Logic of Implicit and Explicit Justifiers

Alessandro Giordani

ABSTRACT. The aim of this paper is to provide an intuitive semantics for systems of justification logic which allows us to cope with the distinction between implicit and explicit justifiers. The paper is subdivided into three sections. In the first one, the distinction between implicit and explicit justifiers is presented and connected with a proof-theoretic distinction between two ways of interpreting sequences of sentences; that is, as sequences of axioms in a certain set and as proofs constructed from that set of axioms. In the second section, a basic system of justification logic for implicit and explicit justifiers is analyzed and some significant facts about it are proved. In the final section, an adequate semantics is proposed, and the system is proved to be sound and complete whit respect to it.

Keywords: justification logic; epistemic logic; implicit justification; explicit justification; Fitting semantics.

1 Introduction

Justification logic is one of the most interesting developments of epistemic logic[1]. It extends the expressive power of the language of standard epistemic logic by introducing sentences like $t : \varphi$, to be intended as φ is justified in virtue of t, or t is a justifier for φ. Axioms for systems of justification logic can be introduced from different points of view. A first approach is to rest on our basic intuitions concerning how justifiers are related with both propositions and other justifiers. A slightly different approach is to focus on principles that characterize well-known systems of logic which are strictly connected with the structure of justification, such as systems of provability logic[2]. In fact, in standard systems of provability logic, a sentence like $\Box\varphi$ is interpreted as stating that φ is provable in some mathematical base theory, so that there is a proof of φ in that theory. Thus, a sentence stating that t is a justifier for φ is intuitively interpreted as stating that t refers to a proof of φ. However, this is not the sole interpretation of a sentence like that. In particular, if $\mathcal{A}$ is a set of logical and non-logical axioms, then two options concerning the way of interpreting that t is a justifier for φ are available.[3]

[1] [1], [3], and [10] are excellent introductions to this topic. In these works, a number of applications of systems of justification logic for the study of the notions of evidence and justification in epistemology are also provided.

[2] See [7] for an extensive introduction to systems of provability logic and their representation in modal logic.

[3] In what follows, I assume that proofs are constructed in Hilbert style systems where *modus ponens* is the only primitive rule.

Option 1. A sentence stating that t is a justifier for φ says that t refers to a proof of φ from $\mathcal{A}$. In this case, t refers to a finite sequence of sentences, with final sentence φ, where every sentence either is some axiom in $\mathcal{A}$ or is obtained from previous sentences by applying *modus ponens*.

Option 2. A sentence stating that t is a justifier for φ says that a proof of φ is obtainable from a sequence t of theorems. In this case, t refers to a finite sequence of sentences, where every sentence is derivable from axioms in $\mathcal{A}$, from which a proof of φ can be constructed.

Hence, while in the first case t refers to an explicit proof of φ, in the second case it refers to the basic sentences from which such a proof can be constructed and, in particular, to the basic axioms that can be used to prove it.

The first interpretation gives rise to a general notion of *explicit* justifier, which is extremely intuitive, since it is based on the idea that it is possible to identify what sentences are justified by t by just considering the structure of t. In fact, since t refers to a proof, all the sentences that are involved in t are certainly justified by t. In the light of this, I will use the standard notation $t : \varphi$ to say that t refers to a proof of φ. By contrast, the second interpretation gives rise to a general notion of *implicit*, or potential, justifier, according to which t is a justifier for all the sentences that are contained in the logical closure of the axioms contained in t. I will use the notation $[t]\varphi$ to say that t refers to a sequence of sentences from which φ is provable.

Remark 1: The notion of potential justification is to be distinguished from the notion of possible justification. Indeed, every sentence that is provable from $\mathcal{A}$ has a proof exploiting a certain set of axioms in $\mathcal{A}$, but it is not true that every sentence that is so provable has a proof exploiting *the same set* of axioms in $\mathcal{A}$. Thus, the notion of potential justification is more fine-grained than the notion of possible justification.

Remark 2: The notion of explicit justification is distinguished from the notion of potential justification. Indeed, every sentence that is provable from a sequence of theorems from $\mathcal{A}$ is the final sentence in a proof consisting in a sequence of theorems from $\mathcal{A}$, but it is not true that every sequence of theorems from $\mathcal{A}$ gives rise to the proof of a unique sentence. In general, while $t : \varphi$ implies $[t]\varphi$, for every t, it is not true that $[t]\varphi$ implies $t : \varphi$, for every t.

I find both the first and the second interpretations worth of investigation, even if only the first one has received a systematic treatment in the current research on justification logic[4]. In the following sections, I will develop a system of logic where both assertions of explicit justification, like $t : \varphi$, and assertions of implicit justification, like $[t]\varphi$, are treated in a unified framework[5]. In particular, my two main aims are to provide an axiomatization of the previous notions and

[4]See [2], [4], and [9], for a survey of different directions in which the logic of justification can be developed.

[5]In [5], an interesting analysis of the distinction between implicit and explicit justifiers is proposed, but the notion of implicit justification is not distinguished from the notion of possible justification. As a consequence, there is no way of articulating the state described by a sentence like $[t]\varphi$. In [11], the distinction between $t : \varphi$ and $[t]\varphi$ is present, but the semantic analysis of $[t]\varphi$, as we will see, is not completely satisfactory.

to introduce a suitable semantics for them. Accordingly, in the next section, a basic system of justification logic for implicit and explicit justifiers is offered and some significant facts about it are proved, while in the final section a suitable semantics is proposed, and the system is proved to be sound and complete with respect to it.

2 Axiomatic characterization

Let us start with introducing an adequate axiomatic system for capturing both the notion of explicit justification and the notion of implicit justification. Let us call the basic system **IEJ**. The standard language of a system of justification logic is characterized by two set of rules, specifying the set of terms and the set of formulas of the language.[6] The language of **IEJ** is characterized in the same way. The set of terms and formulas are defined according to the following grammar.

$t := \ j \mid c \mid t+s \mid t \times s \mid !t$, where j is a variable and c is a constant for justifiers

$\varphi := \ p \mid \neg\varphi \mid \varphi \wedge \psi \mid [t]\varphi \mid t : \varphi$, where p is a variable for propositions

The operators $+$, $\times$, and ! are used to construct new justifiers from basic ones. As usual, $t+s$ is interpreted as a justifier providing justification for all the sentences that can be justified either by t or by s, while $t \times s$ is interpreted as a justifier providing justification for all the sentences that can be justified by applying *modus ponens* to premises justified by t and by s. In addition, ! is a justification checker that returns a justifier $!t$ for the sentence stating that t is a justifier for φ, provided that t is indeed such a justifier. Finally, a justification sentence like $[t]\varphi$ is interpreted as t is an implicit justifier for φ, whereas a justification proposition like $t : \varphi$ is interpreted as t is an explicit justifier for φ.

2.1 Axioms

The basic system **IEJ** is constituted by three groups of axioms: the first group is a standard system for classical propositional logic, while the two other groups are introduced in order to characterize explicit and implicit justifications. It is worth noting that axioms are considered as a priori justified, so that any epistemic agent accepts logical axioms, including the ones concerning justification, as immediately evident. This intuition is made precise by introducing a *constant specification*, which can be construed according to the following definition.

Definition 1: *Constant specification.*

Let $CS!$ be the set of $c : \varphi$, such that c is a constant for justifiers and φ is an axiom of **IEJ**. Then, a *constant specification* CS is a subset of $CS!$ and an *axiomatically appropriate constant specification* is a constant specification where, for all the axioms φ of **IEJ**, there is a constant c such that $c : \varphi \in CS$.

[6] See [1], [9], and [10], for a detailed exposition, and [3] for the connection between operators on justifiers and operators on proofs within the context of the logic of provability.

In particular, we will only work with axiomatically appropriate constant specifications. In this way, every logical axiom is associated with a justification constant, witnessing that the axiom is accepted as justified. Thus, let CS be an axiomatically appropriate constant specification. Then, **IEJ** is characterized, relative to CS, by the following axioms.

Group 1: classical tautologies and *modus ponens*.

For the notion of explicit justification, let us use the standard axioms provided in [8] and [10].

Group 2: axioms concerning explicit justification and *internalization rule*.

EJ1: $t : (\varphi \to \psi) \to (s : \varphi \to t \times s : \psi)$

EJ2: $t : \varphi \vee s : \varphi \to t{+}s : \varphi$

EJ3: $t : \varphi \to !t : (t : \varphi)$

RJ: $c : \varphi$, where φ is an axiom in **IEJ** such that $c : \varphi \in CS$.

Group 2 includes the axioms which characterize the standard notion of explicit justification. **EJ1** states that, given two justifiers, t and s, a justifier like $t \times s$ provides justification to any sentence that can be derived from implications justified by t and sentences justified by s by applying *modus ponens*. The idea is that *modus ponens* is the basic deduction rule and that propositional deduction is accepted by the epistemic agent as providing justification. **EJ2** states that given two justifiers, t and s, a justifier like $t + s$ provides justification to any proposition justified by either t or s. **EJ3** states that justification is internally accessible, so that all justified propositions can be acknowledged as such. Finally, **RJ** allows us to have axioms justified by basic justifiers.

For the notion of implicit justification, I will use the set of axioms provided in [11].

Group 3: axioms and rules concerning implicit justification.

IJ1: $[t](\varphi \to \psi) \to ([s]\varphi \to [t \times s]\psi)$

IJ2: $[t]\varphi \vee [s]\varphi \to [t{+}s]\varphi$

IJ3: $[t]\varphi \to [!t][t]\varphi$

IJ4: $t : \varphi \to [t]\varphi$

IJ5: $[c]\varphi \to [t]\varphi$, where c is a constant

IJ6: $[t \times t]\varphi \leftrightarrow [t + t]\varphi \leftrightarrow [!t]\varphi \leftrightarrow [t]\varphi$

Group 3 includes the axioms which characterize an intuitive notion of implicit justification. The first three axioms state that the notion of implicit justification is similar to the notion of explicit justification, as far as the basic operations are concerned. In particular, **IJ2** captures the idea that the logical closure of a certain set of sentences is included in the logical closure of any set of sentences that includes that first set. **IJ4** states that what is explicitly justified by t is implicitly justified by the same justifier. Indeed, if t refers to a proof of a certain sentence, then that sentence is certainly contained in the logical closure of the set of sentences in t. Hence, the idea that any set of sentences is included in its logical closure is respected. **IJ5** states that the axioms, which are a priori justified, are always implicitly justified by any justifier, since they are contained in the logical closure of any set of sentences. Finally, **IJ6** says that $t \times t$, $t + t$, $!t$, and t provide implicit justification to the same propositions. This axiom captures the idea that what is implicitly justified by t is precisely what can be inferred from sentences in t, so that nothing new is implicitly justified when inferences are performed from sentences in t. Hence, the idea that the logical closure of the logical closure of a set of sentences is included in the logical closure of that set is respected. In conclusion, the crucial properties of a logical closure operator Cn

1. $X \subseteq Cn(X)$
2. $Cn(Cn(X)) \subseteq Cn(X)$
3. $X \subseteq Y \Rightarrow Cn(X) \subseteq Cn(Y)$

are incorporated in the treatment of any implicit justification operator.[7]

2.2 Theorems

In **IEJ**, some fundamental theorems are derivable, which concern rules for explicit and implicit justification. In particular, we get the following crucial rules.

REJ: $\vdash_{\mathbf{IEJ}} \varphi \Rightarrow \vdash_{\mathbf{IEJ}} t : \varphi$, for *some* term t.

The proof is by induction on the length of the derivation.

Suppose φ is an axiom. Then, $\vdash_{\mathbf{IEJ}} c : \varphi$, for some constant c, by **RJ**. Suppose φ is obtained by an application of **RJ**. Then, $\varphi = c : \psi$, for some c and some axiom ψ. Hence, $\vdash_{\mathbf{IEJ}} !c : (c : \psi)$, by **EJ3**, and so $\vdash_{\mathbf{IEJ}} !c : \varphi$. Suppose φ is obtained by an application of *modus ponens* to $\psi \rightarrow \varphi$ and ψ. Then, by induction hypothesis, $\vdash_{\mathbf{IEJ}} t : (\psi \rightarrow \varphi)$ and $\vdash_{\mathbf{IEJ}} s : \psi$, for some t and s. Hence, $\vdash_{\mathbf{IEJ}} t \times s : \varphi$, by **EJ1**.

REJ is a rule of explicit justification, stating that every theorem of **IEL** is justified by some justifier. **REJ** is a version of a *non-standard* rule of necessitation, since not every theorem is justified by the same term t. Hence, a modality like $t :$ is not a standard modality.

[7] See Tarski [12], chapters V and XII. To be more precise, while property 1 is reflected by axiom **IJ4** and property 2 is reflected by axiom **IJ2**, property 3 is reflected by $[t]\varphi \rightarrow [t][t]\varphi$, which is a consequence of axioms **IJ3** and **IJ6**.

RIJ: $\vdash_{\mathbf{IEJ}} \varphi \Rightarrow \vdash_{\mathbf{IEJ}} [t]\varphi$, for *every* term t.

The proof is again by induction on the length of the derivation.

Suppose φ is an axiom. Then, $\vdash_{\mathbf{IEJ}} [t]\varphi$, for every term t, by **RJ**, **IJ4** and **IJ5**. Suppose φ is obtained by an application of **RJ**. Then, $\varphi = c : \psi$, for some c and some axiom ψ. Hence, $\vdash_{\mathbf{IEJ}} !c : (c : \psi)$, by **EJ3**, $\vdash_{\mathbf{IEJ}} [!c](c : \psi)$, by **IJ4**, $\vdash_{\mathbf{IEJ}} [c](c : \psi)$, by **IJ6**, and so $\vdash_{\mathbf{IEJ}} [t](c : \psi)$, by **IJ5**. Suppose φ is obtained by an application of *modus ponens* to $\psi \rightarrow \varphi$ and ψ. Then, by induction hypothesis, $\vdash_{\mathbf{IEJ}} [t](\psi \rightarrow \varphi)$ and $\vdash_{\mathbf{IEJ}} [t]\psi$, for every t. Hence, $\vdash_{\mathbf{IEJ}} [t]\varphi$, by **IJ1**.

RIJ is a rule of implicit justification, stating that every theorem of **IEL** is justified by every justifier. **RIJ** is a version of a *standard* rule of necessitation, since every theorem is justified by the same term t. Hence, a modality like $[t]$ might be a standard modality. In fact, the next proposition shows that $[t]$ actually is a standard modality.

KIJ: $\vdash_{\mathbf{IEJ}} [t](\varphi \rightarrow \psi) \rightarrow ([t]\varphi \rightarrow [t]\psi)$, for *every* term t.

$\vdash_{\mathbf{IEJ}} [t](\varphi \rightarrow \psi) \rightarrow ([t]\varphi \rightarrow [t \times t]\psi)$ by **IJ1**

$\vdash_{\mathbf{IEJ}} [t](\varphi \rightarrow \psi) \rightarrow ([t]\varphi \rightarrow [t]\psi)$ by **IJ6**

Finally, we are also able to obtain the following propositions.

IJ7: $\vdash_{\mathbf{IEJ}} [s]\varphi \rightarrow [t \times s]\varphi$, for *every* term s.

$\vdash_{\mathbf{IEJ}} \varphi \rightarrow \varphi$ axiom in group 1

$\vdash_{\mathbf{IEJ}} c : (\varphi \rightarrow \varphi)$ by **RJ**

$\vdash_{\mathbf{IEJ}} [c](\varphi \rightarrow \varphi)$ by **IJ4**

$\vdash_{\mathbf{IEJ}} [t](\varphi \rightarrow \varphi)$ by **IJ5**

$\vdash_{\mathbf{IEJ}} [t](\varphi \rightarrow \varphi) \rightarrow ([s]\varphi \rightarrow [ttimess]\varphi)$ by **IJ1**

$\vdash_{\mathbf{IEJ}} [s]\varphi \rightarrow [t \times s]\varphi$ by logic

IJ8: $\vdash_{\mathbf{IEJ}} [t]\varphi \rightarrow [t \times s]\varphi$, for *every* term s.

$\vdash_{\mathbf{IEJ}} \varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi)$ axiom in group 1

$\vdash_{\mathbf{IEJ}} [t](\varphi \rightarrow ((\varphi \rightarrow \varphi) \rightarrow \varphi))$ by **IJ1**

$\vdash_{\mathbf{IEJ}} [t]\varphi \rightarrow [t]((\varphi \rightarrow \varphi) \rightarrow \varphi)$ by **KIJ**

$\vdash_{\mathbf{IEJ}} [t]((\varphi \rightarrow \varphi) \rightarrow \varphi) \rightarrow ([s](\varphi \rightarrow \varphi) \rightarrow [t \times s]\varphi)$ by **IJ1**

$\vdash_{\mathbf{IEJ}}$ $[t]\varphi \to ([s](\varphi \to \varphi) \to [t \times s]\varphi)$ by logic

$\vdash_{\mathbf{IEJ}}$ $[t]\varphi \to [t \times s]\varphi$ by group 1, **IJ4** , **IJ5**, and logic

Hence, by **IJ7** and **IJ8**, a modality like $[t \times s]$ is both stronger than t and stronger than s, in accordance with its intended interpretation.

Now, it is worth noting that, in the light of **RIJ** and **KIJ**, every $[t]$ is a standard modality. This suggests a new semantics for the basic system of logic for explicit and implicit justification, which is more insightful than the semantics proposed in [11]. To be sure, the new semantics fits the intuition that, while explicit operators can be modeled by means of syntactic assignments, implicit operators are to be modeled by means of conditions on the set of possible epistemic states.

3 Semantic characterization

The semantic framework for standard systems of justification logic is due to Fitting [9]. In Fitting semantics, a frame is a tuple $\langle W, \mathcal{R}, \mathcal{E} \rangle$, where W is a non-empty set of states, $\mathcal{R}$ is a transitive relation on W, and $\mathcal{E}$ is a function from states and justifiers to sets of formulas. Within this framework, explicit justification is modeled by introducing a syntactic function that, given a justifier t and a possible world w, selects the set of all formulas for which t provides explicit justification at w. In particular, $\varphi \in \mathcal{E}(w, t)$ states that, at w, t is a justifier that can serve as possible evidence for φ. In a similar way, we might model implicit justification by introducing a function that, given a justifier t and a possible world w, selects the set of all formulas for which t provides implicit justification at w.[8] Hence, a frame is a tuple $\langle W, \mathcal{R}, \mathcal{E}, \mathcal{E}^* \rangle$, where

- W is a non-empty set of states
- $\mathcal{R} \subseteq W \times W$ is transitive
- $\mathcal{E}$ is such that $\mathcal{E}(w, t)$ is a set of formulas, for every w and t
- $\mathcal{E}^*$ is such that $\mathcal{E}^*(w, t)$ is a set of formulas, for every w and t

In addition, $\mathcal{E}$ and $\mathcal{E}^*$ must satisfy the following constraints.

1. Conditions on $\mathcal{E}$.

$\varphi \to \psi \in \mathcal{E}(w, t)$ and $\varphi \in \mathcal{E}(w, s) \Rightarrow \psi \in \mathcal{E}(w, t \times s)$

$\mathcal{E}(w, t) \cup \mathcal{E}(w, s) = \mathcal{E}(w, t + s)$

$\varphi \in \mathcal{E}(w, t) \Rightarrow t : \varphi \in \mathcal{E}(w, !t)$

$\mathcal{R}(w, v) \Rightarrow \mathcal{E}(w, t) \subseteq \mathcal{E}(v, t)$

[8]This is the strategy pursued in [11].

2. Conditions on $\mathcal{E}^*$.

$\varphi \to \psi \in \mathcal{E}^*(w,t)$ and $\varphi \in \mathcal{E}^*(w,s) \Rightarrow \psi \in \mathcal{E}^*(w, t \times s)$

$\mathcal{E}^*(w,t) \cup \mathcal{E}^*(w,s) = \mathcal{E}^*(w, t+s)$

$\varphi \in \mathcal{E}^*(w,t) \Rightarrow [t]\varphi \in \mathcal{E}^*(w, !t)$

$\mathcal{E}(w,t) \subseteq \mathcal{E}^*(w,t)$

$\mathcal{E}^*(w,c) \subseteq \mathcal{E}^*(w,t)$, for every c

$\mathcal{E}(w, t \times t) = \mathcal{E}^*(w, t+t) = \mathcal{E}^*(w, !t) = \mathcal{E}^*(w,t)$

$\mathcal{R}(w,v) \Rightarrow \mathcal{E}^*(w,t) \subseteq \mathcal{E}^*(v,t)$

Once these conditions are posed, one can prove a completeness theorem for **IEJ**.[9] To be sure, the conditions on $\mathcal{E}$ and $\mathcal{E}^*$ are introduced precisely for ensuring the soundness of the axioms in group 2 and 3.

3.1 A new semantics for implicit and explicit justification

An apparent limitation of the previously introduced semantic framework is that implicit and explicit justifications are modeled in the same way. In particular, while it is normal to model the set of explicitly justified propositions by means of a selection function like $\mathcal{E}$, since we do not expect such a set to be closed with respect to any logical rule, it is not intuitive to model the set of implicitly justified propositions by means of a selection function like $\mathcal{E}^*$, since, in this case, we do expect such a set to be closed with respect to the logical rules, and indeed **RIJ** and **KIJ** confirm our expectation. Hence, it should be more appropriate to develop the logic of implicit justification by means of conditions linking epistemic states, which are the standard tools for treating implicit epistemic modalities. The rest of this section is thus dedicated to develop this kind of semantics.

Definition 2: Basic frame for **IEJ**.

A basic frame for **IEJ** is a tuple $\langle W, \mathcal{S}, \mathcal{E} \rangle$, where W is a set of epistemic states, $\mathcal{S}$ is a function that assigns to every $w \in W$ and every term t a set of states $\mathcal{S}(w,t)$, and $\mathcal{E}$ is a function that assigns to every $w \in W$ and every term t a set of formulas $\mathcal{E}(w,t)$. In addition, $\mathcal{S}$ and $\mathcal{E}$ must satisfy the following conditions.

1. Conditions on S

[9] See [11], section 4.2.

S1: $\mathcal{S}(w, t \times s) \subseteq \mathcal{S}(w,t) \cap \mathcal{S}(w,s)$

S2: $\mathcal{S}(w, t + s) \subseteq \mathcal{S}(w,t) \cap \mathcal{S}(w,s)$

S3: $\mathcal{S}(w,t) \subseteq \mathcal{S}(w,c)$, for all c

S4: $\mathcal{S}(w, t \times t) = \mathcal{S}(w, t+t) = \mathcal{S}(w, !t) = \mathcal{S}(w,t)$

S5: $v \in \mathcal{S}(w,t) \Rightarrow \mathcal{S}(v,t) \subseteq \mathcal{S}(w,t)$

2. Conditions on $\mathcal{E}$.

E1: $\varphi, \varphi \to \psi \in \mathcal{E}(w,t) \Rightarrow \psi \in \mathcal{E}(w,t)$

E2: $\mathcal{E}(w,t) \cup \mathcal{E}(w,s) = \mathcal{E}(w, t+s)$

E3: $\varphi \in \mathcal{E}(w,t) \Rightarrow [t]\varphi \in \mathcal{E}(w, !t)$

E4: $v \in \mathcal{S}(w,t) \Rightarrow \mathcal{E}(w,t) \subseteq \mathcal{E}(v,t)$

Definition 3: Basic model for **IEJ**.

A model for **IEJ** is a tuple $M = \langle W, \mathcal{S}, \mathcal{E}, V \rangle$, where

- $\langle W, \mathcal{S}, \mathcal{E} \rangle$ is a frame for **IEJ**
- V is such that $V(p) \subseteq W$ for any propositional variable p

As usual, a valuation function for propositional variables is introduced as a function that assigns to each propositional variable a set of epistemic states, which are the states where the proposition denoted by the variable is true.

Definition 4: Truth at a world in a model for **IEJ**.

The notion of truth of a formula is defined as follows:

$M, w \models p \Leftrightarrow w \in V(p)$

$M, w \models \neg\varphi \Leftrightarrow M, w \models \varphi$

$M, w \models \varphi \wedge \psi \Leftrightarrow M, w \models \varphi$ and $M, w \models \psi$

$M, w \models [t]\varphi \Leftrightarrow M, w \models \varphi$, for all v such that $v \in S(w,t)$

$M, w \models t : \varphi \Leftrightarrow M, w \models \varphi$, for all v such that $v \in S(w,t)$, and $\varphi \in \mathcal{E}(w,t)$

The notions of logical consequence and logical validity are defined as usual.

3.2 Characterization

Let us now show that the previous system can be completely characterized by the class of basic frames. It is not difficult to show that the axioms in groups 2 are valid with respect to the class of all frames and that *modus ponens* preserves validity.[10] Let us then focus on the axioms of group 3 and prove their validity.

IJ1: $\Vdash [t](\varphi \to \psi) \to ([s]\varphi \to [t \times s]\psi)$

Suppose $M, w \models [t](\varphi \to \psi)$ and $M, w \models [s]\varphi$. Suppose, in addition, that $u \in \mathcal{S}(w, t \times s)$. Since $\mathcal{S}(w, t \times s) \subseteq \mathcal{S}(w, t) \cap \mathcal{S}(w, s)$, by conditions *S1*, $u \in \mathcal{S}(w, t)$ and $u \in \mathcal{S}(w, s)$. Since $M, v \models \varphi \to \psi$, for all v such that $v \in \mathcal{S}(w, t)$, and $M, v \models \varphi$, for all v such that $v \in \mathcal{S}(w, s)$, by the definition of truth, $M, u \models \varphi \to \psi$ and $M, u \models \varphi$, and so $M, u \models \psi$. Thus, $M, u \models \psi$, for all u such that $u \in \mathcal{S}(w, t \times s)$, and so $M, w \models [t \times s]\psi$.

IJ2: $\Vdash [t]\varphi \vee [s]\varphi \to [t+s]\varphi$

Suppose either $M, w \models [t]\varphi$ or $M, w \models [s]\varphi$. Then, either $M, v \models \varphi$, for all v such that $v \in \mathcal{S}(w, t)$, or $M, v \models \varphi$, for all v such that $v \in \mathcal{S}(w, s)$. In both cases, $M, v \models \varphi$, for all v such that $v \in \mathcal{S}(w, t + s)$, since $\mathcal{S}(w, t + s) \subseteq \mathcal{S}(w, t) \cap \mathcal{S}(w, s)$, by condition *S2*, and so $M, v \models \varphi$, for all v such that $v \in \mathcal{S}(w, t + s)$. Hence, $M, w \models [t + s]\varphi$.

IJ3: $\Vdash [t]\varphi \to [!t][t]\varphi$

Suppose $M, w \models [t]\varphi$, so that $M, v \models \varphi$, for all v such that $v \in \mathcal{S}(w, t)$, and $u \in \mathcal{S}(v, t)$. Then, $\mathcal{S}(u, t) \subseteq \mathcal{S}(v, t)$ and $\mathcal{S}(v, t) \subseteq \mathcal{S}(w, t)$, by condition *S5*, and so $\mathcal{S}(u, t) \subseteq \mathcal{S}(w, t)$. Hence, $M, u \models \varphi$, for all u such that $u \in \mathcal{S}(v, t)$, and so $M, v \models [t]\varphi$. Since this is so for all v such that $v \in \mathcal{S}(w, t)$, and $\mathcal{S}(w, !t) = \mathcal{S}(w, t)$ by condition *S4*, $M, w \models [!t][t]\varphi$.

IJ4: $\Vdash t : \varphi \to [t]\varphi$

Straightforward, by the definition of $M, w \models t : \varphi$ and $M, w \models [t]\varphi$.

IJ5: $\Vdash [c]\varphi \to [t]\varphi$, where c is a constant

Straightforward, by the definition of $M, w \models [t]\varphi$ and condition *S3*.

IJ6: $\Vdash [t \times t]\varphi \leftrightarrow [t + t]\varphi \leftrightarrow [!t]\varphi \leftrightarrow [t]\varphi$

Straightforward, by the definition of $M, w \models [t]\varphi$ and condition *S4*.

Thus we obtain the following

[10] The proof is a straightforward adaptation of the proof proposed in [9], section 3.

THEOREM 1. ***IEJ** is sound with respect to the class of all basic frames for **IEJ**. (relative to a specific constant specification)*

The proof of the completeness theorem is more involved. As usual, the proof is based on a canonicity argument.[11] Therefore, let us start by defining the canonical model for **IEJ**. Let $w/[t] = \{\varphi \mid [t]\varphi \in w\}$ and $w/t = \{\varphi \mid t : \varphi \in w\}$, for all terms t. Then, the canonical model is the tuple $M = \langle W, \mathcal{S}, \mathcal{E}, V\rangle$, where

- W is the set of maximally **IEJ**-consistent sets of formulas
- $\mathcal{S}$ is such that $v \in \mathcal{S}(w,t) \Leftrightarrow w/[t] \subseteq v$
- $\mathcal{E}$ is such that $\mathcal{E}(w,t) = w/t$

Corollary 1: $v \in \mathcal{S}(w,t) \cap \mathcal{S}(w,s) \Leftrightarrow w/[t] \cup w/[s] \subseteq v$.
Straightforward:
$v \in \mathcal{S}(w,t) \cap \mathcal{S}(w,s) \Leftrightarrow v \in \mathcal{S}(w,t)$ and $v \in \mathcal{S}(w,s)$
$v \in \mathcal{S}(w,t) \cap \mathcal{S}(w,s) \Leftrightarrow w/[t] \subseteq v$ and $w/[s] \subseteq v$
$v \in \mathcal{S}(w,t) \cap \mathcal{S}(w,s) \Leftrightarrow w/[t] \cup w/[s] \subseteq v$

LEMMA 1: M is a model for **IEJ**.
We have to show that the conditions on $\mathcal{S}$ and $\mathcal{E}$ are satisfied.

Part 1: the conditions on $\mathcal{S}$ are satisfied.

- *S1*: $\mathcal{S}(w, t \times s) \subseteq \mathcal{S}(w,t) \cap \mathcal{S}(w,s)$

Suppose $v \in \mathcal{S}(w, t \times s)$, so that $w/[t \times s] \subseteq v$, by the definition of $\mathcal{S}$. Since w is maximal, $[t]\varphi \in w \Rightarrow [t \times s]\varphi \in w$, by **IJ8**, and $[s]\varphi \in w \Rightarrow [t \times s]\varphi \in w$, by **IJ7**. Thus, $\varphi \in w/[t] \Rightarrow \varphi \in w/[t \times s]$ and $\varphi \in w/[s] \Rightarrow \varphi \in w/[t \times s]$. Therefore, $w/[t] \cup w/[s] \subseteq w/[t \times s]$, and so $w/[t] \cup w/[s] \subseteq v$. Hence, $S(w, t \times s) \subseteq S(w,t) \cap S(w,s)$, by corollary 1.

- *S2*: $\mathcal{S}(w, t + s) \subseteq \mathcal{S}(w,t) \cap \mathcal{S}(w,s)$.

Suppose $v \in \mathcal{S}(w, t + s)$, so that $w/[t + s] \subseteq v$, by the definition of $\mathcal{S}$. Since w is maximal, $[t]\varphi \vee [s]\varphi \in w \Rightarrow [t+s]\varphi \in w$, by **IJ2**, and so $[t]\varphi \in w \Rightarrow [t+s]\varphi \in w$ and $[s]\varphi \in w \Rightarrow [t+s]\varphi \in w$. Thus, $\varphi \in w/[t] \Rightarrow \varphi \in w/[t+s]$ and $\varphi \in w/[s] \Rightarrow \varphi \in w/[t+s]$. Therefore, $w/[t] \cup w/[s] \subseteq w/[t+s]$, and so $w/[t] \cup w/[s] \subseteq v$. Hence, $S(w, t + s) \subseteq S(w,t) \cap S(w,s)$, by corollary 1.

- *S3*: $\mathcal{S}(w,t) \subseteq \mathcal{S}(w,c)$.

[11] See [6], chapter 4, for an introduction to modal completeness and, in particular, completeness by canonicity. In what follows I will omit the standard parts and definitions, and focus on the new parts of the proofs.

It is to prove that $w/[c] \subseteq v/[t]$, which follows from **IJ5**.

- *S4*: $\mathcal{S}(w, t \times t) = \mathcal{S}(w, t+t) = \mathcal{S}(w, !t) = \mathcal{S}(w, t)$.

It is to prove that $w/[t \times t] = w/[t+t] = w/[!t] = w/[t]$, which follows from **IJ6**.

- *S5*: $v \in \mathcal{S}(w, t) \Rightarrow \mathcal{S}(v, t) \subseteq \mathcal{S}(w, t)$.

Since $w/[!t] = w/[t]$, by **IJ6**, it suffices to prove that, if $w/[!t] \subseteq v$, then $w/[t] \subseteq v/[t]$. Suppose $w/[!t] \subseteq v$ and $\varphi \in w/[t]$. Then, $[t]\varphi \in w$, so that $[!t][t]\varphi \in w$, by **IJ3** and $w \in W$. Therefore, $[t]\varphi \in v$, and so $w/[t] \subseteq v/[t]$.

Part 2: the conditions on $\mathcal{E}$ are satisfied.

The proof of conditions *E1*, *E2*, and *E3* is well-known.[12] We only check *E4*.

- *E4*: $v \in \mathcal{S}(w, t) \Rightarrow \mathcal{E}(w, t) \subseteq \mathcal{E}(v, t)$.

Suppose $v \in \mathcal{S}(w, t)$, so that $w/[t] \subseteq v$, by the definition of $\mathcal{S}$. Since w is maximal, $t : \varphi \in w \Rightarrow !t : (t : \varphi) \in w$, by **EJ3**. By **IJ4**, $!t : (t : \varphi) \in w \Rightarrow [!t](t : \varphi) \in w$. By **IJ6**, $[!t](t : \varphi) \in w \Rightarrow [t](t : \varphi) \in w$. Hence, $t : \varphi \in w \Rightarrow [t](t : \varphi) \in w$, and so $t : \varphi \in w \Rightarrow t : \varphi \in w/[t] \subseteq v$. Therefore, $w/t \subseteq v/t$, from which the conclusion follows.

LEMMA 2 (Truth Lemma): $M, w \models \varphi \Leftrightarrow \varphi \in w$.

The interesting cases are the modal ones.

1. $M, w \models [t]\varphi \Leftrightarrow [t]\varphi \in w$.

$M, w \models [t]\varphi \Leftrightarrow M, w \models \varphi$, for all v such that $v \in \mathcal{S}(w, t)$

$M, w \models [t]\varphi \Leftrightarrow \varphi \in w$, for all v such that $w/[t] \subseteq v$, by I.H.

$M, w \models [t]\varphi \Leftrightarrow \varphi \in w/[t]$, since $w/[t]$ is a closed set

$M, w \models [t]\varphi \Leftrightarrow [t]\varphi \in w$, by the definition of $w/[t]$

2. $M, w \models t : \varphi \Leftrightarrow t : \varphi \in w$.

Suppose $M, w \models t : \varphi$. Then $\varphi \in \mathcal{E}(w, t)$, by the definition of truth. Thus, $t : \varphi \in w$, by the definition of $\mathcal{E}$. Suppose now $t : \varphi \in w$. Then $[t]\varphi \in w$, by **IJ4**. Thus, $M, w \models [t]\varphi$, by I.H., and $\varphi \in \mathcal{E}(w, t)$, by the definition of $\mathcal{E}$.

This concludes the proof. We then obtain the following

THEOREM 2. ***IEJ** is complete with respect to the class of all basic frames for **IEJ**. (relative to a specific constant specification)*

[12] See, for instance, [9], section 8.

3.3 Developments

In this paper, I have presented a complete basic system of logic of implicit and and explicit justification. This work can be extended in at least three different directions. A first possibility is to introduce a hierarchy of systems of increasing power based on **IEJ**. In effect, it is not difficult to see that systems dealing with consistent and correct justifiers can be obtained by introducing axioms like

EJD: $t : \varphi \to \neg(t : \neg\varphi)$ **EJT**: $t : \varphi \to \varphi$
IJD: $[t]\varphi \to \neg[t]\neg\varphi$ **IJT**: $[t]\varphi \to \varphi$

and modifying the conditions on $\mathcal{S}$ so to account for their validity. Along similar lines, more powerful systems might be developed. A second possibility is to make the system dynamic, by looking at the connections with recent intuitions proposed in [13] and [14]. The idea in this case is to interpret $t : \varphi$ as saying that a proof t of φ has been announced, i.e. discovered and published, and to adapt the semantics of the logic of announcement to the present framework. A final possibility is to connect the idea of implicit justification involved in modalities like $[t]$ with the more usual idea of implicit knowledge provided in [5], and to look for an integrated system, where notions like conclusive evidence and default evidence are also accounted for.

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