Is Validity Circular?

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Abstract

There is an old worry, which dates back to Mill but has analogs even in Sextus, that the very definition of 'validity' implies that all valid arguments are circular. This paper investigates how contemporary formal tools may shed light on that worry. Its main finding is that the worry betrays a genuine puzzle: a difficult-to-avoid correspondence between the definitions of 'valid argument' and 'premise circular argument.'

Keywords Premise Circularity, Validity, Fallacies, Logic

1. Introduction

Validity is a well-defined notion. Premise circularity is not. Attempting to define premise circularity precisely generates a puzzle: a difficult-to-avoid correspondence between the definitions of 'valid argument' and 'premise circular argument.' The thesis of this paper is all and only that avoiding such a correspondence constitutes a genuine puzzle—one worth paying attention to and, presumably, with lessons to teach.

This puzzle has a history. It goes back to at least Mill (Sect. 2), but similar worries can be found in Sextus, Descartes, the Renaissance humanist Lorenzo Valla, and the medieval Islamic philosopher, Ibn Taymiyya.[1] The version of the puzzle I discuss, however, is contemporary, aided by the tools of formal logic and set theory. Its basic structure is that reasonable candidates for defining premise circularity are either too narrow, facing straightforward counterexample, or they are so broad as to include all classically valid arguments, or nearly so. Arguments via Addition ($\varphi \vdash \varphi \vee \psi$) do escape all candidate definitions of premise circularity, but that is not much of a victory.

After some historical impetus, I establish the puzzle (Sect. 3) and define terms and assumptions (Sects. 4-5). The center of the paper generates and tests exhaustive lists of candidate definitions of premise circularity, with the result that only three candidates avoid obvious counterexample (Sects. 6-7). I then quickly define classical validity for both semantic ($\vDash$) and syntactic ($\vdash$) consequence (Sect. 8), followed by the thrust of the paper: a proof of the exact correspondence between the definitions of validity and the three live contenders for defining premise circularity (Sect. 9). Section 10 draws two corollaries that make the puzzle worse. Finally, I address objections and briefly consider the possible lessons one might draw from this puzzle.

2. History

In John Stuart Mill's *System of Logic* one finds a curious argument at the opening of a section titled, *Is the Syllogism a Petitio Principii?* The bulk of that section focuses on a different argument, one that rests on properties unique to categorical syllogisms.[2] The quick argument he opens with, in contrast, merely rests on general claims about the definition of validity:

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[1] Descartes, *Regulae* 10 (AT X.406), Sextus Empiricus, *PH*.II.xiv. Valla, *Dialectical Disputations* 2: 227-9. Ibn Taymiyya, *Against the Greek Logicians*. For Taymiyya, compare 55, 255-6, and 290. Interestingly, Taymiyya also quotes from prior Islamic philosopher who also shared the worry: al-Hasan's *al-Ārā wal-Diyānāt* in passage 283.

[2] This other argument (that occupies Mill, though I find it less interesting) is this: in every categorical syllogism, the universal major premise, like "All men are mortal," presupposes the truth of all the particulars it subsumes, such as "Socrates is mortal." So, any such syllogism from universal to particular looks circular (*System*, 184 *et seq*.).

> It is universally allowed that a syllogism is vicious if there is anything more in the conclusion than was assumed in the premise. But this is, in fact, to say that nothing ever was, or can be, proved by syllogism, which was not already known, or assumed to be known before…[Thus,] those who have…followed out the general theorem respecting the logical value of syllogism to its legitimate corollary, have been led to impute uselessness and frivolity to the syllogistic theory itself, on the ground of the *petitio principii* which they allege to be inherent in every syllogism, (*System*, VII: 183).

The conclusion of this argument—that every valid syllogism is a case of *petitio principii*—Mill puts in the mouth of interlocutors, though he does say that they have inferred a "legitimate corollary." But whatever Mill's view, the argument itself is, like I say, curious.[3] It purports to show that all valid syllogisms are question begging (*petitio principii*) just by definition.[4] Where '**P**' stands for the premise set, and '*C*' the conclusion, Mill's argument can be reconstructed as follows:

Mill's Argument

1.1.	If anything in *C* is not assumed in **P**, then **P** therefore *C* is invalid.	[def.]
1.2.	If **P** therefore *C* is valid, then everything in *C* is assumed in **P**.	[1.1]
1.3.	If everything in *C* is assumed in **P**, then **P** therefore *C* is a case of *petitio principii*.	[def.]
1.4.	If **P** therefore *C* is valid, it is a case of *petitio principii*.	[1.2, 1.3]

Mill's Argument faces two substantial worries. First, while there is some sense of "assumed in" that validity requires, since valid inferences are not ampliative, why think it is the same sense

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[3] See n.41below, and D. Botting (2014) for a discussion of Mill's own views.

[4] Mill's terminology in the first line may mislead, but 'vicious' in his sense must mean 'invalid' in our sense otherwise the next line would be nonsense. For agreement, see the commentary of R.F. McRae in *The Collected Works of John Stuart Mill,* Vol. VII, xxvii.

that question-begging arguments possess? If it is not, then Mill's argument is guilty of equivocation. Second, Mill's argument looks self-undermining. For, if it is sound then it applies to itself, so it too is question begging. These two worries—equivocation and self-undermining—frame much of the puzzle below.

3. Puzzle

The puzzle I wish to discuss is inspired by Mill's Argument, though it is distinct in a few important respects. I will talk of circularity, and specifically premise circularity, instead of *petitio principii*. I broaden the target to validity itself, syllogistic or otherwise. Finally, I strengthen the premises to biconditionals—understanding the crux of the puzzle to be a purported correspondence between the definitions of validity and premise circularity.

Start by stipulating away any risk of equivocation with the following (nearly vacuous) analyses of both "valid argument" and "premise circular argument":

2.1. **P** therefore *C* is valid IFF *C* is validly contained in **P**.

2.2. **P** therefore *C* is premise circular IFF *C* is circularly contained in **P**.

By 'validly-contained' I just mean the precise kind of containment relation that yields a correct analysis of validity, whether semantic or proof-theoretic. To simplify exposition, I treat 'validly contained' as the relation that corresponds to classical validity. By the end, however, I consider the extent to which that relation once specified extends to other logics as well.

Likewise, by 'circularly-contained' I just mean the precise kind of containment relation that yields a correct analysis of premise circularity. As such, (2.1) and (2.2) are true by stipulation.

The question then becomes: can the stipulated terms 'validly contained' and 'circularly contained' be defined as substantive and yet distinct relations? That question is the topic of this paper and the crux of the puzzle. First, I need to clarify terms.

4. Terms

This is a puzzle about arguments. I think of arguments as abstracta. People argue, and they should argue with arguments, but arguments themselves are not the act of arguing. Arguments are abstract sets of sentences.[5]

In contrast, an inference is an activity: the act of drawing a conclusion. As inferences are acts done by agents, analyzing inferences may include claims about the relevant agents' epistemic contexts, aims, doxastic attitudes, and so forth. Whereas, an argument's properties are independent of any facts about any agents.

Not any old set of sentences, however, makes an argument. Arguments are sets containing two parts, a premise set and a conclusion, such that the premise set is prior to its conclusion (in some sense of "prior"). Minimally, then, I will think of an argument as an ordered pair: (**P**, *C*).[6] The first member of the pair is a set of sentences (possibly empty); the second member is a single sentence.[7]

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[5] Sentences are clearly understood and determinate, whereas propositions are poorly understood and vague. Thus, defining arguments as sets of sentences yields a more determinate account of arguments. Nevertheless, propositions will be discussed below when relations among propositions diverges from those that hold among sentences.
[6] Ordered pairs are still sets: sets of sets, given Kuratowski's definition of an ordered pair: $(a, b) = \{\{a\}, \{a, b\}\}$ (Kuratwoski, 1921). Kuratoski worked from a prior proposal that stems from Norbert Wiener (1914).
[7] I bold sets and italicize sentences throughout. By treating arguments as (**P**, *C*), I limit the subject to "single-conclusion logic." If arguments are (**P**, **C**), where both members are sets (possibly empty), then the corresponding subject would be "multi-conclusion logic," treatment of which can be found in Shoesmith and Smiley (2008).

The structure of an argument refers to those relations that hold between **P** and *C* in any such (**P**, *C*). Importantly, given the definition of (**P**, *C*), the structure of an argument is constituted by all and only relations among sentences.

This paper only focuses on premise circularity. Rule circularity and other kinds of circularity are left aside.[8] I cannot give a precise definition of premise circularity at the start, since the paper moves through several distinct proposals for a definition of premise circularity. I can only say, like (2.2) above, (**P**, *C*) is premise circular just when *C* is circularly contained in **P**. The trick is getting a precise account of the circularly-contained-in relation.

I will be using both set theory and formal logic, though I supply informal English translations where appropriate. Like above, I reserve bold capitals for names of sets and italicized capitals for individual sentences. The set-theoretic notation otherwise is quite standard. Nevertheless, a notational appendix is given for reference. At times, it will be crucial to distinguish between operations in the object language (whether logic or set theory) from those in the metalanguage. To help, I reserve English and English abbreviations for metalogical operators, such as "IFF". In contrast, I use the following symbols for logical operators: ~, &, ∨, ≡, →, standing for negation, conjunction, disjunction, equivalency, and material implication, respectively.

5. Assumptions

The puzzle rests on, and can even be thought of as working out the implications of, two methodological assumptions.

First, I assume that to say that *C* is circularly contained in **P** is to say something about the structure of the argument (**P**, *C*). And so, specifying the circularly-contained-in relation is a matter

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[8] Some have found as many as six kinds of circularity. For example, see John Greco, "Epistemic Circularity: Vicious, Virtuous, and Benign," *International Journal for the Study of Skepticism* 1 (2011): 1-8.

of specifying a kind of argument structure—a kind of relation between an abstract set of sentences, **P**, and some one sentence, *C*. In short, premise circularity is a kind of argument structure—a circular structure, which I aim to define below. Thereby, Assumption 1 is as follows:

A1. (**P**, *C*) is premise circular IFF (**P**, *C*) has a premise circular structure.

Some have argued that the fallacy of vicious circular reasoning essentially rests on the epistemic context in which an inference is made: the persons involved, which of their beliefs are justified, which claims they take to be established versus which they are trying to establish, and so forth.[9] If such a view is right, then fallacious circularity is a feature of inferences, not arguments. Thereby, one might object to the assumption that premise circularity is an argument structure.

But it is wholly consistent to think both that fallacious circularity is a feature of inferences and arguments themselves can be premise circular. Unless, of course, one further assumes some connection between argument structure and fallacious circularity (this paper does not). Moreover, to make any such assumption is to have the order of investigation backwards. The question of when, why, and how circular reasoning is fallacious should be postponed until the scope of "circular argument" is made clear.

Second, I assume that the following arguments are all premise circular—indeed, they are paradigmatic cases of premise circularity:

[9] To name but a few: Walter Sinnott-Armstrong (1999), Douglas Walton (1985), Andrew Cling (2002), and Michael Veber (2006). The case of Roy Sorenson (1991) is different in important respects, but presumably goes here as well.

3.1. The earth is flat.

3.2. The earth is flat. [3.1]

4.1. The earth is flat.

4.2. The earth is verdant.

4.3. The earth is flat. [4.1]

5.1. God exists and the Bible says so.

5.2. The Bible says that God exists, and he does. [5.1]

6.1 God exists and inspired the Bible.

6.2 God exists. [6.1]

I refer to these examples as Argument (3), Argument (4), and so forth. Thereby, the second assumption is as follows:

A2. Arguments (3) to (6) are premise circular.

The methodology I use below is particularist: I assume that Arguments (3)-(6) are premise circular without yet having a fully worked out theory of premise circularity.[10] Surely, Arguments (3)-(6) *prima facie* have their conclusions contained in their premise sets. But I say so without an account of any such containment relation. Instead, I start from the paradigmatic examples.

[10] Chisholm, R.M. *The Problem of the Criterion*. Milwaukee, WI: Marquette University Press, 1973

Nevertheless, one might wonder why these arguments are assumed as paradigmatic examples, instead of, say, some others. Good paradigmatic examples should be simple, few, and widely accepted exemplars of the property one is trying to understand. Arguments (3) to (6) are just this: they are simple cases of the sort typically pointed to as illustrations of premise circularity. So while there are no doubt others, Arguments (3) to (6) are at least some of those arguments pretheoretically expected to be covered by a definition of premise circularity. Pre-theoretic intuitions aren't infallible, but they are a reasonable starting point.

6. Candidate Definitions of Circular-Containment

With these assumptions, I can now list and test candidates for defining the circular-containment relation. Candidates are kept as live contenders just in case none of Arguments (3)-(6) provide a counterexample.

To generate a list of candidates, note that admissible candidates must be relations among sentences alone, since circular containment is a relation between **P** and *C* alone, and **P** and *C* are constituted by all and only sentences. Mere sentential relations can be either syntactic or semantic. Syntactic options include identity or an extension thereof via rules, like proof-theory. Semantic options include either relations among meanings, propositions, or truth-values. Thus, admissible candidates for circular containment will fall into one of these five kinds, all of which are considered below, starting with the simplest.

6.1. Sentential Identity

The simplest proposal would analyze "*C* is circularly contained in **P**" as a sentential identity relation. Two sentences are identical if and only if they contain all and only the same words in the same order. In short, sentential identity is syntactic. Such accounts exhaustively subdivide into two options. Either C needs to be identical with all of $\mathbf{P}$ or just a part of $\mathbf{P}$. In other words, the first two candidates to test are as follows:[11]

C is circularly contained in $\boldsymbol{P}$ IFF

i. $\{C\} = \mathbf{P}$

ii. $C \in \mathbf{P}$

Compare both options to Arguments (3)-(6). Option (i) is only consistent with Argument (3). Option (ii) is consistent with Arguments (3) and (4). But Arguments (5) and (6) provide a counterexample to both (i) and (ii). Thus, both sentential identity candidates are too narrow.

6.2. Propositional & Meaning Identity

To fit the paradigmatic Arguments (3) to (6), the circular-containment relation must be broadened beyond sentential identity. A broader identity relation holds between meanings and propositions in the sense that two sentences can be non-identical (syntactically) but still express the same proposition or mean the same thing (if those are different). Consider these two options:

C is circularly contained in $\boldsymbol{P}$ IFF

iii. $\exists x \in \mathbf{P}$ AND x expresses the same proposition as C

iv. $\exists x \in \mathbf{P}$ AND x means the same as C

[11] Exhaustiveness bids one to also consider a third option: (iii) $C \notin \mathbf{P}$, but such an third option denies (instead of affirms) a containment relation.

Options (iii) and (iv) fit Arguments (3), (4) and (5). In the case of (5), the premise and conclusion are equivalent associations of each other, so they both express the same proposition and mean the same thing. However, neither Option (iii) nor (iv) fit Argument (6). The conclusion of (6)—"God exists"—is informationally simpler than its premise. Premise (6.1) says "God exists and inspired the Bible," which both expresses and means something more than the mere assertion "God exists." The latter is part of the former, but they are not identical. Thus, defining circular containment as either meaning or propositional identity is also too narrow.

Presumably, one wants to say that the conclusion of (6) is identical to part of its premise (6.1), and that is why (6) is an example of circular containment. One sentence being part of another can be understood in either semantic or syntactic terms, and the next two subsections explore these options. On the semantic side, I use truth values given their precise definitions. Nevertheless, the possible set-theoretic relations among truth values will correspond to the possible relations among other semantic options, like propositions, *mutatis mutandis*.

6.3. Truth Values

First, a few terms. Let $\mathcal{L}$ be the set of all sentences of the language. Let v be an interpretation function that maps sentences in $\mathcal{L}$ to truth-values, to a unique element of {true, false}. The result is a "valuation": an ordered pair such that either (S, true) or (S, false), for any S in $\mathcal{L}$. Furthermore, let $\mathcal{M}$ be an arbitrary set of valuations, a "model", as I'll use the term, for some specified subset of sentences in $\mathcal{L}$. As there are multiple models for any set of sentences, subscripts are used to differentiate models.[12] For example, $v(S)_{\mathcal{M}1}$ refers to the valuation of S in model 1.

[12] A set of sentences, Σ, has 2^n models, where n = the number of atomic sentences in Σ.

I extend the above standard notation in two ways. First, for any S in $\boldsymbol{\mathcal{L}}$, let $\mathbf{T}_S$ be the set of all models that include the valuation (S, true). In other words,

$$\mathbf{T}_S \quad = \quad \{\boldsymbol{\mathcal{M}} \mid v(S)_{\boldsymbol{\mathcal{M}}} = \text{true}\}.$$

In such a way, for example, $\mathbf{T}_C$ stands for the set of all models that assign "true" to the conclusion.

Second, a similar thought can be extended to entire sets of sentences. Roughly, to say that a set of sentences is true in a model is just to say that every member of that set is true in that model. So, let "$V(\boldsymbol{\Sigma})_{\boldsymbol{\mathcal{M}}}$ = true" be equivalent to "for all S in $\boldsymbol{\Sigma}$, $v(S)_{\boldsymbol{\mathcal{M}}}$ = true". Then, one can use $V(\boldsymbol{\Sigma})$ to define $\mathbf{T}_\Sigma$ as the set of all models in which the set of sentences $\boldsymbol{\Sigma}$ is true:

$$\mathbf{T}_\Sigma \quad = \quad \{\boldsymbol{\mathcal{M}} \mid V(\boldsymbol{\Sigma})_{\boldsymbol{\mathcal{M}}} = \text{true}\}.$$

Thereby, for any argument ($\mathbf{P}$, C), $\mathbf{T}_\mathbf{P}$ stands for the set of all models that assign "true" to every member of the premise set.

With this notation, one can now list the exhaustive options for understanding "C is circularly contained in $\mathbf{P}$" as a relation among truth values. These options subdivide into relations between $\mathbf{T}_C$ and $\mathbf{T}_\mathbf{P}$ on the one hand, and relations between $\mathbf{T}_C$ and at least one member of the premise set, $\mathbf{T}_S$, on the other. I'll start with the former.

6.3.1. Relations Between *C* and the Whole Premise Set

In this group, there are five possibilities to consider:[13]

C is circularly contained in **P** IFF

v. $\mathbf{T}_C = \mathbf{T_P}$

vi. $\mathbf{T}_C \subset \mathbf{T_P}$

vii. $\mathbf{T_P} \subset \mathbf{T}_C$

viii. $\mathbf{T}_C \subseteq \mathbf{T_P}$

ix. $\mathbf{T_P} \subseteq \mathbf{T}_C$

Compare these options to the simple truth tables for Arguments (3) and (4) above:

Table for Argument (3)

	Premise	*Conclusion*
	The earth is flat	The earth is flat
1	*T*	*T*
2	*F*	*F*

Table for Argument (4)

	Premises		*Conclusion*
	The earth is flat	The earth is verdant	The earth is flat
1	*T*	*T*	*T*
2	*T*	*F*	*T*
3	*F*	*T*	*F*
4	*F*	*F*	*F*

[13] Strictly speaking, there is a sixth option, the disjoint option: $\mathbf{T_P} \cap \mathbf{T}_C = \{\}$. But as the disjoint option denies any shared models between **P** and *C*, it does not affirm any sort of containment relation between *C* and **P**.

Using row numbers to refer to models, one can see that in the Table for Argument (3), $\mathbf{T}_\mathbf{P} = \{\mathcal{M}_1\}$ and $\mathbf{T}_C = \{\mathcal{M}_1\}$. So, Table 3 fits Option (v) but provides a counterexample to both proper subset options: (vi) and (vii).

On the Table for Argument (4), in contrast, $\mathbf{T}_\mathbf{P} = \{\mathcal{M}_1\}$ but $\mathbf{T}_C = \{\mathcal{M}_1, \mathcal{M}_2\}$. So, Table 4 provides a counterexample to Option (v), and the first subset option, (viii). Thus, the only option that survives testing against Arguments (3) and (4) is Option (ix).

A quick inspection of the Tables for Arguments (5) and (6) shows that Option (ix) is consistent with those as well:

Table for Argument (5)

	Atomic Statements		*Premise*	*Conclusion*
	God exists	Bible says that God exists	God exists and the Bible says so	The Bible says that God exists and he does
1	T	T	T	T
2	T	F	F	F
3	F	T	F	F
4	F	F	F	F

Table for Argument (6)

	Atomic Statements		*Premise*	*Conclusion*
	God exists	God inspired the Bible	God exists and inspired the Bible	God exists
1	T	T	T	T
2	T	F	F	T
3	F	T	F	F
4	F	F	F	F

On the Table for Argument (5), $\mathbf{T}_\mathbf{P} = \{\mathcal{M}_1\}$ and $\mathbf{T}_C = \{\mathcal{M}_1\}$. Whereas, on the Table for Argument (6), $\mathbf{T}_\mathbf{P} = \{\mathcal{M}_1\}$ but $\mathbf{T}_C = \{\mathcal{M}_1, \mathcal{M}_2\}$. Thus, since option (ix) states $\mathbf{T}_\mathbf{P} \subseteq \mathbf{T}_C$, option (ix) is consistent with both Tables (5) and (6). Therefore, the first candidate that avoids counterexample has been found, which I label (A). If circular-containment is understood as a relation between $\mathbf{T_P}$ and $\mathbf{T}_C$, then,

A. *C* is circularly contained in **P** IFF $\mathbf{T_P} \subseteq \mathbf{T}_C$.

6.3.2. Relations Between *C* and a Member of the Premise Set

Turn now to the second group of truth-value options, which understands the circular-containment relation to be a relation between the truth of *C* and the truth of some member of the premise set (instead of the whole premise set). Again, there are five possible options to consider:[14]

C is circularly contained in ***P*** IFF

x. $\exists s \in \mathbf{P}$ and $\mathbf{T}_C = \mathbf{T}_S$

xi. $\exists s \in \mathbf{P}$ and $\mathbf{T}_C \subset \mathbf{T}_S$

xii. $\exists s \in \mathbf{P}$ and $\mathbf{T}_S \subset \mathbf{T}_C$

xiii. $\exists s \in \mathbf{P}$ and $\mathbf{T}_C \subseteq \mathbf{T}_S$

xiv. $\exists s \in \mathbf{P}$ and $\mathbf{T}_S \subseteq \mathbf{T}_C$

Just as before, compare these options with the Tables for Arguments (3) to (6) above. In the Table for Argument (3), there is just one premise to substitute for "*s*." As such, $\mathbf{T}_S = \{\mathcal{M}_1\}$ and $\mathbf{T}_C = \{\mathcal{M}_1\}$. So again, Table 3 fits option (x) but provides a counterexample to both proper subset options: (xi) and (xii).

[14] Again, strictly speaking there is a sixth option, the disjoint option: $\mathbf{T}_S \cap \mathbf{T}_C = \{\}$. But like before, the disjoint option does not plausibly capture any containment relation between *C* and **P**.

The Table for Argument (4), in this case, provides no additional help and the same is true of the Table for Argument (5). However, the Table for Argument (6) provides further guidance. In (6), there is just one premise to substitute for *"s"* and in that case, $\mathbf{T}_S = \{\mathcal{M}_1\}$ but $\mathbf{T}_C = \{\mathcal{M}_1, \mathcal{M}_2\}$. Therefore, Argument (6) is a counterexample to Options (x) and (xiii). The only option that survives is option (xiv): $\exists s \in \mathbf{P}$ and $\mathbf{T}_S \subseteq \mathbf{T}_C$. Therefore, a second candidate that avoids counterexample has been found, which I label (B). If circular containment is understood as a truth-value relation between *C* and some member of **P**, then

B. *C* is circularly contained in **P** IFF $\exists s \in \mathbf{P}$ and $\mathbf{T}_S \subseteq \mathbf{T}_C$

6.4. Proof Theory

While propositions, meaning, and truth values were all semantic attempts to broaden the notion of circular containment, this last attempt returns to syntax, using rules to broaden beyond the explicit sentences that constitute *C* and **P**.

The task here will be to define some set of rules, $\mathcal{R}$, that permit one to alter or rearrange the explicit members of **P** to yield a new sentence, thereby broadening the relata that may stand in an identity relation to *C*. Since any conception of the circularly-contained-in relation must be consistent with the paradigmatic cases of Arguments (3)-(6), one can reverse engineer rules from those cases. First, consider Arguments (3) and (4)

3.3. The earth is flat.
3.4. The earth is flat.[3.1]

4.1. The earth is flat.
4.2. The earth is verdant.
4.3. The earth is flat. [4.1]

To account for the premise circularity of Arguments (3) and (4), it only needs to be the case that $\boldsymbol{\mathcal{R}}$ = {reiteration}:

Reiteration:

$$\frac{\varphi}{\varphi}$$

But move further down the list of paradigmatic premise circular arguments to Argument (5):

5.1. God exists and the Bible says so.

5.2. The Bible says that God exists, and he does. [5.1]

Since (5.1) and (5.2) are not identical sentences—they are the equivalent commutations of each other—reiteration cannot account for the rearrangement from (5.1) to (5.2). An equivalency rule is needed:[15]

Equivalency Rule:

$$\frac{\varphi}{\psi} \quad \text{When and only when } \varphi \equiv \psi$$

Accounting for Arguments (3), (4), and (5), therefore requires expanding the rule set. Allowing redundancy, $\boldsymbol{\mathcal{R}}$ = {reiteration, equivalency rule}.

Finally, consider the last paradigmatic case, Argument (6):

[15] Strictly, only the rule of Commutation is needed, not all equivalency rules, but drawing the line there seems patently arbitrary.

6.1 God exists and inspired the Bible.

6.2 God exists. [6.1]

(6.2) is neither identical nor equivalent to (6.1). A new rule is needed: Conjunction Elimination.

Conjunction Elimination:

$$\frac{\varphi \,\&\, \psi}{\varphi}$$

Thus, to account for all Arguments (3) to (6), it will be the case that $\boldsymbol{\mathcal{R}}$ = {reiteration, equivalency rule, conjunction elimination}. Letting "$\vdash_{\mathcal{R}}$" stand for "follows by the rules of $\boldsymbol{\mathcal{R}}$," the resulting third proposal for circular-containment is this, which I label (C):

C. *C* is circularly contained in **P** IFF **P** $\vdash_{\mathcal{R}} C$

Of course, a more expansive set of rules would also be consistent with Arguments (3) to (6), but $\boldsymbol{\mathcal{R}}$ is the minimal set that fits Arguments (3) to (6).[16]

7. Interlude on Containment

The prior sections have yielded three candidates for defining circular-containment:

[16] More carefully, $\boldsymbol{\mathcal{R}}$ is the minimal non-arbitrary set. See n. 15.

C is circularly contained in $\mathbf{P}$ IFF either

A. $\mathbf{T_P} \subseteq \mathbf{T}_C$, or

B. $\exists s \in \mathbf{P}$ and $\mathbf{T}_S \subseteq \mathbf{T}_C$, or

C. $\mathbf{P} \vdash_{\mathcal{R}} C$.

One might worry that (A), (B), and (C) lose the intuitive sense of 'C is circularly contained in **P**.' Option (C) is not obviously a containment claim at all. And Options (A) and (B) seem backwards, since, given the direction of '$\subseteq$', they more naturally look like analyses of '**P** is contained in C.'

Both worries are answerable. First, take (A)'s analysis of "C is circularly contained in **P**" as $\mathbf{T_P} \subseteq \mathbf{T}_C$. If $\mathbf{T_P} \subseteq \mathbf{T}_C$, then every model that is an element of $\mathbf{T_P}$ is also an element of $\mathbf{T}_C$. Moreover, $\mathbf{T}_C$ just is the set of models that include the valuation (C, true) as an element. So, if $\mathbf{T_P} \subseteq \mathbf{T}_C$, then the valuation ($C$, true) is an element in every $\mathcal{M} \in \mathbf{T_P}$. In other words, ($C$, true) is "contained in" every element of $\mathbf{T_P}$. The same answer, *mutatis mutandis*, applies for Option (B).

The answer for Option (C)—explaining how $\mathbf{P} \vdash_{\mathcal{R}} C$ expresses a containment claim—is more involved but is essential for the puzzle below. It begins with Tarski who first noticed in 1930 that the notion of mathematical closure (with slight emendation) can be used to model logical consequence.[17] Broadly, a closure operator in mathematics is a function (map) from the powerset of a set to itself. Logical consequence can be thought of as a function, since for any **P**, the content of **P** and the rules that define "$\vdash$" jointly determine a single set of sentences (possibly infinite) that follows from **P**. Furthermore, both **P** and the set consequences that follow from **P** are subsets of

[17] Taking the notion of mathematical closure from topology, Tarski noted that if we omit (i) that the closure of the empty set is empty (since theorems derive from an empty set of premises) and (ii) that closure distributes over unions (since what follows from two sets individually may be less than what follows from their union), then we may formulate an adequate notion of a consequence operator. For Tarski's original paper, see (Tarski, 1930). For a recent, full-length, and very helpful, text on the development of mathematical closure as a consequence operator, see (Citkin, A and A. Muravitsky, 2022).

the relevant language, $\mathcal{L}$. Thus, logical consequence is a function from the powerset of $\mathcal{L}$ to itself. It behaves like a closure operator, "*cl*", where $cl\colon \mathcal{P}(\mathcal{L}) \longrightarrow \mathcal{P}(\mathcal{L})$.

By itself, the mathematical notion of a closure operator is only restricted by three properties: it must be extensive, monotonic, and idempotent.[18] Classical validity meets all three.[19] And since the rules in set $\mathcal{R}$ are a subset of those rules that define syntactic consequence in classical logic, the consequence relation "$\vdash_{\mathcal{R}}$" is also extensive, monotonic, and idempotent, and so is consistent with mathematical closure. And since "$\vdash_{\mathcal{R}}$" is consistent with mathematical closure, one can define restricted closure operators for each of the rules in $\mathcal{R}$:

Reiteration: $cl_{=}(\mathbf{P}) = \{x \in \mathcal{L} \mid x \in \mathbf{P}\}$

Equivalency Rule: $cl_{\equiv}(\mathbf{P}) = \{x \in \mathcal{L} \mid \exists y \in \boldsymbol{P} \text{ and } y \equiv x\}$

Conjunction Elimination: $cl_{\&}(\mathbf{P}) = \{x \in \mathcal{L} \mid \exists y \in \boldsymbol{P} \text{ and } \exists z \text{ s.t. } (x \,\&\, z) \equiv y\}$

Thereby, the relation "$\vdash_{\mathcal{R}}$" can be equivalently stated as set membership in the union of the sets defined by these three closure operations on **P**. Therefore, Option (C) for understanding circular containment does express a containment relation. Namely, this one:

C. *C* is circularly-contained in **P** IFF $C \in \bigcup\{cl_{=}(\mathbf{P}), cl_{\equiv}(\mathbf{P}), cl_{\&}(\mathbf{P})\}$

[18] If $cl\colon \mathcal{P}(\mathcal{L}) \rightarrow \mathcal{P}(\mathcal{L})$ then for any *A* and *B* that are subsets of $\mathcal{L}$, the following conditions hold:

i. $A \subseteq cl(A)$ Extensive
ii. $A \subseteq B \rightarrow cl(A) \subseteq cl(B)$ Monotonic
iii. $cl(cl(A)) = cl(A)$ Idempotent

[19] Classical validity is extensive because every sentence follows from itself. Classical validity is monotonic because whatever a subset of **P** entails will also be entailed by all of **P**. Finally, classical validity is idempotent because what follows from what follows from **P** is still just what follows from **P**.

8. Validity as Containment

The final step is to define the containment relation expressed by "C is validly contained in **P**," for which the machinery above can be helpfully repurposed. Validity divides into one of two more precise relations: semantic consequence ('$\vDash$') and syntactic consequence ('$\vdash$').

Roughly, $\mathbf{P} \vDash C$ just in case C is true whenever every member of **P** is true. In terms of models, $\mathbf{P} \vDash C$ just in case every model that assigns "true" to every member of **P** is also a model that assigns "true" to C. So, where $\mathbf{T}_{\mathbf{P}}$ is the set of models that assign "true" to every member of the premise set, and $\mathbf{T}_C$ is the set of models that assign "true" to the conclusion, semantic consequence can be defined as a subset relation—i.e., a containment relation:

$$\mathbf{P} \vDash C \qquad \text{IFF} \qquad \mathbf{T}_{\mathbf{P}} \subseteq \mathbf{T}_C$$

In contrast, $\mathbf{P} \vdash C$ just in case there is a correct derivation from **P** to C. Here, "correct derivation" is defined by classical proof theory. Again, thanks to Tarski, if a closure operator is defined with rules of classical logic, then that closure operator successfully models classical logic's notion of syntactic consequence. Let "*cn*" be the consequence operator: a closure operator defined by the rules of classical logic. In such a way, $cn(\mathbf{P})$ becomes the set of all sentences that follow proof-theoretically from **P**. That is to say, $cn(\mathbf{P}) = \{x \mid \mathbf{P} \vdash x\}$. Thus, syntactic consequence can be analyzed in terms of set membership—so again, as a containment claim:

$$\mathbf{P} \vdash C \qquad \text{IFF} \qquad C \in cn(\mathbf{P})$$

In sum, one might have thought it was mere metaphor to say that for all valid arguments, "*C* is contained in **P**". But it is not. The relation of validity (at least classically) is a literal containment relation in the sense of either $\mathbf{T}_{\mathbf{P}} \subseteq \mathbf{T}_C$ or $C \in cn(\mathbf{P})$.

9. A Puzzling Argument

Above sections have surveyed specifications of two relations: valid containment and circular containment. The yield from those sections is three live candidates for specifying circular containment along with two conceptions of valid-containment. Once so specified, these notions yield a puzzling argument. Premise (1) states the two ways validity is a containment relation, while Premise (2) lists three competing options for understanding premise circularity:

Validity-Circularity Correspondence (VCC) Argument

1. $(\mathbf{P}, C)$ is valid IFF either $\mathbf{T}_{\mathbf{P}} \subseteq \mathbf{T}_C$ or $C \in cn(\mathbf{P})$
2. Either (A), (B), or (C) is true:
    A. $(\mathbf{P}, C)$ is premise circular IFF $\mathbf{T}_{\mathbf{P}} \subseteq \mathbf{T}_C$
    B. $(\mathbf{P}, C)$ is premise circular IFF $\exists s \in \mathbf{P}$ and $\mathbf{T}_S \subseteq \mathbf{T}_C$
    C. $(\mathbf{P}, C)$ is premise circular IFF $C \in \bigcup\{cl_{=}(\mathbf{P}), cl_{\equiv}(\mathbf{P}), cl_{\&}(\mathbf{P})\}$
3. If (A), then $(\mathbf{P}, C)$ is valid IFF $(\mathbf{P}, C)$ is premise circular.
4. If (B), then if $|\mathbf{P}| = 1$, then $(\mathbf{P}, C)$ is valid IFF $(\mathbf{P}, C)$ is premise circular.
5. If (C), then $\mathbf{P} \vdash_{\mathcal{R}} C$ IFF $(\mathbf{P}, C)$ is premise circular.

Let me comment on lemmas (3)-(5) before drawing the obvious further conclusion at (6).

Lemma (3) follows from the fact that $\mathbf{T_P} \subseteq \mathbf{T}_C$ is also the correct analysis of $\mathbf{P} \vDash C$. Now, ($\mathbf{P}$, C) is valid IFF either $\mathbf{P} \vDash C$ or $\mathbf{P} \vdash C$, but for any sound and complete logic, there are no arguments where $\mathbf{P} \vdash C$ but $\mathbf{P} \nvDash C$. Thus, it follows that an argument is valid if and only if it is premise circular.

Lemma (4) follows from the fact that $\mathbf{T}_S \subseteq \mathbf{T}_C$ is the correct analysis of $S \vDash C$ (single-sentence semantic entailment). Thus, whenever the cardinality of $\mathbf{P}$ is 1, it will be true that $\mathbf{P} \vDash C$ if and only if $\exists s \in \mathbf{P}$ and $\mathbf{T}_S \subseteq \mathbf{T}_C$. Then again, for any sound and complete logic, there are no arguments where $\mathbf{P} \vdash C$ but $\mathbf{P} \nvDash C$. Therefore, it follows that whenever the cardinality of $\mathbf{P}$ is 1, ($\mathbf{P}$, C) is valid if and only if it is premise circular.

Lemma (5) follows from the definition of $cl_=(\mathbf{P})$, $cl_\equiv(\mathbf{P})$, and $cl_\&(\mathbf{P})$. A conclusion follows from $\mathbf{P}$ by a rule of $\mathcal{R}$ just in case that $C \in \bigcup\{cl_=(\mathbf{P}), cl_\equiv(\mathbf{P}), cl_\&(\mathbf{P})\}$. So, if premise circularity is analyzed as $C \in \bigcup\{cl_=(\mathbf{P}), cl_\equiv(\mathbf{P}), cl_\&(\mathbf{P})\}$, then $\mathbf{P} \vdash_{\mathcal{R}} C$ IFF ($\mathbf{P}$, C) is premise circular.

Therefore, it follows from (2)-(5) that:

6. Either (A′), (B′), or (C′):

    A′. ($\mathbf{P}$, C) is valid IFF ($\mathbf{P}$, C) is premise circular

    B′. If $|\mathbf{P}| = 1$, then ($\mathbf{P}$, C) is valid IFF ($\mathbf{P}$, C) is premise circular.

    C′. $\mathbf{P} \vdash_{\mathcal{R}} C$ IFF ($\mathbf{P}$, C) is premise circular.

In sum, either (A′) all valid arguments are premise circular, or a large subset of valid arguments are all premise circular. That subset is either all single-premise arguments, if (B′) is true, or all arguments using an equivalency rule, reiteration, or conjunction elimination, if (C′). Now, consider how many valid derivations pass through lemmas justified by either single-premise arguments or by a rule in $\mathcal{R}$. Indeed, a great many valid derivations must do so.[20] Furthermore, if a derivation passes through a lemma that is justified by a premise circular argument, then the whole derivation is premise circular. Otherwise, any premise-circular argument could be made non-circular just by embedding it in a larger derivation.[21] Thereby, on any of the three options for defining premise circularity, the extension of the term is surprisingly wide.

10. Corollaries

While both (B′) and (C′) render the scope of "premise-circular arguments" to be wider than typically thought, they at least limit premise circularity to some condition, either single-premise arguments or arguments via $\mathcal{R}$.

The limiting condition of either (B′) or (C′) gets slippery, however, when the following two meta-theorems of classical logic are considered:

Theorem 1: $\{P_1, ..., P_n\} \vDash C$ IFF $(P_1 \,\&\, \ldots \,\&\, P_n) \vDash C$

Theorem 2: $\{P_1, ..., P_n\} \vdash C$ IFF $(P_1 \,\&\, \ldots \,\&\, P_n) \vdash C$

[20] In other words, not all derivations that in fact use a rule in $\mathcal{R}$ have to; some could be rewritten to avoid rules in $\mathcal{R}$. On the other hand, any derivation could be rewritten to use a rule in $\mathcal{R}$: reiteration. On behalf of (C′), I suggest: (i) possibly using a rule in $\mathcal{R}$ doesn't make a derivation circular, (ii) actually using a rule in $\mathcal{R}$ does, and (iii) if a derivation can't avoid using a rule in $\mathcal{R}$, then the derivation is unavoidably, or necessarily, premise circular.

[21] Proof: Assume $A \vdash A$ is premise circular. Assume for reductio that an argument is premise circular IFF every lemma in the argument is premise circular. Then, $A \vdash A$ could be made non-circular by embedding it as follows:

1. A [Assumption]
2. A [Reiteration, 1]
3. $A \rightarrow A$ [Theorem Introduction]
4. A [Modus Ponens. 2,3]

Theorem 1 says that every semantically valid argument is equivalent to a single-premise semantically valid argument. Likewise, Theorem 2 says that every proof-theoretically valid argument is equivalent to a single-premise proof-theoretically valid argument.

The meta-theoretic equivalency (the "IFF") in these theorems is not just deductive equivalency. All theorems are deductively equivalent in the sense that each is a theorem if and only if the others are, since they all rest on the same set of axioms. The specific sense of equivalency here is stronger.

In either theorem, the left and right arguments—call the left argument in each case (L) and the right argument (R)—have identical conclusions, identical atomic sentences, and the premises of (L) are all and only identical to the conjuncts of the premise in (R). The only non-identity between (L) and (R) is due to a transformation that seems nearly trivial: conjoining multiple premises into a single premise. (L) and (R) aren't just deductively equivalent, they are expressions of nearly the same argument. I will say that (L) and (R) are "equivalent expressions" of each other.

10.1. Corollary from (B′)

Look at both theorems. The right side of each theorem is an argument where $|\mathbf{P}| = 1$. Thus, by option (B′) for defining premise circularity, the right side of each theorem is premise circular. Therefore, for any sound and complete logic, since (B′) requires as much, it follows that:

7. If (B′), then ($\mathbf{P}$, C) is valid IFF it is either premise circular or an equivalent expression of a premise circular argument.

10.2. Corollary from (C′)

A similar, but specifically proof-theoretic, corollary follows from option (C′) and Theorem (2). The proof-theoretic validity of the right side of Theorem (2) is constituted by the derivation from (P_1 & … & P_n) to C. That derivation will include either rules that apply to single lines of a proof (like equivalency rules) or rules that apply to multiple lines (like modus ponens). If the derivation uses multiple-line rules, then conjunction elimination will need to be applied first, and so the derivation would use a rule in $\mathcal{R}$ and be premise circular according to (C′). If the derivation uses only single-line rules, it will use either reiteration, an equivalency rule, or addition (assuming the standard rules of natural deduction). The first two rules are in $\mathcal{R}$ and so the proof would be premise circular according to (C′). The only single-line rule not in $\mathcal{R}$ is addition: $\varphi \vdash \varphi \vee \psi$. Thus, by (C′) and Theorem (2), every proof-theoretically valid argument is either premise circular (by using $\mathcal{R}$) or is an equivalent expression of a premise circular argument (since its single-premise equivalent uses $\mathcal{R}$), unless its single premise equivalent uses all and only addition. Thus, the corollary from (F) does leave one loophole—but it's not much to write home about. The loophole would be all and only arguments with this form: $\{P_1, ..., P_n\} \vdash (P_1 \,\&\, \dots \,\&\, P_n) \vee Q$, since such an argument would neither use $\mathcal{R}$ nor be an equivalent expression of an argument that uses $\mathcal{R}$. Therefore,

8. If (C′), then (**P**, C) is valid IFF it is either premise circular, an equivalent expression of a premise circular argument, or $C = (P_1 \,\&\, \dots \,\&\, P_n) \vee Q$.

11. Final Result

The options for specifying the circularly-contained-in relation are either too narrow or are one of the following three:

A. $(\mathbf{P}, C)$ is premise circular IFF $\mathbf{T}_{\mathbf{P}} \subseteq \mathbf{T}_C$

B. $(\mathbf{P}, C)$ is premise circular IFF $\exists s \in \mathbf{P}$ and $\mathbf{T}_S \subseteq \mathbf{T}_C$

C. $(\mathbf{P}, C)$ is premise circular IFF $C \in \bigcup\{cl_{=}(\mathbf{P}), cl_{\equiv}(\mathbf{P}), cl_{\&}(\mathbf{P})\}$

If (A) is the correct analysis of premise circularity, then (A′) follows: valid arguments are all and only premise circular arguments. If (B) is the correct analysis of premise-circularity, then (B′) follows: valid arguments are all and only either premise-circular or equivalent expressions of premise-circular arguments. If (C) is the correct analysis of premise circularity, then (C′) follows: valid arguments are all and only either premise circular, equivalent expressions of a premise circular arguments, or arguments of the form $\{P_1, ..., P_n\} \vdash (P_1 \,\&\, \dots \,\&\, P_n) \vee Q$. By Premise (2), (A), (B), and (C) are the only options. Therefore,

9. An argument $(\mathbf{P}, C)$ is valid IFF it is either premise circular, an equivalent expression of a premise circular argument, or $C = (P_1 \,\&\, \dots \,\&\, P_n) \vee Q$.

Nearly every single valid argument is either premise circular or an equivalent restatement of a premise circular argument. That is an odd result.

12. Objections

I'll start with the straightforward denials of assumption (A1) and (A2), followed by two more creative attempts to articulate a missing alternative and expand the loophole beyond Addition.

12.1. Against (A1)

(A1), recall, assumes that premise circular arguments are premise circular in virtue of their structure. But one might charge that the whole project to define premise circularity as an abstract structure of arguments is wrong-headed.

If it is wrong-headed, one needs a reason to show why. If the reason is that no abstract structure distinguishes fallacious premise circularity from non-fallacious valid arguments, such a reason is consistent with accepting the VCC Argument as sound. Indeed, one might take the VCC Argument to support that very reason.

If instead, one denies that a purely structural account of premise circularity makes sense, that is clearly wrong. Arguments of the form $P \vdash P$ exist. Such arguments end right where they began, just as one would in drawing a circle (without needing to introduce any angles along the way). So, some arguments are properly called circular by the aptness of that geometric metaphor—a structural metaphor.

12.2. Against (A2)

The more formidable objection is about the scope of arguments with premise circular structures. In other words, one might accept that arguments of the form $P \vdash P$ are structurally premise circular, but deny (A2)—that each and every of Arguments (3) to (6) are also structurally premise circular.

Such a view is possible, but I find it untenable. Argument (3) has the structure $P \vdash P$, and Arguments (4) to (6) are merely slight alterations on that structure, as can be seen below:[22]

3. $P \vdash P$

4. $\{P, Q\} \vdash P$

5. $P \;\&\; Q \vdash Q \;\&\; P$

6. $P \;\&\; Q \vdash P$

Argument (4) is merely $P \vdash P$ with the addition of a superfluous premise. But if one accepts that (3) is premise circular but denies that (4) is, then any premise circular argument could be made non-circular by adding a superfluous premise.

As to Argument (5), take the following instance of (3), call it (3*): $P \;\&\; Q \vdash P \;\&\; Q$. If (3) is premise circular, then so is (3*). But the difference between (3*) and (5) is only the order of the conjuncts. It would be quite odd if something so trivial distinguished between premise circular and non-premise circular structures.

Finally, as to Argument (6), the difference between (6) and (4) is all and only that in (6) the premises are conjoined. So, if (4) counts as a premise circular structure but (6) does not, then any premise circular argument with a structure like (4) could be made non-circular after one application of Conjunction Introduction. That too would be quite odd.

Thus, since $P \vdash P$ is structurally premise circular, it is difficult to avoid accepting that each and every one of Arguments (3) to (6) is structurally premise circular as well.

[22] The structures follow from mere basic sentential logic translations. But the same argument holds for standard predicate logic translations.

12.3. Missing Option

One might worry that the argument above neglects to consider some more plausible account of structural premise circularity. While I have considered identity relations and proof-theoretic expansions of identity (on the syntactic side) and relations among meanings, propositions, and truth values (on the semantic side), the VCC argument fails to consider semantic analyses at a level of structure "lower" than the sentences. Consider the following.

Let "MST" stand for a set of minimal sub-formulas whose truth is necessary for the truth of a whole sentence, where the MST of a sentence, φ, is written MST(φ). For example, if A and B are atomic, then the three formulas below all have a single MST:

$$\mathrm{MST}(A) = \{A\}$$

$$\mathrm{MST}(A \,\&\, B) = \{A, B\}$$

$$\mathrm{MST}({\sim}\mathrm{A}) = \{{\sim}\mathrm{A}\}$$ [23]

Disjunctions, however, are more involved since they do not require a single minimal sub-formula to be true. I will return to them below. For now, sticking to the simple cases, we seem to have a way to analyze the MST of a set of sentences in terms of a union relation. For example, in the case where $\mathbf{X} = \{A, B\}$,

$$\begin{aligned} \mathrm{MST}(\mathbf{X}) &= \\ &= \mathrm{MST}(A) \cup \mathrm{MST}(B) \\ &= \{\mathrm{A}\} \cup \{\mathrm{B}\} \\ &= \{A, B\} \end{aligned}$$

[23] While the falsity of A is necessary for the truth of ~A (assuming bivalence), MST is defined as the minimal set of sub-formulas whose *truth* is necessary for the truth of the whole sentence. There is no more minimal true sub-formula than ~A for the truth of ~A.

Thereby, MSTs would yield an additional semantic option for defining premise circularity. Namely,[24]

MST Proposal: $(\mathbf{P}, C)$ is premise circular IFF $\text{MST}(C) \subseteq \text{MST}(\mathbf{P})$

The MST proposal passes the first test, fitting all of the paradigmatic cases, Arguments (3) to (6):

Argument Form	MST(C)		MST($\mathbf{P}$)
3. $P \vdash P$	$\{P\}$	$\subseteq$	$\{P\}$
4. $\{P, Q\} \vdash P$	$\{P\}$	$\subseteq$	$\{P, Q\}$
5. $P \,\&\, Q \vdash Q \,\&\, P$	$\{P, Q\}$	$\subseteq$	$\{P, Q\}$
6. $P \,\&\, Q \vdash P$	$\{P\}$	$\subseteq$	$\{P, Q\}$

And, importantly, it seems to avoid declaring other valid forms premise circular, like Modus Ponens. After all in the sequent $\{P, P \rightarrow Q\} \vdash Q$, the truth of Q is necessary for neither the truth of P nor for the truth of the conditional, $P \rightarrow Q$, since the latter can be true without Q being true.[25]

This case for avoiding Modus Ponens, however, rests on a misstep created by only considering the simple cases. First, since the MST of a set is the union of the MSTs of its members, then for the Modus Premise case, $\text{MST}(\mathbf{P}) = \text{MST}(P) \cup \text{MST}(P \rightarrow Q)$.

Second, while $\text{MST}(P) = \{P\}$, $\text{MST}(P \rightarrow Q)$ is more involved, as the truth of $(P \rightarrow Q)$ does not require a single minimal sub-formula to be true, but it does require something to be true. On

[24] See Botting, 2011 for a similar proposal. For the MST language, I am grateful to an anonymous referee.
[25] Compare: Botting 2011, 33.

material implication, for example, it requires an exclusive disjunction: either $\sim P$ is true or Q is true (the inclusive option is not minimal). As such, the full MST analysis of a Modus Ponens premise set would be as follows:

$$\begin{aligned} \text{MST}\{\mathbf{P}\} &= \\ &= \text{MST}\{P, P \rightarrow Q\} \\ &= \text{MST}(P) \cup \text{MST}(P \rightarrow Q) \\ &= \{P\} \cup (\{\sim P\} \text{ or } \{Q\}) \\ &= \{P, \sim P\} \text{ or } \{P, Q\} \\ &= \{P, Q\} \end{aligned}$$

In short, there is only one possible MST for the truth of the whole premise, **P**. After all, the left-disjunct on the penultimate lemma is an inconsistent set, and so cannot be true. Whereas, MSTs are defined as the minimal set of *true* sub-formulas required by the truth of the whole (sentence or union of sentences). Hence, the only possible MST for $\{P, P \rightarrow Q\}$ is $\{P, Q\}$. Thereby, the MST proposal does declare Modus Ponens to be premise circular, despite initial appearances to the contrary, as the completed table shows:

Argument Form	MST(C)		MST(**P**)
$\{P, P \rightarrow Q\} \vdash Q$	$\{Q\}$	$\subseteq$	$\{P, Q\}$

The same result can be easily seen to hold for other fundamental rules of natural deduction: Modus Tollens, Disjunctive Syllogism, Conjunction Elimination, and Conjunction Introduction. Addition, however, would again be a loophole.[26]

The natural response at this point would be analogous to the one attempted in the section on truth-value relations (Sect. 6.3). That is to say, instead of analyzing premise circularity as a relation between the conclusion and the whole premise set, try the same relation but between C and one member of **P**:

MST Proposal*: $(\mathbf{P}, C)$ is premise circular IFF $\exists S \in \mathbf{P}$ and $\text{MST}(C) \subseteq \text{MST}(S)$

The updated MST* Proposal would expand the loophole of non-premise circular valid arguments considerably beyond Addition. No longer would Modus Ponens, Modus Tollens, Disjunctive Syllogism, or even Conjunction Introduction be premise circular, though Conjunction Elimination and all equivalency rules still would. But since Conjunction Elimination is still premise circular, the MST* Proposal runs into the same problem generated by Theorems (1) and (2) above:

Theorem 1: $\{P_1, ..., P_n\} \vDash C$ IFF $(P_1 \,\&\, \ldots \,\&\, P_n) \vDash C$

Theorem 2: $\{P_1, ..., P_n\} \vdash C$ IFF $(P_1 \,\&\, \ldots \,\&\, P_n) \vdash C$

[26] Botting (2011,34) remarks that on this sort of proposal Addition should also count as premise circular. Here's a proof to the contrary: Given the sequent, $P \vdash P \vee Q$, MST(C) = $\{P\}$ or $\{Q\}$ but MST(**P**) = $\{P\}$, thus Addition yields two possible subset relations between MST(C) and MST(**P**): either $\{P\} \subseteq \{P\}$ or $\{P\} \nsubseteq \{Q\}$, and the second avoids the MST definition of premise circularity. Thus, Addition is still a loophole. This further shows another way the initial MST Proposal is imprecise. An improvement would be: (**P**, C) is premise circular IFF every MST(C) is a subset of every MST(**P**).

Therefore, while on the MST* Proposal, Modus Ponens is not itself premise circular, Modus Ponens is an equivalent expression of a premise circular argument. For, the single-premise equivalent of Modus Ponens is $(P \& P \rightarrow Q) \vdash Q$, and by parity of reasoning above, the only possible set of *true* sub-formulas for the whole sentence $(P \& P \rightarrow Q)$ is the set $\{P, Q\}$.

To take stock, MSTs do provide another contender for defining structural premise circularity, but it is a contender that ends in the same place: nearly all (at least classically) valid arguments are either premise circular or equivalent expressions of premise circular arguments.

12.4. Expanding the Loophole

If substantively distinct alternatives are elusive, one might instead wonder whether the correspondence between validity and structural premise circularity can be ameliorated by expanding the loophole beyond Addition. Intuitively, structural premise circularity seems to require a kind of relevance between **P** and *C*. Perhaps, then, the loophole can be expanded beyond Addition to patently non-relevant argument forms.

A non-relevant argument, let's say, is the kind of argument form that Relevant Logics are specifically designed to avoid validating. The core examples are the so-called "paradoxes of material implication," which correspond to the sequents: $P \vdash (Q \rightarrow P)$ and $\sim P \vdash (P \rightarrow Q)$. But consider the derivation of the second "paradox":

1. $\sim P$ [Assumption]
2. $\sim P \vee Q$ [Add, 1]
3. $P \rightarrow Q$ [Conditional Exchange, 2]

The proof uses Addition and Conditional Exchange, but Conditional Exchange is an equivalency rule, hence it is in the set of rules $\boldsymbol{\mathcal{R}}$.[27] The same holds for the derivation of $P \vdash (Q \rightarrow P)$. Therefore, the paradoxes of material implication are premise circular on the proof-theoretic definition of premise circularity, and they are also premise-circular on both the whole and single-premise semantic options considered in the paper, since $|\mathbf{P}| = 1$. However, neither paradox of material implication would be premise-circular on the MST options, since the conclusion of each paradox has a possible MST that is not a subset of its respective MST(**P**).[28] Thus, on the MST options the loophole does expand slightly.

Consider another patently non-relevant inference: *ex falso quodlibet*: $(P \,\&\, {\sim}P) \vdash Q$. Here's the derivation:

1.	$P \,\&\, {\sim}P$	[Assumption]
2.	P	[&-Elim., 1]
3.	${\sim}P$	[&-Elim., 1]
4.	$P \vee Q$	[Add, 2]
5.	Q	[D.S., 3, 4]

In short, the derivation requires Conjunction Elimination, which is in $\boldsymbol{\mathcal{R}}$, and $|\mathbf{P}| = 1$, so on all three options considered in the main body of the paper, $(P \,\&\, {\sim}P) \vdash Q$ is premise circular.[29] What of the MST option? Like above, the premise does not have an MST: neither $(P \,\&\, {\sim}P)$ itself nor the set

[27] One can derive the paradoxes of material implication while avoiding Conditional Exchange by using a sub-proof for →-Introduction. That sub-proof strategy, however, would use Reiteration, and so still require a rule in $\boldsymbol{\mathcal{R}}$.
[28] (1) $P \vdash (Q \rightarrow P)$ has MST(**P**) = $\{P\}$ and MST(C) = $\{{\sim}Q\}$ or $\{P\}$. Since, $\{{\sim}Q\} \not\subseteq \{P\}$, it is not the case that every MST(C) is a subset of every MST(**P**). Similar reasoning holds for (2) ${\sim}P \vdash (P \rightarrow Q)$.
[29] Of course, one might instead assume ${\sim}Q$ at line 2 for *reductio*, but that strategy will require either Reiteration or Conjunction Elimination. So, either way one must use a rule in $\boldsymbol{\mathcal{R}}$.

$\{P, \sim P\}$ can be true. But if there is no MST for **P** in this case, then MST(C) cannot be a subset of MST(**P**), so *ex falso quodlibet* would be non-premise-circular (though, trivially so) if either MST definition of premise circularity is correct.

Surprisingly, however, the MST loophole expansion would not extend further to *reductio ad absurdum*, despite its similarity to *ex falso quodlibet*. Reductio arguments have the structure $\{P \rightarrow \bot\} \vdash \sim P$.[30] So, in that case, MST(C) = $\{\sim P\}$ while MST(**P**) = $\{\sim P\}$ or MST(⊥). As the second MST disjunct for MST(**P**) does not exist, the only option for MST(**P**) is $\{\sim P\}$. Thus, MST(C) ⊆ MST(**P**). Thereby reductio is premise circular on both MST options.

In any case, while these minor expansions of the loophole for the MST options are interesting, they do little to ameliorate the general puzzle. Even on the MST definitions of premise circularity, the overwhelming majority of valid arguments would still be premise-circular or equivalent expressions of premise-circular arguments.

13. Self-Undermining

Like Mill's original argument, there remains the worry that if the VCC Argument is sound then it looks self-undermining. For it too is a valid argument, and so would be either premise circular or an equivalent expression of a premise circular argument.

I begin with three clarifications. First, self-application is not self-undermining. For example, a theory of universal grammar would apply to any and all linguistic expressions of that theory.[31] So, a theory of universal grammar self-applies in that sense. But there is nothing wrong with that theory thereby.

[30] Using "⊥" as the falsum symbol, standing for a constant that is always false, an absurdity.
[31] Sosa (2009, 174).

Second, an argument self-applies (in the relevant sense) just in case its conclusion states a property had by the logic used to prove that conclusion, as in a classic case of rule-circularity:

1. If Modus Tollens is a valid rule, then so is Modus Ponens.
2. Modus Tollens is a valid rule
3. Modus Ponens is a valid rule. [MP. 1, 2]

When turning to the VCC Argument, its conclusion, in shortened form, is as follows:[32]

9. An argument (**P**, *C*) is valid* IFF it is either premise circular or an equivalent expression of a premise circular argument.

Let "valid*" be any definition of validity for which (9) is true and $\mathbf{L_9}$ be any logic that defines logical consequence as validity*. Moreover, Let $\mathbf{L_{\vdash 9}}$ be the logic used to prove (9). Thereby, the VCC Argument self-applies just in case $\mathbf{L_9} = \mathbf{L_{\vdash 9}}$, and it necessarily self-applies just in case $\mathbf{L_9}$ is the only possible logic that could prove (9).

Third, self-undermining is a special kind of self-application. It occurs when the self-applied property inhibits the function of that to which it applies. For example, "sentences written in black font should not be heeded." That sentence token self-applies, and relative to the general function of communicating a proposition, it also self-undermines. But notice, relative to the function of serving as an example, it does not.

[32] I drop the loophole disjunct from (9) at this point since it is irrelevant to the self-undermining worry. Afterall, the structure of the VCC argument is neither Addition, *ex falso quodlibet*, nor one of the sequents corresponding to the paradoxes of material implication.

Thereby, the self-undermining worry can be approached by two questions: (i) does or even must $\mathbf{L_9} = \mathbf{L_{\vdash 9}}$? And (ii) if so, is the VCC arguments' function inhibited thereby? A "no" answer to either question would suffice to avoid the self-undermining objection, and it is possible to answer "no" to both.

13.1. First Question: $\mathbf{L_9} = \mathbf{L_{\vdash 9}}$?

Throughout, I have been treating validity* as classical validity, and thereby $\mathbf{L_9}$ as first-order classical logic. Strictly speaking, however, classical logic is not identical to $\mathbf{L_9}$. To see as much, consider how at different points in the argument, I relied upon different properties of a logical consequence relation. Borrowing terminology from Read (1988), I relied upon the following three "generic" properties of a consequence relation (Sects. 7-8): [33]

1. Reflexivity: $A \vdash A$
2. Monotonicity: if $X \vdash A$, then $\{B, X\} \vdash A$
3. Transitivity: if $X \vdash A$, and $\{A, Y\} \vdash B$, then $\{X, Y\} \vdash B$

The analysis of all semantic attempts to define premise circularity assumed bivalence:

4. Bivalence: every sentence has exactly one truth value, true or false.

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[33] These three correspond to the properties of mathematical closure (Citkin, A and A. Muravitsky, 2022). Namely, if $cl: \mathcal{P}(\mathcal{L}) \rightarrow \mathcal{P}(\mathcal{L})$ then for any A and B that are subsets of $\mathcal{L}$, the following conditions hold:

i.	$A \subseteq cl(A)$	Extensive
ii.	$A \subseteq B \rightarrow cl(A) \subseteq cl(\mathrm{B})$	Monotonic
iii.	$cl(cl(A)) = cl(A)$	Idempotent

As well as the truth-functional definitions of conjunction and negation (Sects. 6, 12). And at other times, I relied upon soundness and completeness (Sects. 9-10):

5. Soundness: if $A \vdash B$, then $A \vDash B$

6. Completeness: if $A \vDash B$, then $A \vdash B$

Beginning in Section 7, I relied upon the specific proof-theoretic rules in $\mathcal{R}$:

7. $\mathcal{R}$ = {Reiteration, Conjunction Elimination, Equivalency Rules}

Of the equivalency rules relied upon, one should be made explicit: Conditional Exchange:

8. Conditional Exchange: $(A \rightarrow B) \dashv\vdash (\sim A \vee B)$

In doing so, at one point in the argument, I relied on material implication (Sect. 12).

Strictly speaking then, $\mathbf{L_9}$ is the set of arguments that adhere to properties (1)-(8), and the VCC Argument targets that whole set. While classical logic is among that set, it is not coextensive with $\mathbf{L_9}$.[34] Moreover, not all of the properties (1) to (8) are equally essential to the argument. The three generic properties are essential for giving an account of logical consequence as a literal containment relation, and the rules in set $\mathcal{R}$ are essential for addressing proof-theoretic accounts of premise circularity. The reliance on material implication, however, is not. I used it to show that

[34] Intuitionistic logic can should also likely count, given that the intuitionistic conditional behaves like necessary material implication (Priest 2008, 105). But as I say next, material implication isn't essential to $\mathbf{L_9}$.

Modus Ponens would still be premise circular on the MST proposal. But all that is truly needed to show as much are two conditions: (i) a conditional can be true on multiple sets of sub-formulas, and (ii) the union of those sub-formula sets will either be inconsistent or validate Modus Ponens. Many alternative conceptions of the conditional should fit those conditions: connexive conditionals, for example.

Now to the point: can $\mathbf{L}_{\vdash 9}$ fail to adhere to properties (1)-(8), especially the more essential ones? Plausibly, the answer is yes. Perhaps a non-monotonic logic can, or one that is not both sound and complete. In fact, it seems plausible that the VCC argument need not be given as a formal proof at all: it can be an informal metatheoretic argument comparing disjunctions of definition candidates. I can see no reason why one of these options is not viable. Therefore, $\mathbf{L}_9$ need not be identical to $\mathbf{L}_{\vdash 9}$.

13.2. Section Question: Would $\mathbf{L}_9 = \mathbf{L}_{\vdash 9}$ Inhibit VCC's Function?

Nevertheless, assume that $\mathbf{L}_9 = \mathbf{L}_{\vdash 9}$. Would that be sufficient to dismiss the VCC argument as self-undermining? Again, I think there is room for doubt.

Standardly, the function of an argument is to demonstrate its conclusion. Grant for the moment that (i) if an argument is premise circular, then it is fallacious and (ii) if an argument is fallacious, then it does not demonstrate its conclusion. Thereby, if $\mathbf{L}_9 = \mathbf{L}_{\vdash 9}$, then the VCC Argument does self-undermines relative to that standard function of arguments.

Nevertheless, consider the status of classical logic in that case. It would still be true that the definition of classical validity can be shown to correspond to the definition of a fallacy by classically valid steps. Surely, pointing out that such an argument self-undermines does not give classical logic a clean bill of health. It seems to not resolve the problem in the slightest, for a logic ought not self-undermine in that way.[35] Thereby, even if the VCC argument self-undermines with regard to proving (9), the puzzle remains intact. More generally, a problem persists whenever plausible premises lead to an absurd conclusion, even when no one can coherently endorse the argument.[36]

I have argued that there are two ways that the VCC Argument can avoid self-undermining. First, it can plausibly avoid self-application altogether, since $\mathbf{L}_9$ need not be $\mathbf{L}_{\vdash 9}$: one logic (formal or informal) can prove a fact about some other logic. Second, even if $\mathbf{L}_9 = \mathbf{L}_{\vdash 9}$, the fact that the argument would self-undermine relative to proving (9) does not entail that it undermines relative to demonstrating a puzzle—since any logic that self-undermines in that way has a serious problem.

14. Lessons

If the VCC Argument rests on reasonable assumptions, cannot be dismissed as self-undermining, and can meet the objections posed to it, then what ought one to say about it? I see two broad options.

14.1. Reductio of Classical Validity

Perhaps the correspondence between premise circularity and classical validity (or, more broadly, $\mathbf{L}_9$) should be taken to show why classical logic should be abandoned for some alternative. After all, classical logic's definition of logical consequence leads to the absurdity that nearly all valid arguments are premise circular. So, by reductio, one infers that the right logic cannot be classical (or $\mathbf{L}_9$).

[35] Compare Sextus' remarks that a self-undermining argument can still serve its function as can a purgative, which expels itself along with the malady (*PH* 1.206, 2.188; *M* 8.480). Or again, they can function like a ladder that one kicks away after climbing up, which is the Sextan metaphor Wittgenstein was fond of (*M* 8.481; *TLP* 6.54).
[36] Greco (2008) makes a similar criticism of "dismissive responses" to skeptical arguments (116-17).

The reductio lesson faces a further task. One's preferred non-classical account of logical consequence must disagree with classical validity in ways relevant to the VCC argument. How this plays out will vary for different logics, and I leave the work to those who wish to pursue this option.[37] Nevertheless, to give one an idea of the kind of work required, consider Nelson's (at least early) formulation of connexive logic. Conjunction Elimination is invalid in Nelon's connexive logic, since he defines conjunctions as non-truth-functional wholes.[38] So, Argument (6) from the assumed list of paradigmatically premise circular arguments is not connexively valid, and by the same token, classical theorems (1) and (2) are not connexive theorems. Nevertheless, some of the arguments in Assumption (2) are connexively valid; namely, Arguments (3), (4), and (5).[39] The work, then, for the reductio option is to determine how much of the VCC argument carries over for any chosen non-classical logic.[40] If no non-classical logic could be shown to be a substantial improvement, then the reductio lesson would broaden to a reductio on deduction writ large in favor of induction. I doubt, however, that something so radical is required.[41]

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[37] The closest in print is (Botting 2011), who redefines validity to require truth preservation but prohibit the preservation of what he calls "unassertedness" (34). The resulting view, however, corresponds to the MST option above.

[38] Here's Neslon: "I do not take $p \wedge q$ to mean "p is true and q is true", but simply "p and q", which is a unit or whole…[therefore] it cannot be asserted that the conjunction of p and q entails p," (1930, 444-447). But Nelson wavers on his rejection of Simplification a few years later (Nelson, 1936).

[39] Nelson (1929): 138-141. *Opt. cit.* Mares and Paoli (2019): 416.

[40] Two historical figures I mentioned, while not having a conception of non-classical logics, do draw an analogous lesson, thinking that they show only some forms of deductive argument to be circular. Descartes argues that the syllogistic reasoning of the dialecticians is inherently circular, but not his deductive method of enumeration (CSM I.36-7; AT X.406; on the deductive status of Descartes' method of enumeration, see: Dan Garber, 1978. Taymiyya argues that only categorical syllogisms with empirical major premises are inherently circular, not those with *a priori* major premises. In my view, this is why Taymiyya never writes the sentence "all categorical syllogisms are circular" (Hallaq, xli, xxix, xxx; Lagerlund, 68). Yet, even Hallq concedes Taymiyya is committed to that thesis that all empirical categorical syllogisms are circular (xxxiv). See *Against the Greek Logicans* (55, 255-6, 290).

[41] Mill himself seems to take the radical reductio lesson, arguing that the circularity involved in deductive syllogisms shows that all genuine proof is inductive (*System,* VII: 184-193, 283). A contemporary view that might amenable to the radical reductio would be William Lycan (1994).

14.2. Structural Circularity isn't Fallacious

A second possible lesson, to which I am more inclined, is to reject the idea that structural premise circularity is a fallacy. The impetus here rests on the otherwise innocuous assumption that sometimes it is wholly correct to infer a conclusion using a classically valid argument (beyond the loophole cases). Thereby, if all classically valid arguments (beyond the loopholes) are premise circular, or equivalent expressions thereof, then the mere structure of premise circularity is not sufficient to determine when a fallacy has occurred. In other words, fallacious premise circularity would be a matter of inferences, not arguments.

The resulting project would require finding a property of inferences (not arguments) that distinguishes fallacious from non-fallacious premise circularity. The nearby literature on the fallacy begging the question already contains several views to consider. One camp sees begging the question as a pragmatic fallacy: a violation of the rules of persuasion-dialogues or, in the original version, of the ancient question-and-answer game of the *elenchus* (among others).[42] The other camp sees begging the question as an epistemic fallacy: when premise circularity is fallacious, one believes the premise(s) because one already believes the conclusion, whereas epistemic norms dictate that the dependence relation should go the other way. These epistemic norms can be further specified as bearing on the actual beliefs of the relevant subject (the subjective epistemic option) or on the epistemic states of an ideal agent (on the objective epistemic option).[43]

Lest one feels too sanguine, the lesson here affirms that nearly all valid arguments are either premise-circular or equivalent expressions of premise-circular arguments. That sounds odd. And

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[42] "Persuasion-dialogue" comes from Walton (1994, 2005). The discussion of the *elenchus* game dates to Plato (*Sophist*: 231b7-8) followed by Aristotle (*SE*, 181a15-22). Others, broadly in the pragmatic camp, include Hamblin (1970), Hintinkka (1987), and Sinnot-Armstrong (1999).

[43] In the subjective epistemic camp, one finds Sanford (1988), Ritola (2001, 2003), and Sinnott-Armstrong (1999). In the objective epistemic camp, one finds and Brio and Siegel (2006).

it does so, I take it, for the same reason, that the VCC argument itself smacks as puzzling: it is common to speak as if premise circularity is inherently fallacious. But perhaps we are wrong to do so. In any case, what lesson one ought to draw from the VCC argument is, strictly speaking, ancillary to the goal of this paper. My goal, and only goal, is to demonstrate that the VCC argument constitutes a genuine puzzle that one must face.

15. Conclusion

I began with a stipulation: (**P**, *C*) is premise circular IFF *C* is circularly contained in **P**. The first attempt to specify the circular-containment relation was in terms of explicit membership in the premise set. That quickly proved too narrow as there are paradigmatically premise circular arguments that do not have their conclusions as explicit members of their premise sets. The only avenue, then, is to understand premise circularity to be some kind of implicit containment structure. But both semantic and syntactic attempts to specify implicit containment produce a slippery slope that slides nearly to all valid arguments. Therefore, if Arguments (3)-(6) are premise circular, and if premise circularity is a structure, then all valid arguments are either premise circular or equivalent expressions of a premise circular argument—save a few odd loopholes.

Thereby, unless the puzzle can be unraveled another way, the VCC argument forces one to choose between two options that otherwise would not appear so stark. Either classical validity (and more broadly, $\mathbf{L}_9$) is epistemically normative beyond a few loopholes, or premise circularity is a fallacious argument structure, but one cannot accept both.

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Appendix: Set-Theoretic Notation

A set, **X**, is a non-ordered collection of objects. When **X**'s objects can be explicitly presented, it is written **X** = {*a*, *b*, *c*}. In this example, it is true to say that $a \in$ **X** and $d \notin$ **X**; *a* is an element or member of **X** but *d* is not. The set with no elements is denoted by ∅ or {}.

Sets can also be defined implicitly by specifying a condition that all of its members fulfill. This is done using set-builder notation: {*x* |*x* is blue} reads the set of all *x* such that *x* is blue. If context does not make it clear, the broader set on which the condition is applied can be added like this: {$x \in$ **O** |*x* is blue}, which reads the set of all *x* in **O** such that *x* is blue.

An ordered pair, (*x*, *y*), is a specific kind of set: one with a cardinality of 2 and ordered members. Its defining feature is that (*x*, *y*) = (*a*, *b*) IFF *x* = *a* and *y* = *b*. Such a feature of (*x*, *y*) is provably equivalent to being the set {{x}, {x, y}}.

Sets can be compared in various ways. A few are important here: **X** = **Y** if and only if **X** and **Y** have the same members. **X** ⊆ **Y**, which reads "**X** is a subset of **Y**", just in case every member of **X** is a member of **Y**. Notice that by these definitions, equal sets are subsets of each other and every set is a subset of itself. In contrast, **X** ⊂ **Y** reads "**X** is a proper subset of **Y**," which requires that some members of **Y** are not in **X**.

The union of **X** and **Y** is written: **X** ∪ **Y**. Informally, union is the joining of sets. So, if **X** = {*a*, *b*, *c*} and **Y** = {*d*, *e*}, then **X** ∪ **Y** = {*a*, *b*, *c*, *d*, *e*}. One can take the union of multiple sets at once as the union of a set of sets. So, ⋃{**X**, **Y**, **Z**} = **X** ∪ **Y** ∪ **Z**. Conversely, the intersection of **X** and **Y** is written: **X** ∩ **Y**. Informally, it is the overlap of sets. So, if **X** = {*a*, *b*, *c*} and **Y** = {*d*, *e*}, then **X** ∩ **Y** = ∅. But if **Z** = {*e*,*f*}, then **Y** ∩ **Z** = {*e*}.

The powerset of any set **X** is the set of all the ways of selecting members from **X**. In other words, it is the set of all the subsets of **X**. I use "$\mathcal{P}(\mathbf{X})$" for the powerset of **X**. Thus, $\mathcal{P}(\mathbf{X}) = \{\mathbf{Y} \mid \mathbf{Y} \subseteq \mathbf{X}\}$. Powersets are used to define an important function in the paper: a closure operator "$cl(\mathbf{X})$" and the more specific kind of closure operator, the consequence operator, "$cn(\mathbf{X})$." In general, a function, f, from **X** to **Y** is a map from an element of **X** to a unique element of **Y**, it is written $f\colon \mathbf{X} \longrightarrow \mathbf{Y}$.