# ON THE IMPOSSIBILITY OF USING ANALOGUE MACHINES TO CALCULATE NON-COMPUTABLE FUNCTIONS

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#### Introduction

A number of examples have been given of physical systems (both classical and quantum mechanical) which when provided with a (continuously variable) computable input will give a non-computable output. It has been suggested that these systems might allow one to design analogue machines which would calculate the values of some number-theoretic non-computable function. Analysis of the examples show that the suggestion is wrong. In §4 I claim that given a reasonable definition of analogue machine it will always be wrong. The claim is to be read not so much as a dogmatic assertion, but rather as a challenge.

In §'s 1 and 2 I discuss analogue machines, and lay down some conditions which I believe they must satisfy. In §3 I discuss the particular forms which a paradigm undecidable problem (or non-computable function) may take. In §'s 5 and 6 I justify any claim for two particular examples lying within the range of classical physics, and in §7 I justify it for two (closely connected) examples from quantum mechanics, and discuss, very briefly, other possible quantum mechanical situations. §8 contains various remarks and comments. In §9 I consider the suggestion made by Penrose that a (future) theory of quantum gravity may predict non-locally-determined, and perhaps non-computable patterns of growth for microsopic structures. My conclusion is that such a theory will have to have non-computability built into it.

#### 1. Analogue machines

By a continuously variable quantity ('CVQ') I mean a physical quantity which is represented mathematically by a point in a metric space - e.g., by a real number, or a point of Hilbert space. This is not put forward as an exact definition, but as an indication of how I use the term. For CVQ's very natural definitions of 'computable' have been given in Pour-El & Richards (1989); I shall to this book as CAP. Roughly speaking 'x is computable' means that x is the limit of a sequence of finitely presented approximations and a modulus of convergence for the sequence can be computed.

In the theoretical treatment of a physical device the CVQ's have exact values, and no bound is place, a priori on their magnitude. But when such a device is to be used as an analogue machine to perform some calculation then there will be an upper limit x on the

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size of a CVQ (an electric circuit will melt if the current is too large) and a lower limit  $\epsilon$  on the accuracy with which it can be controlled or measured. Numerical values for x and  $\epsilon$  depend, of course, on the choice of units for the particular CVQ considered, but the ratio  $x/\epsilon$  does not; so we define the <u>precision ratio</u> (PR for short) of the CVQ to be  $x/\epsilon$ . In the theory of the machines there may be different variables having the same physical dimension; these are to be counted as distinct CVQ's and may have different precision rations. The 'independent' variable time is also a CVQ and has a PR; when an analogue machine is to be <u>used</u> some limit must be placed on its run-time.

We are concerned with matters of principle rather than of practice, so although a given analogue machine will have definition precision ratios, we do not place any bound on the PR's that may be attained by some machine.

We are primarily - sometimes only - concerned with those CVQ's which are inputs and outputs of the machine. We shall be interested in cases where these may be continuously controlled or continuously recorded functions; in such cases the relevant x and  $\epsilon$  will be given by some norm for the functions. Most usually the uniform norm will be appropriate, but some machines one might want to say, the  $L^2$  norm.

Even discretely varying quantities such as natural numbers have precision ratios attached to them; perfect accuracy (say  $\epsilon < 1/2$ ) may be attainable, but still there is a bound on the size: one cannot place more than N balls in a given box nor record more than N events with a given geiger counter. In particular if an analogue machine incorporates a battery of digital computers then a PR (which depends both on the programme used and on the hardware) can be assigned to each of them; note that it does not depend on the placing of the decimal point.

In what follows we shall be concerned with the orders of magnitude of PR's rather than with precise values or upper bounds.

# 2. Specification of analogue machines

A specification for an analogue machine is a finite list of instructions which would, in principle, enable a technician or engineer to construct it; descriptions of the apparatus used in a (published) account of an experiment, do, although greatly abbreviated, have this form. If the correct operation of the machines requires particular precision ratios for certain quantities, then the instructions will specify tolerances for certain components<sup>1</sup>. For example a machine might require a cam whose ideal shape ideal shape would be given by  $r = f(\theta)$  where f is some mathematical function. Then the instructions would indicate how the function f could be computed (e.g.,  $f(\theta) = 2 + \sin^2 \theta$  cms for  $0 \le \theta \le 360^{\circ}$ ) and give a permitted tolerance (e.g.,  $\pm 10^{-3}$  cms). Tolerances can be given as precision ratios (3.10<sup>3</sup> in the example). A specification will determine either explicitly or implicitly the PR's in the quantities (including outputs and inputs) occurring in the machine.

<sup>&</sup>lt;sup>1</sup>When A.M. Turing was building his speech encoder ('Delilah') he found that if it was to work, some of the components had to have a tighter than usual tolerance on their values; these were more expensive than the standard components and - at least in the case of resistances - had a gold spot to indicate that they were accurate to within (I think) 1%.

### 3. Undecidable problems

In the examples known to me it is proposed that there might be an analogue machine which with input  $j \in \mathbb{N}$  would output 'Yes' or 'No' to questions of the form ?  $j \in A$ ? where A is some standard recursively enumerable non-recursive set - for example the set which represents the halting problem. I shall only consider proposed machines of this kind. I describe two ways of representing the set A.

3.1. There is a total computable function  $a : \mathbb{N} \to \mathbb{N}$  which enumerates A without repetitions. (This is the notation used throughout CAP).

The waiting-time function  $\nu$  is defined by

(3.1) 
$$\nu(j) \simeq \mu n. \ a(n) = j.$$

This is a partial recursive function whose domain is A and which is not bounded by any total computable function. For any particular analogue machine there is an upper bound J on the inputs it can accept. I define

(3.2) 
$$\beta(J) = \text{Max}\{\nu(j) : j < J \& j \in A\}$$

(with  $\text{Max} \emptyset = 0$ ). This is a total function which is not computable; indeed it eventually majorises every computable function.

3.2. There is a polynomial  $P_A(y, \vec{x})$  such that

$$(3.3) j \in A \leftrightarrow (\exists \vec{m}) \ P_A(j, \vec{m}) = 0,$$

where the variables of  $\vec{m}$  (=  $m_1, m_2, ..., m_k$ ) range over the natural numbers. In this case we define

(3.4) 
$$\nu(j) \simeq (\mu n) (\exists \vec{m} < n) \ P_A(j, \vec{m}) = 0,$$

and

(3.5) 
$$\beta(J) = \text{Max}\{\nu(j) : j \in A \& j < J\}.$$

Then  $\nu$  and  $\beta$  have the same properties as in 3.1. Observe that, if  $j \in A$ , then

(3.6) 
$$\forall \vec{m} < \nu(j) \ P_A(j, \vec{m}) \neq 0.$$

Various explicit definitions of suitable polynomials have been given. For each of these, if  $P_A(j, \vec{m}) = 0$  then at least one of the  $m_i$  encodes a <u>particular</u> sequence which lists the first so many values of some recursive function. So, taking i = 1, we may suppose that

(3.7) 
$$P_A(j, \vec{m}) = 0 \text{ and } P_A(j, m', m_2, \dots, m_k) \neq 0$$

where |m'-m|=1.

## 4. The Claim

Since a given machine cannot handle numbers greater than some bound we consider a given J and the questions  $?j \in A$ ? for j < J. Now I make the following

**CLAIM.** Let J be given. Then one cannot design an analogue machine (whose behaviour is governed by standard physical laws) which will give correct answers to all the questions  $?j \in A?$  for j < J unless one knows a bound  $\beta$  for  $\beta(J)$ .

I call this a claim rather than a conjecture because I do not think one could prove it unless one placed severe restrictions on the notion of 'analogue machine', and this I do not wish to do.<sup>2</sup> But I believe that if someone proposes an analogue machine for settling  $?j \in A$ ? for j < J then it can be shown that either they have (surreptiously?) made use of a bound for  $\beta(j)$ , or that not all the given answers will be correct. To illustrate the significance of the wording of the claim, suppose (what is quite plausible) that someone proves that  $j \notin A$  for all j < J = 10; then he can design a machine which always outputs 'NO' for j < J. But, because of his proof he does in fact know that  $\beta(J) = 0$ .

Of course if one knows a B as above then one does not need an analogue machines to settle  $?j \in A?$  One simple computes a(n) (as in 3.1) on  $P_A(j, m_1, \ldots, m_k)$  (as in 3.2) for all n < B or for all  $m_1, \ldots, m_k < B$ .

# 5. First example (see CAP pp 51-53)

Let

(5.1) 
$$\phi(x) = \begin{cases} e^{-\frac{x^2}{1-x^2}} & \text{for } |x| \le 1\\ 0 & \text{for } |x| \ge 1. \end{cases}$$

 $\phi$  is an infinitely differentiable function  $(\in C^{\infty})$  though it is not analytic. Let

(5.2) 
$$\psi_n(x) = 4^{-a(n)} \phi \left( 2^{-(n+a(n)+2)} \left( x - 2^{-a(n)} \right) \right)$$

where a is as in §3.1. The graph of  $\psi_n(x)$  is a blip of height  $4^{-a(n)}$  centred on  $2^{-a(n)}$ , and having a width of  $2^{-(n+a(n)+1)}$ . If  $m \neq n$  then the supports of  $\psi_m$ ,  $\psi_n$  do not intersect. Set

(5.3) 
$$f'(x) = \sum_{n=0}^{\infty} \psi_n(x);$$

f' has a continuous but unbounded derivative, and f'(x) = 0 for x > 5/4. Since

(5.4) 
$$f'(2^{-j}) = \begin{cases} 4^{-j} & \text{if } j \in A, \\ 0 & \text{if } j \notin A, \end{cases}$$

<sup>&</sup>lt;sup>2</sup>Pour-El in her (1974) gives a definition (based on differential analysers) of 'General Purpose Analogue Computers' and characterizes the class of continuous functions which they can generate. She is not concerned with questions of precision, but I believe that the methods used in §5 and §6 can be applied to justify my claim for all machines of the type she considers.

f' is not a computable function.

Let

(5.5) 
$$\Phi_n(x) = \int_0^x \psi_n(x) dx.$$

The graph of  $\Phi_n$  is a smoothed out step function with initial value 0 (at x = 0) and a final value lying between 0 and  $2^{-n}$ .

Now take

$$f(x) = \sum_{n=0}^{\infty} \Phi_n(x).$$

f is a computable function and its derivative is indeed the f' given by (5.3). Note that ||f||, the uniform norm of f, is less than 2. To settle  $j \in A$  the idea is to feed f into an (analogue) differentiator, and then to observe whether the output f'(x) is zero or not at  $x = 2^{-j}$ . For definiteness let us suppose that we control the current  $i_1$  in a circuit  $C_1$  inductively to a passive circuit  $C_2$  and observe whether the current  $i_2$  in  $C_2$  is zero or not at time  $2^{-j}$ . The claim for this machine is justified on two counts.

- 5.1. Because of the narrowness of the blip  $\psi_n$ , the measurement of the time  $2^{-j}$  at which  $i_2$  is observed must have, for  $j \in A$ , a precision ratio of order  $2^{-\nu(j)}$  if the observed value of  $i_2$  is to be different from zero.
- 5.2. For  $i \in A$ , let

$$f_j(x) = f(x) - \Phi_{\nu(j)}(x).$$

Then  $f'_i(2^{-j}) = 0$ . So if the machine is to give the answer YES for this j, then  $i_1$  must satisfy

$$|i_1(t) - f(t)| < \Phi_{\nu(i)}(t) < 2^{-\nu(j)}$$
.

So unless the precision ratio for the uniform norm of  $i_1$  is better than  $2^{\beta(J)}$  the machine will give wrong answers for some j < J.

5.3. Thus to design a machine which will give correct answers for all j < J we need to know  $\beta(J)$ .

### 6. Second Example

In their (1991) Doria & Costa showed how a function defined in Richardson (1968) could theoretically be used in the construction (based solely on classical dynamics) of a device which would settle questions of the form  $?i \in A$ ?. They write

'Our example is intended to be seen as a <u>Gedanken</u> experiment, as we do not wish to consider at the moment the certainly formidable question of its implementation.'

I shall show that its implementation by an analogue machine requires knowledge of a bound for  $\beta(J)$ .

- 6.1. Let  $k \ge 1$  be given and let  $\mathscr{L}$  be the class of all real-valued functions of k+1 or fewer real variables which can be get by composition from the following initial functions:
  - (i) + and  $\times$ ;
  - (ii) sin;
- (iii) projection functions  $\lambda \vec{x}.x_i$ ;
- (iv) constant functions  $\lambda \vec{x}.c$ , where c is either  $\pi$  or a rational number.

Let  $P_A$  be the polynomial of §2 (3.3). Richardson shows how one can define a function  $F(u, x_1, \ldots, x_k)$  in  $\mathcal{L}$  having the following properties.

- (1) F is an even function of each of the  $x_i$ .
- (2)  $F(u, x_1, \dots, x_k) \ge 0$
- (3)  $F(j, x_1, ..., x_k) > 1$  if  $j \notin A$ .
- (4) If  $F(j, x_1, ..., x_k) \leq 1$  then  $P_A(j, \langle x_1^2 \rangle, ..., \langle x_k^2 \rangle) = 0$  and  $F(j, \langle x_1^2 \rangle, ..., \langle x_k^2 \rangle) = 0$  where  $\langle x_i^2 \rangle$  denotes the natural number nearest to  $x_i^2$ . Hence in this case  $j \in A$ .
- (5) To calculate  $F(j, x_1, \ldots, x_k)$  it is necessary first to calculate  $P_A(j, x_1^2, \ldots, x_k^2)$
- 6.2. Let  $\rho$  either be the function  $\phi$  of §5, or be given by

$$\rho(x) = \frac{1}{2}(|x-1| - (x-1)).$$

In either case  $\rho(x) = 0$  for  $x \ge 1$  and  $\rho(0) = 1$ . If we extend  $\mathcal{L}$  to  $\mathcal{L}^+$  by taking  $\rho$  as a further initial function then either all the functions in  $\mathcal{L}^+$  belong to  $C^{\infty}$  or they are all continuous piecewise analytic functions.

Now set

(6.1) 
$$H(u, \vec{x}) = \rho(F(u, \vec{x})) \ (\vec{x} = x_1, \dots, x_k),$$

and write  $H_j(\vec{x})$  for  $H(j, \vec{x})$ . Then by 6.1 (3), (4), we have

(6.2) 
$$H_j(\vec{x}) = 0 \text{ for all } \vec{x} \text{ if } j \notin A,$$

(6.3) 
$$\exists \vec{x} \ H_j(\vec{x}) = 1 \text{ if } j \in A.$$

But, by (3.6) and 6.1 (4) we see that, for  $j \in A$ ,

(6.4) 
$$H_j(\vec{x}) = 0 \text{ if } x_1^2, \dots, x_k^2 < \nu(j) - 1.$$

Thus if an analogue machine is going to use  $H_j$  to settle  $?j \in A?$  and if  $j \in A$ , then the machine will have to calculate  $P(j, y_1, \ldots, y_k)$  for some values  $y_1, \ldots, y_k$  one at least of which - say  $y_i$  - is greater than  $\nu(j) - 1$ . And by (3.7) the value of one of the y's -  $y_i$ , say - must be accurate to within 1. Hence, for  $j \in A$ , the inputs  $y_1, \ldots, y_k$  for the calculation of  $H_j(y_1, \ldots, y_k)$  need to have a precision ratio of at least  $\nu(j)^3$ . This is also true if  $H_j$  is calculated by a digital computer. Thus the claim is proved for this example.

<sup>&</sup>lt;sup>3</sup>Even if different PR's were used for  $y_1, \ldots, y_k$  I believe the claim would stand: for the  $m_1$  in (3.7) codes a computation sequence, so its size will certainly increase with  $\nu(j)$ .

6.3. Richardson, and following him, Da Costa and Doria make the problem look simpler by coding the k-plot  $\vec{x}$  by a single real number t. Richardson defines decoding functions  $(t)_1, \ldots, (t)_k$  (in  $\mathcal{L}$ ) with the following property:

Given  $\epsilon > 0$  and  $x_1, \ldots, x_k$  one can find t so that

$$(6.5) |x_i - (t)_i| < \epsilon \text{ for } 1 \le i \le k.$$

The functions he defines also satisfy

$$(6.6) (t)_i \le t.$$

Now define a function  $B_i$  by

(6.7) 
$$B_{j}(t) = H_{j}((t)_{1}, \dots, (t)_{k}).$$

Then

(6.8) 
$$B_j(t) = 0 \text{ for all } t, \text{ if } j \notin A,$$

while if  $j \in A$  then for any z < 1

$$(6.9) \exists t \ (B_j(t) > z).$$

But, by (6.4) and (6.6) above we also have

(6.10) 
$$B_i(t) = 0 \text{ if } t^2 < \nu(j) - 1.$$

Any attempt to distinguish between (6.8) and (6.9) will yield further justifications for my claim. For example, Da Costa and Doria define

(6.11) 
$$K(j) = \int_0^\infty B_j(t)\gamma(t)dt$$

where  $\gamma(t)$  is a cut off factor inserted to ensure that the integral converges. (The exact nature of  $B_j$  depends both on the distribution of the zeros of  $P_A$  and on the particular decoding functions; in any case  $B_j$  will be highly oscillatory, and, if  $P_A$  has 'rather few' zeros I think it likely that  $\int_0^\infty B_j(t)dt$  will be of order  $\nu(j)^{-1}$ ).

To specify and analogue machines which, for j < J and  $j \in A$  will output a non zero approximate value for K(j) one will have to specify a value B say, to replace  $\infty$  as the upper limit of integration. But, by (6.9) above, one will then be able to compute a bound for  $\beta(J)$  from B. And because of the cut off factor  $\gamma$ , (6.9) shows that K(j) will be small of order  $\nu(j)^{-1}$ . Da Costa and Doria propose switching from one dynamical system to another, according to whether K(j) = 0 or K(j) > 0. An analogue machine which will correctly effect this switching will thus require, for the CVQ corresponding to K(j) a precision ratio of order  $\beta(J)$ . Thus, in all, there are three different factors in the specification of the proposed machine which requires a knowledge of a bound for  $\beta(J)$ .

# 7. QUANTUM MECHANICAL MACHINES

7.1. Both my examples depend on specifying a self-adjoint operator T on, say, Hilbert space (e.g. specifying the Hamiltonian for some quantum-mechanical system) and making observations on its spectrum to settle  $?j \in A$ ?.

The first example is due to Pour-El and Richards (CAP pp. 190-191). They show that a

certain T may be constructed as a computable limit of a sequence of computable operators  $T_n$  with the following properties.

(1) Let  $\lambda_j$   $(j \geq 0)$  be a computable bounded sequence of real numbers. Then if  $j \notin A$  the spectrum of T has  $\lambda_j$  as an eigenvalue (corresponding to a line in spectranalytic terms), while if  $j \in A$  the spectrum has a continuous band of width  $2 \cdot 2^{-\nu(j)}$  centered on  $\lambda_j$ . The factor  $2^{-\nu(j)}$  ensures that the sequence  $T_n$  has a computable modulus of convergence. To make observation easy one could take

$$\lambda_j = 5 - 4.2^{-j},$$

and then there will be a gap between the bands (if present) around  $\lambda_j$  and  $\lambda_{j+1}$  to separate the lines or bands around  $\lambda_j$  and  $\lambda_{j+1}$  one only needs a precision of the order  $2^j$ ; but to distinguish a <u>line</u> at  $\lambda_j$  and a <u>band</u> around  $\lambda_j$ ; one needs a precision of order  $2^{\gamma(j)}$ . Thus as in the previous examples, to settle  $?j \in A?$  correctly for j < J one needs to know a bound on  $\beta(J)$  in order to ensure that the measyrements made will have the required precision. Another justification for my claim in this example is best illustrated by another example, which is a simplification of one given in Gandy (1991). Namely let the sequence  $\{\lambda_n\}$  be defined by

$$\lambda_n = 2^{-a(n)},$$

and let S be a compact operator with these values of  $\lambda_n$  as its eigenvalues. To decide  $?j \in A?$  it is only necessary to observe, with say, a precision  $2^{j+1}$ , whether or not there is a line at  $2^{-j}$ . (Of course, on physical spectroscopy what one observes is transitions from one  $\lambda$  to another, but this does not affect the argument.) So the question becomes: could one design a quantum mechanical device which would have, for some observable, an approximation S' to S whose eigenvalues for j < J would be close to S? IT will be recalled that a design must allow one to compute approximate values for all relevant parameters and must specify allowed tolerances. I do not know, except in particular cases like atomic and molecular spectra, how one might construct a system which would approximate a given operator for a given observable. But it is obvious, for both S and T, that one would need to know, at least approximately, the entries in the first  $\beta(J)$  rows of their representing matrices (wrt some chosen orthonormal basis). But this justifies the claim<sup>4</sup>.

- 7.2. The wave functions for a quantum mechanical system may result from the superposition of infinitely many more easily defined wave functions and so correspond to the parallel working of infinitely many separate machines. This suggests a possible method for designing a quantum-mechanical device which would give correct answers to the questions  $?j \in A$ ? However the quantum computer described by Deutsch (1985) cannot do this, although it can use superposition greatly to reduce the run time for certain decidable problems.
- 7.3. Refinements in experimental technique allow one to build analogue machines whose behaviour depends on a single quantum (e.g., a single photon). Experiments with such

 $<sup>^4</sup>$ Both S and T are 'effectively determined' operators. The interest of this concept lies not in examples like those given above but in the fact that the authors can (with considerable labour) give a general characterization, in terms of computability, for the spectra of such operators.

devices confirm the often counter-intuitive predictions of standard quantum theory. Could they provide a disproof of my claim? I do not know of any example for this.

## 8. Discussion

- 8.1. When one shows that a given number-theoretic function is computable, or that a given number-theoretic problem is decidable, one does not place bounds on the run-time or the size of the memory unless, of course, one is concerned with problems of complexity. That is, one is not concerned with precision ratios. So it may look as if I have placed unfair restrictions on analogue machines. But suppose one has proved that a certain programme will give correct answers to a problem  $?j \in X?$ . Then, given J, one can compute bounds on the time and space required to settle  $?j \in X?$  correctly for all j < J. But this is exactly what I claim cannot be done for analogue machines intended to settle non-decidable problems.
- 8.2. Cascades of events and chain reactions allow one (as in a photon multiplier) greatly to amplify the scale of an event. This is, in effect, a reduction of precision ratios. Could this be used to overcome the objections raised by my claim? The answer is 'No', because only when one knows a bound for  $\beta(J)$  can one determine how much amplification is needed.
- 8.3. In CAP (and Pour-El & Richards (1979)) other examples are given of differential equiations (in particular the wave equations) which will give a non-computable output for a computable input. The claim can be justified for these using the ideas of §5.
- 8.4. Kreisel has discussed calculation by analogue machines in a number of place; see, in particular, his (1974), (1982), and (199). Some of his comments and analysis are illuminating, and have helped me in getting my ideas stragith. But one of his points is that there are more interesting, more sensible, and more relevant questions to ask than the (logical) question with which I am concerned.
- 8.5. Penrose, in his (1989) and (1994), has argued that the human brain can be thought of as an analogue machine which can, in principle, settle undecidable problems. Firstly, he believes that mathematical results which can, at least in principle, be produced by human intelligence, cannot, even in principle, be produced by artificial intelligence that is by some fixed programme P. Note that P need not be itself directly responsible for the mathematical statements which the machine outputs. P may be like an operating system, for example it may, by a process similar to natural selection, use mutations and tests of fitness to direct the (continual) evolution of subprogrammes for doing mathematics. But this possibility does not, straightforwardly, invalidate Penrose's argument justifying his belief. A concise version of Penrose's argument is given in Gandy (1994). Secondly Penrose believes that the sentences uttered or written by people are caused by physical and chemical events in their brains.

To allow for non-algorithmic actions in the brain, Penrose postulates a - not yet completely formulate - future theory which he calls CQG (for Correct Quantum Gravity). This will have consequences both for cosmology (concerning the direction of time's arrow) and for quantum theory (accounting for the collapse of real (not subjective) wave functions). He suggests ways

in which such a theory may allow for the growth of microscopic structures (such as quasicrystals, synapses and micro tubules in neurons) in ways which are not locally determined nor computable. It seems worthwhile to consider (rather naively) such patterns of grwoth from a mathematical point of view.

#### 9. Patterns of Growth

I consider a pattern of possible growth as being displayed on a tree. At each node P there is a finite label which represents a particular structure  $S_P$  at a particular stage of growth for example, a particular quasi-crystal. If this structure  $S_P$  is capable of growth then there will be a finite number of nodes  $P_1, \ldots, P_k$  immediately below P; each of the structures  $S_{P_1}, \ldots, S_{P_k}$  arises from  $S_P$  by a single step of growth (for example, by the addition of a single molecule). Two distinct structures  $S_P$  and  $S_Q$  may, in one step, grow into the same structure. Hence a node may have two different immediate predecessors; these trees are not the same as those standardly used in recursion theory. A node P and the corresponding structure  $S_P$  are fertile if there is an infinite path through P. If P is not fertile then, however  $S_P$  may grow, after a finite number of steps it will become a structure which can grown no more.

Now we suppose that the label representing any structure S is (coded by) a finite sequence u of 0's and 1's. We may suppose that the significant features of S can be computed from u. An infinite path gives an infinite sequence  $u_1, u_2, \ldots$ , of binary sequences. We define the growth function  $\gamma$  along the path by  $\gamma(u_n) = u_{n+1}$ . If the sequence is computable then so is  $\gamma$ ; in particular there is a Turing machine M which, when presented with  $u_n$  on its tape, will eventually replace it by  $u_{n+1}$ . Now the action of M is certainly locally determined; it will, for example, in general, inspect each of the digits in  $u_n$ . We shall say that  $\gamma$  (and the infinite sequence) are potentially locally determined.

- 9.1. Suppose we are given a tree of structures and a growth function  $\gamma$  which satisfies the following conditions:
  - (i) If u codes a fertile structure S, then  $\gamma(u)$  codes a fertile structure into which S can grown in a single step.
- (ii) The function  $\gamma$  is not potentially locally determined.

Then, starting from any fertile structure S and iterating  $\gamma$  will produce a non-computable infinite sequence of structures.

If one could examine, say, the first J structures in this sequence one could compute the first J values of some non-computable function. The precision ratio of observation has to be sufficiently large to enable one to determine the codes u for these J structures; it might well be a computable function of J.

9.2. Since quasi-crystals have been observed which contain a very large number of molecules, Penrose suggests that their growth is not a matter of chance, but is governed by some - as yet unformulated - laws of non-local actions. If, further, the theory involved actions which were not even potentially locally determined, then it would allow analogue machines to produce non-recursive functions. One would not expect the theory to be totally deterministic; indeed it is plausible that there are at least two distinct infinite paths through any fertile point of

the tree, and hence continuum many such. Although each path yields a non-computable function, one cannot use it to settle a specified undecidable problem.

But for the growth of microstructures in the brain, which determine how neurons behave and how they affect each other, one would expect that certain particular paths would be selected on would be permitted.

- 9.3. The definition of 'potentially locally determined' can be made quite general by considering, in place of the Turing machine M, any mechanism which satisfies the principles of Gandy (1980) in particular, of course, the principle of 'local causation'. And then one has a converse to 9.2 if the growth function along an infinite path is potentially locally determined, then the sequence of structures along it is computable.
- 9.4. It is well-known that there are binary trees whose nodes form a recursive set, which have infinite paths but no computable infinite paths; using this fact one can for example describe a finite set of tiles which can tile the whole plane, but only in a non-computable way (see Hanf (1974)). Using the notion of trial and error predicates (see Putnam (1965)) we can see how the lattermost infinite path,  $\lambda$  say, might be grown. A node is specified by a finite binary sequence u which describes (with 0 for 'Left' and 1 for 'Right') the path from the vertex leading to it, and we consider u also as the structure starting at u. The size of this is just the length of u. Now we define a computable sequence  $u_n$  of nodes on the tree as follows.
  - (i)  $u_0 = ()$  (the vertex of the tree).
- (ii) If  $u_n$  is not terminal (has nodes of the tree below it) then

$$u_{n+1} = u_n 0$$

(iii) Suppose  $u_n$  is terminal and has the form v0 or v011...1 then

$$u_{n+1} = v1$$

Since no node on  $\lambda$  is terminal, none of the  $u_n$  can lie on the right of  $\lambda$ . Below any node v which lies to the left of  $\lambda$  (e.g.; 10 if  $\lambda(1) = 1$   $\lambda(2) = 1$ ) there can only be finitely many nodes of the tree (since v cannot be fertile). Hence for some n we must have a  $u_n$  lying to the right of v. Thuse for any J there will be an  $n_J$  such that  $u_{n_J} = \lambda(1), \lambda(2), \ldots, \lambda(J-1)$ .

- 9.5. At first sight it might look as if this process of trial and error growth could be accomodated in some reasonable physical theory. But this is an illusion; for not only is  $n_J$  not computable from J, but there can be no computable bound on the lengths of the sequences  $u_n$  with  $n < n_J$  which have to be explored before  $u_{n_J}$  is arrived at. And so the process considered is analogous to a trial and error process for deciding if  $j \in A$  (as in §3) one simple looks ahead to see if, for some n, a(n) = j.
- 9.6. Penrose suggests that in a theory of quantum gravity the process of growth would be represented by a superposition of wave functions each corresponding to a particular pattern of growth, and that the effect of gravity would be to collapse the wave function, so that only constituents corresponding to patterns of growth capable of producing large structures would survive. To picture this process on the binary tree let the <u>potential size</u>,  $\pi(v)$  of a node v be the maximum length of all nodes u extending (or lying below) v. If v is fertile

we set  $\pi(v) = \infty$ . Then the proposed theory would ensure that any permitted vertex would grow to some node of great size, though (in the simple form in which I stated it) it would not guarantee growth along an infinite path. It would well be that for a given J there would be a  $k_J$  such that any node of size greater than  $k_J$  would agree with  $\lambda$  at the first J places. But this fact will not allow us to compute values of  $\lambda$  from observations on large structures which have developed, unless we know some (necessarily non-computable) bounds for  $k_J$ . If a theory of growth of the kind considered is to stand up against our claim it looks as if some kind of non-computability must be built into the theory - for example into the way in which gravity determines the collapse of wave functions.

# References

Da Costa & Doria (1991). N.C.A. da Costa & F.A. Doria (1991), <u>Classical Physics and Penroses' thesis</u>, Foundations of Physics Letters, 4, 363-373.

Deutsch (1985). D. Deutsch (1985), Quantum theory, the Church-Turing principle and the universal quantum computer, Proc Roy Soc. (Lond.) A **400**, 97-117.

Gandy (1980). R.O. Gandy (1980), <u>Church's thesis and principles for mechanisms</u>, in the Kleene Symposium, eds J. Barwise J. J. Keisler and K. Kunen, North-Holland Publ. Co., Amsterdam, pp. 123-145.

Gandy (1991). R.O. Gandy (1991), Review of Pour-El & Richards (1989) in Bull. Lond. Math. Soc. 23, 303-305.

Gandy (1994). R.O. Gandy (1994), in preparation.

Hanf (1974). W. Hanf, Non-recursive tilings of the plane I, J. Symb. Logic 39, 283-285.

Kreisel (1974). G. Kreisel (1974), A notion of mechanistic theory, Synthese 29, 143-162.

Kreisel (1982). G. Kreisel (1982), review of papers by Pour-El & Richards, J. Symb. Logic 47, 900-903.

Kreisel (199). G. Kreisel (199), review of Pour-El & Richards (1989) in Zentralblatt.

Penrose (1989). R. Penrose (1989), The Emperor's New Mind, O.U.P., Oxford.

Pour-El (1974). M. Pour-El (1974), Abstract computability and its relation to the general purpose analog computer, Trans. Amer. Math. Soc. 199, 1-28.

Pour-El & Richards (1979). M. Pour-El & I. Richards (1979), A computable ordinary differential equation which possesses no computable solution, Ann. Math. Logic 17, 61-90.

Pour-El & Richards (1989). M. Pour-El & I. Richards (1989), Computability in Analysis and Physics, Perspectives in Mathematical Logic, Springer-Verlag, Berlin. (CAP)

Putnam (1965). H. Putnam (1965), <u>Trial and error predicates and the solution to a problem of Mostowski</u>, J. Symb. Logic, **30**, 49-57.

Richardson (1968). D. Richardson (1968), Some undecidable problems involving elementary functions of a real variable, J. Symb. Logic **33**, 514-520.

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On the impossibility of using analogue machines to calculate non-computable functions.

(R.O. Gandy, 9 Squitchey Lane, Oxford OX2 740)
Sep 93.

Introduction A number of examples have been given of physical systems (both classical and quantum mechanical) which when provided with a (continuously variable) which when provided with a (continuously variable) to apput to ill give a non-computable output. It computable input will give a non-computable output. It has been suggested that such systems might allow one to design analogue machines which would calculate the volves of some analogue machines which would calculate the volves of Ar number-theoretic non-computable function. Analysis of Ar number-theoretic non-computable function. Analysis of Ar examples show that he suggestion is wrong. In 34 I example show that he suggestion is wrong. In 34 I example it will always be wrong. The claim is to be machine it will always be wrong. The claim is to be read not so much as a dogmatic assertion, but rather read not so much as a dogmatic assertion, but rather

In 3: 1 and 2 I discuss analogue machines, and lay down some conditions which I believe they must lay down some conditions which I believe they must satisfy. In 33 I discuss the particular forms which a paradigm undecidable problem for non-computable a paradigm undecidable problem for lastify my function) may take. In 3's 5 and 6 I justify my claim for two particular examples lying within the claim for two particular examples lying within the claim for two particular examples from quaktoum it for two (closely connected) example from quaktoum it for two (closely connected) example from quaktoum and decrease, very briefly, other possible mechanics, and decrease, very briefly, other possible mechanics and decrease, very briefly, other possible made by another mechanical situations. So contains various remarks and comments. In 39 I consider the suggestion made by Pennose that a (future) theory of quantum gravity

Comments. In 39 I consider the suggestion growity

Pennose that a (future) theory of quantum growity

may predict thom-locally-determined, and perhaps non-computable

patterns of growth for microscopic strailares. My conclusion

is that such a theory will have to have non-computability bruit

into it.

21 Analogue machines. By a continuously variable quantity (6CVQ) I mean a physical quantity which is represented matematically by a point in a metric space; e.g., by according to a point of Hilbert space. This is not put forward as an exact deforition, but as an indication

of how I use the term. For CVQ's very natural definitions of computable have been given directly four-E( & Richards ([1989]); I shall lim Pour-E( & Richards ([1989]); I shall very to the limit of the limit is a computable means that re in the limit is a computable means that re in the limit is a sequence of finitely presented approximations of a sequence of finitely presented approximations and a modelus of convergence for the sequence

In a theoretical treatment of a physical can be computed. device the CVQ's have exact values, act no bound is placed, a priorie on their magnitude. But when such a device is to be used as an analogue machine to perform some calculation then there will be an apper limit X on the size of a CVQ (an electric circuit will melt if the current is to large ) and a lower limit, on Are accuracy with which it can be controlled on me as uned. Numerical values for X ad & depend, of course, on the choice of units for the porticular CVQ considered, but the natio X/E does not; so me define the precinion nation of the CVQ to he X/E. In the theory of the machine there may he different variables having the same physical dimension; there are to be counted as distinct (V6's al may have different precision votios. The 'independent variable time is also a CVQ od hus a PR; when an analogue machine is to be used some limit newst be placed on its run-time.

We are concerned with molters of principle rather and practice, so although precion ratios, we do not place a sing or bounder on the PR's allibrate may be attained by some machine.

We are primarily sometimes only concerned with those CVQ's which are
in pate and outputs of the machine. We shall
be interested in comes where these may be
continuously controlled or continuously recorded
continuously controlled or continuously recorded
functions; in such cases the relevant X ad E
will be given by some norm for the functions.
Most usually the uniform norm will be
appropriate, but for some machines one might
wont to use, say, the L2 norm.

Even discretely barying quantities such as natural numbers howe precision hatios attached to them; perfect accounting (say E<1) (may be attained the, but skill the in a bound on size; one cannot the plate more than N events with a por record more than N events with a power given geiger counter. In particular given geiger counter. In particular of an analogue machine incorporates a battery of digital computers them a PR (which depends both on the programme used and on the hardware) can be assigned to each of them: note that it does not depend on the placing of the decimal point.

In what follows we shatt be concerned with the orders of magnitude of PR's rather

The with precise values or apper bounds.

A specification for an analogue machine in a finite lint of instructions which would, in principle, enable a technicion or engineer to construct principle, enable a technicion or engineer to construct it; descriptions of the apparatus used in a (published) accant of an experiment der, although (published) accant of an experiment der, although correct operation of the machine requires particular correct operation of the machine requires particular precinien rations for certain quantities. Then the precinien rations will specify tolerances for certain instructions will specify tolerances for certain components. For example a machine might

2 When A.M. Turing was building his speech encoder ('Delilah') he found that if it was becoder ('Delilah') he found that if it was a lower some of the components had to home a tighter Non usual tolerance on their values; these tighter Non usual tolerance on their values; these were more expensione Non the standard components were more expensione non the standard components were more expensione non the standard components and - of least in the case of resistances - had a gold and - of least in the case of resistances - had a gold and - of least in the case of resistances - had a gold (I thought 1%).

require a cam whose ideal shape would be given by  $r = f(\theta)$  where f is some mathematical function. Then the instructions would indicate how the function of could be computed (e.g.,  $f(\theta) = 2 + \sin^2\theta$ , cms for  $0 \le \theta \le 3(0^\circ)$  and give a permitted to because for  $0 \le \theta \le 3(0^\circ)$  and give a permitted to because  $(e.g., \pm 10^{-3} \, \mathrm{cm} \, \mathrm{s})$ . To because can be given  $(e.g., \pm 10^{-3} \, \mathrm{cm} \, \mathrm{s})$ . To because can be given as precision ratios (3.10° in the example). A as precision will determine either emplicitly specification will determine either emplicitly or implicitly the PR's of the quantities or implicitly the PR's of the quantities and imputs) occurring in the landwing outputs and imputs) occurring in the machine.

# 3 Undecidable problems

In the examples known to me it is proposed that there might be an analogue machine which with input jell would output Yes or No to questions of the form ? jet? when A is some standard recursively enumerable non-recursive
set - for example the set which represents the hilting
set - for example the set which represents of the kind
problem. I shall only proposed machines of the kind
I describe two ways of representing the set A

There is a lotal computable function a: IN -> IN which enumerales A without repetitions. (This is the notation used throughout CAP) The waiting-time function is in defined

2) (j) ~ pun. a(n) = 0.

This is a partial recursive function whose domain in A and which is not bounded by any total computable function. Le for any particuleur analogue machine there is an appear bound J on the imputs it can accept. I de fine

(2) B(J) = Mare { 20(j): j < J & j + A} (with More \$ =0). This is a total fametion which is not computable; indeed it eventually majorists every computable functions

32. There is a polynomial PA(4, se)

(3) jeA (3) P(j, m) = 0,

who the variables of m (AM, Mg) ... , mg range toverof the instrument numbers.

In this case we define

(4) 2(j) ~ (mn)(3 m < n) P(j, m) = 0,

(5) B(T) = Man (v(j): jeAxj<T).

When Nad & home the some properties as in 3.1.
Observe that, if jet, then

(6) Ym < >(i) PA (i, m) #0.

Various explicit definitions of suitable polynomials horn been given. For each of Rese, it

at least  $P_A(j, m) = 0$ then, one of mi encodes a particular sequence,

which lists the first so many values of some necarsine function. So, tehing i=1, we

may suppose that

(7) PA (1, m) = 0 and PA (1, m, m2, -, mx) +0 where |m'-m|=1.

34 the Claim Sice a ginn machine cannot handle numbers greater than some bound me consider a given J and the questions 27 EA? For jeJ.

sections 5 and 6 herous could be somethed 6 7 mgs.

an analogue machine (whose be haviour is governed by standard physical lows) which will give correct consumers to all the questions? jet? for jet unless one knows a bound B for p(J).

I call this a claim rather than a conjecture because I do not think one carled prove it unless one placed severe restrictions on the notion of analogue machine, and fin I do not wish to do? But I believe that if someone proposes

1 Pour-El in her (1974) gives a definition (based on clifferential analyseus) conficement Purpose Analogue Computers differential analyseus) continuous functions which they and characterizes the class of continuous functions of precision but can generate. She is not concerned with questions of precision but can generate. She is not concerned with questions of could be applied to believe that the methods used in 35 and 6 could be applied to justify my claim for all machines of the type she considers a to justify my claim for all machines of the type she considers.

an analogue machine for settling ? jeA? for jeJ then it can be shown that either they have (surreptionsly?) made use be shown that either they have (surreptionsly?) made as of a bound for  $\beta(J)$ , or that not all the given answers will be correct. To illustrate the significance of the wording of the correct. To illustrate the significance plausible) that sometime proves that j & A for all j < J = 10; then the can design a that j & A for all j < J = 10; then the can design a machine which always outputs (NO for j < J. But, because of this proof he does in fact know that  $\beta(J) = 0$  machine which always on fact know that  $\beta(J) = 0$  of this proof he does in fact know that  $\beta(J) = 0$  does not need an analogue machine to settle?  $\beta(J) = 0$  does not need an analogue machine to settle?  $\beta(J) = 0$  does not need an analogue machine to settle?  $\beta(J) = 0$  does not need an analogue machine to settle?  $\beta(J) = 0$  does not need an analogue machine to settle?  $\beta(J) = 0$  does not need an analogue machine to settle?  $\beta(J) = 0$  does not need an analogue machine to settle?  $\beta(J) = 0$  does not need an analogue machine to settle?  $\beta(J) = 0$  does not need an analogue machine to settle?  $\beta(J) = 0$  does not need an analogue machine to settle?  $\beta(J) = 0$  does not need an analogue machine to settle?  $\beta(J) = 0$  does not need an analogue machine to settle?

End 134

35 First example (see CAPPP 51-58) (1) Let  $\phi(x) = e^{-\frac{x^2}{1-x^2}}$  for  $|x| \le 1$ φ is an infinitely differentiable function (∈ C°) Mough itienst analytic.

(2) Let  $\Psi_n(x) = 4$   $\phi(2^{-(n+a(n)+2)}(x-2^{-a(n)}),$ where a is as in 33.1. The groph of  $\psi_n$  is a blip of height  $4^{-a(n)}$  centred on  $2^{-a(n)}$  and howing a width of  $2^{-(n+a(n)+1)}$  If  $m\neq n$  then the supports of  $\psi_m$ ,  $\psi_n$  do not intersect.

(3) Set  $f'(x) = \sum_{n=0}^{\infty} \psi_n(x)$ : f hus a continuous but unbounded detrivation, and ). f((x) = 0 for x > 5/4. Since (4)  $f'(z^{-0}) = 4^{-0}$  if  $j \in A$ , = 0 it it A; Thus f'is not a computable function. The graph of In a smoothed out step function with initial value O (at x = 0) and a final value lying between o and 2. fina computable function and its derivative is indeed Now take Ate f' guin by (3). Note that II follow the auni for m monim spren, is less Priz je to settle sell te ulea is to feed of into anti-lamalogue) differentiator heard then to observe

whether the output f'(x) is zero on not at  $x = 2^{-\delta}$ . For definiteness let us suppose that we control the current i, in a circuit C, to be f(t) (t for time) and couple C, inductively to a passive circuit  $C_2$  and observe whether the current i, in  $C_2$  is zero or not at time  $\mathbf{Z}^{-\delta}$ . The claim for this machine is justified on two counts

5.1 Because of the narrowners of the blip

Un, the measurement of the time 2 detailing

is observed must have, for jet, a precision

ratio of order 2->(i) if the wisserved value

of is to be different from zero.

5.2 For  $j \in A$ , let  $f_j(x) = f(x) - \frac{1}{2}p(j)$ Then  $f'(2^{-j}) = 0$ . So if the machine is to give the answer  $Y \in S$  for this j, then j in the mast satisfy  $|i_j(t) - f(t)| \leq \frac{1}{2}p(j)$ So anless the precision ratio for the uniform norm sf(i, j) is better than  $2^{p(j)}$  the machine will give wrong answers for some j < J.

5.300 Thus to design a machine

how a function edefineded they Richardson (1968) could themetically be used for Atthemstructions Abos edy solely on classical dynamics) of a device which would settle questions of the form? jtA?.

They write Sp Our example is intended to be seen as a Gedonken experiment, as we do not wish to consider at the moment the certainly formidable

quest un of its implementation?.

I shall that its implementation by an analogue, machine requires knowledge of a bound for po (J).

Let k > 1 be given end bet & be the class of all realer valued functions of k+1 or fewer real variables which can be get by composition from the following initial functions:

(i) + ad X;

(111) projection functions 1 72. Xi (IV) constant functions lie a, where

a is either IT or a national number.

Let PA be the polynomial of 32 (3). Richardson shows how one can define a function F(u, x,,..,xk) in & having the following properties.

(1) F is an even function of each of the Xi.

(2) F(u,x1,...,xb) >0

if j & A.  $(3) F(j, \kappa_1, \dots, \kappa_k) > 1$ 

(4) If F(j, x,, -, xk) <1 then PA (j,(x?),..., (xx)) = 0 and  $F(j, \langle x_1^2 7, ..., \langle x_k^2 \rangle) = 0$ where <x; >> denotes the natural number rearest to x?. Hence in this case j EA.

(5) To calculate F(j, x1, -, xk) it is necessory first to calculate Px(0, x2, ..., x2)

6.2. Let p either be the function \$ of 35, or be given by p(x) = 1/2(|x-1|-(x-1)).

In either case p(x) = 0 for  $x \ge 1$  and p(0) = 1.

If we extend f to f by taking p as a further p. initial function then either all Al functions in &+ belong to coo or they are all continuous piecewire analytic functions.

(1)  $H(u, \overline{z}) = \rho(F(u, \overline{z})) (\overline{x} = x_1, ..., x_k),$ and write H; (\$\vec{x}\$) for H(1, \$\vec{x}\$). Then by 6.1(3),(4),

(2) H; (2) = 0 for all 22. if j (A)

(3) 32 H; (2)=1 of j EA. But, by 3.2(6) and 6.1(4) we see that, for jet, (4) H; (x)=0 if x,,,x, x, < 25(j)-1.

Thus if an analogue machine in going to use? H; to settle ? jeA? and if i ElA; then Ac machine will have to calculate P(i, v1, , Vk) for some values y,, , y a one at beast of which - say yi - is greater than 2 (j)-1. And by 3.2(7) the value work lone of the y's - yi, say must be accurate to within 1. Hence, for jet,

the inputs y,,..., y, for the calculation of H; (y,,..., yh) need to have a precision ratio of at least w (i). This is also time if it's in calculated by a digital computer. Thus the claim is proved for this example. 6-3. Richardson, and following him, Da Costa ad Donia make the problem look sampler by coding the k-plet si by a single real number t. Richardson defines decoding functions (t),, (t) k (mf) with the following property: given E>0 and x,,..., x & one can find t (1) |x; -(t); | < E for 150 Em. The functions he defines also satisfy (2) (t); \le t. Now define a function Bj by (b) B3(t) = Hj((t),,...,(t)). Then (4)  $B_{j}(t) = 0$  for all t, if  $j \notin \Lambda$ , while if jet then for any 3 < 1 (5) It (B; (+) > 3). But, by 6.2 (4) and (2) above we doo home (6) Bj (t) = 0 if t2 < >(j)-1 Any attempt to distinguish between (4) and (5) will yield farther justifications for my claim. For example, De Costa ad Dorid

I Even if different PR's were used for 91,..., yn I believe the claim would stank; for the M, in 3.2(7) codes a computation sequence, so its size will, certainly increase with w(i).

define ab (6)  $K(i) = \int B_{i}(t) \gamma(t) dt$ 

when y(t) is a cut off factor inserted to ensure But he integral converges. (The exact nature of B. depends both on the distribution of Religences of PA and on the porticular be lightly oscillatory, and give B. will be highly oscillatory, and fifth PA has be highly oscillatory, and I likely that rather few zeros I think it likely that

To specify an analogue machine which, for j < J and j & A will output a non-zero approximate volue for K(5) one will have to specify a value B suy to replace as as the appear limit of integration. Bat, by (5) above, were will then he able to compute a bound for B(J) from B. And because of the cut off factor vg (5) shows tet K (i) will be small of order »(j) Da Costal ade Dovia propose switching from one dynamical system to another, according to whether K(j)=0 or K(i)>0. An analogue machine which will effect the switching will thus require, for the CVQ corresponding to K(i) a precion ratio of order B(J) if it has, in all, there are three different factors in the specifical of the proposed machine which require to knowledge of abound for B(5).

# 37 Quantum Mechanical machines

7 I Both my examples depend on specifying a self-adjoint quater T for, sous, Hilbert space (e.g. specifying the Homiltonian for some quantum-mechanical system) and making diservations on its spectrum t settle? je A?.

The first example is due to Pour-El ad Richards (CAP PP 190-191). They show that a certain T may be constructed as a compulable limit of a sequence of compatalle operators In with the following properties.

(1) Let 2; (970) be a computable bounded sequence of real numbers. Then it if A the spectrum of T has I, as an eigen-value (corresponding to a line in spectroscopii terms), while if is A the spectrum has a continuous band of width 2.2 ruli) and continuous band of width 2.2 ruli) and centred on  $\lambda_j$ . The factor  $2^{-\nu(j)}$  ensumes. that the sequence In has a compatable modules of convergence. To make observation easy one miruld take

 $\lambda_{1} = 5 - 4.2$ and then there will be a gap between the bonds (if present) around his addite To seperate the lines or bands around \(\lambda\) and \(\lambda\); to one only needs a precining te order 2; but to distinguish between a line at his ood a bend arround his one needs a preisien of order 2<sup>r(i)</sup>. Thus

as in A previous examples, & settle

?; cA? (correctly fr ; 2 J one needs to

know a bound on 78 (5) in order to

ensure that A measurements made will

have the required precision. Another justification

for my claim in this example is best illustrated

for my claim in this example is best illustrated

for my claim in this example is a simplification

by another escample, which is a simplification

of one given in Gandy (1991). Numely

let the sequence (1,1) be defined by

-a(n)

\[
\lambda\_n = 2
\]

one let 5 be accompact operator with these values of he are its eigenvalues. To decide ? je A? it is only necessary to observe, with say, a precision 2; whether or not theme in a line at 2. (of course on physical spectroscopy alat on observes is transitions from one I to another, but this does not affect the argument.) So the question becomes: could one designa quantum mechanical device Mid would have, for some observable, on approximation 5 to 5 whose eigenvolues for j < 5 would he close to those of 5? It will be recalled Aut a design must allow one to compute approximate values for all relevant parameters ad specify allowed tolerances. I do not know, except in particular cases like atomic ad mølewler spectra, how one might construct a system which would approximate a given operator for a given observable. But it is

reed to know, at least approximately, the entries in the first  $\beta(J)$  rows of their representing matrices (cent some chosen or Pronounal book). But Piri justifies the claum.

Both Sand T are effective by determined operators. The interest of Phis concept hies not in examples like those gain above, but in the fact that the authors can but in the fact that the authors can be considerable labour) gone a general levith considerable labour) gone a general characterization, in terms of computability, thereacterization in terms of computability, for the spectra of such operators.

The wave function

for a quantum mechanical system may result from the superposition of infinitely result from the superposition of infinitely many move deasily defined wave functions many move deasily defined wave functions and so correspond to the parallel working of infinitely many seperate machines. This of infinitely many seperate method for designing suggests a possible method for designing a quantum-mechanical device which would a quantum-mechanical device which would a quantum to correct answers to the questions? Jet? I described thousand the guantam compater described the problems of the run time for certain decidable problems. It run time for certain decidable problems. The reduce allow one to build analogue machines whose allow one to build analogue machines whose behaviour depends on a single quantum (e.g., a single photon). Experiments

often counter-intuitive predictions of often counter-intuitive predictions of standard quantum thony. Could they standard quantum thony. Could they provide a disproof of my claim? I provide a disproof of my claim? I clo not know of any example for this.

# 38 Discussion

8.1 When one shows Aut a given nanter. Repretie fametin is competable, or Pat a given humber. Manglie problem is decidable, one does not place bounds on the run time or the size of the memory -unless of course, one is concerned with problems of complexity. That is, one is not concerned with precision vatios. So it may lock as of I have placed an unfair restrictions on analogue machines But sappose one has proved that a certain programme will gine correct answers to a problem ? j & X?. Then, given J, one can compute bounds on the time and space required to settle?; cx? correctly for all j & J. But this is exactly what I claim cannot be done for an alogue machines intended i tot settle a non-decidable problems.

Ast he resententh non-transit theory may and require the experiments facts; and dillat such i theory might allow one

\$.2 Cascades of events al chain reactions allow one (as in a photon-multiplier) greatly te amplify the scale of an event. This is, in effect, a reduction of precision ratios. Could thin be used to overcome the objections raised by my claim? The answer in 'No', because only when one knows a bound for ps (I) con me determine how much amplification is needed.

6.3 InICAP AP other examples are given of differential equations (in particular to wove, equation) which will give a non-computable output for a computable input. The claim can be justified for these using the ideas of 35. 8.4. Kreisel has discussed calculation by analogue machines in a number of places; see, in porticular, his (1974), (1982) and (199). Some of his comments and analogies are illuminating, ad how helped me in getting my ideas straight. But one of his points in that there are more interesting, more sensible al more relevant questions to ask than the (logical) questin with which I am home concerned. \$.5. Penrose, Thas argued that the human brain can be thought of as an analogue machine which can, in principle, settle undecidable. problems. Firstly, he believes that makenatical results which can, at least in principle, be produced by human intelligence, cannot, even in principle, be produced by artificial in telligence - test in by some finced

programme. P. Note that P inced nots bely itself directly responsible for the matternatical itself directly responsible for the matternatical statements which the machine outputs. P may be statements which the machine outputs. P may be like an operating system; for example it may, like an operating system; for example it may, use by a process somilar to natural selection, use mutations and tests of fitness to direct the mutations and tests of fitness to direct the continual) evolution of subprogrammes for (continual) evolution of subprogrammes for doing mathematics. But this possibility closes not, doing mathematics. But this possibility closes not, doing mathematics. But this possibility closes organism to straightforwardly invalidate (Renrose's argument straightforwardly invalidate (Penrose's argument). Secondly argument, is given in Gondy (1994). Secondly argument, is given in Gondy (1994). Secondly argument, is given in Gondy (1994). Secondly argument their brains.

in the brain, penvose postulates a in their brains. - not get completely formulated - future Theory, which he calls CQG (for Correct Quantum Gravity). This will have consequences both for cosmology (concerning the direction of time's arrow) and for quantum theory (accounting for the collapse of real (not subjective) wave functions.) He suggests ways in which such a theory may allow for the growth of microscopie structures (such as quasicrystale, synapses and micro-tubules in neurone) in ways which are not weally determined nor computable. It seems worth while to consider (nather naively) such patterns of growth from a matternatical point of view. 3 d. Patterns of growth I consider at pattern of possible growth as being displayed on a tree. At each node P there is a finite label which represents a positivity a porticular structure Sp at a particular stage

of growth - for example, a particular quasi-crystal.

If this structure Sp is capable of

growth then there will be a finite number of nodes Pi,..., Pk immediately below P; each of the structures Sp.,.., St. arises from 5 by a single step of growth ( for example, by the addition of a single molecule). Two distinct structures Sp al Sa may, in one step, given inthe some structure, hence, a nocle may hove two different immediate predecessors; these thees one not the same on those standardly used in recersion Heavy. A node Prod He corresponding staudane Sp are fertile of there is an in frite path through P. If Pin not fertile then, however sp may grow, after a first number of steps it will become a structure which can grow no more.

Now me suppose that the label representing ong structure S is leaded by a finite seguence u ong structure S is leaded by a finite seguence u of S can be computed from u. An infinite path II guies an infinite seguence u, uz, or, of binory sequences. We define the growth function y along sequences. We define the growth function y along sequences. We define the growth function y along sequences we define the growth function y along the path by Y(un) = un+1. If the sequence is computable than so is y; in porticular three is a Tuning computable than so is y; in porticular three is a Tuning will, eventually, when presented with un on its tape, machine M which, when presented with un on its tape, will, eventually, replace it by until Now the action of will, eventually locally determined; it will, for excemple, in general, inspect each of the digits in excemple, in general, inspect each of the digits in excemple, in general, inspect each of the infinite sequence) are potentially locally determined.

9.1 Suppose we are given a tree of structures and a growth function of which satisfies to fellowing conditions: -

(i) If a codes a fertile structure, then r (a) ocodes a fertile structure int which S can grow in a single step.

(ii) The function r is not potentially locally determined.

Then, starting from any fertile structure 5 and iterating of will produce a non-computable in finite sequence of structures; It one could examine, say, the first J structures in this sequence one could compate the first J values of some non-computable function. The precessor ratio of observation has to be sufficiently large to enable one to determine the codes: in for these J structures; itempet well codes: in for these J structures; itempet well

9.2 Since quasi-engstals have been observed which contain a very large number of molecules, which contain a very large number of molecules, Penrose suggests that their growth is not a matter of chance, but is governed by some - as get an formulated - laws of non-local actions. If, an formulated - laws of non-local actions which were not even potentially locally determined, then were not even potentially locally determined, then it would allow anabogue machines to produce non-recursine functions. One would not expect the theory to be totally deterministic; indeed it is plausible that there are at least two distinct in plausible that there are at least two distinct

ew pord

19.3 (A) The definition of potentially. locally determined can be made quite general by considering, in place of the Turing machine M, any mechanism which satisfies The principles of Gardy (1980) - in particular, I course, the principle of Geal causation. And then one his a converse to 92 - if the growth function along an infinite path is potentially Cocally determined. Hen the sequence of structures along it is computable. 9.4 It is well known that there are

binary trees whose rodes form a recursive set which how infinit paths but not computable infinit paths; using his fact one can for example describe a set of tiles which can tile the whole plane, but only in a non-computable way. Using the notion

of trial aclievror predicates (see Putnam (1965)) we can see how the deftermost unfunite

speath at say, wight be agranted At node is specified by a finite binary sequence, which describes

(with 0 for 'Left' and I for 'Right') the pull from the vertex leading

to it, and we consider we atoth as the structure standing 23 at u. The size of this is just the length of u. Now we define a computable sequence un of nodes on Re tree as follows. Uo = () ( Fe vertere of the tree). (i)) If un is not terminal (has nochs of Re tree below it) then Unri = UnO (iii) Suppose un is terminal and has the form voor volk---Untl = Ul then Soice no noise on I is terminal, none of Re Un can lie on Re right of A. Below any mode videoù lies to the left of & (eg:10 if A(n=1  $\lambda(2)=1)$  there can only be finitely many nodes of the tree (since or connect be fertile). Hence for some n we must have then lying to the right of U. Thus for mg J there will be an no such that  $u_{n_3} = \lambda(1), \lambda(2), \lambda(3-1).$ 9.5 At first sight it might look as if Ann process of trial at ernor growth could be accomodated in some reasonable physical theory. Bat Plen is an illusion; for not only is no mult computable from J, but there can be no computable bound on the lengths of The exequencescian with n < no which have to he explored before unj is arrived at.

The explored before Un is arrived at.

The explored before Un is arrived at.

The explored before un analogous

And so the process consulered in analogous

to a trial ad error process for deciding if

to a trial ad error process for deciding if

JEA (ass in 33) - one simply looks ahead

jeA (ass in 33) - one simply looks ahead

jeA (ass in 33) - one simply looks ahead

9.6 Penrose suggests that in a theory would of quantum gravity to process of grown would be represented by a superposition of wove functions each corruponding to a particular pattern of growth and that the effect of growity wruld be to collapse the wave function, so that only constituents corresponding to patterns of growth capable of producing large structures would survive. To picture this procession The binary tree let the potential size, IT (U) 7. a noile v be the mass in um length of all snodess a certatending (or lying below) v. If v is fertile me set TT (v) =00. Then the proposed theory would ensure that army permitted vertex would grow to a one node of great size, though (in the simple form in which I have stated it) it would not guarantee growth along an infinite puth. It could well be that for a give I there would be a ky such that my node of 15ize greats Par ky would agree with I at the first I places. But this fact will not allow as to compute volues of I from observations on large structures which have developed, unless we know some (necessarily non-computable) bounds for kJ. If a theory of growth of the brind considered is to stand up against our claim it looks as if some kind of non-computability must be built into Ao Pleoney - for example into the way in which growity determines the collopse of wave functions.

(1991),

N.C.A. da Costa & F.A. Dorid / Classical Physics and Penrose's thesis, Foundations of Physics Letters, 4, 363-373

D. Deutsch (1988), Quantum theory, the Church-Turing principle and the universal quantum computer, Proc Roy. Soc. (Lond.) A 400, 97-117.

P.O.Gandy (1980), Church's thesis and principles for mechanisms, in The Kleene Sumposium, eds J. Barwise J.J. Keisler ad K. Kunen, Worth - Holland Publ. Co., Amsterdam, pp 123-145

Lond. Math. Soc. 23; (303-305.)

(1994), in preparation.

W. Hanf (1974), Non-recursive tilings of the plane I,

G. Wreisel (1974), Anotion of mechanistic theory, Synthose 29, 143-162. (1982), review of papers by Pour. Elx Richards, J. Sgmb. Logic

47,900-903

(199), review of Pour. El & Richards (1989) in

R. Penrose (1989), The Emperor's New Mind, O. U. P., Oxford.

M. Pour-El (1974), Abstract computability and its relation to the general parpose analog computer, Trans. Amer. Matz. Sac. 199, 1-28.

M. Pour-ELV I. Richards (1979), A computable ordinary differential equation which possesses no computable solution, Ann. Math. Logic 17, 61-90.

— (1989), Computability in Analysic and Physics, Perspectives in Mathemetical Logic, Springer-Verlag, Berlin: (CAP) It. Putnam (1965), Triol ad error predicates and the solution to a problem of Mostavski T Sint Logic 20 100-17

of Mostavski, J. Symb. Logic, 30, 49-57.

D. Richardson (1968), Some undecidable problems involving elementory

D. Richardson (1968), Fome undecidable problems involving elementory

To a real variable, J. Symb. Logic 33, 514-520.