Goodbye, Kolmogorov!

Silvère Gangloff

During the course of the XXth century, the idea of *deterministic chaos*, that the evolution of a system which is completely determined by certain fixed rules can exhibit behaviors which would intuitively qualified as random, has been central for the study of dynamical systems. Together with the idea of a world in which living organisms are the result of adaptation through time of initially simple ones, themselves randomly generated collections of elementary particles, this led straightforwardly to the question: by which mechanisms precisely can this increase in complexity be made possible ? It has been natural to search for a quantification of the notion of complexity in order to study this question mathematically. Let me notice here another strategy which is and has been the definition of *hierarchies* of dynamical systems defined by the quantification of a natural property satisfied by some dynamical systems, along which systems get intuitively more and more complex, and find in the description of these systems a definition for complexity in the specified framework.

Several quantities have been constructed in order to define complexity. However I believe that they fail the initial explanatory purpose. In this text, I would like to come back to the intuitive roots of one particular definition which I think is the closest to the concept of complexity and relied on the definition of computing machines, *Kolmogorov complexity*[1], in order, on this ground, to propose other ways to think about complexity in ultimately mathematical terms.

—— The definition of algorithmic complexity ——

In a nutshell, the idea behind this definition is that an *object* in general is complex when it is difficult to describe[2]. While it would be difficult to build a definition for all possible mental object, this is possible for a particular class of objects that I shall call *data displays.* In this setting, it is an interpretation of the terms '*diffult*' and '*description*' which allowed a certain formalization of complexity.

1. *Data displays.* — By '*data display*' I mean the result of a collection: first of a space, which can correspond to a space 'out-there', an area occupied

[1]The concept was defined by A.Kolmogorov, not in order to formalize the notion of *complexity*, but rather randomness (as I discuss later in this text). I am referring here to the later interpretation as a formalization of complexity.

[2]In this sense, the idea of chaos mentioned in the introduction is related to complexity, in the sense that the chaoticity of a dynamical systems is intuitively related to the difficulty of describing the state of the system at ulterior time.

by an object (say a blackboard for instance) or can be thought as a mental space, including in particular mathematical objects which can be conceived as a space - especially mental when it is four dimensional for instance - such as an empty matrix; second, a set of fixed identifiable positions in this space which can be conceived as separate units; third, identifiable objects which occupy one position each and can be conceived as the '*state*' of the corresponding unit, and are '*simple*' in the sense that they are not thought as decomposed into other objects.

Typically the objects considered theoretically are 0 or 1, or the absence or presence of a certain stimulus. The simplicity of these objects makes possible a systematic non ambiguous identification - we may say '*objective*' - of what are the objects (distinguishing one from the others) and which object is '*this object*' in the set of possible ones. This allows one to consider, for the verification of a certain statement on these objects, or in order to collect '*information*' on the these objects, to consider each one separately (inhibiting the stimuli generated by the others) in order to remove the influence during this process of the *conceptualization* of the display as a whole. The properties of this kind of this object characterize what we call '*data*' and explain the interest (sometimes obsession) that we culturally put in them. The properties of data often lead us to believe that the conceptualization of the data is straightforward, implying that they way it is done does not need to be thought through. But this is another story.

2. *Description.* — A description of a data display is the process of reproducing this data display in another similar space and positions, first by placing the positions in a certain order and then objects in each of these positions one by one, following the same order. It is also possible to change the space and the configuration of positions if a one-to-one correspondence between positions in the two spaces is agreed upon beforehand. With such a correspondence in mind, the relative difficulty of describing a data display in the first space and provided a set of positions is similar to the relative difficulty of corresponding data displays in the second space and set of positions. Hence, in order to define this relative difficulty, it is sufficient to consider only one simple space and positions. For the example, we consider the space constituted by the paper and positions placed in a line with constant distance between two consecutive of them. We will consider also that the objects to be displayed in these positions are simply 0 and 1.

3. *Difficulty of describing a data display.* — Let us assume I have a particular data display in mind, or in a document that the reader does not have access to. After we agreed that the data display that I have in mind consists in 0s and 1s displayed on seven positions in a row, I could describe this display by saying: '*first is displayed* 1*, then* 0*, then* 0*, then* 1*, then* 1*, then* 0*, then* 1'. Then the reader has also in mind the data display 1001101. Let us consider another example. If the data display was 0000000, I could have repeated the same procedure in order to describe it. However I could also have simply said '*all the objects displayed are* 0', ultimately providing the reader with the representation of 0000000 in mind. Such a simplification is not possible in the case of the data display 1001101, which makes it more *difficult* to describe in the sense that it takes more effort to me in order to describe it.

4. *Quantification.* — In this context, the complexity of a data display quantifies this effort, and its mathematical *definition* is based on the observation that a description *is* an algorithm. The more time I have to use in order to formulate this algorithm, the more effort the description requires. This difficulty, however, is not intrinsic to the data display itself, for the reason that I have a choice between various possible algorithms. In the example 0000000, I could also have chosen to describe it by saying '*first is displayed* 0*, then* 0*, then* 0*, then* 0*, then* 0*, then* 0*, then* 0'. In order to define the intrinsic complexity of a data display, I need to consider the algorithm which requires the shortest time to formulate.

5. *Universal interpreter of programs.* — This provides a quantification of complexity, but it is relative to a subject taking a data display as an object, as well as another subject to which it provides a description. If we like to have a quantification which is not dependent upon subjects, in particular leaving open the possibility to replace subjects with machines, there is more precaution to take, for it is in principle possible to find, for any data display, two communicating machines for which the '*description*' of the data display by the first one to the second can be done in '*instant time*'. If we were about to define complexity intrinsically to the object by replacing descriptive effort between two human subjects by the least effort taken by two communicating machines, we would loose any relativity of this complexity. This problem disappears when considering the first subject to be reduced to a universal interpreter of programs, and the second one to be reduced to any '*observer*', receiving the result of a description done by the universal interpreter. By this universal interpreter of programs, I mean a machine which, provided a way to signify any algorithm by a data display of a fixed set of objects (called program), interpret this program, execute the algorithm, display the data which result from the executing of the algorithm and then stops. The theoretical existence of such a universal interpreter was provided by A.Turing.

——— Algorithmic complexity and randomness ———

1. *Randomness as absence of structure.* — As a matter of fact, the exposition of the last section corresponds to an interpretation of some ideas of A.Kolmogorov that he exposed in his article *On tables of random numbers*[3], whose purpose was to describe a way to apply the rigorous mathematical framework which had been developed for probability theory to '*real random phenomena*'. As he had argued, the concept of probability is intrinsically related to some frequency computation. While in the developed mathematical framework the computations of frequency are done on infinite series of trials, only finite series are possible in the reality.

[3] **A.N.Kolmogorov**, *On Tables of Random Numbers*,Sankhyā: The Indian Journal of Statistics, Series A, Vol. 25, No. 4 (Dec. 1963), pp. 369-376

It is for this purpose that A.Kolmogorov defined his notion of complexity. He himself did not use the term *complexity* as a property of the data display, but rather of any program which describes it[4].

It is not needed to compute Kolmogorov complexity to foresee that it is lower on a sequence such as 0000000000000000 than on the sequence 0100101001000111, for the first one can be described as '*the sequence of length sixteen containing only zeros*' while the second one could be described by '*zero, then one, then zero, ...*', possibly grouping the last three ones for instance, without changing much the length of the description though. A sequence such as 1010101010101010 has an intermediate Kolmogorov complexity, for it consists in the repetition of the pattern 10, eight times. We could think that this complexity simply measures the '*periodicity*' of a sequence, the minimal length of a pattern such as 10 out of which repetition it can be obtained. However, the sequence 1011101010111010 is not obtained by the repetition of a small length pattern but can still be described by '*one every other position, zero on every other remaining position, then one on every remaining position, and then zero on the remaining positions*', obtained by the superposition of the *patterns* 1 1 1 1 1 1 1 1 , 0 0 0 0 , 1 1 and 0 . In a sense, the complexity is lower when it is possible to find '*structures*' in the sequence on which the description can rely in order to simplify. A sequence with high complexity is *random* in the sense that it is far from having any internal structure - rather than being the result of a sequence of pickings according to a probability law, reflecting frequencies over infinite sequences of trials.

2. *Negative definition of meaning.* — Of course it is not clear what should count as '*pattern*' or '*structure*'. As a matter of fact, what we call pattern or structure is usually the result of a purpose and a design for, an intention towards, a meaning, which are all relative to a subject. In a sense, Kolmogorov complexity defines them negatively, by pointing at data displays do not appear as the result of an intention, a design, etc. The search for a positive definition could seem hopeless. Not if we understand what in the subject makes meaning specific to it. For me, this is the *world* of the subject in the sense of the set of experiences it holds as *possible*. In such a world, what makes possible patterns *significant* is their relation to the causal structure of the world.

——— Descriptions of the subject's experience ———

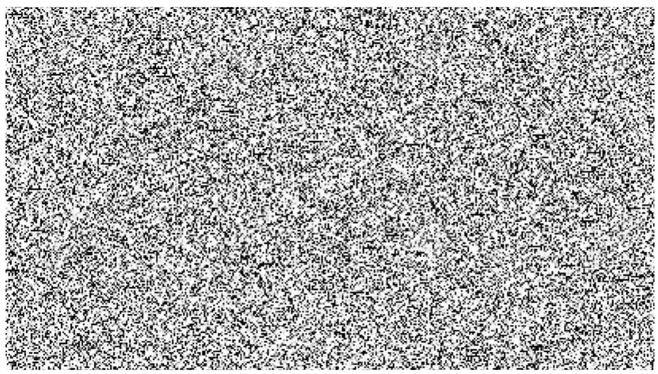
Let us take a step back. We have seen in the first section how it is possible to construct a notion of complexity for data displays - in the sense that complexity is the difficulty to describe such a data display - which constitute a particular model of *experiencing*. By experience, we know that descriptions are possible for more general forms of experiencing.

[4]Let us notice that what Kolmogorov called complexity of a program is simply its length. Of course the more complex an algorithm, the longer it is; however the converse is not true. One could take the algorithm as an object, and then try to define its complexity, but the problem would thus be left intact.

1. *Objects of description in general.* — I distinguish at least three types of descriptions of experience: *(i)* descriptions of worlds as sets of possible experiences, extracted from the subject's experience and grouped together; *(ii)* descriptions of singular experiences relatively to a world they are in; *(iii)* descriptions of worlds as extracted from experiences of another world.

The second type is relatively close to descriptions of data displays, although an understanding of the other types shall shed a different light on it.

In order to illustrate what I call '*description of a world as sets of possible experiences*', I like the example of '*screen snow*'. Sometimes on old television screens - at least - we could observe pictures as the following one:



What we refer to as '*screen snow*' is in fact the set of experiences corresponding to these pictures. How is it constructed as a concept ? In order to answer this question, let me notice first that even the concept of '*image on the screen*' is not so simple: why is the image '*on*' the screen ?; why do we conceive such an image as separated from the remainder of the vision field ? I would say that what we call '*screen*' refers first to the experience of it when it is turned off, and here the word '*on*' inherits its meaning from the same word used in a situation in which $\mathcal{I}$ look at an object resting on another one. Let us say a lemon on a table for instance. $\mathcal{I}$ can always take the lemon in my hand and move it away, thus observing again the part of the table that it has hidden, and then move it back, hiding it again, at will. In the same way the image hides the screen, and can be '*moved away*' by turning off the screen. Of course the image is not '*really*' on the screen: for instance I can look at the screen from the side, and will not observe any image which could be on the screen. However this meaning of the word '*on*' is relative to the typical situation in which I look at the screen, which is determined by its '*function*'. The reason why $\mathcal{I}$ conceive separately the images on the screen from the background is that in the typical situation in which $\mathcal{I}$ am looking at the screen, $\mathcal{I}$ can act on what appears on the screen in ways which are different from the ways $\mathcal{I}$ can act on what $\mathcal{I}$ see in the background. While $\mathcal{I}$ can always hide parts of my visual field with my hand for instance, $\mathcal{I}$ can also act on what appears on the screen by pressing a button on the remote control. Among *the pictures which can appear on the screen*, the '*screen snow*' pictures are separated and grouped together as a consequence of a common property: the absence of any meaningful pattern which can also be seen in the world outside of the screen.

The words '*screen snow*' are a designation of this set of experiences. The same con-cept could be designated by another string of symbols such as $'rty * phg * 1a2x'$ for instance, if we decided to adopt this convention. On the other hand, while the designation '*screen snow*' is not completely determined by the con-cept itself, it is partially because it acts as a description. In the above typology, it is mainly of type *(iii)* because it indicates what it is relatively to the subject's world: '*screen*' because it can be observed on a screen, and '*snow*' for the absence of meaningful pattern.

The reason why I am writing about snow screen is that we are naturally driven to qualify it as '*simple*', by opposition to a computer, or a living organism for instance, in a similar way as *air* or *water*, a simplicity which is relative to description of type *(i)*. The difference of complexity between two worlds as sets of experiences is related to the number of meaningful patterns and their different possible (causal) interactions, or equivalently the logical layouts of worlds as sets of experiences which can be extracted from them - corresponding, for machines or living organisms, to parts serving a certain '*function*', such as memory or computing unit. This complexity corresponds to the idea of '*objectal complexity*' that I proposed in my other text[5]. If we have a definition of what counts as meaningful pattern, we should be able to construct a proper definition for this form of complexity. I believe that these patterns are related to the causal structure of the corresponding world.

Provided a description of a world based on its meaningful patterns, a description of type *(ii)* is the description of an experience in this world which informs how these meaningful patterns are displayed in this experience. For instance such a description tells about the *states* of the various components in the case of a machine. A priori it is not clear that all descriptions of type *(ii)* are descriptions of states.

Overall, while a concept similar to Kolmogorov complexity could be defined in principle for descriptions of type *(iii)*, which would rely on the length '*algorithms*' by which $\mathcal{I}$ come to isolate sets of experiences from my world, the natural concept of complexity relates to descriptions of type *(i)*.

2. *Description for oneself.* — The description of the subject's experience in general depends on the subjects involved - in particular the possibility to describe a certain experience or the possible ways to describe it. It also depends on whether the subject who describes and the one who listens are different or one and the same, for the means by which a description is possible are dependent upon this fact. The nature of the description also seems to differ. When on subject described an experience to itself, such a description consists first into the identification of patterns. Let us notice that there is no possible description without this identification, and that in the case of data displays, it is encompassed into the *model* of experiencing that this term refers to, whose purpose

[5]**S.Gangloff**, *A formal window on phenomenal objectness*

is in fact to systematize this identification. Possible patterns in data displays come after a construction, a synthesis, while in the experience of a subject, elementary, or simple objects are constructed out of identification. Describing an experience to oneself is actually making present to one's mind the various patterns which *compose* together this experience. The nature of the description consists in a transformation of this experience which makes parts of it appear which are connected mentally to other mental objects that we may call *names*, and through this connection are also connected to a set of transcendental operations which can be performed on the patterns - out of which, for instance, one can make a decision to act in a certain way. As a matter of fact, when I say that the patterns '*compose*' the experience, what I really mean is that they compose the experience constructed out of the identification of these patterns, precisely because the constructed experience is so out of composition, faithful to the initial experience, of the identified patterns.

3. *Description for another person.* — The description of an experience to another person consists in the description of the experience which is the result of description to oneself. Practically speaking, when $\mathcal{I}$ describe an experience to another person, $\mathcal{I}$ *name* patterns present in this experience and relative positions in which these patterns are displayed, in such a way that the other person can re-construct an experience - externally or by imagination - an experience which is equivalent to the one $\mathcal{I}$ have, in the sense that the other can operate on the reconstructed experience in a way that is faithful to the original one. Let's say for instance that I have planed to visit an apartment with someone who unfortunately cannot be here with me. I will probably describe my visual experience of the apartment and out of a reconstructed experience this other person can *add* mentally some furniture, personal items, etc., in order to answer the question '*would I like to live in this apartment ?*' and communicate to me the answer.

When two persons do not share the same language, descriptions becomes difficult, for the reason that in different languages, patterns are named in distinct ways. Of course for natural languages and when the description is about an experience of the material world, *translation* dissolves this difficulty. However the term '*language*' can be thought in a more abstract way as referring to a conventional correspondence between mental patterns and names. Two persons who do not live in the same world may have experiences of different nature, and one of them may not have a name for some patterns in the other's experience. When considering visual experience, there is a possible strategy in order to counter this phenomenon which relies on the general structure of visual experience: one can transform in principle any visual experience into a data display which consists in a grid of positions on which are uniformly colored squares are placed, such that the color corresponds to the one of the corresponding position in the visual experience I am having. Such a data display can be communicated using a code for colors with a simple algorithm. For a large picture it is impossible for the other to reconstructed it purely mentally when it is described this way, however it can be reconstructed externally. This allows in particular the designation and definition of a name for a pattern contained in the picture.

This way of describing is of course not optimal when two persons share the same language, or at least segments of the language which relate to the experience to be described.

4. *Aside: mathematization of experience.* — In the terms I have used above, the mathematization of the world that we know, concerned mainly with visual experience, has consisted in a descriptive enrichment of experience via some transformation rather than a reduction of experience, as usually thought, to its '*structure*', in such a way that the experience can be recovered from the constructed experience - this recovery implies in particular to '*forget*' intermediate constructions. In principle, a method for recovering the initial experience from the transformed one makes possible to reason on the former one via the last one and operations on it - which are specific to this one. A secondary interest of this *mathematization* is that the product of the construction - mathematical objects - is simple enough to be described to others in a way that is not affected by the subject of culture, and for this reason the mathematization makes possible a systematization of intersubjective operability on mathematical experiences in the synthesis of a collective conceptualization of the world.

In the field of mathematics, only mathematical experiences are studied, without necessarily considering how these experiences were constructed initially out of natural experiencing - in particular without thinking about the subject of this experience. The will to model the collective discourse on the mathematical field comes from the efficiency of intersubjective operability coming itself from the nature of mathematical description of the world. However, it implies loosing the sense of how mathematical objects are meaningfully related to natural experiencing. Although we do know that the constitution of mathematical objects is based upon this natural experiencing, we tend, in order to understand real phenomena, to extract mathematical structures then considered in themselves without considering if the reasoning operated on these mathematical structures actually informs about the real world.

There is no principle which prevents the mathematization of consciousness as such - despite the fact that the current form that the mathematical field has taken -, in particular when considering it through the spectrum of transcendental operations which constitute the relation of a subject to its world. However it matters for this mathematization to be possible to generate mathematical objects which faithfully describe the reality they refer to. The analysis of descriptions done above reveals one criterion for this is the following: that the possible operations on the mathematical object reflect the possible operations naturally done on what it is meant to represent. This mode of generation of mathematical objects is purely subjectively objective in principle, although this does not prevent intersubjective operability between subjects qua subjects, provided a sensibility to the conditions under which mathematical objects can meaningfully represent real phenomena.

5. *Difficulty relative to mental effort.* — The complexity of an object is related to the difficulty of describing it to oneself - whereas the difficulty of describing it to another person comes from constraints of language. Provided this, the mental effort that describing an object takes is in principle a quantification of the complexity of this object, although it has no evident formal counterpart. Considering the *reasons* for the mental effort may lead to such a formal counterpart in particular cases. I see at least two possible reasons: the quantity of meaningful patterns which are necessary to form a description - for instance in the case of a machine which has many distinct components-, and the indistinctness of the patterns - mental contents for instance. In any case, the mental effort is taken by transcendental operations by which a description to oneself is constructed, and its quantity is related to the allocation of *resources* - spatial, and material (the structure of mechanisms which realize these transcendental operations - that the subject has. Defining complexity in the sense presented here requires overall to answer the question: *what exactly can be computed purely mentally ?*, and *why ?*. I begin exploring this question in the next section.

Let me only notice here that complexity of an object is sensitive to the description, in particular if it is a description for oneself or for another person. For instance, while spatial locations in the visual field necessitate an encoding of the space in the description to another person, they are immediately accessible to oneself, thus the description in this case requires less effort. Also, the proof of a mathematical theorem can sometimes be difficult to reconstitute mentally from the way it is written, and thus to describe to another person, while it is ultimately simple in itself, once this reconstitution is done.

——— What can be computed purely mentally ?[6]———

The purpose of this section is not to propose an answer to this question, but rather to define it more precisely and to propose some threads of thoughts that the question motivate.

By '*purely mentally*' I mean that the computation is done using only mental *resources*, in particular not using a mechanical computer or using the hand to write symbols on a piece of paper or board - the purpose being to understand better what these mental resources are in general and how they are constrained. Furthermore, by '*computation*' I mean any mental process by which a human subject transforms an experience into a mental content. This can be the truth of a mathematical statement or the difference of color of two uniformly colored areas in the vision field. This makes this notion of computation different from the one of informatics, because a computer can recognize colors only if it is fed with a code which makes colors correspond to *symbols* of these colors - which in order to be constituted necessitates a subject of visual experience[7]

[6]From the point of view of philosophy of mathematics, and in particular the philosophy of mathematical practice, I believe that this question is of major technical interest, in order to separate - and conceptualize this separation - between what in the practice is relative to experience qua experience, and what is not.

[7]One simple observation, which might be interested in order to analyze purely mental computation: the operation by which two colors are compared purely mentally is intuitively irreducible to other more elementary operations, contrary to forms for instances. It is possible that the intuitive irreducibility of an element of experience corresponds to the irreducibility of operations about them, and through this could be ultimately formalized mathematically in

There is a natural objection to the possibility of answering the question *what can be computed purely mentally ?* because computational capacity varies across subjects, in particular this is obviously influence by training. However there are limits to what training can do. Furthermore, the point is not really to search for a precise answer to the question, but rather to build a clear conceptualization of related intuitions, and to study subsequently the mind through a set of forms conceived this way rather than as a unique fixed form.

1. *Vanishing information, locality.* — As a mathematician, I sometimes don't need to use a board or piece of paper in order to construct a mathematical reasoning. While computation without visual support is thus possible in principle, this context provides immediate limitations of purely mental computing. For instance, I can draw some triangle on my '*mental board*' by positioning three points and connecting them with straight segments. While I focus on this constructed triangle, it stays present, '*accessible*'. However, if I construct another triangle aside of it, and another one, I will *forget* where I placed the points defining the triangle, although I remember the relative position of a restricted area surrounding it. On the other hand, it is possible in certain cases to hold two objects relatively far from one another when I 'maintain' this presence by considering them alternatively and continuously. It is not immediate to characterize sets of objects which can be '*co-present*' on this mental board, thus there is more exploration left to be done.

2. *Classes of objects and construction schemata.* — The possibility of mathematical reasoning without visual support proves that visual phenomenal experience does not participate to the nature of mathematical objects, even though mathematics are in general learned using visual experience. Classes of objects are identified *a posteriori* - in a learning process which may be similar to the one implemented in machines - but they are *a priori* in nature - once they are identified, they are not simply probability distributions, but plain mental objects. I distinguish *fundamental classes* as the ones which are characterized by a construction schemata and *derived classes* as the ones characterized by an identification process over objects of a fundamental class.

For instance, a triangle is constructed by a sequence of mental operations: placing three non-aligned points and connecting them by straight segments. I think that the information that two operations are similar results from a non-voluntary process which derives mechanically from the operations themselves. The fact that a triangle in a plane - of the vision field for instance - can be constructed out of the data of six real numbers which satisfy some condition corresponding to non-alignment derives from a construction derives from the construction schema of triangles. From a purely mental point of view, the *complexity* of objects of the triangle category corresponds to the construction schema of this category. This is also the number of elementary objects which compose them, themselves results of elementary operations which compose the construction schema.

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a more phenomenologically faithful way than in *Integrated information theory.*

The class of equilateral triangles is derived from the one of triangles: considering a triangle, one can identify it as equilateral by rotating twice, in such a way that extreme points are permuted, and each time checking if the segments departing from any point of the initial triangle coincide with one of the rotated triangle and departing from the same point. Let us also notice that it is as well a fundamental class, as there is a construction schema for equilateral triangles, which is more complex than the one of triangles in general. However it is intuitive that an object of this class in itself is as complex as any triangle, which indicates that the number of elementary objects which compose it characterizes more precisely the notion of complexity.

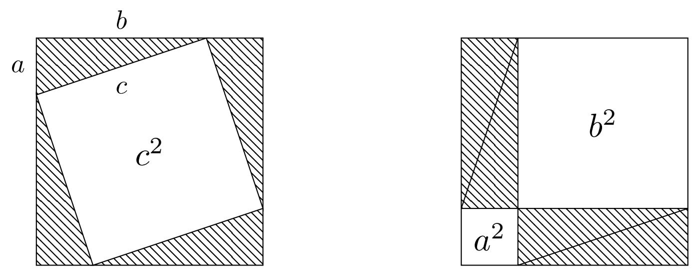
3. How are classes of objects stored ? — When $\mathcal{I}$ recognize an instance of a class of objects, $\mathcal{I}$ am usually not thinking about the process by which it is recognized. $\mathcal{I}$ only use the mental image that is the result of the construction process of objects in this class in order to compare it with the identified object, unless $\mathcal{I}$ doubt that the object $\mathcal{I}$ identify indeed corresponds to this class. I suspect that what is present in my mind when $\mathcal{I}$ instanciate an object of a certain class corresponds to '*neurons*' - or a certain mechanism - which are currently '*active*'. When I am considering mentally the object, $\mathcal{I}$ constantly maintain this activity - countering vanishing of information. Probably, $\mathcal{I}$ am better used to maintain this activity for objects related to vision, which would explain why it is difficult to maintain the activity of neurons underlying construction operations. However, a residual activity of these neurons right at the moment of the construction may well allow me to '*train*' this capacity, and subsequently make present in my mind the construction operations themselves and '*observe*' them. The term '*observing*' may not seem to be adapted, however I believe that it is possible to extend its meaning from objects which we usually think as '*observable*' to mental operations. Whenever an object appears in my vision field, observing it means letting it appear, without acting upon its presence. $\mathcal{I}$ still do actively participate in the way the object, or parts of it alternatively, appears, in particular isolating it from the background in which it appears. However the initial presence of such object is not actually caused be me. This is how mental operations differ from visual objects: I cannot just '*let it appear*' - precisely because it is an operation, but also because it does not *appear*, in the same way as visual objects do. Observing it consists on the other hand in the causal relations in which it is involved.

I think that describing how classes of objects are '*stored*' - where storing makes possible to instanciate any object of a class at will - can only be done by understanding how operations are realized. It is reasonable to think that only classes of objects are stored and not instances themselves, for the reason that there too many different possible such instances. Furthermore, a class is likely to be stored under the form of its construction scheme. Each of the operations in this scheme can be in principle realized by a mechanism - whether it is composed only of neurons or also other types of components - which, in order for me to apply it, has to be *activated* - by '*visiting*' it. It is left to understand *where* each of these mechanism may be located in the brain and how they are *assembled* in order to instanciate an object. Operations composing a construction scheme are involved in a variable number of different other construction schemes. The location of a mechanism should reflect this number - which corresponds to the degree of universality or specialization of an operation. For instance, the '*placement of point*' operation is specifically related to visual processing and the corresponding mechanism is likely to be closer to areas specialized in this, contrary to the mechanisms underlying the instanciation of integers. While singular concepts which contain essentially a certain number - such as for instance the one of triangle or square - may be realized by a collection of information, including a material encoding of this number, not all possible number can be encoded in this way. Furthermore it is for me unlikely that the information of the integer is '*wired*' directly to the visual processing area when instanciating a triangle or a square for instance. This suggests the existence of a *universal counter* constantly communicating with this visual processing area, through which transits the integer information in specific concepts. Through this universal counter can also be constructed locally any integer - for instance when $\mathcal{I}$ want to construct a regular polygon with 28 sides. Now, how could this universal counter be materially implemented, and how can it communicate to the visual processing area ?

With the observation of purely mental counting of large numbers, I come to think that only few '*natural*' numbers - three, four - are stored as such, and that large numbers are processed in general in a different way - although they are all called numbers. When I use the number *three* in the construction of a triangle, I do not need to consider it as integer which is algebraically related to other integers: it is only and simply *three*. On the other hand, *integers* form a class of objects which are the initialization result of a counter, itself an instance of the class *counters*, and what I designated as *universal counter* is rather the construction scheme for the class of counters. This construction scheme is the following: a finite set of symbols - for instance 0 to 9-, the placement of some symbols in this set on fixed positions, the designation of a nihil symbol - usually 0 - and sequences of operations and rules for applying them sequentially. A number is the result of the placement on positions, while the object counter consists in the sequential application of rules which '*decreases*' the number: for instance from 128, to 127, to 126, etc.. until 000. Again, only the construction scheme of the counter class is stored.

I leave for further reflection the question of how precisely a counter is instanciated and connected to other construction processes. Let us only make some observations. In principle, other forms of counters are possible besides the one which correspond to decimal representation of numbers, for instance displaying a series of points in line which are suppressed one by one. Purely mentally though, because of information vanishing, it is easier to use counters corresponding to decimal representation. The number of symbols involved, contrary to the encoding of counters in machines - in which the binary representation is the simplest to use -, seem to realize a trade-off between the area occupied by the representation of usual large numbers - which subjects it to information vanishing - and the number of symbols involved - a large such number makes it more difficult to implement the counter purely mentally. A second observation is that because counting with large numbers uses visual processing, it is difficult to construct purely mentally visual objects which involve large numbers. When $\mathcal{I}$ use a visual support on which $\mathcal{I}$ can write in order to construct a regular polygon with 28 sides, $\mathcal{I}$ can use visual processing in order to count, because information contains in the visual support does not vanish. Third observation: although a counter is systematically instanciated in order to count, it is made easier by uttering the words which correspond to each symbol, for instance 128 by one hundred and eight, because the association between symbols and words make these symbols more present to the mind, and thus easier to manipulate.

4. Visual proofs and generality. — How is it possible to grasp the generality of a mathematical truth by looking at a unique drawing ? An example of visual proof of a theorem is the following one, for pythagorean theorem:



The drawing itself does not contain a proof. Rather, it *suggests* it, in such a way that the sentiment of truth of the theorem relies plainly on arguments suggested by the drawing. A rough description of the suggested proof goes like this: **(i)** $\mathcal{I}$ can always display four copies of any right-angled triangle in such a way that the longest side of each copy form a square, and the other sides form another square, such that the first is included in the area delimited by the second. **(ii)** After $\mathcal{I}$ copy the exterior square elsewhere, $\mathcal{I}$ can assemble the four triangles by pairs to form two rectangles, and display these rectangles in order to cut the square into two smaller squares. **(ii)** In the two constructions, the exterior square has the same area, because each one is a copy of the other. In the first construction, the area of the inner square is c^2, where c is the length of the largest side of each triangle copy. Furthermore, the difference of area between these two squares is four times the area occupied by a triangle. In the second construction, the area of the exterior square differs by four times the area occupied by a triangle from the sum of areas of the squares into which the rectangles cut this exterior square. The area of the smallest one is a^2 and the one of the largest one is b^2. We deduce that $c^2 = a^2 + b^2$.

Visual proofs - proofs which rely on a visual support - are often considered as less *rigorous* than purely formal proofs for the same statement - which rely, for the pythagorean theorem, on inner product. However, ultimately, every mathematical reasoning relies on space relies on the *sense* of space, grasped through vision or touch for instance: the fact that one can encode points in a two-dimensional space with two real numbers does not even make sense without a sense of space. Therefore the use of senses cannot be what makes visual proofs less rigorous. In a purely formal proof, spatial objects - such as triangles, points or segments - are put in correspondence with formal objects in such a way that any spatial object can be *identified*, in the context, with its formal counterpart. The point of this correspondence is ultimately to make possible the consideration of these objects *separately*: in particular the perception (or conception ?) of such an object is not affected by the presence of other ones. This applies also to operations which *combine* objects, in the sense that is only present to the mind the collection of objects to be combined, as a collection, the operation itself, or the result of this operation. The reason why this *separation* is important comes from the existence of visual illusions, which consist in the distortion of the visual experience under the action of certain operations, distortion which is due to the influence of the presence of other objects.[8] It is in principle possible to conceive that a visual proof can be as rigorous as a purely formal proof as long as the visual experience keeps outside of the domain of illusion.

This comparison between visual proofs and purely formal proofs leaves some invariant visible: the fact that a proof consists ultimately in a series of mental operations of which mathematical languages and pictures are only a tool in order to talk about them. It is possible to grasp the truth of a mathematical statement in a single drawing when all the operations necessary for a proof and the sequence in which they are done are represented in the drawing. Is a meta-mathematical truth that every mathematical statement can be formulated as a causal relation. A proof of a statement relies on simple causal relations whose truth comes from a *pure* sentiment of truth which can be considered as data - axioms - and logic rules which are concerned with properties of causal relations - in particular the parallelization or composition of causal relations.[9] The sentiment of truth results from of a statement is caused by the sequence of operations which are concerned with elementary causal relations. On the other hand, even if the truth of a statement is graspable through a drawing, a visual proof leaves some *ghost doubt* which is not supported positively but comes from the non-purity of the sentiment of truth (it is co-present with the visual support). Through a rigorous proof, it is the pure sentiment of truth which of the statement is *computed*.

[8] For instance, in E.H.Adelson's checker-shadow illusion, if one considers successively each of the squares together with its neighborhood, the colors appear to be different. When isolating the pair of squares before comparing them, the colors appear to be the same.

[9] This interpretation of mathematical reasoning in terms of causal relation leads to two threads of thoughts: first, that meaningful mathematical concept are tied, and could be formally tied, with the notion of causal structure; second, a practice of mathematical nature can be developed in order to understand experiential concepts such as the *form* of the space we live in, by relating the formation of such concepts out of the causal structure of experience qua experience of the world - contrary to the mathematical field which is concerned with causal relation between objects and operations on them abstracted from experience. This search is not concerned with the truth of causal relation, but rather how concepts derive from them. Here the generalization of mathematical practice is not about the form of the discourse (which I have criticized in On faith in the mathematical practice), or its properties (which I have explored in *A formal window on phenomenal objectness*), but rather the generalization of *purely mental computation* which root the cognitive practice of mathematics and *explain* the form of mathematical discourse and its properties.

Let me notice further that in visual proofs, the operations which are involved in a purely formal proof are collected into global operations - such as for instance the equality of the area delimited by two triangles such that one obtained by translation and rotation of the other, which is verified by actually translating and rotating one triangle in order to superimpose it to the other[10] - which add to the influence of present objects - here in particular an operation - over the perception of other ones.

——— Complexity as behavioral capacity ———

I would like to end this text with the proposition that the reflection I led above may shed some light on the question of the complexity of the living, as well as the complexity of consciousness: how is it possible, from a set of data, to detect the presence of life - for instance in data collected on other planets - or the presence of consciousness - for instance by measuring the activity of the brain of a patient in coma state ? Intuitively both phenomena imply complexity, and it is and has been natural to search for a quantity, in principle computable from the data, which measures *how much complex* this set of data is, in a sense of complexity which is tightly related to the phenomenon in question: a '*high*' (what does determine what high should mean ?) of this quantity would reveal its presence.

What being *tightly related* to the phenomenon in question for a sense of complexity is to agree with the judgment of complexity, from the point of view of a subject. As I have discussed above, it is possible to think of complexity from this point of view as *difficulty of description.* Hence the question: in what sense is the living, or the phenomenon of consciousness, difficult to describe ?

As I have exposed it above, the concept of complexity applies to worlds as a sets of experiences. In particular a living being is such set of experiences and it is to this being as such that the concept of complexity applies - whereas the *organism* is similar to a '*screen*' which supports these experiences, the place where they appear and the matter which supports them, and group them together. The complexity of such living being lies in the fact that in order to describe its behavior, one has to rely on several meaningful distinct patterns - organs, cells, hormones, etc - as well as several relative displays. In the same way as for describing the possible behaviors of a machine, $\mathcal{I}$ can transform in principle in faithful way the experiences $\mathcal{I}$ have of a living being into a set of data displays - property that I would like to call *displayability* - in order to understand it.

[10] This brings the question: how do precisely do we '*move*' purely mentally an object onto another, and exactly what is so ? I leave this question for further reflection. Let me only notice that two examples to consider are: the experiential '*verification*' that two straight segments are parallel, which consists in a '*projection*' of one segment onto the other (more precisely, a segment is first reduced to its extremities which are projected on the extremities of the other segment, and because they are respectively at the same distance, these parallelized projections are perceived as identical, then the projected is reconstructed from the projected extremities and perceived to coincide with the second segment); the comparison of the length of two such segments. Each of the examples corresponds to a well-known form of visual illusion: respectively the Hering and Muller-Lyer illusions.

On the other hand, the complexity of the phenomenon of consciousness comes from the difficulty to identify mental contents - what $\mathcal{I}$ judge as conscious is what is subject of experience and therefore mental contents - which in general can only be sensed and designated, conceived in another person as a best explanation of a behavior similar to one I may have. In particular mental contents are actually not displayable altogether. While consciousness is complex in this sense, this complexity is not essential to consciousness it should not be identified with it.

A.Turing's definition of computing machines - in particular the idea of universal computing machine - had several known consequences; despite this, I believe that important philosophical consequences of his work have been missed[11]. One of them is the efficiency, in defining meaningful mathematical concepts, of the method through which A.Turing arrived at his definition - considering the mathematician's mind in the process of computing. Another one is the possibility of characterizing the human mind with its computational universality. There are two objections to this conception, which I shall counter. The first one is that it is possible to implement universal computation in machines. However in such a machine, the causal relations which constitute any process simulated by all derive from an initial action of a human being. What differenciates universality of the human mind from universality in machines is autonomy[12] - processes are caused partially from within; on the other hand, autonomy alone is not sufficient to characterize the singularity of the human mind. The second one would question the identification of consciousness in another subject: how is universality recognizable ? As a matter of fact, I think that it possible to grasp intuitively and purely - although not explicitly - the mechanics of thoughts which underlies - allows - this universality in the same way as mathematical truth is grasp, in oneself and in the other as a best explanation of the other's behavior - in particular the possibility to agree on any particular algorithm to execute collectively.

Complexity of the human mind as a form of behavioral capacity - precisely, universality - does not correspond mathematically to a quantity: should it ? This idea has however the following practical implications: in order to determine a way to detect consciousness, one possibility is to understand the conditions of possibility of an autonomous form of universality in a physical system; in relation

[11] Sometimes it is believed that the purpose of mathematical practice is to prove theorems. Other times it is thought that theorems have an epistemological role of confirming definitions which proposed out of intuition but reveal to be naturally involved in complex proofs, meaning in particular that the concept which underlies the definition has a reality beyond the mathematician who proposes the definition. In the sense, theorems are the sign of an epistemological fertility of the definition. However, I believe that definitions themselves are not the purpose of mathematical practice; they are only a useful tool in order to grasp the structure of the world, to understand it. Whenever the definition of a concept can be mathematical, this makes possible to grasp it in a systematical way, through simple, defined, controlled operations. Sometimes, however, there is no natural mathematical formulation of the concept; this should not forbid the search for an understanding of it through non mathematical means, for precisely it is this understanding which is the point of mathematical practice.

[12] In a sense, universality in machines is only the reflect of human mind's universality.

with the idea of an increasing complexity through evolution, this leads to the question: how can a physical system evolve into one which is autonomously universal ?