Proofs of valid categorical syllogisms in one diagrammatic and two symbolic axiomatic systems

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Abstract

Gottfried Leibniz embarked on a research program to prove all the Aristotelic categorical syllogisms by diagrammatic and algebraic methods. He succeeded in proving them by means of Euler diagrams, but didn't produce a manuscript with their algebraic proofs. We demonstrate how key excerpts scattered across various Leibniz's drafts on logic contained sufficient ingredients to prove them by an algebraic method –which we call the Leibniz-Cayley (LC) system– without having to make use of the more expressive and complex machinery of first-order quantificational logic. In addition, we prove the classic categorical syllogisms again by a relational method –which we call the McColl-Ladd (ML) system– employing categorical relations studied by Hugh McColl and Christine Ladd. Finally, we show the connection of ML and LC with Boolean algebra, proving that ML is a consequence of LC, and that LC is a consequence of the Boolean lattice axioms, thus establishing Leibniz's historical priority over George Boole in characterizing and applying (a sufficient fragment of) Boolean algebra to effectively tackle categorical syllogistic.

KEYWORDS

algebra of logic; categorical syllogistic; Leibniz's logic; Christine Ladd-Franklin; Hugh MacColl; O.H. Mitchell; Arthur Cayley's logic; William Stanley Jevons' logic; calculus of classes; term logic; Aristotelian logic; Boolean lattice; Boolean algebra; monadic first-order logic; monadic predicate calculus; Johann Christoph Sturm; Euler diagram.

1. Introduction

This paper is about some interesting ways, aligned with the "algebra of logic" tradition, one can prove the 24 classic categorical syllogistic moods. It offers the following contributions:

(1) A reproduction of short *diagrammatic* proofs of the categorical syllogisms, pioneered by Leibniz (ca. 1686a) and didactically expressed in modern form by Piesk (2017). (This is not a novel contribution¹ of this paper, but is useful to introduce key concepts and motivate the following sections, which contain novel contributions.)

- (2) The compilation in two Tables (1a and 1b) of the various ways one can algebraically represent the fundamental Aristotelic relations (and De Morgan's extensions) by means of elementary algebraic operations and relations: union, intersection, complementation; the empty class, the universe; equality, subset and superset, disjointness and exhaustion, and the negation of those relations.
- (3) The characterization of two *symbolic* axiomatic systems, LC (for Leibniz-Cayley) and ML (for McColl-Ladd), and the systematic proof of all the 24 classic categorical syllogistic moods in them. LC, an algebraic system, is built upon intersection, complementation, identity, non-identity, and the empty class. ML, a relational system, is built upon complementation and the subset and conjointness relations.
- (4) Tables pointing out which syllogistic moods require which axioms of LC and ML (Sections 6.4 and 7.4).
- (5) A discussion of key excerpts of Leibniz's drafts which demonstrates that, in the late 17th century, he presented nearly all building blocks for algebraically proving the 24 classic categorical from axioms doing away with the misconception that the solution of categorical syllogistic in the algebra of logic tradition had to wait the invention of Boolean algebra in the 19th century.
- (6) A proof that ML is strictly less expressive than LC, which itself is strictly less expressive than Boolean algebra thus forming a hierarchy of logical systems based on expressiveness.
- (7) Historical notes throughout the text, which can guide History of Logic researchers to key primary literature for their investigations.

The categorical syllogism proofs expressed by means of Euler and Venn diagrams by Piesk (2017) (reproduced in Section 3) sparkled our motivation for the research that led to this paper. Faced with those elegant diagrammatic proofs, we were irresistibly attracted to the intellectual exercise of proving the same 24 categorical syllogisms in the (Boolean) algebra of (term²) logic. As we delved deep into the literature of the field of algebraic term logic, we learned that this is one of the historical goals of the Leibnizian-Boolean research program –a research goal which dates back to at least April 1679 (Leibniz, 1679b, pp. 43-44)–, which is, surprisingly, only partially completed to this day – by neglect, not because (with tools available nowadays) it is a tough problem. We stand on the shoulders of giants to comprehensively document solutions to this goal by employing modern notation and concepts refined and matured over centuries of hard work by symbolic logicians.

Most elements required to establish the axioms for all the diagrammatic and algebraic systems we describe here were anticipated in unpublished drafts by Gottfried Wilhelm Leibniz in the 17th century – long before George Boole (1847) and his intellectual successors came to the scene. A few of these elements had already been published and were in principle accessible to 19th century mathematical logic researchers

 $^{^{1}}$ We provide a minor contribution here by suggesting that interpreting the boundaries (and not only the shaded areas) of classes in Euler and Venn diagrams as the empty class is convenient, as it perfectly fits some elementary laws of the algebra of sets.

²In this paper, we use "term" to designate an extensional class, which secondarily happens to be associated to a linguistic entity such as subject or predicative. A (classificatory) term is a reference –a label– to a class in extension. We follow De Morgan's (1846) extensional approach: "A term, or name, is merely the word which it is lawful to apply to any one of a collection of objects of thought [...].". (For a dissenting view on "terms" in logic, see Waragai and Oyamada (2007, pp. 124,125-126), Kulicki (2012), and Łukasiewicz (1957, p. 130).)

(Erdmann, 1840; Gerhardt, 1890; Peckhaus, 2018), but others came to the public eye thanks to Louis Couturat's examination and publication, in the 20th century, of selected drafts on logic which were on Leibniz's Nachlass in Hanover (Couturat, 1903), which have fortunately been preserved to the present day. Because many relevant drafts of Leibniz's were unknown in the 19th century, there were a lot of independent reinventions of his ideas by various pioneers in the Boolean research program³.

The tradition of term logic was initiated circa 350 BCE by the founding master Aristotle of Stagira (ca. 350 BCEa)(ca. 350 BCEc), who introduced four fundamental categorical relation forms and enumerated (and proved by means of logical methods) most of the classic categorical syllogisms we algebraically prove in this paper. Aristotle had key insights such as earliest recorded use of literal *placeholders* ("variables") in what would millennia later be recognized as a branch of algebra⁴ (Łukasiewicz, 1957, pp. 7-8)(Patzig, 1968, p. 12)(Bar-Am, 2008, pp. 23,34,39)(Braem, 1475).

The 24 classic categorical syllogisms employ four possibilities of fundamental categorical relations originally investigated by Aristotle in [gr] "Peri Hermeneias" / [la] "De Interpretatione" / [en] "On Interpretation" (ca. 350 BCEa, Chapter 7) and in [gr] "Analytica Protera" / [la] "Analytica Priora" / [en] "Prior Analytics" (Aristotle of Stagira, ca. 350 BCEc), and popularized by Boethius (Parsons, 2021) through linguistic expressions loosely similar to the following:

- **b** A **c**: Every **b** is **c**.
- **b** E **c**: No **b** is **c**. (Every **b** is not **c**.)
- **b** I **c**: At least one⁵ **b** is **c**.
- $\mathbf{b} \mathbf{O} \mathbf{c}$: At least one \mathbf{b} is not \mathbf{c} .

Aristotle's syllogistic was the earliest simultaneous treatment of *generality* ("Every **b** is c.") and existence ("At least one b is c.") in logic, and he had already recognized how those notions are intimately related in that generality means the lack (or non-existence) of a counterexample (von Plato, 2021)⁶⁷.

⁶This ancient observation by Aristotle is precisely formalized (with some abuse of notation) in modern first-order logic as

 $\begin{array}{ll} \forall: \ x \in \overrightarrow{\mathbf{b}} \Rightarrow x \in \mathbf{c} \quad |=| \quad \overline{\exists}: \ x \in \mathbf{b} \land x \notin \mathbf{c} \\ \text{In term logic, it is expressed much more simply as} \end{array}$

 $\mathbf{b} \mathbf{A} \mathbf{c} \mid = \mid \mathbf{b} \overline{\mathbf{O}} \mathbf{c}$

that is, A and O are contradictory.

 $^7\mathrm{In}$ this paper, we decided to adopt the following notation:

• "|=" is the metalogical relation of equivalence, which indicates that the syntactic derivation is bidirectionally valid. It was intentionally chosen to be an "equals" symbol sandwiched between two vertical bars. It is most often expressed in sequent calculus and proof theory by the "⊣⊢" symbol.

³Leibniz explored both extensional and intensional interpretations of the same abstract, deductive logical system. The intensional interpretation, actually favored by the rationalist Leibniz across many of his manuscripts (Lewis, 1918, pp. 13-18,32,35-37,73-74,186-187,213-215,231,322-323,327-330,377,382-385) (Leibniz, 1679a, points 11-12,7,17), is very interesting in itself. In this paper, however, we are concerned with characterizing noteworthy properties of (fragments of) purely deductive, formal, extensional term logics, supportive of arbitrarily assembled classes, whose constituent elements are left implicit in the systems.

The brand of algebra of extensional categorical syllogistic we discuss at length in this paper assumes the existence of logical classes/"categories". Some philosophers find them problematic and sometimes don't accept (or at least attempt to work around) them (Boolos, 1985; Ongley and Carey, 2013; Klement, 2010).

⁴The use of literal placeholders in numerical algebra would have to wait the independent reinvention by Jordanus de Nemore circa 1,225 CE, that is, approximately 1,600 years after Aristotle (de Nemore, ca. 1225; Turner, 1983).

⁵We follow Béziau (2012, pp. 6-11,18) in adopting "At least one" rather than "Some". It represents the set-theoretically precise notion of "inhabitation" or "presence" (often miscalled "existence", even -by habit- in this paper.). Moreover, unlike "Some", "At least one" invites the generalization toward numerical syllogisms (Pratt-Hartmann, 2023, pp. 5-6,225-228,249), an exciting and active research topic pioneered by De Morgan (1846, pp. 384,406).

More than two millennia after Aristotle, Leibniz, knowledgeable in both numerical algebra and term logic, decided to research the possibility of turning term logic into an algebra, making use of the fact that both exact sciences already employed literal placeholders back in his days. These were pioneering feats in what we nowadays call "Boolean algebra".

Key ideas beyond Leibniz's for achieving a symbolic algebra of term logic powerful enough to offer various insightful ways to prove classic categorical syllogisms were advanced by Cayley (1871), McColl (1877), Ladd (1883), and Mitchell (1883), and are summarized in this paper.

Our terminology divides the categorical syllogism proof methods into:

- (1) *Diagrammatic*, where representations and proofs are topologically visual rather than symbolic;
- (2) Algebraic, where transformations involving "=", "≠" and dyadic operations (functions) on classes are employed in a way somewhat familiar to numerical algebra students at middle school;
- (3) Relational, where axioms involving dyadic relations among classes other than "=" and "≠" are used, and proofs employ either free-form logical entailment (from logic) or strict-form composition of relations (from relation algebra);
- (4) *Refutatory*, where concepts and tools typical in propositional logic or involving the Theorem K from relation algebra predominate;
- (5) *Quantificational*, where concepts and tools from first-order monadic quantificational logic are employed.

Aristotle was the earliest to prove many classic categorical syllogisms, using logical *refutatory* methods involving consequence denial such as "*reductio ad absurdum*" and what was later called "proof by regression"⁸. Leibniz proved some categorical syllogisms with diagrammatic methods (Leibniz, ca. 1686a) and beautifully explained Aristotle's method of proof by regression (Leibniz, 1682), where a premise and the conclusion of a valid assertion are transposed, generating a new valid assertion. De Morgan (1847, pp. 87-89) repeated this application of regression to categorical syllogistic. An evolved variant of regression is the method of "inconsistent triads" or "antilogisms" by Ladd

- "|=" is the metalogical relation of *uni*directional syntactic derivation, most often expressed in sequent calculus and proof theory by the turnstile symbol ("⊢"). The previous symbol ("|=]") was a "sandwich"; this one ("|=") is "half a sandwich". We find it an unfortunate historical accident that the double turnstile symbol ("⊢"), that visually looks like our "|=", is commonly used in the model theory literature to indicate semantic, not syntactic entailment. It would have been nice if the denotation of "⊢" and "⊨" were swapped.
- "," is the metalogical "and" operation.
- "M" is the metalogical "or" operation, following our convention of going to the meta, metameta, metameta level and so on by progressively adding vertical bars around a symbol.

Hilbert (1922, pp. 174-175) employed the term "metamathematics", back then in the narrow sense of (finitary) proof theory ("*Beweistheorie*"). In the Polish school of logic, Łukasiewicz and Tarski (1930) Łukasiewicz and Tarski (1930) employed the words [de]"*Metalogik*" ([en]"metalogic") and [de]"*metalogischen*" ([en]"metalogical") as synonyms to "metamathematics" and "metamathematical", in the wider sense of "(pertaining to the) theory of deduction". The "object language vs. metalanguage" distinction was made explicit by Tarski (1933, pp. 167-168,154{fn. 1}), who, in a later article (1936, p. 402), credited his doctoral advisor Stanisław Leśniewski with pioneering it. Moreover, in the latter article, the words "metalogical" and "metalanguage" appear in the same page: 407.

⁸Useful in various contexts, for instance to show that *modus ponens* and *modus tollens* can be derived from each other, and to establish alternative definitions for antisymmetric relation in order theory:

 $\begin{array}{cccc} b \preceq c, & b \neq c & \mid = & c \not\preceq b \\ b \preceq c, & c \preceq b & \mid = & b = c \end{array}$

(1883)(Lewis, 1918, pp. 108-110)(Green, 1991, pp. 2-3). In another evolutionary direction from proof by regression, De Morgan proved the Theorem K in relation algebra (De Morgan, 1860, p. 344)(Maddux, 1991, pp. 434-435)(Schröder, 1895, pp. 242-243,416-417). But all these methods intentionally use tools of propositional logic instead of algebraic transformations.

Leibniz made across private drafts various insightful attempts to devise an *algebraic* method for proving categorical syllogisms⁹, and even correctly proved the categorical syllogism Barbara-1 in an algebraic fashion (Leibniz, ca. 1690, pp. 229-230, Axioma 1). With a masterful ability, he correctly identified most concepts and notions needed for this task, and thus almost completed the goal on his own. Unfortunately, he missed a small piece of the puzzle – one of the suitable algebraic representations of particular categorical relations which was provided almost two centuries later by Cayley (1871)(Valencia, 2004, p. 473). Moreover, he identified individual axioms needed to complete this task, but they are scattered across some of his manuscripts – we organize them in a centralized fashion in this paper.

The earliest *relational* system we could find whose explicit goal (successfully achieved in the same paper) was to prove the set of classic categorical syllogisms was devised by Hugh McColl $(1877)^{10}$. Thus, McColl deserves the credit of being the earliest to satisfactorily achieve this historical goal of the Leibnizian-Boolean research program (though with a method different from the algebraic one elected by Leibniz and Boole). Much remains to be said in this regard – McColl's is not the only relational method possible, as we will show. We can obtain further insights at the problem by looking at other methods.

Proofs of all the 24 classic categorical syllogisms in monadic first-order quantificational logic are known; this exercise has been done countless times¹¹. But we claim that first-order logic is a too heavy machinery for tackling those simple 3-sentence argument forms. Although insightful and very welcome to our portfolio of knowledge, we should not feel satisfied by that solution; it feels like using a bazooka to kill a fly. Instead of invoking all the power and complexity of quantificational-functional reasoning, we offer alternative methods of algebraic proofs instead. According to Anellis, "[...] early efforts" at algebraizing Aristotelic syllogistic after Gottfried Leibniz and before George Boole "proved to be incomplete and abortive. Contemporary efforts to arithmeticize or algebraize Aristotelic syllogistic still persist" to this day (Anellis, 2007). Ours is such a solution, or rather a catalog of various alternative solutions.

Unlike the original Aristotelic tradition of term logic and Leibniz's advanced attempts at devising an algebra of term logic, we will adopt in this paper term logic *without* existential import, rather than assuming that a "term" class is necessarily inhabited by default. As a consequence, whenever a class is inhabited, we will have to explicitly declare so through a premise.

Throughout the paper we provide copious citations and footnotes for historically relevant materials as early as we could find to the origins of key insights and fundamental building blocks to construct the symbolic approaches to classic categorical syllogistic which we consider in this paper. Our remarks are not intended to present the history

 $^{^{9}}$ Leibniz not only algebraized Aristotelic logic; in fact, Leibniz's logic goes beyond categorical syllogistic, as Malink and Vasudevan (2019, p. 10-14,18-19,34,36,39) show.

¹⁰Almost one century later, Tamaki (1974) employed McColl's relations $\{\subseteq, \not\subseteq\}$ and proved the classic categorical syllogisms again, without adopting Boethius' connexive thesis for " \subseteq " and assuming existential import for the categorical relations $\{A, I\}$ though not for $\{E, O\}$.

¹¹See, for example, Tennant (2014) and Metamath (2021). We can even find an implementation of the proof search of the first-order logic representation of categorical syllogistic in a programming language (Koutsoukou-Argyraki, 2019).

of the concepts, tools and ideas envisioned by the pioneers on their own terms for achieving their own goals, but to record instead the origins of the ingredients we use and repurpose for the categorical syllogistic theorems we have the goal of proving in this paper. Here, historical remarks put into context the ingredients of our modernized presentation of proofs in algebraic categorical syllogistic.

2. Preliminaries – diagrams involving one or two terms

2.1. A subclass from a universe

The "smallest" independent regions in a Venn diagram (1880) are called *minterms*. Each minterm is either *inhabited* or *empty*, that is, it either has or does not have at least one element. Figure 1 shows the Venn diagram possibilities involving either an inhabitation or an emptiness mark for each minterm inside a universe of discourse¹² I with one specially designated subclass/term **b**. Here the minterms happen to be **b** itself and its complement, **b**'. When we do not know whether a minterm is inhabited or empty, we leave it blank, in order to indicate lack of information on our part.



Figure 1.: Four possibilities of fundamental relations involving a single term of a universe class – Venn and Euler diagrams.

2.2. Representing two terms from a universe

Let's draw a Venn (and also Euler) diagram representing a universe class that has two terms **b** and **c** as subclasses (Figure 2). In a translation from set algebra to term logic, we will call the sets/classes **b** and **c** the "subject" and the "predicative" terms, respectively. The Venn diagram shows that this configuration gives rise to four minterms¹³: bc, bc', b'c, b'c'. Here the notion of complement is again demonstrated to be fundamental. And again a blank minterm indicates that we do not have the knowledge whether that minterm is inhabited or empty.

Figure 3 enumerates the two possibilities (emptiness or inhabitation mark) for each of the four minterms independently considered for both Venn¹⁴ and Euler diagrams,

¹²In this paper, "**I**" was intentionally chosen to represent the un**I**verse of discourse class to avoid using the initial "U", which might be confused with the union operator " \cup ", and also because "**I**" resembles the digit "1", just like the empty set symbol " $\boldsymbol{\varnothing}$ " resembles the digit "0". Both digits play a major role in Boolean algebra and in anything nowadays referred to as "digital".

¹³Throughout this paper, we assume for notational convenience that the juxtaposition of two terms, **bc**, represents the intersection of the classes they refer to, $\mathbf{b} \cap \mathbf{c}$.

¹⁴Some introductory textbooks include these Venn diagrams and some corresponding algebraic symbology. See for instance Copi, Cohen, and Flage (2016, pp. 106-109).



Figure 2.: Four minterms of a universe class that has two generating terms as subclasses.

showing how they are distinct by direct contrast¹⁵. It shows eight dyadic relations in the form **b**? **c**, where "?" is the relation symbol, which are the fundamental components of what is called De Morgan's syllogistic¹⁶ – an extension of Aristotle's syllogistic¹⁷ (De Morgan, 1846, p. 381)(De Morgan, 1847, pp. 60-61).

Venn and Euler diagrams make it visible that every class being depicted is a subclass of the universe class ($\mathbf{b} \subseteq \mathbf{I}$). However, a challenge in drawing both Euler and Venn diagrams is how to represent the facts that the empty class is a subclass of every class being depicted ($\boldsymbol{\varnothing} \subseteq \mathbf{b}$) and that the intersection of any class with the empty one is the latter ($\mathbf{b} \cap \boldsymbol{\varnothing} = \boldsymbol{\varnothing}$)¹⁸. We propose a solution which does not require modifying diagrams (for instance, by adding a new marker), but merely requires us to change how we look at them.

It is convenient to consider the boundary¹⁹ of the representation of a given class (say, **b**) as the intersection between what is "inside" (**b**) and what is "outside" (**b**') that class. As $\mathbf{bb'} = \boldsymbol{\varnothing}$, we feel we are justified in adopting the convention that the boundary of a class stands for the empty class. The boundary of a class is reasonably considered as an integral part of its visual representation; this is a convenient representation for the fact that $\boldsymbol{\varnothing} \subseteq \mathbf{b}$. Moreover, in a Venn or Euler diagram, the intersection between a class and its border –which, as we have said, stands for the empty class– is visually realized as the border itself. Thus, the fact that $\mathbf{b} \cap \boldsymbol{\varnothing} = \boldsymbol{\varnothing}$ is also neatly represented.

As borders are present in any Venn or Euler diagram, the empty class is always vis-

¹⁸The empty class is the only class simultaneously "included in" and "excluded from" every conceivable class, that is, $\mathbf{b} \subseteq \mathbf{c}$ and $\mathbf{b} \not \in \mathbf{c}$ if and only if $\mathbf{b} = \mathbf{\emptyset}$. Proof:



Therefore, "every **b** is **c**" and "no **b** is **c**" simultaneously if and only if $\mathbf{b} = \mathbf{\emptyset}$.

Notice that, as a consequence of the definition of " β ", $\boldsymbol{\varnothing} \not \in \mathbf{c} = \boldsymbol{\vartheta}$.

¹⁹In another mathematical context, topos theory, Lawvere deals with a strongly related concept named "intrinsic boundary" or "co-Heyting boundary" (nLab authors, 2016)(Lawvere, 1991)(Pagliani, 1998, p. 127).

 $^{^{15}}$ Those possibilities could be portrayed by Carroll diagrams (1886, pp. 44,28) as well, which we decided to leave out of the scope of this paper.

¹⁶In order to preserve the order adopted for our Venn and Euler diagrams, we will enumerate in our Tables the fundamental categorical relations in De Morgan's syllogistic in the order I, E, O, A, Ö, Ä, Ï, Ë rather than in the more conventional order A, E, I, O, Ë, Ä, Ö, Ï. The convention of adopting the umlaut to represent the inverse relation (e.g.: $\mathbf{b} \,\mathbf{I} \,\mathbf{c}$ |=| $\mathbf{b}' \,\mathbf{I} \,\mathbf{c}'$) is from Menne (1957)(1962).

¹⁷Aristotle's syllogistic, which include the 24 classic categorical syllogisms we prove in this paper in Sections 3, 6 and 7, involve the initial four possibilities only, which deal with the two minterms that are subclasses of the subject (**b**): bc and bc'.

Historically, however, Aristotle also alluded to the possibility of categorical relations with a negated subject, such as "Every non-man is just" (Aristotle of Stagira, ca. 350 BCEa, Chapter 10) – millennia later they would become the interest of systematic study.



Figure 3.: Eight possibilities of fundamental relations between two terms of a universe class, one minterm at a time – Venn and Euler diagrams.

ibly depicted, and usually more than once, for boundaries are non-contiguous among classes –most notably the boundary of the universe I and the boundary of any of its subclasses (**b**, **c** and so on). This would make the representation of the empty class "fragmented" in a diagram. This may at first sight look inelegant, but it actually portrays in an elegant fashion an interesting property of the empty class: assuming idempotence and associativity of union (\cup), this "discontiguous border line interpretation"²⁰ is algebraically supported by the identity $\boldsymbol{\varnothing} = \boldsymbol{\varnothing} \cup \boldsymbol{\varnothing} = \boldsymbol{\varnothing} \cup \boldsymbol{\varnothing} \cup \dots \boldsymbol{\varnothing} = bb' \cup cc' \cup \dots zz'$. We can see also see that the empty class is made of pure boundaries when it is represented by the notation "{}", and, in the representation of an inhabited class such as "{x}", the borders (braces) are discontiguous.

In addition, in this interpretation a shaded region in a Venn diagram can be seen as a thicker expansion of a boundary, engulfing an entire region. (A diagram designer who wants to stress this would select the same color for class boundaries and for shaded regions – which usually isn't done for aesthetic reasons only.)

The interaction between complementary classes and their shared border can be modelled by a logical hexagon of opposition – Figure 4, loosely inspired by Béziau (2012, pp. 38-39, picture 53).



Figure 4.: A logical hexagon of opposition modelling the interaction between complementary classes and their shared border.

As shown in Figures 5a and 5b, the notational choices for the Venn diagram were deliberately made in order to visually represent the following desired properties (which are true propositions in set algebra, in its generalization, Boolean algebra, and even in multiset algebra):

i. $b\boldsymbol{\varnothing} = \boldsymbol{\varnothing}$. Therefore, $c = \boldsymbol{\varnothing} \mid = bc = \boldsymbol{\varnothing}$.

²⁰In a mereotopological analysis of Euler and Venn diagrams, as boundaries represent the empty class (whether they touch each other or not), RCC-5 (Cohn and Gotts, 1994) becomes a valid degeneration of RCC-8 (Randell, Cui, and Cohn, 1992; Bennett, 1994) in this context.

ii. The contrapositive of the previous logical assertion: $bc \neq \emptyset \mid = c \neq \emptyset$.



(b) bc is inhabited, therefore c is inhabited. (Venn diagram.)

Figure 5.: Emptiness and inhabitation markers for a minterm in a universe class that has two generating terms.

In order to achieve this, in Venn diagrams the emptiness marker –the shade– was intentionally chosen because it occupies a region as "wide" as possible (a whole class) and spreads all the way inside, but not outside that region (that is, the marker fills all of its subclasses down to the empty class, but not superclasses). In contrast, the inhabitation marker –X– was purposely chosen due to having the opposite properties: it occupies a "minimum" area inside a minterm and affects a region as large as possible –all the regions that contain it (that is, all the superclasses of that class, up to the universe class). These smart design decisions²¹ make them semiotically appropriate notations to graphically represent the two properties we want.

Notice that the arrows in Figure 5 are unidirectional. The converse is not necessarily true.

 $^{^{21}}$ The earliest explicit textual description we could find of the *rationale* for the desired features of emptiness *and* inhabitation marks (including alternative inhabitation marks) for term logic diagrams is by Venn (1883, pp. 599-600).

Euler (1770, p. 126, "Lettre CV" from February 24th, 1761) employed a special marker, " \star ", to indicate that a classificatory term is inhabited. He therefore deserves credit for the earliest use we could find of the inhabitation marker in logical diagrams.

In a draft circa 1903 -CP 4.359-4.363 (Hartshorne and Weiss, 1960, pp. 307-312), referred to as MS. 479 in the "Robin catalog" at https://peirce.sitehost.iu.edu/robin/robin_fm/logic.htm">https://peirce.sitehost.iu.edu/robin/robin_fm/logic.htm)-, Peirce drew Venn diagrams for O, I and I relations using a cross as inhabitation marker, and on CP 4.349 he describes the procedure of representing existence in a Venn diagram (using a dot rather than a cross, however). Peirce's draft CP 4.359-4.363 is also the earliest source we could find for the graphic display of the alternative inhabitation representation, X—X, used in this paper only later, in Figure 8. (See also Hammer (1995), Pietarinen (2016)(2021, pp. 84-100) and Shin, Lemon, and Mumma (2018).)

Decades later, the inhabitation mark and the alternative inhabitation representation are employed for proving categorical syllogisms by means of Euler and Venn diagrams by Lewis (1918, pp. 176,183-184), and the inhabitation mark is employed for Venn diagrams by Quine (1950, p. 70)(1982, pp. 98,102-110).

3. Euler system

In this paper, we are concerned with categorical syllogisms that comply to a rigid form that follows these rules:

- Three terms are involved: \mathbf{s} , \mathbf{m} and \mathbf{p}^{22} .
- There are two premises, where one involves **s** and **m**, and the other involves **m** and **p**.
- There may be an additional premise asserting that a given term among **s**, **m** and **p** is necessarily inhabited.
- There are one or two conclusions involving **s** and **p**.

In the limited logic we are concerned with, Euler diagrams are expressive enough for our needs. Informal diagrammatic proofs^{23} of the 24 classic categorical syllogisms by means of Euler diagrams, taken from Piesk $(2017)^{2425}$, are shown in Figure 6 (split into three parts).

All diagrammatic categorical syllogism proofs include a *logical elimination* step – the dropping of information irrelevant to the conclusion (Venn, 1883, p. 602).

Leonhard Euler, arguably the greatest mathematician ever²⁶, has the merit of popularizing this kind of diagram, after a series of didactic tutorials he wrote in French for educating a young princess was published in 1770 in the form a textbook, titled "Lettres a une princesse d'Allemagne sur divers sujets de physique & de philosophie" (Euler, 1770, pp. 99–131, "Lettre CII" from February 14th, 1761 – "Lettre CV" from February 24th, 1761), which became a best-seller at the time. However, he was definitely **not** the inventor of the kind of diagram which nowadays bears his name. The earliest occurrence we could find of this kind of diagram is a book from 1661 –a century earlier than Euler's letters– by Sturm (1661, p. 86). Also, around 1686 –decades before Euler was born–, Leibniz applied this kind of diagram to deduce categorical syllogisms, in a draft written in Latin and nowadays known as "De Formae Logicae comprobatione per linearum ductus" (Leibniz, ca. 1686a)²⁷, with a presentation that loosely resembles the modern one by Piesk (2017)²⁸. He also realized that the final

 24 A variant of this Euler diagram representation of categorical syllogisms is offered by Flage (2002).

²⁷Vacca (1899) rediscovered in the library of Hannover the draft where Leibniz anticipated Euler in the use of "Euler diagrams" for logic reasoning. Later, on Vacca's advice (Couturat, 1903, Préface, p. I)(Luciano, 2012), Couturat went to the library of Hannover to research Leibniz's manuscripts on logic and then published that insightful draft by Leibniz (ca. 1686a).

²⁸Most of Leibniz's proofs are correct. (His diagrammatic Ferio-1 configuration, for instance, is almost correct, although he made up for it later in Ferison-3; and his "Fessapmo", or Fesapo-4 configuration, is not fully correct, although Felapton-3 is).

In his diagrammatic notation, a missing refinement adopted by Euler decades later would have made Leibniz's configurations clearer and less ambiguous: the employment of a symbol analogous to "X" to mark a classificatory term as inhabited (Euler, 1770, p. 126, "Lettre CV" from February 24th, 1761) – which perhaps the great Leibniz would never have thought of because, in alignment with Aristotle, his logic assumed that all classificatory terms of interest were necessarily inhabited (Leibniz, ca. 1686a):

"Undes patet omnes imperfectos alterutro modo ex perfectae figurae modis derivari vel addendo praemissae superfluam quantitatem, vel demendo conclusioni utilem." (Hence it is clear that all imperfect moods can be

 $^{^{22}}$ Standing for subject of the conclusion, mediator (or middle term), which does not appear in the conclusion, and predicative of the conclusion.

 $^{^{23}}$ These proofs are informal not because they are diagrammatic, but because we have not explicitly enumerated here the axioms and inference rules required by this logic system.

 $^{^{25}}$ For two of the moods, we adopt the names "Baroko-2" and "Bokardo-3" with 'k' rather than 'c' to preserve compatibility with the rationale for the name of De Morgan's "Theorem K" (1860, p. 344).

 $^{^{26}}$ To see just a single example of Euler's impressive achievements, an easy-to-understand problem he devised and solved, popularly called the "Seven Bridges of Königsberg", is the founding point of two major branches of mathematics at once: Graph Theory and Topology.

diagrammatic configuration is the same for similar figures from different moods, which vary from each other only by conversion of premise relations. At the end of his draft essay, Leibniz cites Sturm and mentions he had read his book when he was young. (For further historical remarks on "Euler" diagrams, see Lemanski (2017)(2018) and Bennett (2015).)

Euler (1770, p. 126, "Lettre CV" from February 24th, 1761) began using a notational device to explicitly mark inhabited classificatory terms in his diagrams, and in the next page (127) he adopted the interpretation that, for a blank minterm/region inside a term circle, it is uncertain whether it is inhabited or not. He had the same agnostic position for the blank minterms/regions inside a term circle in the next example (ibid., pp. 128-130). Those pieces of evidence combined suggest that Euler didn't assume existential import for classificatory terms, and didn't consider that universal assertions declared terms to be inhabited – only particular assertions did.

However, there are other pieces of evidence contradicting this conclusion. In the next letter, Euler (1770, pp. 136–139, "Lettre CVI" from February 28th, 1761) enumerated 19 classic categorical syllogisms in the following order: Barbara-1, Darii-1, Celarent-1, Ferio-1, Camestres-2, Baroko-2, Cesare-2, Festino-2, Darapti-3, Disamis-3, Datisi-3, Felapton-3, Ferison-3, Bokardo-3, Bamalip-4, Dimatis-4, Calemes-4, Fesapo-4, and Fresison-4. He did not provide their proofs, though; it seems he decided to leave the proofs as an exercise to the student, since he provided examples of proofs in his previous letter (Euler, 1770, pp. 124–139, "Lettre CV" from February 24th, 1761). Of these syllogisms, Darapti-3, Felapton-3, Bamalip-4, and Fesapo-4 require an additional existential premise in a logical system lacking existential import.

Missing from Euler's enumeration are all and only the classic categorical syllogisms having a "weakened"/subaltern (from universal to particular) conclusion obtained from other syllogisms: Barbari-1, Celaront-1, Cesaro-2, Camestros-2, and Calemos-4. All these categorical syllogisms require an additional existential premise in a logical system lacking existential import.

We suspect that Euler simply copied the enumeration of 19 classic categorical syllogisms from another source and trusted the enumeration to be correct, rather than trying to prove them all using his diagrammatic notation. Had Euler tried to prove them all, he would have discovered that some categorical syllogisms in the two-premise form are invalid when existential import is not implicitly assumed for universal categorical assertions. We don't know what would have been his reaction to this information: would he have embraced the lack of existential import as an improvement on Aristotelic logic –like Brentano (1874, pp. 283-286)(Land, 1876) did more than a century later–, or would he have tried to "fix" his system to accommodate tradition?

The Euler diagrammatic system has been shown here as a motivation for introducing the algebraic and relational systems that follow. The purpose is to show that, while diagrammatic proofs for all 24 classic categorical syllogisms have long been known, we also need algebraic proofs of the same theorems for new insights. As our focus is on justifying and providing the algebraic proofs, we will explicitly describe in this paper neither the axioms nor the inference rules for the Euler diagrammatic system. (See the list of open problems in Section 9.)

derived from the moods of a perfect figure, either by adding the superfluous quantity to the premises, or by weakening the useful conclusion.)

⁽The inhabitation assumption, or existential import, appears in various other drafts, for instance in Leibniz (1690b, p. 233).)

Nevertheless, Leibniz's diagrammatic configurations are very good for the rigor standards from that age.



(e) Camestres-2 and Calemes-4.

(f) Camestros-2 and Calemos-4.

Figure 6.: Proofs from Piesk (2017) of syllogisms by means of Euler diagrams (part 1 of 3).

4. Algebraic and relational representations

Tables 1a and 1b show various alternative algebraic or relational representations for each fundamental categorical relation²⁹. The symbols have their usual meanings in set algebra. Juxtaposition of terms means intersection of classes.

The symbols " \mathbb{M} "/" \mathbb{M} " and " \mathbb{U} "/" \mathbb{V} " mean that their terms are conjoint/disjoint and exhaustive/exclusionary, respectively. They are so defined:

 $\begin{array}{lll} \mathbf{b} \Cap \mathbf{c} & |=| & \mathbf{b} \sqcap \mathbf{c} \neq \boldsymbol{\varnothing} \\ \mathbf{b} \And \mathbf{c} & |=| & \mathbf{b} \cap \mathbf{c} = \boldsymbol{\varnothing} \\ \mathbf{b} \And \mathbf{c} & |=| & \mathbf{b} \cup \mathbf{c} = \mathbf{I} \\ \mathbf{b} \And \mathbf{c} & |=| & \mathbf{b} \cup \mathbf{c} \neq \mathbf{I}. \end{array}$

 $^{^{29}{\}rm The}$ identities for ${\bf b}\,{\rm A}\,{\bf c}$ and ${\bf b}\,{\rm E}\,{\bf c}$ in Table 1a were enumerated by Robert Grassmann (1872, p. 20, points 40-41).

The coining of some of the column names for Table 1b was loosely inspired by Ladd Franklin (1890, p. 79). Reverse is the inverse of the obverse, like in numismatics.

Missing from Table 1b (for space reasons) are the representations " $\mathbf{c}' ? \mathbf{b}$ " and " $\mathbf{c} ? \mathbf{b}'$ ", which are the "converse of the obverse" and the "converse of the reverse", respectively.



Figure 6.: Proofs from Piesk (2017) of syllogisms by means of Euler diagrams (part 2 of 3).

The corresponding categorical relations are mutually connected by the De Morgan's laws:

$$\begin{split} \mathbf{b} & \cap \mathbf{c} = \mathbf{b}' \not \bowtie \mathbf{c}' \\ \mathbf{b} & \not \bowtie \mathbf{c} = \mathbf{b}' \oslash \mathbf{c}' \\ \mathbf{b} & \oslash \mathbf{c} = \mathbf{b}' \not \bowtie \mathbf{c}' \\ \mathbf{b} & \not \bowtie \mathbf{c} = \mathbf{b}' \not \bowtie \mathbf{c}'. \end{split}$$

The analogous relation to "``M" in propositional logic is an assertion involving Sheffer's stroke ("nand") operation (Janssen-Lauret, 2023, pp. 9,10). The analogous relation to "`M" in propositional logic is an assertion involving Peirce's arrow ("nor") operation.

The " \cap " and " \cup " relations are not as often used as " \subseteq " and " \supseteq "³⁰ in the literature about set algebra, but are just as important. Ladd developed in her Doctoral thesis (1883) the earliest in-depth study we could find about the " \cap " and " β " relations³¹. In

³⁰Leibniz developed a logic of containment which employed the " \subseteq " relation (Malink and Vasudevan, 2019, p. 1), writing it as "*est*". von Segner (1740, pp. 71-72) adopted symbols that meant " \subset ", " \supset " and "=" – though not symbols corresponding to " \subseteq " and " \supseteq ". (He also adopted a symbol corresponding to the modern " $\widehat{\square}$ ", and a symbol standing for the monadic operation of class complementation.) As we can see, von Segner was more fond of the symbolic tradition than Leibniz, despite offering a superficial treatment of logic which doesn't come close to Leibniz's deep conceptual analyses. von Segner's novel contributions were simply symbolic notations for some categorical relations.

³¹Ladd actually adopted the symbols " \bigvee " and " $\overline{\bigvee}$ ", perhaps influenced by Boole's (1847, pp. 21–22) use of "v" –which was also cited by Wundt (1880, p. 229)– to represent "some" ("at least one") in his unsuccessful attempt to algebraically treat particular categorical relations; there is at least a curious resemblance among both forms (Halsted, 1883)(Mitchell, 1883, p. 97). This might be confusing for an uninitiated reader of Ladd's thesis since, in modern notation, " \lor " is often used in Logic with the meaning "or".

Like Ladd's original notation, the modern one has the semiotic advantage of suggesting symmetrical relations: $\mathbf{b} \cap \mathbf{c} \mid = \mid \mathbf{c} \cap \mathbf{b}$

 $[\]mathbf{b} \not \cap \mathbf{c} \mid = \mid \mathbf{c} \not \cap \mathbf{b}.$

She was not the earliest adopter of symbols for the categorical relations " \mathscr{M} " and " \mathbb{N} ", though. In 1646 –more than two centuries before Ladd's Doctoral thesis–, Mounyer and Fabri (1646, pp. 254-263) had already adopted



(m) Ferio-1, Festino-2, Ferison-3 and Fresison-4.

Figure 6.: Proofs from Piesk (2017) of syllogisms by means of Euler diagrams (part 3) of 3).

the same book where Ladd's Doctoral thesis was published (Peirce, 1883)³², Mitchell's Doctoral thesis (1883)(Green, 1991, p. 5)(Venn, 1883, p. 601) was published, defining the " \square " relation and its complement, " \square "³³. Some researchers have used the modern notation which we adopt here for the relation symbols, which is beneficial to humans as corroborated by empirical cognitive psychology research (Wege et al., 2020). For instance, Icard, III (2014, pp. 11,5-7) uses " \cap " and a not too different symbol for " \cup ". Ladd's relations " \mathbb{O} " and " \mathbb{O} " were decades later also employed by Rescher (1954,

a dedicated symbol for the """ relation – they employed "X". It is used, for instance, in "Theorema 52" (ibid., p. 257), where they state Celarent-1, albeit with the converse conclusion. (In the same book [ibid., p. 254], by the way, there appears the earliest occurrence of a truth table we could find.) von Segner (1740, pp. 71-72,83) adopted the same symbol "X" for the relation which we represent by the modern notation "⋒". Wundt (1880, pp. 244-248) chose iconic (semiotic-considerate) symbols for disjointness and conjointness: ")(", "≬". The same conjointness symbol was adopted much later by Menne (1962, p. 59) (Novak, 1980, p. 238). In 1881, Ladd was already aware of Wundt's writings on logic (Pietarinen and Chevalier, 2015, p. 10) and developed her logic upon the two mentioned relations which Wundt had assigned dedicated symbols to (Ladd, 1883, p. 17, fn. 1).

Ladd's notation did not distinguish the "?" relation between terms from the metalogical "nand" relation between formulae; the object level vs. metalevel distinction was not typical in that era. The metalogical "nand" relation is a noteworthy alternative to the "|=" assertion, having many interesting properties, many of which have been discovered by Ladd (1883), such as symmetry and free transposition; we feel that, after the publication of Ladd's thesis, the community of logicians has not explored that relation as deeply as they should have done.

 $^{^{32}}$ Ladd and Mitchell were both supervised by Charles Sanders Peirce, the most important and influential American logician of the 19th century, and the editor of the book which contains their Doctoral theses, among others.

³³For " \mathbb{U} ", Mitchell (1883, p. 75) originally adopted the syntax " $(\mathbf{b} + \mathbf{c})_1$ "; for " \mathbb{Q} ", he (p. 97) adopted the syntax " $(\mathbf{b} + \mathbf{c})_q$ ". Mitchell also made use of Ladd's relations " \mathbb{O} " and " \mathbb{P} ", though he adopted the syntax " $(\mathbf{bc})_u$ " and " $(\mathbf{bc})_0$ ", respectively (pp. 75,97), where "u" means "at least one in the **u**niverse of discourse (I)". Decades earlier than Mitchell, De Morgan (1846, p. 381)(1847, pp. 60-61) had presented all the 8 relations of the extended Aristotelic syllogistic. McColl (1877, p. 184) also had mentioned the categorical relations which later became the focus of Mitchell's investigation. More than a century after Mitchell, Dekker (2015) explored De Morgan's syllogistic and adopted "x" and "y" for "" and "y", respectively.

pp. 11-12), although with their meanings exchanged. As he noticed, Leibniz in a sense anticipated later researchers (such as Ladd) in understanding the importance of the relations "∩" and "∩" for logic, through his notions of "communicating" and "incommunicating" terms and a few theorems he enunciated which employ these notions (Leibniz, ca. 1686b)(Leibniz, ca. 1686d, pp. 268–269)(Leibniz, ca. 1686f)(Lenzen, 2014)(Lewis, 1918, pp. 17-18). However, unlike Ladd's, Leibniz's approach is pre-symbolic: he did not dedicate any specific logical symbol to represent these complementary relations.

In the preface to the book he edited, Peirce (1883, p. v) remarks (annotations are ours):

«Miss Ladd and Mr. Mitchell also use two signs expressive of simple relations involving existence and non-existence; but in their choice of these relations they diverge both from McColl and me, and from one another. In fact, of the eight simple relations of terms signalized by De Morgan, Mr. McColl and I have chosen two (" \subseteq " vs. " $\not\subseteq$ "), Miss Ladd two others (" \emptyset " vs. " \square "), Mr. Mitchell a fifth and sixth (" \bigcup " vs. " \emptyset "). (Missing: " \supseteq " vs. " $\not\supseteq$ ", the converse relations to "⊆" and " $\not\subseteq$ ", respectively.) The logical world is thus in a situation to weigh the advantages and disadvantages of the different systems.»

For each row, the representations in Tables 1b and 1a correspond as follows:

- Representation with "E"/"⋒" or "I"/"⋒" (highlighted in Table 1b): relation in terms of intersection and "Ø" (highlighted in Table 1a). (These represent the inhabitation/emptiness mark in the appropriate minterm of the respective Venn diagram in Figure 3.)
- Representation with "Ë"/"⊎" or "Ï"/"♥": relation in terms of union and "I".
 Representation with "Ä"/"⊇" or "Ö"/"⊉": relation in terms of intersection and predicative (in both sides), or union and subject (in both sides).
- Representation with "A"/" \subseteq " or "O"/" $\not\subseteq$ ": relation in terms of union and predicative (in both sides), or intersection and subject (in both sides).
- The representations which mnemonically correspond to plain symbology are highlighted in the respective rows of Tables 1b and 1a.

Thus the 17th and 19th centuries gifted us with distinct systems of logic in their genesis which, once integrated and harmonized, are actually complementary points of view of the same algebra of term logic (Moktefi, 2019)³⁴:

- Diagrammatic (Leibniz, Venn, Carroll): Euler, Venn, and Carroll diagrams;
- Equational (Leibniz, Boole, Jevons, Cayley): $=, \neq$;
- Subsumptive (McColl, Peirce): $\subseteq, \not\subseteq, \not\supseteq, \supseteq$;
- Semicomplementary (Ladd, Mitchell): ∩, ∅, ∅, ⊎.

After translating to relation algebra notation, the "converse representation" column from Table 1b becomes:

• $\breve{I} = I$	• $\ddot{\mathbf{O}} = \mathbf{O}$
• $\breve{\mathbf{E}} = \mathbf{E}$	• $\breve{A} = A$
• $\check{\mathrm{O}} = \check{\mathrm{O}}$	• $\breve{I} = \ddot{I}$
• $\breve{A} = \ddot{A}$	• $\breve{\ddot{E}} = \ddot{E}$.

³⁴Jevons argued that the relation "=" is the most fundamental one. Peirce (1870, p. 2) argued instead that "⊆" is more fundamental than "=" – McColl (1877, p. 177) would likely agree. And Ladd claimed the primacy of "⋒" (and "𝒜"). For Leibniz's views, see Malink and Vasudevan (2019, pp. 28-33). We are free to be agnostic and consider them as complementary viewpoints that shed light on different aspects of the same term logic.

Catego-	In terms of	In terms	In terms of	In terms of	In terms of in-	In terms of
rical	intersection	of union	intersection	union and	tersection and	union and
relation	and $\boldsymbol{\varnothing}$	and \mathbf{I}	and subject	subject	predicative	predicative
bla	$\mathbf{bc} eq \mathbf{\emptyset}$	$\mathbf{b}' \cup \mathbf{c}' \neq \mathbf{I}$	$\mathbf{b}\mathbf{c}' \neq \mathbf{b}$	$\mathbf{b}' \cup \mathbf{c} \neq \mathbf{b}'$	$\mathbf{b'c} \neq \mathbf{c}$	$\mathbf{b} \cup \mathbf{c}' \neq \mathbf{c}'$
DIC	$\mathbf{bc} \supset \mathbf{\emptyset}$	$\mathbf{b}' \cup \mathbf{c}' \subset \mathbf{I}$	$\mathbf{bc}' \subset \mathbf{b}$	$\mathbf{b}' \cup \mathbf{c} \supset \mathbf{b}'$	$\mathbf{b'c} \subset \mathbf{c}$	$\mathbf{b} \cup \mathbf{c}' \supset \mathbf{c}'$
h E a	$\mathbf{bc} = \mathbf{\emptyset}$	$\mathbf{b}' \cup \mathbf{c}' = \mathbf{I}$	$\mathbf{b}\mathbf{c}' = \mathbf{b}$	$\mathbf{b}' \cup \mathbf{c} = \mathbf{b}'$	$\mathbf{b'c} = \mathbf{c}$	$\mathbf{b} \cup \mathbf{c}' = \mathbf{c}'$
DEC	$\mathbf{bc}\subseteq \mathbf{ extsf{0}}$	$\mathbf{b'} \cup \mathbf{c'} \supseteq \mathbf{I}$	$\mathbf{bc'} \supseteq \mathbf{b}$	$\mathbf{b}' \cup \mathbf{c} \subseteq \mathbf{b}'$	$\mathbf{b'c} \supseteq \mathbf{c}$	$\mathbf{b} \cup \mathbf{c}' \subseteq \mathbf{c}'$
hOa	$\mathbf{bc}' eq \mathbf{Ø}$	$\mathbf{b'} \cup \mathbf{c} \neq \mathbf{I}$	$\mathbf{bc} \neq \mathbf{b}$	$\mathbf{b}' \cup \mathbf{c}' \neq \mathbf{b}'$	$\mathbf{b'c'} \neq \mathbf{c'}$	$\mathbf{b} \cup \mathbf{c} \neq \mathbf{c}$
000	$\mathbf{bc}' \supset \mathbf{ extsf{ extsf extsf{ extsf} extsf{ extsf{ extsf} extsf{ extsf{ extsf} extsf{ extsf{ extsf} extsf{ extsf{ extsf{ extsf{ extsf} extsf{ extsf{ extsf} extsf{ extsf} extsf} extsf} extsf} extsf} extsf} ex}$	$\mathbf{b}' \cup \mathbf{c} \subset \mathbf{I}$	$\mathbf{bc} \subset \mathbf{b}$	$\mathbf{b}' \cup \mathbf{c}' \supset \mathbf{b}'$	$\mathbf{b'c'} \subset \mathbf{c'}$	$\mathbf{b} \cup \mathbf{c} \supset \mathbf{c}$
h A a	$\mathbf{bc}' = \mathbf{\emptyset}$	$\mathbf{b}' \cup \mathbf{c} = \mathbf{I}$	$\mathbf{b}\mathbf{c} = \mathbf{b}$	$\mathbf{b}' \cup \mathbf{c}' = \mathbf{b}'$	$\mathbf{b}'\mathbf{c}'=\mathbf{c}'$	$\mathbf{b} \cup \mathbf{c} = \mathbf{c}$
DAC	$\mathbf{bc}'\subseteq \mathbf{ extsf{D}}$	$\mathbf{b'} \cup \mathbf{c} \supseteq \mathbf{I}$	$\mathbf{bc}\supseteq\mathbf{b}$	$\mathbf{b}' \cup \mathbf{c}' \subseteq \mathbf{b}'$	$\mathbf{b'c'} \supseteq \mathbf{c'}$	$\mathbf{b} \cup \mathbf{c} \subseteq \mathbf{c}$
hÖa	$\mathbf{b'c} eq \mathbf{\emptyset}$	$\mathbf{b} \cup \mathbf{c}' eq \mathbf{I}$	$\mathbf{b'c'} \neq \mathbf{b'}$	$\mathbf{b} \cup \mathbf{c} \neq \mathbf{b}$	$\mathbf{bc} \neq \mathbf{c}$	$\mathbf{b}' \cup \mathbf{c}' \neq \mathbf{c}'$
000	$\mathbf{b'c}\supset \mathbf{arnothing}$	$\mathbf{b} \cup \mathbf{c}' \subset \mathbf{I}$	$\mathbf{b'c'} \subset \mathbf{b'}$	$\mathbf{b} \cup \mathbf{c} \supset \mathbf{b}$	$\mathbf{bc} \subset \mathbf{c}$	$\mathbf{b}' \cup \mathbf{c}' \supset \mathbf{c}'$
hÄe	$\mathbf{b'c} = \mathbf{\emptyset}$	$\mathbf{b} \cup \mathbf{c}' = \mathbf{I}$	$\mathbf{b}'\mathbf{c}' = \mathbf{b}'$	$\mathbf{b} \cup \mathbf{c} = \mathbf{b}$	$\mathbf{bc} = \mathbf{c}$	$\mathbf{b}' \cup \mathbf{c}' = \mathbf{c}'$
DAC	$\mathbf{b'c}\subseteq \mathbf{ extsf{0}}$	$\mathbf{b} \cup \mathbf{c}' \supseteq \mathbf{I}$	$\mathbf{b'c'} \supseteq \mathbf{b'}$	$\mathbf{b} \cup \mathbf{c} \subseteq \mathbf{b}$	$\mathbf{bc} \supseteq \mathbf{c}$	$\mathbf{b}' \cup \mathbf{c}' \subseteq \mathbf{c}'$
hÏa	$\mathbf{b'c'} eq \mathbf{\emptyset}$	$\mathbf{b} \cup \mathbf{c} eq \mathbf{I}$	$\mathbf{b'c} \neq \mathbf{b'}$	$\mathbf{b} \cup \mathbf{c}' \neq \mathbf{b}$	$\mathbf{bc}' \neq \mathbf{c}'$	$\mathbf{b}' \cup \mathbf{c} \neq \mathbf{c}$
DIC	$\mathbf{b'c'}\supset \mathbf{ extsf{0}}$	$\mathbf{b} \cup \mathbf{c} \subset \mathbf{I}$	$\mathbf{b'c} \subset \mathbf{b'}$	$\mathbf{b} \cup \mathbf{c}' \supset \mathbf{b}$	$\mathbf{bc}' \subset \mathbf{c}'$	$\mathbf{b}' \cup \mathbf{c} \supset \mathbf{c}$
hËa	$\mathbf{b'c'} = \mathbf{\emptyset}$	$\mathbf{b} \cup \mathbf{c} = \mathbf{I}$	$\mathbf{b'c} = \mathbf{b'}$	$\mathbf{b} \cup \mathbf{c}' = \mathbf{b}$	$\mathbf{bc}' = \mathbf{c}'$	$\mathbf{b}' \cup \mathbf{c} = \mathbf{c}$
DEC	$\mathbf{b'c'}\subseteq \mathbf{arnothing}$	$\mathbf{b} \cup \mathbf{c} \supseteq \mathbf{I}$	$\mathbf{b'c} \supseteq \mathbf{b'}$	$\mathbf{b} \cup \mathbf{c}' \subseteq \mathbf{b}$	$\mathbf{bc'} \supseteq \mathbf{c'}$	$\mathbf{b}' \cup \mathbf{c} \subseteq \mathbf{c}$

(a) Algebraic representations of categorical relations with identity and non-identity.

Catego-	Plain	Comple-	Obverse	Reverse	Inverse	Converse	Contra-
rical	symbol-	mented	represen-	represen-	represen-	represen-	positive
relation	ogy:	represen-	tation:	tation:	tation:	tation:	represen-
	b ? c	tation:	$\mathbf{b}?\mathbf{c}'$	$\mathbf{b}' ? \mathbf{c}$	$\mathbf{b}' ? \mathbf{c}'$	c ?b	tation:
		\sim (b ? c)					$\mathbf{c}' ? \mathbf{b}'$
bIc	b ⋒ c	\sim (b $\not \land$ c)	$\mathbf{b} \nsubseteq \mathbf{c}'$	$\mathbf{b}' ot \supseteq \mathbf{c}$	$\mathbf{b}' \not \!\!\! \not \!\!\! \not \mathbf{c}'$	$\mathbf{c} \cap \mathbf{b}$	$\mathbf{c}' \not \!\! \not \!\! \mathbf{b}'$
$\mathbf{b} \to \mathbf{c}$	b 🕅 c	$\sim \! (\mathbf{b} \Cap \mathbf{c})$	$\mathbf{b}\subseteq\mathbf{c}'$	$\mathbf{b}'\supseteq\mathbf{c}$	$\mathbf{b}' ! \mathbf{c}'$	c∦b	$\mathbf{c}' ! \mathbf{b}'$
$\mathbf{b} \mathbf{O} \mathbf{c}$	$\mathbf{b} ot \subseteq \mathbf{c}$	\sim ($\mathbf{b} \subseteq \mathbf{c}$)	$\mathbf{b} \cap \mathbf{c'}$	$\mathbf{b'} \not \!\!\! \not \!\!\! \mathbf{b'} \mathbf{c}$	$\mathbf{b'} \nsupseteq \mathbf{c'}$	$\mathbf{c} igz \mathbf{b}$	$\mathbf{c}' \nsubseteq \mathbf{b}'$
b A c	$\mathbf{b}\subseteq\mathbf{c}$	\sim (b \nsubseteq c)	b ∦ c′	$\mathbf{b'} ! \mathbf{c}$	$\mathbf{b}'\supseteq\mathbf{c}'$	$\mathbf{c}\supseteq\mathbf{b}$	$\mathbf{c}' \subseteq \mathbf{b}'$
bÖc	b⊉ c	\sim ($\mathbf{b} \supseteq \mathbf{c}$)	$\mathbf{b} \not \!\! arpsi \mathbf{c}'$	$\mathbf{b'} \otimes \mathbf{c}$	$\mathbf{b}' \nsubseteq \mathbf{c}'$	$\mathbf{c} \nsubseteq \mathbf{b}$	$\mathbf{c}' \not\supseteq \mathbf{b}'$
bÄс	$\mathbf{b} \supseteq \mathbf{c}$	\sim (b $\not\supseteq$ c)	$\mathbf{b} \Cup \mathbf{c}'$	b′ ∦ c	$\mathbf{b}' \subseteq \mathbf{c}'$	$\mathbf{c}\subseteq \mathbf{b}$	$\mathbf{c}'\supseteq\mathbf{b}'$
bÏc	$\mathbf{b} \not \!\! \! / \mathbf{c}$	\sim ($\mathbf{b} lacksim \mathbf{c}$)	$\mathbf{b} ot \supseteq \mathbf{c}'$	$\mathbf{b}' \nsubseteq \mathbf{c}$	$\mathbf{b'} \otimes \mathbf{c'}$	c∦b	$\mathbf{c'} \Cap \mathbf{b'}$
bËс	$\mathbf{b} lewprimes \mathbf{c}$	$\sim (\mathbf{b} \not \! \! \not \! \! \mathbf{b} \mathbf{c})$	$\mathbf{b}\supseteq\mathbf{c}'$	$\mathbf{b}' \subseteq \mathbf{c}$	$\mathbf{b}' \not \bowtie \mathbf{c}'$	$\mathbf{c} lewbox{} \mathbf{b}$	$\mathbf{c}' \not \bowtie \mathbf{b}'$

(b) Equivalent relational representations of each categorical relation with a single term on each side of a single relation symbol.

Table 1.: Algebraic and relational representations of categorical relations.

These converse representations can be employed to transform categorical syllogisms in De Morgan's syllogistic into "perfect", composition-friendly syllogisms in the "first" figure.

Table 1a shows that at least one representation is available for each of the 8 categorical relations which avoids dealing with the complementation operation; we just have focus on this sublattice of the Boolean lattice generated by the atoms $\{b'c', b'c, bc', bc\}$ and structured by the $\{\cap, \cup\}$ operations:

 $\mathcal{O} \subseteq \mathbf{bc} \subseteq \{\mathbf{b}, \mathbf{c}\} \subseteq \mathbf{b} \cup \mathbf{c} \subseteq \mathbf{I}$

This partial order is also displayed by means of a Hasse diagram in Figure 7. The representations using only the elements of that Hasse diagram and the " \subseteq " relation (or its converse, " \supseteq ", to put the isolated term on the right for uniformity) are in Table 2.

It is interesting to notice that what are often (and controversially) called for some reason the three (or four) "laws of thought"³⁵ (Ladd Franklin, 1890, pp. 86-

 $^{^{35}}$ Various other fundamental laws of classical logic are enumerated in Section 8.2.1.



Figure 7.: A sublattice of the Boolean lattice generated by the atoms $\{b'c', b'c, bc', bc\}$ where complementation is not employed.

87,77)(Peirce and Ladd-Franklin, 1901)(Leibniz, ca. 1679)(Ladd, 1883, p. 31)(Richeri, 1761, p. 48)(Leibniz, ca. 1686d, p. 259, point 8; p. 261, point 3) are just different representations of " $\mathbf{b} \mathbf{A} \mathbf{b}$ and $\mathbf{b} \ddot{\mathbf{A}} \mathbf{b}$ ", as shown in Table 3.

From Table 3, and by looking up in Table 1b which plain relation corresponds to the contrapositive or converse representation, we can also see that

- (1) **b** and **c** are *identical*³⁶ (**b** = **c**) if and only if **b** A **c** (**b** \subseteq **c**) and **b** Ä **c** (**b** \supseteq **c**).
- (2) **b** and **c** are complementary ($\mathbf{c} = \mathbf{b}'$) if and only if $\mathbf{b} \in \mathbf{c}$ ($\mathbf{b} \not \in \mathbf{c}$) and $\mathbf{b} \stackrel{.}{\to} \mathbf{c}$ ($\mathbf{b} \cup \mathbf{c}$).
- (3) **b** and **c** are simultaneously *identical and complementary* ($\mathbf{b} = \mathbf{c} = \mathbf{b}'$) if and only if the universe is empty ($\mathbf{I} = \boldsymbol{\varnothing}$)³⁷, and thus in such a special case everything degenerates into emptiness (monovalent algebra).

Finally, we can combine the symbolic and diagrammatic notations (Venn diagrams) to draw a logical "hexagon" that highlights mutually contradictory relations (Figure 8). **Red** bidirectional arrows in a straight line indicate mutual contradiction, whereas **black** unidirectional arrows indicate implication. Moreover, " $\mathbf{b} = \boldsymbol{\mathscr{O}}$ " is equivalent to

³⁶By employing Boolean algebra laws, Jevons (1864, pp. 42-43, points 112-113) offers an interesting proof that positive identity entails negative identity and vice-versa ($\mathbf{b} = \mathbf{c}$ |=| $\mathbf{b}' = \mathbf{c}'$). Leibniz offered an elegant (and simpler) proof by using only interchangeability of identicals and involution of complementation (Leibniz, 1690a, point 11)(Lenzen, 2018b, p. 266).

³⁷ To prove this	we also assume	e idempotence	of jı	uxtaposition	$/ \cap$	and	U.
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Relation	Extracted from the sublattice relation(s)	Sublattice relation(s), rewritten (\subseteq/\supseteq)	Respective re- presentation(s)
bIc bEc	$\boldsymbol{\varnothing}\subseteq\mathbf{bc}$	$\mathbf{bc}\supseteq\boldsymbol{\varnothing}$	$bc \supset \emptyset$ $bc = \emptyset$
bOc bAc	$\mathbf{bc} \subseteq \mathbf{b}, \mathbf{c} \subseteq \mathbf{b} \cup \mathbf{c}$	$\mathbf{bc} \subseteq \mathbf{b}, \mathbf{b} \cup \mathbf{c} \supseteq \mathbf{c}$	$\mathbf{bc} \subset \mathbf{b}, \mathbf{b} \cup \mathbf{c} \supset \mathbf{c}$ $\mathbf{bc} = \mathbf{b}, \mathbf{b} \cup \mathbf{c} = \mathbf{c}$
bÖc bÄc	$\mathbf{b}\mathbf{c} \subseteq \mathbf{c}, \mathbf{b} \subseteq \mathbf{b} \cup \mathbf{c}$	$\mathbf{bc} \subseteq \mathbf{c}, \mathbf{b} \cup \mathbf{c} \supseteq \mathbf{b}$	$\begin{aligned} \mathbf{b}\mathbf{c} \subset \mathbf{c}, \mathbf{b} \cup \mathbf{c} \supset \mathbf{b} \\ \mathbf{b}\mathbf{c} = \mathbf{c}, \mathbf{b} \cup \mathbf{c} = \mathbf{b} \end{aligned}$
bÏc bËc	$\mathbf{b} \cup \mathbf{c} \subseteq \mathbf{I}$	$\mathbf{b} \cup \mathbf{c} \subseteq \mathbf{I}$	$ \begin{aligned} \mathbf{b} \cup \mathbf{c} \subset \mathbf{I} \\ \mathbf{b} \cup \mathbf{c} = \mathbf{I} \end{aligned} $

Table 2.: Representations avoiding class complementation.

Identity laws:	Plain symbology $\mathbf{b} \subseteq \mathbf{b}, \ \mathbf{b} \supseteq \mathbf{b}$ Positive identity ("Identity")	Inverse representation $\mathbf{b}' \supseteq \mathbf{b}', \ \mathbf{b}' \subseteq \mathbf{b}'$ Negative identity (Usually omitted)
Complementarity/ Contradiction laws:	Obverse representation $\mathbf{b} \not \otimes \mathbf{b}', \mathbf{b}' \not \otimes \mathbf{b}$ Disjointness ("[Non-]Contradiction")	Reverse representation $\mathbf{b'} \sqcup \mathbf{b}, \ \mathbf{b} \sqcup \mathbf{b'}$ Exhaustion ("Excluded third")

Table 3.: "Laws of thought": variant forms of ${}_{*}\mathbf{b} \mathbf{A} \mathbf{b}$, $\mathbf{b} \ddot{\mathbf{A}} \mathbf{b}_{*}$ according to the $\mathbf{b} \mathbf{A} \mathbf{c}$ and $\mathbf{b} \ddot{\mathbf{A}} \mathbf{c}$ rows (where $\mathbf{c} \leftarrow \mathbf{b}$) from Table 1b (Ladd Franklin, 1890, pp. 86-87,77).

" $\mathbf{b} \subseteq \mathbf{c} \bowtie \mathbf{b} \not \bowtie \mathbf{c}$ ", and " $\mathbf{b} \neq \mathbf{0}$ " is equivalent to " $\mathbf{b} \bowtie \mathbf{c} \bowtie \mathbf{b} \not \subseteq \mathbf{c}$ ".

5. Replacement and inference rules

In the description of each algebraic system, we make the axioms explicit. As a pedantic remark in the interest of precision, we describe here the replacement and inference rules adopted by the algebraic systems discussed in this paper.

An inference rule they assume is substitution of equivalent expressions³⁸. If we know that $expr_1 = expr_2$, then we can replace $expr_1$ for $expr_2$ (or vice-versa) anywhere the other expression appears:

 $expr_1 = expr_2, rel$ |= $rel[expr_2 \leftarrow expr_1]$ For instance, suppose that $(\mathbf{bc})\mathbf{d} = \mathbf{fg}$ and $(\mathbf{fg})\mathbf{h} \neq \mathbf{j}$; then $((\mathbf{bc})\mathbf{d})\mathbf{h} \neq \mathbf{j}$. In particular, by making $rel = expr_2$? $expr_3$, where "?" is a dyadic relation³⁹: $expr_1 = expr_2$, $expr_2$? $expr_3$ |= $expr_1$? $expr_3$

(From which the transitivity of "=" is straightforwardly derived 40 .)

Specializing it even more, we obtain Leibniz's principle of *interchangeability of identicals* (Leibniz, ca. 1686f, p. 94, *Definitio* 1)(Forrest, 2020) (Boole, 1847, p. 18)(Jevons, 1864, pp. 16-18):

 $x_1 = x_2, f(x_2) = y$ $\mid = f(x_1) = y$ or, more succinctly,

 $x_1 = x_2$ |= $f(x_1) = f(x_2)$

Another applicable inference rule is *substitution of placeholder terms*: if we take any axiom or theorem having the term \mathbf{x} as a "generic" placeholder (where it is explicitly

³⁸An informal, example-driven treatment is given by Jevons (1864, pp. 9{point 23},18-19,27).

³⁹ Jevons –the modern founder of what we nowadays call "Boolean algebra", which differs in some key aspects from Boole's original algebra (Hailperin, 1986, pp. 61,82-83,119-121,139-140)(Lewis, 1918, pp. 74,78)– called this specialized rule the "law of sameness", "substitution of equals" or "*substitution of similars*" (Jevons, 1869, pp. V,16-27)(Jevons, 1864, pp. 7-13,29-30,41; pp. 53-54, point 138; pp. 73-74, point 172)(Lewis, 1918, p. 75)(Malink and Vasudevan, 2019, pp. 28-33).

Years before Jevon's inference adopted a generic, arbitrary dyadic relation which we have designated by "?", Boole (1847, p. 18), likely repeating Whately (Jevons, 1869, p. 74), discussed more specialized versions of this 3-expression inference rule, in which the concrete relations "=" and " \neq " instantiate the generic relation.

 $^{^{40}}$ Two other notable properties of "=" -as it is an equivalence relation- are symmetry (which we use in many of our proofs) and reflexivity (which we didn't need to employ in any occasion). Symmetry, reflexivity and transitivity of the "=" relation were explicitly stated in logic by Jevons (1864, pp. 9-11, points 22,24,27). More than two millennia earlier, Euclid of Alexandria (ca. 300 BCE) explicitly stated reflexivity and transitivity (though not symmetry) of "=" through his "common notions" 4 and 1, respectively, and also stated some corollaries of the principle of interchangeability of identicals through his "common notions" 2 and 3.



Figure 8.: A logical "hexagon" (which almost looks like a "cube" due to optical illusion) showing the categorical relations \mathbf{bIc} , \mathbf{bOc} , \mathbf{bAc} , and \mathbf{bEc} , in the absence of existential import for universal ones.

or implicitly understood that it is universally quantified) and replace it by the term or expression \mathbf{w} –in effect, a relabelling, which we are always free to do⁴¹, as long as we consistently do it for all occurrences of \mathbf{x} –, the result will also be true:

 $x \leftarrow w \qquad \mid = \qquad rel[x \leftarrow w]$

For instance, as we will see later, the Leibniz-Cayley system has the axiom $\mathbf{bc} = \mathbf{cb}$; a consequence is that $\mathbf{bm} = \mathbf{mb}$, by consistently relabelling \mathbf{c} as \mathbf{m} .

More generally, this can be done with various placeholder terms at $once^{42}$:

 $x \leftarrow w, z \leftarrow y, ..., q \leftarrow p \qquad \qquad |= \qquad rel[x \leftarrow w, z \leftarrow y, ..., q \leftarrow p]$

Another inference rule we adopt is *modus ponens*, *implication elimination* or *detachment*⁴³:

 $(r1 \mid = r2) \mid, \mid r1 \mid \mid = r2.$

For instance, the Leibniz-Cayley system has the axiom $\mathbf{bc} \neq \boldsymbol{\emptyset} \mid = \mathbf{b} \neq \boldsymbol{\emptyset}$. Thus, in the cases where we know that $\mathbf{bc} \neq \boldsymbol{\emptyset}$, we are justified in deducing $\mathbf{b} \neq \boldsymbol{\emptyset}$.

⁴¹This is analogous to α -conversion in λ -calculus (Church, 1932, p. 355, postulate I).

 $^{^{42}}$ On the other hand, in general we cannot replace a term for a generic placeholder *expression*, even if we do it consistently. For instance:

 $^{(\}mathbf{bc})\mathbf{b} = \mathbf{bc}$ (Universally true.)

 $[\]mathbf{d} \ \mathbf{b} = \ \mathbf{d} \quad \{\mathbf{d} \leftarrow \mathbf{b}\mathbf{c}\} \ (\text{Not true in general!})$

 $^{^{43}\}mathrm{Or}$ its generalization, the cut rule.

In summary, the only inference rules we need are are substitution of equivalent expressions, substitution of placeholder terms, and *modus ponens*.

In the proofs that follow, all required proof steps are made explicit; no shortcuts are taken.

6. Leibniz-Cayley (LC) system

6.1. Leibniz-Cayley system representations

The representations of categorical assertions in the Leibniz-Cayley system make use only of one dyadic operation (which we interpret as intersection), one monadic operation (complement), two dyadic relations (equality, difference), and one constant object (the empty class). The following representations of fundamental categorical assertions are favored in the Leibniz-Cayley system as we define it:

Α	Every \mathbf{b} is \mathbf{c} .	bc = b
Έ	No \mathbf{b} is \mathbf{c} . (Every \mathbf{b} is not- \mathbf{c} .)	bc' = b
Ι	At least one \mathbf{b} is \mathbf{c} .	$bc \neq \mathbf{Ø}$
0	At least one \mathbf{b} is not \mathbf{c} . (At-least-one \mathbf{b} is not- \mathbf{c} .)	$bc' eq {m arnothing}$
*	At least one b exists. (b is-not empty.)	$b eq \mathcal{O}$

By means of the "substitution of similars" inference rule (Section 5) –and with no need of any axiom–, these representations straightforwardly lead to the "subalternation laws" from traditional Aristotelic logic when the subject term is inhabited:

Leibniz has the merit of being the earliest logician we could find to algebraically represent the universal assertions as we do here, and came very, very close to representing particular assertions in the LC fashion, as we will show in the next paragraphs. His various strategies for representing particular assertions are very insightful: a short summary is made by Brown (2012, p. 165), and further comments are made by Lewis (1918, p. 15) and Malink and Vasudevan (2016, pp. 691-692). In 1871, Cayley devised a proper way of representing particular assertions by means of non-identity relations with respect to " $\boldsymbol{\varnothing}$ " (Cayley, 1871)(Valencia, 2004, p. 473), in a break with his predecessor Boole (1847).⁴⁴.

Leibniz offered algebraic representations of categorical assertions in various drafts⁴⁵.

⁴⁴In 1870 – one year before Cayley–, Peirce (1870, pp. 57-58), in a minor remark in the context of his paper, proposed representing particular categorical assertions such as "At least one **b** is **c**." as " $bc \supset \boldsymbol{\varnothing}$ ". Cayley represented it instead as " $bc \neq \boldsymbol{\varnothing}$ ". So, Peirce deserves the credit of providing before Cayley a proper way of representing particular assertions by means of comparison relations involving " $\boldsymbol{\varnothing}$ ". We found no evidence that Cayley was aware of Peirce's discovery when he published his paper.

We prefer to adopt Cayley's representation here because for this section we want a system with only "=" and " \neq " (coincidence and non-coincidence), not " \subseteq ", " \supset ", or other super-/sub-classhood relations. In addition, we feel it is fair to pay special homage to Cayley in the Leibnizian research program on term logic because his paper, unlike Peirce's, adopts as its central concern the algebraic representation of classic categorical assertions.

 $^{^{45}}$ In his research over decades, Leibniz toyed with various attempts of logical systems. As a faithful Aristotelic logic traditionalist, he gave preference to constructing an *intensional* logic of *concepts* rather than an extensional logic of classes (a preference later shared by Frege, who invented a brand of quantificational logic that modelled concepts as Boolean-valued functions – which he employed to enunciate his Basic Law V that inspired the discovery of Russell's paradox and led to intense research towards axiomatic set theory, type theory, lambda calculus and combinatory logic) and often assumed *existential import*. (These are interesting

In a representative draft, Leibniz (1690b, pp. 235-236) wrote (the symbolic decoding **highlighted** inside square brackets are ours):

«(5)A ∞ non non A. $[\mathbf{a} = (\mathbf{a}')']$ [...] (7) AB ∞ BA $[\mathbf{ab} = \mathbf{ba}]$. [...] (12) Coincidunt A ∞ AB et non B ∞ non B non A. $[\mathbf{a} = \mathbf{ab} |=| \mathbf{b}' = \mathbf{b}'\mathbf{a}']$ (13) [...] Universalis affirmativa $[\mathbf{a} A \mathbf{b}:]$ sic exprimi potest: A ∞ AB $[\mathbf{a} = \mathbf{ab}]$ [...] Particularis affirmativa $[\mathbf{a} I \mathbf{b}:]$ sic: [...] AB est Ens $[\Im(\mathbf{ab})]$ [...] vel $[\mathbf{or}]$ A non∞ A non B $[\mathbf{a} \neq \mathbf{ab}']$. Universalis negativa: Nullum A est B $[\mathbf{a} E \mathbf{b}:]$, sic: [...] A ∞ A non B $[\mathbf{a} = \mathbf{ab}']$ seu $[\mathbf{or}]$ AB est non Ens $[\Im(\mathbf{ab})]$. Particularis negativa: Quoddam A est non B $[\mathbf{a} O \mathbf{b}:]$, A non∞ AB $[\mathbf{a} \neq \mathbf{ab}]$, vel $[\mathbf{or}]$ A non B est Ens $[\Im(\mathbf{ab}')]$. [...]»

In Leibniz's era, the symbols " ∞ " and "=" were adopted by different authors to represent the equality relation; Leibniz adopted the former convention. For our modern " \neq " he adopted " $non\infty$ "⁴⁶. In both sides of each (in)equation, literal symbols, accompanied or not by a negation particle ("non"), are employed. In contrast, in other important points of this excerpt, Leibniz consciously adopted the expressions "*est Ens*" and "*est non Ens*" after literal symbols. This is strong evidence that by " $[non]\infty$ " and "*est [non] Ens*" Leibniz meant distinct relations in this and some other excerpts of his drafts. In our interpretation, Leibniz adopted "*est Ens*" ("is an entity"⁴⁷) to indicate

A quote mining exercise does not make a sound historical research, however. The reader is warned that Leibniz's drafts as a whole are far *more* nuanced than the details we focus on: the real Leibniz the logician is far richer in insights and thoughts than our "extensional Leibniz" (Lewis, 1918, pp. 13-14). (The same can be said about our convenient quotations from other early symbolic logicians, such as Jevons, Ladd and Mitchell.) Fortunately, professional historians of logic have plenty of rich material to explore and comment on all the nuances of the real Leibniz for many decades to come.

Nevertheless, even our impoverished, extensional Leibniz is enough for us to appreciate how prolific Leibniz was as a source of great insights. We provide strong evidence that Leibniz is a tremendously skilled founding master of the *algebra* of logic. Indeed, every time we revisit Leibniz's drafts, the master teaches us something new about logic which we passed over in previous readings.

⁴⁶The employment of symbols which are nowadays nonstandard was not unusual up to the 19th century. For instance, in Robert Grassmann's treatise on Logic (1872, p. 8), the symbol " \geq " –a combination of ">" and "<" (Grattan-Guinness, 2000, p. 158)– was employed instead of " \neq " to represent non-equality.

⁴⁷The literal translation of [la] "ens" is [pt,es,it] "ente", [en] "entity".

features on their own, however they are not within the scope of this paper.) He was a 17th-century rationalist Germanic/continental logician concerned with *organized* concepts or ideas, not a 19th-century British logician committed to working with *arbitrarily formed* extensional classes (Lewis, 1918, p. 14,35-37). He also drew symbolic treatment parallels of term logic and propositional logic –anticipating Boole (1847)–, and dealt with some notions of modal logic.

There is no single manuscript where Leibniz does *everything* the way *we* want. A typical situation is that, in a manuscript, Leibniz often has an insight that represents a progress towards our desired end stage, and then, after not having completed the entire puzzle, backtracks to try another direction, undoing the progress towards what we want. Then in another manuscript he documents another important insight, but does not combine it with a good insight he previously abandoned. It was state-of-the-art research at that age. He invents virtually all the required pieces of our puzzle, but the pieces are scattered across different boxes and in each box they are mixed with pieces that are incompatible to our puzzle. In fairness, he wasn't trying to achieve exactly our goals. But this means that, in order to understand how Leibniz contributed so much to the *extensional* algebra of categorical syllogisms involving *classes* which *lack* existential import by default, we are forced to cherry-pick particularly noteworthy passages from his drafts, ignoring much of the original context surrounding those snippets, and assemble excerpts from different drafts to bring a Frankenstein's monster alive, adopting our 21st-century prejudices as a guide to picking and choosing and combining.

that its subject was an inhabited term, and "*est non Ens*" ("is a non-entity") to indicate that its subject was a non-inhabited (that is, extensionally empty) term. These are monadic relations, which we respectively represent in symbolic notation by " $\Im\langle s \rangle$ " and " $\overline{\Im}\langle s \rangle$ "⁴⁸.

In this excerpt, Leibniz correctly proposed the following representations⁴⁹:

• **a** A **b**: **a** = **ab**

- $\mathbf{a} \to \mathbf{b}$: $\mathbf{a} = \mathbf{a}\mathbf{b}'$; $\overline{\mathfrak{I}}\langle \mathbf{a}\mathbf{b} \rangle$
- a I b: $\mathbf{a} \neq \mathbf{a}\mathbf{b}'; \, \mathfrak{I}\langle \mathbf{a}\mathbf{b} \rangle$
- $\mathbf{a} \mathbf{O} \mathbf{b}$: $\mathbf{a} \neq \mathbf{a} \mathbf{b}$; $\Im \langle \mathbf{a} \mathbf{b}' \rangle$.

Leibniz didn't mention in this particular excerpt " $\overline{\mathfrak{I}}\langle \mathbf{ab'} \rangle$ " as an alternative representation for " $\mathbf{a} \mathbf{A} \mathbf{b}$ ", although he fixed this omission in other drafts (ca. 1691a; ca. 1691b). Also noteworthy is that he didn't shy away of using the " \neq " relation in the representation of particular categorical assertions, unlike Boole (1847, pp. 21–22) more than 150 years later, who only attempted to represent all categorical assertions with equations (using "=").

As we can see, in this short excerpt Leibniz identified the importance of the term combination/intersection and complementation operations, the equality/coincidence (" ∞ ") and difference/non-coincidence (" $non\infty$ ") relations, the involution of complementation (a=(a')'), the commutativity of intersection (ab=ba), inhabitation ("Ens") and non-inhabitation/emptiness ("non-Ens", nonexistent, in a loose, non-literal translation⁵⁰).

Later, Cayley (1871) proposed the following representations⁵¹:

- $\mathbf{a} \mathbf{A} \mathbf{b}$: $\mathbf{a} \mathbf{b}' = \boldsymbol{\emptyset}$
- $\mathbf{a} \to \mathbf{b}$: $\mathbf{a} = \mathbf{\emptyset}$
- a I b: $ab \neq \emptyset$
- a O b: $ab' \neq \emptyset$.

Notice that Leibniz's monadic relations " $\Im\langle s \rangle$ " and " $\overline{\Im}\langle s \rangle$ " respectively correspond⁵²

Almost two centuries after Leibniz, Ladd (1883, pp. 29-30)(Venn, 1883, p. 598) reinvented these monadic relations by employing the representations " $s \cap$ " and " $s \not \cap$ " to respectively stand for " $s \cap$ I" and " $s \not \cap$ I". If one asserts some thing exists (does not exist), then one asserts it exists (does not exist) within the universe.

⁴⁹The **highlighted** symbolic encodings are the ones we would select for a pure Leibniz's system, since they would fit the axioms enumerated in Section 6.2.

 50 Ramon Llull (1993, p. 162), a major intellectual influence on the young Leibniz, wrote the following on the contrast between "*Ens*" and "*non-Ens*" and the power of imagination:

"Si extra intellectum nullum non ens est ens, solus intellectus facit non ens."

"As there is no non-being outside of the intellect, then only intellect creates non-being."

(Literally: "If outside the intellect no non-entity is [an] entity, only intellect makes [a] non-entity.")

⁵¹The **highlighted** algebraic encodings of particular categorical assertions are the ones we have selected for the Leibniz-Cayley system instead of Leibniz's relational representations of particular categorical assertions.

«[...] there is abundant textual evidence to show that at least as applied to terms, i.e. to concepts, Leibniz always uses 'est Ens' as synonymous with 'est Possibile' [...]. Accordingly 'est non-Ens' means the same

In some manuscripts, e.g. (Leibniz, ca. 1686e, pp. 391-395, points 144-146,148-155,165,167-169,171), Leibniz adopts the word "res" ("thing") rather than or in alternation with "Ens" ("entity"). Sometimes, e.g. (Leibniz, ca. 1686e, pp. 398-399, points 199-200), Leibniz simply adopts "est" ("is") rather than "est Ens", and "non est" ("is not") rather than "est non Ens".

⁴⁸We chose the character " \Im " for this monadic relation because it is the initial character of " \Im nhabitātus"/" \Im nhabitā ϑ " and also the vowel corresponding to the particular affirmative categorical dyadic relation "I", ensuring that "**b**Ic |=| \Im (**b**c)". The complement of " \Im " is the monadic relation " $\overline{\jmath}$ ". One may think of " \Im " and " $\overline{\jmath}$ " by the mnemonics "is" and "is not" in English, respectively.

 $^{^{52}}$ Of course, here we are projecting our 21-century extensional goals on Leibniz's excerpts. Leibniz was actually dealing with modal logic concepts when he was talking about "*est Ens*" and variants according to Lenzen (1987, p. 5):

to Cayley's dyadic relations " $s \neq \mathbf{\emptyset}$ " and " $s = \mathbf{\emptyset}$ ". Moreover, notice that all of Cayley's representations are about either the emptiness or the inhabitation of some combination of terms.

We will show that a pure Leibniz's system –which adopts Leibniz's algebraic representations for universal categorical assertions and Leibniz's monadic relational representations for particular categorical assertions, together with taking as axioms some laws stated by Leibniz, which are enumerated in Section 6.2– suffices to prove all 24 categorical syllogisms (see the proofs in Section 6.3). For this narrow purpose, Cayley's (1871) representations of the fundamental categorical assertions are superfluous.

Nevertheless, the Leibniz-Cayley system adopts Cayley's algebraic representations (with " $\neq \varnothing$ ") for particular categorical assertions rather than Leibniz's monadic relational representations (with " \Im ") because: we prefer to construct a system with only (in)equations rather than a hybrid algebraic/relational system; it makes it easier to algebraically justify the axiom LC3 (Section 6.2); and, more importantly, because LC3 in the representation adopted by LC (rather than as represented a pure Leibniz's system) is straightforward to prove as a theorem in Boolean algebra (Section 8.2.2).

Unfortunately, Leibniz chose to keep the note containing that key excerpt as a private draft, perhaps feeling his system hadn't yet achieved far enough results to his liking; it was published only in 1903 by the diligent editor Couturat (1903), decades after the results of Boole (1847, p. 21) and Cayley (1871), taken in combination, were published containing the algebraic representations of categorical assertions in the LC system⁵³.

as 'non est Ens' or 'est impossibilis' [...].

[...]

Reinforcing this preference for a logic of possibility over a logic of actuality, Leibniz also justified the validity of "conversion by limitation (*per accidens*)" (an Aristotelic logic law akin to "subalternation") in terms of possibility (Leibniz, ca. 1691a, p. 101)(Leibniz, ca. 1691b, p. 211-212).

We bother providing our unorthodox interpretation because our goal is to extract from Leibniz's manuscripts concepts and tools that lead us to our modern algebra of categorical syllogisms rather than understanding Leibniz's logic on its own terms.

⁵³We stress that Cayley explicitly called attention to the correspondence between "**a**I**b**" and "**ab** $\neq \emptyset$ ", and that of "**a**O**b**" and "**ab**" $\neq \emptyset$ ". Leibniz arguably pioneered such correspondences only for those –e.g. Couturat (1901, p. 358, point 17), Marciszewski (1984, pp. 527,532) and Sotirov (1999, p. 199)– who perform a reconstruction of his logic that treats "*est*" as synonym to "*aequivalent*" ("="), "*non-Ens*" as the empty *class*, and forces the leap "*est Ens* $\mid=\mid$ *non est non-Ens*" for particular categorical assertions, which Leibniz **didn't** perform in the drafts we consulted*.

Both Boole and Cayley algebraically represented " $\mathbf{b} \to \mathbf{c}$ " as " $\mathbf{b} = \boldsymbol{\varrho}$ ". In Boolean algebra (Section 8.2.1), we can derive the LC representation as follows:

= $\mathbf{bc} \cup \mathbf{bc'} = \mathbf{\emptyset} \cup \mathbf{bc'}$ $bc = \emptyset$ |=| $\mathbf{b}(\mathbf{c} \cup \mathbf{c}') = \mathbf{b}\mathbf{c}'$ $\mathbf{bI} = \mathbf{bc}$ $\mathbf{b} = \mathbf{b}\mathbf{c}'$ And the converse: $\mathbf{b}=\mathbf{b}\mathbf{c}'$ |=bc = (bc')c|=|bc = b(c'c)|=| $bc = b\emptyset$ |=| $\mathbf{bc} = \mathbf{\emptyset}$ Therefore,

 $\mathbf{bc} = \mathbf{\emptyset} \quad |\mathbf{c}| = \mathbf{bc'}.$

(*) It is easy to be misled because the copular verb "*est*" is polysemous. Leibniz himself took advantage of –and sometimes was confused (Lenzen, 2018a, pp. 67-68,74) by– the reuse of "*est*" with different meanings (Levey, 2011, pp. 118-119)(Lenzen, 1986)(Rescher, 1954, pp. 4,9).

Regarding uses of "*est Ens*" with distinct meanings, Leibniz on some occasions explicitly employed "*Ens*" and "*Nihil*" ("nothing"/"empty") as complementary, such as in the following excerpt on metaphysics, discussed in detail by Koszkało (2017, pp. 14-16):

"Essentia ablata existentia aut est ens reale aut nihil. Si nihil, aut non fuit in creaturis, quod absurdum; aut non distincta ab existentia fuit, quod intendo."

("Essence taken away from existence is **either a real entity or nothing**. If it is nothing, either it was not in creatures, which is absurd; or it was not distinct from existence, which I intend.")

^[...] on the whole, there is overwhelming evidence showing that Leibniz expresses the possibility-operator 'A est possibile' equally by means of 'A est Res', 'A est Ens' or even 'A est'. [...]»

Lenzen (2004b, p. 94) reinforces this assertion of the synonymy of "est", "est Ens", "est res" and "est possibile", and he complements the previous quote with a short commentary (2004a, p. 74) on some metaphysical goals to which Leibniz applied his brand of modal reasoning.

This historical accident might lead to the impression that the tradition of the algebra of logic was pioneered by Boole in the chronology of publications, if not in the chronology of ideas. However, some other drafts on logic by Leibniz were published in a book (Erdmann, 1840) years before Boole's 1847 pioneering treatise on logic. In particular, a draft by Leibniz (ca. 1691a; ca. 1691b) contained the following excerpt (the symbolic decoding **highlighted** inside square brackets is ours):

«Reductio mea vetus talis fuit:

Universalis Affirmativa: Omne A est B $[\mathbf{a} A \mathbf{b}:]$, id est acquivalent AB et A $[\mathbf{a}\mathbf{b} = \mathbf{a}]$ seu $[\mathbf{or}]$ A non B est non-Ens $[\overline{\mathfrak{I}}\langle \mathbf{a}\mathbf{b}' \rangle]$.

Particularis Negativa: Quoddam A non est B $[\mathbf{a} O \mathbf{b}:]$ seu non aequivalent AB et A $[\mathbf{a}\mathbf{b}\neq\mathbf{a}]$ seu $[\mathbf{or}]$ A non B est Ens $[\Im(\mathbf{a}\mathbf{b}')]$.

At Universalis Negativa: Nullum A est B $[\mathbf{a} \mathbf{E} \mathbf{b}:]$, erit AB est non-Ens $[\overline{\mathfrak{I}}\langle \mathbf{a} \mathbf{b} \rangle]$.

Et Particularis Affirmativa: Quoddam A est B [a I b:], erit AB est Ens $[\Im(ab)]$.

[...] acquivalent AB et BA [ab = ba]. [...]»

Unlike the previous excerpt, here Leibniz didn't adopt the symbols " ∞ " and "non ∞ ", preferring instead to write "aequivalent" and "non aequivalent" in full. Notice that, again, before "Ens" and "non-Ens" he adopted neither " ∞ " nor "aequivalent", but the verb "est", certainly to stress the distinction between the [in]equality relation and the relation indicated by "est [non-]Ens".

In this excerpt, Leibniz correctly proposed the following representations⁴⁹:

• $\mathbf{a} \mathbf{A} \mathbf{b}$: $\mathbf{a} \mathbf{b} = \mathbf{a}$; $\overline{\mathfrak{I}} \langle \mathbf{a} \mathbf{b}' \rangle$

• $\mathbf{a} \to \mathbf{b}: \overline{\mathfrak{I}} \langle \mathbf{a} \mathbf{b} \rangle$

- a I b: $\Im \langle ab \rangle$
- $\mathbf{a} \mathbf{O} \mathbf{b}$: $\mathbf{a} \mathbf{b} \neq \mathbf{a}$; $\Im \langle \mathbf{a} \mathbf{b}' \rangle$.

For our purposes, in this excerpt Leibniz only missed the representation of " $\mathbf{a} \to \mathbf{b}$ " that would be required for a pure Leibniz's system compatible with the axioms in Section 6.2: " $\mathbf{ab'} = \mathbf{a}$ ". But the main point is that Leibniz's research program on categorical syllogistic, some algebraic representations and laws, and some attempts at a complete system for proving all the classic categorical syllogisms had already been published –by the editor Erdmann (1840)– years before Boole's 1847 book. This excerpt alone would suffice, in our view, to establish Leibniz as the founding master of the *algebra* of logic in the chronology of publications too – and not only in the chronology of ideas.

It is also worth it to point out that, despite doing his research almost two centuries earlier, Leibniz went farther than Boole (1847) towards LC in this excerpt. He correctly identified a proper use of " \neq " in term logic (for instance, **a** O **b**: **ab** \neq **a**, as the quote shows), whereas Boole only employed "=" in his logic.

In other excerpts where Leibniz gets it wrong (from the perspective of extensional term logic *without* existential import) by employing "= an inhabited term" instead of " $\neq \boldsymbol{\varnothing}$ ", Boole miserably fails in the same way: Leibniz sometimes employed the letter "Y", "Z" or "W" to stand for a not-yet-determined class –e.g. in (Leibniz, 1690b, p. 234;

We can also exercise our creativity and explore the polysemy of "est" by performing the following loose (and historically inaccurate) interpretations:

	Leibniz, Lac	ld	McColl, Cayley		
Universal	bc « <i>non</i> est [Ens]».	$\overline{\mathfrak{I}}^{\langle \dots \rangle}_{\langle \mathbf{bc} \rangle, \mathbf{bc} \mathbf{M} }$	bc «est» non-Ens.	$\mathbf{bc} \subseteq \boldsymbol{\emptyset}, \mathbf{bc} = \boldsymbol{\emptyset}$	
Particular	bc « est [Ens]».	$\Im \langle \mathbf{bc} \rangle, \mathbf{bc} \in \mathbb{N}$	bc « <i>non</i> est» non-Ens.	$\mathbf{bc} \not\subseteq \boldsymbol{\varnothing}, \mathbf{bc} \not= \boldsymbol{\vartheta}$	

Different choices of primary notion: «est Ens» ("exists") vs. «est» non-Ens ("is empty").

p. 236, point 13)–, whereas Boole (1847, pp. 21–22) usually represented an arbitrary inhabited class by the letter "v". This corroborates Couturat's remark (1901, p. 386) that Leibniz possessed almost all principles of the Boole-Schröder logic, and in some points he was even more advanced than Boole himself⁵⁴.

One of the likely reasons why both Leibniz and Boole insisted on employing "=" to particular assertions was to preserve the validity of the "subalternation" laws⁵⁵

 $\mathbf{b} \mathbf{A} \mathbf{c} \models \mathbf{b} \mathbf{I} \mathbf{c}$

 $\mathbf{b} \to \mathbf{c} \models \mathbf{b} \to \mathbf{c}$

from Aristotelic logic – after all,

 $\mathbf{bc} = \mathbf{b} \mid = \mathbf{bc} = v$

 $\mathbf{bc'=b} \mid = \mathbf{bc'=}v$

make some sense in algebraic reasoning⁵⁶. (For some supporting evidence, see Leibniz (1690b, p. 234), Leibniz (ca. 1691b, pp. 213-214), Leibniz (ca. 1691a, p. 102), Couturat (1901, pp. 358-361, point 18)⁵⁷, Boole (1847, pp. 21-25) and Jevons (1864, pp. 55, points 140-141; 57, point 144).)

Boole's desire to preserve existential import for all categorical assertions⁵⁸ may have encouraged him to think of inhabitation as primary and of emptiness (non-inhabitation) as a derived, subordinate notion in term logic. This seems to be a reasonable and pragmatic choice of default condition for terms at first sight when we consider, like Aristotle, that we most often care to reason about existent things in the world, not nonexistent ones. However, a mathematical fact is that there are infinitely many inhabited classes, but (extensionally) a unique empty class⁵⁹: his "v" doesn't have a well-determined referent, whereas " $\boldsymbol{\varnothing}$ " does. Thus, equality ("=") and difference (" \neq ") assertions are well-defined with " $\boldsymbol{\varnothing}$ ", but not with "v". Algebraic manipulations in LC benefit from this important property of " $\boldsymbol{\varnothing}$ " by considering emptiness as the primary notion, and inhabitation as a synonym for non-emptiness, thus subverting our initial disposition

⁵⁵An opinion we share with Marciszewski and Murawski (1995, p. 140).

 56 In first-order quantificational logic, it corresponds to (an instantiation of) the *existential introduction* axiom:

 $\mathbf{b} \in \mathcal{P}(\mathbf{I}) \mid = \exists v. \ v \in \mathcal{P}(\mathbf{I})$

where $\mathcal{P}(\mathbf{I})$ is the powerclass of \mathbf{I} .

⁵⁷Despite Couturat's strong stance against existential import of universal categorical assertions.

 $^{59}\mbox{Leibniz}$ (ca. 1686d, points 15-22,28-30,39) knew this; he enunciated this and other important facts about the empty class.

Since the origination of axiomatic set theory (if not earlier), the fact that the empty class is unique is stated as a theorem (with various proofs), not as an axiom.

By the Boolean lattice axioms shown in Section 8.2.1, the empty class is the identity element with respect to the union operation. One can easily prove that the identity element associated to a dyadic operation is always unique. See https://proofwiki.org/wiki/Identity_is_Unique.

It is also easy to prove that the empty class is unique by employing the properties of " \subseteq ", or alternatively by employing the extensionality axiom from set theory. See "https://proofwiki.org/wiki/Empty_Set_is_Unique".

 $^{^{54}}$ On the other hand, Leibniz was in hindsight too conservative, clinging too much to the traditional Aristotelic paradigm of grammatically inspired *term* logic. This may seem ironic, given that the algebra of logic he pioneered would surely be considered by 17th-century logicians a radical innovation in his time, had he published his drafts – and indeed it was. However, it wasn't as predominantly symbolic as the 19th-century tradition initiated by Boole. Even for the algebraic logician Leibniz, logic still was more philosophical than algebraic, given features such as the handling of many sentences in verbose prose (instead of adopting purely symbolic representations), his decades-long dedication to the study of the logical meanings of "*est*" ("is"), the absence of the insight of dealing with inhabited terms as extensionally non-empty ("*non est non Ens*"), his bias toward intensionality and even modality, and the stubborn conservation of existential import for universal categorical assertions.

 $^{^{58}}$ Boole (1847, pp. 26-30) embraced "conversion by limitation or *per accidens*" of "A" into "I" and of "E" into "O". This gives rise to difficulties and to a clunky algebraic system for categorical syllogistic, as shown by Makinson (2022, pp. 168-169,171).

for considering inhabitation as a primitive notion⁶⁰. In addition, Boole's obsession with equational reasoning may have led him to overlook that, just like "**b** E **c**" is contradictory to "**b** I **c**", "**b c** = $\boldsymbol{\varnothing}$ " is contradictory to "**b c** = $\boldsymbol{\vartheta}$ ", not to "**b c** = $\boldsymbol{\vartheta}$ ". Cayley (1871) realized what Boole (1847; 1854) overlooked⁶¹.

For comparison, if we want to preserve the validity of the subalternation laws in LC, they would have to be respectively expressed – at the cost of a more complicated formula, roughly following Venn (1881, pp. 167-168) – as

 $\mathbf{bc} = \mathbf{b}, \ \mathbf{b} \neq \boldsymbol{\varnothing} \quad \mid = \quad \mathbf{bc} \neq \boldsymbol{\varnothing}$

$$\mathbf{b}\mathbf{c}'=\mathbf{b}, \ \mathbf{b}\neq\mathbf{\emptyset} \quad \mid = \quad \mathbf{b}\mathbf{c}'\neq\mathbf{\emptyset}$$

Alternatively, by transposing the latter premise:

 $\mathbf{bc'=b} \mid = \mathbf{b} \neq \boldsymbol{\varnothing} \bowtie \mathbf{bc'} \neq \boldsymbol{\varnothing}$ {where " \bowtie " is the meta-level "or"}.

Thus, we would have to

- adopt a definition of **b** A **c** which explicitly adds existential import of the subject as a constraint (so the definition of **b** A **c** would have to be composed of both left-hand-side premises); and also
- either add existential import of the subject to the definition of **b** E **c** –like McColl (1877, p. 180, rule 18) did⁶²– or abandon existential import of the subject for the definition of **b** O **c** –like Tamaki (1974, pp. 191-192) did. Pick your poison.

Alternatively, we could adopt algebraic definitions for " $\mathbf{b} \mathbf{A} \mathbf{c}$ ", " $\mathbf{b} \mathbf{E} \mathbf{c}$ " and " $\mathbf{b} \mathbf{O} \mathbf{c}$ " formed by a single premise only, with the consequence that the subalternation "laws" would no longer be universally applicable, but subject to an additional inhabitation

- *affirmative*, having the "=" copula meaning "equals", "extensionally coincides with", or, in Jevons' parlance, "is the same as".
- *negative*, having the "≠" copula meaning "does *not* equal", "does *not* extensionally coincide with", or "is *not* the same as".

One can intuitively understand Jevons' point of view, which generalizes two important relations from numerical algebra (Jevons, 1869, pp. 5,8,15-26,73)(Jevons, 1864, p. 7, point 15; p. 6, point 13; p. 86, point 203). Cayley's insight, however, is that, in the traditional jargon of categorical syllogistic, sentences with these copulae don't correspond to affirmative and negative ones, but respectively to *universal* and *particular* ones – an insight later explained in prose by Venn (1883, p. 596). In categorical syllogistic jargon, "affirmative" versus "negative" is a distinction in "quality" (in contrast to "quantity") revealed to be about (predicative) obversion –with some resemblance to the opposition of qualities in Jevons (1864, p. 83, point 193)–, and thus these adjectives as used in categorical syllogistic have different meanings from Jevons' usage.

Indeed, considering yet another distinct meaning, Jevons' "affirmative" sentences are in another sense always "negative", if by "negative" we now mean that they assert a **non**existence stance $(b = \mathbf{0} / \overline{\mathfrak{I}} \langle b \rangle)$ (Jevons, 1864, p. 71, point 167), as Brentano (1874, pp. 283-286)(Land, 1876) later noticed. Likewise, "negative" sentences in Jevons' sense are in another sense always "affirmative": they assert something does exist ($b \neq \mathbf{0} /$ $\mathfrak{I} \langle b \rangle$). (Which should be the "primitive/affirmative/default" notion: emptiness/nonexistence ("est Nihil") or inhabitation/existence ("est Ens")? Revisit the table in footnote⁵³.)

Given the three distinct meanings of "affirmative"/"negative", we should always make it clear which one we are referring to.

 62 Storrs McCall (1967, p. 349,347-348) explained that, to go from Barbara-1 to Barbari-1 without adding any further premise, we should accept the validity of A \rightarrow I subalternation, which in a symbology from Table 1b would be

 $\mathbf{s} \subseteq \mathbf{p} \quad \mid = \quad \mathbf{s} \nsubseteq \mathbf{p}'$

("est P" est "non est non P", Boethius' connexive thesis),

and he discussed the difficulties caused by this.

Hugh McColl (1877, p. 180, rule 18) also embraced this thesis, which entails existential import for universal categorical assertions.

⁶⁰Notice that, whereas a purely algebraic treatment with "= $\boldsymbol{\varnothing}$ " and " $\neq \boldsymbol{\varnothing}$ " requires subverting Aristotle's reasonable choice of primary notion, Leibniz's relation " \mathfrak{I} " preserves it, as the table in footnote⁵³ shows.

⁶¹Leibniz (ca. 1686f, p. 94), and most emphatically Jevons (1864, pp. 2-4,8-13,29-30), in the pamphlet that founded modern Boolean algebra, anticipated Cayley (1871) in identifying the importance of the negation of "=" –namely, the " \neq " relation– for general deductive reasoning (though not for the representation of particular categorical relations with " $\boldsymbol{\varnothing}$ "). Jevons divided relations into:

constraint for the subject – as we do in this paper.

6.2. Leibniz-Cayley system axioms

In order to prove all the classic categorical syllogisms, we have extracted from Leibniz's drafts on logic the following axioms to form LC:

The following is a convenient lemma to shorten the proofs of some valid mood/figure pairs:

(LC6) $\begin{array}{c} bc \neq \boldsymbol{\varnothing} \mid = b \neq \boldsymbol{\varnothing} \\ \text{Proof: } bc \neq \boldsymbol{\vartheta} \mid = cb \neq \boldsymbol{\vartheta} \mid b \neq \boldsymbol{\vartheta} \end{array} \quad \{ \text{sub} \$

{subject inhabitation}

Thus, it is fair to say that by the end of 1690 Leibniz had already figured out all the laws required to work as axioms of an algebraic system of categorical syllogistic – a research he seriously undertook from at least 1679 (Leibniz, 1679b, pp. 43-44) on⁶⁷.

Regarding subject inhabitation (LC6) and predicative inhabitation (LC3), each one

 $^{64} \mathrm{In}$ Leibniz's manuscripts on logic, we failed to find a direct assertion like

A Nihil = Nihil $\{a \boldsymbol{\varnothing} = \boldsymbol{\varnothing}\},\$

although we cannot discard it is present in some form somewhere. Nevertheless, we can deduce it from other assertions scattered throughout his manuscripts:

 $a \cup \emptyset = a$ (Leibniz, ca. 1686d, p. 267, point 28)

 $\pmb{\varnothing} \cup a = a \ \{ b \cup n = n \cup b \ (\text{Leibniz, ca. 1687, p. 237, Axiom. 1}) \}$

 $\mathcal{Q} \subseteq a \{ a \cup y = c \mid = \mid a \subseteq c \text{ (Leibniz, ca. 1686d, p. 265, points 9-10)} \}$

 $\boldsymbol{\varnothing}a = \boldsymbol{\varnothing} \{a \subseteq b \mid = \mid ab = a \text{ (Leibniz, 1690b, p. 236, point 13)} \}$

 $a \mathbf{\emptyset} = \mathbf{\emptyset} \{ ab = ba \text{ (Leibniz, 1690b, p. 235, point 7)} \}$

From this, we can deduce the predicative inhabitation law:

 $b \mathbf{\emptyset} = \mathbf{\emptyset}$ {placeholder relabelling: $a \leftarrow b$ }

 $c = \emptyset \mid = bc = \emptyset$ {substitution of similars}

 $bc \neq \mathbf{\emptyset} \models c \neq \mathbf{\emptyset} \{ \text{transposition (from propositional logic)} \}.$

In Leibniz's original system, which adopts for particular assertions the representations " $\Im(s)$ " and " $\overline{\Im}(s)$ " instead of " $s \neq \mathbf{0}$ " and " $s = \mathbf{0}$ " respectively, we can prove an equivalent law to LC3, though the proof is different. Unlike LC, Leibniz (ca. 1686c, point 5) assumed existential import of the generating terms: "A est, id est A est Ens." (" $\Im(a)$ "). Thus,

 $\mid = \Im \langle c \rangle$ {existential import}

 $\Im \langle bc \rangle \mid = \Im \langle c \rangle$ {antecedent introduction before a true consequent}.

In a system like Leibniz's original one but without existential import, one could simply declare the latter law as an alternative axiom to LC3 by fiat, diagrammatically justified by Figure 5b, although if we didn't also assume " $\Im(s) \models |s \neq \emptyset$ ", the axiom would be wanting of a satisfactory *algebraic* justification. Ladd (1883, p. 34) also states this law.

⁶⁵In the representation of categorical assertions by means of single vowels, it corresponds to "A-contraposition" ($\mathbf{b} \mathbf{A} \mathbf{c} \mid = \mid \mathbf{c}' \mathbf{A} \mathbf{b}'$). It symbolically represents contravariance/antitonicity of subclasshood under complementation.

⁶⁶In the representation of categorical assertions by means of single vowels, it corresponds to "E-conversion" ($\mathbf{b} \in \mathbf{c} \mid = \mid \mathbf{c} \in \mathbf{b}$). Leibniz stated that $(\mathbf{b}')' = \mathbf{b}$, which, together with the LC4 axiom, would suffice to deduce LC5 as a theorem.

By adopting both subsumption contraposition and disjointness conversion as axioms, we make the involution law, $(\mathbf{b}')' = \mathbf{b}$, superfluous for the strict purpose of proving classical categorical syllogisms.

 67 LC1 and LC2 are two of the three algebraic axioms of semilattices (Section 8.2.1). Interestingly, idempotence (bb = b), which Leibniz (1690b, p. 235, point 6) also explicitly stated, is not required in LC. These three laws, together with LC3, are the bounded (meet-)semilattice axioms.

⁶³We haven't found an explicit statement of this law by Leibniz, although he implicitly made use of it in a proof (Leibniz, ca. 1690, pp. 229-230, Axioma 1). The earliest explicit statement we could find of the associative law in an algebra is by William Rowan Hamilton (1843, p. 430). In the context of the algebra of logic, it was explicitly stated by Peirce (1867, p. 251).

can be proved from the other by using commutativity (LC1). Thus, any of them might have been chosen as an axiom. The motivation for LC choosing LC3 rather than LC6 is explained in Section 6.4.

Notice that LC didn't adopt axioms of subalternation:

 $bc = b \quad |=| \quad bc \neq \emptyset$

 $bc' = b \quad |=| \quad bc' \neq \emptyset,$

which would imply (by LC6) existential import for universal categorical assertions: bc = b |=| $b \neq \emptyset$

 $bc' = b \quad |=| \quad b \neq \emptyset.$

The lack of this axiom makes LC fully compatible to Boolean algebra, as we will prove in Section 8.2.2. However, due to this intentional omission, 9 of the 24 classic categorical syllogisms require not just two but three premises for them to be valid in this system, just like in the Euler system (Section 3).

These axioms stand on their own as relations between formal, abstract objects which can be manipulated according to the inference rules the system is subject to. In this purely formal sense, the operands can be seen as just "terms" (in mathematical expressions), "names" or individuating, arbitrary "labels". Given these axioms, a skilled middle-school algebra student could trace and maybe even derive on her own the proofs of our theorems without being informed of the context (what these operands and operations refer to). If we had no intended direction and just wanted to derive the theorems of any formal system to see where they lead us to, the chosen axioms would be as arbitrary as any other. For us, the legitimacy of the system as a *logic* which we are interested in studying is justified by the fact that the Euler system, involving *classes* represented by the Euler (and Venn) diagrams shown in Section 3 and by Piesk (2017), is a fully compliant model or interpretation of the axiomatic system we defined, since we want to explore different approaches to prove the same theorems, the classic categorical syllogisms without existential import on universal assertions.

6.3. Syllogism proofs in the Leibniz-Cayley system

Here are the proofs of categorical syllogisms in LC. None of them has more than 5 steps.

Syllo	gism 1.	Barbara-1:
P1.	sm = s	{Every s is m.}
P2.	mp = m	{Every m is p.}
C1.	sp = s	{Every s is p.}
Pro	OF.	
(by L	eibniz (ca. 1	690, pp. 229-230, Axioma 1),
once	we make ass	ociativity explicit:)
S3.	s(mp)=s	(P1), (P2)
S4.	(sm)p = s	(S3), $\begin{cases} b(cd) = (bc)d \\ b \leftarrow s, c \leftarrow m, d \leftarrow p \end{cases}$
S5 .	$\mathbf{sp}=\mathbf{s}$	(S4), (P1). Therefore: (C1).

Syllo	gism 2.	Barbari-1:
P1.	sm = s	{Every s is m.}
P2 .	mp = m	{Every m is p.}
P3.	$s \neq \mathbf{Ø}$	$\{At \text{ least one s exists.}\}$
C1.	sp = s	{Every s is p.}
C2.	$sp eq \mathbf{Ø}$	$\{At \text{ least one s is p.}\}$
Pro	OF.	
S4 .	s(mp) = s	(P1), (P2)
S 5.	(sm)p = s	(S4), $\begin{cases} b(cd) = (bc)d \\ b \leftarrow s, c \leftarrow m, d \leftarrow p \end{cases}$
S6.	$\mathbf{sp}=\mathbf{s}$	(S5), (P1). Therefore: (C1).
S7 .	$\mathbf{sp} \neq \boldsymbol{\varnothing}$	(S6), (P3). Therefore: (C2).

Syllogism 3. Celarent-1:

Р1. Р2.	sm = s mp' = m	{Every s is m.} {No m is p.}
C1.	sp' = s	$-\{No s is p.\}$
Pro	OF.	

(by Whitehead (1898, pp. 102), once we make the use of associativity explicit:)

S 3.	s(mp') = s	(P1), (P2)
S4 .	(sm)p' = s	(S3), $\begin{cases} b(cd) = (bc)d \\ b \leftarrow s, c \leftarrow m, d \leftarrow p' \end{cases}$
S5 .	$\mathbf{sp}' = \mathbf{s}$	(S4), (P1). Therefore: (C1)

* * *

Syllogism 4. Celaront-1:

v	0	
P1.	sm = s	{Every s is m.}
P2.	mp' = m	{Nom is p.}
P3.	$s \neq \mathbf{Ø}$	{At least one s exists.}
C1.	sp' = s	{No s is p.}
C2.	$sp' eq {m arnothing}$	$\{At \text{ least one s is not p.}\}$
Pro	OF.	
S4 .	s(mp')=s	(P1), (P2)
S 5.	(sm)p' = s	(S4), $\begin{cases} b(cd) = (bc)d \\ b \leftarrow s, c \leftarrow m, d \leftarrow p \end{cases}$
S6.	$\mathbf{sp}' = \mathbf{s}$	(S5), (P1). Therefore: (C1)
S7.	$\mathbf{sp}' \neq \boldsymbol{\varnothing}$	(S6), (P3). Therefore: (C2)

* * *

Syllogism 5. Camestres-2:

P1.	sm' = s	{No s is m.}
P2.	pm = p	{Every p is m.}
C1.	sp' = s	{No s is p.}
Pro	OF.	
S 3.	m'p'=m'	(P2), $\begin{cases} bc = b \mid = \mid c'b' = c' \\ b \leftarrow p, c \leftarrow m \end{cases}$
S4 .	s(m'p')=s	(P1), (S3)
S 5.	(sm')p'=s	(S4), $\begin{cases} b(cd) = (bc)d \\ b \leftarrow s, c \leftarrow m', d \leftarrow p' \end{cases}$
S6.	$\mathbf{s}\mathbf{p}'=\mathbf{s}$	(S5), (P1). Therefore: (C1).
		* * *

Syllogism 6. Camestros-2:

•	0	
P1.	sm' = s	{No s is m.}
P2.	pm = p	{Every p is m.}
P3.	$s \neq \mathbf{Ø}$	$\{At \text{ least one s exists.}\}$
C1.	sp' = s	{No s is p.}
C2.	$sp' eq {m arnothing}$	$\{At \text{ least one s is not p.}\}$
Pro	OF.	
S4.	m'p'=m'	(P2), $\begin{cases} bc = b \mid = \mid c'b' = c' \\ b \leftarrow p, c \leftarrow m \end{cases}$
S5 .	s(m'p')=s	(P1), (S4)
S6.	(sm')p'=s	(S5), $\begin{cases} b(cd) = (bc)d \\ b \leftarrow s, c \leftarrow m', d \leftarrow p' \end{cases}$
S7.	$\mathbf{sp}' = \mathbf{s}$	(S6), (P1). Therefore: (C1).
S8.	$\mathbf{sp}' \neq \boldsymbol{\varnothing}$	(S7), (P3). Therefore: (C2).
		* * *

Syllogism 7. Bamalip-4:

•	-	-
P1.	ms = m	{Every m is s.}
P2.	pm = p	{Every p is m.}
P3.	$p eq {m arnothing}$	{At least one p exists.}
C1.	$sp \neq \mathbf{Ø}$	${At least one s is p.}$
Pro	OF.	
S4 .	p(ms) = p	(P2), (P1)
S 5.	(pm)s = p	(S4), $ \begin{cases} b(cd) = (bc)d \\ b \leftarrow p, c \leftarrow m, d \leftarrow s \end{cases} $
S6.	ps = p	(S5), (P2)
S7.	sp = p	$(S6), \begin{cases} bc = cb \\ b \leftarrow p, c \leftarrow s \end{cases}$
S8.	$\mathbf{sp}\neq \boldsymbol{\varnothing}$	(S7), (P3). Therefore: (C1)
		* * *

Syllogism 8. Darapti-3:

	0	1
P1.	ms = m	{Every m is s.}
P2.	mp = m	{Every m is p.}
P3.	$m \neq \mathbf{Ø}$	$\{At \text{ least one } m \text{ exists.}\}$
C1.	$sp eq {m arnothing}$	${}^{\text{At least one s is p.}}$
Pro	DOF.	
S4 .	(ms)p = m	(P2), (P1)
S 5.	m(sp) = m	(S4), $\begin{cases} b(cd) = (bc)d \\ b \leftarrow m, c \leftarrow s, d \leftarrow p \end{cases}$
S6.	$m(sp)\neq \textit{Ø}$	(S5), (P3)
S7.	$\mathbf{sp} eq m{arphi}$	$ \begin{cases} bc \neq \boldsymbol{\varnothing} \mid = c \neq \boldsymbol{\varnothing} \\ b \leftarrow m, c \leftarrow sp. \end{cases} $ Therefore: (C1).

* * *

Syllogism 9. Felapton-3:P1. ms = m {Every m is s.}P2. mp' = m {No m is p.}P3. $m \neq \emptyset$ {At least one m exists.}C1. $sp' \neq \emptyset$ {At least one s is not p.}PROOF.S4. (ms)p' = m (P2), (P1)S5. m(sp') = m (S4), $\begin{cases} b(cd) = (bc)d \\ b \leftarrow m, c \leftarrow s, d \leftarrow p' \end{cases}$ S6. $m(sp') \neq \emptyset$ (S5), (P3)S7. $sp' \neq \emptyset$ (S6), $b \leftarrow m, c \leftarrow sp'$. Therefore: (C1).

* * *

Syllo	gism 11.	Bokardo-3:
P1.	ms = m	{Every m is s.}
P2.	$mp' eq m{ extsf{ extsf extsf{ extsf} extsf{ extsf{ extsf{ extsf} extsf{ extsf} extsf{ extsf{ extsf{ extsf{ extsf} extsf{ extsf{ extsf{ extsf} extsf{ extsf} extsf{ extsf} extsf{ extsf} extsf{ ex}$	$\{At \text{ least one } m \text{ is not } p.\}$
C1.	$sp' eq \mathbf{Ø}$	${}^{\text{At least one s is not p.}}$
Pro	OF.	
S 3.	$(ms)p' \neq 0$	(P1), (P2)
S 4.	$m(sp')\neq \textit{Ø}$	(S3), $ \begin{cases} b(cd) = (bc)d \\ b \leftarrow m, c \leftarrow s, d \leftarrow p' \end{cases} $
S5.	$\mathbf{sp}' \neq \boldsymbol{\varnothing}$	$ \begin{array}{l} \{bc \neq \pmb{\varnothing} \mid = c \neq \pmb{\varnothing}\} \\ (S4), b \leftarrow m, c \leftarrow sp'. \\ \text{Therefore: (C1).} \end{array} $

* * *

Syllo	gism 12.	Darii-1:
P1.	$sm \neq \emptyset$	$\{At \text{ least one s is m.}\}$
P2.	mp = m	{Every m is p.}
C1.	$sp \neq \mathbf{Ø}$	${At least one s is p.}$
Pro	OF.	
S 3.	$s(mp)\neq \pmb{\varnothing}$	(P1), (P2)
S4.	$s(pm)\neq \pmb{\varnothing}$	(S3), $\begin{cases} bc = cb \\ b \leftarrow m, c \leftarrow p \end{cases}$
S 5.	$(sp)m \neq \mathbf{Ø}$	(S4), $ \begin{cases} b(cd) = (bc)d \\ b \leftarrow s, c \leftarrow p, d \leftarrow m \end{cases} $
S6.	$\mathbf{sp} \neq \boldsymbol{\varnothing}$	$ \begin{array}{l} \{bc \neq \varnothing \mid = b \neq \varnothing\} \\ (\text{S5}), b \leftarrow sp, c \leftarrow m. \\ \text{Therefore: (C1).} \end{array} $
		* * *

Syllogism 13. Ferio-1:

P1.	$sm eq \mathbf{Ø}$	{At least one s is m.}
P2.	mp' = m	{Nom is p.}
C1.	$sp' eq {m arnothing}$	${At least one s is not p.}$
Pro	OF.	
S 3.	$s(mp')\neq \pmb{\varnothing}$	(P1), (P2)
S4 .	$s(p'm) \neq \mathbf{Ø}$	(S3), $\begin{cases} bc = cb \\ b \leftarrow m, c \leftarrow p' \end{cases}$
S 5.	$(sp')m\neq {\it \varnothing}$	(S4), $\begin{cases} b(cd) = (bc)d \\ b \leftarrow s, c \leftarrow p', d \leftarrow m \end{cases}$
S6.	$\mathbf{sp}' \neq \boldsymbol{\varnothing}$	$ \begin{array}{l} \{bc \neq \pmb{\varnothing} \mid = b \neq \pmb{\varnothing}\}\\ (\text{S5}), b \leftarrow sp', c \leftarrow m.\\ \text{Therefore: (C1).} \end{array} $

* * *

Syllogism 14. Baroko-2:

P1.	$sm' eq \mathbf{Ø}$	$\{At \text{ least one s is not m.}\}$
P2.	pm = p	{Every p is m.}
C1.	$sp' eq {m arnothing}$	${At least one s is not p.}$
Pro	OF.	
S3.	m'p'=m'	(P2), $\begin{cases} bc = b \mid = \mid c'b' = c' \\ b \leftarrow p, c \leftarrow m \end{cases}$
S4 .	$s(m'p')\neq \textit{Ø}$	(S3), (P1)
S 5.	$s(p'm')\neq \pmb{\varnothing}$	(S4), $\begin{cases} bc = cb \\ b \leftarrow m', c \leftarrow p' \end{cases}$
S6.	$(sp')m' \neq 0$	(S5), $ \begin{cases} b(cd) = (bc)d \\ b \leftarrow s, c \leftarrow p', d \leftarrow m' \end{cases} $
S7.	$\mathbf{sp}' eq \mathbf{arnothing}$	$ \begin{cases} bc \neq \boldsymbol{\varnothing} \mid = b \neq \boldsymbol{\varnothing} \\ b \leftarrow s, c \leftarrow p'. \\ \text{Therefore: (C1).} \end{cases} $
		* * *

Syllogism 15. Dimatis-4: P1. ms = mP2. $pm \neq \emptyset$ {Every m is s.} {At least one p is m.} C1. $sp \neq \emptyset$ ${At least one s is p.}$ Proof. $(\mathbf{P2}), \ \begin{cases} bc = cb \\ b \leftarrow p, c \leftarrow m \end{cases}$ P2a. $mp \neq \emptyset$ **S3.** $(ms)p \neq \emptyset$ (P2a), (P1) $m(sp) \neq \pmb{\varnothing} \quad (\mathrm{S3}), \ \begin{cases} b(cd) = (bc)d \\ b \leftarrow m, c \leftarrow s, d \leftarrow p \end{cases}$ **S4**. $\{bc \neq \mathbf{\emptyset} \mid = c \neq \mathbf{\emptyset}\}$ (S4), $b \leftarrow m, c \leftarrow c_r$. Therefore: (C1). **S5**. $b \leftarrow m, c \leftarrow sp.$ $\mathbf{sp} eq \mathbf{ extsf{ extsf extsf{ extsf} extsf{ extsf} extsf{ extsf{ extsf{ extsf{ extsf} extsf{ extsf{ extsf} extsf{ extsf{ extsf{ extsf{ extsf{ extsf{ extsf extsf} extsf{ extsf} extsf{ extsf} extsf{ extsf} extsf{ ex$

* * *

Syllogism 16. Datisi-3: {At least one m is s.} $\mathbf{\tilde{P1}}. \quad ms \neq \boldsymbol{\varnothing}$ **P1.** $ms \neq \emptyset$ **P2.** mp = m **C1.** $sp \neq \emptyset$ PROOF. {Every m is p.} _ {At least one s is p.} **S**3. $(mp)s \neq \emptyset$ (P1), (P2) $m(ps) \neq \emptyset \quad (S4), \ \begin{cases} b(cd) = (bc)d \\ b \leftarrow m, c \leftarrow p, d \leftarrow s \end{cases}$ **S4**. (S4), $\{bc \neq \emptyset \mid = c \neq \emptyset\}$ **S5**. $ps \neq \mathbf{Ø}$ $b \leftarrow m, c \leftarrow ps$ $\{bc = cb\}$ (S5), $b \leftarrow p, c \leftarrow s$. S6. $\mathbf{sp} \neq \boldsymbol{\varnothing}$ Therefore: (C1).

* * *

Syllogism 17. Ferison-3:

P1.	$ms eq \mathbf{Ø}$	$\{At \text{ least one } m \text{ is } s.\}$
P2.	mp' = m	{No m is p.}
C1.	$sp' \neq \mathcal{O}$	{At least one s is not p.}
Pro	OF.	
S 3.	$(mp')s\neq \textit{Ø}$	(P1), (P2)
S4 .	$m(p's)\neq \textit{Ø}$	(S3), $\begin{cases} b(cd) = (bc)d \\ b \leftarrow m, c \leftarrow p', d \leftarrow s \end{cases}$
S 5.	$p's eq \mathbf{Ø}$	(S4), $\begin{cases} bc \neq \boldsymbol{\varnothing} \mid = c \neq \boldsymbol{\varnothing} \\ b \leftarrow m, c \leftarrow p's \end{cases}$
S6.	$\mathbf{sp}' \neq \boldsymbol{\varnothing}$	$ \begin{cases} bc = cb \\ (S5), b \leftarrow p', c \leftarrow s. \\ Therefore: (C1). \end{cases} $

Syllo	gism 18.	Festino-2:
P1.	$sm \neq \mathbf{Ø}$	$\{At \text{ least one s is m.}\}$
P2.	pm' = p	{No p is m.}
C1.	$sp' eq {m extsf{ extsf extsf{ extsf} extsf{ extsf{ extsf{ extsf{ extsf} extsf{ extsf{ extsf} extsf{ extsf} extsf{ extsf} extsf{ extsf} extsf{ extsf} extsf{ extsf{ extsf{ extsf} extsf{ extsf{ extsf} extsf{ extsf} extsf{ extsf} extsf{ extsf} extsf} extsf} ex$	{At least one s is not p.}
Pro	OF.	
P2a.	mp' = m	(P2), $\begin{cases} bc' = b \mid = \mid cb' = c \\ b \leftarrow p, c \leftarrow m \end{cases}$
S3 .	$s(mp')\neq \pmb{\varnothing}$	(P1), (P2a)
S4 .	$s(p'm)\neq \pmb{\varnothing}$	(S3), $\begin{cases} bc = cb \\ b \leftarrow m, c \leftarrow p' \end{cases}$
S 5.	$(sp')m\neq {\it Ø}$	(S4), $ \begin{cases} b(cd) = (bc)d \\ b \leftarrow s, c \leftarrow p', d \leftarrow m \end{cases} $
S6 .	$\mathbf{sp}'\neq \boldsymbol{\varnothing}$	$\begin{cases} bc \neq \emptyset \mid = b \neq \emptyset \\ (S5), b \leftarrow sp', c \leftarrow m. \\ Therefore: (C1). \end{cases}$

* * *

Syllogism 19. Fresison-4:

P2. $pm' = p$	{No p is m.}
C1. $sp' \neq \emptyset$	${}^{-}{}{At least one s is not p.}$
Proof.	
P2a. $mp' = m$	$(P2), \begin{cases} bc' = b \mid = \mid cb' = c \\ b \leftarrow p, c \leftarrow m \end{cases}$
S3. $(mp')s \neq \emptyset$	(P1), (P2a)
S4. $m(p's) \neq \emptyset$	(S3), $\begin{cases} b(cd) = (bc)d \\ b \leftarrow m, c \leftarrow p', d \leftarrow s \end{cases}$
S5. $p's \neq \emptyset$	(S4), $\begin{cases} bc \neq \boldsymbol{\varnothing} \mid = c \neq \boldsymbol{\varnothing} \\ b \leftarrow m, c \leftarrow p's \end{cases}$
S6. $sp' \neq \emptyset$	$ \begin{cases} bc = cb \\ (S5), & b \leftarrow p', c \leftarrow s. \\ \text{Therefore: (C1).} \end{cases} $

* * *

Syllogism 20. Fesapo-4:

•	0	-
P1.	ms = m	{Every m is s.}
P2.	pm' = p	{No p is m.}
P3.	$m eq {m arnothing}$	{At least one m exists.}
C1.	$sp' \neq \mathbf{Ø}$	{At least one s is not p.}
Pro	OF.	
P2a.	mp' = m	$(P2), \begin{cases} bc' = b \ = \ cb' = c \\ b \leftarrow p, c \leftarrow m \end{cases}$
S4 .	(ms)p' = m	(P2a), (P1)
S 5.	m(sp') = m	(S4), $\begin{cases} b(cd) = (bc)d \\ b \leftarrow m, c \leftarrow s, d \leftarrow p' \end{cases}$
S6.	$m(sp') \neq \mathbf{Ø}$	(S5), (P3)
S7.	$\mathbf{sp}' eq oldsymbol{arphi}$	$ \begin{array}{l} \{bc \neq \varnothing \mid = c \neq \varnothing\} \\ (\text{S6}), b \leftarrow m, c \leftarrow sp'. \\ \text{Therefore: (C1).} \end{array} $
		* * *

Syllo	gism 21.	Cesare-2:
Ď1.	sm = s	{Every s is m.}
P2.	pm' = p	{No p is m.}
C1.	sp' = s	{Nos is p.}
Pro	OF.	
P2a.	mp' = m	(P2), $\begin{cases} bc' = b \mid = \mid cb' = c \\ b \leftarrow p, c \leftarrow m \end{cases}$
S3 .	s(mp')=s	(P1), (P2a)
S4 .	(sm)p' = s	$(\text{S3}), \begin{array}{l} \{b(cd) = (bc)d\} \\ b \leftarrow s, c \leftarrow m, d \leftarrow p' \end{array}$
S5 .	$\mathbf{s}\mathbf{p}'=\mathbf{s}$	(S4), (P1). Therefore: (C1).
		* * *
Syllo	gism 22.	Cesaro-2:
Syllo P1.	gism 22. sm = s	Cesaro-2: {Every s is m.}
Syllo P1. P2.	gism 22. sm = s pm' = p	Cesaro-2: {Every s is m.} {No p is m.}
Syllo P1. P2. P3.	gism 22. sm = s pm' = p $s \neq \emptyset$	Cesaro-2: {Every s is m.} {No p is m.} {At least one s exists.}
Syllo P1. P2. P3. C1.	gism 22. sm = s pm' = p $s \neq \emptyset$ sp' = s	Cesaro-2: {Every s is m.} {No p is m.} {At least one s exists.} {No s is p.}
Syllo P1. P2. P3. C1. C2.	$sm = s$ $pm' = p$ $s \neq \emptyset$ $sp' = s$ $sp' = s$ $sp' \neq \emptyset$	Cesaro-2: {Every s is m.} {No p is m.} {At least one s exists.} {No s is p.} {At least one s is not p.}
Syllo P1. P2. P3. C1. C2. Pro	gism 22. sm = s pm' = p $s \neq \emptyset$ sp' = s $sp' \neq \emptyset$ OF.	Cesaro-2: {Every s is m.} {No p is m.} {At least one s exists.} {No s is p.} {At least one s is not p.}
Syllo P1. P2. P3. C1. C2. PRO P2a.	gism 22. sm = s pm' = p $s \neq \emptyset$ sp' = s $sp' \neq \emptyset$ OF. mp' = m	Cesaro-2: {Every s is m.} {No p is m.} {At least one s exists.} {At least one s is not p.} {At least one s is not p.} (P2), $\begin{cases} bc' = b \ = \ cb' = c \\ b \leftarrow p, c \leftarrow m \end{cases}$
Syllo P1. P2. P3. C1. C2. PRO P2a. S4.	gism 22. sm = s pm' = p $s \neq \emptyset$ sp' = s $sp' \neq \emptyset$ oF. mp' = m s(mp') = s	Cesaro-2: {Every s is m.} {No p is m.} {At least one s exists.} {No s is p.} {At least one s is not p.} (P2), $\begin{cases} bc' = b \ = \ cb' = c \\ b \leftarrow p, c \leftarrow m \end{cases}$ (P1), (P2a)
Syllo P1. P2. P3. C1. C2. PRO P2a. S4. S5.	gism 22. sm = s pm' = p $s \neq \emptyset$ sp' = s $sp' \neq \emptyset$ or. mp' = m s(mp') = s (sm)p' = s	Cesaro-2: {Every s is m.} {No p is m.} {At least one s exists.} {No s is p.} {At least one s is not p.} (P2), $\begin{cases} bc' = b \mid = \mid cb' = c \\ b \leftarrow p, c \leftarrow m \end{cases}$ (P1), (P2a) (S4), $\begin{cases} b(cd) = (bc)d \\ b \leftarrow s, c \leftarrow m, d \leftarrow p' \end{cases}$

Therefore: (C2). * * *

(S6), (P3).

Syllogism 23. Calemes-4:

S7. $sp' \neq \emptyset$

P1.	ms' = m	{Nomiss.}
P2.	pm = p	{Every p is m.}
C1.	sp' = s	{No s is p.}
Pro	OF.	
S 3.	p(ms')=p	(P2), (P1)
S4.	(pm)s' = p	$ (S3), \begin{array}{l} \{b(cd)=(bc)d\}\\ b\leftarrow p, c\leftarrow m, d\leftarrow s' \end{array}$
S5 .	ps' = p	(S4), (P2)
S6.	$\mathbf{s}\mathbf{p}'=\mathbf{s}$	$ \begin{cases} bc' = b \ = \ cb' = c \\ (S5), \ b \leftarrow p, c \leftarrow s. \\ Therefore: (C1). \end{cases} $
		* * *

Syllogism 24. Calemos-4: Synogism 24.Calentos-
P1.P1.ms' = m{No m is s.}P2.pm = p{Every p is mP3. $s \neq \emptyset$ {At least oneC1.sp' = s{No s is p.}C2. $sp' \neq \emptyset$ {At least one {Every p is m.} {At least one s exists.} {At least one s is not p.} Proof. **S4.** p(ms') = p (P2), (P1) **S5.** (pm)s' = p (S4), $\begin{cases} b(cd) = (bc)d \\ b \leftarrow p, c \leftarrow m, d \leftarrow s' \end{cases}$ (S5), (P2) $\mathbf{S6}$ ps' = p

S7.	$\mathbf{s}\mathbf{p}'=\mathbf{s}$	$\{bc' = b \mid = cb' = c\}$ (S6), $b \leftarrow p, c \leftarrow s$. Therefore: (C1).
S8 .	$\mathbf{sp}' \neq \boldsymbol{\varnothing}$	(S7), (P3). Therefore: (C2).

Thus, we have shown that the LC axioms are sufficiently powerful for proving all 24 classic categorical syllogisms.

6.4. Syllogism-axiom matrix for LC

Table 4 summarizes the axioms that have been adopted in the proof of each valid categorical syllogism in the preceding subsection.

Syllogism	(LC2)	(LC3)	(LC1)	(LC5)	(LC4)
	associativity	predicative	commutativity	disjointness	subsumption
		inhabitation		conversion	contraposition
Barbara-1	\checkmark				
Barbari-1	\checkmark				
Celarent-1	\checkmark				
Celaront-1	\checkmark				
Camestres-2	\checkmark				\checkmark
Camestros-2	\checkmark				\checkmark
Bamalip-4	\checkmark		√		
Darapti-3	\checkmark	\checkmark			
Felapton-3	\checkmark	\checkmark			
Disamis-3	\checkmark	✓			
Bokardo-3	\checkmark	\checkmark			
Darii-1*	\checkmark	√	√		
Ferio-1*	\checkmark	√	\checkmark		
Baroko-2*	\checkmark	\checkmark	√		\checkmark
Dimatis-4	\checkmark	\checkmark	√		
Datisi-3	\checkmark	\checkmark	√		
Ferison-3	√	\checkmark	√		
Festino-2*	\checkmark	\checkmark	√	\checkmark	
Fresison-4	\checkmark	\checkmark	√	\checkmark	
Fesapo-4	√	√		\checkmark	
Cesare-2	\checkmark			\checkmark	
Cesaro-2	\checkmark			√	
Calemes-4	\checkmark			\checkmark	
Calemos-4	\checkmark			\checkmark	

^{*} The subject inhabitation (LC6) lemma, convenient for the proof of this mood-figure pair, requires the axioms of predicative inhabitation (LC3) and commutativity (LC1).

Table 4.: Syllogism-axiom matrix for the Leibniz-Cayley system.

Notice that the proof of every mood requires LC2. In addition, the proof of every mood having a particular premise requires LC3.

LC5 is not needed for proving syllogisms in the "basic" set, and is only used where there is an E-conversion that maps certain syllogisms in the "derived" set into syllogisms in the "basic" set.

It is remarkable that the proof of 14 categorical syllogisms in LC requires only bounded (meet-)semilattice axioms (L1 to L3). Among them, all the 8 affirmative categorical syllogisms –which employ only "A" or "T' relations in their premises and conclusions– require in their proof neither LC4 nor LC5; for them, the monadic complementation operation "/" is superfluous. (In contrast, categorical syllogisms which employ "E" or "O" already require complementation in the premise or conclusion, whether their proofs make use of LC4/LC5 or not.)

Table 4 would have been less parsimonious in the application of axioms had LC adopted subject inhabitation (LC6) as an axiom instead of predicative inhabitation (LC3): 5 categorical syllogisms –Darapti-3, Felapton-3, Disamis-3, Bokardo-3, Fesapo-4– would require commutativity (LC1).

Many logical facts were **not** needed to prove all the 24 categorical syllogisms in LC, such as:

- (a) " \neq " is the negation of "=": this fact is not used in any proof. Formally, we could have replaced " \neq " by an arbitrary dyadic relation "R" which we know no property of the proofs would have been the same.
- (b) The representation of universal categorical assertions in terms of " $\boldsymbol{\varnothing}$ ": $b \operatorname{A} c \mid = bc' = \boldsymbol{\varnothing}; \quad b \operatorname{E} c \mid = bc = \boldsymbol{\varnothing}.$
- (c) The symbolization and characterization of the properties of the universe class **I**.
- (d) Involution of complementation: (b')' = b.
- (e) Disjointness of complements: $bb' = \mathbf{\emptyset}^{68}$.
- (f) Idempotence of combination/intersection: $bb = b^{69}$.
- (g) The operation " \cup " (union of two classes) and its properties.

Interestingly, the two operations adopted in LC, intersection and complementation, form together a functionally complete set of Boolean operations.

7. McColl-Ladd (ML) system

7.1. McColl-Ladd system representations

The McColl-Ladd system makes use only of symbols that represent particular or universal relations that are *affirmative* for both the subject and the predicative (" \subseteq " and " \mathfrak{m} ")⁷⁰. Each fundamental categorical relation is represented in ML as follows:

\mathbf{A}	Every \mathbf{b} is \mathbf{c} .	$b \subseteq c$
Б	No \mathbf{b} is \mathbf{c} .	$h \subset a'$
Ľ	(Every b is not- \mathbf{c} .)	$0 \subseteq c$
Ι	At least one \mathbf{b} is \mathbf{c} .	$b \Cap c$
0	At least one \mathbf{b} is not \mathbf{c} .	$h \otimes d$
	(At-least-one \mathbf{b} is not- \mathbf{c} .)	$O \parallel \parallel C$
*	At least one \mathbf{b} exists.	$h \otimes h$
	(At least one \mathbf{b} is \mathbf{b} .)	0 111 0

The symbolic representations of universal categorical assertions are by McColl (1877, p. 181) and reproduced by Ladd (1883, p. 24), whereas the representations of the particular ones are by Ladd (1883, p. 26). In addition, Ladd (1883, p. 29) employs " $b \cap \mathbf{I}$ " for "*"; we employ instead the equivalent assertion " $b \cap b$ " for economy of concepts – we are not strictly required to postulate a universe class, and the McColl-Ladd system as we present it is saved from an extra axiom " $b \cap \mathbf{I}$]= $b \cap b$ ".

⁶⁸This is the algebraic form of the "law of thought" known as non-contradiction (Section 4). The fact that a supposedly fundamental "law of thought" is superfluous for proving classic categorical syllogisms cannot escape our attention.

⁶⁹This is remarkable. Boole (1854, p. 49) claims this is **the** fundamental law of thought. This is the special law that distinguishes his algebra of logic (subordinated to numerical algebra with 0 and 1 only) from numerical algebra over \mathbb{N} or \mathbb{Z} . He even derives non-contradiction –widely held by many logicians up to the 19th century to be one of the fundamental laws of thought (see Section 4), but also unnecessary in categorical syllogistic–from it. Nevertheless, we have shown here that idempotence is a superfluous law for proving classic categorical syllogisms.

 $^{^{70}}$ It would be fair to argue that this system is relational, instead of algebraic in a strict sense, since its object of study is a relational structure, not an algebraic structure with operations/functions only. In general, the signature of a mathematical structure can include special values and/or operations/functions and/or relations, and our position is that we consider their study algebraic in a wider sense. The study of the interplay between operations and relations is not uncommon in algebra, for instance, in lattice theory.

7.2. McColl-Ladd system axioms

In order to prove all the classic categorical syllogisms, we have selected the following axioms to form the McColl-Ladd system⁷¹:

The following is a convenient lemma to shorten the proofs of some valid mood/figure pairs:

(ML6) $c \cap b$, $c \subseteq d \models d \cap b$ {Disamis-3}

Proof.S1.
$$b \cap d \mid = \mid d \cap b$$
 $\{(ML1): b \cap c \mid = \mid c \cap b\}$
 $c \leftarrow d$ Proof.S2. $c \cap b, \ c \subseteq d \mid = b \cap d$ $(ML5), \{(ML1): b \cap c \mid = \mid c \cap b\}$
S3. $c \cap b, \ c \subseteq d \mid = d \cap b$ $(S2), (S1)$

These ML axioms form the subset⁷⁶ of the axioms presented by Moss (2007, p. 21, Figure 9)(2010, p. 31, Figure 3.4)(2011, p. 181, Figure 11.1) –and reused by Hemann, Swords, and Moss (2015, p. 3)– that is needed for proving all the classic categorical syllogisms. In addition, Reichenbach (1952, pp. 7-8) informally justifies why Barbara-1 and Darii-1 are the "primitive" categorical syllogisms which the other 22 ones are reducible to when the the four classic categorical relations are expressed in terms of A/\subseteq and I/\square by obversion.

The comma (",") is typically interpreted as the metalogical "*and*" operator. The relational character of the ML system is enhanced if, in the ML4 and ML5 axioms, we reinterpret the "," as the operator for composition of relations (from relation algebra) instead⁷⁷. Intriguingly, the metalogical "*and*" is the operator for a commutative operation, whereas composition of relations is not necessarily commutative. On the other hand, the fact that ML4 and ML5 are "composition-friendly", with the middle term occupying the position of a "bridge" between two "endpoints", is perhaps a reason why syllogism moods from the first figure were seen by Aristotle as "perfect" (Patzig, 1968, pp. 50-59)(Locke, 1700, book IV, chapter 17, §§ 4 and 8, pp. 405–413)(De Morgan, 1858, p. 217)(Lorenzen, 1957)⁷⁸.

⁷⁴It symbolically represents transitivity of subclasshood.

⁷¹McColl's original system (1877) –which represented categorical assertions by means of the relations " \subseteq " and " $\not\subseteq$ "– adopted the following laws: ML2 (p. 177, rule 11); ML3 (p. 181); ML4 (p. 180, rule 15); " $b \not\subseteq c \mid = c' \not\subseteq b'$ " (p. 180, rule 16) rather than ML1; Bokardo-3 –" $b \subseteq c, b \not\subseteq d \mid = c \not\subseteq d$ " (p. 180, rule 17)– rather than ML5.

 $^{^{72}}$ In the representation of categorical relations by means of single vowels, it corresponds to "A-contraposition" (**b** A **c** |=| **c**' A **b**').

⁷³In the representation of categorical relations by means of single vowels, it corresponds to "E-conversion" ($\mathbf{b} \in \mathbf{c} \mid = \mid \mathbf{c} \in \mathbf{b}$).

Leibniz (ca. 1686e, p. 399, point 200) also stated (in intensional/contravariant language) both ML3 and ML2, with the explicit *algebraic* employment of the term negation (class complement) operation.

 $^{^{75} \}mathrm{It}$ symbolically represents covariance/monotonicity of conjointness: if two classes are conjoint, then one of them is conjoint with any superclass of the other.

⁷⁶A-transposition, taken here as an axiom, is a consequence of the axioms of E-transposition and involution of complementation from the mentioned sources.

 $^{^{77}}$ De Morgan (1860, pp. 331,355) is to be credited for noticing that the deduction of the conclusion in a syllogism can be seen as an application of composition of relations, the premises.

 $^{^{78}}$ Once we recognize all the categorical relations in De Morgan's syllogistic, any categorical syllogism can

Notice that ML didn't adopts axioms of subalternation:

 $b \subseteq c \quad |=| \quad b \cap c \\ b \subseteq c' \quad |=| \quad b \cap c'.$

The lack of this axiom makes ML fully compatible to Boolean algebra, as we will prove in Section 8. However, due to this intentional omission, 9 of the 24 classic categorical syllogisms require not just two but three premises for them to be valid in this system, just like in the Euler system (Section 3) and in LC (Section 6).

ML can straightforwardly derive the subalternation laws of traditional Aristotelic logic from Darii-1 (ML5) if the subject term is inhabited:

 $\begin{array}{ll} b \cap b, \ b \subseteq c & |=| & b \cap c \\ b \cap b, \ b \subseteq c' & |=| & b \cap c'. \end{array}$

As ML is a relational system, it is closer in spirit to the original Aristotelic syllogistic (Aristotle of Stagira, ca. 350 BCEa)(ca. 350 BCEc)⁷⁹ than LC –an algebraic system– is. The following categorical syllogism proofs –none of which has more than 3 steps– reinforce this point.

7.3. Syllogism proofs in the McColl-Ladd system

Here are the proofs of categorical syllogisms in ML.

Syllogism 3. Celarent-1: Syllogism 1. Barbara-1: P1. $s \subseteq m$ {Every s is m.}P2. $m \subseteq p$ {Every m is p.}C1. $s \subseteq p$ {Every s is p.} **P1.** $s \subseteq m$ {Every s is m.} **P2.** $m \subseteq p'$ {No m is p.} **C1.** $s \subseteq p'$ C1. $s \subseteq p$ $\{No s is p.\}$ PROOF. PROOF. (P1), (P2), (by Lewis (1918, p. 194):) $\{b\subseteq c,c\subseteq d\ \mid=b\subseteq d\}$ (P1), (P2), S3. $s \subseteq p$ $b \leftarrow s, c \leftarrow m, d \leftarrow p.$ $\{b\subseteq c,c\subseteq d\ \mid=b\subseteq d\}$ S3. $s \subseteq p'$ Therefore: (C1). $b \leftarrow s, c \leftarrow m, d \leftarrow p'.$ Therefore: (C1). * * * * * * Syllogism 2. Barbari-1: Syllogism 4. Celaront-1: **P1.** $s \subseteq m$ {Every s is m.} **P2.** $m \subseteq p$ {Every m is p.} **P1.** $s \subseteq m$ {Every s is m.} $\begin{array}{c} \cdot \subseteq \\ \mathbf{1} & \mathbf{3} & \mathbf{5} \\ \hline \mathbf{C1} & \mathbf{5} \end{array}$ {At least one s exists.} **P2.** $m \subseteq p'$ {No m is p.} **P3.** $s \cap s$ {Every s is p.} {At least one s exists.} C2. $s \cap p$ C1. $s \subseteq p'$ {At least one s is p.} $\{No s is p.\}$ C2. $s \cap p'$ Proof. {At least one s is not p.} (P1), (P2), Proof. $\{b \subseteq c, c \subseteq d \mid = b \subseteq d\}$ (P1), (P2), **S4**. $\mathbf{s} \subset \mathbf{p}$ $\{b\subseteq c,c\subseteq d\ \mid=b\subseteq d\}$ $b \leftarrow s, c \leftarrow m, d \leftarrow p.$ **S4**. $\mathbf{s} \subseteq \mathbf{p}'$ Therefore: (C1). $b \leftarrow s, c \leftarrow m, d \leftarrow p'.$ Therefore: (C1). (P3), (S4), $\{b \cap c, c \subseteq d \mid = b \cap d\}$ (P3), (S4), $\mathbf{s} \cap \mathbf{p}$ S5. $\{b \Cap c, c \subseteq d \mid = b \Cap d\}$ $b \leftarrow s, c \leftarrow s, d \leftarrow p.$ **S5**. s∩p′ Therefore: (C2). $b \leftarrow s, c \leftarrow s, d \leftarrow p'.$ Therefore: (C2). * * * * * *

be reduced into a first-figure syllogism by applying the conversion operation, e.g. $\mathbf{p} \wedge \mathbf{m} \models \mathbf{m} \ddot{\mathbf{x}} \mathbf{p}$, as $\ddot{\mathbf{A}} = \ddot{\mathbf{A}}$. ⁷⁹Though one important departure is that, in ML, intermediate proof steps which are the result of Atranspositions are not directly expressed as traditional Aristotelic relations {A, E, I, O} with only positive subjects, namely, "Every p is m. |= Every non-m is non-p" in Camestres-2, Camestros-2 and Baroko-2, as we will see in Section 7.3.

* * *

Syllogism 6. Camestros-2:

P1.	$s\subseteq m'$	{Nosism.}
P2.	$p \subseteq m$	{Every p is m.}
P3.	$s \Cap s$	{At least one s exists.}
C1.	$s \subseteq p'$	{No s is p.}
C2.	$s \Cap p'$	$\{At \text{ least one s is not p.}\}$
Pro	OF.	
S4 .	$m'\subseteq p'$	(P2), $\begin{cases} b \subseteq c \mid = \mid c' \subseteq b' \\ b \leftarrow p, c \leftarrow m \end{cases}$
S5.	$\mathbf{s} \subseteq \mathbf{p}'$	$\begin{array}{l} (\mathrm{P1}), (\mathrm{S4}), \\ \{b \subseteq c, c \subseteq d \ = b \subseteq d\} \\ b \leftarrow s, c \leftarrow m', d \leftarrow p'. \end{array}$ Therefore: (C1).
S6.	$\mathbf{s} \Cap \mathbf{p}'$	$\begin{array}{l} (\mathrm{P3}), \ (\mathrm{S5}), \\ \{b \Cap c, c \subseteq d \ = b \Cap d\} \\ b \leftarrow s, c \leftarrow s, d \leftarrow p'. \\ \mathrm{Therefore:} \ (\mathrm{C2}). \end{array}$

* * *

Syllogism 7. Bamalip-4:

P1.	$m\subseteq s$	{Every m is s.}
P2.	$p\subseteq m$	{Every p is m.}
P3.	$p \Cap p$	{At least one p exists.}
C1.	$s \Cap p$	${}^{-}{At least one s is p.}$
Pro	OF.	
S 4.	$p\subseteq s$	$\begin{array}{l} (\mathrm{P2}), \ (\mathrm{P1}), \\ \{b \subseteq c, c \subseteq d \ = b \subseteq d\} \\ b \leftarrow p, c \leftarrow m, d \leftarrow s \end{array}$
S5.	$\mathbf{s} \Cap \mathbf{p}$	$\begin{array}{l} (\mathrm{P3}), \ (\mathrm{S4}), \\ \{c \cap b, c \subseteq d \ \mid = d \cap b\} \\ c \leftarrow p, b \leftarrow p, d \leftarrow s. \end{array}$ Therefore: (C1).

* * *

Syllogism 8. Darapti-3:

P1.	$m\subseteq s$	{Every m is s.}
P2.	$m\subseteq p$	{Every m is p.}
P3.	$m \Cap m$	{At least one m exists.}
C1.	$s \Cap p$	${}^{\text{At least one s is p.}}$
Pro	OF.	
S4 .	$m \Cap p$	$\begin{array}{l} (\mathrm{P3}), \ (\mathrm{P2}), \\ \{b \Cap c, c \subseteq d \ \mid = b \Cap d\} \\ b \leftarrow m, c \leftarrow m, d \leftarrow p \end{array}$
S 5.	$\mathbf{s} \Cap \mathbf{p}$	$\begin{array}{l} (\mathrm{S4}), \ (\mathrm{P1}), \\ \{c \cap b, c \subseteq d \ \mid = d \cap b\} \\ c \leftarrow m, b \leftarrow p, d \leftarrow s. \end{array}$ Therefore: (C1).

* * *

* * *

Syllogism 10. Disamis-3:

Ρ1.	$m \subseteq s$	{Every m is s.}
P2.	$m \Cap p$	${At least one m is p.}$
C1.	$s \Cap p$	${}^{\text{At least one s is p.}}$
Pro	OF.	
S3.	$\mathbf{s} \Cap \mathbf{p}$	$ \begin{array}{l} (\mathrm{P2}), (\mathrm{P1}), \\ \{c \Cap b, c \subseteq d \ = d \Cap b\} \\ c \leftarrow m, b \leftarrow p, d \leftarrow s. \\ \mathrm{Therefore:} \ (\mathrm{C1}). \end{array} $

* * *

* * *

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* * *

Syllogism 15. Dimatis-4:

P1.	$m \subseteq s$	{Every m is s.}
P2.	$p \Cap m$	{At least one p is m.}
C1.	$s \Cap p$	${At least one s is p.}$
Pro	OF.	
S3.	$p \Cap s$	$\begin{array}{l} (\mathrm{P2}), \ (\mathrm{P1}), \\ \{b \Cap c, c \subseteq d \ = b \Cap d\} \\ b \leftarrow p, c \leftarrow m, d \leftarrow s \end{array}$
S 4.	$\mathbf{s} \Cap \mathbf{p}$	(S3), $\begin{cases} b \Cap c \mid = \mid c \Cap b \\ b \leftarrow p, c \leftarrow s. \end{cases}$ Therefore: (C1).

* * *

Syllogism 16. Datisi-3:

P1.	$m \Cap s$	$\{At \text{ least one m is s.}\}$
P2.	$m\subseteq p$	{Every m is p.}
C1.	$s \Cap p$	${At least one s is p.}$
Pro	OF.	
P1a.	$s \Cap m$	(P1), $\begin{cases} b \Cap c \mid = \mid c \Cap b \\ b \leftarrow m, c \leftarrow s \end{cases}$
S3.	$\mathbf{s} \Cap \mathbf{p}$	$\begin{array}{l} (\text{P1a}), (\text{P2}), \\ \{b \Cap c, c \subseteq d \ = b \Cap d\} \\ b \leftarrow s, c \leftarrow m, d \leftarrow p. \\ \text{Therefore: (C1).} \end{array}$

* * *

* * *

Syllogism 18. Festino-2:

```
P1. s \cap m {At least one s is m.}

P2. p \subseteq m' {No p is m.}

C1. s \cap p' {At least one s is not p.}

PROOF.

P2a. m \subseteq p' (P2), \{b \subseteq c' \mid = \mid c \subseteq b'\}

b \leftarrow p, c \leftarrow m

(P1), (P2a),

\{b \cap c, c \subseteq d \mid = b \cap d\}

b \leftarrow s, c \leftarrow m, d \leftarrow p'.

Therefore: (C1).

* * *
```

Syllogism 19. Fresison-4:

P1.	$m \Cap s$	$\{At \text{ least one } m \text{ is } s.\}$
P2.	$p\subseteq m'$	{No p is m.}
C1.	$s \Cap p'$	${}^{-}{At least one s is not p.}$
Pro	OF.	
P1a.	$s \Cap m$	$(P1), \begin{array}{l} \{b \cap c \mid = \mid c \cap b\} \\ b \leftarrow m, c \leftarrow s \end{array}$
P2a.	$m\subseteq p'$	(P2), $\begin{cases} b \subseteq c' & = \ c \subseteq b' \\ b \leftarrow p, c \leftarrow m \end{cases}$
S3.	$\mathbf{s} \Cap \mathbf{p'}$	$\begin{array}{l} (\text{P1a}), \ (\text{P2a}), \\ \{b \Cap c, c \subseteq d \ = b \Cap d\} \\ b \leftarrow s, c \leftarrow m, d \leftarrow p'. \\ \text{Therefore: (C1).} \end{array}$

* * *

Syllogism 20. Fesapo-4:

-	-	=
P1.	$m \subseteq s$	{Every m is s.}
P2.	$p\subseteq m'$	{No p is m.}
P3.	$m \Cap m$	{At least one m exists.}
C1.	$s \Cap p'$	{At least one s is not p.}
Pro	OF.	
P2a.	$m\subseteq p'$	(P2), $\begin{cases} b \subseteq c' \mid = \mid c \subseteq b' \\ b \leftarrow p, c \leftarrow m \end{cases}$
	~ /	(P3), (P2a), $(h \otimes h \otimes h)$

S4.	$m \Cap p'$	$ \begin{array}{l} \{b \Cap c, c \subseteq d \mid = b \Cap d\} \\ b \leftarrow m, c \leftarrow m, d \leftarrow p' \end{array} $
S 5	s⊚n′	$ \begin{array}{l} (\mathrm{S4}), \ (\mathrm{P1}), \\ \{c \cap b, c \subseteq d \ \mid = d \cap b\} \end{array} $

S5. $\mathbf{s} \cap \mathbf{p}'$ $c \leftarrow m, b \leftarrow p', d \leftarrow s.$ Therefore: (C1).

* * *

Syllogism 21. Cesare-2: P1. $s \subseteq m$ {Every s is m.} P2. $p \subseteq m'$ {No p is m.}

P2.	$p \subseteq m'$	{No p 1s m.}
C1.	$s \subseteq p'$	{No s is p.}
Pro	OF.	
P2a.	$m\subseteq p'$	$(P2), \begin{cases} b \subseteq c' \mid = \mid c \subseteq b' \\ b \leftarrow p, c \leftarrow m \end{cases}$
S3.	$\mathbf{s}\subseteq \mathbf{p}'$	(P1), (P2a), $\{b \subseteq c, c \subseteq d \mid = b \subseteq d\}$ $b \leftarrow s \ c \leftarrow m \ d \leftarrow n'$

b.
$$\mathbf{s} \subseteq \mathbf{p}$$
 $b \leftarrow s, c \leftarrow m, d \leftarrow p'.$
Therefore: (C1).

Syllogism 22. Cesaro-2:				
P1.	$s\subseteq m$	{Every s is m.}		
P2.	$p\subseteq m'$	{No p is m.}		
P3.	$s \Cap s$	{At least one s exists.}		
C1.	$s \subseteq p'$	{No s is p.}		
C2.	$s \Cap p'$	$\{At \text{ least one s is not p.}\}$		
Pro	OF.			
P2a.	$m\subseteq p'$	$(P2), \begin{cases} b \subseteq c' & = \ c \subseteq b' \\ b \leftarrow p, c \leftarrow m \end{cases}$		
S4.	$\mathbf{s} \subseteq \mathbf{p}'$	$\begin{array}{l} (\mathrm{P1}), \ (\mathrm{P2a}), \\ \{b \subseteq c, c \subseteq d \ = b \subseteq d\} \\ b \leftarrow s, c \leftarrow m, d \leftarrow p'. \end{array}$ Therefore: (C1).		
S5.	$\mathbf{s} \Cap \mathbf{p}'$	$\begin{array}{l} (\mathrm{P3}), \ (\mathrm{S4}), \\ \{b \Cap c, c \subseteq d \mid = b \Cap d\} \\ b \leftarrow s, c \leftarrow s, d \leftarrow p'. \\ \text{Therefore: (C2).} \end{array}$		
		* * *		



Thus, we have shown that the ML axioms are sufficiently powerful for proving all 24 classic categorical syllogisms.

7.4. Syllogism-axiom matrix for ML

Table 5 summarizes the axioms that have been adopted in the proof of each valid categorical syllogism in the preceding subsection.

Notice that the proof of every mood requires ML4 (when there is a universal conclusion) or ML5 (when there is a particular conclusion). If the proof of a mood requires both ML4 and ML5, then the mood has more than two premises (not assuming existential import for universal assertions). Therefore, the proof of every mood having exactly two premises requires either ML4 or ML5. In other words, for our proofs of 2-premise moods, ML4 and ML5 are mutually exclusive and collectively exhaustive.

If ML1 is required in the proof of a mood, then ML5 is also required.

ML3 is not needed for proving syllogisms in the "basic" set, and is only used where there is an E-conversion that maps certain syllogisms in the "derived" set into syllogisms in the "basic" set.

All the 8 affirmative categorical syllogisms, which employ only "A" or "I" relations in their premises and conclusions, require in their proof neither ML2 nor ML3; for them, the monadic complementation operation "" is superfluous. (In contrast, categorical syllogisms which employ "E" or "O" already require complementation in the premise or conclusion, whether their proofs make use of ML2/ML3 or not.)

Column ML3 has the same mood-figure pairs as column LC5 (Section 6.4), which

Syllogism	(ML4)	(ML5)	(ML1)	(ML3)	(ML2)
	Barbara-1	Darii-1	I-conversion	E-transposition	A-transposition
Barbara-1	\checkmark				
Barbari-1	\checkmark	\checkmark			
Celarent-1	\checkmark				
Celaront-1	\checkmark	\checkmark			
Camestres-2	\checkmark				\checkmark
Camestros-2	\checkmark	\checkmark			√
Bamalip-4*	\checkmark	\checkmark	√		
Darapti-3*		\checkmark	√		
Felapton-3*		\checkmark	√		
Disamis-3*		\checkmark	√		
Bokardo-3*		\checkmark	√		
Darii-1		\checkmark			
Ferio-1		\checkmark			
Baroko-2		\checkmark			\checkmark
Dimatis-4		\checkmark	√		
Datisi-3		\checkmark	√		
Ferison-3		\checkmark	√		
Festino-2		\checkmark		\checkmark	
Fresison-4		\checkmark	√	\checkmark	
Fesapo-4*		√	✓	\checkmark	
Cesare-2	\checkmark			\checkmark	
Cesaro-2	\checkmark	\checkmark		\checkmark	
Calemes-4	\checkmark			 ✓ 	
Calemos-4	\checkmark	\checkmark		 ✓ 	



Table 5.: Syllogism-axiom matrix for the McColl-Ladd system.

corresponds to the same law expressed in a different form; likewise, column ML2 and column LC4 perfectly coincide.

Many logical facts were **not** needed to prove all the 24 categorical syllogisms in ML, such as:

- (a) The postulation and symbolization of the empty class ($\boldsymbol{\varnothing}$) and its properties: $\boldsymbol{\varnothing} \subseteq \mathbf{b}, \ \boldsymbol{\varnothing} \not \bowtie \mathbf{b}$ for any \mathbf{b} .
- (b) The symbolization and characterization of the properties of the universe class I.
- (c) Reflexivity and antisymmetry of " \subseteq ".
- (d) How " \subseteq " is connected to " \cap ": $\mathbf{b} \subseteq \mathbf{c} \mid = \mid \mathbf{b} \not \cap \mathbf{c}'; \quad \mathbf{b} \cap \mathbf{c} \mid = \mid \mathbf{b} \not \subseteq \mathbf{c}'.$
- (e) The term inhabitation law (which is neither an axiom nor a theorem in ML): $\mathbf{b} \cap \mathbf{c} \models \mathbf{b} \cap \mathbf{b}.$
- (f) Laws involving the term combination (or class intersection) operation, such as: $\mathbf{bc} \subseteq \mathbf{b}; \quad \mathbf{b} \subseteq \mathbf{c} \models \mathbf{bd} \subseteq \mathbf{cd}^{80}; \quad \mathbf{b} \subseteq \mathbf{c}, \ \mathbf{b} \subseteq \mathbf{d} \models \mathbf{b} \subseteq \mathbf{cd}.$
- (g) Involution of complementation: (b')' = b.
- (h) The 4 misnamed "laws of thought" (Section 4)⁸¹.

 $^{^{80}\}mathrm{Stated}$ by McColl (1877, p. 178, Rule 12).

 $^{^{81}}$ Aristotle himself recognized that the non-contradiction law is superfluous for proving the classic categorical syllogisms:

[«]The law that it is impossible to affirm and deny simultaneously the same predicate of the same subject is not expressly posited by any demonstration except when the conclusion also has to be expressed in that form [...].»(Aristotle of Stagira, ca. 350 BCEb, 77a10)

8. Connection between the symbolic axiomatic systems

An important characteristic of the symbolic representations selected for the fundamental relations in LC and ML is that they make obverse relations evident (the pairs A/E and I/O), but curiously they don't make contradictory relations evident (the pairs A/O and E/I). In contrast, for instance, in a pure Cayley system (with $=/\neq \emptyset$), in a pure Ladd system (with \bigcirc/\bowtie), or in a pure McColl system (with $\subseteq/\nsubseteq)$, contradictory pairs would also be made evident. Thus we have shown that the information about contradictory relations is superfluous to prove the 24 classic categorical syllogisms – an information-poorer context is sufficient for that.

How can we compare the expressive power of the ML and LC systems? This is what we show in the next subsections.

8.1. Deriving ML from LC

Let's study the relation between the McColl-Ladd and Leibniz-Cayley algebraic axiomatic systems. Are these two axiomatic systems capable of proving exactly the same theorems?

These systems adopt very different representations, so we need definitions to bridge them. Let's adopt the following definitions⁸²:

(D1) $b \subseteq c \mid = \mid bc = b$

 $(\mathbf{D2}) \quad b \cap c \quad |=| \quad bc \neq \emptyset$

These definitions are enough to almost perfectly⁸³ derive the mapping from the McColl-Ladd system representation to the Leibniz-Cayley system representation:

		IVIL	LC
Α	Every \mathbf{b} is \mathbf{c} .	$b \subseteq c$	bc = b
Ε	No \mathbf{b} is \mathbf{c} . (Every \mathbf{b} is not- \mathbf{c} .)	$b\subseteq c'$	bc' = b
Ι	At least one \mathbf{b} is \mathbf{c} .	$b \Cap c$	$bc \neq \mathbf{Ø}$
0	At least one \mathbf{b} is not \mathbf{c} . (At-least-one \mathbf{b} is not- \mathbf{c} .)	$b \Cap c'$	$bc' \neq \mathcal{O}$
*	At least one b exists. (At least one b is b .) (b is-not empty.)	$b \cap b$	$bb eq {m arnothing}$

Does $LC \models ML$? In other words, can we derive all ML axioms from LC axioms? Let's derive each individual ML axiom from LC.

ML1:	$b \cap c \mid = \mid bc \neq \emptyset \mid = \mid cb \neq \emptyset \mid = \mid c \cap b$	$\{(D2), (LC1)\}$
ML2 :	$b \subseteq c \mid = \mid bc = b \mid = \mid c'b' = c' \mid = \mid c' \subseteq b'$	$\{(D1), (LC4)\}$
ML3:	$b \subseteq c' \mid = \mid bc' = b \mid = \mid cb' = c \mid = \mid c \subseteq b'$	$\{(D1), (LC5)\}$
ML4:	(D1), Barbara-1 proof in LC	$\{(D1), (LC2)\}$
ML5.	(D2) (D1) Darii-1 proof in LC	$\{(D2), (D1), (D1$
WILD.	(D2), (D1), Dam 1 proor m Le	$(LC1), (LC2), (LC3)\}$
Therefor	e as long as we additionally assume D1 an	d D2 as bridge definitio

Therefore, as long as we additionally assume D1 and D2 as bridge definitions enriching LC, $LC \models ML$.

What about the converse? Does $ML \models LC$?

 $^{^{82} \}mathrm{Intuitively}$ justifiable by Venn and Euler diagrams.

⁸³When mapping the ML representation for "At least one **b** exists." to the LC representation which we have chosen in Section 6.1 ($b \neq \emptyset$), we have to make use of definition D2 and the idempotence law from Boolean algebra (bb = b). This is a minor detail; had we represented "At least one **b** exists." in LC by $bb \neq \emptyset$ rather than $b \neq \emptyset$ —with the corresponding minor adaptation to the LC3 and LC6 laws—, the proofs in LC would have remained the same, and there would be no need of assuming the idempotence law, at the (small) cost of the loss of a more direct, more intuitive justification for the alternative LC representation ($bb \neq \emptyset$).

If it is true, we should be able to prove every LC axiom from the ML axioms. If it is false, we should be able to find a consequence/theorem from LC which we cannot prove true or false (that is, which is undecidable) in ML.

It turns out that " $b \otimes c \models c \otimes c$ " cannot be proved from ML axioms alone⁸⁴, whereas it is a straightforward consequence of LC3 and the correspondence of definitions of "At least one **b** is **c**." and "At least one **c** exists." in LC and ML:

 $bc \neq \emptyset \mid = c \neq \emptyset$

 $b \ \Cap \ c \quad \mid = \quad c \ \Cap \ c$

ML has no expressive apparatus to state sentences involving the intersection operation, such as " $\mathbf{b} \subseteq \mathbf{cd}$ ". In contrast, in the more expressive LC system we can prove some theorems which are translatable to laws involving " \subseteq " and intersection, such as greatest lower bound:

 $\mathbf{b} \subseteq \mathbf{c}, \ \mathbf{b} \subseteq \mathbf{d} \quad \mid = \quad \mathbf{b} \subseteq \mathbf{cd}$

(Here, **b** is a lower bound of both **c** and **d**, and **cd** is the *greatest* lower bound of **c** and **d** taken together.)

Proof:

- 1. $\mathbf{b} = \mathbf{bc} \{ \mathbf{b} \subseteq \mathbf{c} \}$
- 2. $\mathbf{b} = \mathbf{bd} \{ \mathbf{b} \subseteq \mathbf{d} \}$
- 3. $\mathbf{b} = (\mathbf{bc})\mathbf{d} \{(2), (1)\}$
- 4. $\mathbf{b} = \mathbf{b}(\mathbf{cd}) \{(3), \text{LC2. Therefore: } \mathbf{b} \subseteq \mathbf{cd} \}$
 - Therefore, $\mathbf{ML} \not\models \mathbf{LC}$.

By taking together the facts that LC \models ML and ML $\not\models$ LC, we conclude that LC is strictly more "powerful" than ML, in the sense that we can prove more theorems in LC than in ML.

Note that Barbara-1 (subclasshood transitivity – ML4) is a consequence of the associativity of intersection (one of the semilattice laws, as we will see in Section 8.2). Associativity is more general than transitivity because the involved classes are not required to participate together in a subclasshood relation.

With the axioms we have chosen, every categorical syllogism proof in ML has shown to be shorter than (or at least as short as) the corresponding proof in LC.

8.2. Deriving LC from Boolean algebra

8.2.1. Boolean lattice

There are various equivalent axiomatic systems –entry points– for Boolean algebra. One of them is the Boolean lattice (BL) axiomatic system, which we will adopt in this paper⁸⁵. "Boolean lattice" is usually defined as "complemented distributive lattice" (Birkhoff, 1940, p. 88), with the signature $\langle S, \sqcap, \sqcup, ', \bot, \top \rangle$. Does BL |= LC? To answer this, let's first enumerate the BL axioms, that is, the lattice, distributive lattice, and complemented lattice axioms⁸⁶:

 84 By the way, we would need ML4 and that extra assumption to prove covariance/monotonicity of inhabitation – if a class is inhabited, then any of its superclasses is also inhabited:

S1. $b \cap b, b \subseteq c \mid = b \cap c$ $\{b \cap c, c \subseteq d \mid = b \cap d\}$ $c \leftarrow b, d \leftarrow c$

S2. $b \cap b, b \subseteq c \models c \cap c$ (S1), $\{b \cap c \models c \cap c\}$ The componenting theorem in $I \subseteq C$ ($b \neq \alpha$ has $b \models c \neq \alpha$) can be

⁸⁶Boolean lattice, as a formal (abstract) algebraic structure, was *axiomatically* defined by Ernst Schröder (1877, pp. 8-12), who called it simply "*Logikkalkul*", with an axiomatic system very close to the one we adopt

The corresponding theorem in LC $(b \neq \emptyset, bc = b \mid = c \neq \emptyset)$ can be proved by the application of substitution and LC3.

⁸⁵One could adopt instead the definition of Boolean ring (Stone, 1935), one of Huntington's axiomatic systems (1933; 1904), or one of the two alternative versions of Wolfram's (2018)(McCune et al., 2002) axiom, among various other equivalent axiomatic systems.

Semilattice laws ⁸⁷ :				
Idempotence ⁸⁸	$b \sqcap b = b$	$b \sqcup b = b$		
Commutativity	$b\sqcap c=c\sqcap b$	$b \sqcup c = c \sqcup b$		
Associativity	$(b \sqcap c) \sqcap d = b \sqcap (c \sqcap d)$	$(b \sqcup c) \sqcup d = b \sqcup (c \sqcup d)$		
Lattice laws:		. ,		
$Absorption^{89}$	$b \sqcap (b \sqcup c) = b$	$b \sqcup (b \sqcap c) = b$		
Distributive lattice laws:				
Distributivity ⁹⁰	$b \sqcap (c \sqcup d) = (b \sqcap c) \sqcup (b \sqcap d)$	$b \sqcup (c \sqcap d) = (b \sqcup c) \sqcap (b \sqcup d)$		
Bounded lattice laws				
Identity element	$b \sqcap \top = b$	$b \sqcup \bot = b$		
Complemented lattice laws				
$\begin{array}{c} \text{Complementarity} / \\ \text{Contradiction}^{91} \end{array}$	$b\sqcap b'=\bot$	$b \sqcup b' = \top$		

here (with the minor difference that Schröder was more parsimonious, since he proved the bounded lattice law " $b \sqcup \bot = b$ " and the two absorption laws as theorems), long before Lattice Theory was systematized as a topic of study.

The axiomatic system as presented here is didactically clear (since it follows the typical progression of the study of algebraic structures in abstract algebra) but redundant. For instance, each idempotence law becomes a theorem when both absorption laws hold:

$$b \mid b = b \mid (b \sqcup (b \mid b)) = b;$$

 $b \sqcup b = b \sqcup (b \sqcap (b \sqcup b)) = b.$

For further examples of redundancy in this axiomatic system, see the Section "§ 1. The First Set of Postulates" by Huntington (1904).

⁸⁷The (semi)lattice axioms were collected together into lattice theory –a branch of both abstract algebra and order theory– by Birkhoff (1938, p. 795).

⁸⁸Each idempotence law was algebraically enunciated by Leibniz, respectively in (Leibniz, ca. 1679) and (Leibniz, ca. 1686f, Axioma 1 & Scholium) – drafts published years before Boole's (1847) pamphlet. Boole restated $b \sqcap b = b$ but did not tolerate $b \sqcup b = b$ because $\top \sqcup \top = \top$ would translate to 1 + 1 = 1 in his syntax, which attempted to imitate as closely as possible certain ordinary operations and values from numerical algebra. Few years after Boole's pamphlet was published, Jevons (1864, p. 26, point 69; pp. 82-83, points 191-193) restated $b \sqcup b = b$.

⁸⁹The earliest recognition we could find for an absorption law in Boolean algebra, $b \sqcup (b \sqcap c) = b$, is by Jevons (1864, p. 26, point 70)(Valencia, 2004, p. 454)(Lewis, 1918, p. 74). Boole impeded himself from discovering it, since his partial "union" operation was valid only for disjoint classes, and it is not necessarily the case that b and $b \sqcap c$ are mutually disjoint. For the other absorption law, $b \sqcap (b \sqcup c) = b$, the obstacle for its universal validity in Boole's original algebra is that b and c (in the partial "union" inside parentheses) are not necessarily disjoint.

We didn't succeed in finding an excerpt of some draft where Leibniz makes this law explicit or at least makes use of it implicitly, but we may have inattentively passed over it in our reading. Malink and Vasudevan (2016, p. 709) studied an important manuscript of Leibniz's on logic and couldn't find this law there either – though the possibility that it might be present somewhere else remains.

⁹⁰One of the distributive laws $-b \sqcap (c \sqcup d) = (b \sqcap c) \sqcup (b \sqcap d)$ - was stated by Boole (1847, p. 17). The other one was stated by Peirce (1867, p. 251), and wasn't anticipated by Boole likely because it is not universally valid for numerical algebra with "+" and "*", to which his algebra of logic is subordinated; it is not universally valid for non-trivial Boolean rings either.

Schröder (1890, pp. 280,282,285-287,643) discovered that not all lattices are distributive. By constructing an example of non-distributive lattice, we can prove the independence of the distributive laws from the other lattice axioms. The (modular, bounded) diamond lattice M_5 and the (non-modular, bounded) pentagon lattice N_5 , both with 5 points, are examples of non-distributive lattices. In addition, set partition lattices and noncrossing partition lattices are not distributive in general.

Schröder's (1890, pp. 643) diagrammatic example of non-distributive lattice contains two non-modular pentagon sublattices N_5 : {AB, B, AB + AC, A(B + C), B + C} and {AC, C, AB + AC, A(B + C), B + C}.

⁹¹Lodovico Ignazio Richeri (1761, p. 48), in an article where he attempts to construct a *characteristica universalis* along lines similar to Leibniz's, adopted semiotically opposite symbols similar to " \mathfrak{O} " and " \mathfrak{O} " for false/empty and true/universe, respectively –to which the modern lattice theory symbols " \perp " and " \top " are analogous–, in order to symbolically represent the complementarity laws – disjointness and exhaustion

From these axioms, we can derive the following theorems.

Dominating element for \sqcap / Subsumption with \sqcap and \sqcup : Least element: $b \sqcap c = b \mid = \mid b \sqcup c = c$ $b \sqcap \bot = \bot$ PROOF. Proof. $b \sqcap c = b \mid = b \sqcup c = c$ $b \sqcap \bot$ $b \sqcup c$ $= b \sqcap (b \sqcap b') \quad \{b \sqcap b' = \bot\}$ $= (b \sqcap c) \sqcup c$ {assumption: $b \sqcap c = b$ } $= (b \sqcap b) \sqcap b' \quad \begin{cases} (b \sqcap c) \sqcap d = \\ b \sqcap (c \sqcap d) \end{cases}$ $= c \sqcup (b \sqcap c) \quad \{b \sqcup c = c \sqcup b\}$ $= c \sqcup (c \sqcap b) \quad \{b \sqcap c = c \sqcap b\}$ $= b \sqcap b'$ $\{b \sqcap b = b\}$ $\{b \sqcup (b \sqcap c) = b\}$ = c $\{b \sqcap b' = \bot\}$ $= \bot$ $b \sqcup c = c \mid = b \sqcap c = b$ $b \sqcap c$ Dominating element for \Box / $= b \sqcap (b \sqcup c)$ {assumption: $b \sqcup c = c$ } Greatest element: $\{b \sqcap (b \sqcup c) = b\}$ -h $b \sqcup \top = \top$ Proof. Therefore, $b \sqcap c = b \mid = \mid b \sqcup c = c$. $b \sqcup \top$ $= b \sqcup (b \sqcup b') \quad \{b \sqcup b' = \top\}$ Subsumption and supersumption: $b \sqcap c = b, \ b \sqcup c = b \mid = b = c$ Proof. $\{b \sqcup b' = \top\}$ = Tb $= b \sqcup c$ {assumption: $b \sqcup c = b$ } $= (b \sqcap c) \sqcup c$ {assumption: $b \sqcap c = b$ } $= (c \sqcap b) \sqcup c \ \{b \sqcap c = c \sqcap b\}$ $= c \sqcup (c \sqcap b) \ \{b \sqcup c = c \sqcup b\}$ $\{b \sqcup (b \sqcap c) = b\}$ = c

(Ladd Franklin, 1890, pp. 86-87)(Peirce and Ladd-Franklin, 1901):

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What his symbolic representation of the complementarity laws had: literal placeholder for a term; a symbol for "and"; symbols for false/contradiction and true/necessity.

What was missing: a symbol for "or"; a symbol for "non-"; a symbol for equality/identity.

With some notational adaptations to better exploit mirror reflection symmetry in order to show a kind of "partial cancellation" and "combination" of symbols, we obtain: $\bigcirc \qquad \Theta$

 $\bigcap_{\underline{i}} \bigcap_{\underline{i}} =$

This judicious choice of semiotically opposite symbols for " \perp " and " \top " stressed a symmetry that later would suggest the duality principle of Boolean algebras (nLab authors, 2019) and the two De Morgan's laws – features Leibniz **didn't** seem to have grasped the importance of (Levey, 2011, p. 127)(Lenzen, 2018b, p. 263-265).

It is a pity that Richeri missed the opportunity to present the identity element laws with his layout and a nice symmetrical notation where a kind of "cancellation" of almost antagonistic symbols becomes apparent (Whitehead and Russell, 1910, p. 218, 22:05:IIa-IIb):

$$\frac{1}{\omega} = \frac{1}{\omega}$$

The character "T" (from which the " \top " symbol surely comes from) was adopted to represent "Totalität"

Involution: (b')' = bPROOF. $b \sqcup b' = \top$ $\{b = c \mid | = f(b) = f(c)\}$ $(b')' \sqcap (b \sqcup b') = (b')' \sqcap \top$ $f(x) = (b')' \sqcap x$ $(b')' \sqcap (b \sqcup b') = (b')'$ $\{b \sqcap \top = b\}$ $((b')' \sqcap b) \sqcup ((b')' \sqcap b') = (b')' \quad {b \sqcap (c \sqcup d) = (b')' \quad (c \sqcup$ $(b \sqcap c) \sqcup (b \sqcap d) \}$ $((b')' \sqcap b) \sqcup (b' \sqcap (b')') = (b')' \quad \{b \sqcap c = c \sqcap b\}$ $((b')' \sqcap b) \sqcup \bot = (b')'$ $\{b \sqcap b' = \bot\}$ $(b')' \sqcap b = (b')'$ $\{b \sqcup \bot = b\}$ $b \sqcap b' = \bot$ $\{b = c \mid | = f(b) = f(c)\}$ $(b')' \sqcup (b \sqcap b') = (b')' \sqcup \bot$ $f(x) = (b')' \sqcup x$ $(b')' \sqcup (b \sqcap b') = (b')'$ $\{b \sqcup \bot = b\}$ $((b')' \sqcup b) \sqcap ((b')' \sqcup b') = (b')' \quad {b \sqcup (c \sqcap d) = b' \atop (b \sqcup (c \sqcap d) = b') = b' \atop (b \sqcup (c \sqcap d) = b' \atop (b \sqcup (c \sqcap d) = b') = b' \atop (b \sqcup (c \sqcap d) = b' \atop (b \sqcup (c \sqcap d) = b') = b' \atop (b \sqcup (c \sqcap d) = b' \atop (b \sqcup (c \sqcap d) = b') = b' \atop (b \sqcup (c \sqcap d) = b' \atop (b \sqcup (c \sqcap d) = b') = b' \atop (b \sqcup (c \sqcap d) = b' \atop (b \sqcup (c \sqcap d) = b') = b' \atop (b \sqcup (c \sqcap d) = b' \atop (b \sqcup (c \sqcap d) = b') = b' \atop (b \sqcup (c \sqcap d) = b' \atop (b \sqcup (c \sqcap d) = b') = b' \atop (b \sqcup (c \sqcap d) = b' \atop (b \sqcup (c \sqcap d) = b') = b' \atop (b \sqcup (c \sqcap d) = b')$ $(b \sqcup c) \sqcap (b \sqcup d) \}$ $((b')' \sqcup b) \sqcap (b' \sqcup (b')') = (b')' \quad \{b \sqcup c = c \sqcup b\}$ $((b')' \sqcup b) \sqcap \top = (b')'$ $\{b \sqcup b' = \top\}$ $(b')' \sqcup b = (b')'$ $\{b \sqcap \top = b\}$ Therefore,

(b')' = b

 $\{b \sqcap c = b, \ b \sqcup c = b \mid = b = c\}$

Complementation of equivalent terms: $b = c \mid = \mid b' = c'$ PROOF (by Leibniz³⁶). $\begin{cases} b=c & \mid = f(b) = f(c) \\ f(x) = x' \end{cases}$ $b = c \mid = b' = c'$

Conversely,

$$\begin{array}{l} \text{conversely,} \\ b' = c' & |= (b')' = (c')' \quad \begin{cases} b = c & |= & f(b) = f(c) \\ b \leftarrow b', c \leftarrow c', f(x) = x' \\ b' = c' & |= b = c \\ \qquad \{ (b')' = b \} \end{array}$$

Therefore, $b = c \mid = \mid b' = c'$ Unique complementation 1: $b \sqcap c = \bot, \ b \sqcup c = \top \mid = c = b'$ PROOF⁹². In a Boolean lattice, an element b necessarily has a complement, b', by the complemented lattice laws. Suppose c is also a complement of b, that is: $b \sqcap c = \bot$ $b \sqcup c = \top$ Then c $\{b \sqcap \top = b\}$ $= c \sqcap \top$ $\{b \sqcup b' = \top\}$ $= c \sqcap (b \sqcup b')$ $= (c \sqcap b) \sqcup (c \sqcap b') \quad \{b \sqcap (c \sqcup d) = (b \sqcap c) \sqcup (b \sqcap d)\}$ $= (b \sqcap c) \sqcup (c \sqcap b') \quad \{b \sqcap c = c \sqcap b\}$ $= \perp \sqcup (c \sqcap b')$ {assumption: $b \sqcap c = \bot$ } $= (b \sqcap b') \sqcup (c \sqcap b') \quad \{b \sqcap b' = \bot\}$ $= (b' \sqcap b) \sqcup (c \sqcap b') \quad \{b \sqcap c = c \sqcap b\}$ $= (b' \sqcap b) \sqcup (b' \sqcap c) \quad \{b \sqcap c = c \sqcap b\}$ $\{b \sqcap (c \sqcup d) = (b \sqcap c) \sqcup (b \sqcap d)\}\$ $= b' \sqcap (b \sqcup c)$ {assumption: $b \sqcup c = \top$ } $= b' \sqcap \top$ = b' $\{b \sqcap \top = b\}$

Unique complementation 2: $b \sqcap c' = \bot, \ b \sqcup c' = \top \quad \mid = \quad c = b$ Proof. $b \sqcap c' = \bot, \ b \sqcup c' = \top \quad |= \quad c' = b' \qquad \begin{cases} b \sqcap c = \bot, \ b \sqcup c = \top \quad |= \quad c = b' \\ c \leftarrow c' \end{cases}$ $b \sqcap c' = \bot, \ b \sqcup c' = \top \quad |= \quad (c')' = (b')' \quad \{b = c \ |=| \ b' = c'\}$

De Morgan's law⁹³ for $\overline{\sqcap}$: $(b \sqcap c)' = b' \sqcup c'$ $PROOF^{94}$. $f \leftarrow (b \sqcap c)', \ d \leftarrow b' \sqcup c'$

⁹²Adapted from:

«https://proofwiki.org/wiki/Complement_in_Boolean_Algebra_is_Unique».

See also:

The De Morgan's (1847, pp. 118,59) laws were anticipated in prose by William of Ockham (ca. 1323). ⁹⁴Adapted from:

[«]https://proofwiki.org/wiki/Complement_in_Distributive_Lattice_is_Unique»

[«]https://math.stackexchange.com/questions/3239464/how-to-prove-the-uniqueness-of-complement-in-the-algebra-of-sets-wi 3239493»

⁹³The two De Morgan's laws state or imply that "'" is a monoid homomorphism from " \sqcap " to " \sqcup " and vice

versa – or, more precisely, from $\langle \mathbb{B}, \sqcap, \top \rangle$ to $\langle \mathbb{B}, \sqcup, \bot \rangle$ and vice versa, as it is an involution.

[«]https://www.geeksforgeeks.org/proof-of-de-morgans-laws-in-boolean-algebra/».

$$\begin{aligned} d \sqcap f' \\ &= (b' \sqcup c') \sqcap ((b \sqcap c)')' \\ &= (b' \sqcup c') \sqcap (b \sqcap c) \\ &= (b \sqcap c) \sqcap (b' \sqcup c') \\ &= (b \sqcap c \sqcap \Box b') \sqcup ((b \sqcap c) \sqcap c') \\ &\{b \sqcap c \amalg c \amalg b\} \\ &= ((c \sqcap b) \sqcap b') \sqcup ((b \sqcap c) \sqcap c') \\ &\{b \sqcap c \amalg c \amalg b\} \\ &= ((c \sqcap b \sqcap b') \sqcup ((b \sqcap c) \sqcap c') \\ &\{b \sqcap c \amalg c \amalg b\} \\ &= (c \sqcap (b \sqcap b')) \sqcup ((b \sqcap c) \sqcap c') \\ &\{b \amalg c \amalg c \amalg d\} \\ &= (c \sqcap \bot \sqcup ((b \sqcap c) \sqcap c') \\ &\{b \amalg L \amalg L\} \\ &= (b \sqcap c) \sqcap c' \\ &\{b \amalg L \amalg L\} \\ &= (b \sqcap c \sqcap c') \\ &\{b \amalg L \amalg L\} \\ &= (b \sqcap c \sqcap c') \\ &\{b \amalg L \amalg L\} \\ &= b\} \\ &= b \sqcap (c \sqcap c') \\ &= b \sqcap (c \sqcap c') \\ &= b \sqcap (c \amalg c') \\ &= b \amalg \bot \\ &\{b \sqcap b' \amalg L\} \\ &= L\} \end{aligned}$$

Therefore,

$$\begin{aligned} f &= d \\ (b \sqcap c' = \bot, \ b \sqcup c' = \top \ \mid = \ c = b \\ b \leftarrow d, \ c \leftarrow f \\ f \leftarrow (b \sqcap c)' = b' \sqcup c' \end{aligned}$$

De Morgan's law for $\overline{\Box}$: $(b \sqcup c)' = b' \sqcap c'$ PROOF. $(b' \sqcap c')' = (b')' \sqcup (c')'$ {De Morgan's law for $\overline{\sqcap}$ } $b \leftarrow b', c \leftarrow c'$ $(b' \sqcap c')' = b \sqcup c$ {(b')' = b}

$$\begin{array}{ll} b \sqcup c = (b' \sqcap c')' & \{b = c \mid = \mid c = b\}\\ (b \sqcup c)' = ((b' \sqcap c')')' & \{b = c \mid = \mid b' = c'\}\\ (b \sqcup c)' = b' \sqcap c' & \{(b')' = b\} \end{array}$$

8.2.2. Deriving LC from Boolean lattice axioms

Does $BL \models LC$? Let's derive LC axioms from BL axioms and theorems.

The first thing to do is to map the abstract Boolean lattice signature $\langle S, \Box, \sqcup, ', \bot, \top \rangle$ to the signature we need: $\langle \mathcal{P}(\mathbf{I}), \cap, \cup, ', \boldsymbol{\varnothing}, \mathbf{I} \rangle$, where " \mathcal{P} " is the "powerclass-of" function. After this mapping, we obtain the facts that follow.

LC1 (commutativity) and LC2 (associativity) are semilattice axioms.

LC3 (predicative inhabitation) can be proved as follows:

 $\begin{aligned} \mathbf{b}\boldsymbol{\varnothing} &= \boldsymbol{\varnothing} \quad \{b \sqcap \bot = \bot\} \\ \mathbf{c} &= \boldsymbol{\varnothing} \quad |= \quad \mathbf{b}\mathbf{c} = \boldsymbol{\varnothing} \quad \{\text{substitution of similars}\} \\ \mathbf{b}\mathbf{c} &\neq \boldsymbol{\varnothing} \quad |= \quad \mathbf{c} \neq \boldsymbol{\varnothing} \quad \{\text{transposition (from propositional logic})} \end{aligned}$

LC4 (subsumption contraposition) can be proved as follows:

Suppose $b \sqcap c = b$. Then $(b \sqcap c)' = b' \quad \{b = c \mid = \mid b' = c'\}$ $b' \sqcup c' = b' \quad \{(b \sqcap c)' = b' \sqcup c'\}$ $c' \sqcup b' = b' \quad \{b \sqcup c = c \sqcup b\}$ $c' \sqcap b' = c' \quad \{b \sqcap c = b \mid = \mid b \sqcup c = c\}$ Therefore, $\mathbf{bc} = \mathbf{b} \mid = \mid \mathbf{c'b'} = \mathbf{c'}$.

LC5 (disjointness conversion) can be proved as follows:

Suppose $b \sqcap c' = b$. Then $(b \sqcap c')' = b' \quad \{b = c \mid = \mid b' = c'\}$ $b' \sqcup (c')' = b' \quad \{(b \sqcap c)' = b' \sqcup c'\}$ $b' \sqcup c = b' \quad \{(b')' = b\}$ $c \sqcup b' = b' \quad \{b \sqcup c = c \sqcup b\}$ $c \sqcap b' = c \quad \{b \sqcup c = b \mid = \mid b \sqcup c = c\}$ Therefore, $\mathbf{bc'} = \mathbf{b} \mid = \mid \mathbf{cb'} = \mathbf{c}$.

Therefore, $\mathbf{BL} \models \mathbf{LC}$.

What about the converse? Does LC \models BL? No. The BL theorem (b')' = b is neither an axiom nor a theorem that can be proved in LC. Its axioms are not sufficient to prove bb = b either – nor simple theorems like (bc)b = cb = (cb)c.

Therefore, $\mathbf{LC} \not\models \mathbf{BL}$.

By taking together the facts that $BL \models LC$ and $LC \not\models BL$, we conclude that BL is strictly more "powerful" than LC, in the sense that we can prove more theorems in BL than in LC.

As we have shown, with the LC representation of an inhabited term it is straightforward to prove the predicative inhabitation law LC3 from Boolean lattice axioms. To our purposes here, this is an important advantage of LC over a pure Leibniz's system, which adopts Leibniz's representation of an inhabited term (Section 6.1). No harm is done, since we can easily convert from such a pure Leibniz's system to LC and vice-versa through the following correspondence:

 $\Im\langle s \rangle \quad |=| \quad s \neq \mathcal{Q}.$

Therefore, Leibniz's sytem and LC have equivalent expressive power.

On a note that fits the importance of Cayley's insight for the harmonization between Boolean algebra and the algebra of categorical syllogistic, Green (1991, p. 2) remarked that

 $<\![...]$ It was because of this difficulty of dealing with particular statements that a generally accepted solution of the elimination problem sufficient for a complete treatment of the syllogism came so late in the development of the algebra of logic.» 95

On the other hand, we should have in mind that Leibniz's system is conceptually more parsimonious than LC, as the former does not require postulating:

- The complementary relation to "=". In no proof of a categorical syllogism in LC (Section 6.3) one needs to use the knowledge that " \neq " is the complement of "="; one could even have replaced it by a dyadic relation "R" with unknown properties, and the proofs would have remained the same.
- The concept of empty class ("Ø"). Nowhere in the proofs of categorical syllogisms in LC we make use of properties of "Ø", such as "bØ = Ø" many of these properties Leibniz (ca. 1686d, points 15-22,28-30,39) knew, by the way.

Indeed, in LC " \neq " and " $\boldsymbol{\varphi}$ " are only used together –namely in the representations of particular categorical assertions and in LC3–, and can be safely replaced by a monadic relation " \mathfrak{I} " ("*est Ens*") from Leibniz's original logic. Cayley's contribution, valuable as it is, is not needed in order to prove the classic categorical syllogisms by an equational algebra of logic. His contribution is, above all, a bridge to what is external to the system: it enables an easy correspondence between ideas from Leibniz's pure system and Boolean algebra, allowing us to prove LC3 from BL.

Figure 9 shows the connections we have proved between BL, LC and ML. Since the theorems of ML are a subset of the theorems of LC, the latter theorems are a subset of the theorems of BL, and BL axioms don't lead to (mutually) contradictory conclusions, it follows that ML and LC don't lead to (mutually) contradictory conclusions either.

BL (if interpreted as the Boolean algebra of terms, or of term logic) has the same expressive power as a fragment of *monadic* first-order quantificational logic (Simons, 2020)(Green, 1991, p. 7)(Pratt-Hartmann, 2023, pp. 25-30). Thus, we have proved how the hierarchy of expressiveness is constituted from some axiomatizations of classic Aristotelic categorical syllogistic (but without existential import for universal assertions) –a "toy" logic with a finite, small number of interesting theorems– up to first-order logic.

9. Future work

All proof techniques have their value to illuminating different aspects of categorical syllogistic, and we feel that the "Euler system" diagrammatic proof technique deserves

⁹⁵When put in another context, this remark, in our view, would also be fitting to pay homage to McColl's insight on the opposition " \subseteq " vs. " $\not\equiv$ ", to Ladd's insight on the complementary relations " $\not\equiv$ " vs. " $\not\equiv$ ", to Mitchell's " \cup " vs. " $\not\equiv$ ", and to the opposition " \supseteq " vs. " $\not\equiv$ ", as we saw in Section 7.



Figure 9.: Euler diagram representing the relations between the Boolean lattice, Leibniz-Cayley, and McColl-Ladd axiomatic systems in terms of the set of theorems that can be proved in them.

as much respect as the algebraic proof techniques shown in this paper and the first-order logic proof technique shown in other catalogs. It has the didactic advantage of being easier to understand –almost intuitive– and less intimidating for beginners in logic. Although the "semiformal" proofs we have reproduced here are well-known, appearing even in Wikimedia Commons (Piesk, 2017), what to the best of our knowledge is missing in the literature (and in this paper as well) is the description of all the axioms and inference rules that make the Euler system work. This would be useful for the full formalization of the proofs as a gapless sequence of steps, and would ensure a level of respectability of this diagrammatic proof technique similar to that of the fully formalized, gapless algebraic techniques.

Kraszewski (1956, pp. 54,16) noticed that, if we rewrite all the moods of the 24 traditional Aristotelic categorical syllogisms into the first figure, we see that only 6 fundamental categorical relations from De Morgan's syllogistic are used in the premises (A, Ä, E, I, O, Ö for the premises in the form $\mathbf{s} ? \mathbf{m}$ and $\mathbf{m} ? \mathbf{p}$, with in some cases the addition of a premise \mathbf{sIs} , \mathbf{mIm} or \mathbf{pIp}) and only 4 are used in the conclusion (A, E, I, O in the form $\mathbf{s} ? \mathbf{p}$). Missing in the enumeration are valid categorical syllogisms with premises in the remaining 2 fundamental categorical relations (Ë, Ï) and the conclusion in the 4 fundamental categorical relations with umlaut. In a future work, they should be enumerated, and we should check whether the axioms presented in this paper – perhaps with the replacement of one of the axioms in each symbolic system by the axiom of involution of complementation: $(\mathbf{b}')' = \mathbf{b}$ – are sufficient to prove all the valid categorical syllogisms in De Morgan's syllogistic.

With our proof methods, whenever we are able to construct a proof, we do know a conclusion necessarily follows from the premises (if the proof is correct). However, when we are unable to construct a proof, our proof methods do not provide us tools to know whether there exists a proof (and we only lacked the skill to construct it) or whether the conclusion does not necessarily follow from the premises (in which case no proof of the necessity of that conclusion is possible). For instance:

P1: Every \mathbf{s} is \mathbf{m} .

P2: Every **m** is **p**.

C1: Therefore: At least one \mathbf{s} is \mathbf{p} .

The conclusion is not incompatible with the premises, though it is not necessary either. It might be the case that "No s is p." instead – namely, when $\mathbf{s} = \boldsymbol{\varnothing} / \mathbf{s} \not \cap \mathbf{s}$. This limitation of our proof methods implies that they should also be complemented by

techniques for rejection proofs of invalid categorical syllogisms, for completeness⁹⁶. It would be even better if the rejection proofs helped us to discriminate invalid conclusions which are incompatible with the premises (and thus the complement of the former relations would be necessary conclusions) from invalid conclusions which are compatible with the premises but not necessary (and thus the complement of the former relations would neither be incompatible nor necessary conclusions as well). For instance, with the premises of Barbara-1 –**s** A **m**, **m** A **p**–, any conclusion **s** ? **p** where "?" is not "A" is invalid; "O" (the complementary relation to "A") would be an incompatible conclusion, and remaining relations would be compatible but non-necessary conclusions – we wish those facts to be proved in an style similar to our LC and ML systems. A system is said to be refutationally complete if each formula of its language is either a theorem or a rejected formula (Kulicki, 2020, p. 10)(Wybraniec-Skardowska, 2018, pp. 578– 582). Since this logic –categorical syllogistic– is finite, given that we can enumerate all possible combinations of the 8 basic relations in the strict format "premise 1 - premise 2 - candidate conclusion", what we wish is defining decidable, refutationally complete systems extending LC and ML.

10. Conclusion

We have described two symbolic axiomatic systems –Leibniz-Cayley (Section 6) and McColl-Ladd (Section 7)– which are sufficiently powerful to prove all the 24 classic categorical syllogisms in the term logic tradition founded by Aristotle, as made visual by the diagrammatic Euler system (Section 3), thus totalling three systems presented by this paper (one diagrammatic and two symbolic ones). Our main novel result is summarized in Figure 9.

We have also unveiled new proofs of known theorems – the 24 classic categorical syllogisms. All proofs are short and don't fly over lay people's heads. We claim that, unlike first-order logic proofs, the diagrammatic, algebraic and relational techniques we have adopted from the literature on the topic are understandable to mathematically curious students who are finishing the middle school and that, by providing them all the axioms and three or four proofs of sample theorems, they should be able to prove all remaining theorems⁹⁷. This is because our algebraic proofs can piggyback on their familiarity to elementary numerical algebra, which has many analogous properties to the algebra of classic logic (such as substitutability of equals, commutativity and associativity of certain operations), and they are short and don't impose much cognitive burden, thus reducing the likelihood that the learners will get lost. Moreover, we hypothesize that the Euler diagram technique is immediately "intuitive" even to ordinary students not skilled enough to reproduce the proofs, who would at a minimum be able to visually track the proof steps and grasp them. Thus, these diagrammatic proofs are particularly effective to mathematically-averse Philosophy students who are required to deal with Aristotelic logic in their course curriculum.

The conventional proofs in monadic first-order quantificational logic à la Frege and Peirce (Tennant, 2014; Metamath, 2021; Koutsoukou-Argyraki, 2019) are a great way

⁹⁶The logician David Makinson, in private correspondence with the main author, claimed that "it can be done semantically by considering models of up to 8 elements".

⁹⁷The categorical syllogisms Bamalip-4, Disamis-3, Camestres-2 and Cesare-2 have representative proofs that cover together all axioms proposed in this paper for LC. (For ML, Camestres-2, Dimatis-4 and Cesare-2.) As an exercise, a teacher could show her students the axioms and those proofs and ask them to prove the remaining classic categorical syllogisms.

(or a gateway) to introduce categorical syllogistic students into first-order logic. Our proofs dispel the misconceptions that (a) the mathematical treatment of categorical syllogisms requires methods from first-order logic, and that (b) first-order logic makes Leibnizian/Boolean methods "obsolete".

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Division of work: AGR wrote this survey and proved the results; EMD was AGR's formal research advisor and authorized the submission of the paper. This arrangement is typical and expected in the Brazilian academic culture, which may differ in particular points from established cultural practices in some other places. Without the advisor's kind invitation for AGR to join the graduate program to advance pharmaceutical drug traceability technology, this paper on Logic would never have been written; AGR is very thankful to EMD for the opportunity to join his research group.

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