Basic Action Deontic Logic

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Abstract

The aim of this paper is to introduce a system of dynamic deontic logic in which the main problems related to the definition of deontic concepts, especially those emerging from a standard analysis of permission in terms of possibility of doing an action without incurring in a violation of the law, are solved. The basic idea is to introduce two crucial distinctions allowing us to differentiate (i) what is ideal with respect to a given code, which fixes the types of action that are abstractly prescribed, and what is ideal with respect to the specific situation in which the agent acts, and (ii) the transitions associated with actions and the results of actions, which can obtain even without the action being performed.

Keywords: dynamic deontic logic; deontic paradoxes; ought-to-be logic; ought-to-do logic.

1 Introduction

Systems of deontic logic aim at modeling our intuitions concerning prescriptive concepts, such as prohibition, permission, and obligation, so as to provide appropriate formal frameworks for analyzing deontic problems, conceiving deontically constrained procedures, and assessing existing deontic systems. It is well-known that different kinds of deontic systems can be introduced in the light of the position one assumes with respect to the following non-exclusive options:

(i) developing a deontic logic of states [1,7,14] (ought-to-be logic, *sein-sollen* logic) or carrying the analysis to a deontic logic of actions [5,9,12] (ought-to-do logic, *tun-sollen* logic);

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(ii) developing a static logic of actions [4,9,10] (where what is crucial is to characterize the structure of a system of actions and their basic properties) or carrying the analysis to a dynamic logic of actions [6,8,11] (where it is also crucial to characterize the sequential composition of actions and the properties of such sequences).

It is also well-known that, while the descriptive power of systems of dynamic logic of actions allows us both to solve some traditional paradoxes and to highlight important distinctions which would be otherwise neglected, these systems are still subject to difficulties [2,11], thus appearing inadequate to account for our basic deontic judgements.

The aim of this paper is to introduce a system of dynamic deontic logic in which the main problems related to the definition of deontic concepts, especially those emerging from a standard analysis of permission in terms of possibility of doing an action without incurring in a violation of the law, are solved. Our proposal is based on the idea that, in order to account for the intuitions which generate the paradoxes, more distinctions than those which can be drawn within a standard dynamic deontic system are to be made. In particular, we think that it is crucial to consider (i) a distinction between what is ideal with respect to a given code, i.e., the abstract ideal allowing us to determine the types of action which are permitted or prohibited, and what is ideal with respect to a specific situation, i.e., the concrete ideal determined by the context of the agent [3,7]; and (ii) a distinction between the transitions associated with an action and the result of the action, which possibly obtains without the action being performed. Accordingly, we propose a system constituted of

- an *ontic part*, which includes both a logic of states and a logic of actions, where states are represented, as usual, by sets of possible worlds, and actions, more precisely action types, are represented by relations between worlds;
- a *deontic part*, which includes both a logic of an abstract deontic ideal, represented by a set of worlds satisfying the prescriptions of a code, and an actual deontic ideal, represented by an ordering of the worlds accessible by performing some action.

In this way, we hope to provide a deeper perspective on what is prescribed in a certain context, by constructing a very general modal system for handling traditional problems. The plan of the paper is then as follows. In the next section, we briefly discuss the basic intuitions that our system aims at capturing as they emerge from a discussion of the main deontic paradoxes derivable in a dynamic logic of action. In section 3 we introduce our system of deontic logic of states and actions. Finally, in the last section, we define four groups of deontic concepts and provide solutions to the problems discussed in section 2.

2 Difficulties in defining deontic concepts

In a dynamic deontic logic, where action terms can be combined by using suitable operators, like negation ($\bar{\cdot}$), alternative execution ($\sqcup$), simultaneous execution ($\sqcap$), and sequential execution (;), the deontic operators of prohibition, permission, and obligation can be defined in terms of a propositional constant I, representing an ideal state of law satisfaction, and of the dynamic operator $[\cdot]$, which takes an action term α and a formula φ and returns a new formula $[\alpha]\varphi$, stating that all ways of doing α lead to a φ-state. In fact, an action is

(i) prohibited iff it necessarily results in a violation of the law ($F(\alpha) := [\alpha]\neg I$)

(ii) permitted iff it is not prohibited ($P(\alpha) := \neg[\alpha]\neg I$)

(iii) obligatory iff not doing it is prohibited ($O(\alpha) := [\bar{\alpha}]\neg I$)

Although these definitions seem to be unproblematic, together with some intuitive principles on the action operators, they imply several counter-intuitive conclusions. We especially focus on three groups.

Group 1: standard paradoxes of obligation and permission.

- Ross's paradox: $O(\alpha) \to O(\alpha \sqcup \beta)$ (if it is obligatory to mail a letter, then it is obligatory to mail-the-letter-or-burn-it).
- Permission paradox: $P(\alpha) \to P(\alpha \sqcup \beta)$ (if it is permitted to mail a letter, then it is permitted to mail-the-letter-or-burn-it).

Group 2: paradoxes of permission and prohibition of sequential actions.

- van der Mayden's paradox: $\neg[\alpha]\neg P(\beta) \to P(\alpha;\beta)$ (if there is a way of shooting the president after which it is permitted to remain silent, then it is permitted to shoot-the-president-and-then-remain-silent)
- Anglberger's paradox: $F(\alpha) \to [\alpha]F(\beta)$ (if it is forbidden to shoot the president, then shooting the president necessarily leads to a state in which remaining silent is forbidden).

Group 3: contrary to duties obligations [3].

Paradoxes of group 1 can be avoided by introducing strong notions of obligation and permission, according to which, for an action to be obliged or permitted, it is necessary both that no way of performing it leads to a state of violation and that there is at least a way to perform it which does not lead to a state of violation. Paradoxes of group 2 are more difficult to solve. If we think of an action as characterized by a starting state, a final state, and a transition leading from the first to the second state, then these paradoxes can be seen as the result of disregarding the deontic relevance of the starting state and the process of an action. To be sure, van der Mayden's paradox follows from neglecting the difference between the fact that the final state is safe and the fact that the transition which leads to this state is safe, in the sense that no step in the transition infringes the law, or fails to be the best the agent can do from a deontic perspective, given the initial conditions. Similarly, Anglberger's paradox follows from neglecting the difference between the absolutely ideal states, in which no norm is violated, and the relatively ideal states, in which the best conditions realizable by the agent in the actual conditions are in fact realized. Interestingly, once these distinctions are taken into account, also paradoxes of group 3 turn out to find a solution (but more on this below).

3 Action deontic logic

The language $\mathcal{L}$ of the system **ADL** of action deontic logic contains a set $Tm(\mathcal{L})$ of terms and a set $Fm(\mathcal{L})$ of formulas. Assuming a standard distinction between action types and individual actions, let A be a countable set of action types variables. Then $Tm(\mathcal{L})$ is defined according to the following grammar:

$\alpha ::= a_i \mid 1 \mid \overline{\alpha} \mid \alpha \sqcup \beta \mid \alpha \sqcap \beta \mid \alpha;\beta$ where $a_i \in A$

Intuitively, 1 is the action type instantiated by any action whatsoever; $\bar{\alpha}$ is the action type instantiated by any action which does not instantiate the type α; $\alpha \sqcup \beta$ is the action type instantiated by any action which instantiates either the type α or the type β or both; $\alpha \sqcap \beta$ is the action type instantiated by any action which instantiates the types α and β in parallel; $\alpha;\beta$ is the action type instantiated by any action which instantiates the types α and β in sequence. We assume that an individual action can instantiate different action types. Accordingly, when we say that an action is a token of a_i we do not exclude the possibility that it is also a token of a different type a_j.

Turning to the set of formulas of $\mathcal{L}$, let P be a countable set of propositional variables. Then $Fm(\mathcal{L})$ is defined according to the following grammar:

$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Box\varphi \mid [\alpha]\varphi \mid \mathbf{R}(\alpha) \mid [\uparrow]\varphi \mid I$ where $p \in P$, and $\alpha \in Tm(\mathcal{L})$.

The other connectives and the dual modal operators, $\Diamond\varphi$, $\langle\alpha\rangle\,\varphi$, $\langle\uparrow\rangle\,\varphi$, are defined as usual. The intended interpretation of the modal formulas is as follows: "$\Box\varphi$" says that φ holds in any possible world; "$[\alpha]\varphi$" says that φ holds in any world that can be accessed by performing action α, i.e., that φ holds as a consequence of α; "$\mathbf{R}(\alpha)$" says that the state which is the result of action α is realized [2]; "$[\uparrow]\varphi$" says that φ holds in all the best worlds that can be accessed by performing some action; and, finally, "I" says that the ideal of deontic perfection is realized. It is worth noting that, since 1 is the action type instantiated by any action, "$\langle 1\rangle\,\varphi$" says that φ can be realized by doing an action. Hence, the crucial distinction between *what is possible* and *what is realizable* is captured by the distinction between $\Diamond\varphi$ and $\langle 1\rangle\,\varphi$.

3.1 Semantics

The conceptual framework we adopt is based on the following notion of frame.

Definition 3.1 frame for $\mathcal{L}(\mathbf{ADL})$.
A frame for $\mathcal{L}(\mathbf{ADL})$ is a tuple $F = \langle W, R, \{R_w \mid w \in W\}, r, S, Ideal\rangle$

As mentioned above, frames for $\mathcal{L}(\mathbf{ADL})$ can be subdivided into two parts.

Ontic part: $\langle W, R, \{R_w \mid w \in W\}, r\rangle$, where
(i) $R : W \to \wp(W)$
(ii) $R_w : Tm(\mathcal{L}) \to \wp(W)$, for all $w \in W$
(iii) $r : Tm(\mathcal{L}) \to \wp(W)$

[2] Hence, the formulas $\mathbf{R}(\alpha)$ and $[\alpha]\varphi$ allow us to capture von Wright's distinction between the result and the consequences of an action [13].

We assume that an agent is endowed with a set of primitive actions and think of these actions as ways of obtaining specific resulting states, represented as subsets of a set of possible worlds W. Since the same result can be obtained in different ways, every primitive action corresponds to a set of transitions between worlds in W [3]. More specifically, R, R_w and r are characterized by the following conditions.

Conditions on R

(a) $w \in R(w)$
(b) $v \in R(w) \Rightarrow R(v) = R(w)$

Hence, R models a standard $S5$ notion of ontic modality [4]

Conditions on R_w:

(a) $R_w(\alpha \sqcup \beta) = R_w(\alpha) \cup R_w(\beta)$
(b) $R_w(\alpha; \beta) = \bigcup_{v \in R_w(\alpha)} R_v(\beta)$
(c) $R_w(\alpha) \subseteq R(w)$

Here, R_w is a function that, for each action term, returns the outcomes of the transitions associated with the action performed at w, so that $R_w(\alpha)$ is the set of worlds that are accessible by doing α at w. While conditions (a) and (b) characterize the notions of alternative and sequential actions, (c) captures the intuition that every *realizable state* is a *possible state.* Hence, R and R_w allow us to account for the distinction between what is possible and what is realizable by acting at a world. In fact, it might be the case that reaching a world is beyond the power of the agent, even if that world is possible.

Conditions on r:

(a) $r(\bar{\alpha}) = W - r(\alpha)$
(b) $r(\alpha \sqcap \beta) = r(\alpha) \cap r(\beta)$
(c) $r(\alpha \sqcup \beta) = r(\alpha) \cup r(\beta)$
(d) $r(\alpha) \subseteq r(1)$
(e) $r(\alpha; \beta) \subseteq r(\beta)$
(f) $R_w(\alpha) \subseteq r(\alpha)$
(g) $R(w) \cap r(\alpha) \subseteq r(\beta) \Rightarrow R_w(\alpha) \subseteq R_w(\beta)$
(h) $w \in r(\alpha) \Rightarrow R_w(1) \cap r(\beta) \subseteq r(\alpha; \beta)$

Here, r is a function that, for each action term, returns the state corresponding to the *result* of the action, so that $r(\alpha)$ is the result of α. The conditions connect the intuitive algebra of action results to a corresponding algebra on sets and connect actions with their results. Intuitively:

(a) realizing $\bar{\alpha}$ coincides with not realizing α;
(b) realizing $\alpha \sqcap \beta$ coincides with realizing both α and β;
(c) realizing $\alpha \sqcup \beta$ coincides with realizing either α or β;
(d) realizing any action α is a way of realizing action 1;

[3] Notice that we use the terms "world" and "state" for expressing different concepts, while in the literature about transition systems they are interchangeable with each other. In particular, we use "world" for the complete state which can be reached by performing an action (hence, a world w is an element of W), and "state" for the state of affairs that is the result of an action, as in [13] (hence, a state is in general a subset of W, i.e. a set of worlds).

[4] Using a universal modality would simplify the semantics, but the use of an $S5$ modality gives us a more flexible framework, since the stock of necessary states of affairs can change across the worlds.

(e) realizing any sequence $\alpha;\beta$ is a way of realizing the last action β.

Finally, every realized action realizes its result, by (f); every action whose result involves the result of another action counts as a realization of the latter action, by (g); and, if the result of β is realized after the result of α, then the result of $\alpha;\beta$ is realized as well, by (h).

It is important to note that $r(\alpha)$ does not coincide with $\bigcup_{w\in W} R_w(\alpha)$, since we allow for the possibility that a state of affairs, which is the result of an action, obtains even if no action has brought it about. Indeed, it is possible for a door to be open, even if it was not opened by an agent. As a consequence, $r(1)$, which is W, does not coincide with $\bigcup_{w\in W} R_w(1)$, which is the set of worlds the agent can reach by performing some actions. In addition, we do not assume that $R(w)$ coincides with $R_w(1)$, since, as mentioned above, we allow for a difference between what is possible at a world and what is achievable by acting at it. This is crucial to account for cases where the ideal of perfection, although possible, is not realizable by performing any action.

Deontic part: $\langle W, R, S, Ideal\rangle$, where
(i) $S : W \to \wp(W)$
(ii) $Ideal \subseteq W$

We introduce a deontic function S on W, so that $S(w)$ is the set of the best accessible worlds relative to w, which are the worlds where the conditional ideal that can be achieved in w is realized. In contrast, $Ideal$ is the subset of W containing the best possible worlds from a deontic point of view, which are the worlds where the ideal of deontic perfection is realized.

Conditions on S:	**Conditions on** $Ideal$:
(a) $\varnothing \neq S(w)$	(a) $R(w) \cap Ideal \neq \emptyset$
(b) $S(w) \subseteq R_w(1)$	(b) $R_w(1) \cap Ideal \subseteq S(w)$
(c) $v \in S(w) \Rightarrow S(v) \subseteq S(w)$	(c) $R_w(1) \cap Ideal \neq \emptyset \Rightarrow S(w) \subseteq Ideal$

According to the conditions on S, the set of worlds that can be accessed by the agent always contains a non-empty subset of realizable best options, such that the best options that are accessible by acting in a world that can be reached by w are accessible by w itself. According to the conditions on $Ideal$, the set of accessible worlds always contains a non-empty subset of best possible options. In addition, no accessible world is strictly better, according to S, than any world in $Ideal$, which coincides with the set of the best options if some ideal world is accessible. It is worth noting that a conditional ideal is achievable even if the ideal of perfection cannot be possibly achieved, since $R_w(1) \cap S(w) = S(w)$ is non-empty even if $R_w(1) \cap Ideal$ is empty.

Definition 3.2 model for $\mathcal{L}(\mathbf{ADL})$.
A model for $\mathcal{L}(\mathbf{ADL})$ is a pair $M = \langle F, V\rangle$, where (i) F is a frame for $\mathcal{L}(\mathbf{ADL})$ and (ii) V is a function that maps propositional variables in $\wp(W)$.

Definition 3.3 truth in a model for $\mathcal{L}(\mathbf{ADL})$. The definition of truth is as follows:

$M, w \models p_i \Leftrightarrow w \in V(p_i)$
$M, w \models \neg\varphi \Leftrightarrow M, w \not\models \varphi$
$M, w \models \varphi \wedge \psi \Leftrightarrow M, w \models \varphi$ and $M, w \models \psi$
$M, w \models \Box\varphi \Leftrightarrow \forall v \in W(v \in R(w) \Rightarrow M, v \models \varphi)$
$M, w \models [\alpha]\varphi \Leftrightarrow \forall v \in W(v \in R_w(\alpha) \Rightarrow M, v \models \varphi)$
$M, w \models \mathbf{R}(\alpha) \Leftrightarrow w \in r(\alpha)$
$M, w \models [\uparrow]\varphi \Leftrightarrow \forall v \in W(v \in S(w) \Rightarrow M, v \models \varphi)$
$M, w \models I \Leftrightarrow w \in Ideal$

3.2 Axiomatization

The system **ADL** is defined by the following axioms and rules. The first three groups of axioms take into account the pure modal part of the system, while groups 4, 5 and 6 characterize actions and their results. On the way, we define deontic operators in the Andersonian style.

Group 1: axioms for $\Box$
$\Box$K: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$
$\Box$T: $\Box\varphi \to \varphi$
$\Box$5: $\Diamond\varphi \to \Box\Diamond\varphi$
$\Box$R: φ / $\Box\varphi$

Group 2: axioms for $[\uparrow]$
$[\uparrow]$K: $[\uparrow](\varphi \to \psi) \to ([\uparrow]\varphi \to [\uparrow]\psi)$
$[\uparrow]$D: $[\uparrow]\varphi \to \langle\uparrow\rangle\varphi$
$[\uparrow]$4: $[\uparrow]\varphi \to [\uparrow][\uparrow]\varphi$
$[\uparrow]$I: $[1]\varphi \to [\uparrow]\varphi$

Group 3: axioms for I
I1: $\Diamond I$
I2: $[\uparrow]\varphi \to [1](I \to \varphi)$
I3: $\langle 1\rangle I \to [\uparrow]I$

Definition 3.4 Deontic operators on states based on I.
$[I]\varphi := \Box(I \to \varphi)$ and $\langle I\rangle\varphi := \Diamond(I \wedge \varphi)$.

$[I]\varphi$ is a standard concept of obligation for states [5], as proposed in [1]. It is not difficult to see that $[I]$ is a $KD45$ modality, since we can derive:

(i) $[I](\varphi \to \psi) \to ([I]\varphi \to [I]\psi)$
(ii) $[I]\varphi \to \langle I\rangle\varphi$
(iii) $[I]\varphi \to [I][I]\varphi$
(iv) $\langle I\rangle\varphi \to [I]\langle I\rangle\varphi$
(v) $\varphi/[I]\varphi$

The fundamental distinction we want to highlight here concerns $\langle I\rangle\varphi$ and $\langle\uparrow\rangle\varphi$. While $\langle I\rangle\varphi$ states that φ holds in some ideal world, $\langle\uparrow\rangle\varphi$ states that φ holds in some of the best accessible worlds. As we will see, this distinction gives rise to two different operators of permission.

Let us now introduce the axioms concerning actions and their results.

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[5] Letting $O\varphi$ be $[I]\varphi$ and $P\varphi$ be $\langle I\rangle\varphi$, the choice of an $S5$ modal logic gives us theorems like $O\varphi \to \Box O\varphi$ and $P\varphi \to \Box P\varphi$. In our setting, these principles are justified by the intended interpretation of a formula like $[I]\varphi$. I is an ideal state determined by a specific legal code, and we assume that the distinction between what is prescribed and what is not prescribed is also fixed by that same code. Hence, given that "$O\varphi$" is interpreted as φ is prescribed by the code that fixes I, the previous principles turn out to be intuitive, since it is impossible to change what is prescribed according to the code without changing that code as well.

Group 4: axioms for $[\alpha]$

$[\alpha]$K: $[\alpha](\varphi \to \psi) \to ([\alpha]\varphi \to [\alpha]\psi)$

$[\alpha]$1: $[\alpha]\varphi \wedge [\beta]\varphi \to [\alpha \sqcup \beta]\varphi$

$[\alpha]$2: $[\alpha;\beta]\varphi \leftrightarrow [\alpha][\beta]\varphi$

$[\alpha]$3: $\Box\varphi \to [\alpha]\varphi$

Group 5: axioms for **R**

R1: $\mathbf{R}(\alpha) \leftrightarrow \neg\mathbf{R}(\bar{\alpha})$

R2: $\mathbf{R}(\alpha \sqcap \beta) \leftrightarrow \mathbf{R}(\alpha) \wedge \mathbf{R}(\beta)$

R3: $\mathbf{R}(\alpha \sqcup \beta) \leftrightarrow \mathbf{R}(\alpha) \vee \mathbf{R}(\beta)$

R4: $\mathbf{R}(\alpha) \to \mathbf{R}(1)$

R5: $\mathbf{R}(\alpha;\beta) \to \mathbf{R}(\beta)$

R6: $[\alpha]\mathbf{R}(\alpha)$

R7: $\Box(\mathbf{R}(\alpha) \to \mathbf{R}(\beta)) \to ([\beta]\varphi \to [\alpha]\varphi)$

R8: $\mathbf{R}(\alpha) \to [1](\mathbf{R}(\beta) \to \mathbf{R}(\alpha;\beta))$

These groups of axioms take into account the operations on actions and results and the connections between actions and results, which is further clarified by the following facts.

(1) $[\bar{\bar{\alpha}}]\varphi \leftrightarrow [\alpha]\varphi$

(2) $[\alpha_1]\varphi \vee [\alpha_2]\varphi \to [\alpha_1 \sqcap \alpha_2]\varphi$

(3) $[\alpha \sqcup \beta]\varphi \to [\alpha]\varphi \wedge [\beta]\varphi$

(4) $[\overline{\alpha_1 \sqcap \alpha_2}]\varphi \leftrightarrow [\overline{\alpha_1}]\varphi \wedge [\overline{\alpha_2}]\varphi$

(5) $[\overline{\alpha_1}]\varphi \vee [\overline{\alpha_2}]\varphi \to [\overline{\alpha_1 \sqcup \alpha_2}]\varphi$

(6) $\langle\alpha\rangle \top \wedge [\alpha]\varphi \to \langle\alpha\rangle \varphi$

(7) $\langle\alpha\rangle \top \to \langle\alpha\rangle \mathbf{R}(\alpha)$

(8) $[1]\varphi \to [\alpha]\varphi$

Proof. Let us prove (4).

$\mathbf{R}(\bar{\alpha} \sqcup \bar{\beta}) \leftrightarrow \mathbf{R}(\bar{\alpha}) \vee \mathbf{R}(\bar{\beta})$, by R3

$\mathbf{R}(\bar{\alpha} \sqcup \bar{\beta}) \leftrightarrow \neg\mathbf{R}(\alpha) \vee \neg\mathbf{R}(\beta)$, by R1

$\mathbf{R}(\bar{\alpha} \sqcup \bar{\beta}) \leftrightarrow \neg\mathbf{R}(\alpha \sqcap \beta)$, by R2

$\mathbf{R}(\bar{\alpha} \sqcup \bar{\beta}) \leftrightarrow \mathbf{R}(\overline{\alpha \sqcap \beta})$, by R1

$[\overline{\alpha \sqcap \beta}]\varphi \leftrightarrow [\bar{\alpha} \sqcup \bar{\beta}]\varphi$, by R7

$[\overline{\alpha \sqcap \beta}]\varphi \leftrightarrow [\bar{\alpha}]\varphi \wedge [\bar{\beta}]\varphi$, by (3), and $[\alpha]\mathbf{1}$ □

Since (1-5) are derivable, our system is powerful enough to interpret the system proposed by Meyer in [8], except for the axiom on the negation of sequential actions. In addition, since (7) is derivable, within **ADL** the performability of an action, expressed by $\langle\alpha\rangle \top$, is to be distinguished from the possibility of the result of the action, i.e., $\Diamond\mathbf{R}(\alpha)$. In fact, while $\langle\alpha\rangle \top \to \langle\alpha\rangle \mathbf{R}(\alpha)$, and, hence, $\langle\alpha\rangle \top \to \Diamond\mathbf{R}(\alpha)$, it is possible that $\Diamond\mathbf{R}(\alpha)$ even if α is not performable. Finally, in this system two intuitive concepts of inclusion between actions or action results are definable.

Definition 3.5 inclusions.

(i) $\beta \sqsubseteq \alpha := [\alpha]\mathbf{R}(\beta)$.

(ii) $\beta \sqsubseteq_R \alpha := \Box(\mathbf{R}(\alpha) \to \mathbf{R}(\beta))$.

As it is easy to check, both $\sqsubseteq$ and $\sqsubseteq_R$ are preorders. As it will become clear below, the introduction of these preorders allows us to represent actions that, while being optimal in their results, are not permitted, due to the fact that they also realize what is prohibited during their course.

3.3 Characterization

The system **ADL** is sound and strongly complete with respect to the class of models introduced above. Soundness is straightforward. Completeness is proved by a canonicity argument. Let us first define $w/\Box := \{\varphi \mid \Box\varphi \in w\}$; $w/[\uparrow] := \{\varphi \mid [\uparrow]\varphi \in w\}$; $w/[\alpha] = \{\varphi \mid [\alpha]\varphi \in w\}$.

Definition 3.6 canonical model for $\mathcal{L}(\mathbf{ADL})$. The canonical model for $\mathcal{L}(\mathbf{ADL})$ is the tuple

$M_C = \langle W, R, S, Ideal, \{R_w \mid w \in W\}, r, V \rangle$, where

(1) W is the set of maximal consistent sets of formulas
(2) $R : W \to \wp(W)$ is such that $v \in R(w) \Leftrightarrow w/\Box \subseteq v$
(3) $S : W \to \wp(W)$ is such that $v \in S(w) \Leftrightarrow w/[\uparrow] \subseteq v$
(4) $Ideal = \{w \mid I \in w\} \subseteq W$
(5) $R_w : Tm(\mathcal{L}) \to \wp(W)$ is such that $v \in R_w(\alpha) \Leftrightarrow w/[\alpha] \subseteq v$
(6) $r : Tm(\mathcal{L}) \to \wp(W)$ is such that $v \in r(\alpha) \Leftrightarrow \mathbf{R}(\alpha) \in v$
(7) $V : P \to \wp(W)$ is such that $v \in V(p) \Leftrightarrow p \in v$

For reason of space, we omit the proofs of the following lemmas.

Lemma 3.7 (*Truth Lemma*): *$M_C, w \models \varphi \Leftrightarrow \varphi \in w$.*

Lemma 3.8 (*Model Lemma*): *M_C is a model for $\mathcal{L}(\mathbf{ADL})$.*

They essentially follow from the definitions of R, S, $Ideal$, R_w, r and from the correspondence between axioms of **ADL** and conditions on models for $\mathcal{L}(\mathbf{ADL})$.

4 Deontic concepts and paradoxes

At this point, we can introduce the definition of four different kinds of deontic concepts [6].

Definition 4.1 deontic concepts on states and actions.

Group 1: ideal on states.	**Group 2**: ideal on results.
1. $\mathbf{P}(\varphi) := \langle I \rangle \varphi$	1. $\mathbf{P}(\mathbf{R}(\alpha)) := \langle I \rangle \mathbf{R}(\alpha)$
2. $\mathbf{F}(\varphi) := [I]\neg\varphi$	2. $\mathbf{F}(\mathbf{R}(\alpha)) := [I]\neg\mathbf{R}(\alpha)$
3. $\mathbf{O}(\varphi) := [I]\varphi$	3. $\mathbf{O}(\mathbf{R}(\alpha)) := [I]\mathbf{R}(\alpha)$
4. $\mathbf{P}^S(\varphi) := \Diamond\varphi \wedge \Box(\varphi \to I)$	4. $\mathbf{P}^S(\mathbf{R}(\alpha)) := \Diamond\mathbf{R}(\alpha) \wedge \Box(\mathbf{R}(\alpha) \to I)$
Group 3: ideal on actions.	**Group 4**: conditional on results.
1. $\mathbf{P}!(\alpha) := \langle \alpha \rangle I$	1. $\mathbf{P}(\alpha) := \langle \uparrow \rangle \mathbf{R}(\alpha)$
2. $\mathbf{F}!(\alpha) := \neg \langle \alpha \rangle I$	2. $\mathbf{F}(\alpha) := \neg \langle \uparrow \rangle \mathbf{R}(\alpha)$
3. $\mathbf{O}!(\alpha) := \neg \langle \bar{\alpha} \rangle I$	3. $\mathbf{O}(\alpha) := \neg \langle \uparrow \rangle \mathbf{R}(\bar{\alpha})$
4. $\mathbf{P}!^S(\alpha) := \langle \alpha \rangle I \wedge [\alpha]I$	4. $\mathbf{P}^S(\alpha) := \langle \uparrow \rangle \mathbf{R}(\alpha) \wedge [\uparrow]\mathbf{R}(\alpha)$

The definition of the conditional deontic concepts can be justified by considering the following equivalences.

$M, w \models \mathbf{F}(\alpha) \;\Leftrightarrow\; M, w \models \neg \langle \uparrow \rangle \mathbf{R}(\alpha)$
$M, w \models \mathbf{F}(\alpha) \;\Leftrightarrow\; \forall v \in W(v \in S(w) \Rightarrow M, v \not\models \mathbf{R}(\alpha))$

[6] Concepts in Group 2 are specific instances of concepts in Group 1. They characterize deontic concepts on actions in terms of action results and are of interest when compared with concepts in Group 3 and Group 4.

$M, w \models \mathbf{F}(\alpha) \Leftrightarrow \forall v \in W(v \in S(w) \Rightarrow v \notin r(\alpha))$
$M, w \models \mathbf{F}(\alpha) \Leftrightarrow r(\alpha) \cap S(w) = \varnothing$

Hence, an action is conditionally prohibited provided that its result only holds in worlds that are worse than the best accessible worlds. Similarly, an action is conditionally permitted (obliged) when its result holds is some (all) of the best accessible worlds.

Fact 4.2 *Relations between different deontic concepts.*
(1) $\mathbf{P}!(\alpha) \rightarrow \mathbf{P}(\alpha)$
(2) $\mathbf{P}!(\alpha) \wedge [\alpha]\varphi \rightarrow \mathbf{P}(\varphi)$*, and so* $\mathbf{P}!(\alpha) \rightarrow \mathbf{P}(\mathbf{R}(\alpha))$
(3) $\langle 1 \rangle\, I \wedge \langle \uparrow \rangle\, \varphi \rightarrow \langle I \rangle\, \varphi$*, by* I3*, and so* $\langle 1 \rangle\, I \wedge \mathbf{P}(\alpha) \rightarrow \mathbf{P}(\mathbf{R}(\alpha))$

As expected, (1) all ideally permitted actions are conditionally permitted and (2) both the result and all the consequences of ideally permitted actions are ideally permitted states. In addition, (3) provided that the ideal can be accessed, the result of conditionally permitted actions are ideally permitted. By contrast, it can be proved that not all actions that are conditionally permitted are ideally permitted. Thus, conditional prescription can be effective even in cases where no action is ideally permitted.

Fact 4.3 *Permission and inclusion.*
(1) $\mathbf{P}!(\alpha) \wedge \beta \sqsubseteq \alpha \Rightarrow \mathbf{P}(\beta)$
(2) $\mathbf{P}(\alpha) \wedge \beta \sqsubseteq_R \alpha \Rightarrow \mathbf{P}(\beta)$

Accordingly, actions including conditionally prohibited actions are prohibited.

Now, our claim is that the best way for capturing the intuitions discussed in section 2 is to use conditional deontic concepts. Thus, we assume them to provide a solution to the three groups of paradoxes mentioned above.

4.1 Paradoxes on standard prescriptions

Within **ADL** standard paradoxes concerning the conditional notions of obligation and permission can be solved in two different ways. Firstly, we can opt for using notions of strong permission and obligation as in [6]. Secondly, and more interestingly, we can define two specific notions of choice permission and choice obligation:

– choice permission: $\mathbf{P}(\alpha + \beta) := \langle \uparrow \rangle\, \mathbf{R}(\alpha) \wedge \langle \uparrow \rangle\, \mathbf{R}(\beta)$
– choice obligation: $\mathbf{O}(\alpha + \beta) := \mathbf{O}(\alpha \sqcup \beta) \wedge \mathbf{P}(\alpha + \beta)$

It is then not difficult to see that:

$\vdash_{\mathbf{ADL}} \mathbf{P}(\alpha + \beta) \rightarrow \mathbf{P}(\alpha) \wedge \mathbf{P}(\beta);\quad \nvdash_{\mathbf{ADL}} \mathbf{P}(\alpha) \rightarrow \mathbf{P}(\alpha + \beta)$
$\vdash_{\mathbf{ADL}} \mathbf{O}(\alpha + \beta) \rightarrow \mathbf{P}(\alpha + \beta);\quad \nvdash_{\mathbf{ADL}} \mathbf{O}(\alpha) \rightarrow \mathbf{O}(\alpha + \beta)$

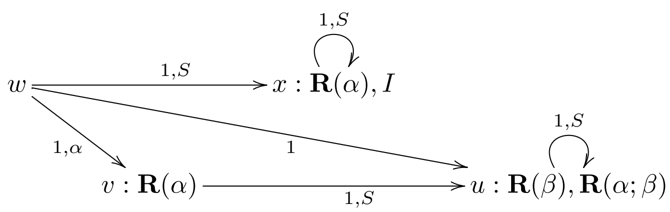
The present solution seems to be more intuitive insofar as both strong permission and strong obligation require that there is no way we can violate the law if we act according to what is strongly permitted or obliged, while ordinary choices can be risky: we are ordinarily allowed to choose between alternative actions even if there are ways of performing such actions that lead to a violation of the law.

4.2 Paradoxes on prescriptions on sequential actions

Within **ADL** paradoxes concerning obligation and permission of sequential actions, when these concepts are fixed according to the conditional definition, find an insightful solution.

As to van der Mayden's paradox, note that both $\langle\alpha\rangle\,\mathbf{P}(\beta) \to \mathbf{P}(\alpha;\beta)$ and the stronger $\mathbf{P}(\alpha) \wedge \langle\alpha\rangle\,\mathbf{P}(\beta) \to \mathbf{P}(\alpha;\beta)$ can fail. Consider the following model:

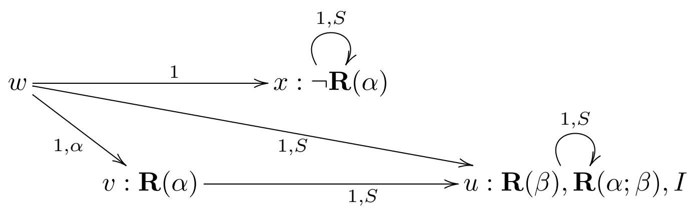
1) $W = R(w) = R(v) = R(u) = R(x) = \{w, v, u, x\}$
2) $R_w(\alpha) = \{v\}$; $R_v(\alpha) = R_u(\alpha) = R_x(\alpha) = \varnothing$
3) $R_w(1) = \{v, u, x\}$; $R_v(1) = R_u(1) = \{u\}$; $R_x(1) = \{x\}$
4) $S(w) = S(x) = \{x\} = Ideal$; $S(v) = S(u) = \{u\}$
5) $r(\alpha) = \{v, x\}$; $r(\beta) = r(\alpha;\beta) = \{u\}$



In this model, $w \models \langle\uparrow\rangle\,\mathbf{R}(\alpha)$ and $w \models \langle\alpha\rangle\,\langle\uparrow\rangle\,\mathbf{R}(\beta)$, but $w \not\models \langle\uparrow\rangle\,\mathbf{R}(\alpha;\beta)$, whence the conclusion. The failure of these principles is due to the fact that, even when α is permitted, $\langle\alpha\rangle\,\mathbf{P}(\beta)$ is not sufficient for $\mathbf{P}(\alpha;\beta)$, since the world we land on by performing α at w may not be one of best options of w. In the previous model, β is permitted in v because the $R(\beta)$-world u is among the best options achievable from v. Still, since this is not sufficient to obtain that u is also among the best options achievable from w, $\alpha;\beta$ is not permitted in w. In addition, note that the converse of the first principle also fails, since $u \models \langle\uparrow\rangle\,\mathbf{R}(\alpha;\beta)$, but $u \not\models \langle\uparrow\rangle\,\mathbf{R}(\alpha)$.

As to Anglberger's paradox, note that both $\mathbf{F}(\alpha) \to \mathbf{F}(\alpha;\beta)$ and $\mathbf{F}(\alpha) \to [\alpha]\mathbf{F}(\beta)$ can fail. Consider the following model:

1) $W = R(w) = R(v) = R(u) = R(x) = \{w, v, u, x\}$
2) $R_w(\alpha) = \{v\}$; $R_v(\alpha) = R_u(\alpha) = R_x(\alpha) = \varnothing$;
3) $R_w(1) = \{v, u, x\}$; $R_v(1) = R_u(1) = \{u\}$; $R_x(1) = \{x\}$
4) $S(w) = S(v) = S(u) = \{u\} = Ideal$; $S(x) = \{x\}$
5) $r(\alpha) = \{v\}$; $r(\beta) = r(\alpha;\beta) = \{u\}$



In this model, $w \not\models \langle\uparrow\rangle\,\mathbf{R}(\alpha)$, but $w \models \langle\alpha\rangle\,\langle\uparrow\rangle\,\mathbf{R}(\beta)$ and $w \models \langle\uparrow\rangle\,\mathbf{R}(\alpha;\beta)$, and the conclusion follows. The failure of these principles is due to the fact that, for α to be prohibited, it is sufficient that α makes the deontic condition of the reference world worse than any of the best accessible worlds. Still, this is not sufficient to exclude that doing $\alpha;\beta$ leads to one of these best accessible worlds.

4.3 Contrary to duty obligations

As a final application, let us consider cases of contrary to duty obligations instantiating these classical schemas:

It ought to be that φ, but $\neg\varphi$	It ought to be that φ, but $\neg\varphi$
It ought to be that if φ then $\neg R(\alpha)$	It ought to be that if φ then $\neg R(\alpha)$
If $\neg\varphi$, then it ought to be that $R(\alpha)$	It ought to be that if $\neg\varphi$ then $R(\alpha)$

In our framework, the most intuitive analysis is:

$[I]\varphi \wedge \neg\varphi$	$[I]\varphi \wedge \neg\varphi$
$[I](\varphi \rightarrow \neg R(\alpha))$	$[I](\varphi \rightarrow \neg R(\alpha))$
$\Box(\neg\varphi \rightarrow [\uparrow]R(\alpha))$	$[\uparrow](\neg\varphi \rightarrow R(\alpha))$

In both cases, we obtain that $[I]\neg R(\alpha)$ and $[\uparrow]R(\alpha)$. Still, no contradiction follows, since in any situation in which the result of α is prohibited, according to the law, the obligation to do α is only conditional. Finally, note that the present interpretation of the conditional leading to a contrary to duty obligation validates both

FD: factual detachment and **DD**: deontic detachment

$$\frac{\Box(\varphi \rightarrow [\uparrow]\psi) \qquad \varphi}{[\uparrow]\psi} \qquad\qquad \frac{\Box(\varphi \rightarrow [\uparrow]\psi) \qquad [\uparrow]\varphi}{[\uparrow]\psi}$$

which is one of the desiderata proposed in [3].

5 Conclusion

In this paper, we have presented a general system of deontic logic of actions in which the main problems related to the definition of deontic concepts in a dynamic framework can be overcome. The solutions we have proposed are based on the introduction of a group of conditional deontic concepts, according to which what is permitted, prohibited and obligatory depends on the best states that the agent can realize, given the conditions in which she is acting. The conceptual apparatus encoded in our system, which allows us to capture these new concepts, includes a twofold distinction on the ontic level. First, a distinction between what is possible and what is realizable by performing an action; and, second, a distinction between the result associated with an action and the consequences of that action. Being based on this conceptually rich framework, our system gives us the possibility of systematically bringing together and comparing in an innovative way Andersonian deontic concepts on states as well as on results of actions, ideal deontic concepts on actions *à la*

Meyer, and conditional deontic concepts on actions. We have shown that the availability of both ideal deontic concepts on states and conditional deontic concepts on actions provides us with a natural solution to the paradoxes of contrary to duty obligations. What is more, the introduction of conditional deontic concepts allows us to define original notions of choice permission and choice obligation that, while not being subject to standard paradoxes, take into account the riskiness of choices. Finally, besides not incurring in paradoxes concerning the sequential execution of actions, the new deontic concepts provides us with a way of making sure that, even in states in which the ideal of deontic perfection is not realizable, the actions of the agent can be deontically qualified in a non-trivial way.

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