MODAL LOGICS FOR TOPOLOGICAL SPACES

by

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Abstract

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We present two bimodal systems, **MP** and **MP***, for reasoning about knowledge and effort.

Knowledge is interpreted as all true statements common to a set of possible worlds which represents our view. Effort corresponds to increase of information and translates to a restriction of our view. Such restrictions are parameterized by the worlds in our view and therefore are neighborhood restrictions. The semantics of these logics consist of pairs of points and their neighborhoods. In this spatial setting basic topological and computation concepts are naturally expressed which make these systems ideal for studying computing knowledge by set-theoretic means.

The system **MP** was introduced and proven complete for the class of sets containing arbitrary neighborhoods by Larry Moss and Rohit Parikh. In this thesis, **MP***, an extension of **MP**, is introduced and proven complete for various class of spaces closed under unions and intersections, among them topological spaces. We also present necessary and sufficient conditions under which a Kripke frame can be turned into a set-theoretic model of ours. Among our results is the finite model property and decidability for $\mathbf{MP}^*$. In addition we present the algebraic models of these systems and discuss further work.

στους γονείς μου

to my parents

Contents

List of Tables

Chapter 1

Introduction

In this thesis we shall present two logical systems, **MP** and **MP***, for the purpose of reasoning about *knowledge* and *effort*. These logical systems will be interpreted in a spatial context and therefore, the abstract concepts of knowledge and effort will be defined by concrete mathematical concepts.

Our general framework consists of a set of possible worlds (situations, scenarios, consistent theories, etc.) A *state of knowledge* is a subset of this set and our knowledge consists of all facts common to the worlds belonging to this subset. This subset of possibilities can be thought as our *view*. Thus two knowers having distinct views can have different knowledge. This treatment of knowledge agrees with the traditional one ([17], [15], [28], [5], [8]) expressed in a variety of contexts (artificial intelligence, distributed processes, economics, etc.)

Our treatment is based on the following simple observation

> "a restriction of our view increases our knowledge."

This is because a smaller set of possibilities implies a greater amount of common facts. Moreover, such a restriction can only be possible due to an increase of information. And such an information increase can happen with spending of time or computation resources. Here is where the notion of effort enters. A restriction of our view is *dynamic* (contrary to the view itself which is a state) and is accompanied by effort during which a greater amount of information becomes available to us (Pratt expresses a similar idea in the context of processes [29].)

We make two important assumptions.

Our knowledge has a subject. We collect information for a specific purpose. Hence we are not considering arbitrary restrictions to our view but restrictions parameterized by possibilities contained in our view, i.e. neighborhoods of possibilities. After all, only one of these possibilities is our actual state. This crucial assumption enables us to express topological concepts and use a mathematical set-theoretic setting as semantics. Without such an assumption these ideas would have been expressed in the familiar theory of intuitionism ([16], [7], [32].) As Fitting points out in [9]

> "Let $\langle \mathscr{G}, \mathscr{R}, \models \rangle$ be a [intuitionistic, propositional] model. $\mathscr{G}$ is intended to be a collection of possible universes, or more properly, states of knowledge. Thus a particular Γ in $\mathscr{G}$ may be considered as a collection of (physical)

> facts known at a particular time. The relation $\mathscr{R}$ represents (possible) time succession. That is, given two states of knowledge Γ and Δ of $\mathscr{G}$, to say $\Gamma\mathscr{R}\Delta$ is to say: if we now know Γ, it is possible that later we will know Δ."

Considering neighborhoods and, inevitably, points which parameterize neighborhoods, the important duality between the facts, which constitute our knowledge, and the possible worlds, where such facts hold, emerges.

The other assumption is that of indeterminacy. Each state of knowledge is closed under logical deduction. Thus an increase of knowledge can happen only by a piece of evidence or information given from outside. Our knowledge is *external* (a term used by Parikh to describe a similar idea in [27].) This fact leads to indeterminacy (we do not know which kinds of information will be available to us, if at all) and resembles indeterminacy expressed in intuitionism through the notion of *lawless* sequence (see [20], [31]) where, not surprisingly, topological notions arise.

To illustrate better these simple but fundamental ideas we present the following examples:

- Suppose that a machine emits a stream of binary digits representing the output of a recursive function f. After time t_1 the machine emitted the stream 111. The only information we have about the function being computed at this time on the basis of this (finite) observation is that

$$f(1) = f(2) = f(3) = 1.$$

As far as our knowledge concerns f is indistinguishable from the constant function $\mathbf{1}$, where $\mathbf{1}(n) = 1$ for all n. After some additional time t_2, i.e. spending more time and resources, 0 might appear and thus we could be able to distinguish f from $\mathbf{1}$. In any case, each binary stream will be an initial segment of f and this initial segment is a neighborhood of f. In this way, we can acquire more knowledge for the function the machine computes. The space of finite binary streams is a structure which models computation. Moreover, this space comprises a topological space. The set of binary streams under the prefix ordering is an example of *Alexandrov topology* (see [33].)

- A policeman measures the speed of passing cars by means of a device. The speed limit is 80 km/h. The error in measurement which the device introduces is 1 km/h. So if a car has a speed of 79.5 km/h and his device measures 79.2 km/h then he knows that the speed of the passing car lies in the interval $(78.2, 80.2)$ but he does not know if the car exceeds the speed limit because *not all* values in this interval are more than 80. However, measuring again and combining the two measurements or acquiring a more accurate device he has the possibility of knowing that a car with a speed of 79.5 km/h does not exceed the speed limit. Note here that if the measurement is, indeed, an open interval of real line and

> the speed of a passing car is exactly 80 km/h then he would never know if such a car exceeded the speed limit or not.

To express this framework we use two modalities K for knowledge and $\Box$ for effort. Moss and Parikh observed in [24] that if the formula

$$A \to \Diamond \mathsf{K} A$$

is valid, where A is an atomic predicate and $\Diamond$ is the dual of the $\Box$, i.e. $\Diamond \equiv \neg\Box\neg$, then the set which A represents is an open set of the topology where we interpret our systems. Under the reading of $\Diamond$ as "possible" and K as "is known", the above formula says that

> "if A is true then it is for A possible to be known",

i.e. A is *affirmative*. Vickers defines similarly an affirmative assertion in [33]

> "an assertion is affirmative iff it is true precisely in the circumstances when it can be affirmed."

The validity of the dual formula

$$\Box \mathsf{L} A \to A,$$

where L is the dual of K, i.e. $\mathsf{L} \equiv \neg\mathsf{K}\neg$, expresses the fact that the set which A represents is closed, and hence A is *refutative*, meaning if it does not hold then it is possible to know that. The fact that affirmative and refutative assertions are represented by opens and closed subsets, respectively, should not come to us as a surprise.

Affirmative assertions are closed under infinite disjunctions and refutative assertions are closed under infinite conjunctions. Smyth in [30] observed first these properties in semi-decidable properties. Semi-decidable properties are those properties whose truth set is r.e. and are a particular kind of affirmative assertions. In fact, changing our power of affirming or computing we get another class of properties with a similar knowledge-theoretic character. For example, using polynomial algorithms affirmative assertions become polynomially semi-decidable properties. If an object has this property then it is possible to know it with a polynomial algorithm even though it is not true we know it now.

Does this framework suffers from the problem of logical omniscience? Only in part. Expressing effort we are able to bound the increase of knowledge depending on information (external knowledge.) Since the modality K which corresponds to knowledge is axiomatized by the normal modal logic of **S5**, knowledge is closed under logical deduction. However, because of the strong computational character of this framework it does not seem unjustified to assume that in most cases (as in the binary streams example) a finite amount of data restricts our knowledge to a finite number of (relevant) formulae. Even without such an assumption we can incorporate the effort to deduce the knowledge of a property in the passage from one state of knowledge to the other.

We have made an effort to present our material somewhat independently. However, knowledge of basic modal logic, as in [6], [19], or [10], is strongly recommended.

The language and semantics of our logical framework is presented in Chapter 2. In the same Chapter we present two systems: **MP** and **MP***. The former was introduced in [24] and was proven complete for arbitrary sets of subsets. It soon became evident that such sets of subsets should be combined, whenever it is possible, to yield a further increase of knowledge or we should assume a previous state of other states of knowledge where such states are a possible. Therefore the set of subsets should be closed under union and intersection. Moreover, topological notions expressed in **MP** make sense only in topological models. For this reason we introduce an extension of the set of axioms of **MP** and we call it **MP***. In Chapter 3, we study the topological models of **MP*** by semantical means. We are able to prove the reduction of the theory of topological models to models whose associated set of subsets is closed under finite union and intersection. Finding for each satisfiable formula a model of bounded size we prove decidability for **MP***. The results of this chapter will appear in [11]. In Chapter 4, we prove that **MP*** is a complete system for topological models as well as topological models comprised by closed subsets. We also give necessary and sufficient conditions for turning a Kripke frame into such a topological model. In Chapter 5, we present the modal algebras of **MP** and **MP*** and some of their properties. Finally, in Chapter 6, we present some of our ideas towards future work.

Chapter 2

Two Systems: MP and MP*

In section 2.1 we shall present a language and semantics which appeared first in [24]. In section 2.2, we shall present the axiom system **MP**, introduced and proven sound and complete with a class of models called *subset spaces* in [24], and the axiom system **MP***, introduced by us, which we shall prove sound and complete for, among other classes, the class of topological spaces.

2.1 Language and Semantics

We follow the notation of [24].

Our language is bimodal and propositional. Formally, we start with a countable set A of *atomic formulae* containing two distinguished elements $\top$ and $\bot$. Then the *language* $\mathscr{L}$ is the least set such that $\mathsf{A} \subseteq \mathscr{L}$ and closed under the following rules:

$$\frac{\phi, \psi \in \mathscr{L}}{\phi \wedge \psi \in \mathscr{L}} \qquad \frac{\phi \in \mathscr{L}}{\neg\phi, \Box\phi, \mathsf{K}\phi \in \mathscr{L}}$$

The above language can be interpreted inside any spatial context.

Definition 1 Let X be a set and $\mathscr{O}$ a subset of the powerset of X, i.e. $\mathscr{O} \subseteq \mathscr{P}(X)$ such that $X \in \mathscr{O}$. We call the pair $\langle X, \mathscr{O} \rangle$ a *subset space*. A *model* is a triple $\langle X, \mathscr{O}, i \rangle$, where $\langle X, \mathscr{O} \rangle$ is a subset space and i a map from A to $\mathscr{P}(X)$ with $i(\top) = X$ and $i(\bot) = \emptyset$ called *initial interpretation*.

We denote the set $\{(x, U) : x \in X, U \in \mathscr{O}, \text{ and } x \in U\} \subseteq X \times \mathscr{O}$ by $X \dot{\times} \mathscr{O}$. For each $U \in \mathscr{O}$ let $\downarrow U$ be the set $\{V : V \in \mathscr{O} \text{ and } V \subseteq U\}$ the lower closed set generated by U in the partial order $(\mathscr{O}, \subseteq)$, i.e. $\downarrow U = \mathscr{P}(U) \cap \mathscr{O}$.

Definition 2 The *satisfaction relation* $\models_{\mathscr{M}}$, where $\mathscr{M}$ is the model $\langle X, \mathscr{O}, i \rangle$, is a subset of $(X \dot{\times} \mathscr{O}) \times \mathscr{L}$ defined recursively by (we write $x, U \models_{\mathscr{M}} \phi$ instead of $((x, U), \phi) \in \models_{\mathscr{M}}$):

$$\begin{array}{lll}
x, U \models_{\mathscr{M}} A & \text{iff} & x \in i(A), \text{ where } A \in \mathsf{A} \\
x, U \models_{\mathscr{M}} \phi \wedge \psi & \text{if} & x, U \models_{\mathscr{M}} \phi \text{ and } x, U \models_{\mathscr{M}} \psi \\
x, U \models_{\mathscr{M}} \neg\phi & \text{if} & x, U \not\models_{\mathscr{M}} \phi \\
x, U \models_{\mathscr{M}} \mathsf{K}\phi & \text{if} & \text{for all } y \in U, \quad y, U \models_{\mathscr{M}} \phi \\
x, U \models_{\mathscr{M}} \Box\phi & \text{if} & \text{for all } V \in \downarrow U \text{ such that } x \in V, \quad x, V \models_{\mathscr{M}} \phi.
\end{array}$$

If $x, U \models_{\mathscr{M}} \phi$ for all (x, U) belonging to $X \dot{\times} \mathscr{O}$ then ϕ is *valid* in $\mathscr{M}$, denoted by $\mathscr{M} \models \phi$.

We abbreviate $\neg\Box\neg\phi$ and $\neg\mathsf{K}\neg\phi$ by $\Diamond\phi$ and $\mathsf{L}\phi$ respectively. We have that

$x, U \models_{\mathscr{M}} \mathsf{L}\phi$ if there exists $y \in U$ such that $y, U \models_{\mathscr{M}} \phi$

$x, U \models_{\mathscr{M}} \Diamond\phi$ if there exists $V \in \mathscr{O}$ such that $V \subseteq U,\ x \in V,$ and $x, V \models_{\mathscr{M}} \phi.$

Many topological properties are expressible in this logical system in a natural way. For instance, in a model where the subset space is a topological space, $i(A)$ is *open* whenever $A \to \Diamond\mathsf{K}A$ is valid in this model. Similarly, $i(A)$ is *nowhere dense* whenever $\mathsf{L}\Diamond\mathsf{K}\neg A$ is valid (cf. [24].)

Example. Consider the set of *real numbers* $\mathbf{R}$ with the usual topology of open intervals. We define the following three predicates:

$$\begin{array}{lll} \mathtt{pi} & \text{where} & i(\mathtt{pi}) = \{\pi\} \\ \mathtt{I}_1 & \text{where} & i(\mathtt{I}_1) = (-\infty, \pi] \\ \mathtt{I}_2 & \text{where} & i(\mathtt{I}_2) = (\pi, +\infty) \\ \mathtt{Q} & \text{where} & i(\mathtt{Q}) = \{q : q \text{ is rational }\}. \end{array}$$

There is no real number p and open set U such that $p, U \models \mathsf{K}\mathtt{pi}$ because that would imply $p = \pi$ and $U = \{\pi\}$ and there are no singletons which are open.

A point x belongs to the *closure* of a set W if every open U that contains x intersects W. Thus π belongs to the closure of $(\pi, +\infty)$, i.e every open that contains

π has a point in $(\pi, +\infty)$. This means that for all U such that $\pi \in U$, $\pi, U \models \mathtt{LI}_2$, therefore $\pi, \mathbf{R} \models \Box \mathtt{LI}_2$. Following the same reasoning $\pi, \mathbf{R} \models \Box \mathtt{I}_1$, since π belongs to the closure of $(-\infty, \pi]$.

A point x belongs to the *boundary* of a set W whenever x belong to the closure of W and $X - W$. By the above, π belongs to the boundary of $(-\infty, \pi]$ and $\pi, \mathbf{R} \models \Box(\mathtt{LI}_1 \wedge \mathtt{LI}_2)$.

A set W is *closed* if it contains its closure. The interval $i(\mathtt{I}_1) = (-\infty, \pi]$ is closed and this means that the formula $\Box \mathtt{LI}_1 \rightarrow \mathtt{I}_1$ is valid.

A set W is *dense* if all opens contain a point of W. The set of rational numbers is dense which translates to the fact that the formula $\Box \mathtt{LQ}$ is valid. To exhibit the reasoning in this logic, suppose that the set of rational numbers was closed then both $\Box \mathtt{LQ}$ and $\Box \mathtt{LQ} \rightarrow \mathtt{Q}$ would be valid. This implies that $\mathtt{Q}$ would be valid which means that all reals would be rationals. Hence the set of rational numbers is not closed.

2.2 MP and MP*

The axiom system **MP** consists of axiom schemes 1 through 10 and rules of table 1 (see page 12) and appeared first in [24].

The following was proved in [24].

Theorem 3 *The axioms and rules of* **MP** *are sound and complete with respect to*

Axioms

1. All propositional tautologies
2. $(A \to \Box A) \wedge (\neg A \to \Box \neg A)$, for $A \in \mathsf{A}$
3. $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
4. $\Box\phi \to \phi$
5. $\Box\phi \to \Box\Box\phi$
6. $\mathsf{K}(\phi \to \psi) \to (\mathsf{K}\phi \to \mathsf{K}\psi)$
7. $\mathsf{K}\phi \to \phi$
8. $\mathsf{K}\phi \to \mathsf{K}\mathsf{K}\phi$
9. $\phi \to \mathsf{K}\mathsf{L}\phi$
10. $\mathsf{K}\Box\phi \to \Box\mathsf{K}\phi$
11. $\Diamond\Box\phi \to \Box\Diamond\phi$
12. $\Diamond(\mathsf{K}\phi \wedge \psi) \wedge \mathsf{L}\Diamond(\mathsf{K}\phi \wedge \chi) \to \Diamond(\mathsf{K}\Diamond\phi \wedge \Diamond\psi \wedge \mathsf{L}\Diamond\chi)$

Rules

$$\frac{\phi \to \psi, \phi}{\psi} \quad \text{MP}$$

$$\frac{\phi}{\mathsf{K}\phi} \quad \mathsf{K}\text{-Necessitation} \qquad \frac{\phi}{\Box\phi} \quad \Box\text{-Necessitation}$$

Table 1: Axioms and Rules of **MP***

subset spaces.

We add the axioms 11 and 12 to form the system **MP*** for the purpose of axiomatizing spaces closed under union and intersection and, in particular, topological spaces.

A word about the axioms (most of the following facts can be found in any introductory book about modal logic, e.g. [6] or [13].) The axiom 2 expresses the fact that the truth of atomic formulae is independent of the choice of subset and depends only on the choice of point. This is the first example of a class of formulae which we are going to call *bi-persistent* and their identification is one of the key steps to completeness. Axioms 3 through 5 and axioms 6 through 9 are used to axiomatize the normal modal logics **S4** and **S5** respectively. The former group of axioms expresses the fact that the passage from one subset to a restriction of it is done in a constructive way as actually happens to an increase of information or a spending of resources (the classical interpretation of necessity in intuitionistic logic is axiomatized in the same way). The latter group is generally used for axiomatizing logics of knowledge.

Axiom 10 expresses the fact that if a formula holds in arbitrary subsets is going to hold as well in the ones which are neighborhoods of a point. The converse is not sound.

Axiom 11 is a well-known formula which characterizes *incestual* frames, i.e. if two points β and γ in a frame can be accessed by a common point α then there is a point

δ which can be accessed by both β and γ. It appeared in the equivalent form (in [24])

$$\Diamond\Box\phi \wedge \Diamond\Box\psi \rightarrow \Diamond\Box(\phi \wedge \psi)$$

and was proved sound in subset spaces closed under (finite) intersection.

Obviously our attention is focused on axiom 12. It is sound in spaces closed under (finite) union and intersection as the following proposition shows.

Proposition 4 *Axioms 1 through 12 are sound in the class of subset spaces closed under finite union and intersection.*

PROOF. Soundness for Axioms 1 through 11 is easy. For Axiom 12, suppose

$$x, U \models \Diamond(\mathsf{K}\phi \wedge \psi) \wedge \mathsf{L}\Diamond(\mathsf{K}\phi \wedge \chi).$$

Since $x, U \models \Diamond(\mathsf{K}\phi \wedge \psi)$, there exists $U_x \subseteq U$ such that

$$x, U_x \models \mathsf{K}\phi \wedge \psi$$

and, since $x, U \models \mathsf{L}(\Diamond\mathsf{K}\phi \wedge \Box\chi)$, there exists $y \in U$ and $U_y \subseteq U$ such that

$$y, U_y \models \mathsf{K}\phi \wedge \chi.$$

We now have that $U_x \cup U_y \subseteq U$ (we assume closure under unions.) Thus

$$x, U_x \cup U_y \models \mathsf{K}\Diamond\phi, \quad y, U_x \cup U_y \models \mathsf{K}\Diamond\phi, \quad x, U_x \cup U_y \models \Diamond\psi, \quad \text{and} \quad y, U_x \cup U_y \models \Diamond\chi.$$

Therefore,

$$x, U \models \Diamond(\mathsf{K}\Diamond\phi \wedge \Diamond\psi \wedge \mathsf{L}\Diamond\chi).$$

∎

With the help of axiom 12 we are able to prove the key lemma 33 which leads to the DNF Theorem (page 45.) and this is the only place where we actually use it. Any formula, sound in the class of subset spaces closed under finite union and intersection, which implies the formula (note the difference from axiom 12)

$$\Diamond(\mathsf{K}\phi \wedge \psi) \wedge \mathsf{L}\Diamond(\mathsf{K}\phi \wedge \chi) \rightarrow \Diamond(\mathsf{K}\phi \wedge \psi \wedge \mathsf{L}\chi)$$

where $\Diamond\phi \rightarrow \Box\phi$, $\Diamond\psi \rightarrow \psi$ and $\chi \rightarrow \Box\chi$ are theorems, can replace axiom 12.

Chapter 3

A Semantical analysis of MP*

In this chapter we prove finite model property, decidability and (strong) reduction of the theory of topological models to that of subset spaces closed under finite union and intersection. The latter was a conjecture in [24]. All these are proved semantically without using any complete axiomatization for these models, i.e. **MP***, and in fact preceded the results of the next chapter. The approach in this chapter seems unrelated to the one of next chapter. We are able to relate both in the last section.

3.1 Stability and Splittings

Suppose that X is a set and $\mathscr{T}$ a topology on X. In the following we assume that we are working in the topological space $(X, \mathscr{T})$. Our aim is to find a partition of $\mathscr{T}$, where a given formula ϕ "retains its truth value" for each point throughout a member of this partition. We shall show that there exists a finite partition of this kind.

Definition 5 Given a finite family $\mathscr{F} = \{U_1, \ldots, U_n\}$ of opens, we define the *remainder* of (the principal ideal in $(\mathscr{T}, \subseteq)$ generated by) U_k by

$$\mathsf{Rem}^{\mathscr{F}} U_k \quad = \quad {\downarrow} U_k - \bigcup_{U_k \not\subseteq U_i} {\downarrow} U_i .$$

Proposition 6 *In a finite set of opens $\mathscr{F} = \{U_1, \ldots, U_n\}$ closed under intersection, we have*

$$\mathsf{Rem}^{\mathscr{F}} U_i \quad = \quad {\downarrow} U_i - \bigcup_{U_j \subset U_i} {\downarrow} U_j ,$$

for $i = 1, \ldots, n$.

Proof.

$$\begin{aligned} \mathsf{Rem}^{\mathscr{F}} U_i &= {\downarrow} U_i - \textstyle\bigcup_{U_i \not\subseteq U_h} {\downarrow} U_h \\ &= {\downarrow} U_i - \textstyle\bigcup_{U_i \not\subseteq U_h} {\downarrow} (U_h \cap U_i) \\ &= {\downarrow} U_i - \textstyle\bigcup_{U_j \subset U_i} {\downarrow} U_i . \end{aligned}$$

∎

We denote $\bigcup_{U_i \in \mathscr{F}} {\downarrow} U_i$ with ${\downarrow} \mathscr{F}$.

Proposition 7 *If $\mathscr{F} = \{U_1, \ldots, U_n\}$ is a finite family of opens, closed under intersection, then*

a. $\mathsf{Rem}^{\mathscr{F}} U_i \cap \mathsf{Rem}^{\mathscr{F}} U_j = \emptyset$, *for $i \neq j$,*

b. $\bigcup_{i=1}^{n} \mathsf{Rem}^{\mathscr{F}} U_i = {\downarrow}\mathscr{F}$, *i.e.* $\{\mathsf{Rem}^{\mathscr{F}} U_i\}_{i=1}^{n}$ *is a partition of* ${\downarrow}\mathscr{F}$. *We call such an* $\mathscr{F}$ *a* finite splitting (of ${\downarrow}\mathscr{F}$),

c. *if* $V_1, V_3 \in \mathsf{Rem}^{\mathscr{F}} U_i$ *and* V_2 *is an open such that* $V_1 \subseteq V_2 \subseteq V_3$ *then* $V_2 \in \mathsf{Rem}^{\mathscr{F}} U_i$, *i.e.* $\mathsf{Rem}^{\mathscr{F}} U_i$ *is convex.*

Proof. The first and the third are immediate from the definition.

For the second, suppose that $V \in {\downarrow}\mathscr{F}$ then $V \in \mathsf{Rem}^{\mathscr{F}} \bigcap_{V \in \downarrow U_i} U_i$. ∎

Every partition of a set induces an equivalence relation on this set. The members of the partition comprise the equivalence classes. Since a splitting induces a partition, we denote the equivalence relation induced by a splitting $\mathscr{F}$ by $\sim_{\mathscr{F}}$.

Definition 8 Given a set of open subsets $\mathscr{G}$, we define the relation $\sim'_{\mathscr{G}}$ on $\mathscr{T}$ with $V_1 \sim'_{\mathscr{G}} V_2$ if and only if $V_1 \subseteq U \Leftrightarrow V_2 \subseteq U$ for all $U \in \mathscr{G}$.

We have the following

Proposition 9 *The relation* $\sim'_{\mathscr{G}}$ *is an equivalence.*

Proposition 10 *Given a finite splitting* $\mathscr{F}$, $\sim'_{\mathscr{F}} = \sim_{\mathscr{F}}$ *i.e. the remainders of* $\mathscr{F}$ *are the equivalence classes of* $\sim'_{\mathscr{F}}$.

Proof. Suppose $V_1 \sim'_{\mathscr{F}} V_2$ then $V_1, V_2 \in \mathsf{Rem}^{\mathscr{F}} U$, where

$$U \quad = \quad \bigcap\{\, U' \mid V_1, V_2 \subseteq U,\ U' \in \mathscr{F} \,\}.$$

For the other way suppose $V_1, V_2 \in \mathsf{Rem}^{\mathscr{F}} U$ and that there exists $U' \in \mathscr{F}$ such that $V_1 \subseteq U'$ while $V_2 \not\subseteq U'$. Then we have that $V_1 \subseteq U' \cap U$, $U' \cap U \in \mathscr{F}$ and $U' \cap U \subseteq U$ i.e. $V_1 \notin \mathsf{Rem}^{\mathscr{F}} U$. ∎

We state some useful facts about splittings.

Proposition 11 *If* $\mathscr{G}$ *is a finite set of opens, then* $\mathsf{Cl}(\mathscr{G})$*, its closure under intersection, yields a finite splitting for* $\downarrow\mathscr{G}$*.*

The last proposition enables us to give yet another characterization of remainders: every family of points in a complete lattice closed under arbitrary joins comprises a *closure system*, i.e. a set of fixed points of a closure operator of the lattice (cf. [12].) Here, the lattice is the poset of the opens of the topological space. If we restrict ourselves to a finite number of fixed points then we just ask for a finite set of opens closed under intersection i.e. Proposition 11. Thus a closure operator in the lattice of the open subsets of a topological space induces an equivalence relation, two opens being equivalent if they have the same closure, and the equivalence classes of this relation are just the remainders of the open subsets which are fixed points of the closure operator. The maximum open in $\mathsf{Rem}^{\mathscr{F}} U$, i.e. U, can be taken as the representative of the equivalence class which is the union of all open sets belonging to $\mathsf{Rem}^{\mathscr{F}} U$.

We now introduce the notion of stability corresponding to what we mean by "a formula retains its truth value on a set of opens".

Definition 12 If $\mathscr{G}$ is a set of opens then $\mathscr{G}$ is *stable for* ϕ, if for all x, either $x, V \models \phi$ for all $V \in \mathscr{G}$, or $x, V \models \neg\phi$ for all $V \in \mathscr{G}$, such that $x \in V$.

Proposition 13 *If* $\mathscr{G}_1$,$\mathscr{G}_2$ *are sets of opens then*

a. if $\mathscr{G}_1 \subseteq \mathscr{G}_2$ *and* $\mathscr{G}_2$ *is stable for* ϕ *then* $\mathscr{G}_1$ *is stable for* ϕ *,*

b. if $\mathscr{G}_1$ *is stable for* ϕ *and* $\mathscr{G}$ *is stable for* χ *then* $\mathscr{G}_1 \cap \mathscr{G}_2$ *is stable for* $\phi \wedge \chi$*.*

Proof. (a) is easy to see while (b) is a corollary of (a). ∎

Definition 14 A finite splitting $\mathscr{F} = \{U_1, \ldots, U_n\}$ is called a *stable splitting for* ϕ, if $\mathsf{Rem}^{\mathscr{F}} U_i$ is stable for ϕ for all $U_i \in \mathscr{F}$.

Proposition 15 *If* $\mathscr{F} = \{U_1, \ldots, U_n\}$ *is a stable splitting for* ϕ*, so is*

$$\mathscr{F}' = \mathsf{Cl}(\{U_0, U_1, \ldots, U_n\}),$$

where $U_0 \in \downarrow\mathscr{F}$*.*

Proof. Let $V \in \mathscr{F}'$ then there exists $U_l \in \mathscr{F}$ such that $\mathsf{Rem}^{\mathscr{F}'} V \subseteq \mathsf{Rem}^{\mathscr{F}} U_l$ (e.g. $U_l = \bigcap\{U_i | U_i \in \mathscr{F}, V \subseteq U_i\}$) i.e. $\mathscr{F}'$ is a *refinement* of $\mathscr{F}$. But $\mathsf{Rem}^{\mathscr{F}} U_l$ is stable for ϕ and so is $\mathsf{Rem}^{\mathscr{F}'} V$ by Proposition 13(a). ∎

The above proposition tells us that if there is a finite stable splitting for a topology then there is a closure operator with finitely many fixed points whose associated equivalence classes are stable sets of open subsets.

Suppose that $\mathscr{M} = \langle X, \mathscr{T}, i\rangle$ is a topological model for $\mathscr{L}$. Let $\mathscr{F}_{\mathscr{M}}$ be a family of subsets of X generated as follows: $i(A) \in \mathscr{F}_{\mathscr{M}}$ for all $A \in \mathsf{A}$, if $S \in \mathscr{F}_{\mathscr{M}}$ then $X - S \in \mathscr{F}_{\mathscr{M}}$, if $S, T \in \mathscr{F}_{\mathscr{M}}$ then $S \cap T \in \mathscr{F}_{\mathscr{M}}$, and if $S \in \mathscr{F}_{\mathscr{M}}$ then $S^\circ \in \mathscr{F}_{\mathscr{M}}$ i.e. $\mathscr{F}_{\mathscr{M}}$ is the least set containing $\{i(A)|A \in \mathsf{A}\}$ and closed under complements, intersections and interiors. Let $\mathscr{F}^\circ_{\mathscr{M}}$ be the set $\{S^\circ|S \in \mathscr{F}_{\mathscr{M}}\}$. We have $\mathscr{F}^\circ_{\mathscr{M}} = \mathscr{F}_{\mathscr{M}} \cap \mathscr{T}$. The following is the main theorem of this section.

Theorem 16 (Partition Theorem) *Let $\mathscr{M} = \langle X, \mathscr{T}, i\rangle$ be a topological model. Then there exists a a set $\{\mathscr{F}^\psi\}_{\psi\in\mathscr{L}}$ of finite stable splittings such that*

1. *$\mathscr{F}^\psi \subseteq \mathscr{F}^\circ_{\mathscr{M}}$ and $X \in \mathscr{F}^\psi$, for all $\psi \in \mathscr{L}$,*
2. *if $U \in \mathscr{F}^\psi$ then $U^\psi = \{x \in U | x, U \models \psi\} \in \mathscr{F}_{\mathscr{M}}$, and*
3. *if ϕ is a subformula of ψ then $\mathscr{F}^\phi \subseteq \mathscr{F}^\psi$ and $\mathscr{F}^\psi$ is a finite stable splitting for ϕ,*

where $\mathscr{F}_{\mathscr{M}}$, $\mathscr{F}^\circ_{\mathscr{M}}$ as above.

Proof. By induction on the structure of the formula ψ. In each step we take care to refine the partition of the induction hypothesis.

- If $\psi = A$ is an atomic formula, then $\mathscr{F}^A = \{X, \emptyset\} = \{i(\top), i(\bot)\}$, since $\mathscr{T}$ is stable for all atomic formulae. We also have $\mathscr{F}^A \subseteq \mathscr{F}^\circ_{\mathscr{M}}$ and $X^A = i(A) \in \mathscr{F}_{\mathscr{M}}$.

- If $\psi = \neg\phi$ then let $\mathscr{F}^\psi = \mathscr{F}^\phi$, since the statement of the proposition is symmetric with respect to negation. We also have that for an arbitrary $U \in \mathscr{F}^\psi$, $U^\psi = U^\neg\phi$.

- If $\psi = \chi \wedge \phi$, let
$$\mathscr{F}^\psi = \mathsf{Cl}(\mathscr{F}^\chi \cup \mathscr{F}^\phi).$$
Now, $\mathscr{F}^\psi$ is a finite stable splitting containing X, by induction hypothesis. Observe that $\mathscr{F}^\chi \cup \mathscr{F}^\phi \subseteq \mathscr{F}^{\chi\wedge\phi}$.

  Now, if $W_i \in \mathscr{F}^\psi$ then there exists $U_j \in \mathscr{F}^\chi$ and $V_k \in \mathscr{F}^\phi$ such that
$$W_i = U_j \cap V_k \quad \text{and} \quad \mathsf{Rem}^{\mathscr{F}^\psi} W_i \quad \subseteq \quad \mathsf{Rem}^{\mathscr{F}^\chi} U_j \cap \mathsf{Rem}^{\mathscr{F}^\phi} V_k$$
(e.g. $U_j = \bigcap\{U_m | W_i \subseteq U_m,\ U_m \in \mathscr{F}^\chi\}$ and $V_k = \bigcap\{V_n | W_i \subseteq V_n,\ V_n \in \mathscr{F}^\phi\}$.) Since $\mathsf{Rem}^{\mathscr{F}^\chi} U_j$ is stable for χ and $\mathsf{Rem}^{\mathscr{F}^\phi} V_n$ is stable for ϕ, their intersection is stable for $\chi \wedge \phi = \psi$, by Proposition 13(b), and so is its subset $\mathsf{Rem}^{\mathscr{F}^\psi} W_i$, by Proposition 13(a). Thus $\mathscr{F}^\psi$ is a finite stable splitting for ψ containing X.

  We have that $\mathscr{F}^\psi \subseteq \mathscr{F}_{\mathscr{M}}$ whenever $\mathscr{F}^\chi \subseteq \mathscr{F}_{\mathscr{M}}$ and $\mathscr{F}^\phi \subseteq \mathscr{F}^\circ_{\mathscr{M}}$. Finally, $W_i^\psi = U_j^\chi \cap V_k^\phi$.

- Suppose $\psi = \mathsf{K}\phi$. Then, by induction hypothesis, there exists a finite stable splitting $\mathscr{F}^\phi = \{U_1, \ldots, U_n\}$ for ϕ containing X. Let
$$W_i = (U_i^\phi)^\circ,$$

for all $i \in \{1, \ldots, n\}$.

Observe that if $x \in U_i - W_i$ then $x, V \models \neg\phi$, for all $V \in \mathsf{Rem}^{\mathscr{F}^\phi} U_i$ and $x \in V$, since $\mathsf{Rem}^{\mathscr{F}^\phi} U_i$ is stable for ϕ, by induction hypothesis.

Now, if $V \in \mathsf{Rem}^{\mathscr{F}^\phi} U_i \cap \downarrow W_i$, for some $i \in \{1 \ldots, n\}$, then $x, V \models \phi$ for all $x \in V$, by definition of W_i, hence $x, V \models \mathsf{K}\phi$ for all $x \in V$.

On the other hand, if $V \in \mathsf{Rem}^{\mathscr{F}^\phi} U_i - \downarrow W_i$ then there exists $x \in V$ such that $x, V \models \neg\phi$ (otherwise $V \subseteq W_i$.) Thus we have $x, V \models \neg\mathsf{K}\phi$ for all $x \in V$. Hence $\mathsf{Rem}^{\mathscr{F}^\phi} U_i \cap \downarrow W_i$ and $\mathsf{Rem}^{\mathscr{F}^\phi} U_i - \downarrow W_i$ are stable for $\mathsf{K}\phi$. Thus, the set

$$F = \{\mathsf{Rem}^{\mathscr{F}} U_i |\ W_i \notin \mathsf{Rem}^{\mathscr{F}} U_i\} \cup \{\mathsf{Rem}^{\mathscr{F}} U_j \downarrow W_j, \mathsf{Rem}^{\mathscr{F}} U_j \cap \downarrow W_j |\ W_j \in U_j\}$$

is a partition of $\mathscr{T}$ and its members are stable for $\mathsf{K}\phi$. Let $\sim_F$ be the equivalence relation on $\mathscr{T}$ induced by F and let

$$\mathscr{F}^{\mathsf{K}\phi} = \mathsf{Cl}(\mathscr{F}^\phi \cup \{\ W_i \mid W_i \in \mathsf{Rem}^{\mathscr{F}^\phi} U_i\}).$$

We have that $\mathscr{F}^{\mathsf{K}\phi}$ is a finite set of opens and $\mathscr{F}^\phi \subseteq \mathscr{F}^{\mathsf{K}\phi}$. Thus, $\mathscr{F}^{\mathsf{K}\phi}$ is finite and contains X. We have only to prove that $\mathscr{F}^{\mathsf{K}\phi}$ is a stable splitting for $\mathsf{K}\phi$, i.e. every remainder of an open in $\mathscr{F}^{\mathsf{K}\phi}$ is stable for $\mathsf{K}\phi$.

If $V_1 \not\sim_F V_2$, where $V_1, V_2 \in \mathscr{T}$, then there exists $U = U_i$ or W_i for some $i = 1, \ldots, n$ such that $V_1 \subseteq U$ while $V_2 \not\subseteq U$. But this implies that $V_1 \not\sim_{\mathscr{F}^{\mathsf{K}\phi}} V_2$. Therefore $\{\mathsf{Rem}^{\mathscr{F}^{\mathsf{K}\phi}} U\}_{U \in \mathscr{F}^{\mathsf{K}\phi}}$ is a refinement of F and $\mathscr{F}^{\mathsf{K}\phi}$ is a finite stable splitting for $\mathsf{K}\phi$ using Proposition 13(a).

We have that $\mathscr{F}^{\mathsf{K}\phi} \subseteq \mathscr{F}^{\circ}_{\mathscr{M}}$ because $W_i \in \mathscr{F}^{\circ}_{\mathscr{M}}$, for $i = 1, \ldots, n$. Now if $U \in \mathscr{F}^{\psi}$ then either $U^{\mathsf{K}\phi} = U$ or $U^{\mathsf{K}\phi} = \emptyset$.

- Suppose $\psi = \Box\phi$. Then, by induction hypothesis, there exists a finite stable splitting $\mathscr{F}^{\phi} = \{U_1, \ldots, U_n\}$ for ϕ containing X.

  Let

  $$\mathscr{F}^{\Box\phi} = \mathsf{Cl}(\mathscr{F}^{\phi} \cup \{U_i \Rightarrow U_j | \ 1 \leq i, j \leq n\}),$$

  where $\Rightarrow$ is the implication of the complete Heyting algebra $\mathscr{T}$ i.e. $V \subseteq U \Rightarrow W$ if and only if $V \cap U \subseteq W$ for $V, U, W \in \mathscr{T}$. We have that $U \Rightarrow W$ equals $(X - (U - W))^{\circ}$. Clearly, $\mathscr{F}^{\Box\phi}$ is a finite splitting containing X and $\mathscr{F}^{\phi} \subseteq \mathscr{F}^{\Box\phi}$. We have only to prove that $\mathscr{F}^{\Box\phi}$ is stable for $\Box\phi$. But first, we prove the following claim:

  Claim 1 *Suppose $U \in \mathscr{F}^{\phi}$ and $U' \in \mathscr{F}^{\Box\phi}$. Then*

  $$U' \cap U \in \mathsf{Rem}^{\mathscr{F}^{\phi}} U \quad \Longleftrightarrow \quad V \cap U \in \mathsf{Rem}^{\mathscr{F}^{\phi}} U \textit{ for all } V \in \mathsf{Rem}^{\mathscr{F}^{\Box\phi}} U'.$$

  Proof. The one direction is straightforward. For the other, let $V \in \mathsf{Rem}^{\mathscr{F}^{\Box\phi}} U'$ and suppose $V \cap U \notin \mathsf{Rem}^{\mathscr{F}^{\phi}} U$ towards a contradiction. This implies that there exists $U'' \in \mathscr{F}^{\phi}$, with $U'' \subset U$, such that $V \cap U \subseteq U''$. Thus, $V \subseteq U \Rightarrow U''$ but $U' \not\subseteq U \Rightarrow U''$. But $U \Rightarrow U'' \in \mathscr{F}^{\Box\phi}$ which contradicts $U' \sim_{\mathscr{F}^{\Box\phi}} V$, by Proposition 10. ∎

Let $U' \in \mathscr{F}^{\Box\phi}$. We must prove that $\mathsf{Rem}^{\mathscr{F}^{\Box\phi}}U'$ is stable for $\Box\phi$.

Suppose that $x, U' \models \neg\Box\phi$. We must prove that

$$x, V' \models \neg\Box\phi$$

for all $V' \in \mathsf{Rem}^{\mathscr{F}^{\Box\phi}}U'$ such that $x \in V'$.

Since $x, U' \models \neg\Box\phi$, there exists $V \in \mathscr{T}$, with $x \in V$ and $V \subseteq U'$, such that $x, V \models \neg\phi$. Since $\mathscr{F}^{\phi}$ is a splitting, there exists $U \in \mathscr{F}^{\phi}$ such that $V \in \mathsf{Rem}^{\mathscr{F}^{\phi}}U$. Observe that $V \subseteq U' \cap U \subseteq U$, so $U' \cap U \in \mathsf{Rem}^{\mathscr{F}^{\phi}}U$, by Proposition 7(c).

By Claim 1, for all $V' \in \mathsf{Rem}^{\mathscr{F}^{\Box\phi}}U'$, we have $V' \cap U \in \mathsf{Rem}^{\mathscr{F}^{\phi}}U$. Thus if $x \in V'$ then $x, V' \cap U \models \neg\phi$, because $\mathsf{Rem}^{\mathscr{F}^{\phi}}U$ is stable for ϕ, by induction hypothesis. This implies that, for all V' such that $V' \in \mathsf{Rem}^{\mathscr{F}^{\Box\phi}}U'$ and $x \in V$, we have $x, V' \models \neg\Box\phi$.

Therefore, $\mathscr{F}^{\Box\phi}$ is a finite stable splitting for $\Box\phi$.

Now $U_i \Rightarrow U_j \in \mathscr{F}^{\circ}_{\mathscr{M}}$ for $1 \leq i, j \leq n\}$, hence $\mathscr{F}^{\Box\phi} \subseteq \mathscr{F}^{\circ}_{\mathscr{M}}$.

Finally, let U belong to $\mathscr{F}^{\Box\phi}$ and $V_1, \ldots, V_m$ be all opens in $\mathscr{F}^{\phi}$ such that $U \cap V_i \in \mathsf{Rem}^{\mathscr{F}^{\phi}}V_i$, for $i = 1, \ldots, m$. Then $x, U \models \Diamond\neg\phi$ if and only if there exists $j \in \{1, \ldots, m\}$ with $x \in V_j$ and $x, V_j \models \neg\phi$ because $x, V_j \cap U \models \neg\phi$ since $V_j \cap U \in \mathsf{Rem}^{\mathscr{F}^{\phi}}V_j$. This implies that

$$U^{\neg\Box\phi} \quad = \quad U^{\Diamond\neg\phi} \quad = \quad U \cap \bigcup_{i=1}^{m} V_i^{\neg\phi}.$$

Since $U, V_1^{\neg\phi}, \ldots, V^{\neg\phi}$ belong to $\mathscr{F}_{\mathscr{M}}$, so does $U^{\neg\Box\phi}$ and, therefore, $U^{\Box\phi} = U - U^{\neg\Box\phi}$.

In all steps of induction we refine the finite splitting, so if ϕ is a subformula of ψ then $\mathscr{F}^{\phi} \subseteq \mathscr{F}^{\psi}$ and $\mathscr{F}^{\psi}$ is stable for ϕ using Proposition 13(a). ∎

Theorem 16 gives us a great deal of intuition for topological models. It describes in detail the expressible part of the topolocical lattice for the completeness result as it appears in Chapter 4 and paves the road for the reduction of the theory of topological models to that of spatial lattices and the decidability result of this chapter.

3.2 Basis Model

Let $\mathscr{T}$ be a topology on a set X and $\mathscr{B}$ a basis for $\mathscr{T}$. We denote satisfaction in the models $\langle X, \mathscr{T}, i\rangle$ and $\langle X, \mathscr{B}, i\rangle$ by $\models_{\mathscr{T}}$ and $\models_{\mathscr{B}}$, respectively. In the following proposition we prove that each equivalence class under $\sim_{\mathscr{F}}$ contains an element of a basis closed under finite unions.

Proposition 17 *Let $(X, \mathscr{T})$ be a topological space, and let $\mathscr{B}$ be a basis for $\mathscr{T}$ closed under finite unions. Let $\mathscr{F}$ be any finite subset of $\mathscr{T}$. Then for all $V \in \mathscr{F}$ and all $x \in V$, there is some $U \in \mathscr{B}$ with $x \in U \subseteq V$ and $U \in \mathsf{Rem}^{\mathscr{F}} V$.*

Proof. By finiteness of $\mathscr{F}$, let $V_1, \ldots, V_k$ be the elements of $\mathscr{F}$ such that $V \not\subseteq V_i$, for $i \in \{1, \ldots, k\}$. Since $V_i \neq V$, take $x_i \in V - V_i$ for $i \in \{1, \ldots, k\}$. Since $\mathscr{B}$ is a basis for $\mathscr{T}$, there exist U_x, U_i, with $x \in U_x$ and $x_i \in U_i$, such that U_x and U_i are subsets of V for $i \in \{1, \ldots, k\}$. Set

$$U = (\bigcup_{i=1}^{k} U_i) \cup U_x.$$

Observe that $x \in U$, and $U \in \mathscr{B}$, as it is a finite union of members of $\mathscr{B}$. Also $U \in \mathsf{Rem}^{\mathscr{F}} V$, since $U \in {\downarrow}V$ but $U \notin \bigcup {\downarrow}V_i$ for $i \in \{1, \ldots, k\}$. ∎

Corollary 18 *Let $(X, \mathscr{T})$ be a topological space, $\mathscr{B}$ a basis for $\mathscr{T}$ closed under finite unions, $x \in X$ and $U \in \mathscr{B}$. Then*

$$x, U \models_{\mathscr{T}} \phi \iff x, U \models_{\mathscr{B}} \phi.$$

Proof. By induction on ϕ.

The interesting case is when $\phi = \Box\psi$. Fix x, U, and ψ. By Proposition 16, there exists a finite stable splitting $\mathscr{F}$ for ϕ and its subformulae such that $\mathscr{F}$ contains X and U. Assume that $x, U \models_{\mathscr{B}} \Box\psi$, and $V \in \mathscr{T}$ such that $V \subseteq U$. By Proposition 7(b), there is some $V' \subseteq U$ in $\mathscr{F}$ with $V \in \mathsf{Rem}^{\mathscr{F}} V'$. By Proposition 17, let $W \in \mathscr{B}$ be such that $W \in \mathsf{Rem}^{\mathscr{F}} V'$ with $x \in W$. So $x, W \models_{\mathscr{B}} \psi$, and thus by induction hypothesis, $x, W \models_{\mathscr{T}} \psi$. By stability, twice, $x, V \models_{\mathscr{T}} \psi$ as well. ∎

We are now going to prove that a model based on a topological space $\mathscr{T}$ is equivalent to the one induced by any basis of $\mathscr{T}$ which is lattice. Observe that this enables us to reduce the theory of topological spaces to that of spatial lattices and, therefore, to answer the conjecture of [24] : a completeness theorem for subset spaces which are lattices will extend to the smaller class of topological spaces.

Theorem 19 *Let $(X, \mathscr{T})$ be a topological space and $\mathscr{B}$ a basis for $\mathscr{T}$ closed under finite unions. Let $\mathscr{M}_1 = \langle X, \mathscr{T}, i\rangle$ and $\mathscr{M}_2 = \langle X, \mathscr{B}, i\rangle$ be the corresponding models. Then, for all ϕ,*

$$\mathscr{M}_1 \models \phi \iff \mathscr{M}_2 \models \phi.$$

Proof. It suffices to prove that $x, U \models_{\mathscr{T}} \phi$, for some $U \in \mathscr{T}$, if and only if $x, U' \models_{\mathscr{B}} \phi$, for some $U' \in \mathscr{B}$.

Suppose $x, U \models_{\mathscr{T}} \phi$, where $U \in \mathscr{T}$, then, by Corollary 18, there exists $U' \in \mathscr{B}$ such that $x \in U'$ and $x, U \models_{\mathscr{T}} \phi$. By Corollary 18, $x, U' \models_{\mathscr{B}} \phi$.

Suppose $x, U \models_{\mathscr{B}} \phi$, where $U \in \mathscr{B}$, then $x, U \models_{\mathscr{T}} \phi$, by Corollary 18. ∎

3.3 Finite Satisfiability

Proposition 20 *Let $\langle X, \mathscr{T}\rangle$ be a subset space. Let $\mathscr{F}$ be a finite stable splitting for a formula ϕ and all its subformulae, and assume that $X \in \mathscr{F}$. Then for all $U \in \mathscr{F}$, all $x \in U$, and all subformulae ψ of ϕ, $x, U \models_{\mathscr{T}} \psi$ iff $x, U \models_{\mathscr{F}} \psi$.*

PROOF. The argument is by induction on ϕ. The only interesting case to consider is when $\phi = \Box\psi$.

Suppose first that $x, U \models_{\mathscr{F}} \Box\psi$ with $U \in \mathscr{F}$. We must show that $x, U \models_{\mathscr{T}} \Box\psi$ also. Let $V \in \mathscr{T}$ such that $V \subseteq U$; we must show that $x, V \models_{\mathscr{T}} \psi$. By Proposition 7(b), there is some $V' \subseteq U$ in $\mathscr{F}$ with $V \in \mathsf{Rem}^{\mathscr{F}} V'$. So $x, V' \models_{\mathscr{F}} \psi$, and by induction hypothesis, $x, V' \models_{\mathscr{T}} \psi$. By stability, $x, V \models_{\mathscr{T}} \psi$ also.

The other direction (if $x, U \models_{\mathscr{T}} \Box\psi$, then $x, U \models_{\mathscr{F}} \Box\psi$), is an easy application of the induction hypothesis. ∎

Constructing the quotient of $\mathscr{T}$ under $\sim_{\mathscr{F}}$ is not adequate for generating a finite model because there may still be an infinite number of points. It turns out that we only need a finite number of them.

Let $\mathscr{M} = \langle X, \mathscr{T}, i \rangle$ be a topological model, and define an equivalence relation $\sim$ on X by $x \sim y$ iff

(a) for all $U \in \mathscr{T}$, $x \in U$ iff $y \in U$, and

(b) for all atomic A, $x \in i(A)$ iff $y \in i(A)$.

Further, denote by x^* the equivalence class of x, and let $X^* = \{x^* : x \in X\}$. For every $U \in \mathscr{T}$ let $U^* = \{x^* : x \in X\}$, then $\mathscr{T}^* = \{U^* : U \in \mathscr{T}\}$ is a topology on X^*. Define a map i^* from the atomic formulae to the powerset of X^* by $i^*(A) = \{x^* : x \in i(A)\}$. The entire model $\mathscr{M}$ lifts to the model $\mathscr{M}^* = \langle X^*, \mathscr{T}^*, i^* \rangle$ in a well-defined way.

Lemma 21 *For all x, U, and ϕ,*

$$x, U \models_{\mathscr{M}} \phi \qquad \textit{iff} \qquad x^*, U^* \models_{\mathscr{M}^*} \phi \ .$$

PROOF. By induction on ϕ. ∎

Theorem 22 *If ϕ is satisfied in any topological space then ϕ is satisfied in a finite topological space.*

PROOF. Let $\mathscr{M} = \langle X, \mathscr{T}, i \rangle$ be such that for some $x \in U \in \mathscr{T}$, $x, U \models_{\mathscr{M}} \phi$. Let $\mathscr{F}^\phi$ be a finite stable splitting (by Theorem 16) for ϕ and its subformulae with respect to $\mathscr{M}$. By Proposition 20, $x, U \models_{\mathscr{N}} \phi$, where $\mathscr{N} = \langle X, \mathscr{F}, i \rangle$. We may assume that $\mathscr{F}$ is a topology, and we may also assume that the overall language has only the (finitely many) atomic symbols which occur in ϕ. Then the relation $\sim$ has only finitely many classes. So the model $\mathscr{N}^*$ is finite. Finally, by Lemma 21, $x^*, U^* \models_{\mathscr{N}^*} \phi$. ∎

Observe that the finite topological space is a quotient of the initial one under two equivalences. The one equivalence is $\sim^\phi_{\mathscr{F}}$ on the open subsets of the topological space, where $\mathscr{F}^\phi$ is the finite splitting corresponding to ϕ and its cardinality is a function of the complexity of ϕ. The other equivalence is $\sim_X$ on the points of the topological space and its number of equivalence classes is a function of the atomic formulae appearing in ϕ. The following simple example shows how a topology is formed with the quotient under these two equivalences

Example: Let X be the interval $[0,1)$ of real line with the the set

$$\mathscr{T} \quad = \quad \{\emptyset\} \cup \{\ [0, \frac{1}{2^n}) \mid n = 0, 1, 2, \dots\ \}$$

as topology. Suppose that we have only one atomic formula, call it A, such that $i(A) = \{0\}$. then it is easy to see that the model $\langle X, \mathscr{T}, i\rangle$ is equivalent to the finite topological model $\langle X^*, \mathscr{T}^*, i^*\rangle$, where

$$\begin{aligned} X^* &= \{\ x_1, x_2\ \}, \\ \mathscr{T}^* &= \{\ \emptyset, \{x_1, x_2\}\ \},\ \text{and} \\ i(A) &= \{\ x_1\ \}. \end{aligned}$$

So the overall size of the (finite) topological space is bounded by a function of the complexity of ϕ. Thus if we want to test if a given formula is invalid we have a finite number of finite topological spaces where we have to test its validity. Thus we have the following

Theorem 23 *The theory of topological spaces is decidable.*

Observe that the last two results apply for lattices of subsets by Theorem 19.

Chapter 4

Completeness for $\mathbf{MP}^*$

Open subsets of a topological space were used in [24] and in the previous section to provide motivation, intuition and finally semantics for $\mathbf{MP}^*$. But in this chapter we shall show that the canonical model of $\mathbf{MP}^*$ is actually a set of subsets closed under arbitrary intersection and finite union, i.e. the *closed* subsets of a topological space. However, these results are not contrary to our intuition for the following reasons: the spatial character of this logic remains untouched. The fact that the canonical model is closed under arbitrary intersections implies strong completeness with the much wider class of sets of subsets closed under finite intersection and finite union. Now, the results of the previous section allow us to deduce strong completeness (in the sense that a consistent set of formulae is simultaneously satisfiable in some model) also for the class of sets of subsets closed under infinite union and finite intersection, i.e. the open subsets of a topological space.

4.1 Subset frames

As we noted in section 2.1, we are not interpreting formulae directly over a subset space but, rather in the pointed product $X \dot{\times} \mathscr{O}$. The pointed product can be turned in a set of possible worlds of a frame. We have only to indicate what the accessibility relations are.

Definition 24 Let $(X, \mathscr{O})$ be a subset space. Its *subset frame* is the frame

$$\langle X \dot{\times} \mathscr{O}, R_{\Box}, R_{\mathsf{K}} \rangle,$$

where

$$(x, U) R_{\Box} (y, V) \quad \text{if} \quad U = V$$

and

$$(x, U) R_{\mathsf{K}} (y, V) \quad \text{if} \quad x = y \text{ and } V \subseteq U.$$

If $\mathscr{O}$ is a topology, intended as the closed subsets of a topological space, we shall call its subset frame *closed topological frame.*

Our aim is to prove the most important properties of such a frame. We propose the following conditions on a possible worlds frame $\mathscr{F} = \langle S, R_1, R_2 \rangle$ with two accessibility relations

1. R_1 is reflexive and transitive.

2. R_2 is an equivalence relation.

3. $R_1R_2 \subseteq R_2R_1$

4. (ending points) $\mathscr{F}$ has ending points with respect R_1, i.e

   for all $s \in S$ there exists $s_0 \in S$ such that for all $s' \in S$ if sR_1s' then $s'R_1s_0$.

5. (extensionality condition) For all $s, s' \in S$, if there exists $s_0 \in S$ such that sR_1s_0 and $s'R_1s_0$ and

   for all $t \in S$ such that tR_2s there exist $t', t_0 \in S$ such that $t'R_2s'$, tR_1t_0 and $t'R_1t_0$, and for all $t' \in S$ such that $t'R_2s'$ there exist $t, t_0 \in S$ such that tR_2s, $t'R_1t_0$ and tR_1t_0,

   then $s = s'$.

6. (union condition) For all $s_1, s_2 \in S$,

   if there exists $s \in S$ such that $sR_2R_1s_1$ and $sR_2R_1s_2$, then there exists $s' \in S$ such that for all $t \in S$ with tR_2s' then $tR_1R_2s_1$ or $tR_1R_2s_2$.

7. (intersection condition) For all $\{s_i\}_{i \in I} \subseteq S$,

if there exists $s \in S$ such that $s_i R_1 s$ for all $i \in I$ then there exists $s' \in S$ such that for all $\{t_i\} \subseteq S$ with $t_i R_2 s_i$ and $t_i R_1 t_0$ for all $i \in I$ and some $t_0 \in S$ then $t_i R_1 R_2 s'$.

8. The frame $\mathscr{F}$ is strongly generated in the sense that

   there exists $s \in S$ such that for all $s' \in S$, $s R_2 R_1 s'$.

We have the following observations to make about the above conditions. Conditions 1 to 6 and 8 are first order, while the intersection condition is not. The extensionality condition implies the following

for all $s, s' \in S$ such that $s R_1 s_0$ and $s' R_2 s_0$ then $s = s'$

which implies that $R_1 \cap R_2$ is the identity in S. In view of the extensionality condition the relation R_1 is antisymmetric. So we can replace condition 1 with the condition that R_1 is a partial order.

Now, we have the following proposition

Proposition 25 *If $(X, \mathscr{T})$ is a topological space then its closed topological frame $\mathscr{F}_{\mathscr{T}}$ satisfies conditions 1 through 8.*

Proof. Let $R_1 = R_\Box$ and $R_2 = R_{\mathsf{K}}$. Conditions 1, 2, 3 are straightforward. For each $(x, V) \in X \dot{\times} \mathscr{T}$ the pair $(x, \bigcap_{x \in U} U)$ is its ending point with respect $R_\Box$ and condition 4 is satisfied. The extensionality condition represents the set-theoretic extensionality of the space. The union and intersection condition is satisfied because $\mathscr{T}$ is closed under finite unions and infinite intersections, respectively. Finally, $\mathscr{F}_{\mathscr{T}}$ is strongly generated by (x, X) for any $x \in X$. ∎

The above proposition could lead to the consequence that topological models are possible worlds models in disguise. But the following theorem shows that this is not the case. There is a duality.

Theorem 26 *Let $\mathscr{F} = \langle S, R_1, R_2 \rangle$ be a frame satisfying conditions 1 through 8. Then $\mathscr{F}$ is isomorphic to a closed topological frame $\mathscr{F}_{\mathscr{T}}$.*

Proof. We shall construct a topological space $(X, \mathscr{T})$ and a frame isomorphism f from $\mathscr{F}$ to $\mathscr{F}_{\mathscr{T}}$. Let

$$X \quad = \quad \{\ s \mid s \in S \text{ is an ending point of } \mathscr{F}\ \}$$

and

$$\mathscr{T} \quad = \quad \{\ U_t \mid t \in S\ \} \cup \{\emptyset\}.$$

We also let

$$s \in U_t \quad \text{if} \quad \text{there exists } s' \text{ such that } s'R_1s \text{ and } s'R_2t.$$

Note that, using conditions 1, 2, 3 we can show that

$$\text{if} \qquad s \in U_t \text{ implies } s \in U_{t'} \qquad \text{then} \qquad U_t \subseteq U_{t'}$$

and, by the extensionality condition,

$$U_t = U_{t'} \quad \text{if and only if} \quad tR_2t'.$$

Therefore the above settings are well defined.

It only remains to show that $\mathscr{T}$ is closed under infinite intersections and finite unions. For the former we must show that $\bigcap_{i\in I} U_{t_i}$ belongs to $\mathscr{T}$, for $U_{t_i} \in \mathscr{T}$, $i \in I$. If $\bigcap_{i\in I} U_{t_i} = \emptyset$ we are done. If not, then there exists $s \in U_{t_i}$, for all $i \in I$. This, by definition, implies that there exist $\{s_i\}_{i\in I}$ such that $s_iR_2t_i$ and s_iR_1s. Now, intersection condition applies and let s' be as in condition 7. We shall show $\bigcap_{i\in I} U_{t_i} = U_{s'}$. For the left to right subset direction, let $r \in \bigcap_{i\in I} U_{t_i}$. This implies, by definition, that there exist $\{r_i\}_{i\in I} \subseteq S$ such that $r_iR_2s_i$, thus $r_iR_2t_i$, and r_iR_1r for all $i \in I$. By the intersection condition $r_iR_1R_2s'$, and therefore $r \in U_{s'}$. For the other subset direction, let $r \in U_{s'}$. Then there exists $r' \in S$ such that $r'R_1r$ and rR_2s. Condition 3 implies that there exist $\{r_i\}_{i\in I} \subseteq S$ such that $r_iR_2t_i$ and r_iR_1r, thus $r \in \bigcap_{i\in I} U_{t_i}$ for all $i \in I$. Therefore $\bigcap_{i\in I} U_{t_i} = U_s' \in \mathscr{T}$.

We can prove similarly that $U_{t_1} \cup U_{t_2} = U_s'$, s' as in the union condition, using the union condition and condition 8.

Let f be the map from S to $X \dot{\times} \mathscr{T}$ defined in the following way

$$f(s) \quad = \quad (s_0, U_s)'$$

where s_0 is the ending point of s in $\mathscr{F}$.

The map f is a frame isomorphism. If $(s, U_t) \in X \dot{\times} \mathscr{T}$ then there exists $s' \in S$ such that $s' R_1 s$ and $s' R_2 t$. We have $f(s') = (s, U_t)$ and f is onto.

Let $f(s) = f(r) = (s_0, U_t)$ for some $s, s_0, r, t \in S$. We have that $s R_1 s_0$, $r R_1 s_0$, $s R_2 t$, and $r R_2 t$. By extensionality property, $s = r$ and f is bijective.

Now observe that

$$\begin{aligned} t R_1 s \quad & \text{if and only if} \quad U_s \subseteq U_t \\ & \text{if and only if} \quad (t_0, U_t) R_\Box (t_0, U_s) \\ & \text{if and only if} \quad f(t) R_\Box f(s), \end{aligned}$$

where t_0 is the common ending point of t and s in $\mathscr{F}$. We have also

$$\begin{aligned} t R_2 s \quad & \text{if and only if} \quad U_t = U_s \\ & \text{if and only if} \quad (t_0, U_t) R_{\mathsf{K}} (s_0, U_s) \\ & \text{if and only if} \quad f(t) R_{\mathsf{K}} f(s). \end{aligned}$$

Therefore f preserves the accessibility relations in both directions and is a frame isomorphism. ∎

Note that, in the above definitions, we could have used equally well the equivalence class of $s \in S$ under the equivalence induced by the symmetric closure of R_1 instead of the ending point of s in $\mathscr{F}$. The above proofs show that the crucial conditions are conditions 1 through 5 and if we are to strengthen or relax the union and intersection conditions we get accordingly different conditions in the lattice of the set of subsets of the space. The same holds for condition 8. We only used this condition to show that there exists a top element, i.e. the whole space, and satisfy the hypothesis of the union condition. If we do not assume this condition the union of two subsets will belong to the set of subsets if they have an upper bound in it. We state this case formally without a proof because we are going to use it later.

Proposition 27 *1. Let $(X, \mathcal{O})$ be a subset space closed under infinite intersections and if $U, V \in \mathcal{O}$ have an upper bound in $\mathcal{O}$ then $U \cup V \in \mathcal{O}$. Then its frame $\mathcal{F}_{\mathcal{O}}$ satisfies conditions 1 through 7.*

2. A frame $\mathcal{F}$ satisfying conditions 1 through 7 is isomorphic to a frame $\mathcal{F}_{\mathcal{O}}$ where $(X, \mathcal{O})$ as in (1).

4.2 On the proof theory of MP*

We shall identify certain classes of formulae in $\mathcal{L}$. This approach is motivated by the results of Chapter 3. In fact, these formulae express definable parts of the lattice of subsets (see section 3.1.)

Definition 28 Let $\mathcal{L}' \subseteq \mathcal{L}$ be the set of formulae generated by the following rules:

$$\mathsf{A} \subseteq \mathcal{L}' \qquad \frac{\phi, \psi \in \mathcal{L}'}{\phi \wedge \psi \in \mathcal{L}'} \qquad \frac{\phi \in \mathcal{L}'}{\neg\phi, \Diamond\mathsf{K}\phi \in \mathcal{L}'}$$

Let $\mathcal{L}''$ be the set $\{\mathsf{K}\phi, \mathsf{L}\phi | \phi \in \mathcal{L}'\}$.

Formulae in $\mathcal{L}'$ have the following properties

Definition 29 A formula ϕ of $\mathscr{L}$ is called *persistent* whenever $\phi \to \Box\phi$ is a theorem (see also [24].)

A formula ϕ of $\mathscr{L}$ is called *anti-persistent* whenever $\neg\phi$ is persistent, i.e. $\neg\phi \to \Box\neg\phi$ (or, equivalently $\Diamond\phi \to \phi$) is a theorem.

A formula ϕ of $\mathscr{L}$ is called *bi-persistent* whenever $(\phi \to \Box\phi) \wedge (\neg\phi \to \Box\neg\phi)$ (or, equivalently $\Diamond\phi \to \Box\phi$) is a theorem.

Thus the truth of bi-persistent formulae depends only on the choice of the point of the space while the satisfaction of persistent formulae can change at most once in any model. We could go on and define a hierarchy of sets of formulae where each member of hierarchy contains all formulae which their satisfaction could change at most n times in all models.

All the following derivations are in **MP*** (Axioms 1 through 12 — see table at page 1.)

Proposition 30 *All formulae belonging to $\mathscr{L}'$ are bi-persistent.*

Proof. We prove it by induction, i.e. bi-persistence is retained through the application of the formation rules of $\mathscr{L}'$.

- If A is atomic then A is bi-persistent because of axiom 2.
- If $\phi = \neg\psi$ then ϕ is bi-persistent by induction hypothesis (IH) and the fact that bi-persistence is a symmetric property with respect negation.

- if $\phi = \Diamond \mathsf{K}\psi$ then we have the following

  1. $\Diamond\mathsf{K}\psi \to \Diamond\mathsf{K}\Box\psi$ by IH
  2. $\Diamond\mathsf{K}\Box\psi \to \Diamond\Box\mathsf{K}\psi$ by Axiom 10
  3. $\Diamond\Box\mathsf{K}\psi \to \Box\Diamond\mathsf{K}\psi$ by Axiom 11
  4. $\Diamond\mathsf{K}\psi \to \Box\Diamond\mathsf{K}\psi$ by 1,2,3

  and

  1. $\neg\Diamond\mathsf{K}\psi \to \Box\neg\mathsf{K}\psi$
  2. $\Box\neg\mathsf{K}\psi \to \Box\Box\neg\mathsf{K}\psi$ by Axiom 5
  3. $\Box\Box\neg\mathsf{K}\psi \to \Box\neg\Diamond\mathsf{K}\psi$
  4. $\neg\Diamond\mathsf{K}\psi \to \Box\neg\Diamond\mathsf{K}\psi$ by 1,2,3

  therefore ϕ is bi-persistent.

- If $\phi = \psi \wedge \chi$ then we have

  1. $\psi \wedge \chi \to \Box\psi \wedge \Box\chi$ by IH
  2. $\Box\psi \wedge \Box\chi \to \Box(\psi \wedge \chi)$ in **S4**

  and

  1. $\neg(\psi \wedge \chi) \to \neg\psi \vee \neg\chi$
  2. $\neg\psi \vee \neg\chi \to \Box\neg\psi \vee \Box\neg\chi$ by IH
  3. $\Box\neg\psi \vee \Box\neg\chi \to \Box(\neg\psi \vee \neg\chi)$ in **S4**
  4. $\Box(\neg\psi \vee \neg\chi) \to \Box\neg(\psi \wedge \chi)$
  5. $\neg(\psi \wedge \chi) \to \Box\neg(\psi \wedge \chi)$ by 1,2,3,4

∎

A faster (semantical) proof would be "the initial assignment on atomic formulae extends to the wider class of $\mathscr{L}'$"! This implies that formulae in $\mathscr{L}'$ define subsets of the topological space.

Formulae in $\mathscr{L}''$ have similar properties as the following lemma show.

Lemma 31 *If ϕ is bi-persistent then $\mathsf{K}\phi$ is persistent and $\mathsf{L}\phi$ is anti-persistent.*

PROOF.

1. $\Diamond\mathsf{L}\phi \rightarrow \mathsf{L}\Diamond\phi$ by Axiom 10
2. $\mathsf{L}\Diamond\phi \rightarrow \mathsf{L}\phi$ by bi-persistence of ϕ.

Similarly $\vdash_{\mathbf{MP}^*} \mathsf{K}\phi \rightarrow \Box\mathsf{K}\phi$.

∎

We prove some theorems of **MP*** that we are going to use later.

Lemma 32 *If ϕ is bi-persistent then $\vdash_{\mathbf{MP}} \Diamond(\phi \wedge \psi) \equiv \Diamond\phi \wedge \Diamond\psi$.*

PROOF. The one implication is straightforward, by normality. For the other

1. $\Diamond\phi \wedge \Diamond\psi \rightarrow \Box\phi \wedge \Diamond\psi$ by bi-persistence of ϕ
2. $\Box\phi \wedge \Diamond\psi \rightarrow \Diamond(\phi \wedge \psi)$ in a normal system.

∎

The following is the key lemma to the DNF Theorem and generalizes Axiom 12

Lemma 33 *For all n,*

$$\vdash_{\mathbf{MP}^*} \Diamond\mathsf{K}\phi \wedge \bigwedge_{i=1}^{n} \mathsf{L}\left(\Diamond\mathsf{K}\phi \wedge \psi_i\right) \rightarrow \Diamond\left(\mathsf{K}\phi \wedge \bigwedge_{i=1}^{n} \mathsf{L}\psi_n\right),$$

where ϕ, ψ_i *are bi-persistent.*

PROOF. By induction on n.

For $n = 1$

$$\vdash_{\mathbf{MP}^*} \Diamond\mathsf{K}\phi \wedge \mathsf{L}(\Diamond\mathsf{K}\phi \wedge \psi) \to \Diamond(\mathsf{K}\phi \wedge \mathsf{L}\psi),$$

follows by Axiom 12 and bi-persistence of ϕ and ψ. Suppose that the lemma is true for $n = k$.

For $n = k + 1$

1. $\Diamond\mathsf{K}\phi \wedge \mathsf{L}(\Diamond\mathsf{K}\phi \wedge \psi_1) \wedge \ldots \wedge \mathsf{L}(\Diamond\mathsf{K}\phi \wedge \psi_k) \wedge \mathsf{L}(\Diamond\mathsf{K}\phi \wedge \psi_{k+1})$

   $\to \Diamond(\mathsf{K}\phi \wedge \mathsf{L}\psi_1 \ldots \wedge \mathsf{L}\psi_k) \wedge \mathsf{L}(\Diamond\mathsf{K}\phi \wedge \psi_{k+1})$ by IH,

2. $\Diamond(\mathsf{K}\phi \wedge \mathsf{L}\psi_1 \ldots \wedge \mathsf{L}\psi_k) \wedge \mathsf{L}(\Diamond\mathsf{K}\phi \wedge \psi_{k+1})$

   $\to \Diamond(\mathsf{K}\phi \wedge \Diamond(\mathsf{L}\psi_1 \wedge \ldots \wedge \mathsf{L}\psi_k) \wedge \mathsf{L}\psi_{k+1})$ by Axiom 12,

3. $\Diamond(\mathsf{K}\phi \wedge \Diamond(\mathsf{L}\psi_1 \wedge \ldots \wedge \mathsf{L}\psi_k) \wedge \mathsf{L}\psi_{k+1})$

   $\to \Diamond(\mathsf{K}\phi \wedge \mathsf{L}\psi_1 \wedge \ldots \wedge \mathsf{L}\psi_k \wedge \mathsf{L}\psi_{k+1})$ by Lemma 31,

4. $\Diamond\mathsf{K}\phi \wedge \mathsf{L}(\Diamond\mathsf{K}\phi \wedge \psi_1) \wedge \ldots \wedge \mathsf{L}(\Diamond\mathsf{K}\phi \wedge \psi_k) \wedge \mathsf{L}(\Diamond\mathsf{K}\phi \wedge \psi_{k+1})$

   $\to \Diamond(\mathsf{K}\phi \wedge \mathsf{L}\psi_1 \wedge \ldots \wedge \mathsf{L}\psi_k \wedge \mathsf{L}\psi_{k+1})$ by 1,2,3.

∎

All formulae of $\mathscr{L}'$ can be expressed in terms of bi-persistent, persistent and antipersistent formulae by means of the following normal form.

Definition 34

1. ϕ is in *prime normal* form (PNF) if it has the form

$$\psi \wedge \mathsf{K}\psi' \wedge \bigwedge_{i=1}^{n} \mathsf{L}\psi_i$$

   where $\psi, \psi', \psi_i \in \mathscr{L}'$ and n is finite.

2. ϕ is in *disjunctive normal form* (DNF) if it has the form $\bigvee_{i=1}^{m} \phi_i$, where each ϕ_i is in PNF and m is finite.

To keep the notation bearable we shall omit the cardinality of (finite) conjunctions and disjunctions, writing, e.g. $\bigvee_i \phi_i$ instead of $\bigvee_{i=1}^{n} \phi_i$. Suppose that ϕ is a formula in the following form

$$\bigwedge_i \left(\psi_i \vee \mathsf{L}\psi_i' \vee \bigvee_j \mathsf{K}\psi_i^j \right),$$

where $\psi_i, \psi_i', \psi_i^j \in \mathscr{L}'$. We shall call such a form *conjunctive normal form* (CNF). Using the distributive laws, we get the equivalent formula

$$\bigvee_k \left(\bigwedge_{l_k} \psi_{i_{l_k}} \wedge \bigwedge_{m_k} \mathsf{L}\psi'_{i_{m_k}} \wedge \bigwedge_{n_k} \mathsf{K}\psi_{i_{n_k}}^{j_{n_k}} \right).$$

Since $\mathscr{L}'$ is closed under negation and conjunction and K distributes over conjunctions, we can express the above formula in the following form

$$\bigvee_k \left(\chi_k \wedge \mathsf{K}\chi_k' \wedge \bigwedge_{m_k} \mathsf{L}\chi_k^{m_k} \right),$$

where $\chi_k, \chi_k', \chi_k^{m_k}$ belong to $\mathscr{L}'$. So ϕ is equivalent to this formula which is in DNF. Therefore DNF and CNF are effectively interchangeable up to equivalence.

We now give the formal analogue of the Partition Theorem.

Theorem 35 (DNF) *For every $\phi \in \mathscr{L}$, there is (effectively) a ψ in DNF such that*

$$\vdash_{\mathbf{MP}^*} \phi \equiv \psi.$$

Proof. By induction on the logical structure of ϕ.

- If $\phi = A$, where A is atomic, the result is immediate because the set of atomic formulae belongs to $\mathscr{L}'$ and A is in PNF.

- Suppose that $\phi = \neg\psi$. Then, by induction hypothesis, ψ is equivalent to a formula in DNF, which implies that ϕ is equivalent to a formula in CNF and, by the above discussion, is equivalent to a formula in DNF.

- If $\phi = \psi \vee \chi$ then ϕ is equivalent to a disjunction of two formulae in DNF, i.e. is itself in DNF.

- If $\phi = \mathsf{K}\psi$ then ψ is equivalent to a formula in CNF, and hence ϕ is equivalent to a formula of the following form

$$\bigwedge_i \mathsf{K}\left(\chi_i \vee \mathsf{L}\chi_i' \vee \bigvee_j \mathsf{K}\chi_i^j\right),$$

since K distributes over conjunctions. Now , since the formula $\mathsf{K}(\phi \vee \mathsf{K}\psi) \leftrightarrow \mathsf{K}\phi \vee \mathsf{K}\psi$ is a theorem of **S5**, the above formula is equivalent to

$$\bigwedge_i \left(\mathsf{L}\chi_i' \vee \left(\mathsf{K}\chi_i \vee \bigvee_j \mathsf{K}\chi_i^j\right)\right),$$

which is in CNF.

- If $\phi = \Diamond\psi$ then, by induction hypothesis, ϕ is equivalent to a formula of the form

  $$\Diamond \bigvee_i \left(\chi_i \wedge \mathsf{K}\chi_i' \wedge \bigwedge_j \mathsf{L}\chi_i^j \right).$$

  Since $\Diamond$ distributes over disjunctions in every normal system, the above formula is equivalent to

  $$\bigvee_i \Diamond \left(\chi_i \wedge \mathsf{K}\chi_i' \wedge \bigwedge_j \mathsf{L}\chi_i^j \right).$$

  By Lemma 32, it is equivalent to

  $$\bigvee_i \left(\Diamond\chi_i \wedge \Diamond \left(\mathsf{K}\chi_i' \wedge \bigwedge_j \mathsf{L}\chi_i^j \right) \right). \tag{1}$$

  Using theorems of **S4** for $\Box$ and **S5** for K, and Lemma 31, formula 1 implies

  $$\bigvee_i \left(\chi_i \wedge \Diamond\mathsf{K}\chi_i' \wedge \bigwedge_j \mathsf{L} \left(\Diamond\mathsf{K}\chi_i' \wedge \chi_i^j \right) \right). \tag{2}$$

  By Lemma 33, formula 2 implies formula 1 and hence they are equivalent. Observe now that $\Diamond\mathsf{K}\chi_i'$ belongs to $\mathscr{L}'$ and, since $\mathscr{L}'$ is closed under conjunctions, the last formula is in DNF. This is the only step of the proof which makes use of Axiom 12. Thus ϕ is equivalent to a formula in DNF.

This completes the proof. ∎

The DNF theorem is the most important property of **MP***. An immediate corollary is that, as far as **MP*** is concerned, we could have replaced the $\Box$ modality with

$\Diamond\mathsf{K}$, since the formulae in normal form are defined using these two modalities. Almost all subsequent proof theoretic properties are immediate or implicit corollaries of the DNF Theorem.

We close this section with the following proposition, which together with Axiom 11 shows that $\Box\Diamond$ is equivalent to $\Diamond\Box$.

Proposition 36 *For all* $\phi \in \mathscr{L}$, $\vdash_{\mathbf{MP}^*} \Box\Diamond\phi \to \Diamond\Box\phi$

Proof. Since (see [6] p.146)

$$\mathbf{S4} \cup \{\Box(\phi \vee \psi) \to (\Diamond\Box\phi \vee \Diamond\Box\psi)\} \vdash \Box\Diamond\phi \to \Diamond\Box\phi,$$

and

$$\mathbf{S4} \cup \{\Box\Diamond\phi \to \Diamond\Box\phi\} \vdash \Box(\phi \vee \psi) \to (\Diamond\Box\phi \vee \Diamond\Box\psi),$$

we have only to show that

$$\vdash_{\mathbf{MP}^*} \Box\Diamond\psi \to \Diamond\Box\psi,$$

where ψ is in prime normal form. For that, consider the following derivation in $\mathbf{MP}^*$

1. $\Box\Diamond\left(\psi \wedge \mathsf{K}\psi' \wedge \bigwedge_{i=1}^{n} \mathsf{L}\psi_i\right)$
   $\to \Box\left(\psi \wedge \Diamond\mathsf{K}\psi' \wedge \bigwedge_{i=1}^{n} \mathsf{L}\psi_i\right)$ by Lemma 32,
2. $\Box\left(\psi \wedge \Diamond\mathsf{K}\psi' \wedge \bigwedge_{i=1}^{n} \mathsf{L}\psi_i\right)$
   $\to \psi \wedge \Diamond\mathsf{K}\psi' \wedge \bigwedge_{i=1}^{n} \Box\mathsf{L}\psi_i,$

3. $\psi \wedge \Diamond \mathsf{K}\psi' \wedge \bigwedge_{i=1}^{n} \Box \mathsf{L}\psi_i$

   $\rightarrow \Diamond (\psi \wedge \mathsf{K}\psi' \wedge \bigwedge_{i=1}^{n} \Box \mathsf{L}\psi_i)$ ϕ and $\Box\mathsf{L}\psi_i$ are bi-persistent,

4. $\Diamond \left(\psi \wedge \mathsf{K}\psi' \wedge \bigwedge_{i=1}^{n} \Box \mathsf{L}\psi_i\right)$

   $\leftrightarrow \Diamond (\psi \wedge \Box\mathsf{K}\psi' \wedge \bigwedge_{i=1}^{n} \Box \mathsf{L}\psi_i)$ by Lemma 32,

5. $\Diamond \left(\psi \wedge \Box\mathsf{K}\psi' \wedge \bigwedge_{i=1}^{n} \Box \mathsf{L}\psi_i\right)$

   $\leftrightarrow \Diamond\Box (\psi \wedge \mathsf{K}\psi' \wedge \bigwedge_{i=1}^{n} \mathsf{L}\psi_i)$.

∎

4.3 Canonical Model

The *canonical model* of **MP*** is the structure

$$\mathscr{C} = (S, \{R_\Box, R_\mathsf{K}\}, v),$$

where

$S = \{s \subseteq \mathscr{L} | s \text{ is } \mathbf{MP}^*\text{-maximal consistent}\}$,

$sR_\Box t$ iff $\{\phi \in \mathscr{L} | \Box\phi \in s\} \subseteq t$,

$sR_\mathsf{K} t$ iff $\{\phi \in \mathscr{L} | \mathsf{K}\phi \in s\} \subseteq t$,

$v(A) = \{s \in S | A \in S\}$,

along with the usual satisfaction relation (defined inductively):

$$
\begin{array}{lll}
s \models_{\mathscr{C}} A & \text{iff} & s \in v(A) \\
s \not\models_{\mathscr{C}} \bot & & \\
s \models_{\mathscr{C}} \neg\phi & \text{iff} & s \not\models_{\mathscr{C}} \phi \\
s \models_{\mathscr{C}} \phi \wedge \psi & \text{iff} & s \models_{\mathscr{C}} \phi \text{ and } s \models_{\mathscr{C}} \psi \\
s \models_{\mathscr{C}} \Box\phi & \text{iff} & \text{for all } t \in S,\ sR_{\Box}t \text{ implies } t \models_{\mathscr{C}} \phi \\
s \models_{\mathscr{C}} \mathsf{K}\phi & \text{iff} & \text{for all } t \in S,\ sR_{\mathsf{K}}t \text{ implies } t \models_{\mathscr{C}} \phi.
\end{array}
$$

We write $\mathscr{C} \models \phi$, if $s \models_{\mathscr{C}} \phi$ for all $s \in S$.

A canonical model exists for all consistent bimodal systems with the normal axiom scheme for each modality (as **MP** and **MP***.) We have the following well known theorems (see [6], or [13].)

Theorem 37 (Truth Theorem) *For all $s \in S$ and $\phi \in \mathscr{L}$,*

$$s \models_{\mathscr{C}} \phi \qquad \textit{iff} \qquad \phi \in s.$$

Theorem 38 (Completeness Theorem) *For all $\phi \in \mathscr{L}$,*

$$\mathscr{C} \models \phi \qquad \textit{iff} \qquad \vdash_{\mathbf{MP}^*} \phi.$$

We shall now prove some properties of the members of $\mathscr{C}$. The DNF theorem implies that every maximal consistent theory s of **MP*** is determined by the formulae in $\mathscr{L}'$ and $\mathscr{L}''$ it contains, i.e. by $s \cap \mathscr{L}'$ and $s \cap \mathscr{L}''$. Moreover, the set

$\{\mathsf{K}\phi, \mathsf{L}\phi | \mathsf{K}\phi, \mathsf{L}\phi \in s\}$ is determined by $s \cap \mathscr{L}''$ alone (this is the K-case of the DNF theorem.)

The following definition is useful

Definition 39 Let $P \subseteq \mathscr{L}'$. We say P is an $\mathscr{L}'$ theory if P is consistent and for all $\phi \in \mathscr{L}'$ either $\phi \in P$ or $\neg\phi \in P$.

Let $S \subseteq \mathscr{L}''$. We say S is an $\mathscr{L}''$ theory if S is consistent and for all $\phi \in \mathscr{L}''$ either $\phi \in S$ or $\neg\phi \in S$.

Hence, $s \cap \mathscr{L}'$ is an $\mathscr{L}'$ theory and $s \cap \mathscr{L}''$ is an $\mathscr{L}''$ theory.

What about going in the other direction? When does an $\mathscr{L}'$ theory and $\mathscr{L}''$ theory determine an **MP*** maximal consistent theory? When their union is consistent because in this case there is a unique maximal extension. To test consistency we have the following lemma.

Lemma 40 *If P and S are an $\mathscr{L}'$ and $\mathscr{L}''$ theory respectively then $P \cup S$ is consistent if and only if*

$$\textit{if} \quad \phi \in P \quad \textit{then} \quad \mathsf{L}\phi \in S.$$

PROOF. Suppose that $P \cup S$ is not consistent then there exists $\phi \in P$ and $\{\mathsf{L}\phi_i\}_{i=1}^n \subseteq S$ such that

$$\vdash_{\mathbf{MP}^*} \bigwedge_{i=1}^{n} \mathsf{L}\phi_i \to \neg\phi,$$

which implies, since K distributes over conjunctions,

$$\vdash_{\mathbf{MP^*}} \bigwedge_{i=1}^{n} \mathsf{L}\phi_i \rightarrow \mathsf{K}\neg\phi.$$

Therefore $\neg\mathsf{L}\phi \in S$ and $\mathsf{L}\phi \notin S$. The other direction is straightforward because $\phi \rightarrow \mathsf{L}\phi$.

∎

It is expected that since $\mathscr{L}'$ and $\mathscr{L}''$ theories determine $\mathbf{MP^*}$ maximal consistent sets they will determine their accessibility relations, as well.

Proposition 41 *For all $s, t \in S$,*

a. $sR_{\Box}t$ if and only if i. $\phi \in t$ if and only if $\phi \in s$, where $\phi \in \mathscr{L}'$,

ii. if $\mathsf{L}\phi \in t$ then $\mathsf{L}\phi \in s$, where $\phi, \psi \in \mathscr{L}'$.

b. $sR_{\mathsf{K}}t$ if and only if $\mathsf{K}\phi \in t$ if and only if $\mathsf{K}\phi \in s$, where $\phi \in \mathscr{L}'$.

Proof. For (a), right to left, let $\phi \in t$ then, by the DNF Theorem, ϕ has the form

$$\bigvee_i \left(\chi_i \wedge \mathsf{K}\chi_i' \wedge \bigwedge_j \mathsf{L}\chi_i^j \right),$$

where $\chi, \chi', \chi_k^j \in \mathscr{L}'$. Then $\Diamond\phi$ has the form

$$\Diamond \bigvee_i \left(\chi_i \wedge \mathsf{K}\chi_i' \wedge \bigwedge_j \mathsf{L}\chi_i^j \right),$$

which is equivalent to

$$\bigvee_i \left(\chi_i \wedge \Diamond\mathsf{K}\chi_i' \wedge \bigwedge_j \mathsf{L}\left(\Diamond\mathsf{K}\chi_i' \wedge \chi_i^j\right) \right),$$

as in the proof of the DNF Theorem. Observe here that, in the case where $\phi \in \mathscr{L}'$, if $\mathsf{K}\phi \in t$ then $\Diamond\mathsf{K}\phi \in t$ which implies that $\Diamond\mathsf{K}\phi \in s$. Thus, by *a(i)* and *a(ii)*, $\Diamond\phi \in s$. Therefore, $sR_{\Box}t$.

For the other direction, *a(i)* is straightforward using the bi-persistence of ϕ. For *a(ii)*, if $\mathsf{L}\phi \in t$ then $\Diamond\mathsf{L}\phi \in s$ and use Lemma 31 to show that $\mathsf{L}\phi \in s$.

For *(b)*, right to left, we proceed as above. Let $\mathsf{K}\phi \in t$, then, by the DNF Theorem, it has the following form

$$\bigwedge_i \left(\mathsf{L}\chi_i' \vee \bigvee_j \mathsf{K}\chi_i^j \right),$$

where $\chi_i', \chi_i^j \in \mathscr{L}'$. Thus $\mathsf{K}\phi \in s$.

The other direction is straightforward by the definition of R_{K}. ∎

From the above proposition we have that for all $s, t \in S$, if $sR_{\Box}t$ then $s \cap \mathscr{L}' = t \cap \mathscr{L}'$ and if $sR_{\mathsf{K}}t$ then $s \cap \mathscr{L}'' = t \cap \mathscr{L}''$.

We write $R_{\mathsf{K}}R_{\Box}$ for the composition of the relation R_{K} and $R_{\Box}$, i.e. if $s, t \in S$, we write $sR_{\mathsf{K}}R_{\Box}t$ if there exists $r \in S$ such that $sR_{\mathsf{K}}r$ and $rR_{\Box}t$. Similarly for $R_{\Box}R_{\mathsf{K}}$.

For the composite relation $R_{\mathsf{K}}R_{\Box}$ and $R_{\Box}R_{\mathsf{K}}$ we have the following corollary of proposition 41, which we stay here without proof

Corollary 42 *For all $s, t \in S$,*

a. $sR_{\Box}R_{\mathsf{K}}t$ *if and only if* *i. if $\phi \in s$ then $\mathsf{L}\phi \in t$, where $\phi \in \mathscr{L}'$,*

ii. if $\mathsf{L}\phi \in t$ then $\mathsf{L}\phi \in s$, where $\phi, \psi \in \mathscr{L}'$.

b. $sR_{\mathsf{K}}R_{\Box}t$ *if and only if* *if $\mathsf{L}\phi \in t$ then $\mathsf{L}\phi \in s$, where $\phi \in \mathscr{L}'$.*

We shall now prove that the canonical model $\mathscr{C}$ of $\mathbf{MP}^*$ satisfies the conditions of Section 4.1 on page 33.

Proposition 43 *The relation $R_\Box$ is reflexive and transitive.*

PROOF. Holds in every system containing **S4**. ∎

Proposition 44 *The relation R_K is an equivalence relation.*

PROOF. Holds in every system containing **S5**. ∎

Lemma 45 *For all $s, t \in S$, if $sR_\Box R_\mathsf{K} t$ then $sR_\mathsf{K} R_\Box t$.*

PROOF. See [24]. It is immediate using Axiom 10. ∎

The relation $R_\Box$ has ending points as shown in the following proposition.

Proposition 46 *For each $s \in S$, there exists $s_0 \in S$ with $sR_\Box s_0$ such that for all $s' \in S$, if $sR_\Box s'$ then $s'R_\Box s_0$.*

PROOF. Let

$$\begin{aligned} A &= s \cap \mathscr{L}', \\ B &= \{\mathsf{L}\phi | \phi \in \mathscr{L}',\ \Box\mathsf{L}\phi \in s\}, \\ C &= \{\mathsf{K}\phi | \phi \in \mathscr{L}',\ \Diamond\mathsf{K}\phi \in s\}. \end{aligned}$$

Now, the set $T = B \cup C$ is an $\mathscr{L}''$ theory. T is consistent. If not, then there exist $\mathsf{L}\phi_1, \ldots, \mathsf{L}\phi_n \in B$ and $\mathsf{K}\phi' \in C$ such that

$$\vdash_{\mathbf{MP}^*} \mathsf{L}\phi_1 \wedge \ldots \wedge \mathsf{L}\phi_n \rightarrow \neg\mathsf{K}\phi,$$

and thus

$$\vdash_{\mathbf{MP}^*} \Box\mathsf{L}\phi_1 \wedge \ldots \wedge \Box\mathsf{L}\phi_n \rightarrow \Box\neg\mathsf{K}\phi.$$

But the formula at the left of the implication belongs to s. Therefore $\Box\neg\mathsf{K}\phi \in s$ so $\neg\Diamond\mathsf{K}\phi \in s$, a contradiction. Now, for $\phi \in \mathscr{L}'$ either $\mathsf{L}\phi \in T$ or $\neg\mathsf{L}\phi \in T$.

Observe that if $\phi \in A$ then $\Box\mathsf{L}\phi \in s$ and therefore $\mathsf{L}\phi \in T$, by definition. So $A \cup T$ has a unique maximal extension, by Lemma 40, call it s_0.

For all $s' \in S$ (s included) such that $sR_\Box s'$ we have that $s \cap \mathscr{L}' = s' \cap \mathscr{L}' = s_0 \cap \mathscr{L}'$ and if $\mathsf{L}\phi \in s_0$ then $\Box\mathsf{L}\phi \in s$ so $\mathsf{L}\phi \in s'$. Thus $s'R_\Box s_0$ using proposition 41. Therefore s_0 is the ending point of s. ∎

The above proposition implies that s and s' have a common ending point if and only if $s \cap \mathscr{L}' = s' \cap \mathscr{L}'$. The one direction comes from proposition 41, while the other from the proof of the above proposition because the construction of the ending point of a maximal consistent theory s depends solely on $s \cap \mathscr{L}'$.

Proposition 47 *The canonical frame of* **MP*** *satisfies the extensionality condition of Section 4.1.*

PROOF. We have to prove that for all $s, s' \in S$, if there exists $s_0 \in S$ such that $sR_\Box s_0$ and $s'R_\Box s_0$ and

> for all $t \in S$ such that $tR_{\mathsf{K}}s$ there exist $t', t_0 \in S$ such that $t'R_{\mathsf{K}}s'$, $tR_\Box t_0$ and $t'R_\Box t_0$, and for all $t' \in S$ such that $t'R_{\mathsf{K}}s'$ there exist $t, t_0 \in S$ such that $tR_{\mathsf{K}}s$, $t'R_\Box t_0$ and $tR_\Box t_0$,

then $s = s'$.

Since s and s' have a common child $s \cap \mathscr{L}' = s' \cap \mathscr{L}'$.

We have only to show that $s \cap \mathscr{L}'' = s' \cap \mathscr{L}''$. For that suppose that $\mathsf{L}\phi \in s$ with $\phi \in \mathscr{L}'$, then there exists $t \in S$ such that $tR_{\mathsf{K}}s$ and $\phi \in t$. By the hypothesis of the condition there exist $t', t_0 \in S$ such that $t'R_{\mathsf{K}}s'$, $tR_\Box t_0$ and $t'R_\Box t_0$. This implies that $t \cap \mathscr{L}' = t' \cap \mathscr{L}'$, so $\phi \in t'$ and $\mathsf{L}\phi \in s'$. the other direction is similar. Therefore $s = s'$. ∎

Proposition 48 *The canonical frame of* **MP*** *satisfies the union condition of section 4.1.*

PROOF. We have to show that for all $s_1, s_2 \in S$,

> if there exists $s \in S$ such that $sR_{\mathsf{K}}R_\Box s_1$ and $sR_{\mathsf{K}}R_\Box s_2$, then there exists $s' \in S$ such that for all $t \in S$ with $tR_{\mathsf{K}}s'$ then $tR_\Box R_{\mathsf{K}}s_1$ or $tR_\Box R_{\mathsf{K}}s_2$.

Let

$$A = \{ \mathsf{K}\phi \mid \mathsf{K}\phi \in (s_1 \cap s_2) \cap \mathscr{L}'' \}$$

and

$$B \;=\; \{\; \mathsf{L}\psi \mid \mathsf{L}\psi \in (s_1 \cup s_2) \cap \mathscr{L}'' \;\}.$$

We shall show that $T = A \cup B$ is an $\mathscr{L}''$ theory. It is clear that for all $\phi \in \mathscr{L}'$, either $\mathsf{L}\phi$ or $\neg\mathsf{L}\phi$ belongs to T. Suppose T is consistent. If not, there exist $\mathsf{K}\phi \in A$ and $\{\mathsf{L}\psi_i\}_{i=1}^n \subseteq B$ such that

$$\vdash_{\mathbf{MP}^*} \neg\left(\mathsf{K}\phi \wedge \bigwedge_{i=1}^{n} \mathsf{L}\psi_i\right).$$

Then

$$\vdash_{\mathbf{MP}^*} \Box\neg\left(\mathsf{K}\phi \wedge \bigwedge_{i=1}^{n} \mathsf{L}\psi_i\right),$$

which implies

$$\vdash_{\mathbf{MP}^*} \neg\left(\Diamond\mathsf{K}\phi \wedge \bigwedge_{i=1}^{n} \mathsf{L}(\Diamond\mathsf{K}\phi \wedge \psi_i)\right).$$

Since $\{\mathsf{L}\psi_i\}_{i=1}^n \subseteq B$ there exist $\{t_i\}_{i=1}^n \subseteq S$ such that either $t_i R_{\mathsf{K}} s_1$ or $t_i R_{\mathsf{K}} s_2$ and $\mathsf{L}\psi_i \in t_i$ for all $i \in \{1,\ldots,n\}$. Since $\mathsf{K}\phi \in A$, we also have that $\mathsf{K}\phi \in t_i$ for all $i \in \{1,\ldots,n\}$. In particular, $\Diamond\mathsf{K}\phi \in t_i$ and $\mathsf{L}(\Diamond\mathsf{K}\phi \wedge \psi) \in t_i$. Now choose $i_0 \in \{1,\ldots,n\}$. Since $sR_{\mathsf{K}}R_{\Box}s_1$ and $sR_{\mathsf{K}}R_{\Box}s_2$ there exists $tR_{\Box}t_{i_0}$ and $tR_{\mathsf{K}}R_{\Box}s_1$ and $tR_{\mathsf{K}}R_{\Box}s_2$. Therefore, $\Diamond\mathsf{K}\phi \in t$ and $\mathsf{L}\psi_i \in t$, by proposition 41 *(a)* and corollary 42. Therefore

$$\left(\Diamond\mathsf{K}\phi \wedge \bigwedge_{i=1}^{n} \mathsf{L}(\Diamond\mathsf{K}\phi \wedge \psi_i)\right) \in t$$

which is a contradiction. This proves that T is an $\mathscr{L}''$ theory.

Observe here that we could have defined T for an infinite number of s_i's, for an infinite version of the union condition, and still get an $\mathscr{L}''$ theory. It would be the rest of this proof that would not work. If it did then the canonical model would have satisfied an infinite version of the union condition.

Now the required s' of the condition is any maximal extension of T. Suppose neither $s' R_\Box R_{\mathsf{K}} s_1$ nor $s' R_\Box R_{\mathsf{K}} s_2$. Then by definition and corollary 42, it must be the case that there exist $\phi_1, \phi_2 \in s' \cap \mathscr{L}'$ such that $\mathsf{L}\phi_1 \notin s_1 \cap \mathscr{L}''$ and $\mathsf{L}\phi_2 \notin s_2 \cap \mathscr{L}''$. But then we have that, for $\phi = \phi_1 \wedge \phi_2 \in s' \cap \mathscr{L}'$, $\mathsf{L}\phi \notin s_1$ and $\mathsf{L}\phi \notin s_2$ but $\mathsf{L}\phi \in T$ which is a contradiction. Similarly for any $t \in S$ such that $t R_{\mathsf{K}} s'$ because $T \subseteq t$ and the rest of the condition is satisfied. ∎

Proposition 49 *The canonical frame of* **MP*** *satisfies the intersection condition of Section 4.1.*

Proof. We have to show that for all $s_i \in S$, $i \in I$,

> if there exists $s \in S$ such that $s_i R_\Box s$ for all $i \in I$ then there exists $s' \in S$ such that for all $\{t_i\} \subseteq S$ with $t_i R_{\mathsf{K}} s_i$ and $t_i R_\Box t_0$ for all $i \in I$ and some $t_0 \in S$ then $t_i R_\Box R_{\mathsf{K}} s'$.

Let $\{t_i^j\}_{i \in I, j \in J}$ all subsets of S such that

> for all $j \in J$, $t_i^j R_{\mathsf{K}} s_i$ and there exists $t_0^j \in S$ such that $t_i^j R_\Box t_0^j$.

This class is not empty since $\{s_i\}_{i \in I}$. Let

$$A = \{ \mathsf{K}\phi \mid \phi \in \bigcup_{i \in I, j \in J} t_i^j \cap \mathscr{L}' \}$$

and

$$B \;=\; \{\; \mathsf{K}\phi \mid \phi \in \bigcap_{i\in I, j\in J} t_i^j \cap \mathscr{L}' \;\}.$$

We shall show that $T = A \cup B$ is an $\mathscr{L}''$ theory. It is clear that either $\mathsf{L}\phi$ or $\neg\mathsf{L}\phi$ belongs to T. Suppose T is consistent. If not, there exist $\mathsf{K}\phi \in A$ and $\{\mathsf{L}\psi_k\}_{k=1}^n \subseteq B$ such that

$$\vdash_{\mathbf{MP}^*} \neg\left(\mathsf{K}\phi \wedge \bigwedge_{k=1}^n \mathsf{L}\psi_k\right).$$

Then

$$\vdash_{\mathbf{MP}^*} \Box\neg\left(\mathsf{K}\phi \wedge \bigwedge_{k=1}^n \mathsf{L}\psi_k\right),$$

which implies

$$\vdash_{\mathbf{MP}^*} \neg\left(\Diamond\mathsf{K}\phi \wedge \bigwedge_{k=1}^n \mathsf{L}(\Diamond\mathsf{K}\phi \wedge \psi_k)\right).$$

Each t_i^j contains $\Diamond\mathsf{K}\phi$ because $\mathsf{K}\phi \in t_0^j$. To see that suppose $\mathsf{K}\phi \notin t_0^j$. Then there exists $t \in S$ such that $tR_{\mathsf{K}}t_0^j$ and $\neg\phi \in t$. Lemma 45 implies that there exist $\{t_i\}_{i\in I} \subseteq S$ such that $t_iR_{\Box}t$ and $t_iR_{\mathsf{K}}s_i$. But $\neg\phi \in t_i$ for all $i \in I$ hence $\mathsf{L}\neg\phi \in T$, a contradiction.

Now, for each k, $1 \le k \le n$, choose $t_{i_0}^{j_k}$ such that $\psi_k \in t_{i_0}^{j_k}$. The choice of i is irrelevant since t_i^j contain the same formulae in $\mathscr{L}'$ for all $i \in I$. We now have $\Diamond\mathsf{K}\phi \wedge \psi_k \in t_{i_0}^{j_k}$ and therefore

$$\left(\Diamond\mathsf{K}\phi \wedge \bigwedge_{k=1}^n \mathsf{L}(\Diamond\mathsf{K}\phi \wedge \psi_k)\right) \;\in\; s_{i_0}.$$

Therefore t is an $\mathscr{L}''$ theory.

Now let s' be any maximal extension of T. If $\phi \in t_i^j \cap \mathscr{L}'$ then $\mathsf{L}\phi \in T$ and thus $\mathsf{L}\phi \in s'$ and if $\mathsf{L}\phi \in s' \cap \mathscr{L}''$ then $\mathsf{L}\phi \in t_i^j$. By Corollary 42, we have that $t_i^j R_\Box R_\mathsf{K} s'$ for all $i \in I$ and $j \in J$. Therefore the intersection condition is satisfied. ∎

Corollary 50 *The canonical frame of* **MP*** *is isomorphic to a subset frame* $\mathscr{F}_{\mathscr{O}_c}$ *where* $(X_c, \mathscr{O}_c)$ *is a subset space closed under infinite intersections and if* $U, V \in \mathscr{O}_c$ *have an upper bound in* $\mathscr{O}_c$ *then* $U \cup V \in \mathscr{O}_c$.

PROOF. By Proposition 27, and Propositions 43 through 49. ∎

By the construction of Theorem 26, X_c consists of the ending points of the members of the domain of the canonical model. We define the following initial assignment i_c

$$i(A) \quad = \quad \{ \; s_0 \mid A \in s_0 \; \}.$$

In this way the model $\mathscr{M} = \langle X_c, \mathscr{O}_c, i_c \rangle$ is equivalent to the canonical model as a corollary of frame isomorphism.

Corollary 51 *For all* $s \in S$ *and* $\phi \in \mathscr{L}$ *we have*

$$\phi \in s \quad \textit{if and only if} \quad s_0, U_s \models_{\mathscr{M}} \phi.$$

Definition 52 A subset X of S, the domain of the canonical model $\mathscr{C}$, is called $\mathsf{K}\Box$-*closed* whenever

$$\text{if } s \in X, \text{ and } sR_\Box t \text{ or } sR_\mathsf{K} t, \quad \text{then} \quad t \in X.$$

The intersection of $\mathsf{K}\Box$-closed sets is still closed, therefore we can define the smallest $\mathsf{K}\Box$-closed containing t, for all $t \in S$. We shall denote this set by S^t. For $t \in S$, we define the model

$$\mathscr{C}^t = \left(S^t, R^t_\Box, R^t_{\mathsf{K}}, v^t\right),$$

where $R^t_\Box = R_\Box \mid_{S^t}$ and $R^t_{\mathsf{K}} \mid_{S^t}$, i.e. the restrictions of $R_\Box$ and R_{K} to S^t. We shall call this model *the submodel of* $\mathscr{C}$ *generated by* t.

Proposition 53 *The frame of a submodel* $\mathscr{C}^t$ *is isomorphic to a closed topological frame.*

Proof. Observe that since the domain is $\mathsf{K}\Box$-closed the frame is strongly generated. The rest of the conditions are inherited from the canonical frame. Now the proposition follows from Theorem 26. ∎

Now as above we have the following corollary

Corollary 54 *A submodel* $\mathscr{C}^t$ *is equivalent to a closed topological model.*

It is a well known fact that a modal system is characterized by the class of generated frames of the canonical frame.

Proposition 55 *The system* **MP*** *is (strongly) characterized by closed topological frames.*

Since the axioms and rules of $\mathbf{MP}^*$ are sound for the wider class of subset spaces with finite union and intersection, we also have the following.

Proposition 56 *The system* $\mathbf{MP}^*$ *is (strongly) characterized by subset frames closed under finite unions and intersections.*

Now by Proposition 55 and 56, Corollary 18 and Theorem 19 of Chapter 3, where we proved the equivalence of a topological model with the model induced by a basis closed under finite unions, we have the following corollary

Corollary 57 *The system* $\mathbf{MP}^*$ *is (strongly) characterized by open topological frames as well as subset frames closed under infinite unions and intersections.*

4.4 Joint models

In this section we are going to prove that the canonical model is strongly generated, in the sense that there is a world in it which access every other using the relation $R_{\mathsf{K}}R_{\Box}$. This translates to the fact that the canonical model as a set of (closed) subsets has a greatest element (a universe), i.e. it represents a topological space. The usual way to proceed in this case (see [19]) is to prove a rule of disjunction but the question is which one. In uni-modal logics we use the primitive modality which determines the accessibility relation. Here $\Box$ determines the partial order but, not surprisingly, we must also use K. It turns out that we do not need such a rule in full generality but only with respect to bi-persistent formulae. What we want to prove is the following rule

$$\text{if} \qquad \vdash_{\mathbf{MP}^*} \mathsf{K}\phi_1 \vee \mathsf{K}\phi_2 \vee \ldots \vee \mathsf{K}\phi_n \qquad \text{then} \qquad \vdash_{\mathbf{MP}^*} \phi_i, \quad \text{for some } i, \quad 1 \leq i \leq n,$$

for $\phi_1, \phi_2, \ldots, \phi_n \in \mathscr{L}'$. Note that the disjunction rule does not hold for **S5**.

In the following we shall assume that $X_1, X_2, \ldots, X_n$ are disjoint. This is without any loss of generality since we can always replace a topological model with an equivalent one using a distinct (but homeomorphic) topological space.

Definition 58 Let $\langle X_1, \mathscr{T}_1, v_1\rangle, \langle X_2, \mathscr{T}_2, v_2\rangle, \ldots \langle X_n, \mathscr{T}_n, v_n\rangle$ be a finite number of topological models. Their *joint model* is $\langle X, \mathscr{T}, v\rangle$ where

$$X = \bigcup_{i=1}^{n} X_i,$$

$$\mathscr{T} \quad \text{is the topology generated by the subbasis} \quad \bigcup_{i=1}^{n} \mathscr{T}_i$$

and

$$v(A) = \bigcup_{i=1}^{n} v_i(A), \quad \text{for each atomic formula } A.$$

As this construction was defined, it brings us from topological models to topological models and the accessibility relations between points and subsets in the old models are transferred to the new one. We only add more by adding more subsets. Observe that the truth assignments for the atomic formulae remain the same and that extends to bi-persistent formulae.

Proposition 59 **MP*** *provides the above rule of disjunction.*

PROOF. By contradiction. Suppose that none of $\phi_1, \phi_2, \ldots, \phi_n$ is a theorem of $\mathbf{MP}^*$. Since topological models characterize the system, there are

$$\langle X_1, \mathscr{T}_1, v_1\rangle, \langle X_2, \mathscr{T}_2, v_2\rangle, \ldots \langle X_n, \mathscr{T}_n, v_n\rangle$$

and $x_1, x_2, \ldots, x_n$ belonging to $X_1, X_2, \ldots, X_n$ respectively such that $x_i \notin v_i(\phi_i)$ for $1 \leq i \leq n$. Let $\langle X, \mathscr{T}, v\rangle$ be the joint model of

$$\langle X_1, \mathscr{T}_1, v_1\rangle, \langle X_2, \mathscr{T}_2, v_2\rangle, \ldots \langle X_n, \mathscr{T}_n, v_n\rangle.$$

Then we have $x_i \notin v(\phi_i)$ and therefore $x, X \models \neg\mathsf{K}\phi_i$ for all $x \in X$ and $1 \leq i \leq n$. Therefore $\mathsf{K}\phi_1 \vee \mathsf{K}\phi_2 \vee \ldots \vee \mathsf{K}\phi_n$ is not a theorem of $\mathbf{MP}^*$. ∎

We can similarly prove a stronger disjunction property, namely

$$\begin{array}{ll} \text{if} & \vdash_{\mathbf{MP}^*} \mathsf{K}\phi \rightarrow \mathsf{K}\phi_1 \vee \mathsf{K}\phi_2 \vee \ldots \vee \mathsf{K}\phi_n \\ \text{then} & \vdash_{\mathbf{MP}^*} \mathsf{K}\phi \rightarrow \phi_i, \quad \text{for some } i, \quad 1 \leq i \leq n, \end{array}$$

for $\phi, \phi_1, \phi_2, \ldots, \phi_n \in \mathscr{L}'$.

Now we are able to prove the following

Theorem 60 *The canonical model of* **MP*** *is strongly generated.*

PROOF. Let

$$T = \{\mathsf{K}\phi \mid \vdash_{\mathbf{MP}^*} \phi,\ \phi \in \mathscr{L}'\} \cup \{\mathsf{L}\phi \mid \nvdash_{\mathbf{MP}^*} \neg\phi, \phi \in \mathscr{L}'\}.$$

The set of formulae T is an $\mathscr{L}''$ theory. For consistency suppose that

$$\vdash_{\mathbf{MP}^*} \neg(\mathsf{L}\phi_1 \wedge \mathsf{L}\phi_2 \wedge \ldots \wedge \mathsf{L}\phi_n),$$

for some $\mathsf{L}\phi_1, \mathsf{L}\phi_2, \ldots, \mathsf{L}\phi_n \in T$. This implies that

$$\vdash_{\mathbf{MP}^*} \mathsf{K}\neg\phi_1 \vee \mathsf{K}\neg\phi_2 \vee \ldots \vee \mathsf{K}\neg\phi_n,$$

and because $\neg\phi_i \in \mathscr{L}'$ for $1 \leq i \leq n$ we can use the rule of disjunction and get

$$\vdash_{\mathbf{MP}^*} \phi_i, \quad \text{for some} \quad 1 \leq i \leq n,$$

which is a contradiction.

Now for any member of the canonical model s, let

$$S = T \cup \{\phi | \phi \in s \cap \mathscr{L}'\},$$

i.e. T plus the bi-persistent formulae of s. The set S is consistent. If

$$\vdash_{\mathbf{MP}^*} \mathsf{L}\phi_1 \wedge \mathsf{L}\phi_2 \wedge \ldots \wedge \mathsf{L}\phi_n \rightarrow \neg\psi,$$

where $\mathsf{L}\phi_i \in T$ for $1 \leq i \leq n$ and $\psi \in s \cap \mathscr{L}'$ then

$$\vdash_{\mathbf{MP}^*} \mathsf{L}\phi_1 \wedge \mathsf{L}\phi_2 \wedge \ldots \wedge \mathsf{L}\phi_n \rightarrow \neg\mathsf{L}\psi.$$

But consistency of T implies that $\neg\psi$ is a theorem, a contradiction.

Moreover, S has a unique maximal extension by the DNF Theorem, call it s'. So we have showed that for all s and t in the canonical model there exists s' and t' such that $s' R_\Box s$ and $s' R_\mathsf{K} t'$. This implies that in the canonical subset model the subset U_s is the required universe. ∎

By Theorem 60 we complete the set of conditions of page 34 which turn the frame of the canonical model into a closed subset frame. To summarize, we have the following corollary (note that the canonical subset model is $\langle X_c, \mathcal{O}_c, i_c \rangle$ of Corollary 50)

Corollary 61 *The canonical subset model of* **MP*** *is a topological space.*

Chapter 5

The Algebras of MP and MP*

In this section we shall give a more general type of semantics for **MP** and **MP***. Every modal logic can be interpreted in an algebraic framework. An algebraic model is nothing else but a valuation of the propositional variables in a class of appropriately chosen algebras. We shall also make connections with the previous chapters.

5.1 Fixed Monadic Algebras

Interior operators were introduced by McKinsey and Tarski [23].

Definition 62 An *interior* operator I on a Boolean algebra $\mathscr{B} = \langle B, \mathbf{0}, \mathbf{1}, \cap, \cup \rangle$ is an operator satisfying the conditions

$$I(a \cap b) = Ia \cap Ib,$$

$$Ia \leq a,$$

$$IIa = Ia,$$

$$I\mathbf{1} = \mathbf{1}.$$

To each interior operator I we associate its dual $C = -I-$ the *closure operator* which satisfies

$$C(a \cup b) = Ca \cup Cb,$$

$$a \leq Ca,$$

$$CCa = Ca,$$

$$C\mathbf{0} = \mathbf{0}.$$

Universal quantifiers were introduced by P. Halmos [14].

Definition 63 A *universal quantifier* $\forall$ on a Boolean algebra $\mathscr{B}$ is an operator satisfying the conditions

$$\forall(a \cup \forall b) = \forall a \cup \forall b,$$

$$\forall a \leq a,$$

$$\forall\mathbf{1} = \mathbf{1}.$$

To each universal quantifier $\forall$ we associate its dual $\exists = -\forall-$ the *existential* *quantifier* which satisfies

$$\exists(a \cap \exists b) = \exists a \cap \exists b,$$

$$a \leq \exists a,$$

$$\exists \mathbf{0} = \mathbf{0}.$$

Definition 64 Let I be an interior operator on a Boolean algebra $\mathscr{B}$. Let $IB = \{a|a \leq Ia\}$ and $CB = \{a|Ca \leq a\}$, i.e. the fixed points of I and C respectively. Let $B^I = IB \cap CB$ then $\mathscr{B}^I = \langle B^I, \mathbf{0}, \mathbf{1}, -, \cap, \cup\rangle$ is a Boolean subalgebra of $\mathscr{B}$.

Definition 65 A *fixed monadic algebra (FMA)* $\mathscr{B}$ is a Boolean algebra with two operators I and $\forall$ satisfying

$$\forall Ia \geq I\forall a.$$

A valuation v on $\mathscr{B}$ is a function from the formulae of **MP** to the elements of B such that

$$\begin{aligned} v(A) &\in B^I, \text{ where } A \text{ is atomic,} \\ v(\neg\phi) &= -v(\phi), \\ v(\phi \wedge \psi) &= v(\phi) \cap v(\phi), \\ v(\phi \vee \psi) &= v(\phi) \cup v(\phi), \\ v(\Box\phi) &= Iv(\phi), \\ v(\mathsf{K}\phi) &= \forall v(\phi). \end{aligned}$$

An *algebraic model* of **MP** is a FMA $\mathscr{B}$ with a valuation v on it. We say ϕ is valid in this model iff $v(\phi) = \mathbf{1}$ and valid in an FMA iff it is valid in all models based on this algebra. Finally, ϕ is FMA-valid if it is valid in all FMA's. The notion of validity can extend to a set of formulae.

Observe that the important part of the algebra is the smallest subalgebra containing B^I and closed under the operators I and $\forall$.

Theorem 66 (Soundness for FMA-validity) *If a formula ϕ is a theorem of* **MP** *then ϕ is FMA-valid.*

PROOF. Let $\langle \mathscr{B}, v\rangle$ be an algebraic model. We must prove that for all axioms ϕ, $v(\phi) = \mathbf{1}$. First observe that in a Boolean algebra $v(\phi \to \psi) = \mathbf{1}$ is equivalent to $v(\phi) \leq v(\psi)$. Take for instance $\mathsf{K}\Box\phi \to \Box\mathsf{K}\phi$. We have that

$$\begin{aligned} \forall I v(\phi) \leq I \forall v(\phi) \quad & \text{implies} \quad v(\mathsf{K}\Box\phi) \leq v(\Box\mathsf{K}\phi) \\ & \text{implies} \quad v(\mathsf{K}\Box\phi \to \Box\mathsf{K}\phi) = \mathbf{1}. \end{aligned}$$

We leave the rest of verifications to the reader. Similarly for rules. ∎

Theorem 67 (Completeness for FMA-validity) *If ϕ is FMA-valid then ϕ is a theorem of* **MP**.

PROOF. The proof is actually the Lindenbaum construction. We define the following equivalence relation on $\mathscr{L}$

$$\phi \sim \psi \quad \text{if and only if} \quad \vdash_{\mathbf{MP}} \phi \equiv \psi.$$

We denote the equivalence class of ϕ with $[\phi]$ and define the following partial order on the set $\mathscr{B}$ of equivalence classes

$$[\phi] \leq [\psi] \quad \text{if and only if} \quad \vdash_{\mathbf{MP}} \phi \rightarrow \psi.$$

All the required properties of an FMA follow from the axioms and rules of **MP**. If we define the valuation on $\mathscr{B}$ with

$$v(\phi) \quad = \quad [\phi]$$

then we have

$$[\phi] = \mathbf{1} \quad \text{if and only if} \quad \vdash_{\mathbf{MP}} \phi.$$

∎

5.2 Generated Monadic Algebras

We shall now define the algebraic models of **MP***

Definition 68 A *generated monadic algebra (GMA)* $\mathscr{B}$ is an FMA satisfying in addition

$$CIa = ICa$$

$$C(\forall a \cap b) \cap \exists C(\forall a \cap c) \leq C(\forall Ca \cap Cb \cap \exists Cc).$$

The concepts of algebraic model, validity, GMA-validity are defined as for FMA's. We used the direct algebraic translation of **MP*** axioms but we could have defined it with a different presentation. Observe that we only need $CIa \leq ICa$ because the other direction is derivable (see Proposition 36.)

We now have the following

Theorem 69 (Algebraic completeness of MP*) *A formula ϕ is a theorem of* **MP*** *if and only if ϕ is GMA-valid.*

PROOF. We omit the proof since it is similar to Theorems 66 and 67. ∎

It is known that a modal algebra determines a (general) frame (see [4].) So, in our case, the canonical algebraic model of **MP***, i.e. its Lindenbaum algebra, must determine a closed topological model (actually its canonical frame.) We shall state only the interesting part of this correspondence: the bijection on the domains. The accessibility relations are defined in the usual way.

Theorem 70 *There is a bijection between the set of the ultrafilters of the canonical algebra of* **MP*** *and the pointed product $X \dot{\times} \mathscr{T}$, where $(X.\mathscr{T})$ is the canonical topology of* **MP****.*

The general theory of modal logic provides for yet another construction. A frame determines a modal algebra. In case of the canonical frame, the modal algebra determined must be isomorphic to the canonical modal algebra. In our case, this algebra (which must be a GMA) has a nice representation. It is the algebra of partitions of the topological lattice as it appeared in Section 3.1. A full account of this result and detailed proofs will appear elsewhere.

Chapter 6

Further Directions

There are several further directions

1. Due to the indeterminacy assumption (see Introduction) $\mathbf{MP}^*$ can be a "core" logical system for reasoning about computation with approximation or uncertainty.

2. A discrete version of our epistemic framework can arise in scientific experiments or tests. We acquire knowledge by "a step-by-step" process. Each step being an experiment or test. The outcome of such an experiment or test is unknown to us beforehand, but after being known it restricts our attention to a smaller set of possibilities. A sequence of experiments, test or actions comprises a *strategy of knowledge acquisition.* This model is in many respects similar to Hintikka's "oracle" (see [18].) In Hintikka's model the "inquirer" asks a series of

questions to an external information source, called "oracle". The oracle answers yes or no and the inquirer increases her knowledge by this piece of additional evidence. This framework can be expressed by adding actions to the language. Preliminary work of ours used *quantales* for modelling such processes. A similar work without knowledge considerations appears in [2].

3. Since we can express concepts like affirmative or refutative assertions, which are closed under infinite disjunctions and conjunctions respectively, it is very natural to add infinitary connectives or fixed points operators (the latter as a finite means to express the infinitary connectives.) This would serve the purpose of specifying such properties of programs as "emits an infinite sequence of ones" (see [1] for a relevant discussion.) An interesting direction of linking topological spaces with programs can be found in [25].

4. Our work in the algebras of $\mathbf{MP}^*$ looks very promising. GMAs (see 5) have very interesting properties. A subalgebra of a GMA corresponds to a complete space and this duality can be further investigated with the algebraic machinery of modal logic (see [21], [22], [3]) or category theoretic methods.

5. Axiom 10 forces monotonicity in our systems. If we drop this axiom, an application of effort no longer implies a further increase in our knowledge. Any change of our state of knowledge is possible. A non-monotonic version of the

systems presented in this thesis can be given along the lines of [26].

6. It would be interesting to consider a framework of multiple agents. Adding a modality K_i for each agent i and assigning a different set of subsets or topology to to each agent we can study their interaction or communication by set-theoretic or topological means.

7. From our work became clear that both systems considered here are linked with intuitionistic logic. We have embed intuitionistic logic to **MP** or $\mathbf{MP}^*$ and it would be interesting to see how much of the expressiveness of these logics can be carried in an intuitionistic framework.

8. Finally, in another direction Rohit Parikh considers an enrichment of the language to express more (and purely) topological properties such as separation properties and compactness.

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