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COPI'S METHOD OF DEDUCTION

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In [1] Bradley pointed out that it was superfluous for Copi to refer to the completeness and analyticity of **RS** to show that the method of deduction set forth in Chapter 3 of *Symbolic Logic*, 3rd ed. [3], is complete. Since in the 4th edition [4] Copi continues to make his proof of completeness depend upon the completeness and analyticity of **RS**, it seems worthwhile to give a proof which clearly stands on its own. To do this, it is necessary to formalize Copi's method of deduction. We will call the formalization **CMD**.* We will let the capital letters, with or without subscripts, from the earlier part of the alphabet be the simple well-formed formulas in **CMD** and the capital letters, with or without subscripts, from the later part of the alphabet be variables in our meta-language which range over the well-formed formulas of **CMD**. The well-formed formulas of **CMD** are defined inductively in the classical way. Well-formed arguments have the form $x \rightarrow Q$, where x is the empty symbol or a well-formed formula of **CMD**. The intended reading of '$P \rightarrow Q$' is Q follows from P; the intended reading of '$x \rightarrow Q$', where x is the empty symbol, is Q follows from the empty premise, or Q follows from any premise. It will become evident that all of the theorems (and axioms) of **CMD** are well-formed arguments. The axiom schema for **CMD** areas follows, where '$\vdash P \leftrightarrow Q$' abbreviates '$\vdash P \rightarrow Q$ and $\vdash Q \rightarrow P$':

Ax1.	$\vdash (P \supset Q) \cdot P \rightarrow Q$	(M.P.)
Ax2.	$\vdash (P \supset Q) \cdot \sim Q \rightarrow \sim P$	(M.T.)
Ax3.	$\vdash (P \supset Q) \cdot (Q \supset R) \rightarrow P \supset R$	(H.S.)
Ax4.	$\vdash (P \vee Q) \cdot \sim P \rightarrow Q$	(D.S.)
Ax5.	$\vdash ((P \supset Q) \cdot (R \supset S)) \cdot (P \vee R) \rightarrow Q \vee S$	(C.D.)
Ax6.	$\vdash ((P \supset Q) \cdot (R \supset S)) \cdot (\sim Q \vee \sim S) \rightarrow \sim P \vee \sim R$	(D.D.)
Ax7.	$\vdash P \cdot Q \rightarrow P$	(Simp.)
Ax8.	$\vdash P \cdot Q \rightarrow P \cdot Q$	(Conj.)
Ax9.	$\vdash P \rightarrow P \vee Q$	(Add.)

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Ax10.	$\vdash \sim(P \cdot Q) \leftrightarrow \sim P \vee \sim Q$	(DeM.)
	$\vdash \sim(P \vee Q) \leftrightarrow \sim P \cdot \sim Q$	
Ax11.	$\vdash P \vee Q \leftrightarrow Q \vee P$	(Com.)
	$\vdash P \cdot Q \leftrightarrow Q \cdot P$	
Ax12.	$\vdash P \vee (Q \vee R) \leftrightarrow (P \vee Q) \vee R$	(Assoc.)
	$\vdash P \cdot (Q \cdot R) \leftrightarrow (P \cdot Q) \cdot R$	
Ax13.	$\vdash P \cdot (Q \vee R) \leftrightarrow (P \cdot Q) \vee (P \cdot R)$	(Dist.)
	$\vdash P \vee (Q \cdot R) \leftrightarrow (P \vee Q) \cdot (P \vee R)$	
Ax14.	$\vdash P \leftrightarrow \sim\sim P$	(D.N.)
Ax15.	$\vdash P \supset Q \leftrightarrow \sim Q \supset \sim P$	(Trans.)
Ax16.	$\vdash P \supset Q \leftrightarrow \sim P \vee Q$	(Impl.)
Ax17.	$\vdash P \equiv Q \leftrightarrow (P \supset Q) \cdot (Q \supset P)$	(Equiv.)
	$\vdash P \equiv Q \leftrightarrow (P \cdot Q) \vee (\sim P \cdot \sim Q)$	
Ax18.	$\vdash (P \cdot Q) \supset R \leftrightarrow P \supset (Q \supset R)$	(Exp.)
Ax19.	$\vdash P \leftrightarrow P \vee P$	(Taut.)
	$\vdash P \leftrightarrow P \cdot P$	

The rules of inference for **CMD** are:

R1. (The Rule of Replacement) If $f(Q)$ is formed by replacing one occurrence of P in $f(P)$ by Q, then

(i) if $\vdash P \leftrightarrow Q$ and $\vdash f(P) \rightarrow R$ then $\vdash f(Q) \rightarrow R$,
(ii) if $\vdash P \leftrightarrow Q$ and $\vdash R \rightarrow f(P)$ then $\vdash R \rightarrow f(Q)$,

and

(iii) if $\vdash P \leftrightarrow Q$ and $\vdash \rightarrow f(P)$ then $\vdash \rightarrow f(Q)$.

R2. (Transitivity of $\rightarrow$) (i) If $\vdash P_1 \ldots P_m \rightarrow Q_1$ and $\vdash R \rightarrow S$, where R is any permutation of $Q_1 \ldots Q_n$, then $\vdash T \rightarrow S$, where T is any permutation of $P_1 \ldots P_m Q_2 \ldots Q_n$.
(ii) If $\vdash \rightarrow Q_1$ and $\vdash R \rightarrow S$, where R is any permutation of $Q_1 \ldots Q_n$, then $\vdash T \rightarrow S$, where T is any permutation of $Q_2 \ldots Q_n$.

R3. (Conditional Proof) If $\vdash P_1 \ldots P_m \rightarrow Q$ then $\vdash R \rightarrow P_i \supset Q$, where R is any permutation of $P_1 \ldots P_{i-1} \cdot P_{i+1} \ldots P_m$.

R4. (Indirect Proof) If $\vdash P_1 \ldots P_m \rightarrow Q \cdot \sim Q$ then $\vdash R \rightarrow S$, where R is any permutation of $P_1 \ldots P_{i-1} P_{i+1} \ldots P_m$ and $\vdash S \leftrightarrow \sim P_i$.

R5. (Adjunction) (i) If $\vdash P \rightarrow Q$ and $\vdash P \rightarrow R$ then $\vdash P \rightarrow Q \cdot R$.
(ii) If $\vdash \rightarrow Q$ and $\vdash \rightarrow R$ then $\vdash \rightarrow Q \cdot R$.

R6. (Introduction of Superfluous Premises) (i) If $\vdash P_1 \ldots P_m \rightarrow Q$ then $\vdash R \rightarrow Q$, where R is any permutation of $P_1 \ldots P_{m+n}$.
(ii) If $\vdash \rightarrow Q$ then $\vdash P_1 \ldots P_m \rightarrow Q$.

For evidence that **CMD** is actually a formalization of Copi's method of deduction we will indicate how proofs in **CMD** can be constructed f[illegible] Copi's proofs in Chapter 3. Consider the proof on p. 61 in [4[illegible] argument with premise $(A \vee B) \supset ((C \vee D) \supset E)$ and conclusion $A \supset$ [illegible]). Copi's proof:

1.	$(A \lor B) \supset ((C \lor D) \supset E)$	
→2.	A	
3.	$A \lor B$	2, Add.
4.	$(C \lor D) \supset E$	1, 3, M.P.
→5.	$C \cdot D$	
6.	C	5, Simp.
7.	$C \lor D$	6, Add.
8.	E	4, 7, M.P.
9.	$(C \cdot D) \supset E$	5-8, C.P.
10.	$A \supset ((C \cdot D) \supset E)$	2-9, C.P.

Proof in **CMD**:

1.	$\vdash A \to A \lor B$	Ax9. (Add.)
2.	$\vdash ((A \lor B) \supset ((C \lor D) \supset E)) \cdot (A \lor B) \to (C \lor D) \supset E$	Ax1. (M.P.)
3.	$\vdash A \cdot ((A \lor B) \supset ((C \lor D) \supset E)) \to (C \lor D) \supset E$	1, 2, R2
4.	$\vdash C \cdot D \to C$	Ax7. (Simp.)
5.	$\vdash C \to C \lor D$	Ax8. (Add.)
6.	$\vdash ((C \lor D) \supset E) \cdot (C \lor D) \to E$	Ax1. (M.P.)
7.	$\vdash C \cdot D \to C \lor D$	4, 5, R2
8.	$\vdash (C \cdot D) \cdot ((C \lor D) \supset E) \to E$	6, 7, R2
9.	$\vdash (C \cdot D) \cdot A \cdot ((A \lor B) \supset ((C \lor D) \supset E)) \to E$	3, 8, R2
10.	$\vdash A \cdot ((A \lor B) \supset ((C \lor D) \supset E) \to (C \cdot D) \supset E$	9, R3. (C.P.)
11.	$\vdash ((A \lor B) \supset ((C \lor D) \supset E) \to A \supset ((C \cdot D) \supset E)$	10, R3. (C.P.)

Consider also his proof on page 54 of [4] for an argument with conclusion E and premises $A \supset (B \cdot C)$, $(B \lor D) \supset E$ and $D \lor A$. Copi's proof:

1.	$A \supset (B \cdot C)$	
2.	$(B \lor D) \supset E$	
3.	$D \lor A$	
4.	$\sim E$	I.P. (Indirect Proof)
5.	$\sim (B \lor D)$	2, 4, M.T.
6.	$\sim B \cdot \sim D$	5, DeM.
7.	$\sim D \cdot \sim B$	6, Com.
8.	$\sim D$	7, Simp.
9.	A	3, 8, D.S.
10.	$B \cdot C$	1, 9, M.P.
11.	B	10, Simp.
12.	$\sim B$	6, Simp.
13.	$B \cdot \sim B$	11, 12, Conj.

Proof in **CMD**:

1.	$\vdash ((B \lor D) \supset E) \cdot \sim E \to \sim (B \lor D)$	Ax2. (M.T.)
2.	$\vdash \sim (B \lor D) \to \sim B \cdot \sim D$	Ax10. (DeM.)
3.	$\vdash ((B \lor D) \supset E) \cdot \sim E \to \sim B \cdot \sim D$	1, 2, R1
4.	$\vdash \sim B \cdot \sim D \leftrightarrow \sim D \cdot \sim B$	Ax11. (Com.)
5.	$\vdash ((B \lor D) \supset E) \cdot \sim E \to \sim D \cdot \sim B$	3, 4, R2
6.	$\vdash \sim D \cdot \sim B \to \sim D$	Ax7. (Simp.)

7.	$\vdash((B \vee D) \supset E) \cdot \sim E \rightarrow \sim D$	5, 6, R2
8.	$\vdash(D \vee A) \cdot \sim D \rightarrow A$	Ax4. (D.S.)
9.	$\vdash((B \vee D) \supset E) \cdot \sim E \cdot (D \vee A) \rightarrow A$	7, 8, R2
10.	$\vdash(A \supset (B \cdot C)) \cdot A \rightarrow B \cdot C$	Ax1. (M.P.)
11.	$\vdash((B \vee D) \supset E) \cdot \sim E \cdot (D \vee A) \cdot (A \supset (B \cdot C)) \rightarrow B \cdot C$	9, 10, R2
12.	$\vdash B \cdot C \rightarrow B$	Ax7. (Simp.)
13.	$\vdash((B \vee D) \supset E) \cdot \sim E \cdot (D \vee A) \cdot (A \supset (B \cdot C)) \rightarrow B$	11, 12, R2
14.	$\vdash \sim B \cdot \sim D \rightarrow \sim B$	Ax7. (Simp.)
15.	$\vdash((B \vee D) \supset E) \cdot \sim E \rightarrow \sim B$	3, 14, R2
16.	$\vdash((B \vee D) \supset E) \cdot \sim E \cdot (D \vee A) \cdot (A \supset (B \cdot C)) \rightarrow \sim B$	15, R6
17.	$\vdash((B \vee D) \supset E) \cdot \sim E \cdot (D \vee A) \cdot (A \supset (B \cdot C)) \rightarrow B \cdot \sim B$	13, 16, R5
18.	$\vdash((B \vee D) \supset E) \cdot (D \vee A) \cdot (A \supset (B \cdot C)) \rightarrow E$	17, R4 (I.P.)

Let '$\vDash P \rightarrow Q$' say that $P \supset Q$ is a truth-table tautology; let '$\vDash\rightarrow Q$' say that Q is a truth-table tautology. To show the completeness of **CMD** we need to show that (i) if $\vDash P \rightarrow Q$ then $\vdash P \rightarrow Q$ and (ii) if $\vDash\rightarrow Q$ then $\vdash\rightarrow Q$. By extrapolating from Canty [2], we can sketch a proof of (i) as follows. (The changes in the proof of (i) required for the proof of (ii) will be put in parentheses.) Suppose $\vDash P \rightarrow Q$. (Suppose $\vDash\rightarrow Q$.)

1.	$\vdash P \cdot \sim Q \leftrightarrow$ a disjunctive normal form of $P \cdot \sim Q$, call it R.	
	($\vdash \sim Q \leftrightarrow$ a disjunctive normal form of $\sim Q$, call it R.)	
2.	$\vdash R \rightarrow A \cdot \sim A$.	
3.	$\vdash P \cdot \sim Q \rightarrow A \cdot \sim A$.	1, 2, R2
	($\vdash \sim Q \leftrightarrow A \cdot \sim A$, by 1, 2 and R2.)	
4.	$\vdash P \rightarrow Q$	3, R4
	($\vdash\rightarrow Q$, by 3 and R4.)	

Call this the **DNF** proof. By using conjunctive normal forms we can give an equally simple proof. Suppose $\vDash P \rightarrow Q$. (Suppose $\vDash\rightarrow Q$.)

1.	$\vdash P \supset Q \leftrightarrow$ a conjunctive normal form of $P \supset Q$, call it S, where the conjuncts of S are $S_1, \ldots, S_n$.	
	($\vdash Q \leftrightarrow$ a conjunctive normal form of Q, call it S, where the conjuncts of S are $S_1, \ldots, S_n$.)	
2.	$\vdash\rightarrow S_1$.	
3.	$\vdash\rightarrow S_2$.	
	⋮	
$n + 1$.	$\vdash\rightarrow S_n$.	
$n + 2$.	$\vdash\rightarrow S_1 \cdot S_2$.	2, 3, R5
	⋮	
$2n$.	$\vdash\rightarrow S_1 \ldots S_n$.	$2n - 1$, $n + 1$, R5
$2n + 1$.	$\vdash\rightarrow P \supset Q$.	1, $2n$, R1
	($\vdash\rightarrow Q$, by 1, $2n$ and R1.)	
$2n + 2$.	$\vdash P \rightarrow Q$.	Ax1., $2n + 1$, R2

Call this the **CNF** proof. To complete the justifications for the steps in the

DNF and **CNF** proofs we need to show that **CMD** is analytic (if $\vdash P \rightarrow Q$ then $\vDash P \rightarrow Q$ and if $\vdash\rightarrow Q$ then $\vDash\rightarrow Q$). By using truth tables we can show that $\vDash P \rightarrow Q$, where $\vdash P \rightarrow Q$ is an axiom. So **CMD** is analytic if R1-R5 do not introduce theorems which are not semantically valid.

R1. By using induction on the number of occurrences of connectives in $f(P)$ other than those in that occurrence of P which is replaced by Q to form $f(Q)$, we can show that if $\vDash P \leftrightarrow Q$ then $\vDash f(P) \leftrightarrow f(Q)$. Now if $\vDash f(P) \leftrightarrow f(Q)$ and $\vDash f(P) \rightarrow R$ then $\vDash f(Q) \rightarrow R$. So, if $\vDash P \leftrightarrow Q$ and $\vDash f(P) \rightarrow R$ then $\vDash f(Q) \rightarrow R$. We can treat the other parts of R1 in the same way.

R2. (i) Suppose $\vDash P_1 \ldots P_m \rightarrow Q_1$ and $\vDash R \rightarrow S$, where R is any permutation of $Q_1 \ldots Q_n$. Suppose not $\vDash T \rightarrow S$, where T is any permutation of $P_1 \ldots P_m \cdot Q_2 \ldots Q_n$. Then there are circumstances, C, in which P_1 - P_m are true, Q_2 - Q_n are true and S false. So Q_1 is false in C since $\vDash R \rightarrow S$. Since $\vDash P_1 \ldots P_m \rightarrow Q_1$, Q_1 is also true in C. So $\vDash T \rightarrow S$.
(ii) Same argument as for (i).

The proofs for R3-R6 are no more complicated than the proof for R2 and will be omitted.

We will now use the analyticity of **CMD** to prove step 2 in the **DNF** proof if we are given step 1. From step 1 it follows that $\vdash R \rightarrow P \cdot \sim Q$. By the analyticity of **CMD** it follows that $\vDash R \rightarrow P \cdot \sim Q$. Since $\vDash P \rightarrow Q$, $P \cdot \sim Q$ is a contradiction. But then R is a contradiction since $R \supset P \cdot \sim Q$ is a tautology. So in each disjunct of R there are at least two conjuncts, one a propositional constant and another its negation. Suppose that B and $\sim B$ occur in one of the disjuncts. By using Ax11, Ax12, $\vdash P \vee (Q \cdot \sim Q) \cdot R \rightarrow P$ (see Canty [2]), Ax7, R1 and R2, we can show that $\vdash R \rightarrow B \cdot \sim B$. But $\vdash B \cdot \sim B \rightarrow A \cdot \sim A$. So $\vdash R \rightarrow A \cdot \sim A$.

To prove step 2 in the **CNF** proof, given step 1, first note that from step 1 $\vdash P \supset Q \rightarrow S$. By the analyticity of **CMD** $\vDash P \supset Q \rightarrow S$. Since $\vDash P \rightarrow Q$, S is a tautology. But then each conjunct in S must be a tautology. So in S_1 there must be at least two disjuncts, one a propositional constant, say A, and the other its negation, $\sim A$. Let T be the disjunction of the other disjuncts, if any, in S_1. By Ax14 and R2 $\vdash A \rightarrow A$. By R3 $\vdash\rightarrow A \supset A$. By Ax16 and R1 $\vdash\rightarrow \sim A \vee A$. By Ax8 $\vdash \sim A \vee A \rightarrow (\sim A \vee A) \vee T$. By R2 $\vdash\rightarrow (\sim A \vee A) \vee T$. Then by Ax11, Ax12 and R1, $\vdash\rightarrow S_1$. We can give the same proof for steps 3-$n + 1$.

To prove step 1 in each of the above proofs, first note that by Ax16, Ax17 and R1, $\vdash P \leftrightarrow P'$, where P' contains no occurrences of the connectives, $\supset$ and $\equiv$. Now let $n(P)$ = the number of occurrences of connectives which are in the scope of a negation sign in P. By using induction on $n(P')$ we can show that $\vdash P' \leftrightarrow P''$, where $n(P'') = 0$.

Suppose $n(P') = 0$. If $P' = P''$ then $\vdash P' \leftrightarrow P''$, where $n(P'') = 0$. The induction hypothesis is that if $n(P') = j$, for $j < k$, then $\vdash P' \leftrightarrow P''$, where $n(P'') = 0$. Suppose $n(P') = k$, for $k > 0$. Let S be the smallest well-formed formula contained in P' such that $n(S) = k$. So $P' = S$, $P' = S \vee T$, $P' = T \vee S$, $P' = S \cdot T$ or $P' = T \cdot S$, where $n(T) = 0$. If we can show that $\vdash S \leftrightarrow S'$, where

$n(S') = 0$, then it is obvious that $\vdash P' \leftrightarrow P''$, where $n(P'') = 0$. There are three cases to consider: (a) $S = \sim S_1$. Subcase (i): $S_1 = \sim S_2$. Since $n(S_2) < n(S)$, by the induction hypothesis $\vdash S_2 \leftrightarrow S_3$, where $n(S_3) = 0$. So $\vdash S \leftrightarrow S_3$, where $n(S_3) = 0$. Subcase (ii): $S_1 = S_2 \vee S_3$. By Ax14, R2, Ax9 and R1 $\vdash S \leftrightarrow \sim S_2 \cdot \sim S_3$. Since $n(\sim S_2) < n(S)$ and $n(\sim S_3) < n(S)$, by the induction hypothesis $\vdash \sim S_2 \leftrightarrow S_4$ and $\vdash \sim S_3 \leftrightarrow S_5$, where $n(S_4) = 0$ and $n(S_5) = 0$. By R1 $\vdash S \leftrightarrow S_4 \cdot S_5$, where $n(S_4 \cdot S_5) = 0$. Subcase (iii): $S_1 = S_2 \cdot S_3$. Use the same argument as for subcase (ii).

(b) $S = S_1 \vee S_2$. Since $n(S_1) < n(S)$ and $n(S_2) < n(S)$, by the induction hypothesis $\vdash S_1 \leftrightarrow S_3$ and $\vdash S_2 \leftrightarrow S_4$, where $n(S_3)$ and $n(S_4) = 0$. By Ax14, R2 and R1 $\vdash S \leftrightarrow S_3 \vee S_4$, where $n(S_3 \vee S_4) = 0$.
(c) $S = S_1 \cdot S_2$. Same argument as for (b).

If P'' is not in disjunctive normal form, then there is a constituent well-formed formula, T, of P'' such that $T = (R_1 \vee \ldots \vee R_m) \cdot (S_1 \vee \ldots \vee S_n)$ where $m > 1$ or $n > 1$. Call such formulas as T disrupting formulas. By Ax14, Ax11, Ax13, R1 and R2 $\vdash T \leftrightarrow U$, where U contains no disrupting formulas. So by R1 $\vdash P'' \leftrightarrow R$, where R is in disjunctive normal form. Since we have shown that $\vdash P \leftrightarrow P'$ and $\vdash P' \leftrightarrow P''$, it follows by R2 that $\vdash P \leftrightarrow R$. If P'' is not in conjunctive normal form, then the disrupting formulas are of the form $(R_1 \ldots R_m) \vee (S_1 \ldots S_n)$, where $m > 1$ or $n > 1$. By using Ax14, Ax11, Ax13, R1 and R2 we can remove all such disrupting formulas. So $\vdash P'' \leftrightarrow S$, where S is in conjunctive normal form, and $\vdash P \leftrightarrow S$.

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