GEORGE GARGOV AND VALENTIN GORANKO

MODAL LOGIC WITH NAMES

ABSTRACT. We investigate an enrichment of the propositional modal language $\mathscr{L}$ with a "universal" modality $\blacksquare$ having semantics $x \vDash \blacksquare\varphi$ iff $\forall y(y \vDash \varphi)$, and a countable set of "names" — a special kind of propositional variables ranging over singleton sets of worlds. The obtained language $\mathscr{L}_c$ proves to have a great expressive power. It is equivalent with respect to modal definability to another enrichment $\mathscr{L}(\boxed{\neq})$ of $\mathscr{L}$, where $\boxed{\neq}$ is an additional modality with the semantics $x \vDash \boxed{\neq}\varphi$ iff $\forall y(y \neq x \rightarrow y \vDash \varphi)$. Model-theoretic characterizations of modal definability in these languages are obtained. Further we consider deductive systems in $\mathscr{L}_c$. Strong completeness of the normal $\mathscr{L}_c$-logics is proved with respect to models in which all worlds are named. Every $\mathscr{L}_c$-logic axiomatized by formulae containing only names (but not propositional variables) is proved to be strongly frame-complete. Problems concerning transfer of properties ([in]completeness, filtration, finite model property etc.) from $\mathscr{L}$ to $\mathscr{L}_c$ are discussed. Finally, further perspectives for names in multimodal environment are briefly sketched.

1. INTRODUCTION

In a propositional setting (modal, temporal, dynamic, etc.) *names* are properties that identify the intensional objects completely, i.e. each of them holds only for a single object (be it a possible world, a temporal state or a data fragment, depending on the semantic interpretation of the language) and can be thus used to name that particular object.

Historically the idea of such naming variables can be traced back to the pioneering works of Prior (1956) and especially of Bull (1970) who introduced temporal reference to particular moments (or to the corresponding state of affairs) by means of special variables (the clock-variables of Prior) which are true at a single time instant. In Bull (1970) one can find an axiomatization of a temporal system in the language extended also by a universal modality (i.e. a modality related semantically to the universal relation).

In the area of application of propositional logic to computer science the idea of names was first explored (independently of the developments in temporal logic) by Passy and Tinchev (1985). In Passy and Tinchev (1991) there are many examples which show that

this enrichment is quite appropriate for the treatment of problems arising from, or justified as arising from, possible applications (cf. Gargov and Passy (1988) as well).

A first attempt to apply named languages to traditional modal logic was made in the main origin of the present paper (Gargov *et al.*, 1987) where the minimal normal modal logic with names was axiomatized and several further properties were stated. Let us note the use of necessity and possibility forms (instead of the universal modality) in the latter paper, borrowed from the treatise on programming logics in Goldblatt (1982). We now find this area ripe for a systematic treatment going from a collection of interesting but loosely connected cases to a kind of a general theory. In our work we try to give a coherent exposition of the theory of names in purely modal environment in order to emphasize just on the effects which names yield. We have chosen to reinstall the universal modality for many reasons, both technical and aesthetical.

After the preliminaries we start with expressiveness of the new language $\mathcal{L}_c$ in Section 3. In Section 4 we investigate modal definability in $\mathcal{L}_c$. The language with names turns out to be equivalent in this respect to another modal language enriched with an additional (recently actively investigated) modality over the inequality relation — the so called difference operator. This equivalence allows for a uniform characterization of modal definability in the spirit of Goldblatt and Thomason (1974). A good demonstration of the expressive power of $\mathcal{L}_c$ is the result that every finite frame is definable in $\mathcal{L}_c$ by means of a single *pure* (not containing propositional variables) formula only. In Section 5 we turn to deductive systems and problems of completeness in $\mathcal{L}_c$. The minimal normal $\mathcal{L}_c$-logic $\mathbf{K}_c$ is axiomatized using an additional rule of inference COV (taken from Passy and Tinchev (1985) where it appears independently from the quite analogous rule in Gabbay (1981); evidently COV originates from the ω-rule in the arithmetic) which enables construction of canonical-like models built from maximal consistent sets each of which contains a name. Thus strong completeness with respect to surjectively named models is settled for all normal $\mathcal{L}_c$-logics. Two classes of $\mathcal{L}_c$-logics are especially considered in the paper, which occupy the two extreme ends of the axis on which one could place logics according to the degree of "involvement with

names". On one hand we have the so-called *minimal extensions* axiomatized by formulae in the basic classical languages (i.e. which do not mention names explicitly), on the other hand the so-called *pure $\mathscr{L}_c$-logics* axiomatized by axiom schemata where only names occur. At the lower end a concern of ours is transfer of properties from a logic to its minimal extension. A simple instance of such a transfer is the conservativeness of the minimal extension (and hence, e.g., the upward transfer of incompleteness). The upward completeness transfer is still an open problem in general but it has been proved for the representative case when the underlying $\mathscr{L}$-logic is canonical. At the upper end we have proved that all pure $\mathscr{L}_c$-logics are strongly frame-complete which gives readily complete axiomatizations in $\mathscr{L}_c$ for a number of logics of particular interest. However, not every first-order definable $\mathscr{L}_c$-logic can be purely axiomatized; a counter-example is $\mathbf{G1}_c = \mathbf{K}_c + \Diamond\Box p \to \Box\Diamond p$, as shown in Section 6. At the end of the paper we briefly discuss several directions of further applications of names in multimodal environment including tense, dynamic etc. languages as well as systems with additional polyadic modalities as Kamp's Since and Until.

To complete these introductory remarks let us just mention a more regular and more promising viewpoint to the names: they are propositional variables with a restriction on the possible valuations. They range over the elements of a *definable* family of subsets of a given frame (in this case singletons). In this vein one could consider other types of restricted variables, ranging over points or subsets with specific definable properties. This idea awaits to be systematically explored in further investigations on intensional logics.

2. PRELIMINARIES

2.1. *Syntax*

We take a classical propositional modal language $\mathscr{L}$ with a countable set $P = \{p_0, p_1, \ldots\}$ of propositional variables, logical signs $\neg$, $\wedge$, truth $\top$, modality $\Box$. The other signs $\vee$, $\to$, $\leftrightarrow$, falsity $\bot$ and $\Diamond$ are classically defined as well as the set of formulae of $\mathscr{L}$.

Now we consider an enrichment of $\mathscr{L}$ with the following syntactical objects:

— an additional *"universal"* *modality* ■ (for a detailed investigation of the role of ■ see Goranko and Passy (1992)) with Kripke semantics over the universal relation W^2 of the universe:

$$\langle W, R, V \rangle \vDash \blacksquare\varphi[x] \text{ iff } \forall y(\langle W, R, V \rangle \vDash \varphi[y]).$$

— countable set $C = \{c_0, c_1, \ldots\}$ of special kind of variables, called *modal constants* or *names*. They have a status similar to that of the propositional variables with an important difference: the names are allowed to be true *at exactly one point of the universe*. The set of formulae of the language $\mathscr{L}_c$ obtained in this way is defined as usual, with the additional clauses: the names are formulae and ■φ is a formula whenever φ is.

We shall use φ, χ, ψ as metavariables for formulae both of $\mathscr{L}$ and $\mathscr{L}_c$ (explicitly pointing the language when necessary) and c, d, e as metavariables for names.

Some notions in $\mathscr{L}_c$: A *closed formula* is a Boolean combination of formulae of the type ■φ. A *pure formula* is a formula not containing propositional variables. An *instant of a formula* is the result of a replacement of propositional variables by arbitrary formulae and names by names.

2.2. Semantics

We assume familarity with all basic semantic notions and constructions connected with the algebraic and relational semantics of classical modal logic. For the technical background, see e.g. Hughes and Cresswell (1984), van Benthem (1984, 1986), Goldblatt (1976), Segerberg (1971). Some of the basic notions are modified here according to the new language.

DEFINITION. A *named model* in $\mathscr{L}_c$ is a quadruple $\langle W, R, V, \chi \rangle$ where $\langle W, R, V \rangle$ is a model in the classical sense and $\chi : C \to W$ is a *valuation of the names*. The extension of V to a valuation of all formulae is based on the clauses

$$(i) \qquad V(\blacksquare\varphi) = \begin{cases} W & \text{if } V(\varphi) = W \\ \phi & \text{otherwise} \end{cases};$$

(ii) $V(c) = \{\chi(c)\}$ for each $c \in C$.   #

Having a valuation we say that $\varphi \in \mathscr{L}_c$ is *true* at a point x of a named model $\mathfrak{N} = \langle W, R, V, \chi \rangle$, denoted $\mathfrak{N} \vDash \varphi[x]$, if $x \in V(\varphi)$; φ is *valid* in $\mathfrak{N}$, denoted $\mathfrak{N} \vDash \varphi$, if $V(\varphi) = W$.

DEFINITION. A named model $\langle W, R, V, \chi \rangle$ is *surjective* if χ is surjective (which, of course, implies $\|W\| \leqslant \aleph_0$).   #

    We say that $\varphi \in \mathscr{L}_c$ is *valid in a model* $\mathfrak{M} = \langle W, R, V \rangle$ if for every $\chi: C \to W$, $\langle \mathfrak{M}, \chi \rangle \vDash \varphi$. Now *validity in a frame* is defined as a validity in all models over the frame. Also the notions of general frame and validity in it are accordingly adapted in $\mathscr{L}_c$.

    Throughout the paper we will freely use metavariables: F, G for frames, $\mathfrak{F}$, $\mathfrak{G}$ for general frames, $\mathfrak{M}$ for models, $\mathfrak{N}$ for named models.

3. EXPRESSIVENESS AND FIRST-ORDER DEFINABILITY IN $\mathscr{L}_c$

There are (at least) two natural extensions to $\mathscr{L}_c$ of the translation ST (see van Benthem (1986)) of the modal formulae into the first-order language L_1 containing a binary predicate R and a countable set of unary predicates $\{P_0, P_1, \ldots \}$:

I. ST' is the standard ST for $\mathscr{L}$ and obeys the following additional clauses (cf. Gargov *et al.*, 1988):
    (i) $ST'(c_i) = (x = y_i)$ where $\{ y_0, y_1, \ldots \}$ is a countable set of individual variables, especially assigned for presenting the names in L_1 and x is a fixed variable, different from $y_0, y_1, \ldots$
    (ii) $ST'(\blacksquare\varphi) = \forall y ST'(\varphi)[y/x]$ where y doesn't occur in $ST'(\varphi)$ and x is the variable, fixed to be the only free variable (if any) besides the y's in $ST'(\varphi)$. Now one can express the validity in a named model. Let $\mathfrak{N} = \langle \mathfrak{M}, \chi \rangle$, where $\mathfrak{M} = \langle W, R, V \rangle$, be a named model. Define an L_1-model $\mathfrak{M}' = \langle W; R, P_0, P_1, \ldots \rangle$ such that $P_i = V(p_i)$ for every $i \in \mathbb{N}$, and a valuation of the individual variables $V: V(y_i) = \chi(c_i)$ and $V(z)$ arbitrary for any z not belonging to $\{ y_0, y_1, \ldots \}$. Then $\langle \mathfrak{M}, \chi \rangle \vDash \varphi$ iff $\langle \mathfrak{M}', V \rangle \vDash \forall x ST'(\varphi)$. Now, for validity in a model we have: $\mathfrak{M} \vDash \varphi$ iff $\mathfrak{M}' \vDash \forall x \forall y_{i1} \ldots \forall y_{ik} ST'(\varphi)$ where $c_{i1}, \ldots, c_{ik}$ are the names occurring in φ. Let us denote the

formula $\forall y_{i_1} \ldots \forall y_{i_1} ST'(\varphi)$ by $ST(\varphi)$. This gives a complete analogy with ST in the standard language $\mathscr{L}$ and validity in a frame is defined in the same way as for $\mathscr{L}$ — with the universal closure of the latter formula over all occurring P's.

II. Define ST'' as follows: let x be a fixed variable;
   (i) $ST''(p_i) = P_{2i+1}x$;
   (ii) $ST''(c_i) = P_{2i}x$;
   (iii) the remaining clauses are the same as above.
Now let $\mathfrak{N} = \langle W, R, V, \chi \rangle$ be a named model. Define a corresponding L_1-model $\mathfrak{N}' = \langle W; R, P_0, P_1, \ldots \rangle$ such that $P_{2i} = \{\chi(c_i)\}$ and $P_{2i+1} = V(p_i)$ for each $i \in \mathbb{N}$. Then $\mathfrak{N} \vDash \varphi$ iff $\mathfrak{N}' \vDash \forall x ST''(\varphi)$. For models ST'' gives: if $\mathfrak{M} = \langle W, R, V \rangle$ and $\mathfrak{M}'$ is obtained as $\mathfrak{N}'$ above, evaluating all P_{2i} arbitrarily, then

$$\mathfrak{M} \vDash \varphi \text{ iff } \mathfrak{M}' \vDash \forall x \forall P_{2i_1} \ldots \forall P_{2i_k}(\exists! z P_{2i_1} z \wedge \ldots \wedge \exists! z P_{2i_k} z \to ST''(\varphi))$$

where $c_{i_1}, \ldots, c_{i_k}$ are the names occurring in φ. Now it is natural to denote by $ST(\varphi)$ the formula

$$\forall P_{2i_1} \ldots \forall P_{2i_k}(\exists! z P_{2i_1} z \wedge \ldots \wedge \exists! z P_{2i_k} z \to ST''(\varphi)).$$

It is provably equivalent to the above ST and thus we have again an analogy with ST in $\mathscr{L}$. Finally, validity in a frame is expressed by the universal closure of $ST(\varphi)$ over all P_{2i+1} such that p_i occurs in φ.

The former translation is preferable from the point of view of simplicity, however it is a bit *ad hoc* while the latter one reflects adequately the idea of names over *definable sets*.

PROBLEM 1. Find model-theoretic characterizations (in the style of van Benthem (1986)) of the fragments of L_1 corresponding to each of the translations above. Is the problem whether an L_1-formula is equivalent to such a translation of a modal formula decidable?

Now, a few words about first-order definability. A formula $\varphi \in \mathscr{L}_c$ is said to be *first-order definable* if the class $FR(\varphi) = \{F/F \vDash \varphi\}$ is definable by a formula of the first-order language L_0 for structures $\langle W, R \rangle$.

First-order definability is not decidable even for $\mathscr{L}$ (see Chagrova (1989)) that is why no exact syntactic characterization of this property exists. However, there are several strong and useful sufficient syntactic conditions, e.g., the so called "Sahlqvist forms" and some generalizations (see Sahlqvist (1975), van Benthem (1986)). It is a routine procedure to check that, as a rule, they apply without further ado to $\mathscr{L}_c$, having in mind the first extension of ST given above. This is because the names behave just as universally bounded individual variables when we consider validity in a frame. Besides we have the following important fact:

PROPOSITION 3.1. *All pure formulae are first-order definable.*

Indeed, if φ is a pure formula, then $ST'(\varphi)$ is the required first-order equivalent.

4. MODAL DEFINABILITY IN $\mathscr{L}_c$

In this section we give a model-theoretic characterization of the expressive power of $\mathscr{L}_c$ with respect to definable classes (properties) of frames.

DEFINITION. (1) Let $\mathfrak{C}$ be a class of frames. $\mathfrak{C}$ is *modally definable* [MD] in $\mathscr{L}_c$ if there exists a set Γ of fomulae of $\mathscr{L}_c$ such that for each frame F, $F \in \mathfrak{C}$ iff $F \vDash \Gamma$.

(2) A (first-order) property of α of frames is *modally definable* [MD] in $\mathscr{L}_c$ iff the class of frames satisfying α is MD in $\mathscr{L}_c$.　　　　#

Modal definability in $\mathscr{L}_c$ is hardly decidable (being not decidable for $\mathscr{L}$), so we can only hope for modal-theoretic characterizations of the properties, definable in this language. Classical results in this direction or $\mathscr{L}$ can be found in Goldblatt and Thomason (1974) and van Benthem (1986); for some enriched modal languages -- in Goranko (1990), Goranko and Passy (1992). Unfortunately, the algebraic semantics which should correspond to $\mathscr{L}_c$ is rather awkward and this is an obstacle to apply the so far developed technique for obtaining such results here. However, it became possible to avoid this drawback due

to the circumstance that $\mathscr{L}_c$ turned out equivalent, with respect to modal definability, to another enrichment of $\mathscr{L}$, viz. $\mathscr{L}(\boxed{\neq})$ (see below). Recently this latter language has been actively investigated (see Gargov *et al.* (1987), Goranko (1990), Koymans (1989), de Rijke (1989)).

4.1. *Equivalence with respect to modal definability between* $\mathscr{L}_c$ *and* $\mathscr{L}(\boxed{\neq})$

The language $\mathscr{L}(\boxed{\neq})$ extends $\mathscr{L}$ by an additional modality $\boxed{\neq}$ with the semantics: $\mathfrak{M} \vDash \boxed{\neq}\varphi[x]$ iff $\forall y(x \neq y \to \mathfrak{M} \vDash \varphi[y])$.

Let us observe that $\blacksquare$ (and hence $\blacklozenge$) is definable in $\mathscr{L}(\boxed{\neq})$:
$\blacksquare\varphi = \boxed{\neq}\varphi \wedge \varphi$.

Note that the notion of valuation does not depend on the language since it initially concerns only the propositional variables. Fixing a language we can readily extend any valuation to a valuation of all formulae of the language, having in mind the intended semantics of the modalities. That is why we will freely consider the same valuation in different languages. For the same reason we can view models as models for different languages.

In this section we shall give a constructive proof that $\mathscr{L}_c$ and $\mathscr{L}(\boxed{\neq})$ are equivalent with respect to modal definability.

I. We shall define a *translation* $\sigma: \mathscr{L}(\boxed{\neq}) \to \mathscr{L}_c$ such that for each model $\mathfrak{M}$, $\mathfrak{M} \vDash \varphi$ iff $\mathfrak{M} \vDash \sigma(\varphi)$.

1. Let us order the formulae from $\mathscr{L}(\boxed{\neq})$ in a sequence: $\varphi_1, \varphi_2, \ldots$. Let $n(\varphi)$ be the number of φ for each $\varphi \in \mathscr{L}(\boxed{\neq})$.

2. Define a translation $\tau': \mathscr{L}(\boxed{\neq}) \to \mathscr{L}_c$ inductively by the complexity of the formulae.

$$\tau'(p) = p;$$
$$\tau'(\neg\varphi) = \neg\tau'(\varphi);$$
$$\tau'(\varphi \wedge \psi) = \tau'(\varphi) \wedge \tau'(\psi)$$
$$\tau'(\Diamond) = \Diamond\tau'(\varphi);$$
$$\tau'(\lozenge\varphi) = (\blacklozenge\tau'(\varphi)) \wedge (c_{n(\varphi)} \to \blacklozenge(\neg c_{n(\varphi)} \wedge \tau'(\varphi))).$$

hence, $\tau'(\boxed{\neq}\varphi) = (\blacksquare\tau'(\varphi)) \vee (c_{n(\neg\varphi)} \wedge \blacksquare(c_{n(\neg\varphi)} \vee \tau'(\varphi))).$

3. Put

$$\tau(\varphi) = \tau'(\varphi) \wedge \bigwedge_{\substack{\diamond\psi \text{ occurs} \\ \text{in } \varphi}} (\blacklozenge\tau'(\psi) \rightarrow \blacklozenge(\tau'(\psi) \wedge c_{n(\psi)})).$$

4. Finally put $\sigma(\varphi) =_{DF} \neg\tau(\neg\varphi)$.

PROPOSITION 4.1.

$$V(\diamondsuit \varphi) = \begin{cases} \emptyset & \text{if } V(\varphi) = \emptyset \\ W \setminus \{x\} & \text{if } V(\varphi) = \{x\} \,. \\ W & \text{otherwise} \end{cases} \qquad \#$$

LEMMA 4.2. *For every model* $\langle W, R, V \rangle$ *there exists a named model* $\langle W, R, V, \chi \rangle$ *such that for every formula* $\varphi \in \mathscr{L}(\boxed{\neq})$, $V(\varphi) = V(\tau'(\varphi)) = V(\tau(\varphi))$.

Proof. Let w be an arbitrary fixed element of W. We shall define a valuation of the names χ as follows: for every c there exists a formula ψ such that $c = c_{n(\psi)}$. If $V(\psi) \neq \emptyset$ then we choose some $v \in V(\psi)$ and put $\chi(c) = v$; otherwise put $\chi(c) = w$.

Now we prove by induction that for each formula φ, $V(\varphi) = V(\tau'(\varphi))$. The only non-trivial case is $\varphi = \diamondsuit\psi$. Provided $V(\psi) = V(\tau'(\psi))$ we shall prove that $V(\diamondsuit\psi) = V(\tau'(\diamondsuit\psi))$:
$V(\tau'(\diamondsuit\psi)) =$

$$V(\blacklozenge\tau'(\psi)) \cap (V(\neg c_{n(\psi)}) \cup (V(\blacklozenge(\neg c_{n(\psi)})) \cap V(\tau'(\psi)))) =$$

$V(\blacklozenge\psi) \cap (V(\neg c_{n(\psi)}) \cup (V(\blacklozenge(\neg c_{n(\psi)})) \cap V(\psi)))$.

We distinguish the following cases:
(a) $V(\psi) = \emptyset$. Then $V(\tau'(\diamondsuit\psi)) \subseteq V(\blacklozenge\psi) = \emptyset$.
(b) $V(\psi) = \{x\}$. Then $V(c_{n(\psi)}) = \{x\}$ hence
$V(\tau'(\diamondsuit\psi)) = W \setminus \{x\} \cup \blacklozenge((W \setminus \{x\}) \cap \{x\}) = W \setminus \{x\}$.
(c) Otherwise. Then $V(\blacklozenge\psi) = W$ and $V(\neg c_{n(\psi)}) \cap V(\psi) \neq \emptyset$ hence $V(\blacklozenge(\neg c_{n(\psi)} \wedge \psi)) = W$ and $V(\tau'(\diamondsuit\psi)) = W$.

This completes the induction. Moreover, by the definition of χ, $V(\blacklozenge\tau'(\psi) \rightarrow \blacklozenge(\tau'(\psi) \wedge c_{n(\psi)})) = W$. Thus for every φ, $V(\varphi) = V(\tau'(\varphi)) = V(\tau(\varphi))$. \qquad \#

LEMMA 4.3. *For every named model* $\langle W, R, V, \chi \rangle$ *and formula* $\varphi \in \mathscr{L}(\boxed{\neq})$, *if* $V(\tau(\varphi)) \neq \emptyset$ *then* $V(\varphi) = V(\tau'(\varphi)) = V(\tau(\varphi))$.

Proof. Let $x \in V(\tau(\varphi))$. Then $x \in V(\blacklozenge\tau'(\psi) \to \blacklozenge(\tau'(\psi) \wedge c_{n(\psi)}))$ for each subformulae $\diamondsuit\psi$ of φ. Hence if $V(\tau'(\psi)) \neq \emptyset$ then $\chi(c_{n(\psi)}) \in V(\tau'(\psi))$, so one can inductively prove, as it was done in the proof of Lemma 4.2, that $V(\varphi) = V(\tau'(\varphi))$; moreover, $V(\tau'(\varphi)) = V(\tau(\varphi))$.   #

THEOREM 4.4. *For every model* $\langle W, R, V \rangle$ *and formula* $\varphi \in L(\boxed{\neq})$, $\mathfrak{M} \vDash \varphi$ *iff* $\mathfrak{M} \vDash \sigma(\varphi)$.

Proof. Let $\mathfrak{M} \nvDash \varphi$. In virtue of Lemma 4.2 there exists χ such that in $\langle \mathfrak{M}, \chi \rangle$ $V(\tau(\neg\varphi)) = V(\neg\varphi) \neq \emptyset$ hence $V(\sigma(\varphi)) \neq W$. Vice versa, let $\mathfrak{M} \nvDash \sigma(\varphi)$, i.e. $V(\tau(\neg\varphi)) \neq \emptyset$. Hence, according to Lemma 4.3, $V(\tau(\neg\varphi)) = V(\neg\varphi)$, and therefore $V(\varphi) \neq W$.   #

COROLLARY 4.5. *For every frame* F *and* $\varphi \in L(\boxed{\neq})$, $F \vDash \varphi$ *iff* $F \vDash \sigma(\varphi)$.   #

So, definability in $\mathscr{L}(\boxed{\neq})$ is not stronger than that in $\mathscr{L}_c$.

II. The opposite is true as well. Indeed, one can easily enforce a propositional variable p to serve as a name in $\mathscr{L}(\boxed{\neq})$, putting as an antecedent $\blacklozenge(p \wedge \diamondsuit\neg p)$. More formally, we shall define a *translation* $\pi: \mathscr{L}_c \to \mathscr{L}(\boxed{\neq})$. First the names are coded with variables, by $\xi: P \cup C \to P$ as follows: $\xi(p_i) =_{DF} p_{2i}$ and $\xi(c_i) =_{DF} p_{2i+1}$ for each $i \in \mathbb{N}$. ξ is accordingly extended to $\xi(\varphi)$ for each formula $\varphi \in \mathscr{L}_c$. For each $q \in P \cup C$, put $v(q) =_{DF} \blacklozenge(q \wedge \boxed{\neq}\neg q)$. Now let $\varphi \in \mathscr{L}_c$ and $d_1, \ldots, d_k$ be the names occurring in φ. Put
$\pi(\varphi) =_{DF} v(\xi(d_1)) \wedge \ldots \wedge v(\xi(d_k)) \to \xi(\varphi)$.

This syntactic translation yields a semantic one. Let $\mathfrak{M} = \langle W, R, V \rangle$ be a model. We define a model $\mathfrak{M}_{\neq} = \langle W, R, V_{\neq} \rangle$ where $V_{\neq}(p) = V(\xi^{-1}(p))$ for each $p \in P$.

The following two propositions are easy exercises.

PROPOSITION 4.6. *Let* $\mathfrak{M}$ *be a model.*

(1) *For every* $p \in P$, $\mathfrak{M} \vDash v(p)$ *if* $V(p)$ *is a singleton and* $\mathfrak{M} \vDash \neg v(p)$ *otherwise.*

(2) *For every* $c \in C$, $\mathfrak{M} \vDash v(c)$.   #

PROPOSITION 4.7. *If $\langle W, R, V, \chi \rangle$ is a named model, $\varphi \in \mathscr{L}_c$ and $V_{\neq}$ is the valuation defined as above then $V(\varphi) = V_{\neq}(\xi(\varphi))$.* #

For convenience we will use a notion of $\mathscr{L}_c$-*valuation* as a valuation both for the variables and names.

THEOREM 4.8. *Let $F = \langle W, R \rangle$ and $\varphi \in \mathscr{L}_c$. Then:*
(1) *for every $\mathscr{L}_c$-valuation V in F, $V(\varphi) = V_{\neq}(\pi(\varphi))$;*
(2) *for every valuation V in F there exists an $\mathscr{L}_c$-valuation V_φ in F such that $V_\varphi(\varphi) \subseteq V(\pi(\varphi))$.*

Proof. (1) Let V be an $\mathscr{L}_c$-valuation. Then $V_{\neq}(\xi(d_i))$ is a singleton, hence $V_{\neq}(v(\xi(d_i))) = W$ for $i = 1, \ldots, k$. Therefore $V_{\neq}(v(\xi(d_1))) \wedge \ldots \wedge V_{\neq}(v(\xi(d_k))) = W$, so $V_{\neq}(\pi(\varphi)) = V_{\neq}(\xi(\varphi)) = V(\varphi)$.

(2) Let $x \notin V(\pi(\varphi))$, i.e. $x \in V(v(\xi(d_1))) \wedge \ldots \wedge V(v(\xi(d_k)))$ and $x \notin V(\xi(\varphi))$ hence $x \in V(v(\xi(d_i)))$ for $i = 1, \ldots, k$ and, in virtue of 4.6, $V(\xi(d_i))$ is a singleton, hence $V(v(\xi(d_i))) = W$. Therefore $V(v(\xi(d_1))) \wedge \ldots \wedge V(v(\xi(d_k))) = W$ whence $V(\pi(\varphi)) = V(\xi(\varphi))$. Then we can define an $\mathscr{L}_c$-valuation V_φ as follows:

$$V_\varphi(q) =_{\mathrm{DF}} \begin{cases} V(\xi(q)) & \text{if } \xi(q) \text{ occurs in } \pi(\varphi), \\ \{w\} & \text{otherwise,} \end{cases}$$

where w is an arbitrary fixed element of W and $q \in P \cup C$. V_φ is an $\mathscr{L}_c$-valuation and $V_\varphi(\varphi) = V(\xi(\varphi)) = V(\pi(\varphi))$. So $x \notin V_\varphi(\varphi)$. #

COROLLARY 4.9. *Let $F = \langle W, R \rangle$, $x \in W$ and $\varphi \in \mathscr{L}_c$. Then:*
(1) $F \vDash \varphi[x]$ *iff* $F \vDash \pi(\varphi)[x]$;
(2) $F \vDash \varphi$ *iff* $F \vDash \pi(\varphi)$. #

The last result shows that the definability in $\mathscr{L}_c$ is not stronger than that of $\mathscr{L}(\boxed{\neq})$. So we have

THEOREM 4.10. *The languages $\mathscr{L}_c$ and $\mathscr{L}(\boxed{\neq})$ are equivalent with respect to modal definability.* #

4.2. Modal definability in $\mathscr{L}(\boxed{\neq})$

We shall characterize modal definability in $\mathscr{L}(\boxed{\neq})$ in the model-theoretic style of Goldblatt and Thomason (1974). All statements

without proofs below are obtained by simple calculations based on corresponding results from Goranko (1990) where languages containing modalities over a relation (in our case equality) and its complement are investigated. First, we shall define the analog in $\mathscr{L}(\boxed{\neq})$ of the notion of SA-construction, introduced by Goldblatt and Thomason.

DEFINITION (cf. Goranko (1990), 3.7, 3.10). $F' = \langle W', R' \rangle$ is an $\neq$-*collapse* of the general frame $\mathfrak{F} = \langle W, R, \mathbb{W} \rangle$ iff F' is a substructure of $F = \langle W, R \rangle$ (i.e. $W \subseteq W'$ and $R = R' \cap W^2$), and there exists a general subframe $\mathfrak{G}$ of $\mathfrak{F}$ such that $F'^+ \cong \mathfrak{G}^+$ and for each $x \in W'$, $R(x) \subseteq [R'(x)]_{\mathfrak{G}^+}$ where $[X]_{\mathfrak{G}^+}$ is the least element of $\mathfrak{G}^+$, containing X.   #

In particular, when $\mathfrak{F} = \langle W, R, \mathscr{P}(W) \rangle$ we obtain a definition of $\neq$-collapse of the frame $\langle W, R \rangle$.

DEFINITION. *General ultraproduct of frames* is an ultraproduct of the corresponding full general frames (see, e.g. van Benthem (1986) or Goldblatt (1976).   #

DEFINITION. Let $\mathfrak{C}$ be a class of frames. The *modally definable closure* of $\mathfrak{C}$ in $\mathscr{L}(\boxed{\neq})$, $[\mathfrak{C}]_{\neq}$, is the smallest MD in $\mathscr{L}(\boxed{\neq})$ class containing $\mathfrak{C}$.   #

THEOREM 4.11. *For every class of frames* $\mathfrak{C}$, $[\mathfrak{C}]_{\neq}$ *consists of all isomorphic copies of* $\neq$-*collapses of general ultraproducts of frames from* $\mathfrak{C}$.   #

COROLLARY 4.12. $\mathfrak{C}$ *is MD in* $\mathscr{L}(\boxed{\neq})$ *iff it is closed under isomorphisms and* $\neq$-*collapses of general ultraproducts of frames.*   #

COROLLARY 4.13. *If* $\mathfrak{C}$ *is a* Δ-*elementary (defined by a set of first-order conditions) class then* $\mathfrak{C}$ *is MD in* $\mathscr{L}(\boxed{\neq})$ *iff it is closed under* $\neq$-*collapses.*   #

In particular, a first-order property is definable in $\mathscr{L}(\boxed{\neq})$ iff it is preserved under $\neq$-collapses. (E.g. all universal first-order conditions are.)

COROLLARY 4.14. *Every finite frame is distinguishable (up to iso-morphism) by a set of $\mathscr{L}(\boxed{\neq})$-formulae.*

Proof. From 4.13, since no finite frame has proper $\neq$-collapses.   #

The essential difference between the above characterization and the classical result of Goldblatt and Thomason is due to the fact that in $\mathscr{L}(\boxed{\neq})$ the notions of generated subframe and disjoint union of frames are trivialized.

Again following Goldblatt and Thomason (1974) and Goranko (1990) we can obtain another characterization of the Δ-elementary classes MD in $\mathscr{L}(\boxed{\neq})$, which is somewhat more convenient to use.

DEFINITION. A bi-relational frame $\langle W, R, S \rangle$ is a *nonstandard $\mathscr{L}(\boxed{\neq})$-frame* if $S \cup \{\langle x, x \rangle / x \in W\} = W^2$.   #

DEFINITION. An $\mathscr{L}(\boxed{\neq})$-*morphism* is any bi-relational p-morphism of a frame of the type $\langle W, R, \neq \rangle$.   #

Note that the image of any $\mathscr{L}(\boxed{\neq})$-morphism f is a nonstandard $\mathscr{L}(\boxed{\neq})$-frame. It is standard iff f is an isomorphism.

DEFINITION. An *ultrafilter extension* of an $\mathscr{L}(\boxed{\neq})$-frame $F = \langle W, R, \neq \rangle$ is the frame $F^* = \langle W^*, R^*, S^* \rangle$, denoted $ue(F)$, where W^* is the set of ultrafilters in W and R^*, S^* are canonically defined on R and $\neq$ respectively.   #

It is easy to see that $\langle W^*, R^*, S^* \rangle$ is a nonstandard $\mathscr{L}(\boxed{\neq})$-frame.

DEFINITION. F is an *ultrafilter contraction* of G iff $G \cong ue(F)$.

THEOREM 4.15 (cf. Goranko (1990)). *A Δ-elementary class $\mathfrak{C}$ is MD in $\mathscr{L}(\boxed{\neq})$ iff $\mathfrak{C}$ is closed under ultrafilter contractions of $\mathscr{L}(\boxed{\neq})$-morphic images.*   #

In virtue of Th. 4.10 all these results directly apply to $\mathscr{L}_c$. Other results concerning definability in $\mathscr{L}(\boxed{\neq})$ can be found in de Rijke (1989).

We shall finish this section with a strengthening of 4.14 which gives an additional evidence on the expressive power of $\mathscr{L}_c$.

Denote

$$\rho_n = \bigvee_{\substack{i \neq j \\ 1 \leqslant i,j \leqslant n+1}} \blacklozenge(c_i \wedge c_j);$$

$$\tau_n = \neg \blacksquare(c_1 \vee \ldots \vee c_{n-1}); \; \sigma_n = \rho_n \wedge \tau_n.$$

It is a standard exercise to see that if $F = \langle W, R \rangle$ then: $F \vDash \rho_n$ iff $\|W\| \leqslant n$; $F \vDash \tau_n$ iff $\|W\| \geqslant n$, and hence $F \vDash \sigma_n$ iff $\|W\| = n$.

DEFINITION. A *well-named frame* is a countable frame $F = \langle W, R \rangle$ provided with a valuation $\chi: C \to |F|$ such that:
  (a) if F is finite and $\|W\| = n$ then $W = \{\chi(c_1), \ldots, \chi(c_n)\}$;
  (b) if F is infinite then χ is a bijection.

DEFINITION. A *diagram* of a finite well-named frame $F = \langle W, R \rangle$ with n points is the formula

$$D(F) = \wedge\{\blacklozenge(c_1 \wedge \Diamond c_j): 1 \leqslant i, j \leqslant n \text{ and } R\chi(c_i)\chi(c_j)\} \wedge$$

$$\wedge \; \{\blacklozenge(c_1 \wedge \neg \Diamond c_j): 1 \leqslant i, j \leqslant n \text{ and } \neg R\chi(c_i)\chi(c_j)\}.$$

$$\#$$

Clearly, there are finitely many non-isomorphic frames with n points. Let us fix some well-named frames over each of them: $F_1^n, \ldots, F_{f(n)}^n$.

Now let $F = \langle W, R \rangle$ and $\|W\| = n$. Denote

$$\kappa'(F) = \blacksquare(c_1 \vee \ldots \vee c_n) \wedge \bigvee\{D(F_i^n): 1 \leqslant i \leqslant f(n) \text{ and } F \not\cong F_i^n\}$$

and $\kappa(F) = \sigma_n \wedge \neg \kappa'(F)$. ($F \not\cong F_i^n$ means that F is not isomorphic to the underlying frame of the well-named frame F_i^n.)

THEOREM 4.16. *For any frame G and finite frame F, $G \vDash \kappa(F)$ iff $G \cong F$.*

Proof. (1) Let $G \vDash \kappa(F)$. Then G consists of n points. Assume $G \not\cong F$. Then there exists a valuation χ in G such that $\langle G, \chi \rangle \cong F_i^n$ for some i. At that $\langle G, \chi \rangle \vDash \blacksquare(c_1 \vee \ldots \vee c_n)$ and $\langle G, \chi \rangle \vDash D(F_i^n)$ hence $\langle G, \chi \rangle \vDash \kappa'(F)$ therefore $\langle G, \chi \rangle \not\vDash \kappa(F)$ − a contradiction.

(2) Vice versa, let $G \nvDash \kappa(F)$. Assume $G \cong F$. Then $G \vDash \sigma_n$, hence for some well-named model $\langle G, V, \chi \rangle \vDash \kappa'(F)[w]$ for some $w \in |G|$. Since $\kappa'(F)$ is a closed pure formula, $\langle G, \chi \rangle \vDash \kappa'(F)$ hence $\langle G, \chi \rangle \vDash \blacksquare(c_1 \vee \ldots \vee c_n)$ and $\langle G, \chi \rangle \vDash D(F_i^n)$ for some i such that $F \ncong F_i^n$. But then $\langle G, \chi \rangle \cong F_i^n$ and so $F_i^n \cong F$ — a contradiction. #

COROLLARY 4.17. *Each finite frame is definable in $\mathscr{L}_c$ by means of a single pure formula.* #

The "pure" definability in $\mathscr{L}_c$ is important from the point of view of axiomatizability — it automatically ensures completeness as we shall see later. That is why we raise a problem:

PROBLEM 2. Find syntactic or (at least) model-theoretic characterization of the L_0-formulae, definable in $\mathscr{L}_c$ with a pure formula. Is the problem of pure definability decidable at all?

5. DEDUCTIVE SYSTEMS IN $\mathscr{L}_c$

5.1. *Necessity and possibility forms in $\mathscr{L}_c$*

Let \$ be a symbol, not belonging to $\mathscr{L}_c$. We define (following Goldblatt (1982) and Gargov *et al.* (1987)) inductively the notions of *necessity* and *possibility forms* in $\mathscr{L}_c$.

DEFINITION. (1) \$ is a *necessity form* (NF) of \$.
  (2) If l is a NF, $\varphi \in \mathscr{L}_c$ and n is either $\square$ or $\blacksquare$ then $\varphi \rightarrow l$ and nl are NF's of \$. #

DEFINITION. (1) \$ is a *possibility form* (PF) of \$.
  (2) If m is a PF, $\varphi \in \mathscr{L}_c$ and p is either $\lozenge$ or $\blacklozenge$ then $\varphi \wedge m$ and pm are PF's of \$. #

We will present each NF and PF of \$ in the following uniform way:
$l(\$) = \varphi_0 \rightarrow n_1(\varphi_1 \rightarrow \ldots n_k(\varphi_k \rightarrow \$) \ldots)$ and $m(\$) = \varphi_0 \wedge p_1(\varphi_1 \wedge \ldots p_k(\varphi_k \wedge \$) \ldots)$ where φ_i is $\top$ when necessary. The number k is the *depth* of the form.

If $l(\$)$ $/m(\$)/$ is a form and $\psi \in \mathcal{L}_c$ by $l(\psi)$ $/m(\psi)/$ we will denote the result of the replacement $\psi/\$$ (of $\$$ by ψ) in the form.

Let $m(\$) = \varphi_0 \wedge p_1(\varphi_1 \wedge \ldots p_k(\varphi_k \wedge \$) \ldots)$, $\mathfrak{N}$ be a named model and $x \in |\mathfrak{N}|$. If $\mathfrak{N} \vDash m(\psi)[x]$ then there exists a chain of points $x_1, \ldots , x_k$ such that $\mathfrak{N} \vDash \varphi_i \wedge p_{i+1}(\varphi_{i+1} \wedge \ldots p_k(\varphi_k \wedge \$) \ldots)[x_i]$ for $i = 1, \ldots , k$ and $x, x_1, \ldots , x_k$ are successively connected by R or W^2 according to $p_1, \ldots , p_k$. Such a chain will be called a *witness of the truth* of $m(\psi)$ in $\langle \mathfrak{N}, x \rangle$.

Note that if $l(\$)$ $/m(\$)/$ is a necessity /possibility/ form then $\neg l(\psi)$ $/\neg m(\psi)/$ is equivalent to a possibility /necessity/ form $l'(\neg \$)$ $/m'(\neg \$)/$ called the *dual* of l $/m/$.

5.2. $\mathbf{K}_c$. *Simple extensions*

The minimal normal logic $\mathbf{K}_c$ of $\mathcal{L}_c$ is axiomatized as follows (cf. with Gargov *et al.* (1987)):

AXIOMS.

> (0)   Enough propositional tautologies.
>
> (1) $\mathbf{K}(\square)$
>
> (2) $S5(\blacksquare)$
>
> (3)   (incl): $\blacksquare p \rightarrow \square p$.
>
> (4)   (nam1): $\blacklozenge c$
>
> (5)   (nam2): $\blacklozenge(c \wedge \varphi) \rightarrow \blacksquare(c \rightarrow \varphi)$

RULES.

$$\text{SUB: } \frac{\varphi}{\text{sub }(\varphi)} \text{ where sub }(\varphi) \text{ is an instant of } \varphi;$$

$$\text{MP: } \frac{\varphi, \varphi \rightarrow \psi}{\psi};$$

$$\text{NEC}_\blacksquare: \frac{\varphi}{\blacksquare \varphi};$$

$$\text{COV: } \frac{l(\neg c) \text{ for each } c \in C}{l(\bot)} \text{ where } l \text{ is a NF in } \mathcal{L}_c.$$

Note that $\text{NEC}_\square$ follows from $\text{NEC}_\blacksquare$, MP and (incl).

DEFINITION. *Simple extensions of* $\mathbf{K}_c$ *($\mathscr{L}_c$-logics)* are extensions of $\mathbf{K}_c$ by means of axioms only.

If L is an $\mathscr{L}_c$-logic, by L^- we will denote the weakening of L obtained by dropping the rule COV.

PROPOSITION 5.1. *Validity in a frame is preserved by COV.*

Proof. Let $F \vDash l(\neg c)$ for each $c \in C$. Suppose $F \nvDash l(\bot)$. Then for some named model $\mathfrak{N} = \langle F, V, \chi \rangle$ and $x \in F$, $\mathfrak{N} \nvDash l(\bot)[x]$ hence $\mathfrak{N} \vDash l'(T)[x]$. Let $x, x_1, \ldots, x_k$ be a witness of this truth. Let $d \in C$ and d does not occur in $l(\bot)$. Define $\chi': C \rightarrow F$ as follows: $\chi'(d) = x_k$, $\chi'(c) = \chi(c)$ for each $c \neq d$. Let $\mathfrak{N}' = \langle F, V, \chi' \rangle$. Then $\mathfrak{N}' \vDash l'(d)[x]$ hence $\mathfrak{N}' \nvDash l(\neg d)[x]$, therefore $F \nvDash l(\neg d)$ — a contradiction.   #

DEFINITION. A frame F is said to be a frame for an $\mathscr{L}_c$-logic L (an *L-frame*) if $F \vDash L$ (i.e. all theorems of L are valid in F). The class of all L-frames is denoted by $FR(L)$.   #

COROLLARY 5.2. *For any* $\mathscr{L}_c$-*logic* L, $FR(L) = FR(L^-)$.   #

Now one can be easily persuaded in the use of COV.

EXAMPLE 1. Consider the $\mathscr{L}_c$-logic $D_c = \mathbf{K}_c + c \rightarrow \Diamond T$. $D_c \vdash \Diamond T$ but $D_c^- \nvdash \Diamond T$ since there exist non-surjective named models for D_c^- in which $\Diamond T$ is not valid, e.g. over the frame $\underset{1}{\bullet} \overset{R}{\longrightarrow} \underset{2}{\bullet}$ with $\chi(c) = 1$ for all $c \in C$. Therefore, by 5.2, D_c^- is incomplete. Indeed, to complete D_c^- it is sufficient to add a weaker (as we shall see) rule COV_0:
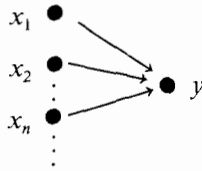
$$\frac{c \rightarrow \varphi \text{ for each } c \in C}{\varphi}.$$

This rule is even weaker than COV_1:

$$\frac{\Box(c \rightarrow \varphi) \text{ for each } c \in C}{\Box(\varphi)}$$

as one can see from the next example.

EXAMPLE 2. Consider the model $\mathfrak{R} = \langle F, V, \chi \rangle$ where F is:

and $\chi(c_i) = x_i$, $V(p_i) = \{x_i\}$, for each $i \in \mathbb{N}$. One can see inductively on φ that $V(\varphi)$ is either finite set of named points or a complement of such a set. (Note that $V(\Box\varphi)$ is $\emptyset$ in the first case and W in the second.) So, COV_0 preserves validity in $\mathfrak{R}$ while COV_1 doesn't since $\mathfrak{R} \models \Box\neg c$ but $\mathfrak{R} \not\models \Box\bot$, hence $(\mathbf{K}_c^- + COV_0 + \Box\neg c) \not\vdash \Box\bot$ but $(\mathbf{K}_c^- + COV_1 + \Box\neg c) \vdash \Box\bot$.

Let us now denote by COV_n the restriction of COV to necessity forms with depth not exceeding n.

PROBLEM 3. (a) Find a syntactic criterion for redundancy of COV in a particular $\mathscr{L}_c$-logic. Is this redundancy problem decidable?

(b) Find a syntactic criterion for restrictibility of COV to some COV_n in particular $\mathscr{L}_c$-logics.

Having Example 1 in mind, it is a plausible guess that the hierarchy $COV_0, \ldots, COV_n \ldots$ cannot be restricted in general to any COV_n. In the temporal language with names, however, the picture is radically simplified: this hierarchy collapses to COV_0. The reason for that is the following observation (see Gabbay and Hodkinson (1990)): over the minimal temporal logic any necessity form $\varphi_0 \rightarrow n_1(\varphi_1 \rightarrow \ldots$ $n_k(\varphi_k \rightarrow \$) \ldots)$ is deductively equivalent to its "converse" $\neg\$ \rightarrow$ $(\varphi_k \rightarrow n_k^*(\varphi_{k-1} \rightarrow \ldots n_1^*\neg\varphi_0) \ldots)$ where $G^* = H$, $H^* = G$, $\blacksquare^* = \blacksquare$.

REMARK (cf. Passy and Tinchev (1985)). Actually, COV is only *prima facie* infinitistic. Consider the finite rule

$$COV^*: \frac{l(\neg c) \text{ for some } c \in C \text{ not occurring in } l(\$)}{l(\bot)}.$$

LEMMA 5.3. *COV* is equivalent to COV on the basis of SUB.*

Proof. Of course, COV* is not weaker than COV. It is not stronger either: suppose L is an $\mathscr{L}_c$-logic and $L \vdash l(\neg c)$ for some c not occurring in $l(\$)$. Then for each $d \in C$: $L \vdash l(\neg d)$ by the substitution d/c. Thus one can infer by COV everything which can be inferred by COV*.   #

Although COV and COV* infer the same formulas, we have to stress, as the referee points out, that the systems with COV differ from the corresponding systems with COV* regarding to *compactness*: due to the infinitary nature of COV is obviously lacks the usual compactness property and the notion of *strong completeness* branches, as the next section shows.

5.3. *Strong completeness of $\mathscr{L}_c$-logics*

Let L be a fixed $\mathscr{L}_c$-logic and Γ be a set of formulae of $\mathscr{L}_c$.

DEFINITION. (1) An *L-theory of* Γ is the least set of formulae $\mathrm{Th}_L(\Gamma)$ which contains $L \cup \Gamma$ and is closed under MP.

(2) A *named L-theory of* Γ is the least set $\mathrm{NTh}_L(\Gamma)$ which contains $L \cup \Gamma$ and is closed under MP and COV.

DEFINITION. Γ is:
  (1) *L-consistent* if $\perp \notin \mathrm{Th}_L(\Gamma)$.
  (2) *surjectively L-consistent* if $\perp \notin \mathrm{NTh}_L(\Gamma)$.

Note that for finite Γ, $\mathrm{NTh}_L(\Gamma) = \mathrm{Th}_L(\Gamma)$ and the last two notions coincide but, e.g. $\{\neg c: c \in C\}$ is $\mathbf{K}_c$-consistent while not surjectively $\mathbf{K}_c$-consistent.

DEFINITION. L is:
  (1) *strongly [surjectively] model-complete* if every [surjectively] *L*-consistent set is satisfiable in a [surjective] named *L*-model.
  (2) *[surjectively] complete* if every *L*-consistent formula is satisfiable in a [surjective] named *L*-model based on an *L*-frame.
  (3) *strongly [surjectively] complete* if every [surjectively] *L*-consistent set is satisfiable in a [surjective] named *L*-model based on an *L*-frame.

THEOREM 5.4. *Every $\mathscr{L}_c$-logic L is strongly surjectively model-complete.*

Proof. We closely follow Gargov *et al.* (1987). Here is a sketch:

Let Γ be a surjectively L-consistent set. The standard deduction lemma holds for named theories: $\psi \in \text{NTh}(\Gamma \cup \{\varphi\})$ iff $\varphi \to \psi \in \text{NTh}(\Gamma)$. Now we shall prove an analog of Lindenbaum's lemma: every surjectively L-consistent set Δ can be included into a maximal consistent named L-theory. We enumerate the formulae of $\mathscr{L}_c$: φ_0, $\varphi_1, \ldots$ and define successively a chain of consistent named L-theories $T_0, T_1, \ldots$ as follows: $T_0 = \text{NTh}_L(\Delta)$. Let T_n be defined. Set $T_{n+1} =_{DF} \text{NTh}(T_n \cup \{\varphi_n\})$ if the latter is consistent. If not, consider two cases:

— φ_n has the form $l(\bot)$ for some NF l. Then there exists a name c such that $l(\neg c)$ does not belong to T_n since T_n is consistent. In this case put $T_{n+1} =_{DF} \text{NTh}(T_n \cup \{\neg l(\neg c)\})$;

— otherwise set $T_{n+1} =_{DF} T_n$.

The union of the chain is the wanted maximal named L-theory. Note that every such theory contains (due to COV) at least one name.

Now, the final step. Let w be a maximal consistent named L-theory containing Γ. Let W be the set of all maximal consistent named L-theories $\blacksquare$-connected with w, i.e. $W = \{x: \blacksquare w \subseteq x\}$ where $\blacksquare x = \{\psi: \blacksquare \psi \in x\}$. Clearly, by the S5 axioms for $\blacksquare$, every two elements of W are $\blacksquare$-connected. Now consider the *canonical surjective model over W*: $\langle W, R, V, \chi \rangle$ where Rxy iff $\Box x \subseteq y$; $V(p) = \{x \in W: p \in x\}$ and $\chi(c)$ is the only $x \in W$ (in virtue of (nam1) and (nam2)) containing c. The truth lemma: $V(\varphi) = \{x \in W: \varphi \in x\}$ is proved by induction as usual. The crucial point: if $\diamondsuit \psi \in x$ then there exists $y \in W$ such that $\Box x \cup \{\psi\} \subseteq y$. It is because $\Box x$ is a named L-theory and hence $\Box x \cup \{\psi\}$ is surjectively L-consistent. The same arguments work for the case $\blacklozenge \psi \in x$.

Thus we obtain a surjective L-model satisfying Γ.   #

REMARK. The proof of the last theorem might be perceived to hint that COV ensures surjectiveness of the models. This is not the case: what COV ensured is that in each *definable* set of the model (set of the points in which some modal formula is true) at least one named

point can be picked. This doesn't imply surjectiveness — witness the following example.

EXAMPLE 3. Let $F = \langle \mathbb{N}, < \rangle$ and $\mathfrak{M} = \langle F, V \rangle$ where $V(p_i) = \{i\}$ for each $i \in \mathbb{N}$. Then, inductively by φ, we can prove that for each φ, $V(\varphi)$ is either finite or co-finite set. Now consider $ue(\mathfrak{M}) = \langle \mathbb{N}^*, <^*, V^* \rangle$. $\mathbb{N}^* = \mathbb{N}_p \cup \mathbb{N}_f$ where $\mathbb{N}_p$ consists of the principal ultrafilters in $\mathbb{N}$: $\mathbb{N}_p = \{u_i : i \in \mathbb{N}\}$ and $\mathbb{N}_f$ consists of the free ultra-filters in $\mathbb{N}$. Then:

(i) for each $u \in \mathbb{N}_f$ and $v \in \mathbb{N}^*$, $v <^* u$;

(ii) for each φ, $V^*(\varphi)$ is finite and $V^*(\varphi) \subseteq \mathbb{N}_p$ or $V^*(\varphi)$ is a complement of such a set.

Now define $\chi : C \to \mathbb{N}^*$ as follows: $\chi(c_i) = u_i$. It is a standard induction on the depth of a necessity form $l(\$)$, using (ii) above, to prove that for each $i \in \mathbb{N}$, $u_i \in V^*(l(\bot))$ whenever $u_i \in V^*(l(\neg c))$ for every c. Therefore, the validity in $ue(\mathfrak{M})$ is preserved by COV.   #

Anyway COV does ensure surjectiveness in finite models.

LEMMA 5.5. *Let* $\mathfrak{N} = \langle F, V, \chi \rangle$ *be a surjective model for an* $\mathscr{L}_c$-*logic L. Then:*

(1) $\mathfrak{M} = \langle F, V \rangle$ *is a model for L.*

(2) *if L is axiomatized by pure formulae over* $\mathbf{K}_c$ *then* $F \vDash L$.

Proof. (1) Suppose $\mathfrak{M} \nvDash \varphi$ for some $\varphi \in L$. Then for some $\mathfrak{N}' = \langle \mathfrak{M}, \chi' \rangle$ and $x \in \mathfrak{M}$, $\mathfrak{N}' \nvDash \varphi[x]$. Let $d_1, \ldots, d_k$ be the names, occurring in φ. Let $\chi(e_1) = \chi'(d_1), \ldots, \chi(e_k) = \chi'(d_k)$. Then $\mathfrak{N} \nvDash \varphi(e_1/d_1, \ldots, e_k/d_k)[x]$ but $\varphi(e_1/d_1, \ldots, e_k/d_k) \in L$ by SUB — a contradiction.

(2) If φ is a pure formula then $\langle F, V \rangle \vDash \varphi$ iff $F \vDash \varphi$.   #

THEOREM 5.6. *Every* $\mathscr{L}_c$-*logic L is strongly model-complete.*

Proof. Let $C' \subseteq C$ be an infinite set of names and $f : C \to C'$ be a bijection. Let $\Gamma \subseteq \mathscr{L}_c$. Call the set $f(\Gamma)$ obtained after replacement of every name c in every formula of Γ by $f(c)$ a *bijective renaming of* Γ. Clearly, every such a renaming preserves L-(in)consistency.

Now let Γ be an L-consistent set. Take $c \in C$ and a bijection $f : C \to C \setminus \{c\}$. Then $f(\text{Th}_L(\Gamma))$ is a consistent named L-theory.

Thus, by 5.4, $f(\Gamma)$ is satisfied in a surjective L-model $\mathfrak{N}$. Now, forget about the valuation of c and rename $\mathfrak{N}$ by f^{-1}. Thus we get a named L-model which satisfies Γ.   #

COROLLARY 5.7 *Every $\mathscr{L}_c$-logic L is strongly [surjectively] complete with respect to its countable models.*

THEOREM 5.8 *Every $\mathscr{L}_c$-logic L, axiomatized over $\mathbf{K}_c$ by pure formulae, is surjectively strongly complete.*

Proof. Every surjective ■-generated canonical model constructed as in the proof of 5.4. is based on an L-frame, by 5.5.   #

The same trick as in the proof of 5.6 shows that every surjectively strongly complete logic is strongly complete.

COROLLARY 5.9. *Every purely axiomatized $\mathscr{L}_c$-logic is strongly complete.*   #

At the end of this section, let us demonstrate the usage of COV in an inference. Let MOD be an arbitrary fixed sequence of modalities □, ◇, ■, ◆ with length k, POS be an arbitrary fixed sequence of ◇ and ◆ and NEC be its dual.

PROPOSITION 5.10. $(\mathbf{K}_c + \text{POS}c \rightarrow \text{MOD}c) \vdash \text{POS}p \rightarrow \text{MOD}p$.

Proof. We will sketch the inference modulo some tautological calculations.

(1) $\text{POS}(p \wedge c) \rightarrow \text{POS}c$;

(2) $\text{POS}c \rightarrow \text{MOD}c$;

(3) $\text{POS}(p \wedge c) \rightarrow \text{MOD}c$ by (1) and (2);

(4) $\text{POS}(p \wedge c) \rightarrow \blacklozenge(p \wedge c)$ by (incl) and S5(■);

(5) $\text{POS}(p \wedge c) \rightarrow \blacksquare(c \rightarrow p)$ by (4) and (nam2);

(6) $\text{POS}(p \wedge c) \rightarrow \blacksquare \ldots \blacksquare(c \rightarrow p)$ /k■'s/ by S5(■);

(7) $\text{POS}(p \wedge c) \rightarrow \text{MOD}(c \wedge p)$ by (3), (6), $K(\square)$, S5(■), (incl), (nam2);

(8) $\text{POS}(p \wedge c) \rightarrow \text{MOD}p$ by (7);

(9) $\neg\mathrm{MOD}p \to \mathrm{NEC}(p \to \neg c)$ by (8);

(10) $\neg\mathrm{MOD}p \to \mathrm{NEC}\neg p$ by (9) and COV;

(11) $\mathrm{POS}p \to \mathrm{MOD}p$ by (10). #

6. MINIMAL EXTENSIONS AND TRANSFER PROBLEMS

DEFINITION. Let L be an $\mathscr{L}_c$-logic, i.e. a simple extension (by means of axioms only) of **K**. The *minimal extension* of L in $\mathscr{L}_c$ is the $\mathscr{L}_c$-logic L_c axiomatized over $\mathbf{K}_c$ by the axioms of L over **K**. #

Let $\mathfrak{D}$ be a property of logics. Then the following $\mathfrak{D}$-*transferring problem* arises:

If L satisfies $\mathfrak{D}$ whether L_c does, too?

It is not difficult to obtain partial answers to such problems (or complete answers for particular logics) but still there are very few general results even for some quite simple and natural enrichments (cf. Goranko and Passy (1990)).

6.1. *Conservativeness of minimal extensions*

THEOREM 6.1. *For every $\mathscr{L}$-logic L, L_c is conservative over L.*

Proof. Let $\varphi \in \mathscr{L}$ and $L \nvdash \varphi$. Then φ is refuted in some L-model $\mathfrak{M} = \langle F, V \rangle$ at some point x. Let p not occur in φ. Consider a valuation V' in F, such that $V'(p) = \emptyset$ and V' coincides with V for all other variables. Then $\mathfrak{M}' = \langle E, V' \rangle$ again is an L-model (indeed, if $\psi(p) \in L$ then $\psi((q \wedge \neg q)/p) \in L$ hence $\mathfrak{M} \vDash \psi((q \wedge \neg q)/p)$ so $\mathfrak{M}' \vDash \psi$) which refutes φ at x. Now let $F_1 = \langle \{w\}, R_1 \rangle$ be a singleton frame for L (it exists due to a well-known result of Makinson (1971)) and V_1 be any valuation in F_1, such that $V_1(p) = \{w\}$. Let $\mathfrak{M}_1 = \langle F_1, V_1 \rangle$ and $\mathfrak{M}_2$ be the disjoint union of $\mathfrak{M}$ and $\mathfrak{M}_1$. Then $\mathfrak{M}_2$ is an L-model refuting φ and such that $V_2(p) = \{w\}$. Now, clearly the named model $\langle \mathfrak{M}_2, \chi \rangle$ such that $\chi(c) = w$ for every name c validates L_c and refutes φ, so $L_c \nvdash \varphi$. #

COROLLARY 6.2. *If L is incomplete then L_c is incomplete, too.*

6.2. *Transferring completeness*

Unlike the above result the completeness-transferring problem in general seems to be rather hard. Here we shall prove an important sufficient condition for a positive solution of this problem.

DEFINITION (Goldblatt (1976)). A general frame $\mathfrak{F} = \langle F, \mathbb{W} \rangle$ is *descriptive* if for every ultrafilter u in $\mathfrak{F}^+$ the following hold:

(i) $\bigcap u$ is a singleton;

(ii) $\bigcap \{ \Diamond X / X \in u \} \subseteq \Diamond (\bigcap u)$.   #

DEFINITION (cf. Goldblatt (1976)). An $\mathscr{L}$-logic L is *canonical* if for every descriptive frame $\mathfrak{F} = \langle F, \mathbb{W} \rangle$, $\mathfrak{F} \vDash L$ implies $F \vDash L$. The same definition holds for canonical $\mathscr{L}_c$-logics.   #

In $\mathscr{L}$ canonicity implies strong completeness since the canonical general frame is descriptive. The same holds in $\mathscr{L}_c$. By a *canonical named model* of an $\mathscr{L}_c$-logic L we shall mean every named model (not necessarily surjective!) constructed as in the proof of 5.4 from maximal L-theories (not only named ones). Every L-consistent set is satisfied in such a model, hence the corresponding general frame which is descriptive. Thus every canonical $\mathscr{L}_c$-logic is strongly complete.

   The opposite is not true in general but, as the next result state, it is true in an important case.

FACT 6.3 (see van Benthem (1979) and Fine (1975)). *If an $\mathscr{L}$-logic L is complete and its axioms are preserved under elementary equivalence then L is canonical.*   #

THEOREM 6.4. (1) *If an $\mathscr{L}$-logic L is canonical then L_c is canonical, too.*

   (2) *If an $\mathscr{L}$-logic L is complete and first-order definable then L_c is surjectively complete.*

   Proof. (1) Every descriptive L_c-frame is a descriptive L-frame, too. (2) Let L be complete and first-order definable. Then, by 6.3 L is canonical, hence L_c is canonical, too. Now, let φ be an L_c-consistent

formula. Then φ is satisfied in a canonical named L_c-model. This model is based on an L_c-frame F. Let T be the first-order theory of the class of L_c-frames $FR(L_c)$ and $c_{i1}, \ldots, c_{ik}$ be the names occurring in φ. Then F is a first-order model of $T' = T \cup \{\exists x \exists y_{i1} \ldots \exists y_{ik} ST'(\varphi)\}$. By downward Löwenheim–Skolem theorem there exists an at most countable model F' of T'. F' is an L_c-frame which yields a countable named model satisfying at most φ. It can be easily renamed into a surjective model satisfying φ.   #

Now we can show that not every first-order property even modally definable, can be defined by pure formulae only. A counter-example is the Church–Rosser property

CR: $\forall x \forall y \forall z(Rxy \ \& \ Rxz \rightarrow \exists t(Ryt \ \& \ Rzt))$,

modally defined by $\Diamond \Box p \rightarrow \Box \Diamond p$. Following Hughes and Cresswell (1984) let us denote $\mathbf{K} + \Diamond \Box p \rightarrow \Box \Diamond p$ by $\mathbf{G1}$. $\mathbf{G1}_c$ is complete since is canonical. If CR were defined by a set of pure formulae Σ then every surjective model for $\mathbf{G1}_c$ would be based on a CR-frame, by 5.5. However, this is not the case, witness the following example, due to Yde Venema (1991).

EXAMPLE 4. Consider the frame $F = \langle W, R \rangle$ as on the picture:
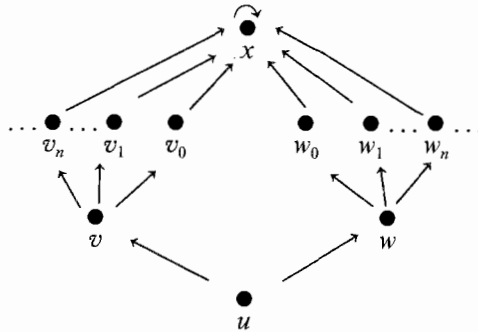
Let $C = \{a, b, c, d\} \cup \{b_i, c_i : i \in \mathbb{N}\}$. Let $V(p) = W$ for every propositional variable p, and $\chi(a) = u$, $\chi(b) = v$, $\chi(c) = w$, $\chi(d) = u$, $\chi(b_i) = v_i$, $\chi(c_i) = w_i$. Thus we obtain a surjective model $\mathfrak{N} = \langle F, V, \chi \rangle$ which is a model for $\mathbf{G1}_c$. Indeed, for every $\varphi \in \mathcal{L}_c$, $\mathfrak{N} \models \Diamond \Box \varphi \rightarrow \Box \Diamond \varphi$. The only non-trivial task is to check this validity at u.

It holds due to the following observation, proved by an easy induction on φ: the sets

$$V_\varphi = \{n \in \mathbb{N}: v_n \in V(\varphi)\} \text{ and } W_\varphi = \{n \in \mathbb{N}: w_n \in V(\varphi)\}$$

are both either finite or co-finite.

Thus F must be a CR-frame which is not true. Hence no set Σ of pure formulae defining CR exists.

6.3. *Transfer of filtrations, finite model property and decidability*

The notion of filtration is smoothly carried out in $\mathscr{L}_c$. Moreover, the property of an $\mathscr{L}$-logic to admit a particular filtration (cf. Hughes and Cresswell (1984)) is transferred to its minimal extension in an obvious way since the names harmlessly go through this construction. Hence, all $\mathscr{L}$-logics proved by a filtration to have the finite model property and to be decidable have their minimal extensions with the same properties. However, the general problems remain still open:

PROBLEM 4. Is the finite model property always transferred?

PROBLEM 5. The same question for decidability.

6.4. *Some axiomatizations in $\mathscr{L}_c$*

1. First of all let us note that, in virtue of 4.17 and 5.9 we have right away strongly complete finite axiomatizations of all logics of single finite frames. Of course, all these logics have the finite model property and hence are decidable.

2. Moreover, the minimal extensions of most of the famous modal logics such as T, K4, B, S4, E, S5 etc. have pure axiomatizations as instances of 5.10 hence are proved to be /surjectively/ strongly complete independently from Th. 6.4. As it follows from the above note on filtrations $\mathbf{K}_c$ and all these logics are finitely complete and decidable.

Here are two more particular $\mathscr{L}_c$-logics of certain interest which are readily strongly complete by 5.9:

The logic of strict linear orderings:

$$L_{\text{slin}} = \mathbf{K}_c +$$
$$\text{(irref) } c \to \Box \neg c$$
$$\text{(tran) } \Diamond \Diamond c \to \Diamond c$$
$$\text{(slin) } \blacklozenge(c \wedge \neg d) \to (\blacklozenge(c \wedge \Diamond d) \vee \blacklozenge(d \wedge \Diamond c)).$$

The above axioms determine strict linear ordering.

The logic of rationals $\langle \mathbb{Q}, < \rangle$:

$$L_{\mathbb{Q}} = L_{\text{slin}} +$$
$$\text{(suc) } \Diamond T$$
$$\text{(pred) } \blacklozenge \Diamond c$$
$$\text{(dens) } \Diamond c \to \Diamond \Diamond c.$$

By Cantor's theorem, $L_{\mathbb{Q}}$ describes up to isomorphism $\mathbb{Q}$ as a countable dense linear ordering without ends.

Let us notice that these two examples show the advantage of $\mathscr{L}_c$ in comparison with $\mathscr{L}(\boxed{\neq})$: although they have the same definability power, the former seems to be much better both to find the axiomatics and to prove its completeness (cf. de Rijke (1989)).

7. POLYMODAL LOGICS AND NAMES

The deductive machine developed here and the main results are easily conveyed to the polymodal setting. Let us just mention several important examples:

1. *Combinatory dynamic logic (dynamic logic with names)*. Actually, this is the origin of our usage of the names in the modal framework. For a quick reference, see Passy and Tinchev (1985) and for a close acquaintance, the excellent and more than comprehensive Passy and Tinchev (1991).

2. *Boolean modal logic with names*. A polymodal logic with a set of modalities over a Boolean algebra of relations (Boolean analog of the dynamic logic) has been axiomatized in Gargov and Passy (1990). Using constants one can give a pure axiomatization of this logic over the polymodal version of $\mathbf{K}_c$ and thus to obtain straightway a completeness theorem.

3. *Tense logic with names.* Extending the usual tense language (with modalities G — "always in the future", H — "always in the past" and their duals F and P) with the universal modality and names, we have the minimal normal tense logic axiomatized analogously to $\mathbf{K}_c$ even without using necessity forms due to the reduction of COV mentioned in Section 5.2. The proof of the completeness of this logic and its minimal extensions follows the same scheme as in $\mathscr{L}_c$.

Let us mention another approach to tense logic with names (called there *nominals*) we had the occasion to acquaint ourselves after completing this paper — that of Blackburn (1989). The universal modality and COV are not used there and the deductive system is simpler. The minimal logic is axiomatized with the only additional scheme SWEEP: $E(c \wedge \varphi) \rightarrow A(c \rightarrow \varphi)$ where E ranges over arbitrary sequences of P's and F's and A ranges over G's and H's. (Of course, in the presence of ■ the scheme SWEEP can be replaced by the single axiom (nam2).) No wonder that COV is redundant there (as in $\mathbf{K}_c$); its essentiality and the usefulness of □ exhibit themselves truly in the simple extensions (the example after 5.2 is only a hint for that). For instance, the extensions with $c \rightarrow \neg Fc$ (irreflexivity) and $c \rightarrow G(Fc \rightarrow c)$ (antisymmetry) in our deductive machinery are readily proved to be complete while in Blackburn's one the proofs require additional efforts and use some involved techniques, e.g. bulldozing. Nevertheless, it is worth investigating where the weaker apparatus suffices.

4. *Since, Until and the general perspective.* Let us finally notice that the languages endowed with names are a fertile soil for axiomatization of more sophisticated modal operators. A famous example are Kamp's operators Since and Until (see, e.g. Burgess (1982)). With the help of names they can be "conditionally" defined, as this has been essentially done in Gabbay and Hodkinson (1990) by means of the formulae $c \rightarrow (S(p, q) \leftrightarrow P(p \wedge G(Fc \rightarrow q)))$ and $c \rightarrow (U(p, q) \leftrightarrow F(p \wedge H(Pc \rightarrow q)))$.

These pseudo-definitions do not enable elimination of S and U from the language but still give readily axiomatizations of the minimal Since–Until logic and all of its pure extensions, and thus avoid the standard but heavy technique involved in the "nameless" languages with Since and Until (cf. Burgess (1982)). Moreover, applying

the equivalence 4.10 to the results from Gabbay (1981) one can obtain a uniform procedure for finite axiomatization in named languages of an arbitrary first-order logical connective.

REFERENCES

van Benthem, J. F. A. K. (1979), Canonical modal logics and ultrafilter extensions. *JSL* **44**(1), 1–8.

van Benthem, J. F. A. K. (1984), Correspondence theory. In: *Handbook of Philosophical Logic*, D. Gabbay and F. Guenthner (eds.), 167–247, vol. II, Reidel, Dordrecht.

van Benthem, J. F. A. K. (1986), *Modal Logic and Classical Logic*, Bibliopolis, Napoli.

Blackburn, P. (1989), Nominal tense logic. Dissertation, Centre for Cognitive Science, University of Edinburgh, 1989.

Bull, R. (1970), An approach to tense logic. *Theoria* **36**(3), 282–300.

Burgess, J. (1982), Axioms for tense logic I: "Since" and "Until", *Notre Dame Journal of Formal Logic* **23**(2), 367–374.

Chagrova, L. (1989), Undecidable problems connected with first-order definability of intuitionistic formulae, *Proceedings of Kalinin State University*, p. 42.

Fine, K. (1975), Some connections between the elementary and modal logic. In: *Proceedings of the Third Scandinavian Logic Symposium*, Uppsala 1973, S. Kanger (ed.), North-Holland, Amsterdam, 15–31.

Gabbay, D. (1981), An irreflexivity lemma with applications to axiomatizations of conditions on tense frames. In: *Aspects of Philosophical Logic*, U. Monnich (ed.).

Gabbay, D. & I. Hodkinson (1990), An axiomatization of the temporal logic with Since and Until over the real numbers. *Journal of Logic and Computation*, Vol. 1.

Gargov, G., S. Passy & T. Tinchev (1987), Modal environment for Boolean speculations. In: *Mathematical Logic and Its Applications* (ed. D. Skordev), 253–263, Plenum Press, New York.

Gargov, G. &. S. Passy (1988), Determinism and looping in Combinatory PDL. *Theoretical Computer Science*, **61**, 259–277.

Gargov, G. & S. Passy (1990), A note on Boolean modal logic. In: *Mathematical Logic*, Proceedings of the Summer School and Conference HEYTING'88, P. Petkov (ed.), Plenum Press, New York and London.

Goldblatt, R. I. (1976), Metamathematics of modal logic. *Reports on Math. Logic* **6**, 41–77 and **7**, 21–52.

Goldblatt, R. I. (1982), *Axiomatizing the Logic of Computer Programming*, Springer LNCS **130**.

Goldblatt, R. I. & S. K. Thomason (1974), Axiomatic classes in propositional modal logic. In: J. Crossley (ed.), *Algebra and Logic*, 163–173, Springer LNM **450**, Berlin.

Goranko, V. (1990), Modal definability in enriched languages. *Notre Dame Journal of Formal Logic* **31**(1), 81–105.

Goranko, V. & S. Passy (1990), Using the universal modality: gains and questions, *Journal of Logic and Computation* **2**(1), 5–30.

Hughes, G. E. & M. J. Cresswell (1984), *A Companion to Modal Logic*, Methuen, London.

Koymans, R. (1989), Specifying Message Passing and Time-Critical Systems with Temporal Logic. Dissertation, Eindhoven.

Makinson, D. (1971), Some embedding theorems for modal logic. *Notre Dame Journal of Formal Logic.* **12**, 252–254.

Passy, S. & T. Tinchev (1991), An Essay in Combinatory Dynamic Logic. *Information and Computation* **93**(2), 263–332.

Prior, A. (1956), Modality and quantification in S5. *Journal of Symbolic Logic* **21**(1), 60–62.

de Rijke, M. (1989), The modal theory of inequality, *Journal of Symbolic Logic* **57**(2), 566–584.

Sahlqvist, H. (1975), Completeness and correspondence in first and second order semantics for modal logic. In: *Proceedings of the Third Scandinavian Logic Symposium*, Uppsala 1973, S. Kanger (ed.), North-Holland, Amsterdam.

Segerberg, K. (1971), *An Essay in Classical Modal Logic*, Filosofiska Studier **13**, Uppsala.

Venema, Y. (1991), Personal correspondence.

George K. Gargov,
40 Shrewsbury Green Drive, apt. F,
Shrewsbury, MA 01545,
U.S.A.

and

Valentin Goranko,
Department of Mathematics,
University of the North,
QwaQwa Branch, Private Bag X13,
Phuthaditjhaba 9866,
Republic of South Africa