Modal Ontology and Generalized Quantifiers*

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Abstract

Timothy Williamson has argued that in the debate on modal ontology, the familiar distinction between actualism and possibilism should be replaced by a distinction between positions he calls contingentism and necessitism. He has also argued in favor of necessitism, using results on quantified modal logic with plurally interpreted second-order quantifiers showing that necessitists can draw distinctions contingentists cannot draw. Some of these results are similar to well-known results on the relative expressivity of quantified modal logics with so-called inner and outer quantifiers. The present paper deals with these issues in the context of quantified modal logics with generalized quantifiers. Its main aim is to establish two results for such a logic: Firstly, contingentists can draw the distinctions necessitists can draw if and only if the logic with inner quantifiers is at least as expressive as the logic with outer quantifiers, and necessitists can draw the distinctions contingentists can draw if and only if the logic with outer quantifiers is at least as expressive as the logic with inner quantifiers. Secondly, the former two items are the case if and only if all of the generalized quantifiers are first-order definable, and the latter two items are the case if and only if first-order logic with these generalized quantifiers relativizes.

1 Logic and Ontology

In this first section, the philosophical and technical background is reviewed and the necessary formal definitions are stated. I start with the distinctions between actualism and possibilism and between contingentism and necessitism, as well as the argument for necessitism given in Williamson (2010). I then describe generalized quantifiers, quantified modal logic, and the class of logics resulting from adding generalized quantifiers to quantified modal logic. The second section will be devoted to the statement and proof of the central theorem of the paper, which I call *the equivalence theorem*. The third section will concern the philosophical relevance of the result, and the fourth section will give a concluding summary.

1.1 Modal Ontology

A central question in the metaphysics of modality is an ontological question. It is often put thus: Are there objects that are possible but not actual? This

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question is then used to distinguish two positions: Those who answer *Yes* are called *possibilists*; those who answer *No* are called *actualists*.

Williamson (2010) criticizes this question, and consequently also this distinction, as obscure. His main complaint is that it is unclear what *actual* and *possible* mean in this instance. Instead of trying to clarify these terms, Williamson reformulates the question as follows: Is necessarily everything necessarily something? He also uses this question to demarcate two positions: Those who answer *Yes* he calls *necessitists*; those who answer *No* he calls *contingentists*. More on these positions can be found in Williamson (2012).

The main aim of Williamson (2010) is to argue in favour of necessitism by showing that there are certain distinctions only necessitists can draw, although they seem sensible without a prior commitment to necessitism. He argues for this along the following lines: For every sentence, there is a second sentence that is neutral in the dispute between necessitism and contingentism such that it follows from contingentism that the two sentences are equivalent. Therefore, the necessitist can map everything the contingentist can say to something the contingentist must consider equivalent, and which is independent of their dispute. Further, the converse is not the case; there are sentences such that no sentence is both neutral in the dispute and equivalent to the first, given necessitism. Therefore, if the necessitist utters such a sentence, the contingentist cannot map it to a sentence the necessitist must consider equivalent, and which is independent of their dispute.

Williamson argues for this using so-called *chunky-style* versions of contingentism and necessitism, which are formulated using a property he calls *chunkiness*. Chunkiness can roughly be understood as being grounded in the concrete, and it plays a similar role in chunky-style contingentism and necessitism as existence plays in actualism and possibilism. Chunky-style contingentism adds to the denial of the claim that necessarily, everything is necessarily something, the assertion that necessarily, everything is chunky, while chunky-style necessitism adds to the assertion that necessarily, everything is necessarily something, the denial of the claim that necessarily, everything is chunky. We can motivate the use of the chunky-style positions by considering how a necessitist can draw the distinction a contingentist can draw with an existential claim. To do so, they have to restrict existential quantification to a domain of things about which they are not in dispute with the contingentist, and this is what the property of chunkiness makes possible. In the following, talk of *necessitism* and *contingentism* will always be about chunky-style necessitism and contingentism.

To be able to give formal proofs for his claims about the existence of mappings for contingentists and necessitists, Williamson uses a quantified modal logic along the lines of Kripke (1963). He represents the positions of contingentism and necessitism using two auxiliary principles called Aux[Con] and Aux[Nec]. These principles are not straightforward formalizations of contingentism and necessitism, as it turns out that necessitists can define their mapping for a position that just assumes that necessarily, everything is chunky. Similarly, to enable contingentists to at least define their mapping in first-order quantified modal logic, necessitists are taken to assume that everything is possibly chunky, and necessarily, whatever has a property or stands in a relation is chunky, in addition to their position that necessarily, everything is necessarily something. Using C to represent *chunkiness*, the auxiliary principles can therefore be stated as follows:

Aux[Con] $\Box\forall x Cx$

Aux[Nec] $\Box\forall x\Box\exists y(x=y)\wedge\forall x\Diamond Cx\wedge\bigwedge_R\Box\forall\bar{x}\left(R\bar{x}\rightarrow\bigwedge_j Cx_j\right)$

In Aux[Nec], the conjunction in the third conjunct ranges over all non-logical relation symbols R – Williamson assumes that we are working in a language with a finite signature – and $\bar{x}$ indicates a sequence of variables.

Being neutral in the dispute between necessitism and contingentism is formalized by defining a formula to be neutral if it is equivalent to one in which all quantifiers and predications are restricted to C. Here, a predication $R\bar{x}$ is restricted to C if it occurs in a conjunction of the form $R\bar{x}\wedge\bigwedge_j Cx_j$. For each formula φ, let φ^{Con} be φ with all of its quantifiers and predications restricted to C. Williamson proves that the function that assigns φ^{Con} to every formula φ can be used as the necessitist's mapping, since for every φ, φ^{Con} is neutral and Aux[Con] entails $\varphi\leftrightarrow\varphi^{Con}$. In the following, I will sometimes say that φ *is equivalent to* φ^{Con}*, given contingentism*, which is meant to say the same.

Williamson notes that an analogous mapping for the contingentist does not exist if the quantified modal logic contains only the modal operators $\Diamond$ and $\Box$. However, he proves that in first-order quantified modal logic, it can be defined once two operators $\uparrow$ and $\downarrow$ are added to the logic. Figuratively speaking, $\uparrow$ allows us to store the world of evaluation, and $\downarrow$ allows us to retrieve it. This is used in the central clause of the recursive definition of the mapping $\cdot^{Nec}$, which assigns to each formula one that is neutral and equivalent to the first, given necessitism:

$$(\exists x\varphi)^{Nec}=\uparrow\Diamond\exists x\left(Cx\wedge\downarrow\varphi^{Nec}\right)$$

A side note on these operators: They derive from the formalization of the temporal indexicals "once" and "then" given in Vlach (1973). Modal analogs of Vlach's operators were used in Bricker (1989) and Forbes (1989) in a way similar to Williamson's use of $\uparrow$ and $\downarrow$. By incorporating ideas presented in Hodes (1984), Williamson gives $\uparrow$ and $\downarrow$ a richer semantics, which solves some problems noted in Forbes (1989). A different set of operators that achieve the same has also been described in Correia (2007). Interestingly, both Williamson's as well as Correia's extensions were anticipated in Vlach (1973, appendix A).

The crucial result in Williamson (2010) is that $\cdot^{Con}$ can be extended to second-order quantifiers on the plural interpretation (as described in Boolos (1984)), but that this is not the case for $\cdot^{Nec}$, even if the logic contains $\uparrow$ and $\downarrow$. That is, in quantified modal logic with second-order quantifiers on the plural interpretation, there is a formula such that there is no neutral formula which is equivalent to the first, given necessitism. One such formula is the following:

(34) $\exists X(\exists xXx\wedge\exists x\neg Xx\wedge\forall x\forall y(\Diamond Rxy\rightarrow(Xx\rightarrow Xy)))$

It says that there are some things, of which some but not every thing is one, which can have R only to themselves (adapted from Williamson (2010, p. 705)). The distinctions necessitists can draw with (34) under given interpretations of R seem sensible independently of necessitism, although they assume necessitism to express it. Since there is no neutral formula equivalent to (34) given necessitism, contingentists are unable to draw these distinctions.

It is important that the second-order quantifiers are interpreted plurally, since Williamson's formal result uses the fact that what a bound second-order variable applies to does not vary from world to world. This is not plausible if second-order quantifiers range over properties: That John has some property does not imply that he necessarily has that property. But it is plausible for plural quantification: If there are some things of which John is one, then he is necessarily one of them.

This concludes my sketch of Williamson's argument. It raises many philosophical questions: Is the metaphysical dispute on modal ontology really better characterized in terms of contingentism and necessitism, rather than in the more traditional terms of actualism and possibilism? If so, are the relevant positions really the chunky-style versions of contingentism and necessitism, rather than the non-chunky-style versions? Is it enough to show that contingentists are unable to draw certain distinctions in a particular formal language, rather than in natural language, or in any one of a philosophically relevant class of formal languages?

Some of these philosophical questions raise related technical questions. In the discussion on actualism and possibilism, results have been used which concern the relative expressivity of quantified modal logics with two different kinds of quantifiers, called *inner* and *outer* quantifiers. In many respects, these results resemble Williamson's results described above. For the philosophical question on the correct characterization of the metaphysical dispute, it may be interesting to ask the technical question how these kinds of results relate. Similarly, the philosophical question concerning Williamson's use of his particular formal language raises the technical question in which quantified modal logics the analogs of his results hold.

It is these technical issues that I will be concerned with in this paper. They can be summed up in the following two questions:

(Q1) What is the relation between Williamson's results and the more traditional results on the relative expressivity of logics with inner and outer quantifiers?

(Q2) How stable are Williamson's results across different quantified modal logics?

It would be nice if we could give a general answer to these questions for all philosophically relevant quantified modal logics. Although nice, this would be very difficult. It would be difficult from a philosophical perspective, since we would have to delineate the philosophically relevant quantified modal logics, and it would also be difficult from a technical perspective, since we would have to work in a framework in which all of them can be represented, and then prove general results about them.

I therefore work with a compromise in this paper; a range of logics that is wide enough to yield an interesting amount of variety, but narrow enough to be manageable. It is the range of logics obtained by adding generalized quantifiers to the first-order modal logic used by Williamson. One important feature of generalized quantifiers that makes this range manageable is that they use first-order variables, whose interpretation is already settled by the first-order case. To be able to state and settle (Q1) and (Q2) for this range of logics, I will now introduce generalized quantifiers as well as quantified modal logic formally.

1.2 Generalized Quantifiers

I start by introducing generalized quantifiers. Generalized quantifiers as I will use them were introduced in Lindström (1966) as a generalization of the definition given in Mostowski (1957), which is why they are sometimes called "Lindström quantifiers". I roughly follow Peters and Westerståhl (2006); for a concise introduction to generalized quantifiers, see Westerståhl (2011).

Syntactically, generalized quantifiers can be used similarly to the first-order quantifiers $\forall$ and $\exists$. Simple generalized quantifiers bind one variable and operate on one formula just as $\forall$ and $\exists$ do, but more complex ones can also bind several variables and operate on several formulas. An example of the simple kind is the generalized quantifier $\mathcal{Q}_0$ which is interpreted in a way such that $\mathcal{Q}_0 x\varphi$ is true in a model if and only if there are are infinitely many things in its domain that satisfy $\varphi(x)$.

To extend first-order logic with this quantifier, it would be straightforward to write down this condition slightly more formally in the form of a clause of recursive truth-conditions. But to give uniform truth-conditions for generalized quantifiers, one can also define $\mathcal{Q}_0$ to be the function that maps every set D to the set $\mathcal{Q}_{0D}$ of infinite subsets of D. Then we can define $\mathcal{Q}_0 x\varphi$ to be true in a model with domain D if and only if the set of elements of D satisfying $\varphi(x)$ is in $\mathcal{Q}_{0D}$. This suggests that we can define a generalized quantifier to be a function that maps every set to a set of its subsets. And in fact, this is almost correct; there are only two things that have to be adjusted: Firstly, we want truth in a model to be invariant under isomorphisms, so we have to make sure that generalized quantifiers preserve bijections between sets. Secondly, we have to generalize the definition to generalized quantifiers binding several variables and operating on several formulas. This gives us the following definition:

A *generalized quantifier of type* $\langle n_1, \ldots, n_k\rangle$ is a function $\mathcal{Q}$ that maps every set D to a k-ary relation over n_j-ary relations over D such that the following condition is satisfied:

(Isom) For any bijection f from a set D to a set D' and relations $R_1, \ldots, R_k$ of arities $n_1, \ldots, n_k$ over D, $\mathcal{Q}_D(R_1, \ldots, R_k)$ if and only if $\mathcal{Q}_{D'}(f(R_1), \ldots, f(R_k))$.

Here, f is lifted to relations in the obvious way: $f(\bar{o}) \in f(R)$ if and only if $\bar{o} \in R$, where $f(\bar{o})$ is $\langle f(o_1), \ldots, f(o_n)\rangle$. $\mathcal{Q}_D(R_1, \ldots, R_k)$ is meant to express that $R_1, \ldots, R_k$ stand in the relation $\mathcal{Q}_D$.

Although we will later be concerned with quantified *modal* logics with generalized quantifiers, I first define the result of adding a set of generalized quantifiers $\mathbb{Q}$ to first-order logic. To fit the following discussion of quantified modal logics best, I only consider signatures containing a finite number of relation symbols of different arities, and no individual constants or function symbols. Formulas are obtained from these relation symbols and a countably infinite set of first-order variables in the usual way, using the logical constants $=$, $\neg$, $\wedge$ and $\forall$. As usual, other common operators such as $\vee$ or $\exists$ will be used as syntactic abbreviations. To add generalized quantifiers, we introduce the following syntactic rule:

> If $\mathcal{Q} \in \mathbb{Q}$ is of type $\langle n_1, \ldots, n_k\rangle$ and for each $j \leq k$, φ_j is a formula and $\bar{x}_j$ is a sequence of n_j variables, then $\mathcal{Q}\bar{x}_1 \ldots \bar{x}_k(\varphi_1, \ldots, \varphi_k)$ is a formula.

For brevity, I will also write $\mathcal{Q}\bar{\bar{x}}\bar{\varphi}$ for $\mathcal{Q}\bar{x}_1 \ldots \bar{x}_k(\varphi_1, \ldots, \varphi_k)$. Note that we just use the metalanguage symbol "$\mathcal{Q}$" which denotes a generalized quantifier again in the formal syntax of our logic, where it is interpreted using that generalized quantifier. In practice, this will not lead to any confusion. With this, we can define the syntax of our language: Call all the expressions that can be constructed using the usual rules for the construction of first-order formulas and the above rule for generalized quantifiers $L_{\mathbb{Q}}$-*formulas*. Such a formula is called *closed* or a *sentence* if all occurrences of variables in it are bound. Finally, let $L_{\mathbb{Q}}$ be the set of closed $L_{\mathbb{Q}}$-formulas.

For the semantics, we interpret $L_{\mathbb{Q}}$-formulas on structures of the form $\mathfrak{A} = \langle D, e\rangle$, where D is a set and e is a function that maps every relation symbol R to a set $e(R) \subseteq D^n$, n being the arity of R. Note that D may be empty. We write $|\mathfrak{A}|$ for D and $R^{\mathfrak{A}}$ for $e(R)$. I call such structures *models*, and use the term *structure* more freely. An $L_{\mathbb{Q}}$-formula φ is interpreted relative to a model $\mathfrak{A}$ and an assignment a for $\mathfrak{A}$. Such an assignment is a partial function that maps variables to elements of $|\mathfrak{A}|$. The partiality is required for models with empty domains. We write $a[o/x]$ for the assignment that maps x to o and every variable y in the domain of a besides x to $a(y)$, and extend this notation to tuples in the obvious way, writing $a[\bar{o}/\bar{x}]$. Truth of a formula is only defined relative to assignments whose domain includes all free variables of the formula in question. A formula φ being *true* in a model $\mathfrak{A}$ relative to an assignment a is written $\mathfrak{A}, a \vDash \varphi$, and the truth relation $\vDash$ is defined recursively in the usual manner, with the following clause for generalized quantifiers $\mathcal{Q} \in \mathbb{Q}$:

$$\mathfrak{A}, a \vDash \mathcal{Q}\bar{\bar{x}}\bar{\varphi} \text{ iff } \mathcal{Q}_{|\mathfrak{A}|}\left(\varphi_1(\bar{x}_1)^{\mathfrak{A},a}, \ldots, \varphi_k(\bar{x}_k)^{\mathfrak{A},a}\right)$$

where $\varphi_j(\bar{x}_j)^{\mathfrak{A},a} = \{\bar{o} \in |\mathfrak{A}|^{n_j} : \mathfrak{A}, a[\bar{o}/\bar{x}_j] \vDash \varphi_j\}$.

Truth of a *sentence* relative to a model is derived from this as follows:

$$\mathfrak{A} \vDash \varphi \text{ iff } \mathfrak{A}, a \vDash \varphi \text{ for all assignments } a$$

Besides the syntax and semantics of first-order logics with added generalized quantifiers, we also need a number of notions to talk about these kinds of logics. As we will need the same notions again for the quantified modal logics discussed later, I will introduce an abstract format in which a logic can be specified, and define these notions for any such logic. In abstract model theory, logics are taken to be specified by a set of sentences and a truth relation; see, e.g., Barwise and Feferman (1985). But for these logics, it is assumed that their sentences are interpreted relative to models as defined above, and the quantified modal logics that we will work with later use a different kind of models. Therefore, we will use a more general notion of a logic here, namely as a triple $\langle L, \mathsf{X}, \vDash\rangle$, where L is a set (the set of sentences), X is a class (the class of structures), and $\vDash$ is a binary relation that holds between members of X and L (the truth relation). Letting M be the class of models, we can define first-order logic with the generalized quantifiers in $\mathbb{Q}$ as $\mathcal{L}_{\mathbb{Q}} = \langle L_{\mathbb{Q}}, \mathsf{M}, \vDash\rangle$. For singleton sets $\{\mathcal{Q}\}$, I will shorten $\mathcal{L}_{\{\mathcal{Q}\}}$ to $\mathcal{L}_{\mathcal{Q}}$, and similarly for many other pieces of notation.

We first define the notions of validity and consequence, for which we re-use the symbol for truth in a structure. For any logic $\mathcal{L} = \langle L, \mathsf{X}, \vDash\rangle$, we write $\vDash \varphi$ for a sentence φ being *valid*, and $\Gamma \vDash \varphi$ for a sentence φ being a *consequence of* a set of sentences Γ, which are defined as follows:

$$\vDash \varphi \text{ iff } \mathfrak{S} \vDash \varphi \text{ for all } \mathfrak{S} \in \mathsf{X}$$

$$\Gamma \vDash \varphi \text{ iff } \mathfrak{S} \vDash \varphi \text{ for all } \mathfrak{S} \in \mathsf{X} \text{ such that } \mathfrak{S} \vDash \psi \text{ for all } \psi \in \Gamma$$

For singleton sets $\{\chi\}$, I write $\chi \vDash \varphi$ instead of $\{\chi\} \vDash \varphi$.

Another notion we will need for various logics is that of relative expressivity, which was already mentioned above. The following definition is based on the idea that a logic is at least as expressive as another if for any sentence of the latter, there is a sentence of the former that expresses the same, while two sentences express the same if they are true in the same structures. More formally, let $\mathcal{L}_1 = \langle L_1, \mathsf{X}, \vDash_1 \rangle$ and $\mathcal{L}_2 = \langle L_2, \mathsf{X}, \vDash_2 \rangle$ be two logics that use the same class of structures X. Then $\mathcal{L}_2$ is *(at least) as expressive as* $\mathcal{L}_1$, written $\mathcal{L}_1 \preceq \mathcal{L}_2$, if for all $\varphi \in L_1$ there is a $\psi \in L_2$ such that for all $\mathfrak{S} \in \mathsf{X}$, $\mathfrak{S} \vDash_1 \varphi$ if and only if $\mathfrak{S} \vDash_2 \psi$.

In the definition of Williamson's mapping $\cdot^{Con}$, first-order quantifiers where treated by restricting them to the predicate C. We can expect that a similar procedure will be necessary for generalized quantifiers if we want to extend $\cdot^{Con}$ to a modal logic containing generalized quantifiers. Fortunately, the concept of relativizing, which is standard in the literature on generalized quantifiers, captures the relevant property. We therefore also define what it is for a logic to relativize, although we only define this for first-order logics with added generalized quantifiers. This definition needs two additional concepts, namely that of definability of a generalized quantifier, and that of relativization of a generalized quantifier. We start with these auxiliary concepts.

Let $\mathbb{Q}$ be a set of generalized quantifiers and $\mathcal{Q}$ a generalized quantifier of type $\langle n_1, \ldots, n_k \rangle$. $\mathcal{Q}$ is *definable* in $\mathcal{L}_{\mathbb{Q}}$ if there is a $\varphi \in L_{\mathbb{Q}}$ containing at most relation symbols $V_1, \ldots, V_k$ of arities $n_1, \ldots, n_k$ such that for all models $\mathfrak{A}$, $\mathfrak{A} \vDash \varphi$ if and only if $\mathcal{Q}_{|\mathfrak{A}|}\left(V_1^{\mathfrak{A}}, \ldots, V_k^{\mathfrak{A}}\right)$. Further, the *relativization* of $\mathcal{Q}$, written $\mathcal{Q}^{\mathrm{rel}}$, is the unique generalized quantifier of type $\langle 1, n_1, \ldots, n_k \rangle$ such that $\mathcal{Q}^{\mathrm{rel}}_D(S, R_1, \ldots, R_k)$ if and only if $\mathcal{Q}_S(R_1 \cap S^{n_1}, \ldots, R_k \cap S^{n_k})$.

With these two notions, we can define which logics relativize: $\mathcal{L}_{\mathbb{Q}}$ *relativizes* if for each $\mathcal{Q} \in \mathbb{Q}$, $\mathcal{Q}^{\mathrm{rel}}$ is definable in $\mathcal{L}_{\mathbb{Q}}$. Another characterization of relativizing can be given with the notion of relative expressivity, using the following equivalence linking definability and relative expressivity: $\mathcal{L}_{\mathbb{Q}} \preceq \mathcal{L}_{\mathbb{Q}'}$ if and only if each $\mathcal{Q} \in \mathbb{Q}$ is definable in $\mathcal{L}_{\mathbb{Q}'}$. (See Peters and Westerståhl (2006, p. 452, Proposition 1) for a proof.) From this, it follows directly that $\mathcal{L}_{\mathbb{Q}}$ relativizes if and only if $\mathcal{L}_{\mathbb{Q}^{\mathrm{rel}}} \preceq \mathcal{L}_{\mathbb{Q}}$, where $\mathbb{Q}^{\mathrm{rel}} = \{\mathcal{Q}^{\mathrm{rel}} : \mathcal{Q} \in \mathbb{Q}\}$.

1.3 Quantified Modal Logic

I will now define the kind of quantified modal logics we will be working with, namely the logics resulting from adding generalized quantifiers to the first-order quantified modal logic used in Williamson (2010). I will deviate slightly from Williamson's presentation to make the definitions in this paper more uniform. Syntactically, the language is like the extensional language used above, except that three unary sentential operators are added to the recursive definition of a formula: $\Box$, $\uparrow$ and $\downarrow$. As usual, we use $\Diamond$ as an abbreviation for $\neg\Box\neg$. Call the formulas that can be constructed in this way the $L_{\Box\mathbb{Q}}$-formulas.

For the semantics, we take a variant of the kind of structures used in Kripke (1963), which I call *Kripke models*. A Kripke model is a structure $\langle W, D, d, i, @\rangle$, where W and D are sets, d is a function from W to subsets of D, i is a function mapping every relation symbol R to a function mapping every $w \in W$ to a relation $i(R)(w) \subseteq D^n$, n being R's arity, and $@ \in W$. W is called the set of *worlds*, D the *outer domain*, d the *domain function*, i the *intepretation function*, and @ the *actual world*. Let K be the class of Kripke models. To identify some subclasses of K, we introduce two conditions on Kripke models. The first requires any object having a property or standing in a relation in a world to be in the domain of that world, while the second requires the outer domain to be the union of the domains of all worlds:

(P) $i(R)(w) \subseteq d(w)^n$ for all relation symbols R and $w \in W$

(D) $D = \bigcup_{v \in W} d(v)$

The assumption of (P) is sometimes called *property actualism* or the *being constraint*. Define P to be the class of Kripke models satisfying (P), D the class satisfying (D), and PD the class satisfying both (P) and (D). While Williamson uses P as his model theory, the other two classes are needed for the discussion of actualism and possibilism.

Truth of an $L_{\Box\mathbb{Q}}$-formula in a Kripke model will be relativized to a world, a finite sequence of worlds, and an assignment. The relativization to a sequence of worlds is not standard in quantified modal logic; this is needed to interpret the operators $\uparrow$ and $\downarrow$. Here, an assignment is a function from variables to members of the outer domain of the Kripke model. As before, we don't require assignments to be total. To be able to specify the truth-conditions, we introduce the convention that for a sequence of world s and a world w, we write $s^\frown w$ for the sequence obtained from appending w to s. Now we can state the conditions for a formula being true in a Kripke model $\mathfrak{M} = \langle W, D, d, i, @\rangle$ relative to a world w, sequence of worlds s and assignment a whose domain includes the free variables in φ, for which we write $\mathfrak{M}, w, s, a \vDash \varphi$:

$\mathfrak{M}, w, s, a \vDash R\bar{x}$ iff $a(\bar{x}) \in i(R)(w)$

$\mathfrak{M}, w, s, a \vDash x = y$ iff $a(x) = a(y)$ and $a(x) \in d(w)$

$\mathfrak{M}, w, s, a \vDash \neg\varphi$ iff not $\mathfrak{M}, w, s, a \vDash \varphi$

$\mathfrak{M}, w, s, a \vDash \varphi \wedge \psi$ iff $\mathfrak{M}, w, s, a \vDash \varphi$ and $\mathfrak{M}, w, s, a \vDash \psi$

$\mathfrak{M}, w, s, a \vDash \Box\varphi$ iff $\mathfrak{M}, v, s, a \vDash \varphi$ for all $v \in W$

$\mathfrak{M}, w, s, a \vDash \uparrow\varphi$ iff $\mathfrak{M}, w, s^\frown w, a \vDash \varphi$

$\mathfrak{M}, w, s^\frown v, a \vDash \downarrow\varphi$ iff $\mathfrak{M}, v, s, a \vDash \varphi$

$\mathfrak{M}, w, \langle\rangle, a \vDash \downarrow\varphi$ iff $\mathfrak{M}, w, \langle\rangle, a \vDash \varphi$

$\mathfrak{M}, w, s, a \vDash \forall x\varphi$ iff $\mathfrak{M}, w, s, a[o/x] \vDash \varphi$ for all $o \in d(w)$

$\mathfrak{M}, w, s, a \vDash \mathcal{Q}\bar{\bar{x}}\bar{\varphi}$ iff $\mathcal{Q}_{d(w)}\left(\varphi_1(\bar{x}_1)^{\mathfrak{M},w,s,a}, \ldots, \varphi_k(\bar{x}_k)^{\mathfrak{M},w,s,a}\right)$

where $\varphi_j(\bar{x}_j)^{\mathfrak{M},w,s,a} = \{\bar{o} \in d(w)^{n_j} : \mathfrak{M}, w, s, a[\bar{o}/\bar{x}_j] \vDash \varphi_j\}$. Note that I use the same symbol "$\vDash$" for the truth relation here as for the truth relation of $\mathcal{L}_{\mathbb{Q}}$. Of course, they are two different relations, but this difference will always be clear from the context since the two relations relate formulas to different kinds of structures, so there is no need to clutter up the notation by further indices.

We now have to define which formulas are sentences, and define the notion of truth in a Kripke model for them. The natural definition of truth in a Kripke model is as truth in the actual world relative to all sequences and assignments. This notion will be most useful if the syntactic characterization of sentences ensures that truth of a sentence is invariant under variation of the sequence of worlds and the assignment. For the assignment, we can require sentences to be closed, which we can define as before as the property of having no free occurrences of variables. But we also need an analogous property for the sequence of worlds, which I call *arrow-closure*. Roughly, a formula is arrow-closed if for every occurrence of $\downarrow$, there is an occurrence of $\uparrow$ in whose scope it lies, which ensures that the initial choice of the sequence of worlds is irrelevant for the truth of the whole formula. To make this precise, we first inductively define the *arrow degree* $\mathrm{ad}(\varphi)$ of an $L_{\Box\mathbb{Q}}$-formula φ as follows:

$$\begin{aligned}
&\mathrm{ad}(R\bar{x}) = \mathrm{ad}(x = y) = 0\\
&\mathrm{ad}(\neg\varphi) = \mathrm{ad}(\Diamond\varphi) = \mathrm{ad}(\forall x\varphi) = \mathrm{ad}(\varphi)\\
&\mathrm{ad}(\varphi \wedge \psi) = \max(\mathrm{ad}(\varphi), \mathrm{ad}(\psi))\\
&\mathrm{ad}(\mathcal{Q}\bar{\bar{x}}\bar{\varphi}) = \max(\mathrm{ad}(\varphi_1), \ldots, \mathrm{ad}(\varphi_k))\\
&\mathrm{ad}(\uparrow\varphi) = \max(\mathrm{ad}(\varphi) - 1, 0)\\
&\mathrm{ad}(\downarrow\varphi) = \mathrm{ad}(\varphi) + 1
\end{aligned}$$

Now we can define a formula to be *arrow-closed* if its arrow degree is 0. By induction on the complexity of formulas, it can be proven that for all Kripke models $\mathfrak{M}$, worlds w, assignments a for $\mathfrak{M}$, and sequences of worlds s, t of length $\geq \mathrm{ad}(\varphi)$ that agree on the last $\mathrm{ad}(\varphi)$ elements, $\mathfrak{M}, w, s, a \vDash \varphi$ if and only if $\mathfrak{M}, w, t, a \vDash \varphi$. As the special case for $\mathrm{ad}(\varphi) = 0$, it follows that arrow-closure does want we want it to do: Truth of an arrow-closed formula is preserved under any variation in the sequence of world. So we can define *sentences* to be formulas that are closed and arrow-closed. Let $L_{\Box\mathbb{Q}}$ be the set of these sentences.

With this, we can define truth of a sentence in a Kripke model as indicated above:

$$\mathfrak{M} \vDash \varphi \text{ iff } \mathfrak{M}, @, s, a \vDash \varphi \text{ for all sequences } s \text{ and assignments } a$$

With the sentences, the semantic structures, and the truth relation defined, we can put our logics in the abstract form introduced above. As mentioned, Williamson uses the class of Kripke structures P satisfying constraint (P). So to discuss his results in the context of quantified modal logics with generalized quantifiers, we define a logic $\mathcal{L}^{\mathsf{P}}_{\Box\mathbb{Q}} = \langle L_{\Box\mathbb{Q}}, \mathsf{P}, \vDash\rangle$ for every set of generalized quantifiers $\mathbb{Q}$. With this, we can immediately apply the notions of validity, consequence, and relative expressivity to these logics, as they were defined for any logic in this form. But before going on, there are two things to note concerning validity and consequence:

Firstly, these notions differ from the ones used in Williamson (2010), since according to Williamson's definition, formulas that are not sentences can be valid and stand in the consequence relation. However, this is not important, since this is neither technically needed for his results, nor philosophically relevant (see Williamson (2010, p. 705) for the latter). Secondly, the re-use of the symbol for truth as the symbol for validity and consequence will introduce some ambiguity, as we will later discuss logics using different classes of Kripke models. Hence I will add the class of Kripke models to the symbol for validity and consequence, e.g., writing $\vDash^{\mathsf{P}} \varphi$ for φ being a valid sentence of $\mathcal{L}^{\mathsf{P}}_{\Box\mathbb{Q}}$. Similarly, stating that two sentences are equivalent will depend on the class of Kripke models considered, as being equivalent is being true in the same structures of the relevant class. However, to keep things readable, I will rely on context to disambiguate in this case.

Now that we have set up logics with generalized quantifiers in which we can discuss Williamson's results, we have to specify analogs of the formal claims he uses to argue that necessitists can draw more distinctions than contingentists in second-order logic. On the one hand, to show that necessitists can draw the distinctions contingentists can draw, Williamson proves that there is a function $\cdot^{Con}$ that maps every sentence to a neutral sentence that is equivalent to the first, given contingentism. On the other hand, to show that there are some distinctions only the necessitist can draw, he shows that there is a sentence such that no neutral sentence is equivalent to the first, given necessitism. These conditions can be applied in the context of a logic with generalized quantifiers as well, except that we have to specify what it means for a sentence of such a logic to be neutral.

In all the languages he considers, Williamson defines neutrality as being equivalent to φ^{Con} for some formula φ, where $\cdot^{Con}$ just restricts predications and quantifiers to C, as explained above. If the generalized quantifiers are such that $\mathcal{L}_{\mathbb{Q}}$ relativizes, there is a natural extension of this definition, using the relativization of each generalized quantifier to restrict the quantificational domain to the chunky things; and in fact, we will use exactly this construction in Proposition 12 below. But if $\mathcal{L}_{\mathbb{Q}}$ does not relativize, this strategy no longer works, although we would still like to differentiate between neutral and non-neutral sentences. We could solve this problem by allowing sentences of $L_{\mathbb{Q}^{\mathrm{rel}}}$ in the condition for neutrality, but we can also solve it in a more general way by replacing Williamson's syntactic criterion of neutrality by a semantic one.

As described in Williamson (2010, p. 675), the motivation behind the syntactic definition of neutrality is the idea that a contingentist and a necessitist do not disagree on what is chunky, and how chunky things are; at least, they do not disagree on this *qua* being a contingentist and a necessitist. So a sentence is neutral if and only if it only says something about what the domain of the chunky is and the properties and relations of chunky things. Williamson captures this syntactically, by requiring neutral sentences to be equivalent to sentences in which all predications and quantifiers are restricted to C. But it can also be captured semantically, by requiring the truth of neutral sentences to be invariant under changes between Kripke models that agree on the chunky things and their properties and relations.

To make this formally precise, define two Kripke models $\mathfrak{M}, \mathfrak{M}' \in \mathsf{P}$ to *chunky-coincide* if they have the same set of worlds W, the same actual world, and their interpretation functions i and i' agree on the chunky things, in the sense that $i(R)(w) \cap i(C)(w)^n = i'(R)(w) \cap i'(C)(w)^n$ for all relation symbols R and $w \in W$. Note that $i(C) = i'(C)$ follows from the case in which R is C. Now we can define a sentence in $L_{\Box\mathbb{Q}}$ to be *neutral* if for any two chunky-coinciding Kripke models $\mathfrak{M}, \mathfrak{M}' \in \mathsf{P}$, $\mathfrak{M} \vDash \varphi$ if and only if $\mathfrak{M}' \vDash \varphi$.

The adequacy of this definition is confirmed by the fact that for all sets of generalized quantifiers for which $\mathcal{L}_{\mathbb{Q}}$ relativizes, the semantic criterion just defined is equivalent to the natural extension of Williamson's syntactic criterion sketched above, as we will see in Proposition 13 below. In particular, this result implies that the two definitions are equivalent in the first-order case considered by Williamson.

With this model-theoretic definition of neutrality, we have made precise what it means for contingentists and necessitists to be able to map every sentence in $L_{\Box\mathbb{Q}}$ to a neutral one that is equivalent, given the other theory. We introduce the following notation for this:

Con $\lhd_{\mathbb{Q}}$ *Nec* iff for all $\varphi \in L_{\Box\mathbb{Q}}$ there is a neutral $\psi \in L_{\Box\mathbb{Q}}$ such that Aux[Con] $\models^{\mathsf{P}}$ $\varphi \leftrightarrow \psi$.

Nec $\lhd_{\mathbb{Q}}$ *Con* iff for all $\varphi \in L_{\Box\mathbb{Q}}$ there is a neutral $\psi \in L_{\Box\mathbb{Q}}$ such that Aux[Nec] $\models^{\mathsf{P}}$ $\varphi \leftrightarrow \psi$.

With this, we can discuss question (Q2), and state and prove things about the analogs of Williamson's results in the context of quantified modal logics with generalized quantifiers. But we also want to consider question (Q1), and say something about the older discussion concerning the relative expressivity of logics with outer and inner quantifiers. In this discussion, one distinguishes two ways of defining the semantics of the ordinary existential and universal quantifiers on Kripke structures. One is the definition for $\forall$ and $\exists$ given above, where the quantifiers range over the domain of the world of evaluation. These are called *inner quantifiers*. On the other definition, universal and existential quantification is interpreted as ranging over the outer domain of the Kripke model. These are called *outer quantifiers*. Analogously, we can distinguish two ways of evaluating generalized quantifiers in Kripke models, as we will see below. To distinguish them syntactically, I will keep using $\forall$ and $\exists$ only as inner quantifiers, and introduce Π and Σ for the universal and existential outer quantifiers. Similarly, for any generalized quantifier $\mathcal{Q}$, I will use $\mathcal{Q}^O$ in the syntax to mark that the quantifier is interpreted as an outer quantifier. We therefore define a quantified modal language just as before, except that $\forall$ is replaced by Π, and any generalized quantifier $\mathcal{Q}$ is replaced by $\mathcal{Q}^O$. Call the set of sentences in this language $L_{\Box\mathbb{Q}^O}$.

This language is also interpreted on Kripke models, and truth is relativized to a world, a sequence of worlds, and an assignment as well. The only semantic difference lies in the interpretation clauses for outer quantifiers. For any Kripke model $\mathfrak{M} = \langle W, D, d, i, @\rangle$, they are as follows:

$$\mathfrak{M}, w, s, a \models \Pi x\varphi \text{ iff } \mathfrak{M}, w, s, a[o/x] \models \varphi \text{ for all } o \in D$$
$$\mathfrak{M}, w, s, a \models \mathcal{Q}^O\bar{\bar{x}}\bar{\varphi} \text{ iff } \mathcal{Q}_D\left(\varphi_1(\bar{x}_1)_O^{\mathfrak{M},w,s,a}, \ldots, \varphi_k(\bar{x}_k)_O^{\mathfrak{M},w,s,a}\right)$$

where $\varphi_j(\bar{x}_j)_O^{\mathfrak{M},w,s,a} = \{\bar{o} \in D^{n_j} : \mathfrak{M}, w, s, a[\bar{o}/\bar{x}_j] \models \varphi_j\}$.

To discuss the relative expressivity of logics with inner and outer quantifiers, we have to use the class PD of Kripke models as the semantics, rather than the class P. So we also require that Kripke models satisfy (D), which means that their outer domain is the union of the domains of all worlds. We do so because this assumption is usually made in the discussion of possibilism and actualism. And making the assumption does make a difference: At least in the case of first-order quantifiers, i.e., if we choose $\mathbb{Q} = \emptyset$, it is often claimed that the logics with inner and outer quantifiers are equally expressive. But if there are Kripke models in which something in the outer domain is not in the domain of any world, the sentence $\Sigma x\Box\neg x = x$ distinguishes these Kripke models from the others, and it is clear that this class cannot be delineated using any sentence of $L_{\Box\mathbb{Q}}$. Hence to make sure that the question of relative expressivity remains interesting, we have to restrict ourselves to Kripke models satisfying (D) as well here.

So we define two logics, one with inner quantifiers, which is supposed to stand in some relation with actualism and which I call $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}}$, and another with outer quantifiers, which is supposed to stand in some relation with possibilism and which I call $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O}$. Let $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}} = \langle L_{\Box\mathbb{Q}}, \mathsf{PD}, \models\rangle$ and $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O} = \langle L_{\Box\mathbb{Q}O}, \mathsf{PD}, \models\rangle$. It should be noted that just replacing inner with outer quantifiers is not the most natural formal representation of possibilism. E.g., since identity statements are only true of objects in the domain of the world of evaluation, the sentence $\Pi x(x = x)$ is not valid in $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O}$. But possibilists may not be prepared to give up such basic laws of identity. A more natural formalization will therefore be discussed in section 3.3, and shown to give equivalent results. To keep things as simple as possible, we stick with the variants of Williamson's logic defined above for the equivalence theorem.

2 The Equivalence Theorem

We can now investigate questions (Q1) and (Q2) for the range of extensions of Williamson's first-order quantified modal logic by generalized quantifiers. The first question concerns the relation between the ability of contingentists and necessitists to map sentences to neutral sentences that are equivalent, given the other theory, and the relative expressivity of logics with inner and outer quantifiers. Since the mappings used by Williamson are very similar to the mappings used to prove the equivalence in expressivity of inner and outer quantification in first-order modal logic, it is natural to conjecture that the two are equivalent for any set of generalized quantifiers $\mathbb{Q}$. That is, it is natural to conjecture that a mapping for necessitists exists if and only if the logic with outer quantifiers is at least as expressive as the logic with inner quantifiers, and a mapping for contingentists exists if and only if the logic with inner quantifiers is at least as expressive as the logic with outer quantifiers. We will see below that this conjecture is correct.

The second question concerns the existence of mappings for necessitists and contingentists in different quantified modal logics. Using the notation introduced above, we would like to know for any set of generalized quantifiers $\mathbb{Q}$ whether *Con* $\lhd_{\mathbb{Q}}$ *Nec* and *Nec* $\lhd_{\mathbb{Q}}$ *Con* hold. From Williamson's definitions of mappings for necessitists in first- and second-order quantified modal logic, it is clear that the crucial feature of a logic that enables necessitists to define their mapping is the ability to restrict its quantifiers to a predicate. To capture this ability, I introduced the notion of relativizing. Therefore, the natural conjecture is that a mapping for necessitists exists in a logic $\mathcal{L}^{\mathsf{P}}_{\Box\mathbb{Q}}$ if and only if $\mathcal{L}_{\mathbb{Q}}$ relativizes, and we will see below that this is correct. When is there a mapping for contingentists? It is clear that there is one if every member of $\mathbb{Q}$ is first-order definable. The interesting question is whether this is also a necessary condition, and we will see below that this is in fact the case.

To complete the picture, we add to these results the well-known facts that $\mathcal{L}_{\mathbb{Q}}$ relativizes if every member of $\mathbb{Q}$ is first-order definable, but not *vice versa*. Establishing all this will answer questions (Q1) and (Q2) for the restricted range of logics considered here. We can summarize the results to be proven with the following theorem:

Theorem 1 (Equivalence Theorem)**.** *For any set $\mathbb{Q}$ of generalized quantifiers, (A1)–(A3) are mutually equivalent, (B1)–(B3) are mutually equivalent, and the* *latter imply the former, while the former do not imply the latter:*

(A1)	$\mathit{Con} \lhd_{\mathbb{Q}} \mathit{Nec}$	*(B1)*	$\mathit{Nec} \lhd_{\mathbb{Q}} \mathit{Con}$
(A2)	$\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}} \preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O}$	*(B2)*	$\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O} \preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}}$
(A3)	$\mathcal{L}_{\mathbb{Q}}$ *relativizes*	*(B3)*	$\mathcal{L}_{\mathbb{Q}} \preceq \mathcal{L}_{\emptyset}$

We will prove this by establishing two circles of implications in parallel; first from 1 to 2, then from 2 to 3, and finally from 3 to 1. That (B1)–(B3) imply (A1)–(A3), but not *vice versa*, follows with these implications from the fact that (B3) implies (A3), but not *vice versa*.

2.1 From 1 to 2

The central idea behind the proof of the implications from (A1) to (A2) and (B1) to (B2) is that the information represented by any contingentist model (a Kripke model in P satisfying Aux[Con]) can be represented in a Kripke model in PD using inner quantification without the chunkiness predicate C, and *vice versa*. Similarly, the information represented by any necessitist model (a Kripke model in P satisfying Aux[Nec]) can be represented in a Kripke model in PD using outer quantification without the chunkiness predicate C, and *vice versa*. This will be made precise and proven below, in terms of mappings between such models and appropriate formulas. These results will enable us to transfer any mapping witnessing $\mathit{Con} \lhd_{\mathbb{Q}} \mathit{Nec}$ to a mapping witnessing $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}} \preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O}$, and analogously any mapping witnessing $\mathit{Nec} \lhd_{\mathbb{Q}} \mathit{Con}$ to a mapping witnessing $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O} \preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}}$. Before going through the formal proof, let me give an illustration of the four different representations of information according to contingentism, necessitism, actualism and possibilism, where actualism and possibilism are represented by logics with inner and outer quantifiers. In the following example, we have two worlds, and we represent things that are in the domain of a world, in the quantifier range, or chunky by vertical lines. In the left world, there are some (chunky) things, but in the right world, there are more (chunky) things. This simple picture of modal space is represented in different ways according to the four positions, but as Figure 1 shows, none of them adds or omits any information.

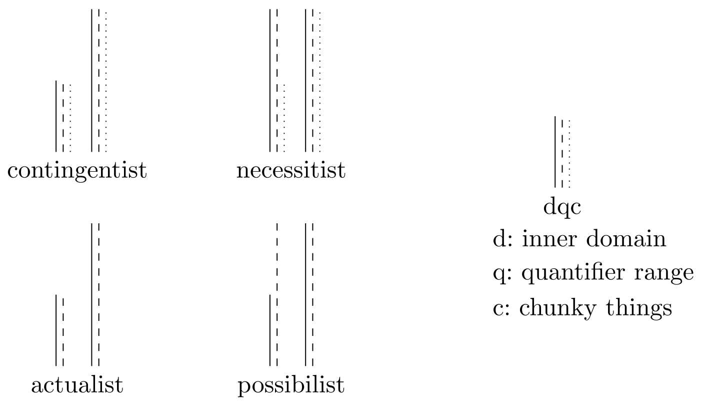

Figure 1: Variants of a Kripke model

I will now define the mappings in detail. We have to go from contingentism to actualism and *vice versa*, and similarly from necessitism to possibilism and *vice versa*. In each direction, we need a mapping for formulas as well as a corresponding mapping for Kripke models. Hence we require eight mappings. They will be denoted by two letters, the first letters of the position from which we map and the position to which we map. E.g., $\mathfrak{M}^{CA}$ is the actualist version of a contingentist model $\mathfrak{M}$. What a contingentist says about $\mathfrak{M}$ with φ, an actualist says about $\mathfrak{M}^{CA}$ with φ^{CA}. We also have to be explicit about signatures, since we will have maps between languages in different signatures – contingentism and necessitism use a *chunky* predicate, which actualism and possibilism don't use. I will use τ for an arbitrary signature, and write $L_1[\tau]$ for the set of sentences in L_1 of signature τ, and "$L_1[\tau]$-formulas" for the corresponding set of formulas. Similarly, I write $\mathsf{X}[\tau]$ for the Kripke models of signature τ in X. Finally, I write τC for the signature that results from adding a new unary predicate C to τ, and $i|\tau$ for the result of restricting the function i to τ. Note that when specifying a recursive mapping from one language to another, I only specify the non-trivial recursion clauses.

To map from contingentism to actualism, we just have to replace predications of chunkiness by statements about self-identity in any formula, and remove the interpretation of C from the interpretation function in any Kripke model, while setting the outer domain to the union of the domains of all worlds to ensure that the resulting Kripke model is in PD:

$.^{CA}$: $L_{\Box\mathbb{Q}}[\tau C]$-formulas$\to L_{\Box\mathbb{Q}}[\tau]$-formulas:

$$(Cx)^{CA} = (x = x)$$

$.^{CA}$: $\mathsf{P}[\tau C] \to \mathsf{PD}[\tau]$:
$\langle W, D, d, i, @\rangle^{CA} = \langle W, D^{CA}, d, i^{CA}, @\rangle$, where $D^{CA} = \bigcup_{v\in W} d(v)$ and $i^{CA} = i|\tau$

To map from actualism to contingentism, we do not have to change any formulas, and can just add the interpretation of C according to the domain function in any Kripke model:

$.^{AC}$: $L_{\Box\mathbb{Q}}[\tau]$-formulas$\to L_{\Box\mathbb{Q}}[\tau C]$-formulas:
the identity function

$.^{AC}$: $\mathsf{PD}[\tau] \to \mathsf{P}[\tau C]$:
$\langle W, D, d, i, @\rangle^{AC} = \langle W, D, d, i^{AC}, @\rangle$, where $i^{AC}|\tau = i$ and $i^{AC}(C) = d$

To map from necessitism to possibilism, we have to replace predications of chunkiness by statements about self-identity in any formula, as well as replacing identity statements by statements of possible identity and inner quantifiers by outer quantifiers. Models are treated as in the mapping from contingentism to actualism, except that we have to set the domain function to the interpretation of C:

$.^{NP}$: $L_{\Box\mathbb{Q}}[\tau C]$-formulas$\rightarrow L_{\Box\mathbb{Q}O}[\tau]$-formulas:

$$(Cx)^{NP} = (x = x)$$
$$(x = y)^{NP} = \Diamond x = y$$
$$(\forall x\psi)^{NP} = \Pi x\psi^{NP}$$
$$\left(\mathcal{Q}\bar{\bar{x}}\bar{\psi}\right)^{NP} = \mathcal{Q}^{O}\bar{\bar{x}}\bar{\psi}^{NP}$$

$.^{NP}$: $\mathsf{P}[\tau C] \rightarrow \mathsf{K}[\tau]$:

$\langle W, D, d, i, @\rangle^{NP} = \langle W, D^{NP}, d^{NP}, i^{NP}, @\rangle$, where $D^{NP} = \bigcup_{v\in W} d(v)$, $d^{NP} = i(C)$ and $i^{NP} = i|\tau$

Note that if $\mathfrak{M} \in \mathsf{P}[\tau C]$ satisfies Aux[Nec], then $\mathfrak{M}^{NP} \in \mathsf{PD}[\tau]$.

To map from possibilism to necessitism, we have to add a predication of chunkiness to any identity statement in any formula, as well as replacing outer quantifiers by inner quantifiers. Models are treated by adding the interpretation of C according to the domain function, and setting the new domain function to be the constant function to the outer domain:

$.^{PN}$: $L_{\Box\mathbb{Q}O}[\tau]$-formulas$\rightarrow L_{\Box\mathbb{Q}}[\tau C]$-formulas:

$$(x = y)^{PN} = (x = y \wedge Cx)$$
$$(\Pi x\psi)^{PN} = \forall x\psi^{PN}$$
$$\left(\mathcal{Q}^{O}\bar{\bar{x}}\bar{\psi}\right)^{PN} = \mathcal{Q}\bar{\bar{x}}\bar{\psi}^{PN}$$

$.^{PN}$: $\mathsf{PD}[\tau] \rightarrow \mathsf{P}[\tau C]$:

$\langle W, D, d, i, @\rangle^{PN} = \langle W, D, d^{PN}, i^{PN}, @\rangle$, where $d^{PN}(w) = D$ for all $w \in W$, $i^{PN}|\tau = i$ and $i^{PN}(C) = d$

First of all, we have to show that these mappings work correctly, in the sense that truth of a sentence in a Kripke model is invariant under mapping both the sentence and the Kripke model with corresponding mappings. More formally, we need the following lemma:

Lemma 2. *Let τ be a signature and $\mathfrak{M} \in \mathsf{K}$.*

(i) If $\mathfrak{M} \in \mathsf{P}[\tau C]$ and $\mathfrak{M} \vDash$ Aux[Con] then for all $\varphi \in L_{\Box\mathbb{Q}}[\tau C]$: $\mathfrak{M}^{CA} \vDash \varphi^{CA}$ if and only if $\mathfrak{M} \vDash \varphi$.

(ii) If $\mathfrak{M} \in \mathsf{PD}[\tau]$ then for all $\varphi \in L_{\Box\mathbb{Q}}[\tau]$: $\mathfrak{M}^{AC} \vDash \varphi^{AC}$ if and only if $\mathfrak{M} \vDash \varphi$.

(iii) If $\mathfrak{M} \in \mathsf{P}[\tau C]$ and $\mathfrak{M} \vDash$ Aux[Nec] then for all $\varphi \in L_{\Box\mathbb{Q}}[\tau C]$: $\mathfrak{M}^{NP} \vDash \varphi^{NP}$ if and only if $\mathfrak{M} \vDash \varphi$.

(iv) If $\mathfrak{M} \in \mathsf{PD}[\tau]$ then for all $\varphi \in L_{\Box\mathbb{Q}O}[\tau]$: $\mathfrak{M}^{PN} \vDash \varphi^{PN}$ if and only if $\mathfrak{M} \vDash \varphi$.

Proof. By inductions on the complexity of formulas. □

With this lemma, we can prove the implication from (A1) to (A2) by transferring any mapping witnessing $Con \lhd_{\mathbb{Q}} Nec$ to a mapping witnessing $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}} \preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O}$, and similarly for the implication from (B1) to (B2). To do so, we need the definitions of two mappings on Kripke models defined in Williamson (2010, pp. 728–729, 731). Let $\mathfrak{M} = \langle W, D, d, i, @\rangle$ be a Kripke model in P. Then $\mathfrak{M}^{Con} = \langle W, D, d^C, i^C, @\rangle$, where $d^C = i(C)$ and $i^C(R)(w) = i(R)(w) \cap i(C)(w)^n$ for every $w \in W$ and relation symbol R. Further, $\mathfrak{M}^{Nec} = \langle W, D, d^N, i^C, @\rangle$, where $d^N(w) = \bigcup_{v \in W} i(C)(v)$ for all $w \in W$, and i^C is as above.

Proposition 3 ((A1)⇒(A2))**.** *If $Con \lhd_{\mathbb{Q}} Nec$ then $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}} \preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O}$.*

Proof. Assume that $Con \lhd_{\mathbb{Q}} Nec$ and consider any $\varphi \in L_{\Box\mathbb{Q}}[\tau]$. Since $Con \lhd_{\mathbb{Q}} Nec$, there is a neutral $\psi \in L_{\Box\mathbb{Q}}[\tau C]$ such that $\text{Aux[Con]} \vDash^{\mathsf{P}} \varphi^{AC} \leftrightarrow \psi$. We prove that $\vDash^{\mathsf{PD}} \varphi \leftrightarrow \psi^{NP}$. So consider any $\mathfrak{M} \in \mathsf{PD}[\tau]$. By Lemma 2 (ii), $\mathfrak{M} \vDash \varphi$ if and only if $\mathfrak{M}^{AC} \vDash \varphi^{AC}$. $\mathfrak{M}^{AC} \vDash \text{Aux[Con]}$, so $\mathfrak{M}^{AC} \vDash \varphi^{AC}$ if and only if $\mathfrak{M}^{AC} \vDash \psi$. $\mathfrak{M}^{AC}$ and $\left(\mathfrak{M}^{AC}\right)^{Nec}$ chunky-coincide and ψ is neutral, so the latter is the case if and only if $\left(\mathfrak{M}^{AC}\right)^{Nec} \vDash \psi$. Since by Williamson (2010, appendix, 1.18), $\left(\mathfrak{M}^{AC}\right)^{Nec} \vDash \text{Aux[Nec]}$, it follows from Lemma 2 (iii) that this in turn is the case if and only if $\left(\left(\mathfrak{M}^{AC}\right)^{Nec}\right)^{NP} \vDash \psi^{NP}$. As $\left(\left(\mathfrak{M}^{AC}\right)^{Nec}\right)^{NP} = \mathfrak{M}$, this is the case if and only if $\mathfrak{M} \vDash \psi^{NP}$. Together, it follows from these equivalences that $\mathfrak{M} \vDash \varphi$ if and only if $\mathfrak{M} \vDash \psi^{NP}$. Since $\psi^{NP} \in L_{\Box\mathbb{Q}O}[\tau]$, this establishes that $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}} \preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O}$. □

Analogously, we can prove the implication from (B1) to (B2):

Proposition 4 ((B1)⇒(B2))**.** *If $Nec \lhd_{\mathbb{Q}} Con$ then $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O} \preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}}$.*

Proof. Analogous to the proof of Proposition 3, using Lemma 2 (i) and (iv). □

2.2 From 2 to 3

We first prove the implication from (A2) to (A3) by contraposition; i.e., assuming that $\mathcal{L}_{\mathbb{Q}}$ does not relativize, we show that $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}} \not\preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O}$. I will first sketch the proof strategy for a single generalized quantifier $\mathcal{Q}$ of type $\langle 1\rangle$: If $\mathcal{L}_{\mathcal{Q}}$ does not relativize, then $\mathcal{Q}^{\text{rel}}$ is not definable in $\mathcal{L}_{\mathcal{Q}}$. By the semantics of inner and outer generalized quantifiers, $\mathcal{Q}x\Diamond Vx$ is equivalent to $\left(\mathcal{Q}^{\text{rel}}\right)^O yx(y = y, \Diamond Vx)$. But if there is a sentence of $L_{\Box\mathcal{Q}O}$ that is equivalent to the latter sentence, then we should be able to define $\mathcal{Q}^{\text{rel}}$ in $\mathcal{L}_{\mathcal{Q}}$. So $\mathcal{Q}x\Diamond Vx$ has no equivalent in $L_{\Box\mathcal{Q}O}$, which means that $\mathcal{L}^{\mathsf{PD}}_{\Box\mathcal{Q}} \not\preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathcal{Q}O}$. To rigorously prove that $\mathcal{Q}x\Diamond Vx$ has no equivalent in $L_{\Box\mathcal{Q}O}$, we map any sentence in this language to one in $L_{\mathcal{Q}}$. Since $\mathcal{Q}^{\text{rel}}$ is not definable in $\mathcal{L}_{\mathcal{Q}}$, we can find a model showing that the latter sentence does not define $\mathcal{Q}^{\text{rel}}$. We can now map this model to a Kripke model which witnesses that the original sentence is not equivalent to $\mathcal{Q}x\Diamond Vx$.

I start making this proof idea precise by defining these mappings for any generalized quantifier $\mathcal{Q}$ of type $\langle n_1, \ldots, n_k\rangle$. Let τ be the signature containing only relation symbols $V_1, \ldots, V_k$ of arities $n_1, \ldots, n_k$, and let U be a unary predicate. For any model $\mathfrak{A} \in \mathsf{M}[\tau U]$, define $\mathfrak{A}^{01} = \langle \{0, 1\}, |\mathfrak{A}|, d, i, 0\rangle$, where $d(0) = U^{\mathfrak{A}}$, $d(1) = |\mathfrak{A}|$, $i(V_j)(0) = \emptyset$ and $i(V_j)(1) = V_j^{\mathfrak{A}}$ for all $j \leq k$. Note that $\mathfrak{A}^{01} \in \mathsf{PD}[\tau]$. Further, we define a mapping $\cdot^{w,s}$ from $L_{\Box\mathbb{Q}O}[\tau]$-formulas to $L_{\mathbb{Q}}[\tau U]$-formulas for any sequence s and element w in $\{0, 1\}$ by simultaneous induction:

$$
\begin{aligned}
&(V_j\bar{x})^{0,s} = \bot && (V_j\bar{x})^{1,s} = V_j\bar{x}\\
&(x=y)^{0,s} = (x=y \wedge Ux) && (x=y)^{1,s} = (x=y)\\
&(\Diamond\varphi)^{w,s} = \varphi^{0,s} \vee \varphi^{1,s}\\
&(\uparrow\varphi)^{w,s} = \varphi^{w,s^\frown w}\\
&(\downarrow\varphi)^{w,s^\frown v} = \varphi^{v,s}\\
&(\downarrow\varphi)^{w,\langle\rangle} = \varphi^{w,\langle\rangle}\\
&(\Pi x\varphi)^{w,s} = \forall x(\varphi^{w,s})\\
&\left(\mathcal{Q}^O\bar{\bar{x}}\bar{\varphi}\right)^{w,s} = \mathcal{Q}\bar{\bar{x}}(\bar{\varphi}^{w,s})
\end{aligned}
$$

To prove the implication, we need a lemma on these mappings:

Lemma 5. *For any $\mathfrak{A} \in \mathsf{M}[\tau U]$ and $\varphi \in L_{\Box\mathbb{Q}O}[\tau]$,*

$\mathfrak{A} \vDash \varphi^{0,\langle\rangle}$ *if and only if* $\mathfrak{A}^{01} \vDash \varphi$.

Proof. By induction on the complexity of formulas. □

Proposition 6 ((A2)⇒(A3)). *If $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}} \preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O}$ then $\mathcal{L}_{\mathbb{Q}}$ relativizes.*

Proof. Assume that $\mathcal{L}_{\mathbb{Q}}$ does not relativize. Then there is a $\mathcal{Q} \in \mathbb{Q}$ such that $\mathcal{Q}^{\text{rel}}$ is not definable in $\mathcal{L}_{\mathbb{Q}}$. We will show that $\mathcal{Q}\bar{\bar{x}}(\Diamond V_1\bar{x}_1, \ldots, \Diamond V_k\bar{x}_k)$ witnesses $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}} \not\preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O}$. So consider any $\varphi \in L_{\Box\mathbb{Q}O}[\tau]$. Since $\mathcal{Q}^{\text{rel}}$ is not definable in $\mathcal{L}_{\mathbb{Q}}$, there is a model $\mathfrak{A}$ such that it is not the case that $\mathfrak{A} \vDash \varphi^{0,\langle\rangle}$ if and only if $\mathcal{Q}^{\text{rel}}_{|\mathfrak{A}|}\left(U^{\mathfrak{A}}, V_1^{\mathfrak{A}}, \ldots, V_k^{\mathfrak{A}}\right)$. Lemma 5 states that $\mathfrak{A} \vDash \varphi^{0,\langle\rangle}$ if and only if $\mathfrak{A}^{01} \vDash \varphi$. Furthermore, $\mathcal{Q}^{\text{rel}}_{|\mathfrak{A}|}\left(U^{\mathfrak{A}}, V_1^{\mathfrak{A}}, \ldots, V_k^{\mathfrak{A}}\right)$ if and only if $\mathfrak{A}^{01} \vDash \mathcal{Q}\bar{\bar{x}}(\Diamond V_1\bar{x}_1, \ldots, \Diamond V_k\bar{x}_k)$. Therefore it is not the case that $\mathfrak{A}^{01} \vDash \varphi$ if and only if $\mathfrak{A}^{01} \vDash \mathcal{Q}\bar{\bar{x}}(\Diamond V_1\bar{x}_1, \ldots, \Diamond V_k\bar{x}_k)$. Since φ was chosen arbitrarily among $L_{\Box\mathbb{Q}O}[\tau]$, it follows that $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}} \not\preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O}$. □

We prove the implication from (B2) to (B3) by contraposition as well. So assuming that $\mathcal{L}_{\mathbb{Q}} \not\preceq \mathcal{L}_{\emptyset}$, we show that $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O} \not\preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}}$. As above, I will first sketch the proof strategy: We map any sentence φ witnessing $\mathcal{L}_{\mathbb{Q}} \not\preceq \mathcal{L}_{\emptyset}$ to a sentence $\varphi^{\Diamond} \in L_{\Box\mathbb{Q}O}$, and prove that it has no equivalent in $L_{\Box\mathbb{Q}}$. To do so, we use the fact that for any $m \in \mathbb{N}$, there are models $\mathfrak{A}$ and $\mathfrak{B}$ that satisfy the same sentences up to quantifier rank m, but evaluate φ differently. We map these models to Kripke models $\mathfrak{A}^n$ and $\mathfrak{B}^n$, for which we show that $\mathfrak{A}/\mathfrak{B}$ satisfies φ if and only if $\mathfrak{A}^n/\mathfrak{B}^n$ satisfies $\varphi^{\Diamond}$. Further, we prove that $\mathfrak{A}^n$ and $\mathfrak{B}^n$ satisfy the same sentences in $L_{\Box\mathbb{Q}}$ up to a certain modal degree determined by m and n, by showing that if $\mathfrak{A}$ and $\mathfrak{B}$ are related by a back-and-forth system, then $\mathfrak{A}^n$ and $\mathfrak{B}^n$ are also related by such a system. Since any sentence in $L_{\Box\mathbb{Q}}$ has a finite modal degree, we can find Kripke models $\mathfrak{A}^n$ and $\mathfrak{B}^n$ witnessing that no such sentence is equivalent to $\varphi^{\Diamond}$.

Carrying out this proof strategy in a general way crucially relies on two ideas. The first is the construction of the Kripke models $\mathfrak{A}^n$ and the corresponding mapping $\cdot^{\Diamond}$ between formulas, which are adapted from Williamson (2010, appendix 3). The second is the idea of using back-and-forth systems to prove that two structures satisfy the same sentences up to a certain complexity. Such systems are common in many branches of logic; see Hodges (1997, section 3.2) for variants for first-order logic, which are sometimes called *potential* or *partial isomorphisms*, or *Ehrenfeucht-Fraïssé games*, when presented in game-theoretic form; see Blackburn et al. (2001, section 2.2) for a variant for propositional modal logic, which they call *bisimulations*; and Peters and Westerståhl (2006, section 13.3) for applications of such systems to results on the definability of generalized quantifiers. Back-and-forth systems are rarely defined for quantified modal logic; van Benthem (2010, p. 123) contains one of many ways of doing so, for which he uses the term *world-object bisimulations.* I will use a definition which is tailor-made for this proof.

First, we define the mappings $\cdot^{\Diamond}$ and $\cdot^{n}$. $\cdot^{\Diamond}$ maps $L_{\mathbb{Q}}$-formulas to $L_{\Box\mathbb{Q}^O}$-formulas, by prefixing predications with $\Diamond$-symbols and replacing inner by outer quantifiers:

$$(R\bar{x})^{\Diamond} = \Diamond R\bar{x}$$
$$(x = y)^{\Diamond} = \Diamond x = y$$
$$(\forall x\varphi)^{\Diamond} = \Pi x\varphi^{\Diamond}$$
$$(\mathcal{Q}\bar{\bar{x}}\bar{\varphi})^{\Diamond} = \mathcal{Q}^O\bar{\bar{x}}\bar{\varphi}^{\Diamond}$$

For any $n \in \mathbb{N}$ and model $\mathfrak{A}$, define $\mathfrak{A}^n = \langle W, |\mathfrak{A}|, d, i, \emptyset\rangle$, where $W = \{S \subseteq |\mathfrak{A}| : |S| \leq n\}$, $d(w) = w$, and $i(R)(w) = R^{\mathfrak{A}} \cap d(w)^n$ for all relation symbols R and $w \in W$. In analogy to Williamson (2010, appendix, 3.1), we can prove that the mappings preserve truth in the following sense:

Lemma 7. *For all $n \in \mathbb{N}$, models $\mathfrak{A}$ and $\varphi \in L_{\mathbb{Q}}$ such that all relation symbols occurring in φ have arity $\leq n$,*

$$\mathfrak{A} \vDash \varphi \text{ if and only if } \mathfrak{A}^n \vDash \varphi^{\Diamond}.$$

Proof. By induction on the complexity of formulas. □

The next notions that need to be defined are those of quantifier rank and modal degree, which measure the complexity of formulas by determining the depth of the deepest nesting of quantifiers or $\Box$ symbols. We define the *quantifier rank* $\mathrm{qr}(\varphi)$ of an $L_{\emptyset}$-formula φ inductively as follows:

$$\mathrm{qr}(R\bar{x}) = \mathrm{qr}(x = y) = 0$$
$$\mathrm{qr}(\neg\varphi) = \mathrm{qr}(\varphi)$$
$$\mathrm{qr}(\varphi \wedge \psi) = \max(\mathrm{qr}(\varphi), \mathrm{qr}(\psi))$$
$$\mathrm{qr}(\forall x\varphi) = \mathrm{qr}(\varphi) + 1$$

The *modal degree* $\mathrm{md}(\varphi)$ of an $L_{\Box\mathbb{Q}}$-formula φ is defined similarly:

$$\mathrm{md}(R\bar{x}) = \mathrm{md}(x = y) = 0$$
$$\mathrm{md}(\neg\varphi) = \mathrm{md}(\uparrow\varphi) = \mathrm{md}(\downarrow\varphi) = \mathrm{md}(\forall x\varphi) = \mathrm{md}(\varphi)$$
$$\mathrm{md}(\varphi \wedge \psi) = \max(\mathrm{md}(\varphi), \mathrm{md}(\psi))$$
$$\mathrm{md}(\Box\varphi) = \mathrm{md}(\varphi) + 1$$
$$\mathrm{md}(\mathcal{Q}\bar{\bar{x}}\bar{\varphi}) = \max(\mathrm{md}(\varphi_1), \ldots, \mathrm{md}(\varphi_k))$$

Using these notions, we introduce two useful pieces of notation, with which we can express that two structures satisfy the same sentences up to a certain complexity. Let $l \in \mathbb{N}$. For any models $\mathfrak{A}, \mathfrak{B}$, we write $\mathfrak{A} \equiv^l_\emptyset \mathfrak{B}$ if for all sentences $\varphi \in L_\emptyset$ such that $\mathrm{qr}(\varphi) \leq l$, $\mathfrak{A} \vDash \varphi$ if and only if $\mathfrak{B} \vDash \varphi$. Similarly, for any Kripke models $\mathfrak{M}, \mathfrak{N} \in \mathsf{PD}$, we write $\mathfrak{M} \equiv^l_{\Box\mathbb{Q}} \mathfrak{N}$ if for all sentences $\varphi \in L_{\Box\mathbb{Q}}$ such that $\mathrm{md}(\varphi) \leq l$, $\mathfrak{M} \vDash \varphi$ if and only if $\mathfrak{N} \vDash \varphi$.

To define back-and-forth systems, we need some additional definitions: For models $\mathfrak{A}$ and $\mathfrak{B}$, we write $f : \mathfrak{A} \cong \mathfrak{B}$ if $f : |\mathfrak{A}| \to |\mathfrak{B}|$ is an *isomorphism*; i.e., if f is a bijection such that for any relation symbol R, $\bar{o} \in R^{\mathfrak{A}}$ if and only if $f(\bar{o}) \in R^{\mathfrak{B}}$. Further, for any model $\mathfrak{A}$ and $S \subseteq |\mathfrak{A}|$, we write $\mathfrak{A}|S$ for the *submodel* of $\mathfrak{A}$ based on S; i.e., the unique model $\mathfrak{B}$ with $|\mathfrak{B}| = S$ and $R^{\mathfrak{B}} = R^{\mathfrak{A}} \cap S^n$ for all relation symbols R. For every Kripke model $\mathfrak{M} = \langle W, D, d, i, @\rangle$ in P and world $w \in W$, we define the *inner model* of $\mathfrak{M}$ at w, written $\mathfrak{M}_w$, to be $\langle d(w), e\rangle$, where $e(R) = i(R)(w)$ for all relation symbols R.

We start with back-and-forth systems for the logic $\mathcal{L}_\emptyset$. Let $\mathfrak{A}, \mathfrak{B}$ be models. A *partial isomorphism from* $\mathfrak{A}$ *to* $\mathfrak{B}$ is a partial injection from $|\mathfrak{A}|$ to $|\mathfrak{B}|$ such that $f : \mathfrak{A}|\mathrm{dom}(f) \cong \mathfrak{B}|\mathrm{im}(f)$, where $\mathrm{dom}(f)$ and $\mathrm{im}(f)$ are the domain and the image of f. For any $l \in \mathbb{N}$, a *back-and-forth system from* $\mathfrak{A}$ *to* $\mathfrak{B}$ *of length* l is a sequence $I = \langle I_j : 0 \leq j \leq l\rangle$ of non-empty sets of partial isomorphisms from $\mathfrak{A}$ to $\mathfrak{B}$ such that for all j such that $0 < j \leq l$:

(i) $I_j \subseteq I_{j-1}$.

(ii) For all $f \in I_j$ and $o \in |\mathfrak{A}|$, there is a $g \supseteq f$ such that $g \in I_{j-1}$ and $o \in \mathrm{dom}(g)$.

(iii) For all $f \in I_j$ and $o' \in |\mathfrak{B}|$, there is a $g \supseteq f$ such that $g \in I_{j-1}$ and $o' \in \mathrm{im}(g)$.

If I is a back-and-forth system from $\mathfrak{A}$ to $\mathfrak{B}$ of length l, we write $I : \mathfrak{A} \cong^l \mathfrak{B}$. We write $\mathfrak{A} \cong^l \mathfrak{B}$ if there is an I such that $I : \mathfrak{A} \cong^l \mathfrak{B}$.

We need a similar definition of back-and-forth systems for logics of the form $\mathcal{L}_{\Box\mathbb{Q}}$. Let $\mathfrak{M} = \langle W, D, d, i, @\rangle$ and $\mathfrak{M}' = \langle W', D', d', i', @'\rangle$ be Kripke models. A *partial isomorphism from* $\mathfrak{M}$ *to* $\mathfrak{M}'$ is a tuple $\langle \sigma, \rho\rangle$ such that σ is a partial injection from W to W' and ρ is a partial injection from D to D' such that $\sigma(@) = @'$ and for all $w \in \mathrm{dom}(\sigma)$, $\rho|d(w) : \mathfrak{M}_w \cong \mathfrak{M}'_{\sigma(w)}$. For any $l \in \mathbb{N}$, a *back-and-forth system from* $\mathfrak{M}$ *to* $\mathfrak{M}'$ *of length* l is a sequence $I = \langle I_j : 0 \leq j \leq l\rangle$ of non-empty sets of partial isomorphisms from $\mathfrak{M}$ to $\mathfrak{M}'$ such that for all j such that $0 < j \leq l$:

(i) $I_j \subseteq I_{j-1}$.

(ii) For all $\langle \sigma, \rho\rangle \in I_j$ and $w \in W$, there is a $\sigma' \supseteq \sigma$ and a $\rho' \supseteq \rho$ such that $\langle \sigma', \rho'\rangle \in I_{j-1}$ and $w \in \mathrm{dom}(\sigma')$.

(iii) For all $\langle \sigma, \rho\rangle \in I_j$ and $w' \in W'$, there is a $\sigma' \supseteq \sigma$ and a $\rho' \supseteq \rho$ such that $\langle \sigma', \rho'\rangle \in I_{j-1}$ and $w' \in \mathrm{im}(\sigma')$.

If I is a back-and-forth system from $\mathfrak{M}$ to $\mathfrak{M}'$ of length l, we write $I : \mathfrak{M} \cong^l \mathfrak{M}'$. We write $\mathfrak{M} \cong^l \mathfrak{M}'$ if there is an I such that $I : \mathfrak{M} \cong^l \mathfrak{M}'$.

We can prove that Kripke models related by back-and-forth systems satisfy the same sentences whose modal degree does not exceed the length of the back-and-forth system. More concisely:

Lemma 8. *If* $\mathfrak{M} \cong^l \mathfrak{M}'$ *then* $\mathfrak{M} \equiv^l_{\Box\mathbb{Q}} \mathfrak{M}'$.

Proof. By induction on the complexity of formulas. □

The final lemma that is needed for this proof shows that being related by a back-and-forth system transfers in a certain way from $\mathfrak{A}$ and $\mathfrak{B}$ to $\mathfrak{A}^n$ and $\mathfrak{B}^n$:

Lemma 9. *If* $\mathfrak{A} \cong^{ln} \mathfrak{B}$ *then* $\mathfrak{A}^n \cong^l \mathfrak{B}^n$.

Proof. Let $\mathfrak{A}, \mathfrak{B}$ be models and $I = \langle I_j : 0 \leq j \leq ln \rangle$ be a back-and-forth system from $\mathfrak{A}$ to $\mathfrak{B}$ of length ln. We use this to construct a back-and-forth system from $\mathfrak{A}^n$ to $\mathfrak{B}^n$ of length l: For any partial isomorphism f from $\mathfrak{A}$ to $\mathfrak{B}$, define $f^{l,n} = \langle \sigma_f, f \rangle$, where $\sigma_f(w) = \{f(o) : o \in w\}$ for all $w \subseteq \mathrm{dom}(f)$ such that $|w| \leq n$. Define $I^{l,n} = \langle I^{l,n}_j : 0 \leq j \leq l \rangle$, where $I^{l,n}_j = \{f^{l,n} : f \in I_{jn}\}$. By checking the conditions of back-and-forth systems, we can prove that $I^{l,n} : \mathfrak{A}^n \cong^l \mathfrak{B}^n$. □

Proposition 10 ((B2)⇒(B3))**.** *If* $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O} \preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}}$ *then* $\mathcal{L}_{\mathbb{Q}} \preceq \mathcal{L}_{\emptyset}$.

Proof. By contraposition. Assume that $\mathcal{L}_{\mathbb{Q}} \not\preceq \mathcal{L}_{\emptyset}$. It follows by the Fraïssé-Hintikka Theorem (see Hodges (1997, p. 84, Theorem 3.3.2)) that there is a $\varphi \in L_{\mathbb{Q}}$ such that for all $m \in \mathbb{N}$, there are models $\mathfrak{A}, \mathfrak{B}$ such that $\mathfrak{A} \vDash \varphi$, $\mathfrak{B} \nvDash \varphi$, and $\mathfrak{A} \equiv^m_{\emptyset} \mathfrak{B}$. To show that $\varphi^{\Diamond}$ has no equivalent in $L_{\Box\mathbb{Q}}$, consider any $\psi \in L_{\Box\mathbb{Q}}$. Let n be the maximal arity of the relation symbols occurring in ψ and $l = \mathrm{md}(\psi)$. Then there are structures $\mathfrak{A}, \mathfrak{B}$ such that $\mathfrak{A} \vDash \varphi$, $\mathfrak{B} \nvDash \varphi$, and $\mathfrak{A} \equiv^{ln}_{\emptyset} \mathfrak{B}$. Since $\mathfrak{A} \equiv^{ln}_{\emptyset} \mathfrak{B}$, by a theorem of Carol Karp (see Karp (1965, Theorem 1)), $\mathfrak{A} \cong^{ln} \mathfrak{B}$, which by Lemma 9 implies $\mathfrak{A}^n \cong^l \mathfrak{B}^n$. By Lemma 8, $\mathfrak{A}^n \equiv^l_{\Box\mathbb{Q}} \mathfrak{B}^n$, so $\mathfrak{A}^n \vDash \psi$ if and only if $\mathfrak{B}^n \vDash \psi$. By Lemma 7, the fact that $\mathfrak{A} \vDash \varphi$ and $\mathfrak{B} \nvDash \varphi$ implies that $\mathfrak{A}^n \vDash \varphi^{\Diamond}$ and $\mathfrak{B}^n \nvDash \varphi^{\Diamond}$. Since $\varphi^{\Diamond} \in L_{\Box\mathbb{Q}O}$, ψ was chosen arbitrarily from $L_{\Box\mathbb{Q}}$, and $\mathfrak{A}^n, \mathfrak{B}^n \in \mathsf{PD}$, it follows that $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O} \not\preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}}$. □

This proof could be simplified by replacing $\varphi^{\Diamond}$ with $\mathcal{Q}^O\bar{\bar{x}}(\Diamond V_1\bar{x}_1, \ldots, \Diamond V_k\bar{x}_k)$ for a generalized quantifier $\mathcal{Q}$ that is not first-order definable, similar to the proof of Proposition 6. However, the proof method used here is more interesting as it indicates how it can be adapted to other extensions of first-order logic.

2.3 From 3 to 1

For the implications in this section, we need a lemma stating that relations of relative expressivity transfer from first-order logics with generalized quantifiers to the corresponding quantified modal logics with generalized quantifiers:

Lemma 11. *If* $\mathcal{L}_{\mathbb{Q}} \preceq \mathcal{L}_{\mathbb{Q}'}$ *then* $\mathcal{L}^{\mathsf{P}}_{\Box\mathbb{Q}} \preceq \mathcal{L}^{\mathsf{P}}_{\Box\mathbb{Q}'}$.

Proof. If $\mathcal{L}_{\mathbb{Q}} \preceq \mathcal{L}_{\mathbb{Q}'}$, then for every $\mathcal{Q} \in \mathbb{Q}$, there is a sentence $\varphi_{\mathcal{Q}} \in \mathcal{L}_{\mathbb{Q}'}[V_1 \ldots V_k]$ that is equivalent to $\mathcal{Q}\bar{\bar{x}}(V_1\bar{x}_1, \ldots, V_k\bar{x}_k)$. Let $\psi \in \mathcal{L}^{\mathsf{P}}_{\Box\mathbb{Q}}$. If we replace any subformula starting with a generalized quantifier $\mathcal{Q} \in \mathbb{Q}$ in ψ by $\varphi_{\mathcal{Q}}$, while replacing $V_1\bar{x}_1, \ldots, V_k\bar{x}_k$ in $\varphi_{\mathcal{Q}}$ by the appropriate subformulas of ψ, then we get a sentence in $\mathcal{L}^{\mathsf{P}}_{\Box\mathbb{Q}'}$ which is equivalent to ψ, as we can prove by induction on the complexity of formulas. This establishes that $\mathcal{L}^{\mathsf{P}}_{\Box\mathbb{Q}} \preceq \mathcal{L}^{\mathsf{P}}_{\Box\mathbb{Q}'}$. □

Proposition 12 ((A3)⇒(A1))**.** *If* $\mathcal{L}_{\mathbb{Q}}$ *relativizes then Con* $\lhd_{\mathbb{Q}}$ *Nec.*

Proof. We first extend the mapping $\cdot^{Con}$ from $L_{\Box\emptyset}$-formulas to $L_{\Box\emptyset}$-formulas defined in Williamson (2010, appendix 1, p. 728) to a mapping $\cdot^{RCon}$ from $L_{\Box\mathbb{Q}}$-formulas to $L_{\Box\mathbb{Q}^{\text{rel}}}$-formulas, by adding the following clause for any $\mathcal{Q} \in \mathbb{Q}$:

$$(\mathcal{Q}\bar{\bar{x}}\bar{\varphi})^{RCon} = \mathcal{Q}^{\text{rel}} y \bar{\bar{x}} \left(Cy, \varphi_1^{RCon}, \ldots, \varphi_k^{RCon}\right)$$

We can prove that $\mathfrak{M} \vDash \varphi^{RCon}$ if and only if $\mathfrak{M}^{Con} \vDash \varphi$ by extending the induction on the complexity of formulas in Williamson (2010, appendix, 1.1) by a clause for generalized quantifiers.

If $\mathcal{L}_{\mathbb{Q}}$ relativizes, then $\mathcal{L}_{\mathbb{Q}^{\text{rel}}} \preceq \mathcal{L}_{\mathbb{Q}}$, and so by Lemma 11, $\mathcal{L}^{\mathsf{P}}_{\Box\mathbb{Q}^{\text{rel}}} \preceq \mathcal{L}^{\mathsf{P}}_{\Box\mathbb{Q}}$. Hence for any $\varphi \in L_{\Box\mathbb{Q}}$, there is a sentence $\psi \in L_{\Box\mathbb{Q}}$ that is equivalent to φ^{RCon}. Define $\cdot^{Con} : L_{\Box\mathbb{Q}} \to L_{\Box\mathbb{Q}}$ to be a function that maps every φ to such a ψ. By choice, $\mathfrak{M} \vDash \varphi^{Con}$ if and only if $\mathfrak{M}^{Con} \vDash \varphi$ for all $\mathfrak{M} \in \mathsf{P}$. With the fact that $\mathfrak{M} \vDash \text{Aux[Con]}$ implies $\mathfrak{M} = \mathfrak{M}^{Con}$, shown in Williamson (2010, appendix, 1.7), it follows that $\text{Aux[Con]} \vDash^{\mathsf{P}} \varphi \leftrightarrow \varphi^{Con}$. That φ^{Con} is neutral follows from the fact that for any chunky-coinciding Kripke models $\mathfrak{M}, \mathfrak{N} \in \mathsf{P}$, $\mathfrak{M}^{Con}$ and $\mathfrak{N}^{Con}$ differ at most in the outer domain, which is irrelevant for the truth of sentences in $L_{\Box\mathbb{Q}}$. So $Con \lhd_{\mathbb{Q}} Nec$. □

Using the extension of $\cdot^{Con}$ in this proof, we can now show that if Williamson's syntactic condition of neutrality can be extended to a set of generalized quantifiers $\mathbb{Q}$, it coincides with the model-theoretic condition used here:

Proposition 13. *Assume that $\mathcal{L}_{\mathbb{Q}}$ relativizes, and let $\cdot^{Con}$ be as in the proof of Proposition 12. Then any $\varphi \in L_{\Box\mathbb{Q}}$ is neutral if and only if there is a $\psi \in L_{\Box\mathbb{Q}}$ such that $\vDash^{\mathsf{P}} \varphi \leftrightarrow \psi^{Con}$.*

Proof. First, assume that φ is neutral, and consider any $\mathfrak{M} \in \mathsf{P}$. Since $\mathfrak{M}$ and $\mathfrak{M}^{Con}$ chunky-coincide, $\mathfrak{M} \vDash \varphi$ if and only if $\mathfrak{M}^{Con} \vDash \varphi$. Since $\mathfrak{M}^{Con} \vDash \varphi$ if and only if $\mathfrak{M} \vDash \varphi^{Con}$, it follows that $\mathfrak{M} \vDash \varphi$ if and only if $\mathfrak{M} \vDash \varphi^{Con}$.

Now assume that there is a $\psi \in L_{\Box\mathbb{Q}}$ such that $\vDash^{\mathsf{P}} \varphi \leftrightarrow \psi^{Con}$, and let $\mathfrak{M}, \mathfrak{N} \in \mathsf{P}$ chunky-coincide. Then, as noted above, $\mathfrak{M}^{Con} \vDash \psi$ if and only if $\mathfrak{N}^{Con} \vDash \psi$. Therefore $\mathfrak{M} \vDash \psi^{Con}$ if and only if $\mathfrak{N} \vDash \psi^{Con}$, from which it follows that $\mathfrak{M} \vDash \varphi$ if and only if $\mathfrak{N} \vDash \varphi$. □

Proposition 14 ((B3)⇒(B1))**.** *If $\mathcal{L}_{\mathbb{Q}} \preceq \mathcal{L}_{\emptyset}$ then $Nec \lhd_{\mathbb{Q}} Con$.*

Proof. By Lemma 11 and the fact that $Nec \lhd_{\emptyset} Con$, which follows from Williamson (2010, appendix, 1.15 and 1.21) and Proposition 13. □

This concludes the proof of the equivalence theorem.

3 What Does It Show?

Now that we have proven the equivalence theorem, we can ask: What does it show? What does this formal result tell us about the metaphysical dispute on modal ontology? Naturally, we have to combine the formal result with some philosophical assumptions to arrive at some interesting conclusions. In the first part of this section, I will extend some of the philosophical conclusions that have been drawn in the literature on similar technical results. Roughly, the upshot will be that in some way, the result supports necessitism or possibilism. The subsequent parts of this section will be devoted to different objections or replies contingentists or actualists might make.

3.1 Necessitism, Possibilism, and Verbal Disputes

In the debate on necessitism and contingentism, the equivalence theorem substantiates Williamson's result on second-order logic: For every set of generalized quantifiers $\mathbb{Q}$ such that $\mathcal{L}_{\mathbb{Q}}$ is more expressive than first-order logic, there is a sentence that has no neutral equivalent, given necessitism. E.g., if it contains the infinity quantifier $\mathcal{Q}_0$ introduced in section 1.2, then the following sentence has no neutral equivalent, given necessitism:

(ω) $\mathcal{Q}_0 x \Diamond Fx$

It says that there are infinitely many things which could be F. If Williamson's reasoning is correct, then this means that the necessitist can use this to draw a distinction the contingentist cannot draw, which is a point in favor of necessitism.

It is worth considering more concretely what the distinction is the contingentist is accused of not being able to make in this instance. As an example, let F stand for the property of being an electron, assume that there are only finitely many electrons, and assume that the contingentist takes being an electron to be an essential property. Then if the necessitist utters "There are infinitely many things that could be electrons", the contingentist will dispute this, since there are only finitely many electrons, and according to them, what is not an electron could not be one. However, it seems that there is some claim the necessitist makes with their utterance, which, although stated in a way that assumes necessitism, is a metaphysical claim that is independent of the ontological dispute between necessitism and contingentism.

To bring this out more clearly, we can consider finitary analogs of (ω). (Similar considerations for (34) are described in Williamson (2010, pp. 710–711).) For every natural number n, let $\exists_{\geq n}$ be the generalized quantifier formalizing "there are at least n things such that ...". With this, we can write down the following finitary analog of (ω):

(n) $\exists_{\geq n} x \Diamond Fx$

Since $\exists_{\geq n}$ is first-order definable, it follows from the equivalence theorem that there is a sentence of first-order quantified modal logic that is neutral and equivalent to (n), given necessitism. This means that contingentists are faced with the following situation: For any natural number n, if a necessitist claims that there are n objects that could be electrons, they can disentangle a metaphysical question from the necessitist's utterance they can engage with, but they can no longer do so if the necessitist claims that there are infinitely many such objects. But if there is a distinction involved that is intelligible to the contingentist in the former case, why shouldn't there be such a distinction involved in the latter case?

A natural response to this challenge is to allow for conjunctions of infinite sets of formulas in the formal syntax, as the conjunction of neutral equivalents, given necessitism, to instances of (n) is a neutral equivalent, given necessitism, to (ω). However, this strategy is specific to the generalized quantifier $\mathcal{Q}_0$, and we will see in section 3.4 that such infinitary devices do not provide a general solution, even if quantification over infinite sets of variables is allowed.

Besides providing more examples for Williamson's claim that some distinctions can only be drawn under the assumption of necessitism, the equivalence theorem and the techniques used to prove it strengthen Williamson's argument in a number of ways. Firstly, some of these examples are simpler than the ones Williamson gives; e.g, compare (ω) with (34) in section 1.1. Secondly, Williamson's result on second-order logic makes the artificial assumption that there is no world in which infinitely many things are chunky. The proof used here for the equivalence theorem can be modified so as to allow for Kripke models that do not make this assumption. The relevant parts of the proof can also be extended to quantified modal logic with plural second-order quantifiers while still dispensing with this assumption. Thirdly, the equivalence theorem shows that Williamson's result does not trade on any peculiarity of plural second-order logic. So even if there are philosophical problems with plural second-order quantifiers, unless they apply to generalized quantifiers as well, contingentism is still faced with the same kind of troubles.

In the debate on possibilism and actualism, the equivalence theorem can be used in arguments in favor of possibilism rather than actualism. E.g., Kit Fine and Saul Kripke have argued against certain arguments for possibilism using the fact that first-order quantified modal logic is as expressive with inner quantifiers as with outer quantifiers. (See Fine (1977, p. 156), Kripke (1983, p. 488) and Fine (2003, section 4); Fine refers to Prior (1967, pp. 149–151) and Kripke to Hazen (1976).) The equivalence theorem shows that this expressivity result does not extend to generalized quantifiers, since for generalized quantifiers that are not first-order definable, quantified modal logic is not as expressive with inner quantifiers as with outer quantifiers. Although this may not show that possibilism is correct, it certainly casts doubt on the viability of Fine's and Kripke's reply to these arguments for possibilism.

Another application of the equivalence theorem concerns the position that the metaphysical dispute at hand – whether put in terms of contingentism and necessitism or actualism and possibilism – is *merely verbal*. Arguments in favor of a metaphysical dispute being verbal are sometimes based on the claim that the different positions in the dispute just come down to the use of different, intertranslatable ways of speaking, e.g., in Hirsch (2009). In the present case, the equivalence theorem indicates that such arguments cannot be given for the position that the dispute between contingentism and necessitism or actualism and possibilism is merely verbal, since it shows that statements made by proponents of the two positions are in fact not intertranslatable – at least not in the languages considered here. (See Williamson (2010, p. 671, fn. 14) for a similar observation concerning second-order logic.)

It seems likely that the equivalence theorem can be extended to a wide range of tense logics, which would mean that the consequences just stated also apply to the temporal analog of the dispute on modal ontology. The temporal analogs of actualism and possibilism are called *presentism* and *eternalism*, and Williamson (2012) calls the analogs of contingentism and necessitism *temporaryism* and *permanentism*. It is interesting to note that in the discussion of presentism and eternalism, generalized quantifiers have already been used in ways that are similar to what I am suggesting, both for the criticism of presentism, as well as for the defense of the substantiality of the dispute about temporal ontology. As an example for criticisms of presentism, Lewis (2004, pp. 6–7) has argued that it is difficult for a presentist to analyze sentences such as "There have been infinitely many kings named John". For an example of a defense of the substantiality of the dispute about temporal ontology, see Sider (2006, pp. 91–

92), who uses sentences such as "Half the objects from all of time that are Ks are Ls" as examples of statements for which a translation from eternalist to presentist discourse is difficult to find. Both of these generalized quantifiers are not first-order definable, so the equivalence theorem provides a way of formally substantiating their arguments.

I now turn to ways in which contingentists and actualists might try to counter these applications of the equivalence theorem.

3.2 Relativizing

As I have presented it above, the equivalence theorem points in favor of necessitism or possibilism, since for any set of generalized quantifiers $\mathbb{Q}$ such that $\mathcal{L}_{\mathbb{Q}}$ is more expressive than first-order logic, contingentists cannot map every sentence to a neutral one that is equivalent to the first, given necessitism, and the logic with inner quantifiers is not as expressive as the logic with outer quantifiers. But contingentists or actualists may point to the other part of the equivalence theorem and ask: What if $\mathcal{L}_{\mathbb{Q}}$ does not relativize? Then the logic with outer quantifiers is also not as expressive as the logic with inner quantifiers, and necessitists cannot map every sentence to a neutral one that is equivalent to the first, given contingentism. Doesn't this show that necessitists and contingentists or possibilists and actualists are in a similar position?

The first thing to note is that, as the equivalence theorem states, whenever first-order logic is as expressive as $\mathcal{L}_{\mathbb{Q}}$, the latter relativizes. Therefore, if the logic with inner quantifiers is as expressive as the logic with outer quantifiers, or contingentists can map every sentence to a neutral one that is equivalent given necessitism, then the logic with outer quantifiers is as expressive as the logic with inner quantifiers, and necessitists can map every sentence to a neutral one that is equivalent given contingentism. So whenever contingentists or actualists don't have a problem, neither do necessitists or possibilists.

This leaves the cases where $\mathcal{L}_{\mathbb{Q}}$ does not relativize. One might ask: Although these cases might be problematic for contingentists or actualists as well, why shouldn't they be problematic for necessitists or possibilists? The reason is that for any set of generalized quantifiers $\mathbb{Q}$, the set of generalized quantifiers $\mathbb{Q}^{\text{rel}}$ is such that $\mathcal{L}_{\mathbb{Q}^{\text{rel}}}$ is at least as expressive as $\mathcal{L}_{\mathbb{Q}}$, and $\mathcal{L}_{\mathbb{Q}^{\text{rel}}}$ relativizes. (See Peters and Westerståhl (2006, p. 454, Facts 5 and 6).) So when considering a set of generalized quantifiers $\mathbb{Q}$ for which $\mathcal{L}_{\mathbb{Q}}$ does not relativize, necessitists or possibilists can claim that this set is unnaturally restrictive. Note that such an answer is unavailable to contingentists or actualists; if $\mathcal{L}_{\mathbb{Q}'}$ is at least as expressive as $\mathcal{L}_{\mathbb{Q}}$ and $\mathcal{L}_{\mathbb{Q}}$ is more expressive than first-order logic, then $\mathcal{L}_{\mathbb{Q}'}$ is more expressive than first-order logic as well.

This can be illustrated using the generalized quantifier $\mathcal{Q}^{\mathcal{R}}$ of type $\langle 1 \rangle$ – the so-called *Rescher quantifier* first discussed in Rescher (1964) – and a generalized quantifier of type $\langle 1, 1 \rangle$ for which I will use the symbol $\mathcal{M}$. We can read $\mathcal{Q}^{\mathcal{R}}$ as formalizing the phrase "most things are ...", and $\mathcal{M}$ as formalizing "most ... are ...". Consequently, they are defined as follows:

$$\mathcal{Q}^{\mathcal{R}}{}_D(S) \text{ iff } |S| > |D \backslash S|$$
$$\mathcal{M}_D(S_1, S_2) \text{ iff } |S_1 \cap S_2| > |S_1 \backslash S_2|$$

As shown in Barwise and Cooper (1981, Theorem C13), $\mathcal{L}_{\mathcal{Q}^{\mathcal{R}}}$ does not relativize, but since $\mathcal{M} = \left(\mathcal{Q}^{\mathcal{R}}\right)^{\text{rel}}$, $\mathcal{L}_{\mathcal{M}}$ does relativize. So if we consider $L_{\Box\mathcal{Q}^{\mathcal{R}}}$, then there are some distinctions only contingentists can draw. An example is the one they can draw using the sentence $\mathcal{Q}^{\mathcal{R}}x\Diamond Nx$, which for the sake of concreteness, we may read as "Most things could be nice". However, by moving to the more expressive language $L_{\Box\mathcal{M}}$, necessitists can draw the same distinction using the sentence $\mathcal{M}yx(Cy, \Diamond Nx)$, which we then read as "Most chunky things could be nice". Hence necessitists might just need slightly more expressive logical resources to draw the distinctions contingentists can draw, but this doesn't seem to be a philosophically relevant point. We can argue similarly in the case of inner and outer quantifiers: Although there is no sentence in $L_{\Box\mathcal{Q}^{\mathcal{R}}O}$ with the same truth-conditions as $\mathcal{Q}^{\mathcal{R}}x\Diamond Nx$, there is one in $L_{\Box\mathcal{M}O}$, namely $\mathcal{M}^{O}yx(y = y, \Diamond Nx)$.

3.3 Without the Being Constraint

Actualists could also object to the use of the equivalence theorem in arguing for possibilism by claiming that the logics $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}}$ and $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O}$ are not the correct formal representations of actualism and possibilism. Indeed, in comparing the expressivity of these logics, we considered a question that is slightly different from one that is often discussed in the literature on actualism and possibilism, e.g. in Forbes (1989) and Correia (2007). These authors consider a semantics in which the being constraint (P) is not enforced. That is, they allow Kripke models in which it is not the case that $i(R)(w) \subseteq d(w)$. Since they allow objects outside the domain of a world to have properties there, they assume that $x = y$ is true in a world under an assignment that maps these variables to the same object, even if it isn't in the domain of the world. Finally, they assume that there is a unary logical predicate E expressing *existence*, whose semantics is given by the domain function. To make sure that this objection by the actualist is not effective, I will now show that the equivalence theorem can be extended to these logics. I will first give the formal definitions, which I will mark by indexing the appropriate symbols by $*$.

Syntactically, these logics differ from the ones considered before only in containing an additional logical unary predicate E. I use $L_{\Box\mathbb{Q}*}$ and $L_{\Box\mathbb{Q}*O}$ for the languages obtained from adding this predicate to the languages $L_{\Box\mathbb{Q}}$ and $L_{\Box\mathbb{Q}O}$. The truth relation is defined as above, except for a new truth-condition for E and an amended truth-condition for $=$:

$\mathfrak{M}, w, s, a \vDash^{*} Ex$ iff $a(x) \in d(w)$

$\mathfrak{M}, w, s, a \vDash^{*} x = y$ iff $a(x) = a(y)$

All other operators are interpreted as before, including inner and outer quantifiers. As models, we use the class of Kripke models D satisfying the requirement that the outer domain is the union of the domains of all worlds. With this, we can define the new logics as follows: $\mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*} = \langle L_{\Box\mathbb{Q}*}, \mathsf{D}, \vDash^{*}\rangle$ and $\mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*O} = \langle L_{\Box\mathbb{Q}*O}, \mathsf{D}, \vDash^{*}\rangle$. We show that it makes no difference whether these logics are considered instead of those used in the equivalence theorem:

Proposition 15. *For any set $\mathbb{Q}$ of generalized quantifiers, (A2) if and only if (A2*), and (B2) if and only if (B2*):*

(A2)	$\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}} \preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O}$	*(B2)*	$\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O} \preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}}$
(A2)*	$\mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*} \preceq \mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*O}$	*(B2*)*	$\mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*O} \preceq \mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*}$

We prove this by showing that (A2*) follows from (A3) and implies (A2), and analogously, that (B2*) follows from (B3) and implies (B2).

We start with the implications from 3 to 2*. As in section 2.3, we need a lemma which states that relations of relative expressivity transfer from first-order logics with generalized quantifiers to the corresponding quantified modal logics with generalized quantifiers:

Lemma 16. *If* $\mathcal{L}_{\mathbb{Q}} \preceq \mathcal{L}_{\mathbb{Q}'}$ *then* $\mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*O} \preceq \mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}'*O}$.

Proof. Analogous to the proof of Lemma 11. □

Proposition 17 ((A3)⇒(A2*))**.** *If* $\mathcal{L}_{\mathbb{Q}}$ *relativizes then* $\mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*} \preceq \mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*O}$.

Proof. We can show that $\mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*} \preceq \mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}^{\mathrm{rel}}*O}$ using the following mapping $\cdot^{E}$ from $L_{\Box\mathbb{Q}*}$-formulas to $L_{\Box\mathbb{Q}^{\mathrm{rel}}*O}$-formulas:

$$(\forall x\varphi)^{E} = \Pi x \left(Ex \to \varphi^{E}\right)$$
$$(\mathcal{Q}\bar{\bar{x}}\bar{\varphi})^{E} = \left(\mathcal{Q}^{\mathrm{rel}}\right)^{O} y\bar{\bar{x}} \left(Ey, \bar{\varphi}^{E}\right)$$

If $\mathcal{L}_{\mathbb{Q}}$ relativizes, then $\mathcal{L}_{\mathbb{Q}^{\mathrm{rel}}} \preceq \mathcal{L}_{\mathbb{Q}}$, so by Lemma 16, $\mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}^{\mathrm{rel}}*O} \preceq \mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*O}$. It follows by the transitivity of $\preceq$ that $\mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*} \preceq \mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*O}$. □

Proposition 18 ((B3)⇒(B2*))**.** *If* $\mathcal{L}_{\mathbb{Q}} \preceq \mathcal{L}_{\emptyset}$ *then* $\mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*O} \preceq \mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*}$.

Proof. We can show that $\mathcal{L}^{\mathsf{D}}_{\Box\emptyset*O} \preceq \mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*}$ using the following mapping $\cdot^{\uparrow\downarrow}$ from $L_{\Box\emptyset*O}$-formulas to $L_{\Box\mathbb{Q}*}$-formulas:

$$(\Pi x\varphi)^{\uparrow\downarrow} = \uparrow\Box\forall x \downarrow\varphi^{\uparrow\downarrow}$$

If $\mathcal{L}_{\mathbb{Q}} \preceq \mathcal{L}_{\emptyset}$, then by Lemma 16, $\mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*O} \preceq \mathcal{L}^{\mathsf{D}}_{\Box\emptyset*O}$. It follows by the transitivity of $\preceq$ that $\mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*O} \preceq \mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*}$. □

Now we prove the implications from 2* to 2. For this, we define two mappings. The first, $\cdot^{+*}$, maps $L_{\Box\mathbb{Q}(O)}$-formulas to $L_{\Box\mathbb{Q}*(O)}$-formulas:

$$(x = y)^{+*} = (x = y \wedge Ex)$$

The second, $\cdot^{-*}$, maps $L_{\Box\mathbb{Q}*(O)}$-formulas to $L_{\Box\mathbb{Q}(O)}$-formulas:

$$(Ex)^{-*} = (x = x)$$
$$(x = y)^{-*} = \Diamond x = y$$

We prove that when considering Kripke models in PD, they both map sentences to equivalent ones:

Lemma 19. *For all* $\mathfrak{M} \in \mathsf{PD}$*:*

(i) For all $\varphi \in L_{\Box\mathbb{Q}}$ *or* $\varphi \in L_{\Box\mathbb{Q}O}$*:* $\mathfrak{M} \vDash \varphi$ *if and only if* $\mathfrak{M} \vDash^{*} \varphi^{+*}$.

(ii) For all $\varphi \in L_{\Box\mathbb{Q}*}$ *or* $\varphi \in L_{\Box\mathbb{Q}*O}$*:* $\mathfrak{M} \vDash^{*} \varphi$ *if and only if* $\mathfrak{M} \vDash \varphi^{-*}$.

Proof. By inductions on the complexity of formulas. □

Proposition 20 ((A2*)⇒(A2))**.** *If* $\mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*} \preceq \mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*O}$ *then* $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}} \preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O}$.

Proof. Assume that $\mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*} \preceq \mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*O}$. Consider any $\varphi \in L_{\Box\mathbb{Q}}$. Then there is a $\psi \in L_{\Box\mathbb{Q}*O}$ such that $\mathfrak{M} \vDash^* \psi$ if and only if $\mathfrak{M} \vDash^* \varphi^{+*}$ for all $\mathfrak{M} \in \mathsf{D}$, and so in particular for all $\mathfrak{M} \in \mathsf{PD}$. By Lemma 19, for all $\mathfrak{M} \in \mathsf{PD}$, $\mathfrak{M} \vDash \varphi$ if and only if $\mathfrak{M} \vDash^* \varphi^{+*}$, and $\mathfrak{M} \vDash^* \psi$ if and only if $\mathfrak{M} \vDash \psi^{-*}$. Hence for all $\mathfrak{M} \in \mathsf{PD}$, $\mathfrak{M} \vDash \varphi$ if and only if $\mathfrak{M} \vDash \psi^{-*}$. Since $\psi^{-*} \in L_{\Box\mathbb{Q}O}$, $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}} \preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O}$. $\Box$

Proposition 21 ((B2*)⇒(B2))**.** *If* $\mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*O} \preceq \mathcal{L}^{\mathsf{D}}_{\Box\mathbb{Q}*}$ *then* $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O} \preceq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}}$.

Proof. Analogous to the proof of Proposition 20. $\Box$

This concludes the proof of the result that (A2*) is equivalent to (A2) and (B2*) is equivalent to (B2). Hence as far as the equivalence theorem and its philosophical implications for actualism and possibilism is concerned, it makes no difference which versions of the logics we consider.

3.4 Infinitary Logics

Confronted with problems like the ones posed by the equivalence theorem, it is natural for actualists to turn to infinitary languages, as it is done, e.g., in Fine (1977, p. 158) and Fine (2003, pp. 226–227). Considering the analogous strategy for contingentism, Williamson (2010, section 10) argues that the specific infinitary constructions proposed by Fine are problematic from a philosophical perspective, and that more standard infinitary constructions do not solve the contingentist's problems with second-order logic. In particular, he considers logics in which conjunctions can range over arbitrarily large sets of formulas, and first- as well as second-order quantifiers can range over arbitrarily large sets of variables. He shows that in the case of second-order logic, the result that some sentences have no neutral equivalent, given necessitism, can be extended to such infinitary variants.

We can ask how the results on generalized quantifiers fare in such a context. Most of the implications in the proof of the equivalence theorem can easily be extended to such infinitary logics; in particular, this is straightforward for the implications from 1 to 2. It is clear that the most interesting implication for the contingentist or actualist is the one which shows that for any set of generalized quantifiers $\mathbb{Q}$ such that $\mathcal{L}_{\mathbb{Q}}$ is more expressive than first-order logic, the logic with inner quantifiers is not as expressive as the logic with outer quantifiers. We might ask whether this still holds in the context of infinitary logic. I will now show that the implication still holds, by proving that in this setting, *no* logic with inner quantifiers is as expressive as the corresponding logic with outer quantifiers. Surprisingly, this means that introducing infinitary devices leaves contingentists and actualists in a worse position than before, since even if no generalized quantifiers are used, there will be sentences that have no neutral equivalent, given necessitism, and sentences with outer quantifiers that have no equivalent with inner quantifiers. So not only do the infinitary constructions not help contingentists or actualists with generalized quantifiers, they introduce problems themselves.

To define the infinitary variants of $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}}$ and $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O}$, we add the syntactic rules that for any set of formulas Φ, $\bigwedge\Phi$ is a formula, and for any set of variables X, $\forall X\varphi$ or $\Pi X\varphi$ is a formula. Note that we do not extend generalized quantifiers in any way; only first-order quantifiers can now bind sets of variables. Call the resulting languages $L_{\Box\mathbb{Q}\infty\infty}$ and $L_{\Box\mathbb{Q}O\infty\infty}$. For these new constructions, we introduce the following truth-conditions:

$\mathfrak{M}, w, s, a \vDash \bigwedge \Phi$ iff $\mathfrak{M}, w, s, a \vDash \varphi$ for all $\varphi \in \Phi$

$\mathfrak{M}, w, s, a \vDash \forall X\varphi$ iff $\mathfrak{M}, w, s, a' \vDash \varphi$ for all assignments a' that agree with a on the variables not in X and for which $a'(x) \in d(w)$ for all $x \in X$

$\mathfrak{M}, w, s, a \vDash \Pi X\varphi$ iff $\mathfrak{M}, w, s, a' \vDash \varphi$ for all assignments a' that agree with a on the variables not in X and for which $a'(x) \in D$ for all $x \in X$

Define $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}\infty\infty} = \langle L_{\Box\mathbb{Q}\infty\infty}, \mathsf{PD}, \vDash\rangle$ and $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O\infty\infty} = \langle L_{\Box\mathbb{Q}O\infty\infty}, \mathsf{PD}, \vDash\rangle$.

To show that $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O\infty\infty} \not\leq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}\infty\infty}$ holds for any set of generalized quantifiers $\mathbb{Q}$, we show that there is a sentence φ_1 containing only first-order outer quantifiers which is true in a Kripke model if the outer domain contains uncountably many things which have the property F in some world. We then show that there is no equivalent sentence using only inner quantifiers, no matter which generalized quantifiers are admitted. Similar to the proof in section 2.2, we use back-and-forth systems for the latter claim. I start with a definition of back-and-forth systems that is appropriate for infinitary logics.

Let $\mathfrak{M} = \langle W, D, d, i, @\rangle$ and $\mathfrak{M}' = \langle W', D', d', i', @'\rangle$ be Kripke models. A *back-and-forth system from* $\mathfrak{M}$ *to* $\mathfrak{M}'$ is a non-empty set I of partial isomorphisms from $\mathfrak{M}$ to $\mathfrak{M}'$ such that:

(i) For all $\langle\sigma, \rho\rangle \in I$ and $w \in W$, there is a $\sigma' \supseteq \sigma$ and a $\rho' \supseteq \rho$ such that $\langle\sigma', \rho'\rangle \in I$ and $w \in \mathrm{dom}(\sigma')$.

(ii) For all $\langle\sigma, \rho\rangle \in I$ and $w' \in W'$, there is a $\sigma' \supseteq \sigma$ and a $\rho' \supseteq \rho$ such that $\langle\sigma', \rho'\rangle \in I$ and $w' \in \mathrm{im}(\sigma')$.

If there is a back-and-forth system from $\mathfrak{M}$ to $\mathfrak{M}'$, we write $\mathfrak{M} \cong^{\infty} \mathfrak{M}'$. Similar to before, we can prove that Kripke models related by a back-and-forth system satisfy the same sentences. To state this, we write $\mathfrak{M} \equiv_{\Box\mathbb{Q}\infty\infty} \mathfrak{M}'$ if for all sentences $\varphi \in L_{\Box\mathbb{Q}\infty\infty}$, $\mathfrak{M} \vDash \varphi$ if and only if $\mathfrak{M}' \vDash \varphi$.

Lemma 22. *If* $\mathfrak{M} \cong^{\infty} \mathfrak{M}'$ *then* $\mathfrak{M} \equiv_{\Box\mathbb{Q}\infty\infty} \mathfrak{M}'$.

Proof. By induction on the complexity of formulas. □

For the proof of the next proposition, we use two Kripke models that will also be useful in the next section. Define $\mathfrak{N} = \langle \mathbb{N}, \mathbb{N}, d_{\mathfrak{N}}, i_{\mathfrak{N}}, 0\rangle$ and $\mathfrak{R} = \langle \mathbb{R}, \mathbb{R}, d_{\mathfrak{R}}, i_{\mathfrak{R}}, 0\rangle$, where $d_{\mathfrak{N}}(m) = \{m\}$ and $i_{\mathfrak{N}}(R)(m) = \{m\}^n$ for all $m \in \mathbb{N}$ and relation symbols R, and $d_{\mathfrak{R}}(r) = \{r\}$ and $i_{\mathfrak{R}}(R)(r) = \{r\}^n$ for all $r \in \mathbb{R}$ and relation symbols R.

Proposition 23. $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O\infty\infty} \not\leq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}\infty\infty}$.

Proof. We first define φ_1 to be the following sentence:

$$\Sigma\{x_\alpha : \alpha < \omega_1\} \left(\left(\bigwedge_{\alpha<\beta<\omega_1} \neg x_\alpha = x_\beta\right) \wedge \left(\bigwedge_{\alpha<\omega_1} \Diamond F x_\alpha\right)\right)$$

By quantifying over a set of variables of size $\aleph_1$, this sentence expresses that $|\bigcup_{w\in W} i(F)(w)| \geq \aleph_1$. We can show that $\mathfrak{N} \cong^{\infty} \mathfrak{R}$ using the back-and-forth system consisting of the partial isomorphisms $\langle\beta, \beta\rangle$ for all partial injections β from $\mathbb{N}$ to $\mathbb{R}$ with a finite domain such that $\beta(0) = 0$. By Lemma 22, it follows that $\mathfrak{N} \equiv_{\Box\mathbb{Q}\infty\infty} \mathfrak{R}$. Since $\mathfrak{N} \nvDash \varphi_1$ and $\mathfrak{R} \vDash \varphi_1$, it follows that $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O\infty\infty} \not\leq \mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}\infty\infty}$. □

3.5 Modal Extensions

As infinitary logics seem not to help them, contingentists and actualists might hope that more expressive modal operators do so. To back this up, contingentists might point out that in first-order modal logic without the modal operators $\uparrow$ and $\downarrow$, they had the same problem of not being able to draw certain distinctions, but once these operators were added, the problem disappeared, and actualists can tell a similar story about relative expressivity. Natural examples for modal operators that might provide a helpful increase in expressivity are propositional quantifiers, which are already used in Fine (1977) and Kripke (1983), or quantifiers over worlds, which are often discussed under the label of *hybrid logic*, e.g., in Areces and ten Cate (2007).

As with infinitary logics, many of the implications in the equivalence theorem can be extended to many such additions. And again, it is the proof showing that if $\mathcal{L}_{\mathbb{Q}}$ is more expressive than first-order logic then the logic with inner quantifiers is not as expressive as the logic with outer quantifiers which does not immediately generalize in all cases. E.g., while it does work for quantifiers over worlds, it doesn't work without modification for propositional quantifiers. But as with infinitary logics, the proof technique using back-and-forth systems can be adapted to accommodate propositional quantifiers. To demonstrate this, I will now show that first-order modal logic with the generalized quantifier $\mathcal{Q}_1$, which formalizes "there are uncountably many", and propositional quantifiers is not as expressive using inner quantifiers as using outer quantifiers. This shows that the problems for contingentists and actualists with generalized quantifiers cannot be solved in general by adding propositional quantifiers.

To add propositional quantifiers to $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}}$ and $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O}$, we use a countably infinite set of propositional variables, and add the syntactic rules that any propositional variable is a formula, and that for any formula φ and propositional variable p, $\forall p\varphi$ is a formula. Of course, being a sentence now also requires having no free propositional variables. Adapting notation from Fine (1970), I call the resulting languages $L_{\Box\mathbb{Q}\pi+}$ and $L_{\Box\mathbb{Q}O\pi+}$. As usual, we formalize propositions as sets of worlds; thus, assignments now also map propositional variables to sets of worlds. Taking any set of worlds to be a proposition, we get the following truth-conditions for the new constructions:

$$\mathfrak{M}, w, s, a \vDash p \text{ iff } w \in a(p)$$

$$\mathfrak{M}, w, s, a \vDash \forall p\varphi \text{ iff } \mathfrak{M}, w, s, a[P/p] \vDash \varphi \text{ for all } P \subseteq W$$

Define $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}\pi+} = \langle L_{\Box\mathbb{Q}\pi+}, \mathsf{PD}, \vDash\rangle$ and $\mathcal{L}^{\mathsf{PD}}_{\Box\mathbb{Q}O\pi+} = \langle L_{\Box\mathbb{Q}O\pi+}, \mathsf{PD}, \vDash\rangle$. Let $\mathcal{Q}_1$ be the generalized quantifier such that $\mathcal{Q}_{1D} = \{S \subseteq D : |S| \geq \aleph_1\}$. To show that $\mathcal{L}^{\mathsf{PD}}_{\Box\mathcal{Q}_1 O\pi+} \not\leq \mathcal{L}^{\mathsf{PD}}_{\Box\mathcal{Q}_1\pi+}$, we adapt the proof idea of Proposition 23, and show that the Kripke models $\mathfrak{N}$ and $\mathfrak{R}$ defined above are indistinguishable using $\mathcal{L}^{\mathsf{PD}}_{\Box\mathcal{Q}_1\pi+}$. As usual, I start by introducing an appropriate kind of back-and-forth systems.

Let $\mathfrak{M} = \langle W, D, d, i, @\rangle$ and $\mathfrak{M}' = \langle W', D', d', i', @'\rangle$ be Kripke models. A *$\pi+$ partial isomorphism from $\mathfrak{M}$ to $\mathfrak{M}'$* is a tuple $\langle \sigma, \Sigma, \rho\rangle$ such that σ is a partial injection from W to W', Σ is a partial injection from $\mathcal{P}(W)$ to $\mathcal{P}(W')$ and ρ is a partial injection from D to D' such that $\sigma(@) = @'$ and for all $w \in \text{dom}(\sigma)$ and $P \in \text{dom}(\Sigma)$, $\rho|d(w) : \mathfrak{M}_w \cong \mathfrak{M}'_{\sigma(w)}$ and $w \in P$ if and only if $\sigma(w) \in \Sigma(P)$.

A *$\pi+$ back-and-forth system from $\mathfrak{M}$ to $\mathfrak{M}'$* is a non-empty set I of $\pi+$ partial isomorphisms from $\mathfrak{M}$ to $\mathfrak{M}'$ such that:

(i) For all $\langle \sigma, \Sigma, \rho \rangle \in I$ and $w \in W$, there is a $\sigma' \supseteq \sigma$ and a $\rho' \supseteq \rho$ such that $\langle \sigma', \Sigma, \rho' \rangle \in I$ and $w \in \mathrm{dom}(\sigma')$.

(ii) For all $\langle \sigma, \Sigma, \rho \rangle \in I$ and $w' \in W'$, there is a $\sigma' \supseteq \sigma$ and a $\rho' \supseteq \rho$ such that $\langle \sigma', \Sigma, \rho' \rangle \in I$ and $w' \in \mathrm{im}(\sigma')$.

(iii) For all $\langle \sigma, \Sigma, \rho \rangle \in I$ and $P \subseteq W$, there is a $\Sigma' \supseteq \Sigma$ such that $\langle \sigma, \Sigma', \rho \rangle \in I$ and $P \in \mathrm{dom}(\Sigma')$.

(iv) For all $\langle \sigma, \Sigma, \rho \rangle \in I$ and $P' \subseteq W'$, there is a $\Sigma' \supseteq \Sigma$ such that $\langle \sigma, \Sigma', \rho \rangle \in I$ and $P' \in \mathrm{im}(\Sigma')$.

If there is a $\pi+$ back-and-forth system from $\mathfrak{M}$ to $\mathfrak{M}'$, we write $\mathfrak{M} \cong^{\pi+} \mathfrak{M}'$. Similar to before, we can prove that Kripke models related by a $\pi+$ back-and-forth system satisfy the same sentences. To state this, we write $\mathfrak{M} \equiv_{\Box\mathbb{Q}\pi+} \mathfrak{M}'$ if for all sentences $\varphi \in L_{\Box\mathbb{Q}\pi+}$, $\mathfrak{M} \vDash \varphi$ if and only if $\mathfrak{M}' \vDash \varphi$.

Lemma 24. *If* $\mathfrak{M} \cong^{\pi+} \mathfrak{M}'$ *then* $\mathfrak{M} \equiv_{\Box\mathbb{Q}\pi+} \mathfrak{M}'$.

Proof. By induction on the complexity of formulas. ☐

Lemma 25. $\mathfrak{N} \cong^{\pi+} \mathfrak{R}$.

Proof. For this proof, we need the notion of a partition, and of a certain kind of functions from one partition to another. A partition of a set D is a set X of non-empty subsets of D such that $\bigcup X = D$ and $A \cap B = \emptyset$ for any distinct $A, B \in X$. For any partitions X and Y, let $f : X \to Y$ be a *finite cardinality bijection* if it is a bijection and for any $A \in X$, A and $f(A)$ are both infinite or $|A| = |f(A)|$.

Let F be the set of finite cardinality bijections from a finite partition of $\mathbb{N}$ to a finite partition of $\mathbb{R}$. For any $f \in F$ from X to Y, define f_s to be the function such that $f_s(\bigcup X') = \bigcup_{A \in X'} f(A)$ for all $X' \subseteq X$, and define f_a to be the function such that $f_a(n) = r$ if $f(\{n\}) = \{r\}$. Now let $I = \{\langle f_a, f_s, f_a \rangle : f \in F$ and $f(\{0\}) = \{0\}\}$. By checking the conditions, it can be verified that I is a $\pi+$ back-and-forth system from $\mathfrak{N}$ to $\mathfrak{R}$. ☐

Proposition 26. $\mathcal{L}^{\mathsf{PD}}_{\Box\mathcal{Q}_1 O\pi+} \not\leq \mathcal{L}^{\mathsf{PD}}_{\Box\mathcal{Q}_1\pi+}$.

Proof. $\mathfrak{N} \nvDash \mathcal{Q}^O_1 x \Diamond Fx$ and $\mathfrak{R} \vDash \mathcal{Q}^O_1 x \Diamond Fx$. By Lemmas 25 and 24, $\mathfrak{N} \equiv_{\Box\mathcal{Q}_1\pi+} \mathfrak{R}$, so $\mathcal{L}^{\mathsf{PD}}_{\Box\mathcal{Q}_1 O\pi+} \not\leq \mathcal{L}^{\mathsf{PD}}_{\Box\mathcal{Q}_1\pi+}$. ☐

While this result shows that propositional quantifiers do not solve all problems with generalized quantifiers for contingentists and actualists, they may still hope that these problems can be solved by adding even richer modal resources. Although we have not seen any reason to think that this can be done, we also have not seen any reason to think that it cannot be done. This emphasizes the importance of the philosophical question which motivated question (Q2): All the technical results discussed here and in Williamson (2010) use particular formal languages, and we have seen that for different languages, we get different results about who can draw which distinctions and whose language is more expressive. Which formal language is the one which is relevant? Does it have to stand in some particular relation to natural language? Or should we consider richer and richer languages? When can we draw philosophical conclusions – do we have to establish that the relevant property holds for all languages that are at least as expressive as a particular language?

4 Conclusion

The problems for contingentism which Williamson (2010) has worked out in the context of plurally interpreted second-order quantified modal logic can also be formulated using quantified modal logics to which certain sets of generalized quantifiers are added, namely those sets $\mathbb{Q}$ which contain a generalized quantifier that is not first-order definable and for which $\mathcal{L}_{\mathbb{Q}}$ (first-order logic with the generalized quantifiers in $\mathbb{Q}$) relativizes. Such extensions of quantified modal logic are also relevant for the older debate between actualism and possibilism, since for exactly these sets of generalized quantifiers, the logic with outer quantifiers is strictly more expressive than the logic with inner quantifiers. This indicates that there is a tight connection between the technical facts relevant for the two debates. It also strengthens Williamson's argument in favor of necessitism and casts doubt on Fine's and Kripke's objections to certain arguments for possibilism. In particular, the results presented here show that Williamson's arguments do not rely on peculiarities of second-order quantifiers on the plural interpretation. Rather, the results suggest that these arguments are based on general features of quantified modal logics with quantifiers whose expressivity exceeds that of first-order logic, as they apply to a wide range of such logics and are to some degree robust under extensions by infinitary devices and propositional quantifiers. While these findings increase our understanding of the technical facts relevant for the metaphysical dispute on modal ontology, I have mentioned some of the many technical as well as philosophical issues that remain open.

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