to speak of double such a quantity. Attempts to convert the ordering of the degrees of the quality into a numerical scale are typically complex and subject to philosophical concerns about possible lack of validity.⁹ There is no prospect of measuring IQ by concatenation of rods.

The nature of intensive quantities, vis-à-vis extensive ones, is to some degree clarified by the old scholastic debate as to whether intensification of qualities occurs by addition of parts. Scotus and Ockham (for the affirmative) held that a blue's becoming more intense, for example, is due to an overlaying of more and more parts of blueness. They pointed to the example of illumination, where addition of candles increases the illumination of a nearby surface, suggesting that illumination itself increases by addition of parts.¹⁰ That would make intensive quantities close to extensive ones (though not identical to them, since it may not be true that the parts are comparable in size and hence quantitatively additive). Aquinas denies the addition theory, at least in many cases. While allowing that it may be true of illumination, he says that charity is a "simple form": there are no such things as numerically distinct miniature pieces of charity which could be added together to produce an intense charity. A more intense charity can only differ from a less intense charity by charity being in the subject more intensely.¹¹

In general, the question as to whether an intensive quantity is intensified by addition is a matter for empirical science. For example, if degree of illumination is found to be analysable in terms of number of incident photons, then illumination is intensified by addition. But it is strange that speed, which we understand so thoroughly, remains an ambiguous case. Although speeds can certainly grow by a kind of addition, as when I walk forward in a moving train and my speed over the ground is the sum of the train's speed and my walking speed, it is doubtful if the two speeds are parts of the resultant speed. The notion of part seems neither clearly applicable nor clearly inapplicable to speeds.

⁹ J. Michell, *Measurement in Psychology: A Critical History of a Methodological Concept*, New York: Cambridge University Press, 1999.

¹⁰ Sylla, Medieval quantifications of qualities (*op. cit.* n. 6), 11-15.

¹¹ Thomas Aquinas, *Summa Theologiae* II-II q. 24 art 5; a modern version of the dispute in M. Eddon, Armstrong on quantities and resemblance, *Philosophical Studies* 136 (2007), 385-404.

Discrete versus continuous quantity

Aristotle's remark, quoted earlier, on the numerable versus the measurable refers to another major distinction within quantity: that between discrete (or atomic) and continuous.

Aristotle explains the difference as "divisible into non-continuous (respectively continuous) parts". "Continuous" could be read as "contiguous", that is, "touching". It thus relies on a quasi-spatial notion, with the parts laid out on some kind of "space" and either having no meaningful space joining them (in the discrete case, such as whole numbers or the syllables of words) or occupying all the intervening space (in the continuous case as in length or mass). That raises the question of whether there is a "topic-neutral" concept of space (in which variation can occur), which is wider than the notion of physical space, and in principle could apply to other categories.

It appears from Aristotle's own statements about other categories that there ought to be such a concept. As we have just seen, he admits "the more and the less" (that is, continuous variation) in the category of quality, which he sharply distinguishes from the categories of quantity and space. Time admits variation and "distances" between instants. Less noticed is that Aristotle also admits continuous variation in the category of substance, when he suggests that there could be a continuous range of primitive species stretching from non-living to living.¹² Aristotle's insights are confirmed by modern mathematics, which has developed formalisations (that is, topic-neutral characterisations) of the notions of "metric space" and "topological space", which can apply to any "space", physical or otherwise, across which variation can occur.¹³

Nevertheless the category of quantity is the one where most of the mathematical interest in the discrete and the continuous has focussed. The interplay of discrete and

¹² J. Franklin, Aristotle on species variation, *Philosophy* 61 (1986), 245-52; debate summarised in H. Granger, Aristotle's natural kinds, *Philosophy* 64 (1989), 245-7.

¹³ W.A. Sutherland, *Introduction to Metric and Topological Spaces*, New York: Oxford University Press, 1975, 19-21, 45; relations of modern mathematical developments to Aristotle described in A. Newstead, Aristotle and Modern Mathematical Theories of the Continuum, in *Aristotle and Contemporary Science II*, ed. D. Sfendoni-Mentzou, J. R. Brown and J. Hattiangadi, Frankfurt am Main: Peter Lang, 2001, 113-129; S. Hegarty Aristotle's notion of quantity and modern mathematics, *Philosophical Studies* (Ireland) 18 (1969), 25-35.

continuous is one of the great themes of mathematics. Mathematical work stemmed from an early Greek discovery about ratios of quantities: the fundamental distinctness of continuous and discrete quantity. It is far from clear initially whether the two kinds of quantity have much in common, for example whether the ratio ‘the double’ has much in common with the counting number 2.¹⁴ Perhaps the first truly surprising result in mathematics was the one attributed (traditionally but without much evidence) to Pythagoras, the proof of the incommensurability of the side and diagonal of a square. It is natural to think that it is possible to convert any continuous quantity to a discrete one by choosing units on a ruler. Given a ruler divided finely enough, it should be possible to compare any continuous quantities, say lengths, by counting exactly how many times the ruler’s unit is needed to measure each quantity. One length might be 127 times the unit and another 41 times, showing that the ratio of the lengths is 127 to 41. Surely by choosing the unit small enough, one could compare exactly any two lengths? But “Pythagoras” proved that for those two naturally occurring lengths, the diagonal and side of a square, this is impossible: there is *no* unit, no matter how small, such that both the diagonal and side are whole-number multiples of it. The diagonal and side of a square are “incommensurable”. So the ratios of continuous quantities are more varied than the relations of discrete quantities. Therefore geometry, and continuous quantity in general, is in some fundamental sense richer than arithmetic and not reducible to it via choice of units. While much about the continuous can be captured through discrete approximations, it always has secrets in reserve.¹⁵

The differing origins of continuous and discrete quantity led to some classical problems in Aristotelian philosophy of quantity. The emphasis on the distinctness of the discrete and the continuous produced a mystery as to why some of the more structural features of the two kinds of ratios should be identical, such as the principle

¹⁴ Newton emphasizes the distinction in one of his magisterial pronouncements, “By *Number* we understand not so much a Multitude of Unities, as the abstracted Ratio of any Quantity, to another Quantity of the same kind, which we take for Unity.” (I. Newton, *Arithmetica Universalis* (1728), 2; similar in L. Euler, *Elements of Algebra*, 3rd ed, London, 1822; both discussed in Bigelow and Pargetter, *Science and Necessity*, 60-61.)

¹⁵ J. Franklin, *What Science Knows: And How It Knows It*, New York: Encounter Books, 2009, 118-122. The continuous *can* be done discretely, in a way, but only with “continuum many” points, that is, a higher order of infinity.

of alternation of ratios (that if the ratio of a to b equals the ratio of c to d , then the ratio of a to c equals that of b to d). Is this principle part of a “universal mathematics”, a science of quantity in general?¹⁶

These questions point to the need to examine closely the most central concept of quantity, ratio.

Ratios

The crucial concept of quantity is *ratio* or *proportion*. It applies, as we saw, to all extensive quantities and those intensive quantities such as speed that are quantitative in the fullest sense – those in which it makes sense to say that one quantity of a kind is twice another. John Bigelow, one of the most Aristotelian of recent philosophers of mathematics, introduces ratios as follows. The Aristotelian language is chosen to keep close to physically real relations:

Physical objects, like elephants and Italians, humming-birds and Hottentots, have many physical properties and relations: volume and surface area, for example. And the physical properties of these objects stand in important relations to one another. In particular, such physical properties stand in relations of proportion to one another. There is a relation between the surface area of the humming-bird and that of the Hottentot; and this may or may not be the same as the relationship that holds between the surface areas of an Italian and an elephant.

Relationships such as proportion will hold not only between surface areas but also between volumes. Conceivably, the relationship between the surface areas of two objects might be the same as the relationship between volumes for two other objects. But it is a fact of considerable biological significance that the relation between surface areas of two objects will not, in general, be the same as the relationship between their volumes. Ignoring differences in shape (say, by supposing an elephant were shaped like an

¹⁶ C.B. Crowley, *Universal Mathematics in Aristotelian-Thomistic Philosophy* (Washington, DC, 1980).

Italian, or vice versa), it turns out that if the elephant has ten times the height then it will have a hundred times the surface area and a thousand times the volume. The volumes of the elephant and the Italian, or the Hottentot and the humming-bird, will be ‘more different’ than their surface areas. There are several distinct relationships present; furthermore, there are distinctive ways in which these relationships differ from one another. There are also distinctive relationships among these relationships. These facts have consequences of physical significance: for instance, with regard to problems of heat regulation. It is from such fertile soil as this that most of mathematics has grown.¹⁷

Thus for example the universal “being 1.57 kilograms in mass” stands in a certain relation, a ratio, to the universal “being 0.35 kilograms in mass”. Pairs of lengths can stand in that same ratio, as can pairs of time intervals. The ratio itself is just what those binary relations between pairs of masses, lengths and time intervals have in common (“A ratio is a sort of relation in respect of size between two magnitudes of the same kind”, as Euclid says.¹⁸)

The nature of ratios has been clarified by another scholastic dispute, this time a more recent one. It is debated whether quantities are monadic or relational. One side (Armstrong, Swoyer) hold that there are basic quantities like lengths, and then there are ratios between them. The other side (Bigelow and Pargetter) hold that only the ratios are absolute, and a quantity is merely a position in the system of ratios: there are no absolute lengths, only ratios of lengths. For comparison, it appears that colours are absolute or monadic (a colour is the particular shade it is, irrespective of its relation to other colours), whereas there may be no absolute positions in time, but only the positions of an instant relative to others (those theses may themselves be debatable, but prima facie they give examples of respectively monadic and relational properties with which quantity can be compared).

¹⁷ J. Bigelow, Sets are haecceities, in *Ontology, Causality and Mind: Essays in Honour of D.M. Armstrong*, ed. J. Bacon, K. Campbell and L. Reinhardt, Cambridge: Cambridge University Press, 1993, ch. 4, at 74-5.

¹⁸ Euclid, *Elements*, book V definition 3.

Bigelow and Pargetter argue, in favour of the relational theory, that just as attributing intrinsic position to points would not explain why one is east of another (since the positions themselves must stand in that relation), so it is with “being twice as massive as”:

You may try to ground this in intrinsic properties of determinate masses. But why should object *a*'s having one property and object *b*'s having another property entail *a*'s being twice as massive as *b*? We must presuppose a relation between the property of *a* and the property of *b*. The property of being *this* mass must stand in a relation of proportion to the property of having *that* mass.¹⁹

So since relationality is unavoidable, they say, it should be regarded as basic: “for an individual to have a particular determinate property is just for it to stand in a particular range of relationships to other individuals.”

Armstrong argues to the contrary that it seems that objects have monadic properties and the relations between them supervene and are true in every possible world: “Is it not the case that, for example *a* has the monadic property of *being two kilograms in mass*, while *b* has the property of *being one kilogram in mass*, and if any two things have these properties, then in every possible world the first is twice as massive as the second?” If the relation were external, as Bigelow and Pargetter think, it would be hard to explain why the ratio of the mass of an object to an identical one must be 1:1.²⁰ Unlike the case of “earlier than”, where objects can retain their intrinsic properties while moving around so as to break the relation, objects cannot change their massiness without changing their mass ratios to other objects. Again, as Armstrong says, what if there were only one mass in universe? In that case, there would be no ratios to other objects to constitute its mass; yet it is hard to believe that it would lack a determinate mass (for example, it would take a certain force to push it

¹⁹ Bigelow and Pargetter, *Science and Necessity*, 55-6; also in J. Bigelow and R. Pargetter, Quantities, *Philosophical Studies* 54 (1988), 287-304; comment in J. Forge, Bigelow and Pargetter on quantities, *Australasian Journal of Philosophy* 73 (4) (1995), 594-605; formalization in B. Mundy, Extensive measurement and ratio functions, *Synthese* 75 (1988), 1-23.

²⁰ D.M. Armstrong, Are quantities relations? A reply to Bigelow and Pargetter, *Philosophical Studies* 54 (1988), 305-316, at 308; similar in C. Swoyer, The metaphysics of measurement, in J. Forge, ed. *Measurement, Realism and Objectivity: Essays on Measurement in the Social and Physical Sciences*, Dordrecht: Reidel, 1987, 235-290. There is undoubtedly one quantity (of any given kind) that is absolute, because it stands in no ratio to the others; namely, the zero quantity. However, it could be

with a certain acceleration, according to the nomic connections of Newton's second law). Finally, if mass were quantized and there were just two atoms in the universe, then the mass ratio of their sum to each of them is determined to be "twice as massive", and it seems clear that that ratio is not freestanding but supervenes on (is true in virtue of) the repetition of the objects (since mass is an extensive quantity).

So there are reasons to favour the theory that quantities such as mass are monadic and that the ratios between them supervene on the determinate quantities.

Characterizing "quantity"

In the light of the above, it is natural to attempt a definition of "quantity". What kinds of properties should count as "quantities"? Given that Aristotle's definition applies only to extensive quantities, and that the quantification of intensities tends to blur the distinction between the categories of quantity and quality, it is unclear if any coherent view of quantity is available in the Aristotelian tradition.

Starting from Aristotle's concept of what is "subject to more and less", a possible alternative can be based on the mathematics of order structures. A partial order (in mathematical terminology) is a binary relation that is reflexive, antisymmetric and transitive. (An example is inclusion among sets: it arranges sets in an ordering of smaller and larger, but not every pair of sets is comparable.) A linear or total order is a partial order in which any two elements *are* comparable (for example, "greater than" among whole numbers).²¹ In the language of measurement theory, the items are said to be comparable on an ordinal scale; however, the "scale", in the sense of a scale of numbers, is not part of the definition but a consequence: if items are linearly ordered, they may be assigned numbers such that items later in the ordering have greater numbers. If items are linearly ordered, it may or may not be that there is a notion of distance between the items being ordered, that is, it is meaningful to compare the interval between *a* and *b* with that between *c* and *d*, as less, equal or more (in the

argued that the zero quantity should be considered a non-being and hence not truly part of the system of quantities. (Debate in Y. Balashov, *Zero-valued physical quantities*, *Synthese* 119 (199), 253-286).

²¹ B.A Davey and H.A. Priestley, *Introduction to Lattices and Order*, 2nd ed., Cambridge: Cambridge University Press, 2002, ch. 2.

language of measurement theory, the items are comparable on an interval scale). If so, it may or may not be that the items have a size such that the ratio between sizes is meaningful (“comparable on a ratio scale”).

The most core or paradigmatic quantities are those comparable on at least an interval scale. That implies that the ordering of items is a system isomorphic to the continuum, or to a piece of it (for example, the interval from 0 to 1, in the case of probabilities) or a substructure of it (such as the rationals or integers). It is not entirely out of the question to call a purely ordinal scale such as the 1-to-10 scale of mineral hardness or IQ a “quantity”,²² but it is stretching the meaning of the term because there is no “quantum” or repeatable atom separating items and care is needed not to attribute meaning to differences between items.

One may more loosely call any (not necessarily linear) order structure a kind of quantity (in that it permits *some* comparisons on a kind of scale). Thus vectors and complex numbers can be called quantities in that all the real-number multiples of a fixed one form a linear order and are thus subject to comparison as “more or less”. Although ones in different directions are not strictly comparable, direction varies continuously and hence a vector is approximately comparable with one in a nearby direction; vectors in different directions are also comparable in respect of length.²³

One might go so far as to allow fuzzy quantities (such as imprecise probabilities) by a family resemblance, as they share the properties of the continuum except for absolute precision.²⁴

Puzzles on the relation of quantity and space

Quantity and space, according to Aristotle, are different categories. That leads to a number of difficult problems on the relation of the spatial quantities (length, area and

²² As commonly done in the social sciences; see N. Cliff and J.A. Keats, *Ordinal Measurement in the Behavioral Sciences*, Mahwah, NJ: Erlbaum, 2003. Another suggestion in B. Hale, Reals by abstraction, *Philosophia Mathematica* 8 (2000), 100-123, at 106.

²³ Discussions in Bigelow and Pargetter, *Science and Necessity*, section 2.6; Bigelow, *Reality of Numbers*, part II(c); S. Leuenberger and P. Keller, Introduction: the philosophy of vectors, *Dialectica* 63 (4) (2009), 369–380 and other papers in the same special issue.

²⁴ On “value indefiniteness” in the quantities of quantum mechanics, see J. Forge, Quantities in quantum mechanics, *International Studies in the Philosophy of Science*, 14 (2000), 43-56.

volume) to real space. These problems are not artifacts of an arbitrary classification of categories, but genuine. We will try to explain what these puzzles are but will not try to solve them fully.

A body of length one metre must, it seems, occupy an extent of space of exactly one metre – though not any particular one-metre part of space. So there is a very close relation between length and the properties of space. Yet it appears also that while truths about quantity are all necessary, it is a contingent matter what shape space has – that was the lesson of the discovery of non-Euclidean geometries. So how do the necessities of quantity “fit” (so to speak) into the contingent truths of space?

A precise version of the problem arises in another context, the continuum. The continuum (now modelled by the real number line) is the essential ingredient in the real functions that are the basic tools of mathematical physics. The continuum was once supposed to be instantiated necessarily to the real space we live in, implying in particular that real space is infinitely divisible. Euclid’s geometry incorporates that assumption. David Hume argued that that could not be right, as our limited sense knowledge cannot support knowledge of the infinite divisibility of space.²⁵ Philosophers and mathematicians alike dismissed him as one ignorant of the mysteries of geometry, but he was right – the geometry that real space has, on the small scale as much as the large, is a contingent matter to be decided by observation and experiment, not a necessary truth to be laid down *a priori*. What, then, is “the continuum”, if it is not the structure of real space?

To answer such questions, let us take possibly the simplest problem of this kind. We have observational knowledge of lengths in the mid-range size – at least from grains of sand to mountains. It is a truth about ratios that twice and half a length is also a length (just as twice a whole number is a number). Lengths do not run out, any more than numbers do. But instantiated lengths may well run out. If the universe is finite in size, then no lengths longer than the diameter of the universe are instantiated, and if space is atomic, no length shorter than the size of an atom of space is instantiated. What then does it mean to speak of the system of *all* lengths and to state necessary truths about the relations within it?

The problem is a particularly clear and urgent case of the “problem of uninstantiated universals” discussed among Aristotelian philosophers. Should Hume’s example of an uninstantiated shade of blue be said to exist in some sense, or must all universals be instantiated in order to have any reality at all? Many Aristotelians argue that admitting uninstantiated universals would be excessively Platonist, in acknowledging a realm of Forms beyond the real world, ungrounded in any true reality.²⁶ They must say, then, that lengths greater than the diameter of the universe (for example) are mere possibilities. The difficulty for that suggestion is that those “mere” possibilities appear themselves to stand in ratios to each other, in ways correctly described by mathematics. The “mere” possibilities thus themselves form a Platonic-like world of forms, of complex structure, the truths of which have no apparent truthmaker. Our knowledge of ratios, such as that three times a length lies between twice and four times that length, applies to lengths beyond the diameter of the universe. Those truths stand ready to be, so to speak, clothed in reality if the universe expands.

Brent Mundy argues for the reality of uninstantiated universals by asking how a general theory of quantity relates to empirical evidence about quantities. A nominalist theory faces the problem that standard postulates of the theory of (extensive) quantity such as that the sum of two quantities is a quantity are literally false (for example, if mass means, operationally, measurement in a balance, then two large enough masses may be too large to fit together in a balance, though they do fit individually). That problem is shared by an Aristotelian realism that admits only instantiated quantities: the sum of two instantiated lengths may not be instantiated. Mundy suggests that with *a posteriori* realism – one which takes it as a matter for science to determine which universals there are – the empirical evidence supports the reality a determinable quantity more than of the collection of those determinates that happen to be instantiated. On grounds of theoretical simplicity, length in general is the theoretical entity that makes sense of the empirical evidence, not lengths-in-the-instantiated

²⁵ J. Franklin, Achievements and fallacies in Hume’s account of infinite divisibility, *Hume Studies* 20 (1994), 85-101.

²⁶ D.M. Armstrong, *Universals: An Opinionated Introduction*, Westview Press, 1989, 75-82.

range.²⁷ To restrict lengths to the instantiated range would be a “simplification” analogous to supposing that only observed bodies exist – it fails to posit the natural range of which the data happen to be a sample.

It is the same with mathematical structures such as the continuum, Euclidean geometry or infinite numbers. Those can be described as (possibly) uninstantiated structures or as (merely) possible structures, but in either case they describe a complex form which may be instantiated in reality – a form about which there can be necessary knowledge. They differ from the Forms of classical Platonism which necessarily lie beyond mundane reality and cannot be literally instantiated in it. Aristotelian forms can be instantiated, but it is for the contingencies of historical reality (or the will of God, or whatever decides such matters) to determine which are in fact instantiated.

Because of the tendency of quantity to apply across vast ranges of size, it is not easy to make sense of in terms of a strict Aristotelian realism that does not admit uninstantiated universals. The best attempt to do so is the combinatorial theory of possibility of David Armstrong. Armstrong holds that possibilities are recombinations of actual elements in the world – there being a unicorn is possible because it is a recombination of parts of actually existing entities. But combination is to allow addition and deletion of actually existing particulars (but not addition of universals): “Combination is to be understood widely. It includes the notion of expansion (perhaps ‘repetition’ is a less misleading term) and also contraction.”²⁸ Individuals are to be allowed to clone themselves indefinitely, indeed infinitely often, to create new possibilities.

The difficulty is that the possibility of very large or infinite numbers is then built into the theory, or presupposed by it, rather than analysed by it. Why are numbers larger than those instantiated in the universe possible? Because the actual individuals in the universe are subject to “indefinite multiplication”.²⁹ (Similarly, the possibility of a length greater than the diameter of the universe is grounded in the possibility of

²⁷ B. Mundy, The metaphysics of quantity, *Philosophical Studies* 51 (1) (1987), 29-54; Mundy calls his position “naturalistic Platonism”, but it is identical to Aristotelian realism with uninstantiated universals.

²⁸ D.M. Armstrong, *A Combinatorial Theory of Possibility*, Cambridge, Cambridge University Press, 37.

replication of actual individuals to give a body of greater total length: an uninstantiated quantity is “combinatorially accessible from actual” quantities.³⁰) But what is the ground of the possibility of indefinite replication of individuals itself? The theory does not say. Instead it has to assume that possibility in order to get started.³¹ What, for example, is the ground of the possibility of some particular infinite cardinal? It is the possibility that actual individuals should be infinitely replicated (at least) that many times (a possibility normally regarded as controversial, in view of Aristotelian doubts about actual infinities). That may indeed be the ground, but the combinatorial theory of possibility has not given an analysis of that possibility, only an assertion of it.

The knotty and irreducible nature of the possibility of indefinite replication is confirmed by the need for the axioms of mathematics to include an “axiom of infinity”. Among the basic axioms of set theory, the most obviously non-logical one – the one that is most directly an obstacle to any attempt to regard mathematics as logic or as analytically true – is the Axiom of Infinity, stating “There is an infinite set” (or equivalently, “The numbers do not run out.”) It is independent of the other axioms.³² There is no passage via logic or simple recombination from the finite to the possibility of the infinite.

The problem of the relation of the necessities of quantity to the contingencies of actual magnitudes does not exhaust the puzzles concerning space and quantity. The theory of the ancients that arithmetic studies discrete quantity and geometry studies continuous quantity encounters the problem that geometry also studies shape. Shape is assigned by Aristotle to the category of quality, and it can vary completely independently of size, in that a given shape can be realised in a figure of any size (in Euclidean space, at least). Yet the relation between size and shape must be more intimate than that suggests, since if the disposition of the points of a body is determined, both the shape and the size of the body supervene. A philosophy of

²⁹ Armstrong, *Combinatorial Theory*, 125.

³⁰ Armstrong, *Combinatorial Theory*, 56.

³¹ Armstrong, *Combinatorial Theory*, 58-60.

³² Originally denied by Bertrand Russell but conceded in *Principia Mathematica* (A.N. Whitehead and B. Russell, *Principia Mathematica*, 2nd ed, Cambridge: Cambridge University Press, 1963, II, 203).

geometry is required to resolve the problem. The field is undeveloped and will not be attempted here.³³

A further problem concerns the “geometry” of quantities themselves. If we take the “space” of vector quantities of a single kind, say the space of all possible forces on a body (in 3D), then that space has a natural geometry. Vectors have length, distance between them, and angles. The geometry is always Euclidean. Does this give Euclidean geometry a special position of privilege in the space of forms, even if that position has been denied to it in the geometry of real space? The problem even has a one-dimensional version. If we keep firing an arrow forward in actual space, it is possible that we may eventually come back to where we started (if space is finite, curved and has no boundary). But if we take a quantity such as a length and keep adding it to itself, we cannot come back to where we started. The “geometry” of the space of lengths is necessarily infinite. It remains unclear why that is so.

Suarez raised a complex of other problems, related to the question of the relation of the length of a body to its occupancy of space. His theory is that length (or area or volume) is “aptitudinally situat”.³⁴ The problem that he has principally in mind is how the body of Christ in the Eucharist can have the dimensions of a human body yet fit in the space of a host.³⁵ He also considers the problem of condensation and rarefaction, where a body occupies different amounts of space over time.³⁶ That raises the somewhat different problem of the relation of the quantity of stuff in a substance (possibly to be identified with the scientific concept of mass) and its spatial dimensions. Further problems concern the exclusion of two bodies from the same space. Whether or not Suarez’s theory solves all those problems, the phrase “aptitudinally situat” is suggestive even when restricted to the problem of the relation of length and space. A rigid body is apt for being situated in any space obtainable by translations and rotations from the space it actually occupies, while a non-rigid body

³³ One realist approach in G. Nerlich, *The Shape of Space*, 2nd ed, Cambridge: Cambridge University Press, 1994; theories surveyed in L. Sklar, *Space, Time and Spacetime*, Berkeley and Los Angeles: University of California Press, 1974.

³⁴ Francisco Suárez, *Metaphysical Disputations* 40, discussed in D.P. Lang, Aquinas and Suarez on the essence of continuous physical quantity, *Laval théologique et philosophique* 58 (3) (2002), 565-595.

³⁵ Francisco Suárez, *Metaphysical Disputations* 40, “On Continuous Quantity” section 2, trans. R. Pasnau, at <http://spot.colorado.edu/~pasnau/research/suarez%20dm40-2.pdf>

³⁶ Lang, 593.

of fixed volume is apt for being situated in a greater range of spaces, and a compressible body in still more spaces.

Discrete quantity, numbers and sets

Discrete quantities arise in quite a different way from ratios. It is characteristic of “unit-making” or “count” universals like “being an apple” or “being a horse” (in Aristotle’s example) to structure their instances discretely. That is what distinguishes them from mass universals like “being water”. A heap of apples stands in a certain relation to “being an apple”. That relation is the number of apples in the heap. The same relation can hold between a heap of shoes and “being a shoe”. The number is just what these binary relations have in common.³⁷

Aristotle emphasized – if a little cryptically – the relativity of number to the universal being used to divide the mass being counted into units:

‘The one’ means the measure of some plurality, and ‘number’ means a measured plurality and a plurality of measures ... The measure must always be some identical thing predicable of all the things it measures, e.g. if the things are horses, the measure is ‘horse’, and if they are men, ‘man’. If they are a man, a horse, and a god, the measure is perhaps ‘living being’, and the number of them will be a number of living beings.³⁸

Thus, suppose there are seven black swans on the lake now. The proposition refers to a part of the world, the black biomass on the lake, and a structuring property, being a black swan on the lake now. Both are necessary to determining that the relation between the mass and the property should be “seven”: if it were a different mass (e.g. the black swans on or beside the lake) or a different unit-making property (e.g. being a swan organ on the lake now) then the numerical relation would be

³⁷ From D.M. Armstrong, *Sketch for a Systematic Metaphysics*, Oxford: Oxford University Press, 2010, ch. 13; originally from P. Forrest and D.M. Armstrong, The nature of number, *Philosophical Papers* 16 (1987), 165-186 and mostly in G. Kessler, Frege, Mill and the foundations of arithmetic, *Journal of Philosophy* 77 (1980), 65-79.

³⁸ Aristotle, *Metaphysics* bk 14 ch. 1, 1088a4-11, further in H.G. Apostle, *Aristotle’s Philosophy of Mathematics*, Chicago: Chicago University Press, 1952; V.E. Smith, *St Thomas on the Object of Geometry*, Milwaukee: Marquette University Press, 1954; D. Bostock, Aristotle’s philosophy of mathematics, in *Oxford Handbook on Aristotle*, to appear.

different. Therefore numbers are not properties of parts of the world simply, but must be properties of the relation between parts of the world and the unit-making properties that structure them.

So the fact that the heap of shoes stands in one such numerical relation to “being a shoe” and another numerical relation to “being a pair of shoes” (made much of by Frege³⁹) does not show that the number of a heap is subjective, or not about something in the world, but only that number is relative to the count universal being considered. For Aristotelians, the universal is real and so is its relation to the heap it structures.

Whereas ratios have nothing to do with sets, numbers are intimately connected with them. Given a set, there is something to count. And conversely, if there is counting, there is a set of entities being counted, and indeed sets are good for little else. Given a heap and a unit-making property structuring it, there is immediately created (there supervenes) both the set of things of which the heap is the mereological sum, and a number of things in that set. If there is no unit-making property – if there is just stuff – there is no number and no set. If there is a unit-making property, there is a set and a number of elements in the set.

So what are sets, from an Aristotelian point of view? The Aristotelian cannot rest content with the Platonist story that sets are a simple Platonist entity at which questions should stop, and that the membership relation is *sui generis*. That conception is problematic, but even if it were intelligible and satisfactory, it would interpose a Platonist entity in a story where there should be no role for it, the story of how unit-making properties structure a heap into something able to be counted.

The Aristotelian desires a theory according to which sets are ontologically nothing over and above there being a unit-making property to structure a heap. Several closely-related theories are available. The leading one is that of David Armstrong. He adopts David Lewis’s proposal that a set is the mereological sum of its singletons, and adds the idea that the singleton of x is simply the state of affairs of

³⁹ G. Frege, *The Foundations of Arithmetic* (1884), trans. J.L. Austin, 2nd revised ed, Oxford: Blackwell, 1980, §22, p. 28 and §54, p. 66.

there being some unit-making universal that singles out x .⁴⁰ The essence of the suggestion is that at the basic philosophical level necessary in these questions, we cannot help ourselves naively to the notion of “object”. When we assert “The cat sat on the mat”, “The”, in “the cat”, indicates that we are dealing with a single unified *object*, cut out from the background. In the apparent continuum of matter that is the universe and the flux it undergoes, what cuts out the single warm furry item, draws its boundaries and points it out as an individual thing deserving a common noun?⁴¹ It is the property, the repeatable unit-making property “being a cat”, that cuts the cat from the background, and in doing so creates a singleton (and when actually repeated creates other sets) and at the same time creates something to be counted.

The epistemology of quantity: perception, measurement, counting and understanding

There is a tension in Aristotelian views on how quantity is known. On the one hand, quantity as a real property of things is easily perceivable, so Aristotelians need no non-naturalist story of access to Platonic entities to account for basic knowledge of quantities. On the other hand, the more emphasis is placed on the perceivability of quantity, the harder it becomes to account for the characteristic certainty of our knowledge about quantity (referred to in the opening of this article), a certainty underpinned by mathematical proof and apparently extending well beyond the scope of the perceivable world – extending even to higher orders of infinity, according to the majority view.

It is impossible here to give a full overview of the problem, but it is possible to give some insight into the basic distinctions that Aristotelians must draw between perceptual and intellectual knowledge – a distinction in fact best illustrated by our knowledge of quantity.

⁴⁰ D.M. Armstrong, Classes are states of affairs, *Mind* 100 (1991), 189-200; several proposals listed in A. Paseau, Motivating reductionism about sets, *Australasian J. of Philosophy* 86 (2008), 295-307.

⁴¹ B.C. Smith, *On the Origin of Objects*, Cambridge, Mass: MIT Press, 1996.

Extensive research on animals and human babies has shown that they have considerable skills in the perception of approximate sizes and numerosities.⁴² For examples, human babies, as soon after birth as it is possible to experiment on them, display the ability to distinguish a group of two sounds from a group of three.⁴³ All but the very simplest estimates are inherently fuzzy and do not involve any operation like counting or measuring. It is interesting that even at this early stage, quantity has an epistemological advantage over other categories in being accessible by more than one sense; as Aristotle remarks, “‘Common sensibles’ are movement, rest, number, figure, magnitude; these are not peculiar to any one sense, but are common to all.”⁴⁴

Later, but still in infancy, humans learn to count exactly and to measure. According to the view of sets sketched above, it should be possible to perceive and hence count sets, once one has recognised the count-universal that structures the heap. As argued by Penelope Maddy, if I open an egg carton and see that there are three eggs in it, I perceive both the pale curved surface of the egg-heap and that it is structured by “being an egg” into three parts, each an egg. That is sufficient to perceive the heap as a set of three eggs.⁴⁵ Such abilities are the ones developed in early mathematical education, usually with great difficulty but eventual success.⁴⁶

Measurement, like counting, requires the addition of a kind of intellectual recognition to simple perceptual rough-and-ready estimation of magnitude. The theory of measurement displays particularly clearly the difference between a Platonist and an Aristotelian approach to quantity. The usual approach to measurement sets up the problem with a Platonist bias, concentrating on “representation theorems” that

⁴² E.g. S. Dehaene, *The Number Sense: How the mind creates mathematics*, New York: Oxford University Press, 1997); A. Bisazza, L. Piffer, G. Serena and C. Agrillo, Ontogeny of numerical abilities in fish, *PLoS ONE* 5(11) (2010), e15516. doi:10.1371/journal.pone.0015516; W. Mack, Numerosity discrimination: Infants discriminate small from large numerosities, *European Journal of Developmental Psychology* 3 (2006), 31-47; S. Cordes and E.M. Brannon, The relative salience of discrete and continuous quantity in young infants, *Developmental Science* 12 (3) (2009), 453–463.

⁴³ R. Bijeljac-Babic, J. Bertoncini and J. Mehler, How do four-day-old infants categorize multisyllabic utterances? *Developmental Psychology* 29 (1993), 711-21; visual parallels in S.E. Antell and D.P. Keating, Perception of numerical invariance in neonates, *Child Development* 54 (1983), 695-701.

⁴⁴ Aristotle, *De Anima* II.6, 418a16-20.

⁴⁵ P. Maddy, *Realism in Mathematics*, 58-67.

⁴⁶ Z.P. Dienes, *Building Up Mathematics*, London: Hutchinson, 1960, esp. ch. 2; A. Baroody, M.-L. Lai and K.S. Mix, The development of young children’s early number and operation sense and its implications for early childhood education, in B. Spodek and O.N. Saracho, eds, *Handbook of Research on the Education of Young Children*, 2nd ed, Mahwah NJ: Lawrence Erlbaum, 2006, ch. 11.

describe the conditions under which quantities can be represented by numbers.⁴⁷ That poses the problem as if it is one about the association of numbers to parts of the world, which inevitably leads to a Platonist or nominalist perspective.

But a closer look suggests an Aristotelian reinterpretation. What is it about the quantitative properties of the measured world that ensures that a representation by numbers exists? The standard treatment (of measurement of length) begins by looking at the properties of concatenating identical rods, and axiomatizing those properties as a basis for showing that a representation by numbers exists.⁴⁸ But the quantitative properties exist prior to the representation and are the condition of its existence: as the Aristotelian maintains, the system of ratios of lengths, for example, pre-exists in the physical things being measured, and measurement consists in identifying the ratios that are of interest in a particular case; the arbitrary choice of unit that allows ratios to be converted to digital numerals for ease of calculation is something that happens at the last step.⁴⁹ That in turn suggests an Aristotelian realist view of the real numbers arising in measurement. As the Joel Michell puts it, in language similar to that used of ratios above:

The commitment that measurable attributes sustain ratios has a further implication, viz., that *the real numbers are spatiotemporally located relations*. It commits us to a realist view of number. If Smith's weight is 90 kg, then this is equivalent to asserting that the real number, 90, is a kind of relation, viz., the kind of relation holding between Smith's weight and the weight of the standard kilogram. Since these weights are real, spatiotemporally located instances of the attribute, any relation holding between them will likewise be real and spatiotemporally located. This kind of relation is what was referred to above as a ratio. So the realist view of measurement implies that *real numbers are ratios*.⁵⁰

⁴⁷ E.g. C.W. Savage and P. Ehrlich, *Philosophical and Foundational Issues in Measurement Theory*, Hillsdale, NJ: L. Erlbaum, 1992.

⁴⁸ D.H. Krantz, R.D. Luce, P. Suppes and A. Tversky, *Foundations of Measurement*, vol. 1, New York: Academic, 1971, ch. 1.

⁴⁹ Similar comments in Bigelow and Pargetter, *Science and Necessity*, 60-61.

⁵⁰ J. Michell, The logic of measurement: a realist overview, *Measurement* 38 (2005), 285-294; relation to Aristotle discussed in C.B. Crowley, *Aristotelian-Thomistic Philosophy of Measure and the International System of Units (SI)*, Lanham, Md: University Press of America, 1996.

Measurement and counting inform us that certain particulars have certain quantities. *General* truths about quantity are another matter entirely, and it is here that Aristotelian epistemology, at least in its traditional form, parts company with naturalism. According to traditional Aristotelianism, the human intellect possesses an ability completely different in kind from anything possessed by animals, an ability to abstract universals and *understand* their relations.

Although philosophically mysterious, it is easy to exhibit this ability in practice – and easiest to do so in cases involving quantity. For example, in this diagram,

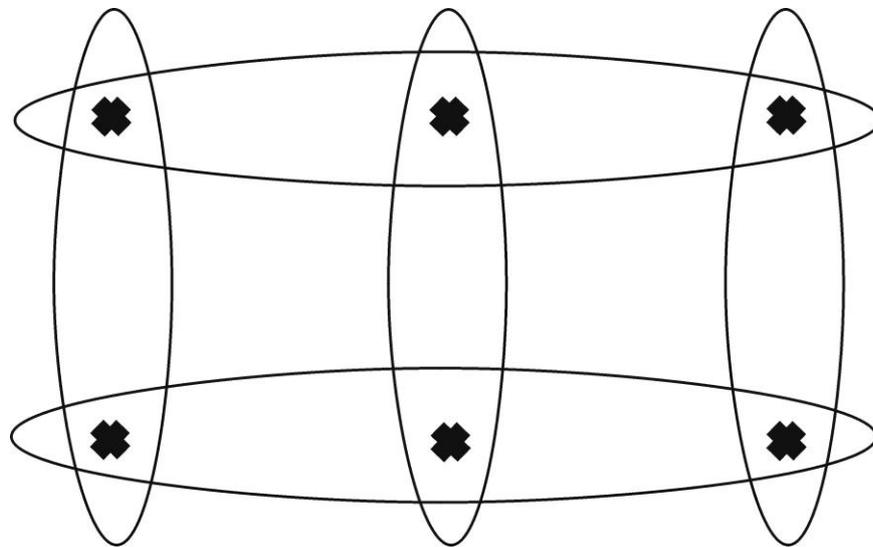


Fig 2 Why $2 \times 3 = 3 \times 2$

the point of the ovals is to guide the visual system so as to group the six objects as alternately two sets of triples and three sets of pairs. That is what allows the intellect to grasp the relation between the parts and hence achieve its certain knowledge of the equation $2 \times 3 = 3 \times 2$. The mind know only knows *that* $2 \times 3 = 3 \times 2$, but has an insight or understanding of why it *must* be so.

According to the model of science in Aristotle's *Posterior Analytics*, a true science differs from a heap of observational facts – even a heap of true empirical generalisations – by being organised into a system of deductions from self-evidently true axioms which express the nature of the universals involved. Ideally, each

deduction from the premises allows the human understanding to grasp why the conclusion must be true. Euclid's geometry, the model for all of pure mathematics since, conforms closely to Aristotle's model.⁵¹ As we saw, although it is an empirical question whether Euclidean geometry applies to physical space, spaces of vector quantities are Euclidean so Euclid's geometry is still a science of reality.

The Aristotelianism of the scholastics maintained that such an ability to grasp pure relations of universals was so far removed from sensory knowledge as to prove that the "active intellect" must be immaterial and immortal.⁵² That is not an idea that has found much favour in modern philosophy, for obvious reasons. But the complete inability of the Artificial Intelligence project to imitate human understanding (as opposed to human calculation, information retrieval or pattern recognition) suggests that providing a naturalistic substitute for the "active intellect" is far from easy.

Quantity and the philosophy of mathematics

As we have seen, a great deal is known about the quantitative properties of things – about lengths, ratios and relations between ratios, about discrete quantities and their addition, and so on. That body of knowledge bears an uncanny resemblance to the subject taught in schools under the name "mathematics". Therefore the existence of quantities, discrete and continuous, counted and measured, properties of real bodies, has suggested to many a realist but non-Platonist philosophy of mathematics. From the time of Aristotle to the eighteenth century, one philosophy of mathematics dominated the field. Mathematics, it was said, is the "science of quantity". Discrete quantity is studied by arithmetic and continuous quantity by geometry. A version of an Aristotelian theory of mathematics as a realist science of quantity, both discrete and continuous, was standard and virtually unchallenged in early modern times.⁵³

⁵¹ R.D. McKirahan, *Principles and Proofs: Aristotle's theory of demonstrative science* (Princeton, 1992), ch. 12.

⁵² Z. Kuksewicz, The potential and the agent intellect, ch. 29 of *The Cambridge History of Later Medieval Philosophy*, ed. N. Kretzmann *et al* (Cambridge, 1988); Y.R. Simon, Nature and the process of mathematical abstraction, *The Thomist* 29 (1965), 117-39; D. McGraw, Intellectual abstraction as incompatible with materialism, *Southwest Philosophy Review* 11 (2) (1995), 23-30.

⁵³ E.g. I. Barrow, *The Usefulness of Mathematical Learning Explained and Demonstrated*, London, 1734, repr. London: Cass, 1970, 10-15; *Encyclopaedia Britannica*, 1st ed., Edinburgh, 1771, article

The quantity theory plainly gives an initially reasonable picture of at least elementary mathematics, with its emphasis on counting, measuring, and calculating with the resulting numbers. It promises direct answers to questions about what the object of mathematics is (certain properties of physical and possibly non-physical things such as their size), and how those properties are known (the same way other natural properties of physical things are known – by perception in simple cases and inference from perception in more complex ones).

The realist quantity theory apparently then died in the nineteenth century, partly from lack of defence but partly from Frege's criticisms of the possibility of mathematics being about properties of the real world. Under Frege's influence, twentieth-century philosophy of mathematics was dominated by an oscillation between Platonism and nominalism in its various forms (including logicism and formalism). Frege and many later authors defended a Platonist view of the reality of the "abstract objects" of mathematics such as numbers and sets,⁵⁴ while nominalists tried to show that mathematics as applied in science can do without reference to such objects.⁵⁵

Needless to say, that created endless difficulties in accounting for applied mathematics, since both Platonism and nominalism make it hard to see how mathematics can be so successful in real-world applications.⁵⁶ It also created an irreconcilable conflict between ontology and epistemology in mathematics, with Platonism taking the well-known objectivity of mathematics seriously but leaving it mysterious how we can access objects in another world, and nominalism making epistemology easy but making the objectivity and applicability of mathematics a

'Mathematics', vol. III 30-1; P. Mancosu, *Philosophy of Mathematics and Mathematical Practice in the Seventeenth Century*, New York: Oxford University Press, 1996, 16, 35-37, 56, 88; D.M. Jesseph, *Berkeley's Philosophy of Mathematics*, Chicago: University of Chicago Press, 1993, ch. 1; late developments in J. Michell, Bertrand Russell's 1897 critique of the traditional theory of measurement, *Synthese* 110 (1997), 257-276.

⁵⁴ Notable recent examples include S. Shapiro, *Philosophy of Mathematics: Structure and Ontology*, New York: Oxford University Press, 1997; J.R. Brown, *Philosophy of Mathematics: An introduction to the world of proofs and pictures*, London: Routledge, 1999; M. Colyvan, *The Indispensability of Mathematics*, Oxford: Oxford University Press, 2001.

⁵⁵ H. Field, *Science Without Numbers: A Defence of Nominalism*, Princeton: Princeton University Press, 1980; J. Azzouni, *Deflating Existential Consequence: a case for nominalism*, Oxford: Oxford University Press, 2004.

mystery.⁵⁷ The neglect of epistemology accounts for two strange absences in the philosophy of mathematics: understanding (and mathematics is where one first goes to experience pure understanding) and measurement (the primary way in which mathematics joins to the world). Lastly, there is the divorce between the philosophy of mathematics, on the one hand, and developmental psychology and mathematics education, on the other – surely the considerable knowledge of infants’ mathematical learning, much of which is about quantity should be compatible with the correct philosophy of mathematics? An Aristotelian realism, centred on a realist understanding of quantity, shows obvious promise of resolving these tensions, by exhibiting real properties of things that can be the objects of learning in children, the objects of understanding in adults, and the basis of the applications of applied mathematics. It is time for a revival of moderate realism in the philosophy of mathematics, starting with the philosophy of quantity.⁵⁸

⁵⁶ S. Körner, *The Philosophy of Mathematics: An introductory essay* (London, 1960), 176-183; M. Steiner, *The Applicability of Mathematics as a Philosophical Problem*, Cambridge, Mass: Harvard University Press, 1998).

⁵⁷ Noted in P. Benacerraf, Mathematical truth, *Journal of Philosophy*, 70 (1973), 661-679.

⁵⁸ A preliminary effort in J. Franklin, Aristotelian realism, in *The Philosophy of Mathematics*, ed. A. Irvine (Handbook of the Philosophy of Science series), Amsterdam: North-Holland Elsevier, 2009, 101-153; overview in J. Franklin, Aristotelianism in the philosophy of mathematics, *Studia Neoaristotelica* 8 (2011), 3-15 (in that theory “quantity” as an object of mathematics is supplemented by “structure”). Other works with Aristotelian tendencies include J. Bigelow, *The Reality of Numbers: A Physicalist’s Philosophy of Mathematics*, Oxford: Clarendon, 1988; J. Bigelow and R. Pargetter, *Science and Necessity*, Cambridge: Cambridge University Press, 1990, ch. 2; A.D. Irvine, ed, *Physicalism in Mathematics*, Dordrecht: Kluwer, 1990.