Contents

Argumentationstheorie. Scholastische Forschungen
en und semantischen Regeln korrekten Folgerns 148
iferimento e intenzionalità. Per una ontologia del
i Francesco) 151
Burdens of proof in modern discourse (D. Felipe) 152

Shorter Notices 154

Announcements 157

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Apodeictic Syllogisms: Deductions and Decision Procedures

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One semantic and two syntactic decision procedures are given for determining the validity of Aristotelian assertoric and apodeictic syllogisms. Results are obtained by using the Aristotelian deductions that necessarily have an even number of premises.

1. Background

McCall's *1963* L–X–M calculus generates the two-premised assertoric and apodeictic syllogisms that are clearly valid for Aristotle. But McCall's account of deduction does not match that found in Aristotle's *Prior Analytics*, A25. One of the purposes of the following discussion is to give a recursive definition of deduction for syllogisms with two or more premises that matches Aristotle's. With it we can make sense of the passages in A25 that have puzzled commentators. So, for example, we show that every deduction has an even number of premises and an odd number of members.

In many respects the system developed below is similar to McCall's. For example, all of McCall's 333 "valid L–X–M moods" are deducible in the following system. But the system diverges from McCall's in that LIaa will not be regarded as valid. Thom *1991* gives good reasons for not regarding it in this way.

The syntax for the system is taken from Johnson *1993*, which makes use of singular sentences to accommodate proofs by *ecthesis*. So the system is unlike McCall's L–X–M, which excludes singular sentences.

The semantics for the system is similar to that in Johnson *1993*. Its set-up commits Aristotle to the truth of certain sentences without making the truth values assigned to sentences a function of assignments that make up the sentence. So, for example, we commit Aristotle to saying that for every predicate term 'a' there is some singular term 'm' such that 'm is a' is true. But we remain neutral on the question of whether by saying this we are committing him to saying that (i) every predicate term denotes a non-empty set or (ii) there is no uninstantiated property.

But the semantics below also differs from that in Johnson *1993*, which accommodated Thom's *1991* claim that, for Aristotle, if something is necessarily an x then anything that is an x is necessarily an x. (In personal correspondence Thom recently told me that, for Aristotle, snow is necessarily white but it is not true that everything that is white is necessarily white. I think Thom's recent remark is right, and the semantics below conforms with it. So, in this sense, the semantics is similar to that in Johnson *1989*.)

Decision procedures for determining the validity of syllogisms are given by: (i) putting a limit on the number of valuations needed to find counterexamples; (ii) putting a limit on the size of deductions; and (iii) listing forms of inconsistent sets for the syllogistic language.

2. Preliminaries

Sentences are built from
Names: $m_1, m_2, \ldots$
Terms: $a_1, a_2, \ldots$
Copulas: $\in, \in_n, \notin, \notin_n$
Quantifiers: A, E, I, O
Operators: L, M

Definition 2.1: (1) m *c* a is a *singular sentence* iff m is a name, *c* is a copula, and a is a term. (2) Qab is an *assertoric sentence* iff Q is a quantifier and both a and b are terms. (3) Lx and Mx are *apodeictic sentences* iff x is an assertoric sentence. (4) Singular sentences, assertoric sentences, and apodeictic sentences are *sentences* and are the only *sentences*. (Read '$m_1 \in a_1$' as 'm_1 is an a_1', '$m_1 \in_n a_1$' as 'm_1 is necessarily an a_1', '$m_1 \notin a_1$' as 'm_1 is not an a_1', '$m_1 \notin_n a_1$' as 'm_1 is necessarily not an a_1', 'LAa_1a_2' as 'Necessarily all a_1 are a_2', and 'MEa_1a_2' as 'Possibly no a_1 are a_2', etc.)

Definition 2.2.: A finite set of sentences is a *chain* iff each number of the set is an assertoric or apodeictic sentence and the members of the set can be arranged as a finite sequence such that each term occurs exactly twice and in consecutive members of the sequence, where the first and last members of a sequence are consecutive members. (A set of sentences with one member is a chain iff its member is a sentence in which one term occurs twice.)

Definition 2.3: $\langle Y, x\rangle$ is a *syllogism* iff Y, x (that is $Y \cup \{x\}$) is a chain. (So, for example, $\langle\{LAa_1a_2\}, LIa_2a_1\rangle$ is a syllogism but $\langle\{LAa_1a_2\}, LIa_1a_1\rangle$ is not. (Set brackets will often be omitted when referring to chains and syllogisms.) A syllogism $\langle Y, x\rangle$ is an *apodeictic syllogism* if one of the members of Y, x is an apodeictic sentence; otherwise, it is an *assertoric syllogism*. (There are 12×24^n syllogisms with exactly n premises.)

Definition 2.4: An $\in$-*evaluation* is a function υ that assigns t or f to sentences, which meets the sixteen conditions stated below. In the statement of these conditions, a, b, and c range over terms and m ranges over names. There are countably many terms and countably many names.

(1) For every a, there is an m such that $\upsilon(m \in a) = t$.
(2) For every m and a, $\upsilon(m \in a) = t$ iff $\upsilon(m \notin a) = f$.
(3) For every m and a, if $\upsilon(m \in_n a) = t$ then $\upsilon(m \in a) = t$.
(4) For every m and a, if $\upsilon(m \notin_n a) = t$ then $\upsilon(m \notin a) = t$.
(5) $\upsilon(Aab) = t$ iff, for every m, if $\upsilon(m \in a) = t$ then $\upsilon(m \in b) = t$.
(6) $\upsilon(Eab) = t$ iff no m is such that $\upsilon(m \in a) = t$ and $\upsilon(m \in b) = t$.
(7) $\upsilon(Iab) = t$ iff $\upsilon(Eab) = f$.
(8) $\upsilon(Oab) = t$ iff $\upsilon(Aab) = f$.

(9) $\upsilon(LAab) = t$ iff
(i) for every m, if $\upsilon(m \in a) = t$ then $\upsilon(m \in_n b) = t$, and
(ii) for every m and c, if $\upsilon(m \notin_n b) = t$ and $\upsilon(Aca) = t$ th
(10) $\upsilon(LEab) = t$ iff
(i) for every m and c, if $\upsilon(m \in a) = t$ and $\upsilon(Acb) = t$ t
(ii) for every m and c, if $\upsilon(m \in b) = t$ and $\upsilon(Aca) = t$ t
(iii) for every c, if, for some m, $\upsilon(m \in a) = t$ and υ(n
some n, $\upsilon(n \notin_n b) = t$ and $\upsilon(n \in_n c) = t$, and
(iv) for every c, if, for some m, $\upsilon(m \in b) = t$ and υ(n
some n, $\upsilon(n \notin_n a) = t$ and $\upsilon(n \in_n c) = t$.
(11) $\upsilon(LIab) = t$ iff either
(i) for some m, $\upsilon(m \in a) = t$ and $\upsilon(m \in_n b) = t$, or
(ii) for some m, $\upsilon(m \in_n a) = t$ and $\upsilon(m \in b) = t$.
(12) $\upsilon(LOab) = t$ iff, for some m, $\upsilon(m \in_n a) = t$ and $\upsilon(m \notin_n b)$
(13) $\upsilon(MAab) = t$ iff $\upsilon(LOab) = f$.
(14) $\upsilon(MEab) = t$ iff $\upsilon(LIab) = f$.
(15) $\upsilon(MIab) = t$ iff $\upsilon(LEab) = f$.
(16) $\upsilon(MOab) = t$ iff $\upsilon(LAab) = f$.

Definition 2.5: υ is an $\in^n$-*valuation* iff υ is an $\in$-valuation (infinite) set of sentences such that υ assigns t to each memb non-members of Y, then no more than n names occur in the is $\in$-*satisfiable* ($\in^n$-*satisfiable*) iff there is an $\in$-valuation (assigns t to each member of Y. Y is *inconsistent* iff Y is $\langle Y, x\rangle$ is *valid* ($Y \vDash x$) iff Y, x* is inconsistent, where LAab* = MIab, LIab* = MEab, LOab* = MAab, Aab* = Oab, Eab* = Oab* = Aab, MAab* = LOab, MEab* = LIab, MIab* = LE LAab.

3. V_n-syllogisms

X/LAab refers to Aab or LAab. X/LAa – b refers to $\varnothing$ i it refers to $\{X/LAb_1b_2, \ldots X/LAb_{n-1}b_n\}$, where $a = b_1$, b = refers to $\varnothing$, if a = b; otherwise, it refers to X/LAa – c, LAcb. $\{LAa_2a_3, Aa_3a_4, LAa_4a_5\}$ has form X/LAa – b and als $\{LAa_2a_3, Aa_3a_4\}$ has form X/LAa – b but not form LAa – b. Eab or LEab; X/LIab refers to Iab or LIab; and X/LOab LOab.

Definition 3.1: A chain has an I_n-*form* iff it has n members, one of the following *I-forms* (inconsistent-forms):

11 X/LAa – b, X/LOab
12 LAa – c, MAcd, LAd – b, LOab
13 X/LAa – c, LAcb, MOab
21 X/LAc – a, X/LAc – b, X/LEab
22 X/LAc – a, X/LAc – d, MAde, X/LAe – b, LEab (or LE
23 X/LAc – a, X/LAc – d, LAdb, MEab (or MEba)
31 X/LAc – a, X/LAd – b, X/LIcd, X/LEab (or X/LEba)
32 X/LAc – a, X/LAd – e, MAef, X/LAf – b, X/LIcd (or X
LEba)

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: ⟨Y, x⟩ is a *syllogism* iff Y, x (that is Y ∪ {x}) is a chain. (So, for $Aa_1a_2\}$, LIa_2a_1⟩ is a syllogism but ⟨{LAa_1a_2}, LIa_1a_1⟩ is not. (Set often be omitted when referring to chains and syllogisms.) A x⟩ is an *apodeictic syllogism* if one of the members of Y, x is an tence; otherwise, it is an *assertoric syllogism*. (There are 12×24^n h exactly n premises.)

An *∈-evaluation* is a function v that assigns t or f to sentences, the sixteen conditions stated below. In the statement of these b, and c range over terms and m ranges over names. There are y terms and countably many names.

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m and a, if $v(m \notin_n a) = t$ then $v(m \notin a) = t$.
t iff, for every m, if $v(m \in a) = t$ then $v(m \in b) = t$.
t iff no m is such that $v(m \in a) = t$ and $v(m \in b) = t$.
iff $v(Eab) = f$.
t iff $v(Aab) = f$.

(9) $v(LAab) = t$ iff
(i) for every m, if $v(m \in a) = t$ then $v(m \in_n b) = t$, and
(ii) for every m and c, if $v(m \notin_n b) = t$ and $v(Aca) = t$ then $v(m \notin_n c) = t$.
(10) $v(LEab) = t$ iff
(i) for every m and c, if $v(m \in a) = t$ and $v(Acb) = t$ then $v(m \notin_n c) = t$,
(ii) for every m and c, if $v(m \in b) = t$ and $v(Aca) = t$ then $v(m \notin_n c) = t$,
(iii) for every c, if, for some m, $v(m \in a) = t$ and $v(m \in c) = t$ then, for some n, $v(n \notin_n b) = t$ and $v(n \in_n c) = t$, and
(iv) for every c, if, for some m, $v(m \in b) = t$ and $v(m \in c) = t$ then, for some n, $v(n \notin_n a) = t$ and $v(n \in_n c) = t$.
(11) $v(LIab) = t$ iff either
(i) for some m, $v(m \in a) = t$ and $v(m \in_n b) = t$, or
(ii) for some m, $v(m \in_n a) = t$ and $v(m \in b) = t$.
(12) $v(LOab) = t$ iff, for some m, $v(m \in_n a) = t$ and $v(m \notin_n b) = t$.
(13) $v(MAab) = t$ iff $v(LOab) = f$.
(14) $v(MEab) = t$ iff $v(LIab) = f$.
(15) $v(MIab) = t$ iff $v(LEab) = f$.
(16) $v(MOab) = t$ iff $v(LAab) = f$.

Definition 2.5: v is an *$\in^n$-valuation* iff v is an ∈-valuation and if Y is the (infinite) set of sentences such that v assigns t to each member of Y and to no non-members of Y, then no more than n names occur in the sentences in Y. Y is *∈-satisfiable* (*$\in^n$-satisfiable*) iff there is an ∈-valuation ($\in^n$-valuation) that assigns t to each member of Y. Y is *inconsistent* iff Y is not ∈-satisfiable. ⟨Y, x⟩ is *valid* (Y ⊨ x) iff Y, x* is inconsistent, where LAab* = MOab, LEab* = MIab, LIab* = MEab, LOab* = MAab, Aab* = Oab, Eab* = Iab, Iab* = Eab, Oab* = Aab, MAab* = LOab, MEab* = LIab, MIab* = LEab, and MOab* = LAab.

3. V_n-syllogisms

X/LAab refers to Aab or LAab. X/LAa − b refers to ∅ if a = b; otherwise, it refers to $\{X/LAb_1b_2, \ldots X/LAb_{n-1}b_n\}$, where $a = b_1$, $b = b_n$, $n > 1$. LAa − b refers to ∅, if a = b; otherwise, it refers to X/LAa − c, LAcb. (So, for example, $\{LAa_2a_3, Aa_3a_4, LAa_4a_5\}$ has form X/LAa − b and also form LAa − b. $\{LAa_2a_3, Aa_3a_4\}$ has form X/LAa − b but not form LAa − b.) X/LEab refers to Eab or LEab; X/LIab refers to Iab or LIab; and X/LOab refers to Oab or LOab.

Definition 3.1: A chain has an *I_n-form* iff it has n members, for n ⩾ 1, and has one of the following *I-forms* (inconsistent-forms):

11 X/LAa − b, X/LOab
12 LAa − c, MAcd, LAd − b, LOab
13 X/LAa − c, LAcb, MOab
21 X/LAc − a, X/LAc − b, X/LEab
22 X/LAc − a, X/LAc − d, MAde, X/LAe − b, LEab (or LEba)
23 X/LAc − a, X/LAc − d, LAdb, MEab (or MEba)
31 X/LAc − a, X/LAd − b, X/LIcd, X/LEab (or X/LEba)
32 X/LAc − a, X/LAd − e, MAef, X/LAf − b, X/LIcd (or X/LIdc), LEab (or LEba)

33 X/LAc – a, X/LAd – b, MIcd, LEab (or LEba)
34 LIab, MEab (or MEba)
35 X/LAc – a, X/LAd – e, LAeb, X/LIcd (or X/LIdc), MEab (or MEba)

So, for example, all of the following sets of sentences have I-form 11: $\{Oa_1a_1\}$, $\{LOa_1a_1\}$, $\{Aa_1a_2, OA_1a_2\}$, $\{Aa_1a_2, LOa_1a_2\}$, $\{LAa_1a_2, Oa_1a_2\}$, $\{LAa_1a_2, LOa_1a_2\}$, $\{Aa_1a_2, Aa_2a_3, Oa_1a_3\}$, $\{LAa_1a_2, LAa_2a_3, LOa_1a_3\}$, among others. In contrast to sets of sentences with I-form 11, sets with I-form 12 have at least two members and sets with I-form 32 have at least three members.

The I-forms are numbered to reflect their relations to the three Smiley-forms, defined below.

Definition 3.2: $\langle Y, x\rangle$ is a *V-syllogism* (valid-syllogism) iff it is a syllogism and Y, x* has an I-form. $\langle Y, x\rangle$ is a *V_n-syllogism* iff $\langle Y, x\rangle$ is a V-syllogism and Y, x has n members.

Since $\{Aa_1a_2, Aa_2a_3, Aa_1a_3{}^*$ (that is, $Oa_1a_3)\}$ has I-form 11, and since $\langle\{Aa_1a_2, Aa_2a_3\}, Aa_1a_3\rangle$ is a syllogism, $\langle\{Aa_1a_2, Aa_2a_3\}, Aa_1a_3\rangle$ (Barbara-XXX) is a V_3-syllogism. Since $\langle\{Aa_1a_2, Oa_2a_3{}^*$ (that is, $Aa_2a_3)\}, Oa_1a_3\rangle$ has I-form 11, and since $\langle\{Aa_1a_2, Oa_1a_3\}, Oa_2a_3\rangle$ is a syllogism, $\langle\{Aa_1a_2, Oa_1a_3\}, Oa_2a_3\rangle$ (Bocardo-XXX) is a V_3-syllogism.

The following table lists the 333 V_3-syllogisms.

	X/L X/L X/M	L L L	L X L	X L L	L M M	M L M	M X M	X M M	L M X	M L X	
Barbara	11	13	13		12	12					
Celarent	31	33	33		32	35			32	35	
Darii	31	35	35		33	32	32	33			
Ferio	31	32	32		33	35	35		33		
Cesare	31	33	33		32	35			32	35	
Camestres	31	33		33	35	32			35	32	
Festino	31	32	32		33	35	35		33		
Baroco	11	12			13	12			13		
Darapti	21	23	23	23	22	22	22	22			
Felapton	21	22	22		22	23	23		22		
Disamis	31	35		35	32	33	33	32			
Datisi	31	35	35		33	32	32	33			
Bocardo	11	12			12	13	13				
Ferison	31	32	32		33	35	35		33		
Bramantip	21	23		23	22	22	22	22			
Camenes	31	33		33	35	32			35	32	
Dimaris	31	35		35	32	33	33	32			
Fresison	31	32	32		33	35	35		33		
Fesapo	21	22	22		22	23	23		22		
Barbari	21	23	23		22	22	22	22			
Celaront	21	22	22		22	23	23		22	23	
Cesaro	21	22	22		22	23	23		22	23	
Camestrop	21	22		22	23	22			23	22	
Camenop	21	22		22	23	22			23	22	
Total	192	24	15	8	24	24	16	7	15	8	= 333

Cells marked with a numeral indicate the I-form that genera
So, for example, I-form 11 generates Barbara-XXX, Bar
cardo-LLM, among others. I-form 22 generates Camenop-M
example of Camenop-MLX: $\langle\{MAa_1a_2, LEa_2a_3\}, Oa_3a_1\rangle$. T
generated by I-form 22, first form the following special case
c = a and e = b: {Aad, MAde, LEea}. Then replace a by a
a_2, forming $\{Aa_3a_1$ (that is, $Oa_3,a_1{}^*$), MAa_1a_2, $LEa_2a_3\}$.)

The 333 V_3-syllogism are precisely the syllogisms that
counts as the 'valid L–X–M moods'. Following McCall, thes
Aristotle recognized as valid or would have regarded as val
out his system in more detail. No empty cell on the table
that Aristotle recognized as valid.

Some features of V_n-syllogisms are obvious. (For exa
V_n-syllogisms in which L occurs in the conclusion and L d
premises. And there is no V_n-syllogism in which I occurs m
premises.) Others are less obvious but are still decidable. (
moods are there such that some V_n-syllogism has this mo
many V_n-syllogisms are there?)

One of our goals will be to show that the V_n-syllo
syllogisms. Half of the argument is found in the next theo
half will use the account of deducibility given in the next sec

Theorem 3.1: If $\langle Y, x\rangle$ is a syllogism, $Y \cup \{x\}$ has n memb
$\langle Y, x\rangle$ is a V_n-syllogism.

Proof. A chain α has a *Smiley-form* iff α has one of the I-f
provided 'L' does not occur in α. (These forms, listed in Th
Smiley *1973*, provide a decision procedure for determi
assertoric syllogisms.) By definition, if α is a chain then
results from deleting all occurrences of L and M in α, and
results from replacing all occurrences of Qab in α^x with
quantifier. (So, if $\alpha = MAa_1a_2, LOa_2a_1$, then $\alpha^x = Aa_1a$
LAa_1a_2, LOa_2a_1. By inspecting the I-forms we note that if
α^x has a Smiley-form. For example, if α has I-form 13 the
Aa-b, Oab. If α has I-form rs then α^x has I-form r1.)

Theorem 3.1 follows immediately from Lemmas 1 and 2,

Lemma 1: If α is a chain that does not have an I-form and
Smiley-form then α is $\in^3$-satisfiable.

Proof. Use the following two lemmas.

Lemma 1.1: If α is a chain and α^x does not have a Sm
$\in^3$-satisfiable.

Proof. The general strategy is to take advantage of J
involving "ordinary" set-theoretic definitions of satisfiability

X/LAd – b, MIcd, LEab (or LEba)
(or MEba)
X/LAd – e, LAeb, X/LIcd (or X/LIdc), MEab (or MEba)

mple, all of the following sets of sentences have I-form 11: $_1a_1\}$, $\{Aa_1a_2, OA_1a_2\}$, $\{Aa_1a_2, LOa_1a_2\}$, $\{LAa_1a_2, Oa_1a_2\}$, $_2\}$, $\{Aa_1a_2, Aa_2a_3, Oa_1a_3\}$, $\{LAa_1a_2, LAa_2a_3, LOa_1a_3\}$, among ast to sets of sentences with I-form 11, sets with I-form 12 have nbers and sets with I-form 32 have at least three members.
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$_2, Aa_2a_3, Aa_1a_3^*$ (that is, Oa_1a_3)} has I-form 11, and since , Aa_1a_3⟩ is a syllogism, ⟨$\{Aa_1a_2, Aa_2a_3\}, Aa_1a_3$⟩ (Barbara-yllogism. Since ⟨$\{Aa_1a_2, Oa_2a_3^*$ (that is, Aa_2a_3)}, Oa_1a_3⟩ has since ⟨$\{Aa_1a_2, Oa_1a_3\}, Oa_2a_3$⟩ is a syllogism, ⟨$\{Aa_1a_2, Oa_1a_3\}$, o-XXX) is a V_3-syllogism.
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X/L X/L /M	L L L	L X L	X L L	L M M	M L M	M X M	X M M	L M X	M L X	
11	13	13		12	12					
31	33	33		32	35			32	35	
31	35	35		33	32	32	33			
31	32	32		33	35	35		33		
31	33	33		32	35			32	35	
31	33		33	35	32			35	32	
31	32	32		33	35	35		33		
11	12			13	12			13		
21	23	23	23	22	22	22	22			
21	22	22		22	23	23		22		
31	35		35	32	33	33	32			
31	35	35		33	32	32	33			
11	12			12	13	13				
31	32	32		33	35	35		33		
21	23		23	22	22	22	22			
31	33		33	35	32			35	32	
31	35		35	32	33	33	32			
31	32	32		33	35	35		33		
21	22	22		22	23	23		22		
21	23	23		22	22	22	22			
21	22	22		22	23	23		22	23	
21	22	22		22	23	23		22	23	
21	22		22	23	22			23	22	
21	22		22	23	22			23	22	
92	24	15	8	24	24	16	7	15	8	= 333

Cells marked with a numeral indicate the I-form that generates the V_3-syllogism. So, for example, I-form 11 generates Barbara-XXX, Barbara-XXM, and Bocardo-LLM, among others. I-form 22 generates Camenop-MLX. (Consider this example of Camenop-MLX: ⟨$\{MAa_1a_2, LEa_2a_3\}, Oa_3a_1$⟩. To recognize that it is generated by I-form 22, first form the following special case of I-form 22, where c = a and e = b: {Aad, MAde, LEea}. Then replace a by a_3, d by a_1, and e by a_2, forming {Aa_3a_1 (that is, Oa_3,a_1^*), MAa_1a_2, LEa_2a_3}.)

The 333 V_3-syllogism are precisely the syllogisms that McCall *1963*, p. 46, counts as the 'valid L–X–M moods'. Following McCall, these are inferences that Aristotle recognized as valid or would have regarded as valid if he had worked out his system in more detail. No empty cell on the table marks an inference that Aristotle recognized as valid.

Some features of V_n-syllogisms are obvious. (For example, there are no V_n-syllogisms in which L occurs in the conclusion and L does not occur in the premises. And there is no V_n-syllogism in which I occurs more than once in the premises.) Others are less obvious but are still decidable. (For any n how many moods are there such that some V_n-syllogism has this mood? For any n how many V_n-syllogisms are there?)

One of our goals will be to show that the V_n-syllogisms are the valid syllogisms. Half of the argument is found in the next theorem. The remaining half will use the account of deducibility given in the next section.

Theorem 3.1: If ⟨Y, x⟩ is a syllogism, Y ∪ {x} has n members, and $Y \vDash x$, then ⟨Y, x⟩ is a V_n-syllogism.

Proof. A chain α has a *Smiley-form* iff α has one of the I-forms 11, 21, and 31, provided 'L' does not occur in α. (These forms, listed in Theorem 2 on p. 143 of Smiley *1973*, provide a decision procedure for determining the validity of assertoric syllogisms.) By definition, if α is a chain then α^x is the chain that results from deleting all occurrences of L and M in α, and α^L is the chain that results from replacing all occurrences of Qab in α^x with LQab, where Q is a quantifier. (So, if $\alpha = MAa_1a_2$, LOa_2a_1, then $\alpha^x = Aa_1a_2$, Oa_2a_1 and $\alpha^L = LAa_1a_2$, LOa_2a_1. By inspecting the I-forms we note that if α has an I-form then α^x has a Smiley-form. For example, if α has I-form 13 then α^x has Smiley-form Aa-b, Oab. If α has I-form rs then α^x has I-form r1.)

Theorem 3.1 follows immediately from Lemmas 1 and 2, below.

Lemma 1: If α is a chain that does not have an I-form and α^x does not have a Smiley-form then α is $\in^3$-satisfiable.

Proof. Use the following two lemmas.

Lemma 1.1: If α is a chain and α^x does not have a Smiley-form then α^x is $\in^3$-satisfiable.

Proof. The general strategy is to take advantage of Johnson's *1991* result involving "ordinary" set-theoretic definitions of satisfiability for chains, where it is shown that any chain that does not have an I-form is satisfiable in a three-membered domain.

As in Johnson *1993*, we use matrices of form

	a_1	a_2	⋯	a_n	a_{n+1}	⋯
m_1						
m_2						
m_3						
m_4						
⋮						

to indicate $\in$-valuations, where each cell is filled with $\in$, $\in_n$, $\notin_n$, or the empty symbol, and where $\in$ or $\in_n$ occurs in each column. So, for example, the matrix

g_1	a_1	a_2	a_3	a_4	a_5	⋯
m_1	$\in$	$\in_n$	$\notin_n$	$\in$	⋯	
m_2		$\in$				
m_3			$\in$			
m_4						
⋮						

indicates an $\in$-valuation, g_1, where $g_1(m_1 \in a_1) = t$, $g_1(m_1 \in a_2) = t$, $g_1(m_1 \in_n a_2) = t$, $g_1(m_1 \in a_2) = t$, $g_1(Aa_1a_4) = t$, $g_1(Aa_1a_5) = t$, $g_1(m_2 \notin a_1) = t$, etc.

To prove Lemma 1.1 we convert Johnson's *1991* matrices for models with three-membered domains that show the satisfiability of chains that do not have Smiley-forms into $\in^3$-valuations. So, for example, the following matrix on p. 186 of Johnson *1991*, shows that the chain $\langle Ea_1a_2, Oa_2a_3, Aa_3a_4, Oa_1a_4\rangle$ is satisfiable:

$$M_1 \begin{bmatrix} a_1 & \varnothing & a_3 & a_4 \\ \varnothing & a_2 & \varnothing & a_4 \\ \varnothing & \varnothing & \varnothing & \varnothing \end{bmatrix}$$

Convert this matrix, M_1, in a natural way, forming matrix

M_2

g_2	a_1	a_2	a_3	a_4	a_5	⋯
m_1	$\in$		$\in$	$\in$	⋯	
m_2		$\in$		$\in$		
m_3						
⋮						

g_2 is an $\in^3$-valuation that also satisfies the chain Ea_1a_2, Oa_2a_3, Aa_3a_4, Oa_1a_4. By following the above procedure we can convert any model with a three-membered domain that shows the satisfiability of a chain of assertoric sentences into an $\in^3$-valuation that shows the satisfiability of the chain.

Lemma 1.2: If α is a chain and α^x does not have a Smiley-from then α^L is $\in^3$-satisfiable.

Proof. Assume the antecedent. By Lemma 1.1 there is a ma

g_3	a_1	a_2	⋯	a_n	a_{n+1}	⋯
m_1						
m_2						
m_3						
m_4						
⋮						

that indicates an $\in^3$-valuation that satisfies α^x, where cells i
empty and the remaining cells are empty or contain '$\in$'. Co
matrix M_4 by changing each occurrence of '$\in$' to '$\in_n$', lea
unchanged. The $\in^3$-valuation, g_4, determined by M_4 satisfie
consider the four types of sentences that could occur in
LIa_ia_j, and LOa_ia_j. Note that for a sentence Lx of an
$g_4(Lx) = f$ then $g_3(x) = f$, though we assumed that $g_3(x) = t$.

Since any $\in^3$-valuation that assigns t to LQab assigns t
Lemma 1 is true, given Lemma 1.2.

Lemma 2: If α is a chain that does not have an I-form and
then α is $\in^3$-satisfiable.

Proof. The lemma follows directly from the following f
consider the number of occurrences of M in α.

Lemma 2.1: If α is a chain that does not have an I-form, α
and M does not occur in α, then α is $\in^3$-satisfiable.

Proof. If α^x has a Smiley-form and M does not occur in
11, 21, or 31. So the lemma is vacuously true.

Lemma 2.2: If α is a chain that does not have an I-form, α
and M occurs exactly once in α, then α is $\in^3$-satisfiable.

Proof. Assume the antecedent. For each of the forms α ma
matrix M such that v_M is an $\in^3$-valuation that assigns t to
chain. The column to the right of the last labeled column ha
the first column. For the following matrices Condition 1 f
met, since in each of the matrices either $\in$ or $\in_n$ occurs in ea

We let 'XQ occurs in α' be short for 'Q, but neither LQ
α', where Q is a quantifier.

Case 1: α^x has I-form 11. There are three subcases to consid

Subcase 1.1: XO occurs in α. Then α has form (1M1)
X/LAd – b, Oab. Use

v_{1M1}	a	⋯	c	d	⋯	b
m_1	$\in_n$	⋯	$\in_n$			
m_2				$\in_n$	⋯	$\in_n$
m_3						
⋮						

Proof. Assume the antecedent. By Lemma 1.1 there is a matrix, M_3, of form

g_3	a_1	a_2	$\cdots$	a_n	a_{n+1}	$\cdots$
m_1						
m_2						
m_3						
m_4						
$\vdots$						

that indicates an $\in^3$-valuation that satisfies α^x, where cells in row m_i $(i > 3)$ are empty and the remaining cells are empty or contain '$\in$'. Convert matrix M_3 into matrix M_4 by changing each occurrence of '$\in$' to '$\in_n$', leaving the other cells unchanged. The $\in^3$-valuation, g_4, determined by M_4 satisfies α^L. To show this, consider the four types of sentences that could occur in α^L: LAa_ia_j, LEa_ia_j, LIa_ia_j, and LOa_ia_j. Note that for a sentence Lx of any of these types if $g_4(Lx) = f$ then $g_3(x) = f$, though we assumed that $g_3(x) = t$.

Since any $\in^3$-valuation that assigns t to LQab assigns t to Qab and MQab, Lemma 1 is true, given Lemma 1.2.

Lemma 2: If α is a chain that does not have an I-form and α^x has a Smiley-form then α is $\in^3$-satisfiable.

Proof. The lemma follows directly from the following four lemmas, which consider the number of occurrences of M in α.

Lemma 2.1: If α is a chain that does not have an I-form, α^x has a Smiley-form, and M does not occur in α, then α is $\in^3$-satisfiable.

Proof. If α^x has a Smiley-form and M does not occur in α then α has I-form 11, 21, or 31. So the lemma is vacuously true.

Lemma 2.2: If α is a chain that does not have an I-form, α^x has a Smiley-form, and M occurs exactly once in α, then α is $\in^3$-satisfiable.

Proof. Assume the antecedent. For each of the forms α may have we specify a matrix M such that υ_M is an $\in^3$-valuation that assigns t to each member of the chain. The column to the right of the last labeled column has the same entries as the first column. For the following matrices Condition 1 for an $\in$-valuation is met, since in each of the matrices either $\in$ or $\in_n$ occurs in each column.

We let 'XQ occurs in α' be short for 'Q, but neither LQ nor MQ, occurs in α', where Q is a quantifier.

Case 1: α^x has I-form 11. There are three subcases to consider.

Subcase 1.1: XO occurs in α. Then α has form (1M1) X/LAa − c, MAcd, X/LAd − b, Oab. Use

υ_{1M1}	a	$\cdots$	c	d	$\cdots$	b	$\cdots$
m_1	$\in_n$	$\cdots$	$\in_n$				$\in_n \cdots$
m_2				$\in_n$	$\cdots$	$\in_n$	
m_3							
$\vdots$							

(If $a \neq c$ and '$\in$' occurred in cell m_1/c then v_{1M1} would assign f to LAac, given condition 10.i.) *Subcase 1.2*: LO occurs in α and α has more than two members. (If α has exactly two members then α has I-form 12.) If α has form (1M2) X/LAa – c, MAcd, X/LAd – e, Aeb, LOab, use

v_{1M2}	a	···	c	d	···	e	b	···
m_1	$\in$	···	$\in_n$				$\notin_n$	$\in_n$ ···
m_2				$\in_n$	···	$\in_n$	$\in$	
m_3								
⋮								

If α has the form (1M3) X/LAa – e, Aec, MAcd, X/LAd – b, LOab, use

v_{1M3}	a	···	e	c	d	···	b	···
m_1	$\in_n$	···	$\in_n$	$\in$	$\notin_n$	···	$\notin_n$	$\in_n$ ···
m_2					$\in_n$	···	$\in_n$	
m_3								
⋮								

(If $d \neq b$ and if no symbol occurred in cell m_1/d then v_{1M3} would assign f to LAdb, given condition 10.ii.) *Subcase 1.3*: MO occurs in α. If α has exactly one member, then α has form (1M4) MOaa. Use

v_{1M4}	a	···	
m_1	$\in$	$\in$	···
m_2			
m_3			
⋮			

If α has more than one member, then α has form (1M5) X/LAa – c, Acb, MOab. Use

v_{1M5}	a	···	c	b	···	
m_1	$\in_n$	···	$\in_n$	$\in$	$\in_n$	···
m_2						
m_3						
⋮						

Case 2: α^x has I-form 21. *Subcase 2.1*: XE occurs in α. Then α has form (2M1) X/LAc – d, MAde, X/LAe – a, X/LAc – b, Eab (or Eba). Use

v_{2M1}	a	···	e	d	···	c	···	b	···	
m_1				$\in_n$	···	$\in_n$	···	$\in_n$		
m_2	$\in_n$	···	$\in_n$						$\in_n$	···
m_3										
⋮										

Subcase 2.2: LE occurs in α. Vacuous, since α has I-form 21. *Subcase 2.3*: ME occurs in α. *Subcase 2.3.1*: α has exactly one member. Then α has form (2M2)

MEaa. Use

v_{2M2}	a	···	
m_1	$\in$	$\in$	···
m_2			
m_3			
⋮			

Subcase 2.3.2: α has more than one member. Define *condi* some a and b X/LAa – b occurs in α and if X/LAc – X/LAc – d $\subseteq$ X/LAa – b.

Subcase 2.3.2.1: Condition i is met. Then α has form (2M MEab (or MEba). Use

v_{2M3}	a	···	d	b	···
m_1	$\in$	···	$\in_n$	$\in$	$\in$
m_2					
m_3					
⋮					

Subcase 2.3.2.2: Condition i is not met. Then α has forn Ada, X/LAc – e, Aeb, MEab. Use

v_{2M4}	a	d	···	c	···	e	b
m_1	$\in$	$\in_n$		$\in_n$	···	$\in_n$	$\in$
m_2							
m_3							
⋮							

Case 3: α^x has I-form 31. *Subcase 3.1*: XE occurs in α. occurs in α. Then α has form (3M1) X/LAc – e, X/LAd – b, X/LIcd (or X/LIdc), Eab (or Eba). Use

v_{3M1}	a	···	f	e	···	c	d	···	b
m_1				$\in_n$	···	$\in_n$	$\in_n$	···	$\in$
m_2	$\in_n$	···	$\in_n$						
m_3									
⋮									

Subcase 3.1.2: MI occurs in α. Then α has the form X/LAd – b, MIcd, Eab (or Eba). Use

v_{3M2}	a	···	c	d	···	b	·
m_1	$\in_n$	···	$\in_n$				$\in$
m_2				$\in_n$	···	$\in_n$	
m_3							
⋮							

Subcase 3.2: LE occurs in α. Vacuous, since α has I-form 3

'∈' occurred in cell m_1/c then v_{1M1} would assign f to LAac, given i.) *Subcase 1.2*: LO occurs in α and α has more than two members. actly two members then α has I-form 12.) If α has form (1M2) MAcd, X/LAd − e, Aeb, LOab, use

v_{1M2}	a	···	c	d	···	e	b	···
m_1	$\in$	···	$\in_n$				$\notin_n$	$\in_n$···
m_2				$\in_n$	···	$\in_n$	$\in$	
m_3								
⋮								

orm (1M3) X/LAa − e, Aec, MAcd, X/LAd − b, LOab, use

v_{1M3}	a	···	e	c	d	···	b	···
m_1	$\in_n$	···	$\in_n$	$\in$	$\notin_n$	···	$\notin_n$	$\in_n$···
m_2					$\in_n$	···	$\in_n$	
m_3								
⋮								

if no symbol occurred in cell m_1/d then v_{1M3} would assign f to condition 10.ii.) *Subcase 1.3*: MO occurs in α. If α has exactly one α has form (1M4) MOaa. Use

v_{1M4}	a	···	
m_1	$\in$	$\in$	···
m_2			
m_3			
⋮			

e than one member, then α has form (1M5) X/LAa − c, Acb,

v_{1M5}	a	···	c	b	···	
m_1	$\in_n$	···	$\in_n$	$\in$	$\in_n$	···
m_2						
m_3						
⋮						

I-form 21. *Subcase 2.1*: XE occurs in α. Then α has form (2M1) Ade, X/LAe − a, X/LAc − b, Eab (or Eba). Use

	a	···	e	d	···	c	···	b	···	
				$\in_n$	···	$\in_n$	···	$\in_n$		
	$\in_n$	···	$\in_n$						$\in_n$	···

E occurs in α. Vacuous, since α has I-form 21. *Subcase 2.3*: ME bcase 2.3.1: α has exactly one member. Then α has form (2M2)

MEaa. Use

v_{2M2}	a	···	
m_1	$\in$	$\in$	···
m_2			
m_3			
⋮			

Subcase 2.3.2: α has more than one member. Define *condition i* as follows. For some a and b X/LAa − b occurs in α and if X/LAc − d occurs in α then X/LAc − d ⊆ X/LAa − b.

Subcase 2.3.2.1: Condition i is met. Then α has form (2M3) X/LAa − d, Adb, MEab (or MEba). Use

v_{2M3}	a	···	d	b	···	
m_1	$\in$	···	$\in_n$	$\in$	$\in$	···
m_2						
m_3						
⋮						

Subcase 2.3.2.2: Condition i is not met. Then α has form (2M4) X/LAc − d, Ada, X/LAc − e, Aeb, MEab. Use

v_{2M4}	a	d	···	c	···	e	b	···	
m_1	$\in$	$\in_n$		$\in_n$	···	$\in_n$	$\in$	$\in$	···
m_2									
m_3									
⋮									

Case 3: α^x has I-form 31. *Subcase 3.1*: XE occurs in α. *Subcase 3.1.1*: MA occurs in α. Then α has form (3M1) X/LAc − e, MAef, X/LAf − a, X/LAd − b, X/LIcd (or X/LIdc), Eab (or Eba). Use

v_{3M1}	a	···	f	e	···	c	d	···	b	···	
m_1				$\in_n$	···	$\in_n$	$\in_n$	···	$\in_n$		
m_2	$\in_n$	···	$\in_n$							$\in_n$	···
m_3											
⋮											

Subcase 3.1.2: MI occurs in α. Then α has the form (3M2) X/LAc − a, X/LAd − b, MIcd, Eab (or Eba). Use

v_{3M2}	a	···	c	d	···	b	···	
m_1	$\in_n$	···	$\in_n$				$\in_n$	···
m_2				$\in_n$	···	$\in_n$		
m_3								
⋮								

Subcase 3.2: LE occurs in α. Vacuous, since α has I-form 32 or 33.

Subcase 3.3: ME occurs in α. If α has exactly two members than α has form (3M3) Iab, MEab (or MEba). Use

υ_{3M3}	a	b	···	
m_1	$\in$	$\in$	$\in$	···
m_2				
m_3				
⋮				

If α has more than two members and Condition i is met then α has form (3M4) X/LAc – d, Adb, X/LIac (or X/LIca), MEab (or MEba). Use

υ_{3M4}	a	c	···	d	b	···	
m_1	$\in$	$\in_n$	···	$\in_n$	$\in$	$\in$	···
m_2							
m_3							
⋮							

If α has more than two members and Condition i is not met then α has form (3M5) X/LAc – e, Aea, X/LAd – f, Afb, X/LIcd, MEab (or MEba). Use

υ_{3M5}	a	e	···	c	d	···	f	b	···	
m_1	$\in$	$\in_n$	···	$\in_n$	$\in_n$	···	$\in_n$	$\in$	$\in$	···
m_2										
m_3										
⋮										

Lemma 2.3: If α is a chain that does not have an I-form, α^x has a Smiley-form, and M occurs exactly twice in α, then α is $\in^3$-satisfiable.

Proof. Assume the antecedent. For each of the forms α may have we either use a matrix given in Lemma 2.2 or provide a new matrix that shows that α is $\in^3$-satisfiable. By familiar observations the new matrices satisfy Condition 1 for an $\in$-valuation. If α is a chain in which there is more than one occurrence of the operator M let α^{1M} be any chain that results from removing all but one occurrence of M from α. We use the fact that if an α^{1M} chain is $\in^3$-satisfiable then chain α is $\in^3$-satisfiable. To show this note that Qab $\vDash$ MQab. (For example, if $\upsilon(\text{Aab}) = t$ and $\upsilon(\text{MAab}) = f$, then, for some m, $\upsilon(m \in_n a) = t$, $\upsilon(m \notin_n b) = t$, and $\upsilon(m \in b) = t$, which is impossible.)

Case 1: α^x has I-form 11. *Subcase 1.1*: XO or MO occurs in α. Then an α^{1M} chain has form 1M1. *Subcase 1.2*: LO occurs in α. Then α has form (1MM1) X/LAa – c, MAcd, X/LAd – e, MAef, X/LAf – b, LOab. Use

υ_{1MM1}	a	···	c	d	···	e	f	···	b	···	
m_1	$\in_n$	···	$\in_n$				$\notin_n$		$\notin_n$	$\in_n$	···
m_2							$\in_n$	···	$\in_n$		
m_3				$\in_n$	···	$\in_n$					
⋮											

(If f $\neq$ b and no symbol occurred in cell m_1/f then υ_{1MM1} wo
given condition 10.ii.)

Case 2: α^x has I-form 21. *Subcase 2.1*: XE or ME occurs
chain has form 2M1. *Subcase 2.1*: LE occurs in α. *Subca*
(2MM1) X/LAc – d, MAde, X/LAe – f, MAfg, X/LAg –
(or LEba). Use

υ_{2MM1}	a	···	g	f	···	e	d	···	c	···
m_1	$\notin_n$	···	$\notin_n$				$\in_n$	···	$\in_n$	··
m_2	$\in_n$	···	$\in_n$				$\notin_n$	···	$\notin_n$	··
m_3				$\in_n$	···	$\in_n$				
⋮										

(Given condition 11.i, since $\upsilon_{2MM1}(m_2 \in a) = t$ and υ_{2MM1}
occur in cell m_2/d. Otherwise, υ_{2MM1} would assign f to LE
11.ii, since $\upsilon_{2MM1}(m_2 \in a) = t$ and $\upsilon_{2MM1}(m_2 \in g) = t$, it mus
some i $(1 \leq i \leq 3)$ υ_{2MM1} $(m_i \notin_n b)$ and υ_{2MM1} $(m_i \notin_n g) = t$
true for i = 2.) Otherwise, υ_{2MM1} would assign f to LEab.)

Subcase 2.1.2: α has form (2MM2) X/LAc – d, MAde, X/
MAfg, X/LAg – b, LEab. Use

υ_{2MM2}	a	···	e	d	···	c	···	f	g	··
m_1				$\in_n$	···	$\in_n$	···	$\in_n$		
m_2	$\in_n$	···	$\in_n$						$\notin_n$	·
m_3	$\notin_n$	···	$\notin_n$						$\in_n$	·
⋮										

Case 3: α^x has I-form 31. *Subcase 3.1*: XE or ME occurs
chain has form 3M1 or 3M2. *Subcase 3.2*: LE occurs i
(3MM1) X/LAc – e, MAef, X/LAf – g, MAgh, X/L
X/LIcd (or X/LIdc), LEab (or LEba), use

υ_{3MM1}	a	···	h	g	···	f	e	···	c	d	·
m_1	$\notin_n$	···	$\notin_n$				$\in_n$	···	$\in_n$	$\in_n$	·
m_2	$\in_n$	···	$\in_n$				$\notin_n$	···	$\notin_n$	$\notin_n$	·
m_3				$\in_n$	···	$\in_n$					
⋮											

If α has form (3MM2) X/LAc – e, MAef, X/LAf – a,
LAh – b, X/LIcd, LEab (or LEba), use

υ_{3MM2}	a	···	f	e	···	c	d	···	g	h	·
m_1				$\in_n$	···	$\in_n$	$\in_n$	···	$\in_n$		
m_2	$\in_n$	···	$\in_n$							$\notin_n$	·
m_3	$\notin_n$	···	$\notin_n$							$\in_n$	·
⋮											

And if α has form (3MM3) X/LAc – e, MAef, X/LAf –
(or MIdc), LEab (or LEba), use

(If f $\neq$ b and no symbol occurred in cell m_1/f then v_{1MM1} would assign f to LAfb, given condition 10.ii.)

Case 2: α^x has I-form 21. *Subcase 2.1*: XE or ME occurs in α. Then an α^{1M} chain has form 2M1. *Subcase 2.1*: LE occurs in α. *Subcase 2.1.1*: α has form (2MM1) X/LAc – d, MAde, X/LAe – f, MAfg, X/LAg – a, X/LAc – b, LEab (or LEba). Use

v_{2MM1}	a	$\cdots$	g	f	$\cdots$	e	d	$\cdots$	c	$\cdots$	b	$\cdots$	
m_1	$\notin_n$	$\cdots$	$\notin_n$				$\in_n$	$\cdots$	$\in_n$	$\cdots$	$\in_n$	$\notin_n$	$\cdots$
m_2	$\in_n$	$\cdots$	$\in_n$				$\notin_n$	$\cdots$	$\notin_n$	$\cdots$	$\notin_n$	$\in_n$	$\cdots$
m_3				$\in_n$	$\cdots$	$\in_n$							
$\vdots$													

(Given condition 11.i, since $v_{2MM1}(m_2 \in a) = t$ and $v_{2MM1}(Adb) = t$, '$\notin_n$' must occur in cell m_2/d. Otherwise, v_{2MM1} would assign f to LEab. Given condition 11.ii, since $v_{2MM1}(m_2 \in a) = t$ and $v_{2MM1}(m_2 \in g) = t$, it must be the case that for some i $(1 \leqslant i \leqslant 3)$ v_{2MM1} $(m_i \notin_n b)$ and v_{2MM1} $(m_i \notin_n g) = t$. (The consequent is true for i = 2.) Otherwise, v_{2MM1} would assign f to LEab.)

Subcase 2.1.2: α has form (2MM2) X/LAc – d, MAde, X/LAe – a, X/LAc – f, MAfg, X/LAg – b, LEab. Use

v_{2MM2}	a	$\cdots$	e	d	$\cdots$	c	$\cdots$	f	g	$\cdots$	b	$\cdots$	
m_1				$\in_n$	$\cdots$	$\in_n$	$\cdots$	$\in_n$					
m_2	$\in_n$	$\cdots$	$\in_n$						$\notin_n$	$\cdots$	$\notin_n$	$\in_n$	$\cdots$
m_3	$\notin_n$	$\cdots$	$\notin_n$						$\in_n$	$\cdots$	$\in_n$	$\notin_n$	$\cdots$
$\vdots$													

Case 3: α^x has I-form 31. *Subcase 3.1*: XE or ME occurs in α. Then an α^{1M} chain has form 3M1 or 3M2. *Subcase 3.2*: LE occurs in α. If α has form (3MM1) X/LAc – e, MAef, X/LAf – g, MAgh, X/LAh – a, X/LAd – b, X/LIcd (or X/LIdc), LEab (or LEba), use

v_{3MM1}	a	$\cdots$	h	g	$\cdots$	f	e	$\cdots$	c	d	$\cdots$	b	$\cdots$	
m_1	$\notin_n$	$\cdots$	$\notin_n$				$\in_n$	$\cdots$	$\in_n$	$\in_n$	$\cdots$	$\in_n$	$\notin_n$	$\cdots$
m_2	$\in_n$	$\cdots$	$\in_n$				$\notin_n$	$\cdots$	$\notin_n$	$\notin_n$	$\cdots$	$\notin_n$	$\in_n$	$\cdots$
m_3				$\in_n$	$\cdots$	$\in_n$								
$\vdots$														

If α has form (3MM2) X/LAc – e, MAef, X/LAf – a, X/LAd – g, MAgh, LAh – b, X/LIcd, LEab (or LEba), use

v_{3MM2}	a	$\cdots$	f	e	$\cdots$	c	d	$\cdots$	g	h	$\cdots$	b	$\cdots$	
m_1				$\in_n$	$\cdots$	$\in_n$	$\in_n$	$\cdots$	$\in_n$					
m_2	$\in_n$	$\cdots$	$\in_n$							$\notin_n$	$\cdots$	$\notin_n$	$\in_n$	$\cdots$
m_3	$\notin_n$	$\cdots$	$\notin_n$							$\in_n$	$\cdots$	$\in_n$	$\notin_n$	$\cdots$
$\vdots$														

And if α has form (3MM3) X/LAc – e, MAef, X/LAf – a, X/LAd – b, MIcd (or MIdc), LEab (or LEba), use

v_{3MM3}	a	...	f	e	...	c	d	...	b	...	
m_1				$\in_n$	...	$\in_n$					
m_2	$\in_n$	...	$\in_n$				$\notin_n$	...	$\notin_n$	$\in_n$	...
m_3	$\notin_n$	...	$\notin_n$				$\in_n$		$\in_n$	$\notin_n$	...
$\vdots$											

Lemma 2.4: If α is a chain that does not have an I-form, α^x has a Smiley-form, and M occurs more than twice in α, then α is $\in^3$-satisfiable.

Proof. Assume the antecedent. If α is a chain with more than two occurrences of M let α^{2M} be any chain that results from deleting all but two occurrences of M from α. So either an α^M or an α^{2M} chain has one of the forms mentioned in Lemmas 2.2 and 2.3. (For example, if α^x has I-form 11 and LO occurs in α then a chain α^{2M} has form 1MM1.) So α is $\in^3$-satisfiable, since Qab $\vDash$ MQab, as noted in the proof of Lemma 2.3.

4. Deductions

The following account of deduction is motivated by Aristotle's discussion of deductions in *Pr. An.* A25. So, for example, we attempt to accommodate the following claim, using Smith's *1989*, pp. 39–41, translations here and below:

> Counting deductions by their main premises ... every deduction will be from an even number of premises and an odd number of terms [sentences] (for the terms [sentences] are more in number by one than the number of premises). (*Pr. An. 42b1-4*)

I think Aristotle used two types of deducibility relationships—those of one type generate the valid two-premised syllogisms, and those of the other type generate the valid polysyllogisms from the valid two-premised syllogisms. For the latter type of deducibility relationship the even-premised feature, mentioned above, holds, though it does not hold for the former.

Definition 4.1: $\{x, y\} \vdash_3$ iff $\langle\{x, y\}, z\rangle$ is a V_3-syllogism.

Definition 4.2: 1) If $\{y_1, y_2\} \vdash_3 y_3$ then $\langle y_1, y_2, y_3\rangle$ is a *deduction* of y_3 *from* $\{y_1, y_2\}$; 2) If $\langle y_1, \ldots y_n, \ldots x\rangle$ is a deduction of x from Y, if $\{x, w\} \vdash_3 z$, and if some term in w does not occur in a member of $\{y_1, \ldots y_n\}$, then $\langle y_1, \ldots y_n, \ldots x, w, z\rangle$ is a *deduction of* z *from* Y, w; 3) If $\langle y_1, \ldots y_n, \ldots x\rangle$ is a deduction of x from Y, if $\{w, z\} \vdash_3 y_i$, and if a term in w and z occurs in no member of Y then $\langle y_1, \ldots w, z, y_i, \ldots y_n, \ldots x\rangle$ is a *deduction of* x *from* $Y \cup \{w, z\} - y_i$; 4) δ is a *deduction* of x from Y only if δ is a deduction of x from Y in virtue of the conditions 1 to 3.

So, for example, $\langle$Aab, Abc, Aac$\rangle$ is a deduction of Aac from {Aab, Abc} (by 1). So $\langle$Aab, Abc, Aac, Acd, Aad$\rangle$ is a deduction of Aad from {Aab, Abc, Acd} (by 2), since {Aac, Acd} $\vdash_3$ Aad and 'd' does not occur in {Aab, Abc}. So $\langle$Aab, Abe, Aec, Abc, Aac, Acd, Aad$\rangle$ is a deduction of Aad from {Aab, Abe, Aec, Acd} (by 3), since {Abe, Aec} $\vdash_3$ Abc and 'e' does not occur

in {Aab, Abc, Acd}. In contrast, note that $\langle$Aab, Abc, Aa
deduction of Aab from {Aab, Abc}. This is one of the
eschewed by Aristotle. (See *Pr. An.* B5.)

Conditions 2) and 3) are in line with the following comm

> ... the term [sentence] will be put either outside or in
> *An. 42b8*)

Condition 2) builds deductions by putting sentences on t
and Condition 3) builds deductions by putting sentenc
sequences.

The above account of deduction squares with

> ... every demonstration will be through three terms
> more, ... (*Pr. An. 41b36*)

Note that $\{x, y\} \vdash_3 z$ figures in each clause of the recursive

Definition 4.3: If $\langle x_1, \ldots x_n\rangle$ is a deduction of x_n from Y
$\langle x_1, \ldots x_n\rangle$ are x_1 to x_{n-1}.

Theorem 4.1: If δ is a deduction of x from Y and Y has n
δ has $2n - 2$ premises and $2n - 1$ members.

Proof. *Basis step*: $n = 2$. Trivial. *Recursion steps*: If
deduction of y from Y, where δ has $2n - 2$ premises and
where Y has n members, is used together with Conditions
is a deduction of y' from Y' then δ' has $2n - 2 + 2$ (
premises and Y' has $2n - 1 + 2$ (that is, $2(n + 1) - 1$) mem

Aristotle also counts conclusions:

> The conclusions will be half as many as the number o
> *42b1-4*).

Definition 4.4: If $\langle x_1, \ldots x_n\rangle$ is a deduction of x_n from Y
of $\langle x_1, \ldots x_n\rangle$ are the members of $\langle x_1, \ldots x_n\rangle$ that are not

Corollary 1 of Theorem 4.1: If δ is a deduction of y f
members ($n \geqslant 2$) then δ has $n - 1$ conclusions (half of 2
premises.).

Corollary 2 of Theorem 4.1: If δ is a deduction of x from
even number of members, then the number of membe
members of Y is odd, and (ii) if the number of membe
members of Y is even, then Y has an odd number of memb

At the end of A25 Aristotle says that there are many r
premises, which is in apparent conflict with preceding p
Aristotle was considering alternative deductions and the va
these alternative deductions, taken *collectively*. So, for
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in {Aab, Abc, Acd}. In contrast, note that ⟨Aab, Abc, Aac, Acb, Aab⟩ is not a deduction of Aab from {Aab, Abc}. This is one of the "circular deductions" eschewed by Aristotle. (See *Pr. An.* B5.)

Conditions 2) and 3) are in line with the following comment:

> . . . the term [sentence] will be put either outside or in the middle . . . . (*Pr. An. 42b8*)

Condition 2) builds deductions by putting sentences on the right of sequences and Condition 3) builds deductions by putting sentences in the middle of sequences.

The above account of deduction squares with

> . . . every demonstration will be through three terms [sentences], and no more, ... (*Pr. An. 41b36*)

Note that $\{x, y\} \vdash_3 z$ figures in each clause of the recursive definition.

Definition 4.3: If $\langle x_1, \ldots x_n \rangle$ is a deduction of x_n from Y then the *premises* of $\langle x_1, \ldots x_n \rangle$ are x_1 to x_{n-1}.

Theorem 4.1: If δ is a deduction of x from Y and Y has n members ($n \geq 2$) then δ has $2n - 2$ premises and $2n - 1$ members.

Proof. *Basis step*: $n = 2$. Trivial. *Recursion steps*: If the fact that δ is a deduction of y from Y, where δ has $2n - 2$ premises and $2n - 1$ members, and where Y has n members, is used together with Conditions 2 or 3 to show that δ' is a deduction of y′ from Y′ then δ' has $2n - 2 + 2$ (that is, $2(n + 1) - 2$) premises and Y′ has $2n - 1 + 2$ (that is, $2(n + 1) - 1$) members.

Aristotle also counts conclusions:

> The conclusions will be half as many as the number of premises. (*Pr. An. 42b1-4*).

Definition 4.4: If $\langle x_1, \ldots x_n \rangle$ is a deduction of x_n from Y then the *conclusions* of $\langle x_1, \ldots x_n \rangle$ are the members of $\langle x_1, \ldots x_n \rangle$ that are not members of Y.

Corollary 1 of Theorem 4.1: If δ is a deduction of y from Y and Y has n members ($n \geq 2$) then δ has $n - 1$ conclusions (half of $2n - 2$, the number of premises.).

Corollary 2 of Theorem 4.1: If δ is a deduction of x from Y then: (i) if Y has an even number of members, then the number of members of δ that are not members of Y is odd, and (ii) if the number of members of δ that are not members of Y is even, then Y has an odd number of members.

At the end of A25 Aristotle says that there are many more conclusions than premises, which is in apparent conflict with preceding passages. I think that Aristotle was considering alternative deductions and the variety of conclusions in these alternative deductions, taken *collectively*. So, for example, we list 10 deductions, with 12 "conclusions added", in virtue of which {Iba, Abc, Acd, Ede} ⊢ Oae, where conclusions are put in square brackets: ⟨Iba, Abc, [Iac or Ica], Acd, [Iad or Ida], Ede, [Oae]⟩; ⟨Abc, Acd, [Abd], Ede, [Ebe or Eeb], Iab, [Oae]⟩; and ⟨Acd, Ede, [Ece or Eec], Abc, [Ebe or Eeb], Iba, [Oae]⟩. Note that in each of these 10 deductions 'conclusions added will be one fewer in number than the terms [sentences] which were already present' (*Pr. An. 42b18*). Note, for example, that for the first of the 10 deductions mentioned above, the four sentences present are Iba, Abc, Acd, and Ede, and the three conclusions added are Iac, Iad, and Oae.

According to the interpretations by Ross *1949*, p. 381, and Smith *1989*, p. 147, of the last paragraph of A25, the number of conclusions should be 6 instead of 12 for the example cited. I think that Ross and Smith are mistakenly treating Aristotle's reference to conclusions as a reference to term pairs (intervals) that may appear in various deductions. (For the example cited there are 6 intervals—ac, ad, ae, bc, be, ce—represented in the 12 conclusions added.). Certainly our interpretation fits more comfortably with Aristotle's claim that conclusions will be *much greater in number* than the premises.

But the last paragraph of A25 suggests that, for Aristotle, if there is a deduction of y from Y then there is a deduction in which each term interval not represented in a premise occurs in a conclusion. This is false, given the above account of deduction. For example, ⟨LAca, Abc, [LAba], LObd, LOad⟩ is a deduction of LOad from {LAca, Abc, LObd}. But there is no deduction in which a conclusion occurs which represents a cd interval. So, for example, the following attempt at such a deduction is blocked: ⟨Abc, LObd, [Ocd], LAca, xxx⟩. Though Oad may be entered after LAca, LOad may not be, since Bocardo-XLL is not a V_3-syllogism. (For Aristotle, Bocardo-XLL is clearly invalid.)

Definition 4.5: x is *deducible from* Y (Y ⊢ x) iff there is a deduction of x from Y.

Decision procedure 4.1 (corollary of Theorem 4.1): A decision procedure for determining whether Y ⊢ x is obtained by considering the (2n − 1)-membered sequences that contain no terms other than those in Y ∪ {x}, where n is the number of terms in Y.

Theorem 4.3 (Completeness): If ⟨Y, x⟩ is a V_n-syllogism, for n ≥ 3, then Y ⊢ x.

Proof. The following two lemmas will be used:

Lemma 1: If Y, X/LAab ⊢ x then Y, X/LAa − c, X/LAcb ⊢ x.

Lemma 2: If Y, LAab ⊢ x then Y, X/LAa − c, LAcb ⊢ x.

Prove these lemmas by using induction on the number n (n ≥ 0) of members of X/LAa − c and by using the relevant Barbaras.

To prove the theorem we use the I-forms to show how a deduction can be constructed for each V_n-syllogism generated by that I-form. We categorize I-forms by categorizing x* in a set of sentences with an I-form. x* may or may not be "indicated by an expression X/LAx − y or LAx − y." So, for example, given the set of sentences $\{Aa_1a_2, Aa_2a_3, Oa_1a_3\}$ with I-form 11, if x* is Aa_1a_2

or Aa_2a_3 then x* is indicated by X/LAa − b, but if x* is Oa_1a_3 the
indicated by an expression X/LAx − y or LAx − y.

If x* is indicated by X/LAx − y or LAx − y, we shall say th
category X/L or category L, respectively, otherwise, x* is in categor
'basic'). So, given the set of sentences $\{LAa_1a_2, LAa_2a_3, MOa_1a_3\}$
13, if x* is LAa_1a_2 then x* is in category X/L, and if x* is LAa_2a_3
then x* is in category B.

If x* is in category L, indicated by LAx − y, then x* is either in
or category L_l ('r' for 'right,' and 'l' for 'left') depending upon wh
obtained by substituting a term for y. So, for example, given the
LAa_2a_3, MAa_3a_4, LAa_4a_5, LAa_5a_6, LOa_1a_6} with I-form 12, if x*
LAa_4a_5 then x* is in category L_l, and if x* is LAa_2a_3 or LAa_5a_6 t
category L_r.

When considering the I-forms we assume that, for each expressio
and LAx − y used to express I-forms, x ≠ y. Arguments for the sim
in which x = y will be omitted.

When constructing the required deductions it is useful to recogni
Y ⊢ x and {x, y} ⊢ z then Y, y ⊢ z, provided a term in y does not
member of Y; and ii) if Y, x ⊢ y and {w, z} ⊢ x, then Y, w, z ⊢ y,
term in w and z does not occur in Y.

In the discussion of the following I-forms the category of x* is m
each case.

I-form 11. *Case 1*: x* has form X/LOab (B). X/LAac, X/LAcb
(Barbara). So X/LAa − b ⊢ X/MAab (Lemma 1). *Case 2*: x* has fo
(X/L, part of X/LAa − b). Since X/LAdb, X/LOab ⊢ Oad (Baroco
X/LAac ⊢ X/MOcd (Bocardo), it follows that X/LAac, X/LAdb,
X/MOcd. So X/LAa − c, X/LAd − b, X/LOab ⊢ X/MOcd (Lemma 1

I-form 12. *Case 1*: x* has form MAcd (B). LAac, LAdb, LO
(Baroco, Bocardo). So LAa − c, LAd − b, LOab ⊢ LOcd
Case 2: x* has form LOab (B). LAac, MAcd, LAdb ⊢ MAab (B
LAa − c, MAcd, LAd − b ⊢ MAab (Lemma 2). *Case 3.1*: x* has
(L_r, part of LAa − c). X/LAae, MAcd, LAdb, LOab ⊢ MOec. So
MAcd, LAd − b, LOab ⊢ MOec (Lemmas 1 and 2). *Case 3.2*: x* has
(L_r, part of LAd − b). LAac, MAcd, X/LAdf, LOab ⊢ MOfb. S
MAcd, X/LAd − f, LOab ⊢ MOfb (Lemmas 1 and 2). *Case 4.1*: x
X/LAef (L_l, part of LAa − c). X/LAae, LAfc, MAcd, LAdb, LOab
So X/LAa − e, LAf − c, MAcd, LAd − b, LOab ⊢ X/MOef (Lemm
Case 4.2: x* has form X/LAgh (L_l, part of LAd − b). LAac, MAc
LAhb, LOab ⊢ X/MOgh. So LAa − c, MAcd, X/LAd − g, LAh −
X/MOgh (Lemmas 1 and 2).

To complete the arguments for the cases listed for the following
a step that uses Lemma 1. So, for example, for case 1 of I-form
step is X/LAa − c, MOab ⊢ MOcb.

I-form 13. *Case 1*: x* has form LAcb (B). X/LAac, MOab ⊢ MOcb
has form MOab (B). X/LAac, LAcb ⊢ LAab. *Case 3*: x* has fo
(X/L, part of X/LAa − c). X/LAad, X/LAec, LAcb, MOab ⊢ X/MO

where conclusions are put in square brackets: ⟨Iba, Abc, [Iac or d or Ida], Ede, [Oae]⟩; ⟨Abc, Acd, [Abd], Ede, [Ebe or Eeb], nd ⟨Acd, Ede, [Ece or Eec], Abc, [Ebe or Eeb], Iba, [Oae]⟩. Note of these 10 deductions 'conclusions added will be one fewer in the terms [sentences] which were already present' (*Pr. An. 42b18*). mple, that for the first of the 10 deductions mentioned above, the s present are Iba, Abc, Acd, and Ede, and the three conclusions , Iad, and Oae.

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x is *deducible from* Y ($Y \vdash x$) iff there is a deduction of x

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Completeness): If ⟨Y, x⟩ is a V_n-syllogism, for $n \geq 3$, then $Y \vdash x$.

llowing two lemmas will be used:

, $X/LAab \vdash x$ then Y, $X/LAa - c$, $X/LAcb \vdash x$.

, $LAab \vdash x$ then Y, $X/LAa - c$, $LAcb \vdash x$.

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or Aa_2a_3 then x* is indicated by $X/LAa - b$, but if x* is Oa_1a_3 then x* is not indicated by an expression $X/LAx - y$ or $LAx - y$.

If x* is indicated by $X/LAx - y$ or $LAx - y$, we shall say that x* is in category X/L or category L, respectively, otherwise, x* is in category B ('B' for 'basic'). So, given the set of sentences $\{LAa_1a_2, LAa_2a_3, MOa_1a_3\}$ with I-form 13, if x* is LAa_1a_2 then x* is in category X/L, and if x* is LAa_2a_3 or MOa_1a_3 then x* is in category B.

If x* is in category L, indicated by $LAx - y$, then x* is either in category L_r or category L_l ('r' for 'right,' and 'l' for 'left') depending upon whether x* is obtained by substituting a term for y. So, for example, given the set $\{Aa_1a_2, LAa_2a_3, MAa_3a_4, LAa_4a_5, LAa_5a_6, LOa_1a_6\}$ with I-form 12, if x* is Aa_1a_2 or LAa_4a_5 then x* is in category L_l, and if x* is LAa_2a_3 or LAa_5a_6 then x* is in category L_r.

When considering the I-forms we assume that, for each expression $L/Ax - y$ and $LAx - y$ used to express I-forms, $x \neq y$. Arguments for the simpler I-forms in which $x = y$ will be omitted.

When constructing the required deductions it is useful to recognize that: i) if $Y \vdash x$ and $\{x, y\} \vdash z$ then Y, $y \vdash z$, provided a term in y does not occur in a member of Y; and ii) if Y, $x \vdash y$ and $\{w, z\} \vdash x$, then Y, w, $z \vdash y$, provided a term in w and z does not occur in Y.

In the discussion of the following I-forms the category of x* is mentioned for each case.

I-form 11. *Case 1*: x* has form X/LOab (B). X/LAac, $X/LAcb \vdash X/MAab$ (Barbara). So $X/LAa - b \vdash X/MAab$ (Lemma 1). *Case 2*: x* has form X/LAcd (X/L, part of $X/LAa - b$). Since X/LAdb, $X/LOab \vdash Oad$ (Baroco) and Oad, $X/LAac \vdash X/MOcd$ (Bocardo), it follows that X/LAac, X/LAdb, $X/LOab \vdash X/MOcd$. So $X/LAa - c$, $X/LAd - b$, $X/LOab \vdash X/MOcd$ (Lemma 1).

I-form 12. *Case 1*: x* has form MAcd (B). LAac, LAdb, $LOab \vdash LOcd$ (Baroco, Bocardo). So $LAa - c$, $LAd - b$, $LOab \vdash LOcd$ (Lemma 2). *Case 2*: x* has form LOab (B). LAac, MAcd, $LAdb \vdash MAab$ (Barbara). So $LAa - c$, MAcd, $LAd - b \vdash MAab$ (Lemma 2). *Case 3.1*: x* has form LAec (L_r, part of $LAa - c$). X/LAae, MAcd, LAdb, $LOab \vdash MOec$. So $X/LAa - e$, MAcd, $LAd - b$, $LOab \vdash MOec$ (Lemmas 1 and 2). *Case 3.2*: x* has form LAfb (L_r, part of $LAd - b$). LAac, MAcd, X/LAdf, $LOab \vdash MOfb$. So $LAa - c$, MAcd, $X/LAd - f$, $LOab \vdash MOfb$ (Lemmas 1 and 2). *Case 4.1*: x* has form X/LAef (L_l, part of $LAa - c$). X/LAae, LAfc, MAcd, LAdb, $LOab \vdash X/MOef$. So $X/LAa - e$, $LAf - c$, MAcd, $LAd - b$, $LOab \vdash X/MOef$ (Lemmas 1 and 2). *Case 4.2*: x* has form X/LAgh (L_l, part of $LAd - b$). LAac, MAcd, X/LAdg, LAhb, $LOab \vdash X/MOgh$. So $LAa - c$, MAcd, $X/LAd - g$, $LAh - b$, $LOab \vdash X/MOgh$ (Lemmas 1 and 2).

To complete the arguments for the cases listed for the following I-forms, add a step that uses Lemma 1. So, for example, for case 1 of I-form 13, the final step is $X/LAa - c$, $MOab \vdash MOcb$.

I-form 13. *Case 1*: x* has form LAcb (B). X/LAac, $MOab \vdash MOcb$. *Case 2*: x* has form MOab (B). X/LAac, $LAcb \vdash LAab$. *Case 3*: x* has form X/LAde (X/L, part of $X/LAa - c$). X/LAad, X/LAec, LAcb, $MOab \vdash X/MOde$.

I-form 21. *Case 1*: x* has form X/LEab (B). X/LAca, X/LAcb ⊢ X/MIab. *Case 2.1*: x* has form X/LAde (X/L, part of X/LAc – a). X/LAcd, X/LAea, X/LAcb, X/LEab ⊢ X/MOde. *Case 2.2*: x* has form X/LAfg (X/L, part of X/LAc – b). X/LAca, X/LAcf, X/LAgb, X/LEab ⊢ X/MOfg.

I-form 22. *Case 1*: x* has form MAde (B). X/LAca, X/LAcd, X/LAeb, LEab (or LEba) ⊢ LOde. *Case 2*: x* has form LEab (or LEba) (B). X/LAca, X/LAcd, MAde, X/LAeb ⊢ MIab (and MIba). *Case 3.1*: x* has form X/LAfg (X/L, part of X/LAc – a). X/LAcf, X/LAga, X/LAcd, MAde, C/LAeb, LEab (or LEba) ⊢ X/MOfg. *Case 3.2*: x* has form X/LAhi (X/L, part of X/LAc – d). X/LAca, X/LAch, X/LAid, MAde, X/LAeb, LEab (or LEba) ⊢ X/MOhi. *Case 3.3*: x* has form X/LAjk (X/L, part of X/LAe – b). X/LAca, X/LAcd, MAde, X/LAej, X/LAkb, LEab (or LEba) ⊢ X/MOjk.

I-form 23. *Case 1*: x* has form LAdb (B). X/LAca, X/LAcd, MEab (or MEba) ⊢ MOdb. *Case 2*: x* has form MEab (or MEba) (B). X/LAca, X/LAcd, LAdb ⊢ LIab (and LIba). *Case 3.1*: x* has form X/LAef (X/L, part of X/LAc – a). X/LAce, X/LAfa, X/LAcd, LAdb, MEab (or MEba) ⊢ X/MOef. *Case 3.2*: x* has form X/LAgh (X/L, part of X/LAc – d). X/LAca, X/LAcg, X/LAhd, LAdb, MEab (or MEba) ⊢ X/MOgh.

I-form 31. *Case 1*: x* has form X/LIcd (B). X/LAca, X/LAdb, X/LEab (or X/LEba) ⊢ X/MEcd. *Case 2*: x* has form X/LEab (or X/LEba) (B). X/LAca, X/LAdb, X/LIcd ⊢ X/MIab (and X/MIba). *Case 3.1*: x* has form X/LAef (X/L, part of X/LAc – a). X/LAce, X/LAfa, X/LAdb, X/LIcd, X/LEab (or X/LEba) ⊢ X/MOef. *Case 3.2*: x* has form X/LAgh (X/L, part of X/LAd – b). X/LAca, X/LAdg, X/LAhb, X/LIcd, X/LEab (or X/LEba) ⊢ X/MOgh.

I-form 32. *Case 1*: x* has form MAef (B). X/LAca, X/LAde, X/LAfb, X/LIcd (or X/LIdc), LEab (or LEba) ⊢ LOef. *Case 2*: x* has form X/LIcd (or X/LIdc) (B). X/LAca, X/LAde, MAef, X/LAfb, LEab (or LEba) ⊢ X/MEcd (and X/MEdc). *Case 3*: x* has form LEab (or LEba) (B). X/LAca, X/LAde, MAef, X/LAfb, X/LIcd (or X/LIdc) ⊢ MIab (and MIba). *Case 4.1*: x* has form XLAgh (X/L, part of X/LAc – a). X/LAcg, X/LAha, X/LAde, MAef, X/LAfb, X/LIcd (or X/LIdc), LEab (or LEba) ⊢ X/MOgh. *Case 4.2*: x* has form X/LAij (X/L, part of X/LAd – e). X/LAca, X/LAdi, X/LAje, MAef, X/LAfb, X/LIcd (or X/LIdc), LEab (or LEba) ⊢ X/MOij. *Case 4.3*: x* has form X/LAkl (X/L, part of X/LAf – b). X/LAca, X/LAde, MAef, X/LAfk, X/LAlb, X/LIcd (or X/LIdc), LEab (or LEba) ⊢ MOkl.

I-form 33. *Case 1*: x* has form MIcd (B). X/LAca, X/LAdb, LEab (or LEba) ⊢ LEcd. *Case 2*: x* has form LEab (or LEba) (B). X/LAca, X/LAdb, MIcd ⊢ MIab (and MIba). *Case 3.1*: x* has form X/LAef (X/L, part of X/LAc – a). X/LAce, X/LAfa, X/LAdb, MIcd, LEab (or LEba) ⊢ X/MOef. *Case 3.2*: x* has form X/LAgh (X/L, part of X/LAd – b). X/LAca, X/LAdg, X/LAhb, MIcd, LEab (or LEba) ⊢ X/MOgh.

I-form 34. Nothing to consider, since our discussion is limited to V-syllogisms with at least two premises.

I-form 35. *Case 1*: x* has form LAeb (B). X/LAca, X/LA
X/LIdc), MEab (or MEba) ⊢ MOeb. *Case 2*: x* has form X/L
(B). X/LAca, X/LAde, LAeb, MEab (or MEba) ⊢ X/MEcd
Case 3: x* has form MEab (or MEba) (B). X/LAca, X/LAde
(or X/LIdc) ⊢ LIab (and LIba). *Case 4.1*: x* has form X/LAf
X/LAc – a). X/LAcf, X/LAga, X/LAde, LAeb, X/LIcd (or X/I
MEba) ⊢ X/MOfg. *Case 4.2*: x* has form X/LAhi (X/L, part
X/LAca, X/LAdh, X/LAie, LAeb, X/LIcd (or X/LIdc), MEa
X/MOhi.

Theorem 4.4 (Soundness): If Y ⊢ x then Y ⊨ z.

Proof. Basis step: Note that each of the I_3-forms is inconsis
steps: If Y ⊨ x and {x, y} ⊨ z then Y, y ⊨ z. And if Y, x ⊨ y and {
w, z ⊨ y.

Theorem 4.5: If ⟨Y, x⟩ is a syllogism and Y has two or more
Y ⊢ x iff Y ⊨ x.

Proof. Immediate consequence of Theorems 3.1, 4.3 and 4.4.

Decision procedure 4.2 (corollary of Theorem 4.5): If ⟨Y, x⟩ is
Y has two or more members, a decision procedure for deter
X ⊨ y is an immediate consequence of decision procedure 4.1 and

5. Decision procedures for V_n-syllogisms (n ≥ 1)

There are exactly 4 V_1-syllogisms: ⟨∅, Aaa⟩, ⟨∅, Iaa⟩, ⟨
⟨∅, MIaa⟩. (These syllogisms are deducible in McCall's L–X–M
is ⟨∅, LIaa⟩. It should be noticed, that McCall *1963*, p. 50, ass
axiom "for convenience", not because he has any reasons f
Aristotle regarded it as valid.)

Theorem 5.1: If ⟨Y, x⟩ is a V_1-syllogism then Y ⊨ x.

Proof. Straightforward.

There are exactly 60 V_2-syllogisms. We list the number
generated by each I-form, where no V_2-syllogism generated by o
generated by another: 11: (8), 12: (2), 13: (2), 21: (16), 22:
(16), 32: (0), 33: (4), 34: (4), 35: (0). So, for example, I-for
⟨{LAab}, Aab⟩ and ⟨{Oab}, MOab⟩. (All V_2-syllogisms ar
McCall's L–X–M.)

Theorem 5.2: If ⟨Y, x⟩ is a V_2-syllogism then Y ⊨ x.

Proof: Straightforward. Note that each I_2-form is inconsistent.

Theorem 5.3: If ⟨Y, x⟩ is a syllogism, then Y ⊨ x iff ⟨Y, x⟩ is a V

I-form 35. *Case 1*: x* has form LAeb (B). X/LAca, X/LAde, X/LIcd (or X/LIdc), MEab (or MEba) ⊢ MOeb. *Case 2*: x* has form X/LIcd (or X/LIdc) (B). X/LAca, X/LAde, LAeb, MEab (or MEba) ⊢ X/MEcd (and X/MEdc). *Case 3*: x* has form MEab (or MEba) (B). X/LAca, X/LAde, LAeb, X/LIcd (or X/LIdc) ⊢ LIab (and LIba). *Case 4.1*: x* has form X/LAfg (X/L, part of X/LAc – a). X/LAcf, X/LAga, X/LAde, LAeb, X/LIcd (or X/LIdc), MEab (or MEba) ⊢ X/MOfg. *Case 4.2*: x* has form X/LAhi (X/L, part of X/LAd – e). X/LAca, X/LAdh, X/LAie, LAeb, X/LIcd (or X/LIdc), MEab (or MEba) ⊢ X/MOhi.

Theorem 4.4 (Soundness): If Y ⊢ x then Y ⊨ z.

Proof. Basis step: Note that each of the I_3-forms is inconsistent. *Recursion steps*: If Y ⊨ x and {x, y} ⊨ z then Y, y ⊨ z. And if Y, x ⊨ y and {w, z} ⊨ x then Y, w, z ⊨ y.

Theorem 4.5: If ⟨Y, x⟩ is a syllogism and Y has two or more members then Y ⊢ x iff Y ⊨ x.

Proof. Immediate consequence of Theorems 3.1, 4.3 and 4.4.

Decision procedure 4.2 (corollary of Theorem 4.5): If ⟨Y, x⟩ is a syllogism and Y has two or more members, a decision procedure for determining whether X ⊨ y is an immediate consequence of decision procedure 4.1 and Theorem 4.5.

5. Decision procedures for V_n-syllogisms ($n \geq 1$)

There are exactly 4 V_1-syllogisms: ⟨∅, Aaa⟩, ⟨∅, Iaa⟩, ⟨∅, MAaa⟩, and ⟨∅, MIaa⟩. (These syllogisms are deducible in McCall's L–X–M calculus, but so is ⟨∅, LIaa⟩. It should be noticed, that McCall *1963*, p. 50, assumes LIaa as an axiom "for convenience", not because he has any reasons for thinking that Aristotle regarded it as valid.)

Theorem 5.1: If ⟨Y, x⟩ is a V_1-syllogism then Y ⊨ x.

Proof. Straightforward.

There are exactly 60 V_2-syllogisms. We list the number of V_2-syllogisms generated by each I-form, where no V_2-syllogism generated by one I-form is also generated by another: 11: (8), 12: (2), 13: (2), 21: (16), 22: (4), 23: (4), 31: (16), 32: (0), 33: (4), 34: (4), 35: (0). So, for example, I-form 11 generates ⟨{LAab}, Aab⟩ and ⟨{Oab}, MOab⟩. (All V_2-syllogisms are deducible in McCall's L–X–M.)

Theorem 5.2: If ⟨Y, x⟩ is a V_2-syllogism then Y ⊨ x.

Proof: Straightforward. Note that each I_2-form is inconsistent.

Theorem 5.3: If ⟨Y, x⟩ is a syllogism, then Y ⊨ x iff ⟨Y, x⟩ is a V-syllogism.

Proof. Immediate consequence of Theorems 3.1, 4.5, 5.1, and 5.2.

Decision procedure 5.1 (corollary of Theorem 5.3): If $\langle Y, x \rangle$ is a syllogism, to decide whether $Y \models x$, ask whether $\langle Y, x \rangle$ is a V-syllogism.

Theorem 5.4: If $\langle Y, x \rangle$ is a syllogism, then $Y \models x$ iff there is no $\in_3$-valuation that assigns t to all members of Y and assigns f to x.

Proof. Immediate consequence of the proof of Theorem 3.1.

Decision procedure 5.2 (corollary of Theorem 5.4): If $\langle Y, x \rangle$ is a syllogism, to decide whether $Y \models x$ examine all of the $4^{(3 \times n)}$ matrices of form

	a_1	a_2	$\cdots$	a_n	a_{n+1}	$\cdots$
m_1					$\in$	$\cdots$
m_2						
m_3						

where a_1 to a_n are all and the only terms that occur in the sentences in $Y \cup \{x\}$, and each m_i/a_j cell (for $1 \leqslant i \leqslant 3$ and $1 \leqslant j \leqslant n$) is either empty or contains one of the following three expressions: '$\in$', '$\in_n$', or '$\notin_n$'. Ask whether any of these matrices indicates an $\in$-valuation that assigns t to all members of Y and assigns f to x.

6. Final remark

The simplest of the above three decision procedures for determining the validity of syllogisms is given by decision procedure 5.1. This decision procedure generalizes Smiley's *1973* simple decision procedure for determining the validity of assertoric syllogisms.

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References

Johnson, F. *1989* 'Models for modal syllogisms', *Notre Dame journal of formal logic*, **30**, 271–284.

——— *1991* 'Three-membered domains for Aristotle's syllogistic', *Studia logica*, **50**, 181–187.

——— *1993* 'Modal ecthesis', *History and philosophy of logic*, **14**, 171–182.

McCall, S. *1963 Aristotle's modal syllogisms*, Amsterdam (North-Holland).

Ross, W. D. *1949 Aristotle's prior and posterior analytics*, Oxford (Clarendon Press).

Smiley, T. J. *1973* 'What is a syllogism?', *Journal of philosophical logic*, **2**, 136–154.

Smith, R. *1989 Aristotle: prior analytics*, Indianapolis (Hackett).

Thom, P. *1991* 'The two Barbaras', *History and philosophy of logic*, **12**, 135–149.

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C. I. Lewis's Calculus of Predicat

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In 1951 C. I. Lewis published a logic of general terms (or properties) that he ca *predicates*. Although this system is of less significance than Lewis's earlier wor modal logic, it has considerable historical interest and does not deserve the almost received. My aim here is to situate this system in the context of Lewis's earlier w several of its central features. After sketching the historical background, I pre Lewis's system, discuss his reasons for preferring it to quantified modal log semantics for it that is suggested by Lewis's informal discussion of his system general views on meaning. I then discuss Lewis's sketchy extension of his quantifiers and examine his claim that it can serve as a foundation for logic in ger noting two minor changes in *CP* that, from today's vantage point, would count as

In a series of works, from the year of his first published pape appearance of *Symbolic logic* in 1932, C. I. Lewis developed sev logics that laid the groundwork for modern propositional modal well-known paper, 'Notes on the Logic of Intension', published for his colleague Henry Sheffer in 1951, Lewis presented a logic functions that he called the *calculus of predicates*. He believed t could provide a foundation for logic in general; in particular, afforded a way of combining modality and propositional func superior to quantified modal logic. History has not vindicated Le about the importance of his system, and its intrinsic interest c great as that of his earlier work. Still, Lewis's calculus of predi beyond the attempts of earlier thinkers to devise an intensional terms; it shows how the architect of modern propositional mod have developed an alternative to quantified modal logic, an several recent logics of properties. Hence, Lewis's calculus of sufficient historical interest to deserve better than the almost tota received. My aim here is to situate this system in the context of work on propositional modal logic and to examine several of its c

In §1 I sketch the historical background of Lewis's calculu (*CP*, for short) and give an informal account of its central n present the syntax of *CP* and in §3 consider why Lewis preferre quantified modal logic. Although the purely syntactical features o great interest, since they so closely parallel those of one of Lewis modal logics, S2, the sorts of interpretations he envisioned fo interesting and fundamental ways from those he had in mind f tional modal logics. Accordingly, in §4 I supply *CP* with a se suggested by Lewis's informal discussion of his system (together v views on meaning). In §5 I discuss Lewis's sketchy extension of quantifiers and in §6 I examine his claim that *CP* can serve as a logic in general. In the final section I note two minor changes in today's vantage point, would count as improvements.