Exclusive Disjunction and the Biconditional: An Even-Odd Relationship

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An elementary truth table argument shows that exclusive disjunction is just the negation of the biconditional: $(P \oplus Q) \equiv \neg(P \Leftrightarrow Q)$. This relationship is sometimes used to explain why inclusive, rather than exclusive, disjunction is the standard disjunction. Either disjunction can be formed from the other $((P \vee Q) \equiv ((P \oplus Q) \oplus (P \wedge Q))$; $(P \oplus Q) \equiv ((P \vee Q) \wedge \neg(P \wedge Q)))$, but only exclusive disjunction is the negation of another simple connective.

However, while $P \oplus Q$ is logically equivalent to the *negation* of $P \Leftrightarrow Q$, $P \oplus Q \oplus R$ is logically equivalent to $P \Leftrightarrow Q \Leftrightarrow R$ itself. (One can omit all parentheses in logical expressions involving only $\oplus$ or $\Leftrightarrow$, since both connectives are commutative and associative.) The reason for this is that $\oplus$ is a mutual exclusivity connective, whereas $\Leftrightarrow$ is an identity connective. Hence, $P \oplus Q \oplus R$ is true precisely when $P \oplus Q$ and R have opposite truth values, which occurs precisely when $P \Leftrightarrow Q$ and R have identical truth values. Generalizing this pattern gives strings of propositions connected by $\oplus$ or $\Leftrightarrow$ that alternate in accordance with the following identities:

$$\text{(A)} \quad \bigoplus_{i=1}^{n} P_i \equiv \overset{n}{\underset{i=1}{\Leftrightarrow}} P_i, \quad \text{for } n \text{ odd;}$$

$$\text{(B)} \quad \bigoplus_{i=1}^{n} P_i \equiv \neg\left(\overset{n}{\underset{i=1}{\Leftrightarrow}} P_i \right), \quad \text{for } n \text{ even.}$$

We now prove these identities by mathematical induction on the number of propositions.

Proof. Basis: The logical equivalence $P_1 \oplus P_2 \equiv \neg(P_1 \Leftrightarrow P_2)$ follows directly from the truth tables for the two expressions.

Induction Step: Assume the identities true for an integer $n \geq 2$. We will show them true for $n + 1$.

(A) n is odd. We begin with $\oplus_{i=1}^{n+1} P_i$, which can be rewritten $(\oplus_{i=1}^{n} P_i) \oplus P_{n+1}$. By the basis, this is equivalent to $\neg((\oplus_{i=1}^{n} P_i) \Leftrightarrow P_{n+1})$. By the induction hypothesis, this is equivalent to $\neg((\Leftrightarrow_{i=1}^{n} P_i) \Leftrightarrow P_{n+1})$. This, in turn, is just $\neg(\Leftrightarrow_{i=1}^{n+1} P_i)$, which concludes the induction step for case (A) and with it the proof of case (A).

(B) n is even. We begin with $\oplus_{i=1}^{n+1} P_i$, which can be rewritten $(\oplus_{i=1}^{n} P_i) \oplus P_{n+1}$. By the basis, this is equivalent to $\neg((\oplus_{i=1}^{n} P_i) \Leftrightarrow P_{n+1})$. By the induction hypothesis, this is equivalent to $\neg(\neg(\Leftrightarrow_{i=1}^{n} P_i) \Leftrightarrow P_{n+1})$. Since $\neg(\neg P \Leftrightarrow Q)$ is true just when P and Q have identical truth values (i.e., $\neg(\neg P \Leftrightarrow Q) \equiv (P \Leftrightarrow Q)$), this in turn yields $(\Leftrightarrow_{i=1}^{n} P_i) \Leftrightarrow P_{n+1}$, which is just $\Leftrightarrow_{i=1}^{n+1} P_i$. This concludes the induction step for case (B) and with it the proof of case (B).

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