Notes on a
course
introducing
logarithms

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(Luis António Freire ...
(winter time
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properties ①

basic definition $\log_b x = y \iff b^y = x$

properties:

(i) $\log_b AB \stackrel{?}{=} \log_b A + \log_b B$

$$\begin{bmatrix} \log_b A = x \\ \log_b B = y \end{bmatrix} \Rightarrow \begin{bmatrix} b^x = A \\ b^y = B \end{bmatrix} \odot$$

$$b^x \cdot b^y = AB$$

$$b^{x+y} = AB$$

$$\log_b AB = x + y$$

$$\log_b AB = \log_b A + \log_y B$$

(ii) $\log_b \frac{A}{B} \stackrel{?}{=} \log_b A - \log_b B$

$$\begin{cases} \log_b A = x \\ \log_b B = y \end{cases} \Rightarrow \begin{cases} b^x = A \\ b^y = B \end{cases} (\div)$$

$$\frac{b^x}{b^y} = \frac{A}{B}$$

$$b^{x-y} = \frac{A}{B}$$

$$\log_b \frac{A}{B} = x - y$$

$$\log_b \frac{A}{B} = \log_b A - \log_b B$$

(iii) $\log_b A^n \overset{?}{=} n \cdot \log_b A$

$$\log_b A = x$$

$$b^x = A$$

$$(b^x)^n = A^n$$

$$b^{nx} = A^n$$

$$\log_b A^n = nx$$

$$\log_b A^n = n \cdot \log_b A$$

properties 4

(iv) $\log_b \sqrt[n]{A} \stackrel{?}{=} \dfrac{\log_b A}{n}$

$$\log_b \sqrt[n]{A}$$

$$= \log_b A^{\frac{1}{n}}$$

$$\stackrel{\text{(by iii)}}{=} \frac{1}{n} \cdot \log_b A$$

$$= \frac{\log_b A}{n}$$

extra 1 :

$$b^{\log_b x} \stackrel{?}{=} x$$

$$\log_b x = y$$

$$b^y = x$$

$$b^{\log_b x} = x$$

(This property will be very useful, later on, when we are going to prove that the exponential function $f(x)=2^x$ and the logarithm function $g(x)=\log_2 x$ are "inverse functions" of one another)

(in order to do so, we need first to study the concept of «composition-of-functions") extra 2 :

$$\log_b x \stackrel{?}{=} \frac{\log_{\square} x}{\log_{\square} b}$$

$$\log_b x = y$$

$$b^y = x$$

$$\log_{\square} b^y = \log_{\square} x$$

$$y \cdot \log_{\square} b = \log_{\square} x$$

$$y = \frac{\log_{\square} x}{\log_{\square} b}$$

important remarks:

related to the proof of property (i):

$$5^3 \cdot 5^4 = 5^{3+4} = 5^7$$

because

$$5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 = 5^{3+4}$$

This illustrates the formula

$$A^p \cdot A^q = A^{p+q}$$

related to the proof of property (ii):

$$\frac{5^8}{5^6} = 5^{8-6} = 5^2$$

because

$$\frac{\cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot 5 \cdot 5}{\cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5} \cdot \cancel{5}} = 5^{8-6}$$

illustrating the general rule

$$\frac{A^p}{A^q} = A^{p-q}$$

Another important remark, is to be able to use the basic definition in both ways:

$$\log_{☾} \bigstar = \square \iff ☾^{\square} = \bigstar$$

It's important to be able to express the "exponential" format

$$☾^{\square} = \bigstar$$

into the "logarithm" format

$$\log_{☾} \bigstar = \square$$

(and vice-versa)

For instance, during the proof of (i),
we had a passage from
this format $b^{x+y} = AB$

into the format $\log_b AB = x+y$

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Likewise, we had to switch
from one "format" to another,
in almost every proof of
the other properties.

remark 4

$$X = Y \qquad 5 = 5$$

$$(X)^n = (Y)^n \qquad 5^3 = 5^3$$

(This observation
was used
in the proof of
proposition (iii))

remark 5

$$\left(A^3\right)^2 = A^{(3)\cdot(2)} = A^6$$

$$= A \cdot A \cdot A \cdot A \cdot A \cdot A$$

illustrating the fact that, in general

$$\left(A^p\right)^q = A^{pq}$$

(we will also need this observation
in order to understand a
certain (crucial) detail
in the proof of
proposition (iii))

remark 6

$$1000^{\frac{3}{15}} = \sqrt[15]{1000^3}$$

$$A^{\frac{p}{q}} = \sqrt[q]{A^p}$$

$$\sqrt[n]{A^1}$$

Remark

$$~~\sqrt[n]{A}~~$$

$$A^{\frac{p}{q}} = \sqrt[q]{A^p}$$

$$A^{\frac{1}{n}} = \sqrt[n]{A^1}$$

$$A^{\frac{1}{n}} = \sqrt[n]{A}$$

(This is fundamental
To understand the proof
of proposition (iv))

remark

8

$$\frac{a}{b} = \frac{c}{d}$$

from The chapter of Ratio-and-Proportion, we know That "a" can "slide" down Through The diagonal and become

$$\frac{1}{b} = \frac{c}{ad}$$

$$y \cdot E = P \qquad \frac{yE}{1} = \frac{P}{1}$$

$$y = \frac{P}{E} \qquad y = \frac{P}{E}$$

(This is related To The very last passage in The proof of proposition "extra 2") remark

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This was a crucial step
in the proof of "extra ②":

if $\quad 5 = 5$

then $\quad \log_{\square} 5 = \log_{\square} 5 \quad$ (where $\square$ means "any base")

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Likewise

if $\quad ☾ = ★$

then $\quad \log_{\square} ☾ = \log_{\square} ★$

(of course!!)... (this is pretty "obvious"..
(and fundamental to understand
the proof of this important
property of (change of bases)

Exercises (page 1)

(These exercises should be done without the calculator)

(otherwise, it wouldn't make sense, because, the purpose of such exercises is to practice the "mechanism" of algebra)

$\left(\frac{1}{10}\right)^4 =$

$7^{-2} =$

$5^{-3} =$

$7^2 =$

$(-7)^2 =$

$5^2 =$

$0^3 =$

Exercises (page 2)

$25^2 =$

$7^3 =$

$(-2)^4 =$

$2^4 =$

$(-2)^5 =$

Exercises (page 3)

$(-1)^6 =$

$5^3 =$

$(-1)^7 =$

$10^3 =$

Exercises (pg. 4)

$$\log_5 125 = \square$$

$$\log_3 81 = \square$$

$$\log_9 81 = \square$$

$$\log_2 \frac{1}{8} = \square$$

$$\log_{10} 0.01 = \square$$

Exercises (page 5)

$$\log_{\frac{5}{7}} \frac{25}{49} = \square$$

$$\log_2 8 = \square$$

$$\log_{10} 1000 = \square$$

$$\log_{10} 0.001 = \square$$

$$\log_{10} \frac{1}{1000} = \square$$

Exercises (page 6)

$\sqrt{16} =$

$\sqrt[3]{125} =$

$\sqrt[3]{343} =$

$\sqrt{49} =$

$2^5 =$

$2^6 =$

$2^7 =$

$2^8 =$

$2^9 =$

$2^{10} =$

Exercises (page 7)

(i) $\frac{2}{3} + \frac{3}{4} =$

(ii) $\frac{2}{3} \cdot \frac{3}{4} =$

(iii) $\frac{2}{3} : \frac{3}{4} =$

(iv) $\frac{2}{3} - \frac{3}{4} =$

Exercises (page 8)

$$\frac{\frac{5}{7}}{\frac{9}{8}} =$$

$$\frac{1}{\frac{17}{20}} =$$

$$\left(\frac{8}{9}\right)^{-2} =$$