On faith in the *practice* of mathematics

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For centuries now, the mathematical discourse has been considered as an ideal of clarity, coherence and certainty for the inquiry of science into the structure of the world as it is in itself, independently of thoughts and beliefs. It has thus been a commonplace to oppose mathematics to religion in general, as characterized by the obscurity, dogmatism, and sometimes self-contradiction of religious discourses. Although this vision is rather straightforwardly contradicted by the possibility of the religious sentiment and scientific rigor coexisting in the same mind - say of I.Newton for instance - this fact has not been, to my knowledge, seriously exploited to question it. I think that it is possible to explain this coexistence by the epistemological complementarity of the two discourses in terms of the properties of their object and subsequently of their *language*. The purpose of this short text, however, is to challenge the above vision in another way, by bringing mathematics and religion closer to each other through the evidence of a common aspect of both *practices*.

Philosophy of mathematical practice — The approach undertaken here thus belongs to what seems (to the author) to be a major trend in contemporary philosophy of mathematics, and which consists, in contradiction with tradition in this field, in focusing on the practice of mathematics rather than what remains of it, what it constructs and has constructed. In other words, the philosophy of mathematical practice focuses on the construction itself. During the last century, the product of mathematical practice has been seen - at least from the point of view of philosophers - as a *language*, in the sense that a language is interpreted as a collection of possible propositions. These propositions have the particular property that they can be considered either as true or false, a property that is ensured by the fact that the '*words*' involved in the propositions - interpreted in a general sense, including mathematical symbols and diagrams - are in stable one-to-one correspondence with mental contents for which these words act as pointers, as well as the stability of the combinatorics of these mental contents. An understanding of these properties has meant the possibility to configure the discourse of other intellectual domains according to the principles underlying them, in order for these fields to *construct* in the same way as the field of mathematics constructs, meaning that what is constructed does not disappear when the attention leaves it, and to *grasp* the truth in the same way as mathematicians do rather than merely *touching* it, before loosing it. For logical positivism, sculpting philosophy on the model of mathematics coincided with removing from its discourse the shadow of metaphysics, and behind it dogmatism. Although the mathematical field had this kind of influence well before the XXth century - noticing Plato's *Timaeus* and the application of the form of mathematical discourse to metaphysics by Spinoza in his *Ethics* - this phenomenon acquired another magnitude with the development of formal logics and subsequently the creation of a whole branch of philosophy known as *analytic philosophy*.

The idea behind this *practical turn* in the philosophy of mathematics seems to be that this field has built since then over an *idealization* of mathematics rather than mathematics *themselves*. In my view, the obsession - visible notably in the philosophy of A.Badiou - for the dead shadow of mathematics as they were at the time of G.Cantor had the effect of reducing the field to metaphysical considerations, for instance about the mode of existence of mathematical objects, how they relate to reality and how a subject has access to them [problems formulated in particular by P.Benacerraf[1]]. Besides, other pernicious effects have been observable in other fields: the removal of the subject from the

[1] **Paul Benacerraf**, *Mathematical truth*, Journal of Philosophy (1973), vol. 70, pp. 661–679.

field of economics; paradoxically, as a consequence of systematization politics on research, the fragmentation of the field of mathematics and the subsequent impossibility of an integrated viewpoint - a fact observable since the period of H.Poincaré, and often unquestioned.

According to the reference collection of texts *The Philosophy of Mathematical Practice*[2], this branch of philosophy of mathematics is also driven by the idea of a renewal of the field, for which "*attention to mathematical practice is a necessary condition*" - in order to re-understand them. This implies in particular re-introducing the subject behind the language - interestingly, in a similar way as done in quantum physics for the consciousness of the observer in relation with the measurement problem[3] -, understanding the process of creation as well as expanding the scope well beyond set theory - including more recent branches of mathematics. However thus far, rather than a re-conceptualisation of mathematics, the outcomes of this approach have been, to my taste, limited to a description and explicitation, somehow formal, using the vocabulary of philosophers, of aspects of mathematical practice which have been known - although often intuitively and implicitely - by mathematicians. One may consider for instance the question *what are mathematical diagrams ?*[4], in other words what is their role in proofs compared to illustrations.

About the mathematician's conscious mind — (i) *On the conditions of meaning creation in the field of mathematics.* —— Contrarily to these philosophers however, I am led to think about mathematical practice for reasons which are substantially different and relate to *consciousness studies*. A recent trend in this field consists in a mathematical approach of phenomenology - as the discourse about phenomenal experience itself - in the perspective of relating, through the language of mathematics, the structure of the point of view of a

[2]**Paolo Mancosu** (Ed. by), *The Philosophy of Mathematical Practice*, Oxford University Press, 2008.

[3]**Michel Bitbol**, *Physique et Philosophie de l'Esprit*, Flammarion, 2000.

[4]**Silvia De Toffoli**, *What Are Mathematical Diagrams?*, Forthcoming in Synthese.

subject of experience and the structure of what is expected to generate this experience: the nervous system. Such a correspondence may ultimately make possible an explanation of why and how certain physical systems have phenomenal experience and not others. This project is not without difficulty. As a matter of fact, the reflexes that have been developed in the fields of science and mathematics for the formalization process may easily distort the reality of lived experience rather than describing it precisely. Subsequently, methods should be developed in order to keep away from this difficulty. Furthermore, if it is possible, a mathematical description of the point of view may use mathematical objects in a way radically different from the use of mathematical objects in the practice of mathematics and in the investigation of science in their current form. Ultimately doubts about the remaining possibility of distortion shall be excused if this process *creates meaning* along its way. As a consequence the above approach necessitates a preliminary work of understanding meaning creation through mathematical practice and the generalization of the conditions of this meaning creation beyond the current form of mathematics - as practiced by mathematicians.

Besides keeping away from distortion, another fundamental requirement on the language which should be constituted is *tractability*, meaning roughly the possibility of meaning creation through mathematical practice on the objects encompassed by the language, manifested by non-trivial proofs of statements on these objects. Because of *undecidability theorems* such as the ones of A.Turing, this requirement is actually constraining.

Because of its manifest development, it is clear that the practice of mathematics has found in the past conditions under which mathematical development is possible, in particular matching both requirements of non-distortion and tractability.

If these conditions were understood properly, this understanding would allow in principle the desired generalization of mathematical practice. As a side result, we would be also able to apply these conditions to other fields in order to ameliorate the epistemological efficiency without forcing the form of the field of mathematics - which is actually doing wrong to these fields -, in fact subsuming the cultures of these fields under the one of the field of mathematics, which we know had percinious effects.

(ii) *Formalizing fundamental aspects of the conscious mind in the practice of mathematics. ——* Another directly related reason is that mathematics may be considered as the practice of a particular relation of the conscious mind to its phenomenal experience, formed for the purpose of a growth of the conscious grasp on the subject's world. As a consequence, the actual form of mathematical practice is related to properties of consciousness as such. In order to have an understanding of these properties, one possible approach is to *formalize* the structure of the conscious mind *while* it is in the practice of mathematics, beginning with fundamental aspects in order to progress later towards more refine ones, building, for the phenomenology itself, on the formalization at the previous step.

As a matter of fact, it is in this way that A.Turing approached the formalization of computation[5] - although he chose to believe that the whole human mind may be seen as the functioning of a sophisticated computing machine - by looking at how the mathematician's mind works while in the process of computing. Furthermore, despite the fact that orthodox mathematicians believe that creation in mathematics is only generated through the tension between the formulation of a conjecture - statements on properly defined mathematical objects which are strongly believed to be true but for which no one is able to provide an answer at the moment of the formulation - and its resolution, several parts of contemporary mathematical language were the outcome of a formalization process similar to the one of A.Turing. For instance: formal logics (reflexion on the notion of truth); set theory (construction of objects out of more elementary ones by an act of collection); category theory (isomorphisms between different domains of mathematics). Each '*layer*' of the functioning of the conscious human mind while in the practice of mathematics comes with a novel form of

[5] **Alan M. Turing**, *On Computable Numbers, with an Application to the Entscheidungsproblem*, Proceedings of the London Mathematical Society (1937), 2, 42 (1), pp. 230–265.

mathematics: if it was not the case, it would not be differenciated from the previous one.

The question is then: what shall be the next aspect of the conscious mathematical mind to be formalized ? For my own part, I believe, in line with J.Searle's intuition[6] that *understanding* is the most manifest difference between a conscious mind and a constructed machine, that this faculty of the mind should be what we are searching for. As understanding an object - thought as a display of data - consists primarily in the *distinction* of certain patterns in this display into which it can be *decomposed* - for instance when the object is a machine, this patterns are the parts which are relevant for the description of the machine's functioning - the formalization of understanding should derive from a proper formalization of the simpler event of distinction.

Although this next aspect should be practically the closest within reach which stays unformalized at the moment, in practice all unformalized aspects of the mathematician's conscious mind should be considered, for each of them may have an impact on perceiving what is within reach and what is not, and what is the most central within this domain.

Faith is fundamental for mathematical creation; as a consequence, an understanding of this field, of its nature, has to come with an understanding of this fact. Furthermore, the relative simplicity of mathematical experience and of the relation between the mathematician and his or her world offers an opportunity to grasp more firmly the mysterious phenomenon of faith. These are the main reasons why I chose to focus on faith in this short text.

Faith *in* mathematical practice.— The incompleteness theorems proven by K.Gödel are well-known for putting an end to D.Hilbert's ambition to construct a formal language which would allow mathematicians to prove, using this language, any mathematical proposition which is true (*completeness*) and such that no contradiction shall appear in this formalism (*consistency*). They led

[6] **John R. Searle**, *Minds, brains, and programs*, Behavioral and Brain Sciences (1980), vol. 3, pp. 417-424.

to the idea that mathematics are a particular form of religion in the following sense:

> *"If a 'religion' is defined to be a system of ideas that contains unprovable statements, then Gödel taught us that mathematics is not only a religion, it is the only religion that can prove itself to be one."* - **John D. Barrow**[7]

As a matter of fact, properties of the actual formal language of mathematics such as consistency or completeness are objects of belief; when thinking about it, even the basis of this language, natural numbers, are also objects of belief and faith, when this belief becomes a choice[8]. It is this metaphysical doubt and the fear of belief which comes with it which underlay the preoccupation, in philosophy of mathematics, about the mode of existence of mathematical objects and their access. As the proof of a mathematical proposition *is* removing any doubt about the truth of this proposition, doubting of one's own belief about this truth is essential to the process of proving a proposition. It is thus natural for mathematicians to fear metamathematical beliefs. However our *faith* in doubt itself rests on the idea that doubt makes appear to one's mind what has been hidden thus far by the belief in a certain truth and contradicts this belief. It appears clear though that this doubt has not been creative in philosophy of mathematics in the past few decades. Therefore we should doubt of our doubt, and subsequently about our doubt of faith. As I mentioned above, creation excuses belief. We shall see that faith actually creates as much as doubt does. As a matter of fact, belief becomes faith when this fact is seen.

As the words of John D. Barrow appear as a concession of the field of mathematics to religion and metaphysics, meant to come with a form of humility that religions implicitely do not have, granting the field a sort of majesty of the exception - contributing to its *idealization* -, they are the sign of an underlying metaphysical wound which is the result of a frustrated ambition of universality within the field.

[7]**John D. Barrow**, *The Artful Universe*, Oxford University Press, 1995.
[8]**Edward Nelson**, *Mathematics and Faith*, Unpublished

I shall also observe that John D. Barrow's definition of religion is symptomatic of the reduction of practice to language and discourse that logical positivism (in particular) brought forward, and operated on mathematics as well as metaphysics, for the purpose of comparing and then opposing them. Only by returning to the practice one may understand the involvement of faith in mathematics and *embrace* it. Furthermore, the way religious discourse is perceived from outside of any religious community is often distorted from what religion really means for such a community; we should thus question this vision.

From this outside, intuitively, the religious discourse revolves around a certain restricted number of statements which are placed at its center, in a repeated manner, and without manifest *progression* in the complexity of the discourse surrounding them. Furthermore, these statements - this is reflected in J.D.Barrow's words - can not be proved. Here it matters to analyze what in practice a *proof* consists in (also a question which belongs to the philosophy of mathematical practice). A proof of a proposition may be defined as a *systematic* way - independent from the subjects involved - to reach intersubjectively a point where doubt is absent about the truth of the proposition, as well as about intersubjective accord on this truth. In practice the effect of systematicity is to optimize the time spent on discussing the truth of the proposition, for by proof it imposes itself on the subject. This is the actual cause of the perceived property of the mathematical field to progress efficiently, indefinitely, and by belief exhaustively, in the faithful description of the subject's world - which roots logical positivism.

If one can not prove a proposition, any position on its truth is arbitrary, as it depends on the point of view of the subject. Not only is it costly to take a position on an non-provable proposition, a discourse which posits such proposition (*dogmatism*) also imposes this cost to the other; let us notice by the way that although the purpose of *Pascal's wager* is precisely to counter this vision, it fails to do so for the reason that it simply does not remove uncertainty. From the outside of religious community, *faith* tends to be identified with the the act of an arbitrary position where *belief* is allowed by the non-existence of contradicting proof, and thus should we keep away from it. This is rather far from the following definition:

"Now faith is confidence in what we hope for and assurance about what we do not see". - **Hebrews 11:1**.

Mathematicians often like to think of their discipline as defined by exactness - in line with the way philosophers characterized it - because it offers them an idealized version of their field, and by transition, of *themselves* (in a way of narcissism). For this reason it is not perfectly clear that mathematicians have authority on the definition of the essence of mathematics. I would rather characterize the discipline of mathematics as a project of understanding the subject's world - an understanding which exactness only serves - which consists in a description indefinitely growing in precision. This description is made clear by the use of stable *concepts*, themselves *created* out of already constructed ones by collection and composition, for its expression, the created concept making accessible to one's own mind and the ones of others certain '*objects*' which have been thus far only intuitive and obscure.

As a matter of fact, initial obscurity is a necessary condition of creation in the field of mathematics: whatever is initially clear is not of interest for a mathematician - it is said to be '*trivial*', in other words straightforwardly accessible to anyone who has what is considered to be common knowledge of the field. As a consequence, what is created was not *seen* before, even if it was sensed, as it actually is (in a sense, mathematical concepts are discovered by an act of creation). The act of creation itself implies, because it necessitates effort and a sacrifice of time, a certain assurance (even if it is only personal, sometimes shared with a few others) of what we do not see. In a sense then, faith, as defined in **Hebrews 11:1**, is essential to mathematical practice. Furthermore, as the field of mathematics consistently creates, it is a *demonstration* of the effect of faith.

The implementation of the principle of rigor itself is also in practice a matter of faith in the sense of choice of belief in the necessity, when evaluating the truth of a proposition, and despite all appearances, to consider any reasonable doubt, even an impression of the possibility of contradiction, for with time and attention what we may not see on the moment shall become visible. Even if the proposition is ultimately true, this time is not wasted, as it leads to a better understanding of *why* it is true. Furthermore, when it needs to be generalized, '*hidden*' factors which make the proposition true shall matter. Mathematicians have faith - against some physicists for instance - of the importance of the implementation of rigor, which is essential to the practice of mathematics, in the project of understanding the subject's world. Perhaps it is even this faith which defines the term '*mathematician*'.

Perhaps religious practice is also defined by what it creates: the conditions of possibility of *stillness* of the soul. The process by which it creates this is unseen by the profane person, and it is precisely why it necessitates faith.

Faith of the *subject* of mathematical experience.— Faith, in the mathematical field, is involved not only with the principles but also with the *process* of creation itself. Although not all mathematical research consists in the search for a definitive answer to a well-formulated problem, for the example let us consider a mathematician in the process of searching for such an answer. Also for the example, let us consider a simple problem (simple, yet remaining unsolved): *Collatz conjecture*. Consider the function T from the set of positive integers, denoted $\mathbb{N}^*$, to itself, such that for all $n \in \mathbb{N}^*$, $T(n) = n/2$ whenever n is even, and $T(n) = 3n + 1$ whenever n is odd. The conjecture is formulated as follows: for all n, there exists an integer m_n such after iterating m_n times the function T on n, one obtains the number 1, fact denoted by $T^{m_n}(n) = 1$.

In *The foundations of science*, Henri Poincaré[9], has described a decomposition into stages of the search for the answer to a problem such as the one posed by the conjecture. I will describe this process in a slightly different way here: **1.** The search begins with what I shall call (mental) *data sampling*, which consists in displaying - on the paper, on the blackboard, or even on a '*mental board*'

[9] **Henri Poincaré**, *The foundations of science: Science and hypothesis, The value of science, Science and method*, The Science Press, 1913.

when possible - data which are relevant for the problem: here for instance, these data would be the steps of computation for the iterates of T on an integer n, until these values reach 1, for n taking a certain number of integer values. **2.** Then the data are, by an unconscious process, *conceptualized*; where the conceptualization may include in particular a confirmation for the formulation of the conjecture itself. **3.** Attempts are then made of answering the problem which are inspired by existing answers to other problems; because of the difficulty of the problem, consequent of its interest, these attempts fail. **4.** The failure of these attempts consist in itself in a collection of data which are then conceptualized through an unconscious process, at the end of which the answer appears in what H.Poincaré has described as a *sudden illumination*.

In the reality though the unconscious process of the fourth point may be decomposed into multiple ones of the same nature, sometimes with as many *illuminations*, but sometimes not - when the result of the search, although not trivial, derives from numerous small steps. Each time the result of the process may be regarded as an indication of how to direct the reflection then, and so on until the answer appears clearly. Whether or not, from the point of view of the subject, and for a particular problem, the whole process will actually end at some point is most of the time obscure. As a matter of fact, this depends on the proper complexity of the problem: if the problem is trivial one can straightforwardly foresee the existence and the form of the answer, although considering it has subsequently no interest; when the problem is too hard, the answer may be highly valuable in principle but the interest of considering the problem is limited, for it is not likely that the answer will actually be found - again from the point of view of the subject.

Characterizing or even designating intuitively the class of problems which balance these two criteria is of major interest for the field of mathematics, although not achievable, in particular for this depends on several factors which are not understood at all. In practice, however, the mathematician has to make a choice, and this choice is based on other factors such as the *meaning* of the problem. Since meaning depends in principle on the sensibility of the subject, the choice of a problem depending on its meaning is a subjective position which involves faith - this does not mean that there is not ultimately an objective counterpart to this subjective position.

I have mentioned in the beginning how the case of mathematics may enlighten the one of religion. For instance, christian discourse uses the word '*sin*' in order to designate a fault against God. If one tries to find a meaning to this definition which does not involve religious vocabulary, one may see that a fault against God may be understood as a fault against oneself, whose nature as a fault is manifested in suffering resulting from the action; as well one may understand God as a force structuring the fabric of reality which *signifies* to the subject, causing its suffering, the fact that the action is *wrong* (for itself). In order to keep one's soul still, it is only logical to listen to it. The concept of *sin* is thus experiencially useful. However outside of the religious community, it is often seen as oppressive. In order to resolve the contradiction between points of view, I found that an analogy between the relation of the mathematical subject to problems and the one of the religious subject to sins may be useful. While the concepts of problem and sin are considered as experiencially useful ones, as much as deciding which problem balances tractability and meaning, determining *which* action is a sin and what is not is a matter of subjective sensibility. Calling the notion of sin oppressive results in fact from a confusion between the notion itself and what religious authority proposes, sometimes imposes, as determination of which actions are sins.

Beyond this, the chosen belief in consistency of the mathematical language may also be seen as analogical to the belief in the coherence and benevolence of the inner *signifier*.

As what H.Poincaré has called *sudden illumination* comes with high intellectual pleasure (of the perception of a certain beauty), it has been natural for mathematicians to focus in their reflection and research on areas where this phenomenon has the higher manifest intensity and frequency; although this is correlated with higher conceptual creation and subsequent understanding, it is not correlated with the value in themselves of the concepts created. In a sense, mathematicians follow intellectual pleasure in the same way as religious persons follow what they call the *voice of God*. What matters is to see that both enrich the mind instead of destructing what is inside of it - in the case of religion, the stillness of the soul actually lies in this enrichment.

Outside of these areas, reigns more self-doubt, hesitation, and creation necessitates more faith, in many various ways - just like, as J.Hadamard[10] has drawn it, against the idea of H.Poincaré, there is a whole spectrum of possible ways for mathematicians to rely on the unconscious part of their mind. This is where radical creation may emerge, and a more chaotic form of beauty which takes more time to appreciate.

Dogmatism in the mathematical field.— No mathematician would take seriously an attempt to answer Collatz conjecture unless it is coming from a mathematician who has proven himself or herself - for the reason that this conjecture is well-known for its absurd difficulty. However Collatz conjecture is seductive; its formulation is so simple that it provides a permanent instance of *misplaced belief* - in one's ability to resolve it. It is misplaced because attempts of resolution are often a loss of time, leading only to a mild despair.

I think that this is what, for several mathematicians, separates religion from the field of mathematics: an excessive confidence in the truth of a restricted set of metaphysical propositions that with time shall reveal themselves misleading, leaving ultimately the '*believer*' with despair. However this is a misconception which may come itself from the excessive confidence on the universality of the actual mode of knowledge acquisition in the field of mathematics. While faith is not necessary in order to believe in mathematical progression, for the reason that the impact of mathematical creation is straightforwardly visible objectively and thus intersubjectively, the meaning of metaphysical propositions take a lot more time to be sensed this way. Furthermore, contrarily to the interest of an

[10] **Jacques Hadamard**, *The psychology of invention in the mathematical field*, Princeton University Press, 1945.

answer to Collatz conjecture, it should be clear that the purpose of religion - understood as the search for conditions of possibility for stillness of the soul - may not be doubted, and therefore excuse the obscurity - coming from its object - of the religious discourse.

On the other hand the relative sealing between the mathematical community and religious ones results in the isolation from any contradiction of the inner conceptualization of what lies outside of it, its omnipresence via the *medium* of the community itself maintains it, and it maintains the sealing in return. It is only by taking the sealing as a phenomenon in itself and breaking through it that it is possible to contradict these beliefs for oneself.

The conceived border between the field of mathematics and religion is then moved *within* the mathematical community, generating ultimately hypertelic beliefs.

Two ideas in this direction: **1.** The fear of undecidability phenomenon has resulted, somehow paradoxically, into the reduction of the difficulty of the problems approached in areas of mathematics where this phenomenon appears, as well and beyond them as the equation of *productivity* with meaning, the exclusion of pure intuition in the creation of mathematical meaning (still needed for the project of formalization of the mathematician's mind that I described above), and the belief that mathematical creation has to come with an unavoidable increase of complexity and specificity. This, in my view, comes from a lack of preoccupation and therefore understanding of *how* precisely mathematical meaning is created. This postulate, which one may find also in the field of natural sciences[11], finds a historical contradiction in the creation by A.Turing of *computing machines* - which, although ultimately the formulation of this definition is simple '*cognitively*', was conditioned on the state of mathematics and science at this time. **2.** In dynamical systems, the project of defining a mathematical notion of *complexity* - or information integration - that corresponds to

[11]**Maurice Merleau-Ponty**, *La Nature - Cours du Collège de France (1956-1960)*, Éditions Seuil, 2021.

the intuition underlying this term in common language - and motivated initially the study of *cellular automata* - has been bounded by the exclusion of pure intuition as a proper way of creating meaning. As a matter of fact, the consequence of this exclusion is that only '*one-shot*' definitions are allowed, which thus consist only in a best formal approximation, provided the limited intuition in the short time context of the definition, of the informal reality. As a consequence of equating productivity and meaning, the question is declared impossible to settle (while this impossibility comes from the belief system of the community), and the domain is subsequently left practically unexplored.

Two answers: **1.** somehow paradoxically, *faith* is what allows the individual mathematician to pursue meaningful creation beyond the artefacts of collective beliefs. **2.** Noam Chomsky has well expressed, in his conference talk titled *The Machine, the Ghost and the Limits of Understanding* that he gave at Oslo university in 2011, that the most significant advances of science have been the result of lowering expectations. Not that resignation is what creates meaning, but rather that when doing so, one may be able to really *see* the reality beyond what we expect to see.

Furthermore.— In mathematics, dogmatism does not appear in the discourse itself, it surrounds it; it lies in how the society of mathematicians itself forms what mathematics actually are over what they are in essence. Beliefs persist in the dark, under the self-idealized image of the field, where mathematicians do not like to look, for they are afraid of it. However only by standing in the dark is it possible to understand oneself, and to understand the real limit between the clear and the obscure. Furthermore, as I had faith and had seen, I can say with assurance:

Most beautiful flowers root in the dark, before they bloom in the light.

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